Time Series Analysis

Basics of Time Series Analysis

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Linear Processes & Prediction

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About This Lesson

Linear Process: Definition

A time series $\{X_t\}$ is a *linear process* if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\,,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

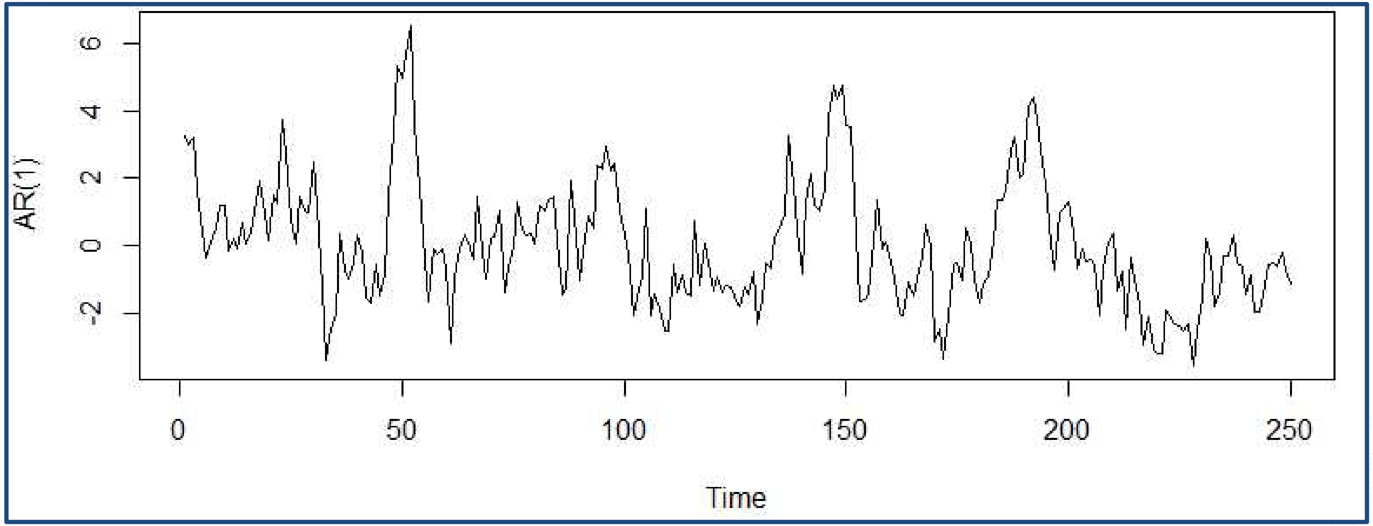
<u>Special case</u>: if $\psi_j = 0$ for all $j < 0$ then $\{X_t\}$ is called a *moving average* or $\mathrm{MA}(\infty)$ process.

Linear Process: Example

An $\mathrm{AR}(1)$ process is defined to be the stationary solution of

$$X_t - \phi X_{t-1} = Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, $|\phi| < 1$ and Z_t is uncorrelated with X_s for all $s < t$.



Linear Processes: Autocovariance

For $\{X_t\}$ a *linear process*, if $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty$, X_t is *stationary* with mean 0 and *autocovariance function*

$$\gamma_Y(h) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\psi_j\psi_k\gamma_Y(h+k-j)\,.$$

In the case where $\{X_t\} \sim \mathrm{WN}(0,\sigma^2)$,

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty}\psi_j\psi_{j+h}\sigma^2$$

Prediction for a Stationary Time Series

Data: $\{X_t\}$ a stationary time series

Objective: Predict X_{n+h} in terms of values $\{X_1, \ldots, X_n\}$.

- The *best linear predictor* of X_{n+h} in terms of $X_1, \ldots, X_n$ is defined to be the linear combination

$$P_n X_{n+h} = a_0 + a_1 X_n + a_2 X_{n-1} + \ldots + a_n X_1$$

which minimizes the mean squared error (MSE)

$$S(a_0, a_1, \ldots, a_n) = \mathrm{E}[(X_{n+h} - a_0 - a_1 X_n - \ldots - a_n X_1)^2]$$

Best Linear Predictor

Data: $\{X_t\}$ a stationary time series with $\mathrm{E}(X_t) = \mu$.

