Physarum polycephalum e il problema del cammino minimo

(*Physarum polycephalum* and the shortest path problem)

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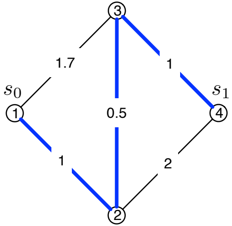
The Shortest Path problem

Given: a connected weighted network; source and sink nodes
Compute: the minimum-weight path connecting source to sink



Our working assumptions (can be relaxed):

- weights are strictly positive
- shortest path is unique

Combinatorial network optimization

Classic algorithms for network problems are combinatorial: manipulate discrete objects (nodes, edges, paths. . .)

Computational complexity expressed in terms of:

- n: number of nodes
- m: number of edges

Hybrid combinatorial-numerical methods

Since 2004, a new generation of fast algorithms is emerging:

Reduce network problems to solving equations of the form

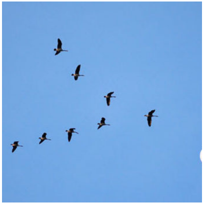
$$\boxed{Lx = b}$$

where $L \in \mathbb{R}^{n \times n}$ is a graph Laplacian matrix (see later)

"Laplacian paradigm": build around this algorithmic primitive

Distributed natural algorithms

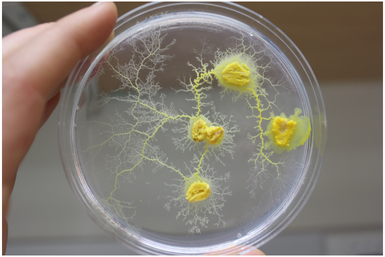
Biological system		Computational problem
Flashing of fireflies	$\Rightarrow$	Synchronization
Flocking of birds	$\Rightarrow$	Consensus
Fly morphogenesis	$\Rightarrow$	Maximal Independent Set
Slime mold foraging	$\Rightarrow$	Shortest Path

Introducing *Physarum polycephalum*

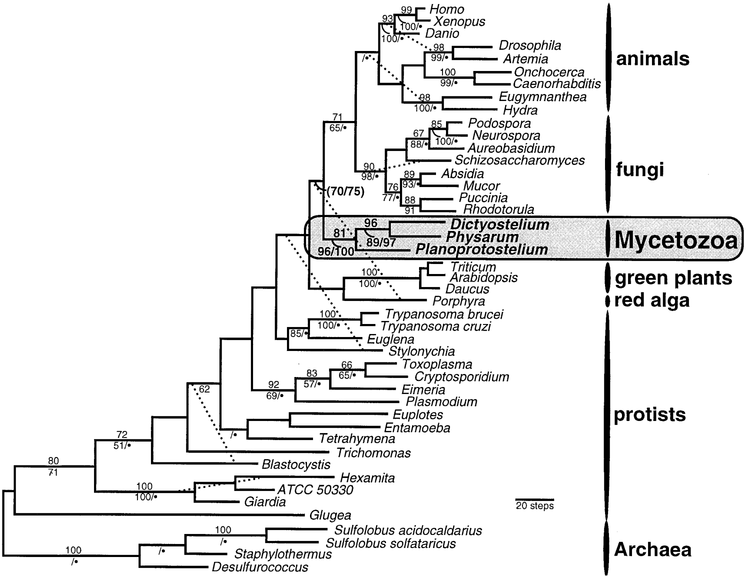
(video)

Introducing *Physarum polycephalum*

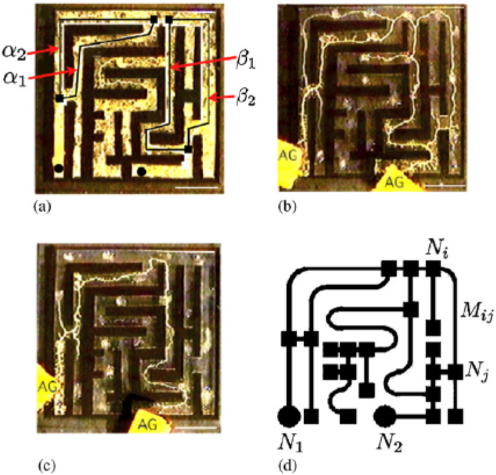


- *poly-cephalum*: "Many-headed"
- Giant cell, multiple nuclei
- Inhabits shady and moist areas, decaying logs, ...

