Dynamics and Community Structure in Networks

Emanuele Natale

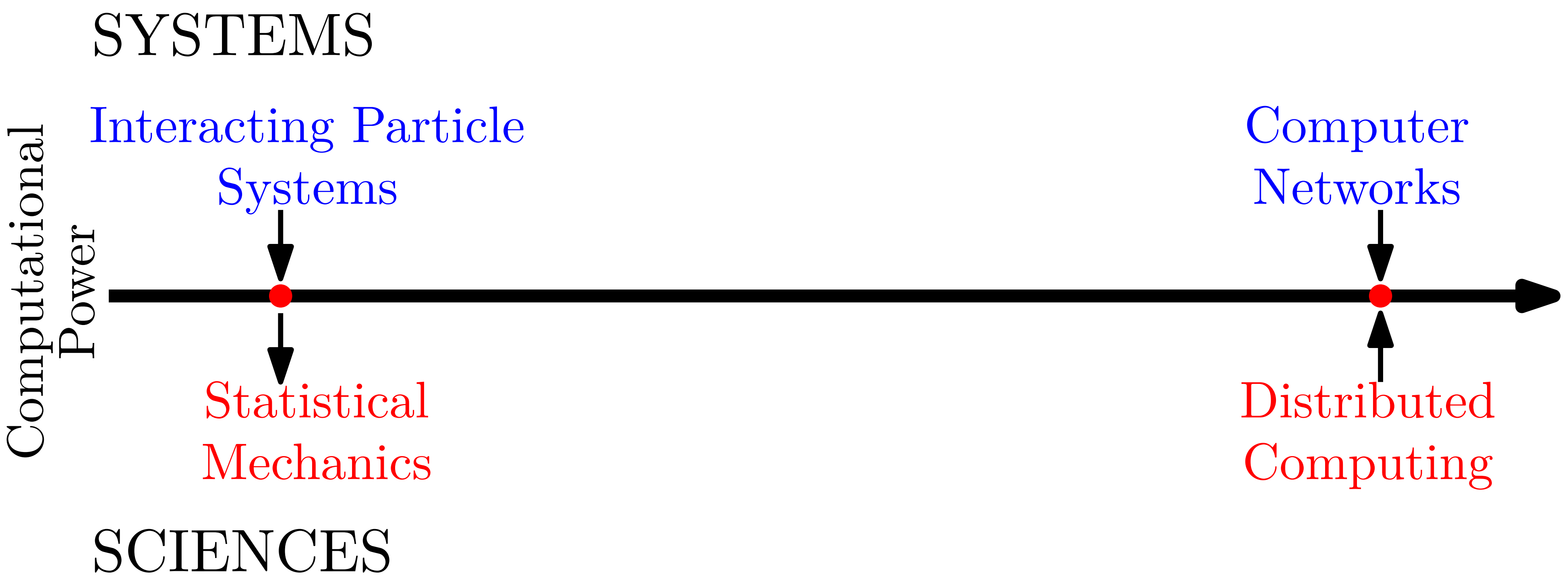
Computational Aspects of Complex Networks

Rome, December 6, 2024

Roadmap

- Intro to Computational Dynamics
- Community Detection via Synchronous Averaging
- Community Detection via Asynchronous Averaging
- 2-Choices on Clustered Graphs & Evolution

Communication in *Simple* Systems



Communication in *Simple* Systems

Schools of fish
[Sumpter et al. '08]

Insects colonies
[Franks et al. '02]

Flocks of birds
[Ben-Shahar et al. '10]

Biological Systems

SYSTEMS

Interacting Particle Systems

Computer Networks

Computational Power

Statistical Mechanics

Distributed Computing

SCIENCES

Dynamics

(informal) *Very simple* distributed algorithms: For every graph, agent and round, states are updated according to fixed rule of current state and symmetric function of states of neighbors.

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To go beyond this talk:

- Becchetti et al. *Consensus Dynamics: An Overview.* 2020.
- Mossel & Tamuz. *Opinion exchange dynamics.* 2017.
- Shah. *Gossip Algorithms.* 2007.

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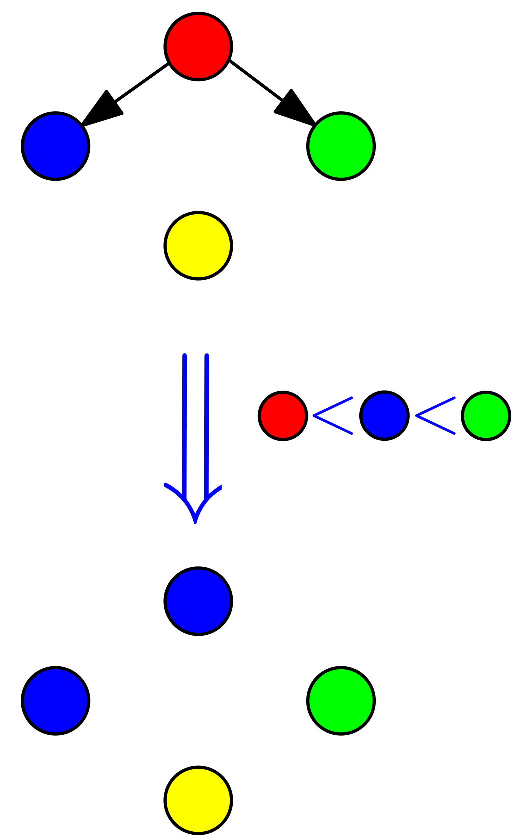
Examples of Dynamics

