Time Series Analysis

ARMA Models

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Parameter Estimation

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About This Lesson

ARMA Model

$\{X_t, t \in \mathbb{Z}\}$ an ARM$A(p,q)$ process,

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = Z_t - \theta_1 Z_{t-1} - \ldots - \theta_q Z_{t-q}$$

where $Z_t \sim$ WN$(0, \sigma^2)$.

- Model Parameters: ϕ_1, .., ϕ_p (AR coefficients), $\theta_1, .., \theta_q$ (MA coefficients), μ (mean) and σ^2 (variance) unknown
- Commonly, de-mean the process X_t (hence μ =0)
- Orders: p, q set fixed but they need to be selected

AR Model: Linear Regression

Let $\{X_t\}$ be an AR(p) process satisfying

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t, \quad Z_t \sim \text{IID}(0, \sigma^2)$$

- Predicting variables: X_{t-1},.., X_{t-p}
- Regression coefficients: ϕ_1,.., ϕ_p
- Model selection: AR order p

➡ **Statistical inference: Normality assumption**

AR Model: The Yule-Walker Equations

Let $\{X_t\}$ be a causal AR(p) process satisfying

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = Z_t, \qquad Z_t \sim \text{WN}(0, \sigma^2)$$

Multiply both sides by $X_{t-j}, j = 0, 1, \ldots, p$ and take expectations to get the *Yule-Walker equations*

$$\gamma(0) - \phi_1 \gamma(1) - \ldots - \phi_p \gamma(p) = \sigma^2$$
$$\gamma(1) - \phi_1 \gamma(0) - \ldots - \phi_p \gamma(p-1) = 0$$
$$\ldots$$
$$\gamma(p) - \phi_1 \gamma(p-1) - \ldots - \phi_p \gamma(0) = 0$$

The Yule-Walker Equations (cont'd)

Alternatively, these can be written as

$$\boxed{\Gamma_p \bar{\phi} = \gamma_p(1)} \text{ and } \quad \sigma^2 = \gamma(0) - \bar{\phi}^T \gamma_p(1),$$

where

$$\Gamma_p = [\gamma(i-j)]_{i,j=1,\ldots,p} \qquad \bar{\phi} = \left(\phi_1, \ldots, \phi_p\right)^T$$

and $\gamma_p(1) = \left(\gamma(1), \ldots, \gamma(p)\right)^T$.

The Yule-Walker Equations: Estimation

Replacing $\boldsymbol{\gamma(h)}$ by $\boldsymbol{\hat{\gamma}(h)}$ gives equations for the *Yule-Walker estimators* $\hat{\bar{\phi}}_p = (\hat{\phi}_{p1}, \ldots, \hat{\phi}_{pp})^T$ of $\bar{\phi}$ and $\hat{v}_p$ of σ^2

$$\hat{\Gamma}_p \hat{\bar{\phi}}_p = \hat{\gamma}_p(1)$$

$$\hat{v}_p = \hat{\gamma}(0) - \sum_{j=1}^{p} \hat{\bar{\phi}}_{pj} \hat{\gamma}(j) \,.$$

Invert $\hat{\Gamma}_p$: $\hat{\bar{\phi}}_p = \hat{\Gamma}_p^{-1} \hat{\gamma}_p(1)$
Alternatively, use Durbin-Levinson Recursive Algorithm (computationally efficient)

The fitted model is

$$X_t - \hat{\phi}_{p1} X_{t-1} - \ldots - \hat{\phi}_{pp} X_{t-p} = Z_t, \qquad Z_t \sim \mathrm{WN}(0, \hat{v}_p).$$

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The Yule-Walker Estimates: Properties

$\{X_t\}$ be a causal AR(p) process:

- $\hat{\bar{\phi}}_p \sim AN(\bar{\phi}, \frac{\sigma^2}{n} \hat{\Gamma}_p{}^{-1})$
- $\hat{\sigma}^2$ consistent estimator for σ^2
- $\hat{\phi}_{hh} \sim AN(0, \frac{1}{n})$ for h>p

➡ We can use the sample PACF to test for AR order, and we can calculate approximate confidence intervals for the parameters

MA Model: The Innovations Algorithm

Let $\{X_t\}$ be an MA(q) process satisfying

$$X_t = Z_t - \theta_1 Z_{t-1} - \ldots - \theta_q Z_{t-q}, \qquad Z_t \sim \mathrm{WN}(0, \sigma^2)$$

Suppose that $\hat{\gamma}(0) > 0$. Then parameter estimates can be obtained recursively by $\hat{v}_0 = \hat{\gamma}(0)$,

$$\hat{\theta}_{m,m-k} = \hat{v}_k^{-1}\left[\hat{\gamma}(m-k) - \sum_{j=0}^{k-1} \hat{\theta}_{m,m-j}\hat{\theta}_{k,k-j}\hat{v}_j\right], \qquad k = 0, \ldots, m-1$$

and

$$\hat{v}_m = \hat{\gamma}(0) - \sum_{j=0}^{m-1} \hat{\theta}_{m,m-j}^2 \hat{v}_j$$

The Innovations Algorithm: Properties

Let $\{X_t\}$ be an MA(q) process:

- Asymptotic behavior of $\hat{\hat{\theta}}_m$ is a little complicated;
- Loosely speaking, for fixed k,

$$\hat{\theta}_{mk} \to \psi_k \text{ (consistency)}$$

as m becomes large, where ψ_k is the term appearing in the causal representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \,.$$

ARMA Model: The Innovations Algorithm

$\{X_t\}$ be a zero-mean causal ARMA(p,q) process:

Let $\hat{\theta}_{mj},\ j = 1, \ldots, p+q$ be the estimates obtained using the innovations algorithm. Then estimates can be obtained by solving

$$\begin{bmatrix} \hat{\theta}_{m,q+1} \\ \hat{\theta}_{m,q+2} \\ \vdots \\ \hat{\theta}_{m,q+p} \end{bmatrix} = \begin{bmatrix} \hat{\theta}_{m,q} & \hat{\theta}_{m,q-1} & \ldots\hat{\theta}_{m,q+p-1} \\ \hat{\theta}_{m,q+1} & \hat{\theta}_{m,q} & \cdots\hat{\theta}_{m,q+p-2} \\ & & \\ \hat{\theta}_{m,q+p-1} & \hat{\theta}_{m,q+p-2} & \cdots \quad \hat{\theta}_{m,q} \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_p \end{bmatrix}$$

and

$$\hat{\theta}_j = \hat{\theta}_{mj} - \sum_{i=1}^{\min(j,p)} \hat{\phi}_i \hat{\theta}_{m,j-i}, \qquad j = 1, 2, \ldots, q.$$

Summary