Introducing *Physarum polycephalum*



Maze-solving by *P. polycephalum*



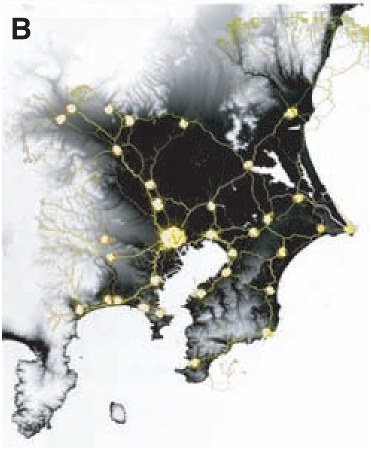
Nakagaki, Yamada, Tóth, *Nature* (2000)

Maze experiment

(video)

Network design by *P. polycephalum*

Tero et al., *Science* (2010)



Physarum network

Rail system around Tokyo

Physarum dynamics and optimization problems

Tero, Kobayashi and Nakagaki (2007) proposed a mathematical model for the network dynamics of the maze experiment

Variants of this model converge to optimal (!) solutions of classic optimization problems:

- Shortest path in undirected graphs
- ℓ_1-norm regression
- Shortest path in directed graphs
- Linear optimization

Valid flows

- Connected graph $G = (N, E)$
- $s_0, s_1 \in N$: source and sink
- Fix an arbitrary orientation $\vec{E}$ on the edges

A function $z : \vec{E} \to \mathbb{R}$ is a valid flow if:

- the net flow out of s_0 is $+1$
- the net flow out of s_1 is -1
- the net flow out of any other node is 0

where the net flow out of $v \in N$ is

$$\sum_{e=(u,v)\in\vec{E}} z_e - \sum_{e'=(v,u)\in\vec{E}} z_{e'} \ (= \text{outflow}(v) - \text{inflow}(v))$$

Incidence matrix and flow conservation

The signed incidence matrix $A \in \mathbb{R}^{N \times E}$ is:
$A_{ve} = +1$ if edge e starts in v, -1 if edge e ends at v, 0 otherwise

The balance vector $b \in \mathbb{R}^{N}$ is:
$b_v = +1$ if $v =$ source, -1 if $v =$ sink, 0 otherwise

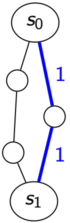
$$A = \begin{bmatrix} +1 & +1 & 0 & 0 & 0 \\ -1 & 0 & +1 & +1 & 0 \\ 0 & -1 & -1 & 0 & +1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} +1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\boxed{z \text{ is a valid flow} \Leftrightarrow A \cdot z = b}$$

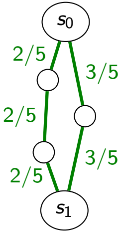
Shortest path flow vs. electrical flow

Among all valid flows z: (assuming all weights $= 1$)

- The shortest path minimizes $\|z\|_1 := |z_1| + |z_2| + \ldots + |z_m|$
- The electrical flow minimizes $\|z\|_2 := (z_1^2 + z_2^2 + \ldots + z_m^2)^{1/2}$



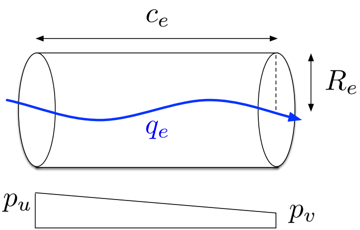
minimize $\|z\|_1$
such that $Az = b$



minimize $\|z\|_2$
such that $Az = b$

The Physarum flow is an electrical flow!
However it evolves as weights are adapted via feedback

