Time Series Analysis

ARMA Models

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Basic Concepts

About This Lesson

Time Series: Definition

A *stochastic process* is a collection of random variables $\{X_t, t \in T\}$, defined on a probability space (Ω, F, P).

A *time series* is a stochastic process in which T is a set of time points, usually $T = \{0, \pm 1, \pm 2, \dots\}, \{1, 2, 3, \dots\}, [0, \infty),$ or $(-\infty, \infty)$

Note: The term “time series” is also used to refer to the realization of such a process (observed time series).

Time Series: Stationarity

The *auto-covariance function* of a time series $\{X_t, t \in \mathbb{Z}\}$:

$$\gamma_X(r,s) = \mathrm{E}\left[\left(X_\gamma - \mathrm{E}\left[X_\gamma\right]\right) \cdot (X_s - \mathrm{E}[X_s])\right].$$

$\{X_t\}$ is *(weakly) stationary* if:

1. $\mathrm{E}\left[\mathrm{X_t}^2\right] < \infty$ for all $\mathrm{t} \in \mathbb{Z}$,
2. $\mathrm{E}[\mathrm{X_t}] = \mu$ for all $\mathrm{t} \in \mathbb{Z}$, and
3. $\gamma_\mathrm{X}(\mathrm{r},\mathrm{s}) = \gamma_\mathrm{X}(\mathrm{r}+\mathrm{t},\mathrm{s}+\mathrm{t})$ for all $\mathrm{r},\mathrm{s},\mathrm{t} \in \mathbb{Z}$.

ARMA Model: Definition

The process $\{X_t, t \in \mathbb{Z}\}$ is said to be an $\mathrm{ARM}A(p,q)$ process if $\{X_t\}$ is stationary and if for every t,

$$\boxed{X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p}} \rightarrow \boxed{\text{Auto Regression}}$$

$$\boxed{= Z_t - \theta_1 Z_{t-1} - \ldots - \theta_q Z_{t-q}} \rightarrow \boxed{\text{Moving Average}}$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

- AR order p and MA order q
- $\{X_t\}$ is said to be an $\mathrm{ARM}A(p,q)$ process with mean μ if $\{X_t - \mu\}$ is an $\mathrm{ARM}A(p,q)$ process.

ARMA Model: Notation

Write the ARMA model in the more compact form

$$\boxed{\phi(B)}X_t = \boxed{\theta(B)}Z_t,$$

where

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

and

$$\theta(z) = 1 - \theta_1 z - \ldots - \theta_q z^q$$

The polynomials are called the autoregressive and moving average polynomials, respectively.

ARMA Model: General

The class of “autoregressive moving average” or ARMA models forms an important family of stationary time series, for a number of reasons, including the following two.

- For *any* autocovariance function $\gamma(\cdot)$ such that $\boxed{\lim_{h\to\infty} \gamma(h) = 0,}$ and any integer $k > 0$, it is possible to find an ARMA process with autocovariance $\gamma_X(\cdot)$ such that $\gamma_X(h) = \gamma(h)$ for $h = 0, 1, 2, \ldots, k$.
- The linear structure of ARMA models makes prediction “easy” to carry out.

Summary

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