Time Series Analysis

ARMA Models

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Causal and Invertible Processes

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About This Lesson

ARMA Model: Notation

We will often write (4) in the more compact form

$$\phi(B)X_t = \theta(B)Z_t,$$

where

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

and

$$\theta(z) = 1 - \theta_1 z - \ldots - \theta_q z^q$$

The polynomials are called the *autoregressive* and *moving average polynomials*, respectively.

ARMA Model: Stationarity

Not all formulations $\phi(B)X_t = \theta(B)Z_t$ model a stationary time series:

A stationary solution to the ARMA equation exists and is unique if and only if:

$$\phi(z) \neq 0 \text{ for all } z \in \mathbb{C} \text{ such that } |z| = 1$$

(Stationarity condition: ARMA stationary exists and is unique if and only if no zeroes of $\phi(z)$ lie on the unit circle.)

Causal ARMA Process

An ARMA process $\{X_t\}$ is a *causal function* of Z_t if there exist constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty}|\psi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad \text{for all} \quad t \in \mathbb{Z}.$$

Suppose $\{X_t\}$ *is an ARMA process for which* $\phi(\cdot)$ *and* $\theta(\cdot)$ *have no common zeroes. Then* $\{X_t\}$ *is causal if and only if*

$$\phi(z) \neq 0 \quad \textit{for all} \quad z \in \mathbb{C} \quad \textit{such that} \quad |z| \leq 1.$$

The coefficients of the causal function are given by

$$\boxed{\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)},} \qquad |z| \leq 1.$$

Causal Process: Example

<u>The AR(1) process:</u> We have $X_t(1 - \phi B) = Z_t$

Thus $\phi(z) = 1 - \phi z$. This has only one zero at $z = 1/\phi$. A unique stationary solution to exists if and only if $|1/\phi| \neq 1$, or

$$|\phi| \neq 1.$$

$\{X_t\}$ is causal if and only if $|1/\phi| > 1$, or

$$|\phi| < 1.$$

AR(1) Causal Process

```
## Causal AR(1) processes (|phi|=0.1<1)
a1 = 0.1
ar1 = filter(w2,filter=a1,method='recursive')
ar1.1 = ar1[1251:1500]
a1 = - 0.1
ar1 = filter(w2,filter=a1,method='recursive')
ar1.2 = ar1[1251:1500]
## Causal AR(1) processes reaching non-causality/stationarity (|phi|=0.9)
a1 = 0.9
ar1 = filter(w2,filter=a1,method='recursive')
ar1.3 = ar1[1251:1500]
a1 = - 0.9
ar1 = filter(w2,filter=a1,method='recursive')
ar1.4=ar1[1251:1500]
```

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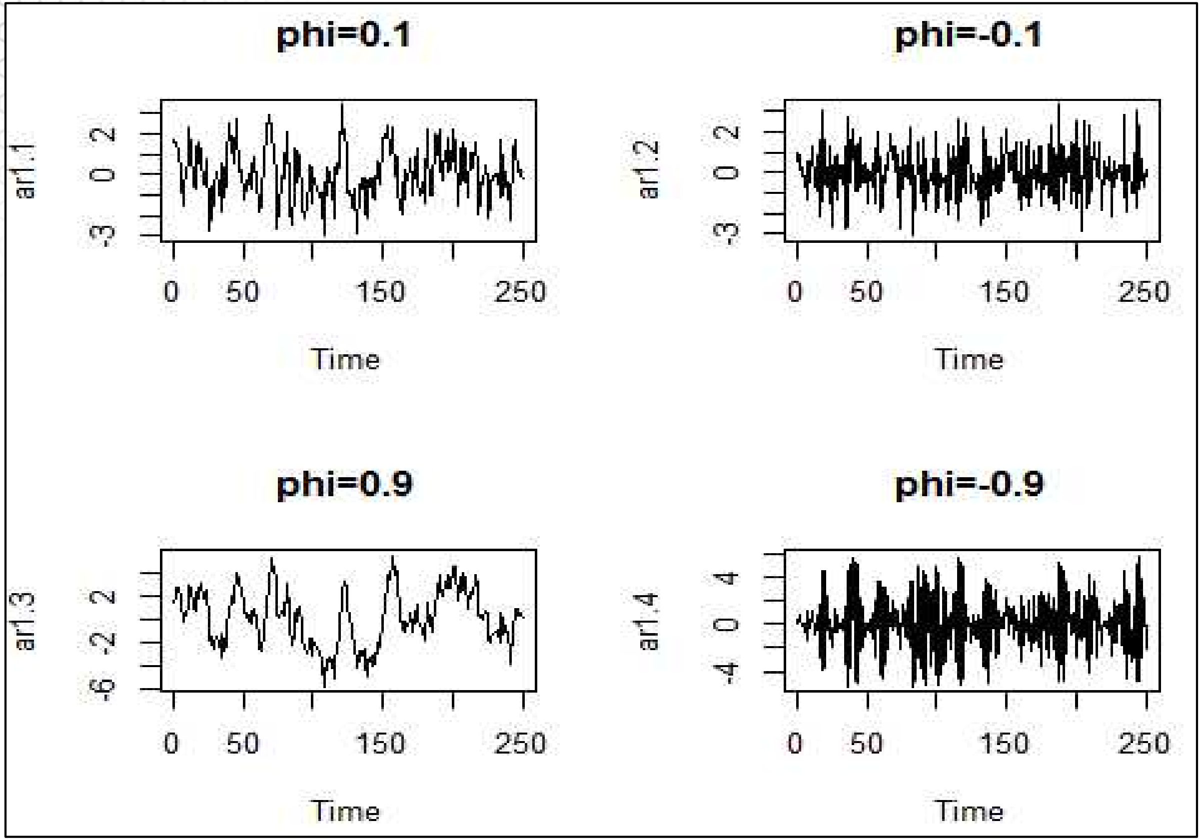
AR(1) Causal Process



Invertible Process

An ARMA process $\{X_t\}$ is *invertible* if there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty}|\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad \text{for all} \quad t \in \mathbb{Z}.$$

Suppose $\{X_t\}$ is an ARMA process for which $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeroes. Then $\{X_t\}$ is invertible if and only if

$$\theta(z) \neq 0 \quad \textit{for all} \quad z \in \mathbb{C} \quad \textit{such that} \quad |z| \leq 1.$$

The coefficients for an invertible process are given by

$$\boxed{\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)},} \qquad |z| \leq 1.$$

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ARMA Model: Properties

Given an ARMA process $\phi(B)X_t = \theta(B)Z_t$ for which $\phi(z) \neq 0 \;\; \forall \, |z| = 1$, it is possible to find $\tilde{\phi}(\cdot)$, $\tilde{\theta}(\cdot)$ and a white noise process $\{\tilde{Z}_t\}$ such that

$$\tilde{\phi}(B)X_t = \tilde{\theta}(B)\tilde{Z}_t$$

and $\{X_t\}$ is a causal function of $\{\tilde{Z}_t\}$.

If, in addition, $\theta(z) \neq 0 \;\; \forall \, |z| \leq 1$, then $\tilde{\theta}(\cdot)$ can be chosen so that $\{X_t\}$ is an invertible function of $\{\tilde{Z}_t\}$.

Summary