Time Series Analysis

Modeling Heteroskedasticity

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ARCH Model

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About This Lesson

Modeling Heteroskedasticity

In a generic ARMA model, the model reduces to $\phi(B)Y_t = \theta(B)Z_t$ with

$$\mathrm{E}[Z_t] = 0, \qquad \mathrm{E}[Z_t^2] = \sigma^2, \qquad \mathrm{E}[Z_t Z_r] = 0, \qquad \text{for } r \neq t.$$

- Under the ARMA modeling the unconditional variance of Z_t is assumed constant. However, in most real data studies, the conditional variance of Z_t could change with time.
- We can rewrite the residual process characteristics as

$$Z_t = \sigma_t R_t \text{ with } \mathrm{E}[R_t] = 0,\ \mathrm{E}[R_t^2] = 1$$

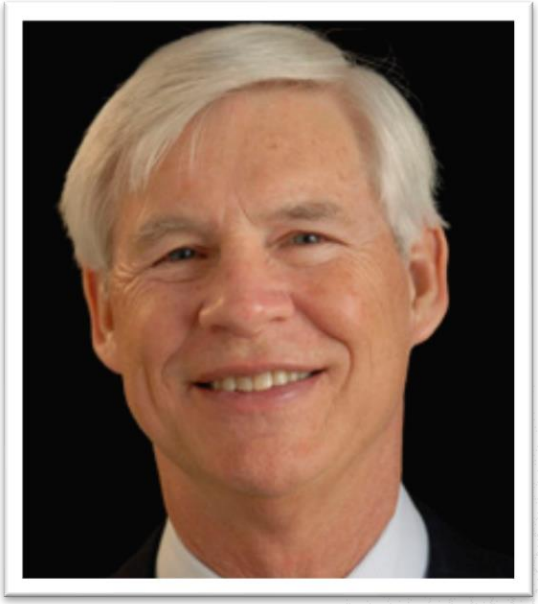
where $\sigma_t = \sigma_{t|t-1,t-2,\dots,1}$ can be a deterministic or random function

R_t is a sequence of iid random variables.

Z_t is white noise hence Y_t is weakly stationary

Modeling Volatility: Simple but Important

- In 1982, Robert Engle developed the **autoregressive conditional heteroskedasticity (ARCH)** models to model the time-varying volatility often observed in economical time series data. For this contribution, he won the **2003 Nobel Prize in Economics** (*Clive Granger shared the prize for cointegration)
- ARCH models assume the variance of the current residual term or innovation to be a function of the actual sizes of the previous time periods' residual terms: often the variance is related to the squares of the previous innovations.



Robert F. Engle (born 1942) currently teaches at NYU.

The ARCH Family of Models

The *Autoregressive Conditional Heteroskedasticity (ARCH)* model:

If σ_t is a linear function with lagged values of the mean equation residuals, then the time-series dynamic of volatility is like an AR process.

AR process

$$Z_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \ldots + \omega_t$$

$$\mathrm{E}[\omega_t] = 0, \qquad \mathrm{E}[\omega_t^2] = \lambda^2, \qquad \mathrm{E}[\omega_t \omega_r] = 0, \qquad \text{for } r \neq t.$$

Conditional on the past history $F_{t-1} = \{Z_{t-1}, Z_{t-2}, \ldots\}$ we have

$$\mathrm{E}[Z_t^2 | F_{t-1}] = \sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \gamma_2 Z_{t-2}^2 + \ldots$$

Reference: Robert Engle. “Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation.” *Econometrica*, 50, pages 987:1007 (July 1982)

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ARCH vs AR Model

For example, we consider the $ARCH(1)$ model:

$$Y_t = \mu + Z_t, \qquad Z_t|F_{t-1} \sim N(0, \sigma_t^2) \text{ and } \sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2$$