- 3-Median dynamics



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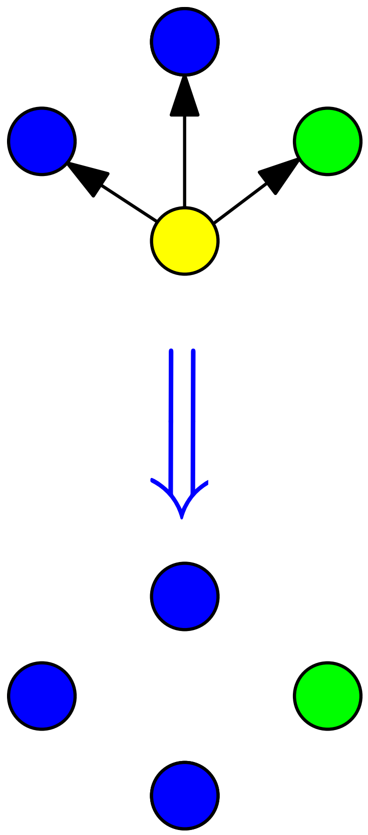

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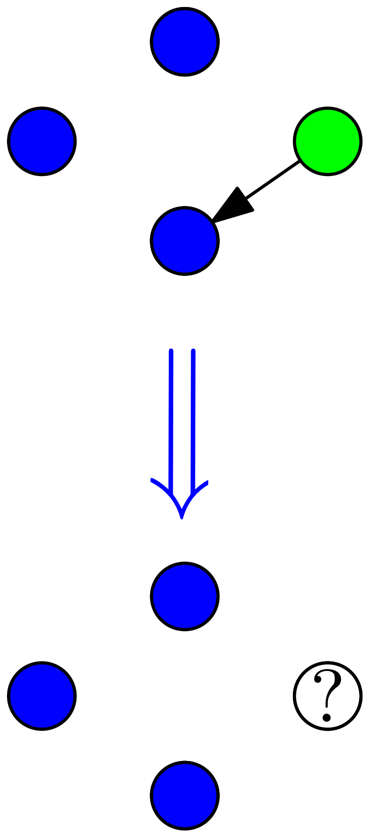

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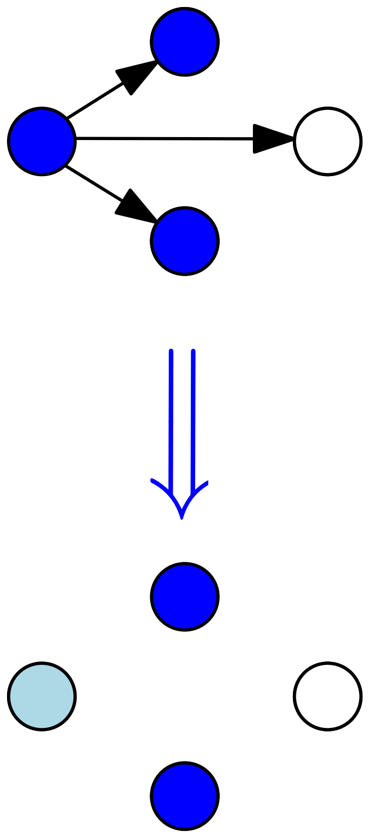
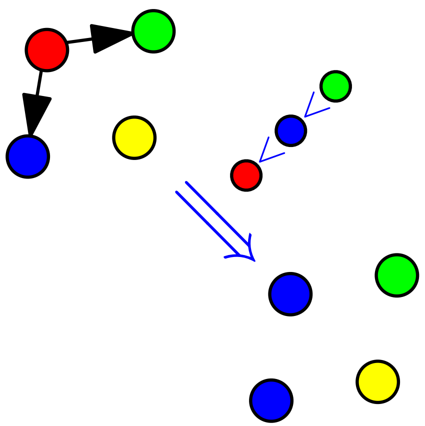
- Averaging dynamics



The Power of Dynamics: Plurality Consensus

Computing the Median



- 3-Median dynamics [Doerr et al. '11]. Converge to $\mathcal{O}(\sqrt{n \log n})$ approximation of median of system in $\mathcal{O}(\log n)$ rounds w.h.p., even if $\mathcal{O}(\sqrt{n})$ states are arbitrarily changed at each round ($\mathcal{O}(\sqrt{n})$-bounded adversary).

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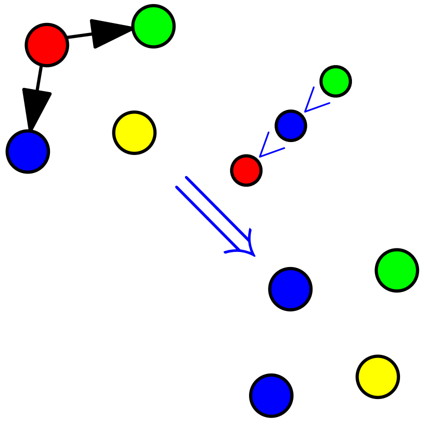

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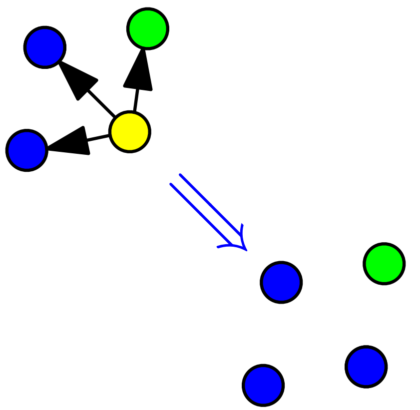

- 3-Majority dynamics [SPAA '14, SODA '16]. If plurality has **bias** $\mathcal{O}(\sqrt{kn \log n})$, converges to it in $\mathcal{O}(k \log n)$ rounds w.h.p., even against $o(\sqrt{n/k})$-bounded adversary. Without bias, converges in $\text{poly}(k)$. h-majority converges in $\Omega(k/h^2)$.