Electric-hydraulic analogy



Poiseuille's law in fluidodynamics states that

$$q_e = \frac{\pi}{8\eta}\frac{R_e^4(p_u - p_v)}{c_e} = constant \cdot \frac{x_e(p_u - p_v)}{c_e}$$

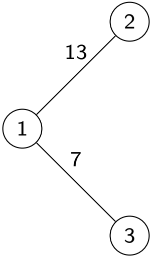
$$\boxed{\text{flow} = \frac{\text{potential difference}}{\text{resistance}}}$$

Analogous to Ohm's law

The Laplacian matrix

Denote by w_e the weight of an edge $e = (u, v)$

$$L_{u,v} := \begin{cases} \sum_{e \in \delta(u)} w_e & \text{if } u = v \\ -w_e & \text{if } (u, v) = e \in E \\ 0 & \text{otherwise.} \end{cases}$$



$$L = \begin{bmatrix} 20 & -13 & -7 \\ -13 & 13 & 0 \\ -7 & 0 & 7 \end{bmatrix}$$

Factorization of the Laplacian

$$\boxed{L = AWA^\top}$$

- A is the signed incidence matrix, for example:

$$A = \overbrace{\begin{bmatrix} +1 & +1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}}^{\text{edges}} \left.\vphantom{\begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix}}\right\} \text{nodes}$$

- W is the diagonal $m \times m$ weight matrix, for example:

$$W = \begin{bmatrix} 13 & 0 \\ 0 & 7 \end{bmatrix} \left.\vphantom{\begin{bmatrix} 13 \\ 0 \end{bmatrix}}\right\} \text{edges}$$

Weights in the Physarum network

The weight of an edge is the ratio $\frac{x_e}{c_e}$ $\left(= \frac{1}{\textbf{resistance}(e)}\right)$

$$W = \begin{bmatrix} \frac{x_1}{c_1} & \dots & 0 \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \frac{x_m}{c_m} \end{bmatrix}$$

The c_e are static, but the x_e will evolve with time

Recap of the static equations

A valid flow is a vector $z \in \mathbb{R}^E$ that satisfies flow conservation:

$$\boxed{A \cdot z = b} \qquad (A = \text{ incidence matrix})$$

The electrical flow $q \in \mathbb{R}^E$ is related to the potentials p by Ohm's law:

$$\boxed{q = W \cdot A^\top \cdot p}$$

The node potentials $p \in \mathbb{R}^N$ are the solutions to Poisson's equation:

$$\boxed{AWA^\top p = L \cdot p = b} \qquad \text{with } b_u = \begin{cases} +1 & \text{if } u = s_0 \\ -1 & \text{if } u = s_1 \\ 0 & \text{otherwise} \end{cases}$$

Energy dissipation

$$\boxed{\text{Energy dissipation } \mathcal{E} := \sum_{e \in E} \frac{\text{flow}(e)^2}{\text{weight}(e)}}$$

Observation. If our flow and weights encode a single path P,

$$\mathcal{E} = \sum_{e \in P} \frac{1^2}{1/c_e} = \text{cost of } P$$

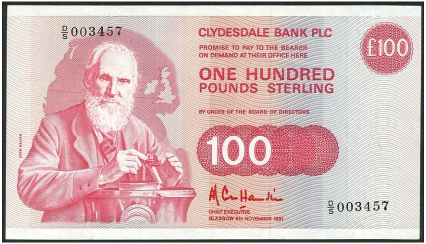
Two important principles of electrical networks

Thomson's principle

The electrical flow q is the unique minimizer of the energy dissipation among all valid flows.

Conservation of energy

The energy dissipation of q equals the potential difference between source and sink.