σ_t is a deterministic estimate of volatility. Since we do not observe volatility with certainty, the true volatility of Z_t is defined as:

$$Z_t^2 = \sigma_t^2 + \omega_t \ (Z_t = \sigma_t R_t) \text{ with } \mathrm{E}[\omega_t] = 0 \text{ and}$$
$$\mathrm{E}[\omega_t^2] = 1 \ (\mathrm{E}[R_t] = 0 \text{ and } \mathrm{E}[R_t^2] = 1)$$

Then the above equation for σ_t can be re-expressed as:

$$Z_t^2 - \omega_t = \gamma_0 + \gamma_1 Z_{t-1}^2$$
$$(1 - \lambda_1 B) Z_t^2 = \gamma_0 + \omega_t$$

This is an AR(1) in Z_t^2 where ω_t is the error between the true conditional variance σ_t^2 and actual volatility Z_t^2.

ARCH Model: Interpretation

Consider the simple equation of ARCH(1):

$$Z_t^2 = \sigma_t^2 + \omega_t \ (Z_t = \sigma_t R_t) \text{ with } \mathrm{E}[\omega_t] = 0 \text{ and}$$
$$\mathrm{E}[\omega_t^2] = 1 \ (\mathrm{E}[R_t] = 0 \text{ and } \mathrm{E}[R_t^2] = 1)$$

The propagation effect:

- If Z_{t-1} has an unusually large absolute value, then σ_t larger than usual;
- When Z_t has a large deviation that makes σ_t^2 large, so that Z_{t+1} tends to be large, and so on
- Volatility in Z_t tends to persist throughout for a long time

Stationarity of ARCH(1)

The stationarity conditions are:

- $\mathrm{E}[Y_t] = \mu$

- $\mathrm{E}[Y_t^2] = \mathrm{E}[Z_t^2]$ computed as below

  $(1 - \gamma_1)\mathrm{E}[Z_t^2] = \gamma_0$

  $\mathrm{E}[Z_t^2] = \gamma_0/(1 - \gamma_1)$

- $Cov(Y_t, Y_{t-1}) = \mathrm{E}[Z_t Z_{t-1}] = 0$

The weak stationarity condition is that $\gamma_1 < 1$.

Kurtosis under ARCH(1)

The fourth moment of Z_t imposes an additional constraint on γ_1

$$\mathrm{E}[Z_t^4] = \frac{3\gamma_0(1+\gamma_1)}{(1-\gamma_1)(1-3\gamma_1^2)},$$

which requires that $1 - 3\gamma_1^2 > 0$.

The unconditional kurtosis of Z_t is defined and is derived in Tsay (2005):

$$\frac{\mathrm{E}[Z_t^4]}{[Var(Z_t)]^4} = 3\left(\frac{1-\gamma_1^2}{1-3\gamma_1^2}\right) > 3$$

This means that an ARCH(1) model for Z_t has ‘fat’ tails and thus it is more likely to produce outliers than simple (normal) white noise (kurtosis = 3).