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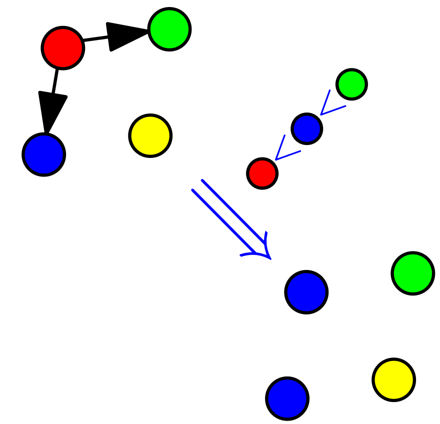

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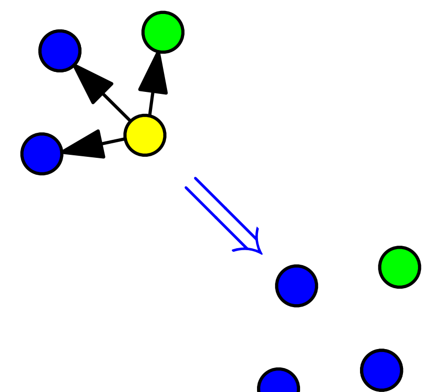
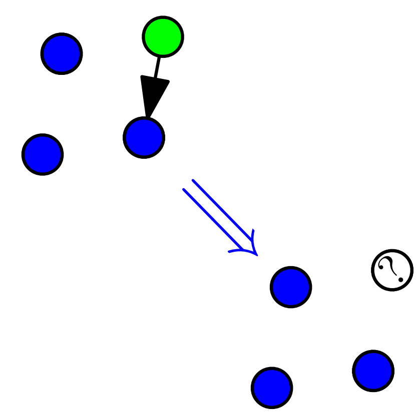


- Undecided-State dynamics [SODA '15]. If majority/second-majority ($c_{maj}/c_{2^{nd}maj}$) is at least $1+\epsilon$, system converges to plurality within $\tilde{\Theta}(\sum_{i=1}^{k} \left(c_i^{(0)}/c_{maj}^{(0)}\right)^2)$ rounds w.h.p.



The Median, the Mode and... the Mean

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Can dynamics solve a problem non-trivial in centralized setting?

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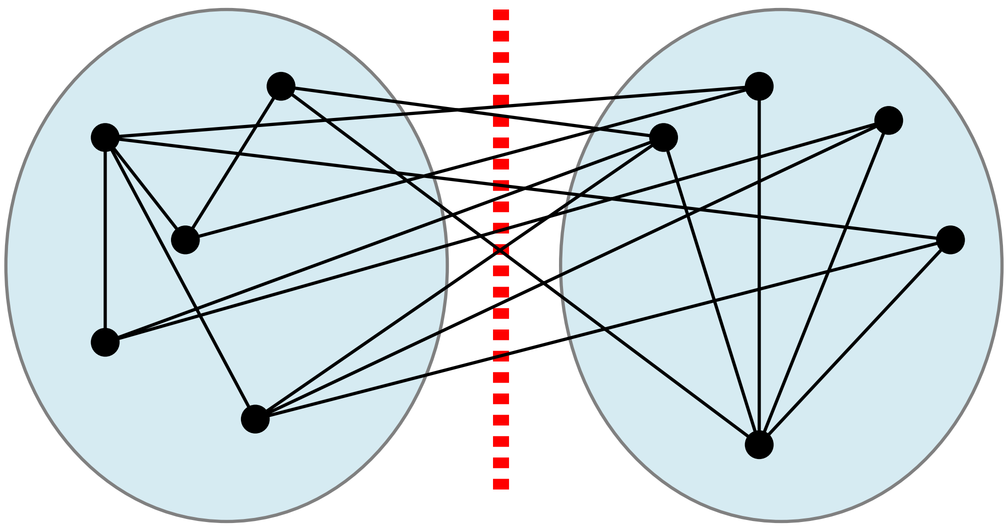
Community Detection as Minimum Bisection

Minimum Bisection Problem.

Input: a graph G with $2n$ nodes.

Output: $S = \arg \min_{\substack{S \subset V \\ |S|=n}} E(S, V - S)$.

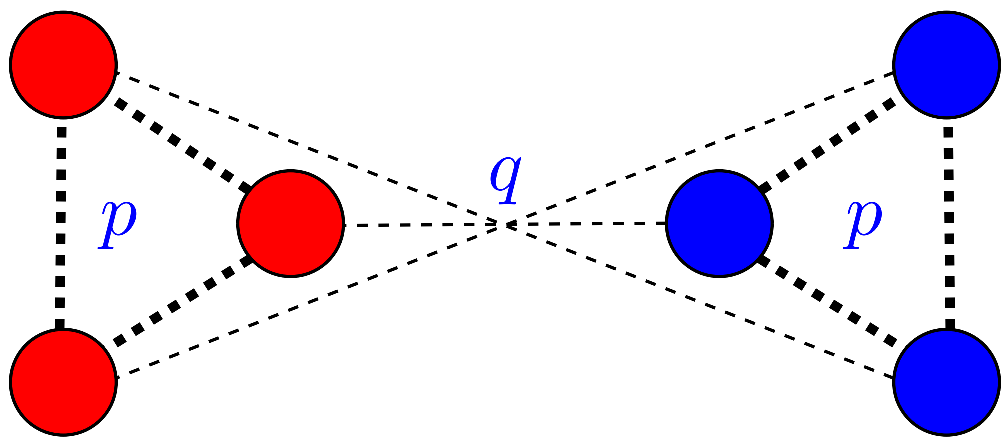


[Garey, Johnson, Stockmeyer '76]:

Min-Bisection is *NP-Complete*.