William Thomson (1824–1907)
aka Lord Kelvin

The weights' dynamics: TKN model

The tubes are elastic and respond to the flow

TKN postulate the flow q_e to be the "driving" variable:

$$\boxed{\frac{dx_e}{dt} = \varphi(|q_e(x)|) - x_e} \qquad \forall e \in E$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function

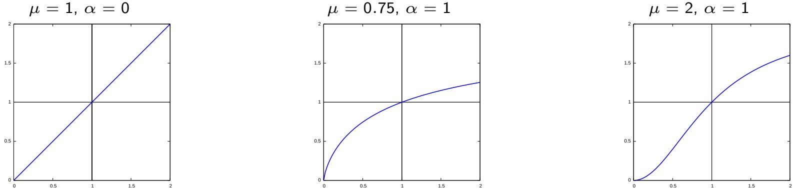
Interpretation as a positive feedback mechanism

- Large flow $\Rightarrow$ tube expands
- Small flow $\Rightarrow$ tube contracts
- Direction of flow does not matter

Forms of response functions

Proposed form for the edge response function φ:

$$\varphi(y) = \frac{(1+\alpha)y^{\mu}}{1+\alpha y^{\mu}}$$



There are no experimental data on what the true response actually looks like

What is the global behavior of the dynamics?

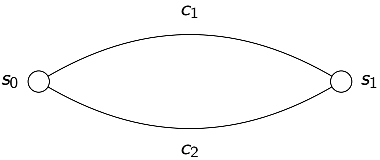
It is not a priori clear whether the local adaptation mechanism

$$\boxed{\dot{x}_e = \varphi(|q_e(x)|) - x_e}$$

gives rise to a globally meaningful behavior

Two parallel links example

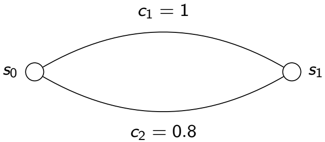
Assume response function $\varphi(y) = y \;\Rightarrow\; \dot{x}_e = |q_e(x)| - x_e$

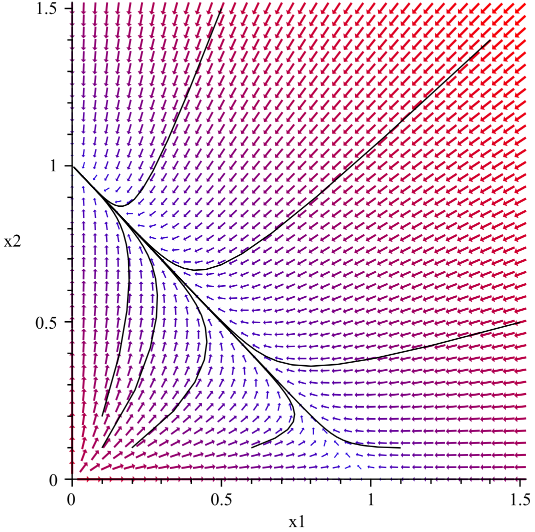


$$\dot{x}_1 = \frac{x_1/c_1}{x_1/c_1 + x_2/c_2} - x_1$$

$$\dot{x}_2 = \frac{x_2/c_2}{x_1/c_1 + x_2/c_2} - x_2.$$

State space for two parallel links





TKN dynamics when $\varphi(y) = y$

Consider

$$\boxed{\dot{x} = |q(x)| - x}$$

under some initial condition $x(0) > 0$

Theorem (B., Mehlhorn, Varma, 2012)

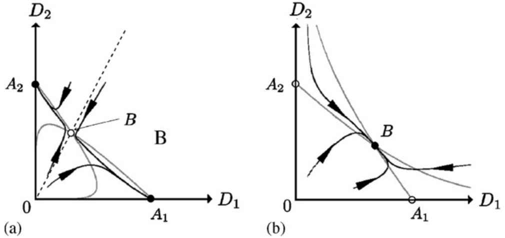
For any graph and any initial condition, as time $t \to \infty$,

$$x_e(t) \to 1 \text{ if } e \text{ belongs to the shortest } s_0 - s_1 \text{ path},$$
$$x_e(t) \to 0 \text{ otherwise.}$$

That is, the model computes the shortest path

TKN dynamics when $\varphi(y) = y^\mu$

$$\boxed{\dot{x} = |q(x)|^\mu - x}, \qquad \mu \neq 1$$



(a) If $\mu > 1$, both paths are stable; initial condition matters

(b) If $\mu < 1$, the stable fixpoint does not correspond to any path

The TKN model computes the shortest path only when $\mu = 1$!

