Time Series Analysis

Basics of Time Series Analysis

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The Concept of Stationarity

About This Lesson

Time Series: Basics

Data: Y_t, where t indexes time, e.g. minute, hour, day, month

Model: Y_t = m_t+s_t+X_t

- m_t is a trend component;
- s_t is a seasonality component with known periodicity d ($s_t = s_{t+d}$) such that $\sum_{j=1}^{d} s_j = 0$
- X_t is a stationary component, i.e. its probability distribution does not change when shifted in time

Approach: m_t and s_t are first estimated and subtracted from Y_t to have left the stationary process X_t to be model using time series modeling approaches.

Time Series: Stationarity

The *auto-covariance* of a time series $\{X_t, t \in \mathbb{Z}\}$:

$$\gamma_X(r, s) = \mathrm{E}[(X_r - \mathrm{E}[X_r]) \cdot (X_s - \mathrm{E}[X_s])].$$

$\{X_t\}$ is *(weakly) stationary* if:

1. $\mathrm{E}[X_t] = m$ for all $t \in \mathbb{Z}$
2. $\mathrm{E}\left[{X_t}^2\right] < \infty$ for all $t \in \mathbb{Z}$, and
3. $\gamma_X(r, s) = \gamma_X(r + t, s + t)$ for all $\mathrm{r}, \mathrm{s}, t \in \mathbb{Z}$.

How realistic is the assumption of stationarity?

- Only when the system is tightly controlled; systems tend to drift away from stationarity.
- Changes (1st order difference) or changes of changes (2nd order difference) of the time series may instead behave stationary

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Examples of Stationary Time Series

1. If $\{X_t\}$ is a sequence of random variables with

$$\gamma_X(r,s) = \begin{cases} \sigma^2, & r = s \\ 0, & otherwise \end{cases}$$

with $\sigma^2 < \infty$ and $\mathrm{E}[X_t] = 0$, then $\{X_t\}$ is called *white noise* and we write

$$X_t \sim \text{WN}(0, \sigma^2).$$

2. IID noise with finite second moment:

If $\{X_t\}$ is a sequence of independent identically distributed random variables with mean zero and second moment equal to $\sigma^2 < \infty$, we write

$$X_t \sim \text{IID}(0, \sigma^2).$$

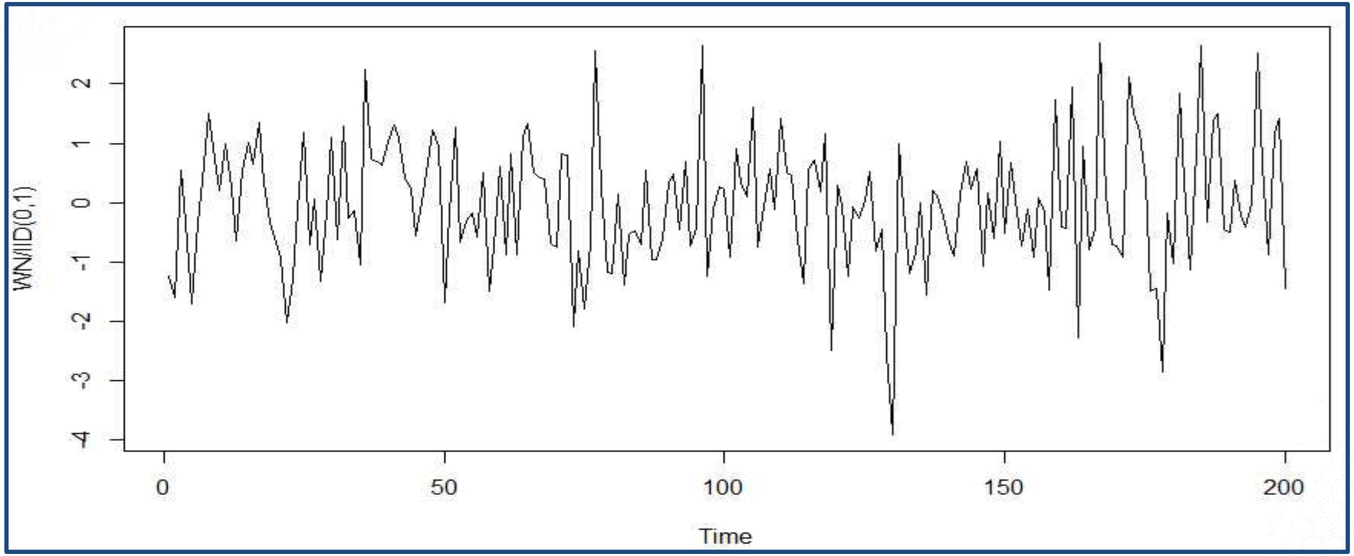
Examples of Stationary Time Series (cont'd)

What is the difference between $WN(0, \sigma^2)$ ***and*** $IID(0, \sigma^2)$***?***

- Uncorrelated versus Independent

How to generate in the R statistical software?

