Time Series Analysis

ARMA Models

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Other ARMA Models

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About This Lesson

ARMA Model

$\{X_t, t \in \mathbb{Z}\}$ an $\mathrm{ARM}A(p, q)$ process,

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

- Model Parameters: ϕ_1, .., ϕ_p (AR coefficients), $\theta_1, \ldots, \theta_q$ (MA coefficients), μ (mean) and σ^2 (variance) unknown
- Commonly, de-mean the process X_t (hence X_t=0)
- Orders: p, q set fixed but they need to be selected

ARIMA Model

If d is a non-negative integer and $\{X_t\}$ is a process for which

$$(1 - B)^d X_t = Y_t$$

$$\phi(B)Y_t = \theta(B)Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2).$$

If Y_t is a causal ARMA(p, q) process, then $\{X_t\}$ is said to be an ARIMA(p, d, q) process. This definition implies that

$$\phi^*(B)X_t = \theta(B)Z_t, \qquad \phi^*(B) = \phi(B)(1 - B)^d.$$

ARIMA models with $d > 0$ usual exhibit slow decay in the autocorrelation function.

Forecasting ARIMA Model

Suppose $\{X_t\}$ is an ARIMA(p, d, q) process. For $d \geq 1$, $\mathrm{E}[X_t]$ and $\mathrm{E}[X_t X_{t+h}]$ are not uniquely determined in ARIMA.

- Data $\{x_{1-d}, x_{2-d}, \ldots, x_0, x_1, \ldots, x_n\}$. We want to find $P_n X_{n+h}$, the best linear predictor of X_{n+h} in terms of $\{1, X_{1-d}, \ldots, X_0, X_1, \ldots, X_n\}$.
- The best linear predictor $P_n X_{n+h}$ is given by

$$P_n X_{n+h} = P_n Y_{n+h} - \sum_{j=1}^{d} \binom{d}{j} (-1)^j P_n X_{n+h-j}$$

where $P_n Y_{n+h}$ is the B.L.P. of Y_{n+h} in terms of $\{1, Y_1, \ldots, Y_n\}$.

- This enables us to compute $P_n X_{n+j}$ recursively, for $j = 1, 2, \ldots$

Forecasting ARIMA Model (cont'd)

For large n, the mean squared error of the predictor $P_n X_{n+h}$ is

$$\sigma_n^2(h) = \mathrm{E}\left[\left(X_{n+h} - P_n X_{n+j}\right)^2\right] \simeq \sum_{j=0}^{h-1} \psi_j^2 \sigma^2 \,,$$

where

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi^*(z)}.$$

Seasonal ARMA Models

- Seasonal MA(Q) with period s:

$$X_t = Z_t + \Theta_1 Z_{t-s} + \Theta_2 Z_{t-2s} + \ldots + \Theta_Q Z_{t-Qs}$$

It is a special case of the non-seasonal MA of order $q = Qs$.

- Seasonal AR(P) with period s:

$$X_t = \Phi_1 X_{t-s} + \Phi_2 X_{t-2s} + \ldots + \Phi_P X_{t-Ps} + Z_t$$

It is a special case of the non-seasonal AR of order $p = Ps$.

Seasonal ARMA Models (cont'd)

<u>Seasonal ARMA$(p,q) \times (P,Q)$</u> with period <u>s</u>: A model with AR characteristic polynomial $\phi(z)\Phi(z)$ and MA characteristic polynomial $\theta(z)\Theta(z)$:

$$\phi(z)\Phi(z)X_t = \theta(z)\Theta(z)Z_t$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p$$
$$\Phi(z) = 1 - \Phi_1 z^2 - \Phi_2 z^{2s} - \ldots - \Phi_P z^{Ps}$$
$$\theta(z) = 1 - \theta_1 z - \theta_2 z^2 - \ldots - \theta_q z^q$$
$$\Theta(z) = 1 - \Theta_1 z^s - \Theta_2 z^{2s} - \ldots - \Theta_Q z^{Qs}$$

It is a special case of the non-seasonal ARMA of AR order $p + Ps$ and MA order $q + Qs$.

Seasonal ARIMA Models

<u>Example</u> Consider the simple model

$$Y_t = M_t + S_t + Z_t \text{ with } S_t = S_{t-s} + \epsilon_t \text{ and } M_t = M_{t-1} + \xi_t$$

where $\{Z_t\}$, $\{\epsilon_t\}$ and $\{\xi_t\}$ mutually independent white noise series.

- The difference process

$$(1 - B)(1 - B^s)Y_t = (\xi_t + \epsilon_t + Z_t) - (\epsilon_{t-1} + Z_{t-1}) - (\xi_{t-s} + Z_{t-s}) + Z_{t-s-1}$$

is stationary with autocorrelation at lags $0, 1, s$ and $s + 1$ which agrees with the seasonal model ARMA$(0, 1) \times (0, 1)$ with seasonal period s.

<u>General Model</u> An ARMA$(p, d, q) \times (P, D, Q)$ is Y_t such that

$$W_t = (1 - B)^d (1 - B^s)^D Y_t$$

is and ARMA $(p, q) \times (P, Q)$ seasonal model with period s.

Summary