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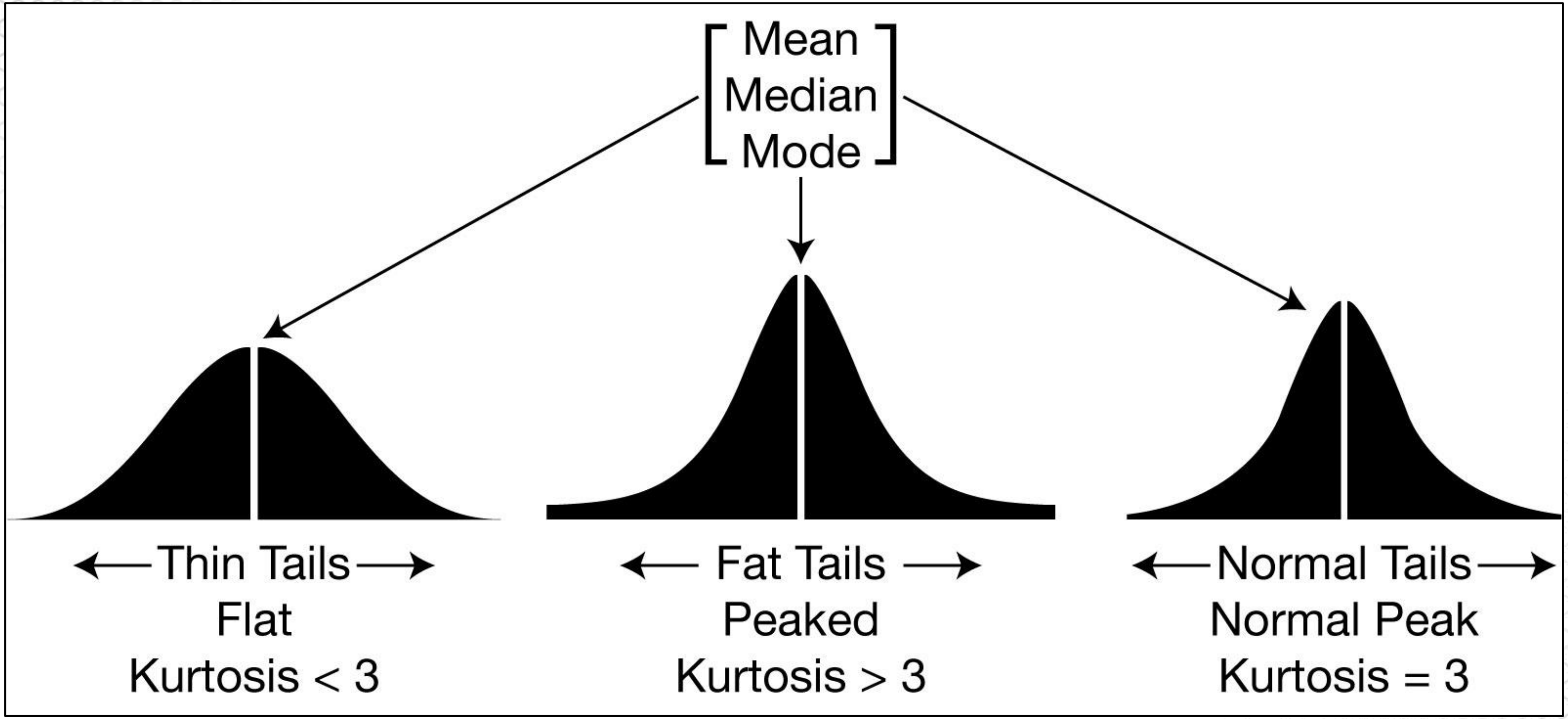
Kurtosis under ARCH(1)



Examples of Joint ARCH models

AR-ARCH models: for example, AR(1)-ARCH(1) is defined by the set of equations below

$$Y_t = \mu + \phi Y_{t-1} + Z_t$$
$$\sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2$$

ARMA-ARCH models: for example, ARMA(1,1)-ARCH(2) is defined by the set of equations below

$$Y_t = \mu + \phi Y_{t-1} + Z_t + \theta Z_{t-1}$$
$$\sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \gamma_2 Z_{t-2}^2$$

Estimation of ARCH Models

The parameter vector for an ARCH(p) model is $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_p\}$.

The *unconditional likelihood* for an ARCH of order p is:

$$f(y_T, \dots, y_{p+1}, y_p, \dots;\ \Gamma) = f(y_T, \dots, y_{p+1} | y_p, \dots;\ \Gamma)\, f(y_p, \dots;\ \Gamma)$$

where the last term is the joint distribution of the first p RV's in the time series. Except for the small order ARCH models, this last term is difficult to express, and therefore, ignored if p is much smaller than T.

The *conditional likelihood* for an ARCH of order p is:

$$f(y_T, \dots, y_{p+1} | y_p, \dots;\ \Gamma) = \prod_{t=p+1}^{T} f(y_t | y_{t-1}, \dots;\ \Gamma)$$

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Summary