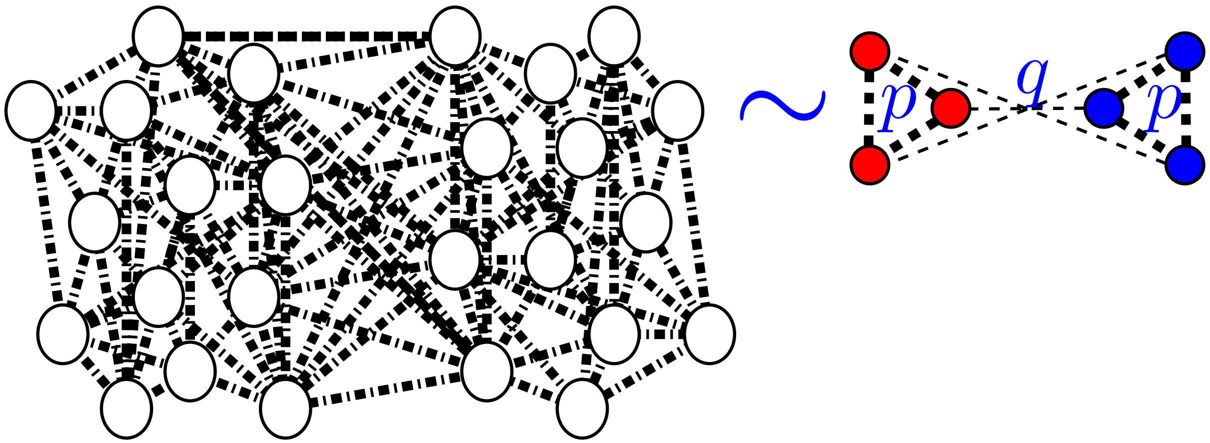
The Stochastic Block Model

Stochastic Block Model (SBM). Two "communities" of equal size V_1 and V_2, each edge inside a community included with probability $p = \frac{a}{n}$, each edge across communities included with probability $q = \frac{b}{n} < p$.



The Stochastic Block Model

Reconstruction problem. Given graph generated by SBM, find original partition.



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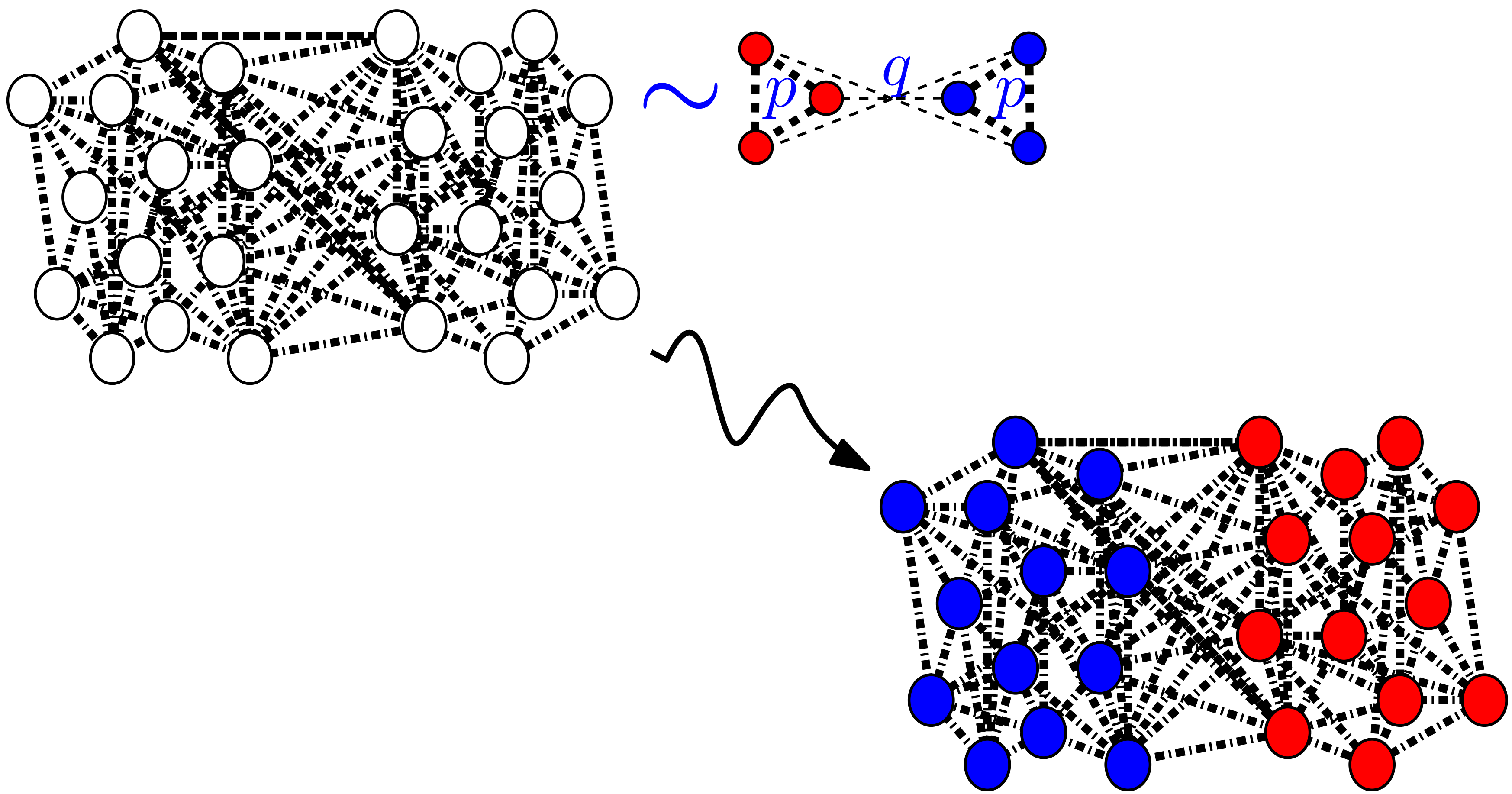
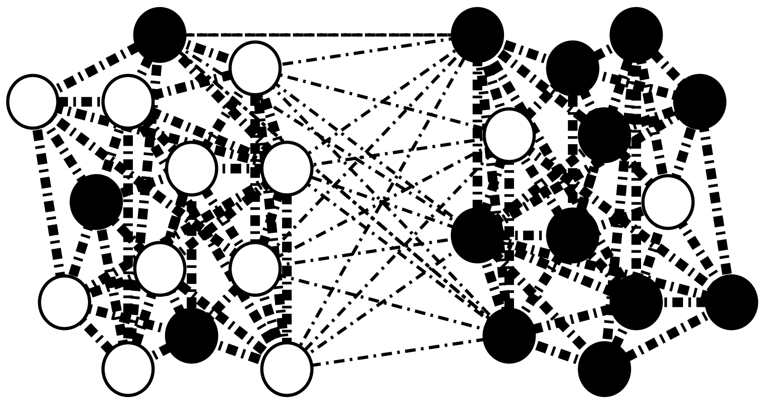


