Time Series Analysis

ARMA Models

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Autocovariance and Partial Autocorrelation Function

About This Lesson

ARMA Model: Notation

We will often write (4) in the more compact form

$$\phi(B)X_t = \theta(B)Z_t,$$

where

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

and

$$\theta(z) = 1 - \theta_1 z - \ldots - \theta_q z^q$$

The polynomials are called the autoregressive and moving average polynomials, respectively.

ARMA Model: Autocovariance Function

Assuming that $\{X_t\}$ is causal, it has a representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \,,$$

We then have

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} \,.$$

But to use this formula, we first need to find the coefficients $\{\psi_j, j = 0, 1, 2, \dots\}$.

Autocovariance Function: Derivation

The coefficients of $\{X_t\}$ causal: $\psi(z)\phi(z) = \theta(z)$

Expanding out the polynomials on both sides and equating coefficients of z^m, we get the system of equations in ψ_m

$$\psi_m - \sum_{0<k\leq m} \phi_k \psi_{m-k} = \theta_m, \qquad m \leq \max(p-1, q)$$

$$\psi_m - \sum_{0<k\leq p} \phi_k \psi_{m-k} = 0, \qquad m > \max(p-1, q)$$

where we define $\theta_m = 1$ and adopt the convention that $\phi_k = 0$ for $k > p$ and $\phi_k = 0$ for $k > q$.

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Autocovariance Function: Estimation

Objective: Given $\{x_1, \ldots, x_n\}$ observations of a stationary time series $\{X_t\}$, estimate the autocovariance function $\gamma_X(\cdot)$ of $\{X_t\}$

- The sample *autocovariance function* is

$$\hat{\gamma}_X(h) = \frac{1}{n}\sum_{j=1}^{n-h}\left(x_{j+h} - \bar{x}\right)\left(x_j - \bar{x}\right), \qquad 0 \le h < n,$$

with $\hat{\gamma}_X(h) = \hat{\gamma}_X(-h), \;\; -n < h \le 0$, where $\bar{x} = \frac{1}{\mathrm{n}}\sum_{j=1}^{n} x_j$.

- The *sample autocorrelation function* is defined by

$$\hat{\rho}_X(h) = \frac{\hat{\gamma}_X(h)}{\hat{\gamma}_X(0)}, \qquad |h| < n.$$

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Partial Autocorrelation Function

Suppose $\{X_t\}$ is a stationary time series with mean zero, for which $\gamma_X(h) \to 0$ as $h \to \infty$. The *partial autocorrelation function (PACF)* $\alpha_X(h)$ is defined by

$$\alpha_X(0) = 1,$$

$$\alpha_X(h) = \alpha_{hh},$$

where α_{hh} is the last component of

$$\alpha_h = \Gamma_h^{-1}\gamma_h(1),$$

with

$$\Gamma_h = [\gamma_X(i-j)]_{i,j=1,\ldots,h} \quad \text{and} \quad \gamma_h(1) = \left(\gamma_X(1), \gamma_X(2), \ldots, \gamma_X(h)\right)^T.$$

PACF and Prediction

$\{X_t\}$ is a stationary time series with mean zero, for which $\gamma_X(h) \to 0$ as $h \to \infty$:

$$P_h X_{h+1} = a_1 X_h + a_2 X_{h-1} + \ldots + \boxed{a_h} X_1$$

the *one-lag linear prediction* given $X_1, \ldots, X_h$. If $a_1, \ldots, a_h$ are selected such that we minimize

$$S(a_1, \ldots, a_h) = \mathrm{E}[(X_{h+1} - a_1 X_h - \ldots - a_h X_1)^2].$$

then $P_h X_{h+1}$ is called the *Best Linear Unbiased Predictor (BLUP)* for X_{h+1}. We define the <u>partial autocorrelation function</u> as

$$\alpha(h) = \boxed{a_h}$$

Sample PACF

The *sample partial autocorrelation function* $\hat{\alpha}_X(h)$ is defined by

$$\hat{\alpha}_X(h) = \text{the last compenent of } \hat{\Gamma}_h^{-1}\hat{\gamma}_h(1),$$

where $\hat{\Gamma}_h$ and $\hat{\gamma}_h(1)$ are obtained by replacing $\gamma_X(\cdot)$ with $\hat{\gamma}_X(\cdot)$ in the expression for Γ_h and $\gamma_h(1)$.

- The sample partial autocorrelation function $\hat{\alpha}(h)$ and the sample autocorrelation function $\hat{\rho}(h)$ are important in identifying a “good” model for a given realization of a time series.

ACF and MA(q) Process

Let $\{X_t\}$ be the stationary solution of $X_t = \theta(B)Z_t$, where $\theta(z) = 1 - \theta_1 z - \ldots - \theta_q z^q$ and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

We have

$$\gamma_X(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h},$$

where $\psi_0 = 1, \psi_1 = -\theta_1, \ldots, \psi_q = -\theta_q, \psi_{q+1} = 0, \ldots,$

It follows that $\gamma_X(h) = 0$ for $|h| > q$. (So $\rho_X(h) = 0$ for $|h| > q$).

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PACF and AR(p) Process

Now suppose that $\{X_t\}$ is the stationary solution of

$$\phi(B)X_t = Z_t,$$

where $\phi(z) = 1 - \phi_1 z - \ldots - \phi_q z^q$ and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

It can be shown that $\alpha_X(h) = 0$ for $|h| > p$.

MA(q) and AR(p) Processes

Summarizing:

An AR(p) process has PACF $\alpha(h) = 0$ for $|h| > p$.

An MA(q) process has ACF $\rho(h) = 0$ for $|h| > q$.

Unfortunately, there are no such simple rules for ARMA(p, q) processes in general.

Summary