A revised model

Let's postulate that the controlling variable is not the flow q_e, but the pressure gradient, $(p_u - p_v)/c_e$

By Poiseuille's law,

$$\frac{p_u - p_v}{c_e} \propto \frac{q_e}{x_e}$$

So we postulate

$$\boxed{\frac{dx_e}{dt} = x_e \left(\varphi \left(\left| \frac{q_e(x)}{x_e} \right| \right) - 1 \right)} \qquad \forall e \in E$$

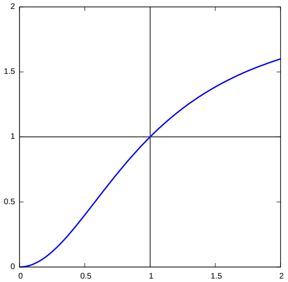
Note: When $\varphi(y) = y$, this model is equivalent to TKN

Robustness of the revised model

Claim: in the revised model, convergence to optimality is guaranteed for any "standard" response function on parallel links networks

A standard response function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:

1. $\varphi(1) = 1$
2. φ is strictly increasing
3. φ is smooth (differentiable)



These include $\varphi(y) = ((1 + \alpha)y^{\mu})/(1 + \alpha y^{\mu})$ and much more

Typical steps of the analysis

1. Characterization of the equilibrium points of the dynamics
2. The dynamics converge to some equilibrium point
3. Energy dissipation converges to the shortest path length

Points 2 and 3 require defining appropriate potential functions

Parallel-links networks

In such a network, the signed incidence matrix A is

$$A = \begin{bmatrix} +1 & +1 & \dots & +1 \\ -1 & -1 & \dots & -1 \end{bmatrix}$$

The Laplacian matrix is

$$L = AWA^\top = \left(\sum_{e \in E} \frac{x_e}{c_e} \right) \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The energy dissipation is

$$\mathcal{E} = \left(\sum_{e \in E} \frac{x_e}{c_e} \right)^{-1}$$

Location of fixed points

Since $\mathcal{E} = p_{s_0} - p_{s_1}$ (conservation of energy), by Ohm's law

$$\frac{q_e}{x_e} = \frac{\mathcal{E}}{c_e} \quad \forall e$$

A fixed point of our dynamics satisfies

$$x_e \cdot \left(\varphi\left(\frac{\mathcal{E}}{c_e} \right) - 1 \right) = 0 \quad \forall e$$

$\Rightarrow$ for each e, either $x_e = 0$ or $\mathcal{E} = c_e$
$\Rightarrow$ there can be at most one nonzero x_e (as costs are distinct)
$\Rightarrow$ exactly one $x_e = 1$

x is a fixed point $\Leftrightarrow$ x is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$

Nature of fixed points

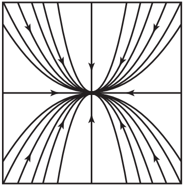
To study the stability of fixed points, use the Jacobian matrix

$$J(x)_{i,j} = \left(\frac{\partial \dot{x}_i}{\partial x_j} \right)$$

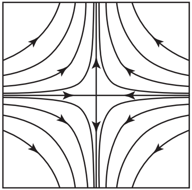
A fixed point x is

- stable if all eigenvalues of $J(x)$ have $Re(\lambda) < 0$
- unstable if some eigenvalues of $J(x)$ have $Re(\lambda) > 0$

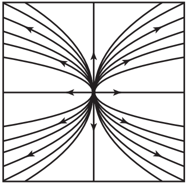
stable



unstable



unstable



Nature of fixed points

If we compute $J(\mathbf{e}_1)$ (equilibrium corresponding to edge 1), we get

$$J(\mathbf{e}_1) = \begin{pmatrix} -\varphi'(1) & -\frac{c_1}{c_2}\varphi'(1) & \dots & -\frac{c_1}{c_m}\varphi'(1) \\ 0 & \varphi\left(\frac{c_1}{c_2}\right) - 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \varphi\left(\frac{c_1}{c_m}\right) - 1 \end{pmatrix}$$