- The *best linear predictor* of X_{n+h} in terms of $X_1, \ldots, X_n$ is

$$P_n X_{n+h} = \mu + \sum_{i=1}^{n} a_i (X_{n+1-i} - \mu),$$

where $\boxed{\alpha_n = (a_1, \ldots, a_n)^T}$ satisfies $\boxed{\Gamma_n \alpha_n = \gamma_n(h) \text{ with}}$

$\boldsymbol{\Gamma_n} = [\gamma_X(i-j)]_{i,j=1,\ldots,n}$ and

$\boldsymbol{\gamma_n}(\boldsymbol{h}) = \left(\gamma_X(h), \gamma_X(h+1), \ldots, \gamma_X(h+n-1)\right)^T$.

- The *mean square prediction error* is

$$\mathrm{E}[(X_{n+h} - P_n X_{n+h})^2] = \gamma_X(0) - \alpha_n^T \gamma_n(h).$$

Best Linear Predictor (cont'd)

If Γ_n is non-singular, then

$$P_n X_{n+1} = \alpha_n^T (X_n, X_{n-1}, \ldots, X_1)^T, \text{ where } \alpha_n = \Gamma_n^{-1} \gamma_n(1).$$

The corresponding mean-squared prediction error is

$$v_n = \gamma_X(0) - \alpha_n^T \gamma_n(1).$$

Problem: When n is large, Γ_n^{-1} can be a nuisance to compute.

The Durbin-Levinson Algorithm

Let $\alpha_n = (a_{n1}, \ldots, a_{nn})$ be the solution to $\alpha_n = \Gamma_n^{-1} \gamma_n(1)$. Then the coefficients $a_{n1}, \ldots, a_{nn}$ can be computed recursively by

$$a_{nn} = \frac{1}{v_{n-1}} \left[\gamma_X(n) - \sum_{j=1}^{n-1} a_{n-1,j} \gamma_X(n-j) \right],$$

$$\begin{bmatrix} a_{n,1} \\ \vdots \\ a_{n,n-1} \end{bmatrix} = \begin{bmatrix} a_{n-1,1} \\ \vdots \\ a_{n-1,n-1} \end{bmatrix} - a_{nn} \begin{bmatrix} a_{n-1,n-1} \\ \vdots \\ a_{n-1,1} \end{bmatrix},$$

and

$$v_n = v_{n-1}(1 - a_{nn}^2),$$

where $a_{11} = \gamma_X(1)/\gamma_X(0)$ and $v_0 = \gamma_X(0)$.

The Innovations Algorithm

If $\{X_t\}$ has zero mean and $\mathrm{E}\left[X_i X_j\right] = \kappa(i,j)$, where the matrix $\mathrm{K} = [\kappa(i,j)]_{i,j=1}^{n}$ is non-singular for each $n = 1, 2, \dots$, then the one-step predictors $P_n X_{n+1}$ and their mean squared errors v_n are recursively given by

$$P_n X_{n+1} = \begin{cases} 0, & n = 0 \\ \sum_{j=1}^{n} \boxed{\theta_{nj}} \boxed{\left(X_{n+1-j} - P_{n-j} X_{n+1-j}\right),} & n \geq 1 \end{cases}$$

$$v_0 = \kappa(1,1)$$

$$\theta_{n,n-k} = v_k^{-1} \left(\kappa(n+1, k+1) - \sum_{j=0}^{k-1} \theta_{k,k-j} \theta_{n,n-j} v_j \right), \qquad k = 0, 1, \dots, n-1,$$

$$v_n = \kappa(n+1, n+1) - \sum_{j=0}^{n-1} \theta_{n,n-j}^2 v_j \, .$$

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Summary