- Use rnorm, rpois, r…

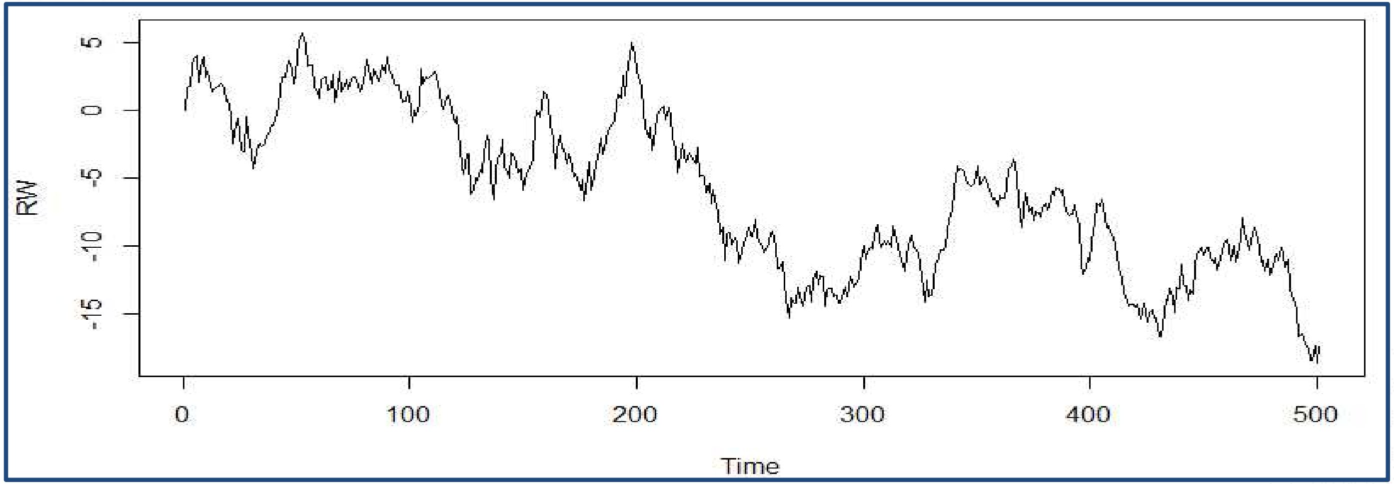


Example of Non-Stationary Time Series

Random walk: Suppose $X_t \sim \mathrm{IID}(0, \sigma^2)$ $S_t = \sum_{j=1}^{t} X_j$.

$\{S_t, t = 1, 2, \ldots\}$ is called a *random walk*.

$\rightarrow$ Not stationary because $V(S_t) = t\,\sigma^2$



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Autocovariance Function

For a stationary time series $\{X_t\}$, the *autocovariance* function is $\gamma_X(h) = \mathrm{Cov}(X_{t+h}, X_t)$ with the following properties:

1. $\gamma(0) \geq 0$,
2. $|\gamma(h)| \leq \gamma(0)$, and
3. $\gamma(h) = \gamma(-h)$.

The *autocorrelation* function of a stationary time series $\{X_t\}$ is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} ,$$

and has all the properties of the autocovariance function, except that $\rho_X(0) = 1$.

Autocovariance Function: Estimation

Objective: Given $\{x_1, \ldots, x_n\}$ observations of a stationary time series $\{X_t\}$, estimate the autocovariance $\gamma_X(\cdot)$ of $\{X_t\}$

- The sample *autocovariance function* is

$$\hat{\gamma}_X(h) = \frac{1}{n}\sum_{j=1}^{n-h}(x_{j+h} - \bar{x})(x_j - \bar{x}), \qquad 0 \le h < n,$$

Why n and not n-h?

with $\hat{\gamma}_X(h) = \hat{\gamma}_X(-h), \;\; -n < h \le 0$, where $\bar{x} = \frac{1}{\mathrm{n}}\sum_{j=1}^{n} x_j$.

- The *sample autocorrelation function* is defined by

$$\hat{\rho}_X(h) = \frac{\hat{\gamma}_X(h)}{\hat{\gamma}_X(0)}, \qquad |h| < n.$$

Don't forget about the uncertainty!

Autocovariance & Stationarity

Do we need stationarity?

- Autocovariance applies generally to all time series, including nonstationary processes: $\mathrm{Cov}(X_{t+h}, X_t)$ can depend on time and lag

- The autocovariance function applies only to stationary processes

- Autocovariance can be estimated using the sample autocovariance function for nonstationary processes: $\hat{\gamma}_X(h)$ can be used to evaluate (non)stationarity

Summary