Observation: $\varphi(y) \geq 1 \Leftrightarrow y \geq 1$, since $\varphi(1) = 1$

Equilibrium $\mathbf{e}_1$ is stable $\Leftrightarrow c_1 < c_j$ for each $j \neq 1$
$\Leftrightarrow$ Edge 1 is the shortest edge

Nothing special about $\mathbf{e}_1$: the same holds for $\mathbf{e}_2, \dots, \mathbf{e}_m$

Convergence to an equilibrium

Does the process converge?
How do we rule out cycles in the dynamics?

Introduce a potential function to track progress

A Lyapunov function is a continuous function $V : \mathbb{R}_+^m \to \mathbb{R}$ such that

1. V is bounded from below
2. $\frac{d}{dt}V(x(t)) \leq 0$
3. $\frac{d}{dt}V(x(t)) = 0 \Leftrightarrow x$ is an equilibrium point

Then $\frac{d}{dt}V(x(t))$ must approach zero as $t \to \infty$
$\Rightarrow x$ must approach an equilibrium point

Lyapunov function for parallel links

$$V(x) := \sum_e x_e + \log \mathcal{E}(x) = \sum_e x_e - \log \left(\sum_e \frac{x_e}{c_e} \right)$$

Property 1: $V(x)$ is bounded from below

1. All $x_e \geq 0$, always
2. All $x_e \leq 2$, after some finite time: as long as $x_e > 2$,

$$\dot{x}_e < x_e \left(\varphi\left(\frac{|q_e|}{2} \right) - 1 \right) \leq x_e \left(\varphi\left(\frac{1}{2} \right) - 1 \right) < 0$$

   hence the dynamics are repelled from the region $x_e > 2$.
3. Since $x_e \in [0, 2]$, $V(x)$ cannot be less than some constant

Convergence to an equilibrium

Property 2

$$\begin{aligned}\dot{V}(x) &= \sum_{e\in E} \frac{\partial V}{\partial x_e} \dot{x}_e \\ &= \sum_{e\in E} \left(1 - \frac{\mathcal{E}}{c_e}\right) x_e \left(\varphi\left(\frac{\mathcal{E}}{c_e}\right) - 1\right) \\ &\leq 0\end{aligned}$$

Property 3

$\dot{V} = 0 \Leftrightarrow \forall e \ (x_e = 0 \text{ or } \mathcal{E} = c_e) \Leftrightarrow x$ is an equilibrium.

Hence V is a Lyapunov function

Convergence to the shortest-edge equilibrium

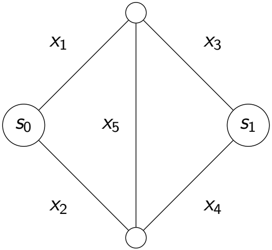
It suffices to show that $\mathcal{E} \to c_i$ where i is the shortest edge

Proof by contradiction: if $\mathcal{E}(t) \geq (1+\delta)c_i$ for all large t, then

$$\begin{aligned}\frac{d}{dt}\log x_i &= \frac{\dot{x}_i}{x_i} \\ &= \frac{x_i}{x_i}\left(\varphi\left(\frac{\mathcal{E}}{c_i}\right) - 1\right) \\ &\geq \varphi(1+\delta) - 1 \\ &> 0,\end{aligned}$$

so $x_i(t) \to \infty$, contradiction to $x_i(t) \leq 2$ for all large enough t. $\square$

A more complex example



$$
\begin{aligned}
\dot{x}_1 &= \frac{x_1}{K}(x_2x_3 + x_3x_4 + x_3x_5 + x_4x_5) - x_1 \\
\dot{x}_2 &= \frac{x_2}{K}(x_1x_4 + x_3x_4 + x_3x_5 + x_4x_5) - x_2 \\
\dot{x}_3 &= \frac{x_3}{K}(x_1x_4 + x_1x_2 + x_1x_5 + x_2x_5) - x_3 \\
\dot{x}_4 &= \frac{x_4}{K}(x_2x_3 + x_1x_2 + x_1x_5 + x_2x_5) - x_4 \\
\dot{x}_5 &= \left|\frac{x_5}{K}(x_1x_4 - x_2x_3)\right| - x_5,
\end{aligned}
$$

where
$K = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_4 + x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_5$
(Kirchhoff polynomial of the network)

What is known for general networks?