Community Detection via Averaging Dynamics



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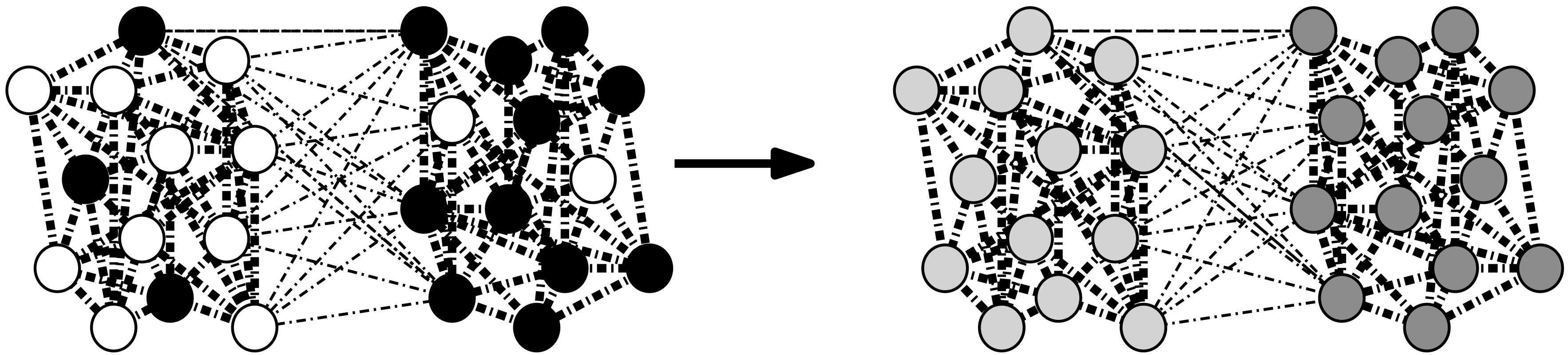

