Time Series Analysis

Modeling Heteroskedasticity

Nicoleta Serban, Ph.D.
Professor
Stewart School of Industrial and Systems Engineering

GARCH Model

About This Lesson

Limitations of ARCH models

- The model assumes that positive and negative shocks have the same effects on volatility because it depends on the square of the previous shocks. The ARCH model is rather restrictive.

- ARCH models are likely to overpredict the volatility because they respond slowly to large isolated shocks to the return series.

- The ARCH model does not provide any new insight for understanding the source of variations of a time series. It merely provides a mechanical way to describe the behavior of the conditional variance. It gives no indication about what causes such behavior to occur.

The GARCH Models

- Generalization of ARCH models

- General intuition: a weighted average of past squared residuals with declining weights which never go completely to zeros

- Conditional Variance ~ Weighted average of the long run average variance & most recent squared residuals

The GARCH Model Formulation

The simplest of the GARCH specification is that of a $GARCH(1,1)$ model. This is defined as:

$$Y_t = \mu + Z_t, \qquad Z_t | F_{t-1} \sim N(0, \sigma_t^2)$$
$$\sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

More generically, there are $GARCH(m,n)$ specifications where σ_t^2 is:

$$\sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \ldots + \gamma_m Z_{t-m}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_n \sigma_{t-n}^2$$

Reference: Timothy Bollerslev. “Generalized autoregressive conditional heteroskedasticity”. Journal of Econometrics, 31, pages 307:327 (1986).

GARCH vs ARMA

GARCH is an ARMA form of heteroscedasticity

$$\sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

However, this is only the deterministic part of volatility. Actual volatility Z_t^2 cannot be observed fully, and is therefore defined as:

$$Z_t^2 = \sigma_t^2 + \omega_t.$$

Then, the above equation in σ_t^2 becomes:

$$Z_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \beta_1 \boxed{(Z_{t-1}^2 - \omega_{t-1})} + \omega_t$$

$$= \gamma_0 + (\gamma_1 + \beta_1) Z_{t-1}^2 + \omega_t - \beta_1 \omega_{t-1}$$

$$\boxed{(1 - (\gamma_1 + \beta_1)B) Z_t^2 = \gamma_0 + (1 - \beta_1 B)\omega_t}$$

This is an $ARMA(1,1)$ model in the AR component of Z_t^2 and the MA component of ω_t.

Stationarity of GARCH(1,1)

The stationarity conditions are satisfied:

- $\mathrm{E}[Y_t] = \mu$

- $\mathbb{V}(Y_t) = \mathrm{E}[Z_t^2]$ computed as below

$$\mathrm{E}[Z_t^2] = \gamma_0 + (\gamma_1 + \beta_1)\mathrm{E}[Z_{t-1}^2] + \mathrm{E}[\omega_t - \beta_1 \omega_{t-1}]$$
$$\boxed{(1 - \gamma_1 - \beta_1)\mathrm{E}[Z_t^2] = \gamma_0}$$
$$\sigma^2 = \gamma_0/(1 - \gamma_1 - \beta_1)$$

The stationarity conditions are that $(\gamma_1 + \beta_1) < 1$ and $0 \leq \gamma_1, \beta_1 \leq 1$.

- $Cov(Y_t, Y_{t-1}) = \mathrm{E}[Z_t Z_{t-1}] = 0.$

Georgia Tech

Characteristics of GARCH(1,1)

1. Weak stationarity under $(\gamma_1 + \beta_1) < 1$ and $0 \leq \gamma_1, \beta_1 \leq 1$

2. Model volatility/variability clusters

3. It has heavy tails: if $1 - 2\beta_1^2 - (\gamma_1 + \beta_1)^2 > 0$ then the corresponding kurtosis is

$$\frac{\mathrm{E}[Z_t^4]}{\left[\mathrm{E}[Z_t^2]\right]^2} = \frac{3(1 - (\gamma_1 + \beta_1)^2)}{1 - 2\beta_1^2 - (\gamma_1 + \beta_1)^2} > 3$$

Examples of Joint GARCH models

- **AR**-GARCH models: for example, AR(1)-GARCH(1,1) is defined by the set of equations below

$$Y_t = \mu + \phi Y_{t-1} + Z_t$$
$$\sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

- **ARMA**-GARCH models: for example, ARMA(1,1)-GARCH(1,2) is defined by the set of equations below

$$Y_t = \mu + \phi Y_{t-1} + Z_t + \theta Z_{t-1}$$
$$\sigma_t^2 = \gamma_0 + \gamma_1 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2$$

MLE for GARCH models

The parameter vector for a GARCH(m, n) model is $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_m, \beta_1, \ldots, \beta_n\}$.
<u>Assumptions</u> Under normality assumptions, the conditional likelihood the GARCH model becomes:

$$f(y_t | y_{t-1}, \ldots ;\ \Gamma) = \frac{1}{\sqrt{2\pi\sigma_t^2}} exp\left[-\frac{1}{2}\left(\frac{y_t^2}{\sigma_t^2}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi(\gamma_0 + \gamma_1 y_{t-1}^2 + \gamma_2 y_{t-2}^2 + \ldots + \beta_1 \sigma_{t-1}^2 + \ldots)}}$$

$$exp\left[-\frac{1}{2}\left(\frac{y_t^2}{\gamma_0 + \gamma_1 y_{t-1}^2 + \gamma_2 y_{t-2}^2 + \ldots + \beta_1 \sigma_{t-1}^2 + \ldots}\right)\right]$$

and the unconditional distribution is a combination of the conditional density and the joint density of $Y_{\max(m,n)}, \ldots, Y_1$.

Summary