Analyzing the revised model on arbitrary topologies is difficult

Theoretical results are known for

- $\varphi(y) = y$ (see B., Mehlhorn & Varma 2012 or B. 2013)
- $\varphi(y) = y^2$ (B. 2019)

Simulations suggest convergence to optimum with any standard response function φ

Equilibrium points for general networks

On parallel links networks, we had a fixed point for each edge:

x is a fixed point $\Leftrightarrow$ x is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$

For general networks, fixed points correspond to source-sink paths

x is a fixed point $\Leftrightarrow$ x is the characteristic vector of a valid path

The case $\varphi(y) = y$

One Lyapunov function that works is

$$\boxed{V(x) := \frac{\sum_e c_e x_e}{cut_*(x)} + (cut_0(x) - 1)^2}$$

where:

- $cut_*(x)$ is the value of a minimum x-weighted cut
- $cut_0(x)$ is the value of the x-weighted cut around the source

V has the required properties

Important ingredients: Thomson's principle + Max-Flow Min-Cut

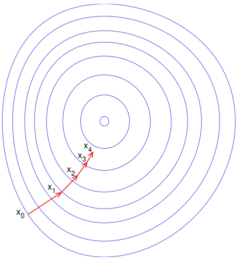
The case $\varphi(y) = y^2$

When $\varphi(y) = y^2$, the dynamics can be written in a very nice form

$$\dot{x}_e = -x_e \frac{\partial f(x)}{\partial x_e}, \quad e \in E$$

which is a rescaled form of gradient descent. Here,

$$f(x) := \sum_{e \in E} c_e x_e + \mathcal{E}(x)$$



This simplifies the analysis considerably since f is automatically a Lyapunov function for the dynamics

From continuous to discrete time

Euler discretization of the dynamics (with $\varphi(y) = y$):

$$\boxed{\frac{x^{k+1} - x^k}{\eta} = \left|q(x^k)\right| - x^k} \qquad k = 0, 1, 2, \ldots$$

η is the discretization step

Implementation:

1. Solve the Laplacian linear system $L(x^k)p = b$
2. Compute $q(x^k) = W(x^k)A^\top p$
3. Compute $x^{k+1} = (1 - \eta)x^k + \eta \left|q(x^k)\right|$

Discrete dynamics with $\varphi(y) = y$

Theorem (Becchetti, B. et al. '13; Straszak-Vishnoi '16)

There is an $x^0 \propto \mathbf{1}$ *such that after* k *steps, with*

$$k = \tilde{O}\left(\frac{n}{\epsilon^3} \cdot \frac{c_{max}}{c_{min}} \cdot\right), \epsilon > 0,$$

the Euler discretization satisfies

$$c^\top x^* \leq c^\top x^k \leq (1+\epsilon) c^\top x^*$$

where x^* *is the optimal solution.*

Is it really a distributed process?

The discrete process iterates this cycle:

1. Solve $L(x^k)p = b$
2. Compute $q(x^k) = W(x^k)A^\top p$
3. Compute $x^{k+1} = (1-\eta)x^k + \eta\left|q(x^k)\right|$

Steps 2 and 3 use only local information

But is there a local formulation of step 1?

Distributed Laplacian solving

Consider a connected network G with weights $x \in \mathbb{R}^E$

Can we solve $L(x)\, p = b$ through a decentralized process?