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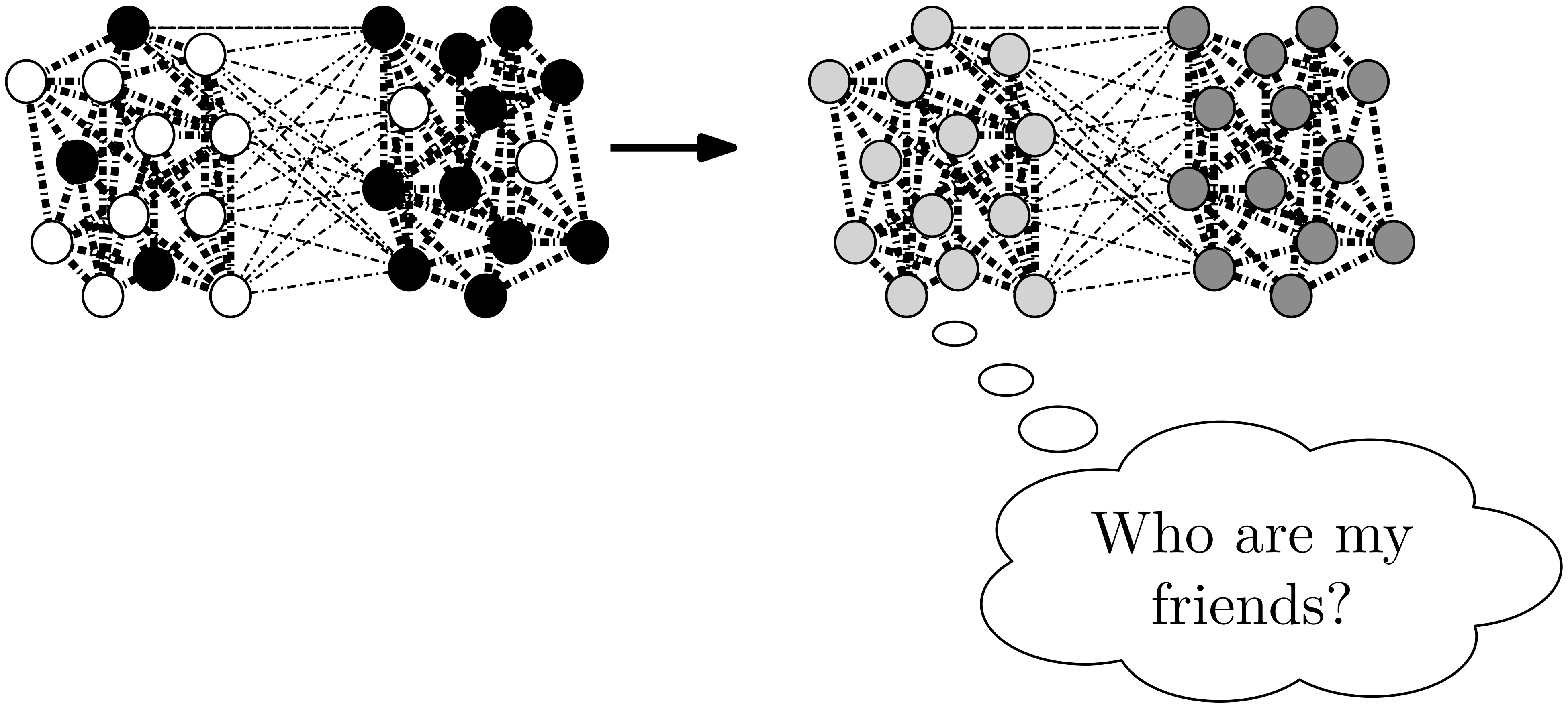

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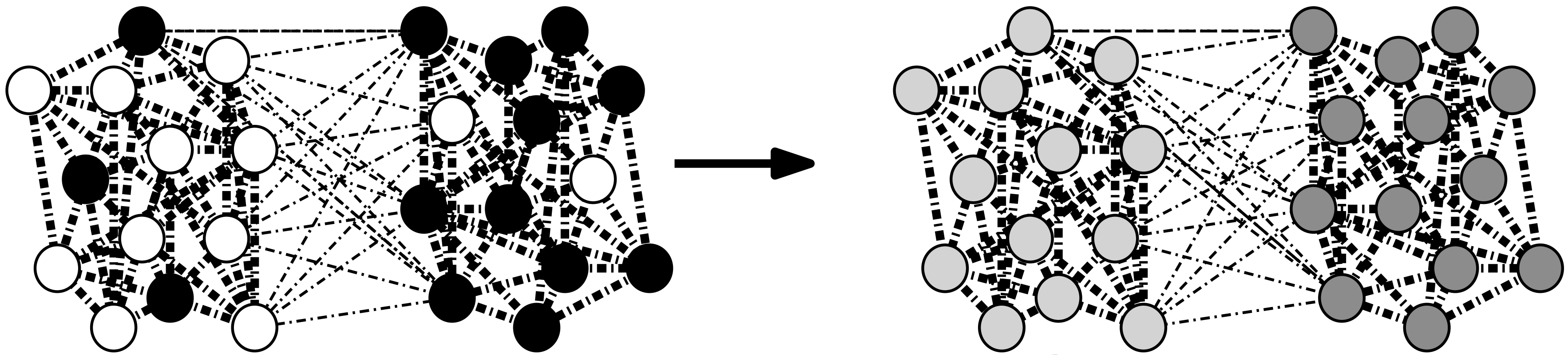
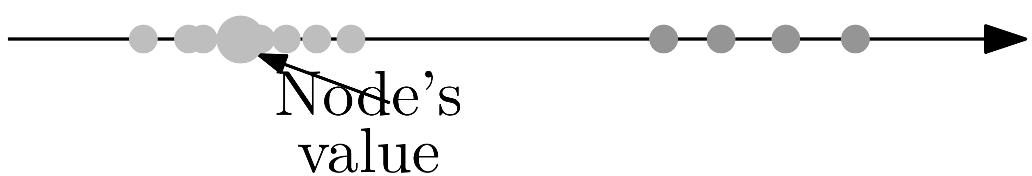


Local view of a node:



