Module Demos.6
Root analysis of an ARMA process.

Conditions:

1. An ARMA(p,q) process is stationary iff the roots of the characteristic AR polynomial $\phi(z)$ do not lie on the unit circle in the complex plane, or that $|z|\neq 1$.
2. An ARMA(p,q) process is causal iff the roots of the characteristic AR polynomial $\phi(z)$ lie outside the unit circle in the complex plane, or that $|z|>1$.
3. An ARMA(p,q) process is invertible iff the roots of the characteristic MA polynomial $\theta(z)$ lie outside the unit circle in the complex plane, or that $|z|>1$.

Important considerations summary:

1. We must check that $\phi(z)$ and $\theta(z)$ polynomials have no common factors.
2. The MA(q) part is irrelevant for determining stationarity, so the MA(q) part can be ignored.
3. MA(q) process is causal and stationary by default.
4. AR(p) process is invertible by default.
5. Some of these roots may be complex numbers. A complex number, a+bi is outside the unit circle if its magnitude is greater than 1, i.e., $\sqrt{a^2+b^2}>1$. A point is inside, on, or outside the unit circle, if its magnitude is <1, =1, or >1 respectively.

Consider that we the following AR process of order p:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

Using the backshift operator, we can re write the equation as:

$$\left(1-\phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right) X_t = Z_t$$

Thus, the AR polynomial is:

$$\phi(B) = \left(1-\phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right)$$

In the case of a MA process of order q:

$$X_t = Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2} - \cdots - \theta_q Z_{t-q}$$

Using the backshift operator, we can re write the equation as:

$$X_t = Z_t(1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)$$

Thus, the MA polynomial is:

$$\theta(B) = (1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)$$

In the case of an ARMA process, we can represent as:

$$\phi(B) X_t = \theta(B) Z_t$$

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t = \psi(B) Z_t$$

From here, we want to check that there are not common factors between the AR and MA polynomial equations. This will ensure correct representation of time dependency, avoiding parameter redundancy.

For example, consider an ARMA(2,2):

$$X_t = X_{t-1} + 0.15X_{t-2} - Z_{t-1} - 0.15Z_{t-2} + Z_t$$
$$(1 - B - 0.15B^2)X_t = Z_t(1 - B + 0.15B^2)$$

Thus:

The AR polynomial: $\phi(B) = (1 - B - 0.15B^2)$

And the MA polynomial: $\theta(B) = (1 - B - 0.15B^2)$

Thus:

$$X_t = \frac{(1 - B - 0.15B^2)}{(1 - B - 0.15B^2)} Z_t = X_t = Z_t$$

This implies that the coefficients are not identified. In other words, the model is incorrectly specified.

Why we check for causality?

This property refers to the fact the time series depends on lagged values, and it has a finite value, assuring a decaying autocovariance.

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t = \psi(B)Z_t = \sum_{j=0}^{\infty} \psi_j \, Z_{t-1}$$

Thus causality implies: $\sum_{j=0}^{\infty} \psi_j < \infty$, and this is accomplished when the roots of the characteristic AR lie outside the unit circle.

Why we check for invertibility?

As for causality, we still need the infinite sum to be truncated to get a unique solution, $\sum_{j=0}^{\infty} \psi_j < \infty$, and this is accomplished when the roots of the characteristic MA polynomial lie outside the unit circle.

For example, consider the following ARMA (1,1):

$$X_t = 1.5X_{t-1} + 0.2Z_{t-1} + Z_t$$
$$(1 - 1.5B)X_t = (1 + 0.2B)Z_t$$

AR characteristic polynomial:

$$\phi(z) = 1 - 1.5z$$
$$z = 2/3$$

MA characteristic polynomial:

$$\theta(z) = 1 + 0.2z$$
$$z = 5$$

From the results we can say that:

- There are no common Factors.
- The AR root does not lie on the unit circle ($|z| = \left|\frac{2}{3}\right| \neq 1$) thus the series is stationary.
- The AR root is inside the unit circle ($|z| = \left|\frac{2}{3}\right| < 1$) thus the series is not causal.
- The MA root is outside the unit circle ($|z| = |5| > 1$) thus the series is invertible.