We consider two approaches:

1. Jacobi's method (deterministic)
2. Token diffusion (stochastic)

Jacobi's method

An iterative method:

$$p_u^{(k+1)} = \frac{b_u + \sum_{v \sim u} x_{uv} p_v^{(k)}}{\sum_{v \sim u} x_{uv}}, \qquad k = 0, 1, \ldots$$

Node u maintains information of $p_u^{(k)}$ and b_u

To update node u, need information only from the neighbors of u

Jacobi's method is convergent and

$$L \cdot p^{(k)} \to b \qquad \text{as } k \to \infty$$

But how fast in terms of the network parameters?

Convergence rate of Jacobi's method

Theorem (Becchetti, B., Natale 2018)

The error in Jacobi's method converges to zero at rate

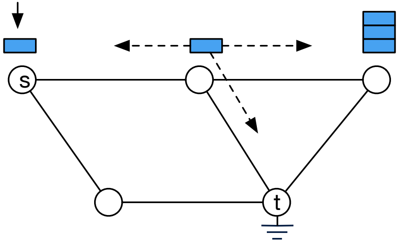
$$O(\max(|1-\lambda_2|^k, |1-\lambda_n|^k))$$

where $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2$ are the eigenvalues of the normalized Laplacian of the network.

However, Jacobi's method presupposes that nodes exchange real numbers – not so realistic in the slime mold context

A stochastic diffusion-based model

Instead of continuous flows, consider flow particles (tokens)



Repeat forever:

1. insert one new token at the source
2. each token moves from node u to neighbor v with probability proportional to the weight x_{uv}
3. remove all tokens at the sink, if any

Convergence of token diffusion

Theorem (Becchetti, B., Natale, 2018)

The number of tokens $Z_u^{(t)}$ on node u, as $t \to \infty$, satisfies

$$\mathbb{E}\left[Z_u^{(t)}\right] \to L_{u,u} p_u$$

The number of tokens at u can be used to estimate the local node potential p_u

Hence, token diffusion also converges to the correct values

The more "well-connected" the graph is, the faster the convergence

Fully local explanation: an open problem

The token diffusion process was analyzed for fixed weights

Does everything still work if we couple the diffusion dynamics with the edge-response dynamics, adapting the weights along the way?

Seems plausible, but no analysis available so far!

Beyond the network setting

The shortest path problem can be expressed as

$$\min \sum_{j=1}^{m} c_j |z_j|$$
$$\text{s.t. } Az = b,\ z \in \mathbb{R}^m$$

where A is the signed incidence matrix of the graph

What if A is not a network matrix?

The problem is called an ℓ_1-norm regression problem

An important problem in statistics and signal processing

Solving ℓ_1-regression

No network interpretation, but the math still works!
We just ask A to be a full-rank matrix (then $L = AWA^\top$ is invertible)

We still use the dynamics

$$\dot{x} = x\left(\left(\frac{|q(x)|}{x}\right)^{\mu} - 1\right)$$

where now $q(x) := W(x)A^\top L^{-1}(x)$

Theorem (Straszak-Vishnoi 2016; B. 2019)

When $\mu = 1$ or $\mu = 2$, the dynamics converges to an optimal solution of the ℓ_1-regression problem.

Convergence is often provably fast (see references)

Problems solved by Physarum dynamics

	Undirected dynamics $\dot{x} = \lvert q(x) \rvert - x$	**Directed dynamics** $\dot{x} = q(x) - x$
Network setting	Undirected shortest path	Directed shortest path
Algebraic setting	$\min c^\top \lvert z \rvert$ $Az = b$ ℓ_1-regression	$\min c^\top z$ $Az = b, z \geq 0$ Linear programming

Conclusions

- The TKN model achieves optimality only for the response function $\varphi(y) = y$
- A revised model achieves optimality for any set of standard response functions
  - Proved analytically for parallel-links graphs
  - Proved analytically for any graph if $\mu \in \{1, 2\}$
  - Supported by simulations for more general topologies
- Process can be truly distributed (complete analysis still missing)
- A simple generalization of the dynamics can solve the more general ℓ_1-regression problem or even linear programming

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