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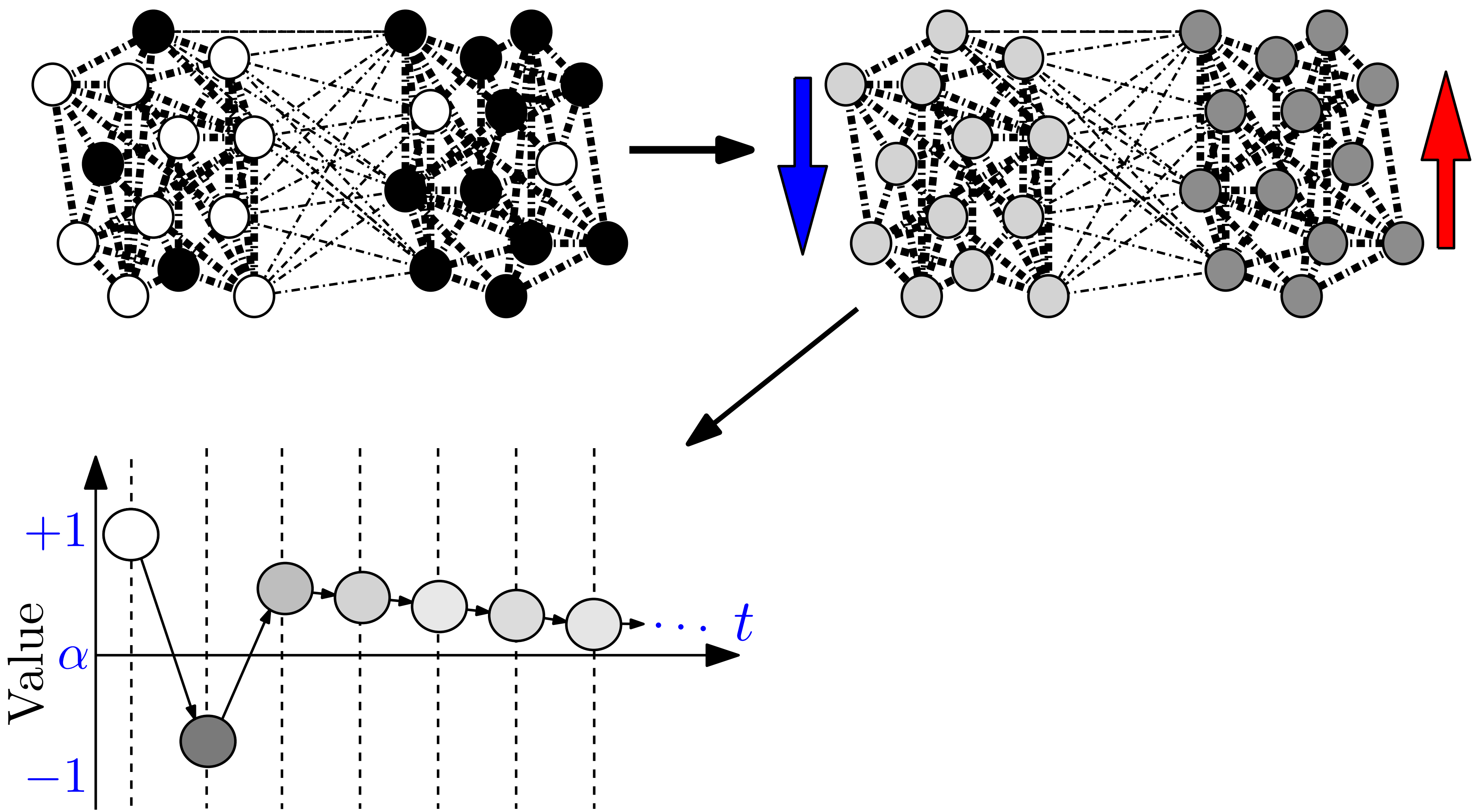

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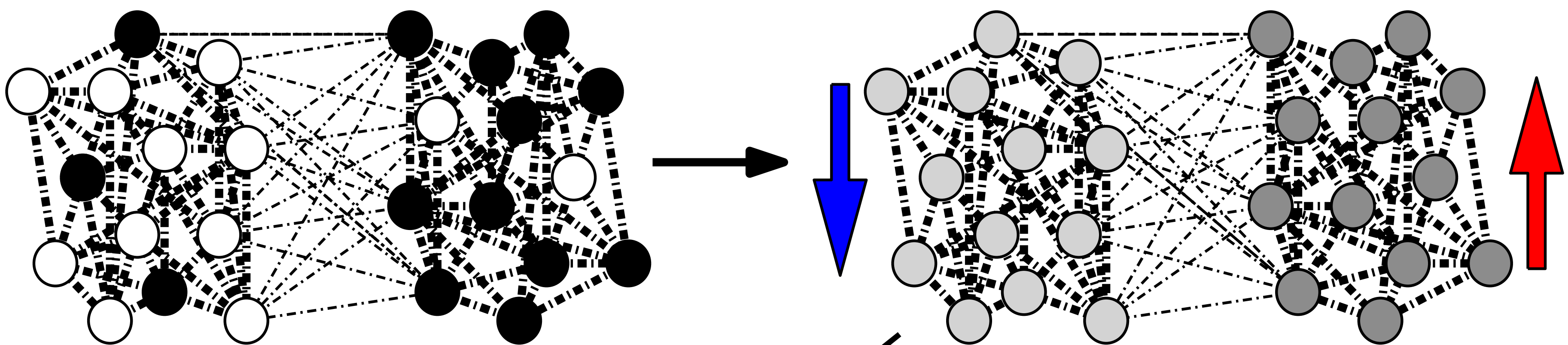
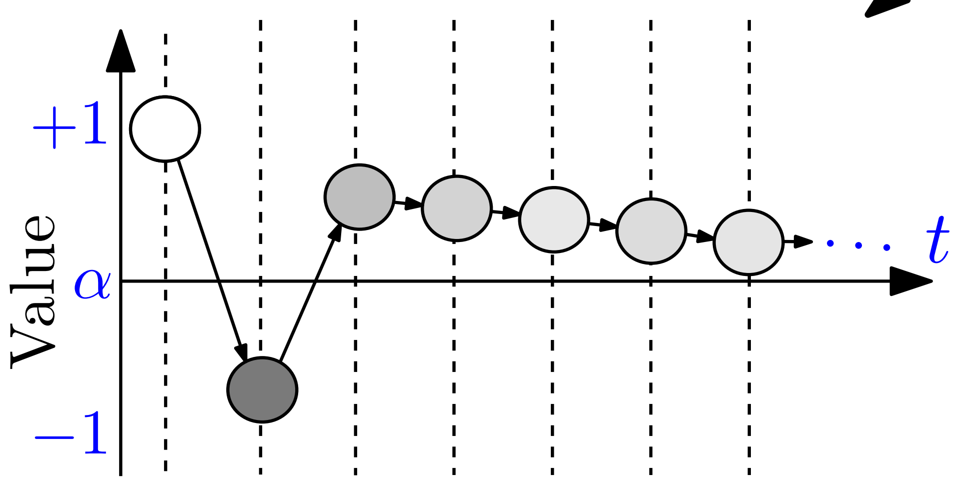




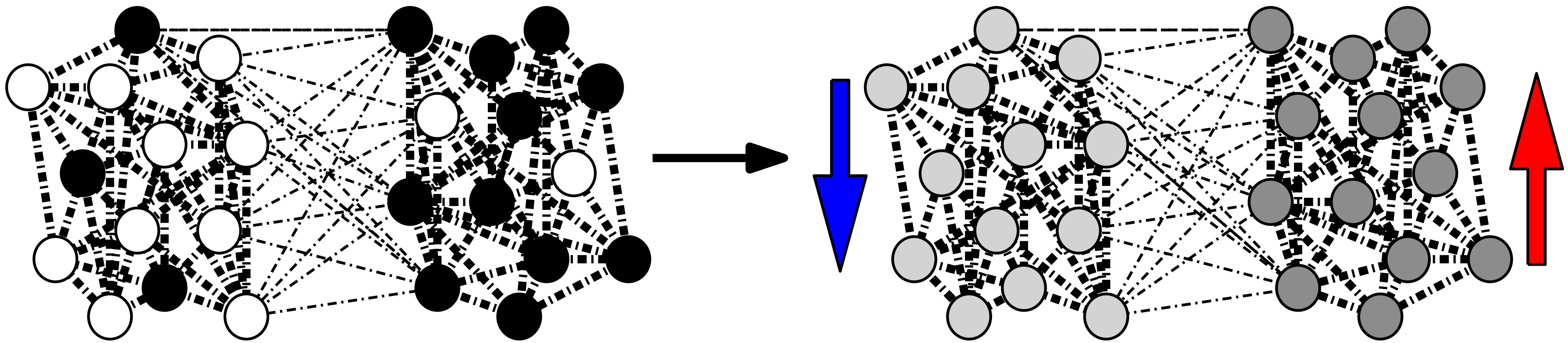
$v_1, \ldots, v_n$ eigenvectors of random walk matrix P:

$v_1 = \mathbb{1} = (1, \ldots, 1)$

$v_2 \approx \chi = (1, \ldots, 1, -1, \ldots, -1)$

"nice" graph

Community Detection via Averaging Dynamics

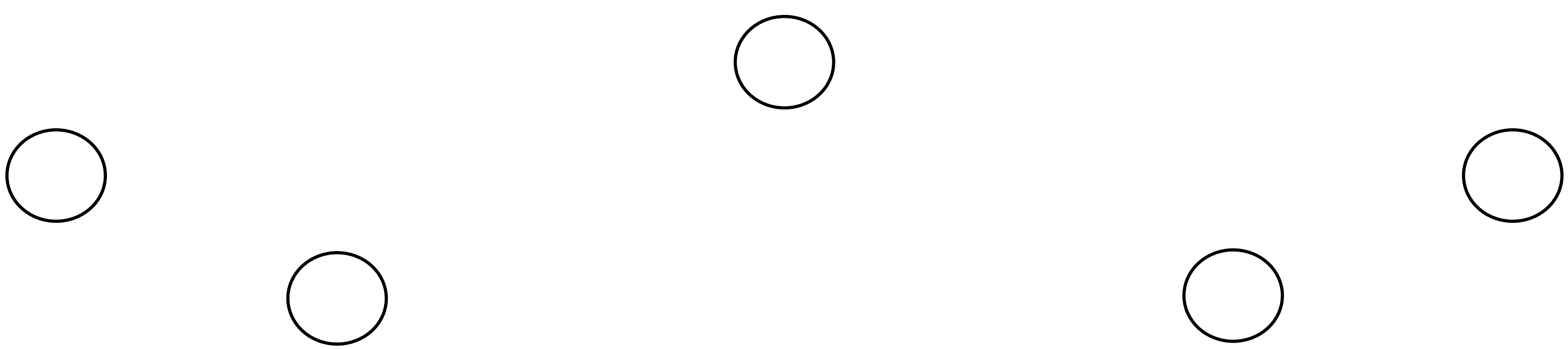


[SODA '17](Informal). $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.
i) $\chi = \mathbf{1}_{V_1} - \mathbf{1}_{V_2}$ close to right-eigenvector of eigenvalue λ_2 of transition matrix of G, and
ii) gap between λ_2 and $\lambda = \max\{\lambda_3, |\lambda_n|\}$ sufficiently large,
then Averaging (approximately) identifies (V_1, V_2).

The Averaging Dynamics in the $\mathcal{LOCAL}$ Model

Al nodes at the same time:

- At $t = 0$, randomly pick value $x^{(t)} \in \{\mathbf{+1}, \mathbf{-1}\}$.
- Then, at each round
  - Set value $x^{(t)}$ to average of neighbors,
  - Set label to **blue** if $x^{(t)} < x^{(t-1)}$, **red** otherwise.



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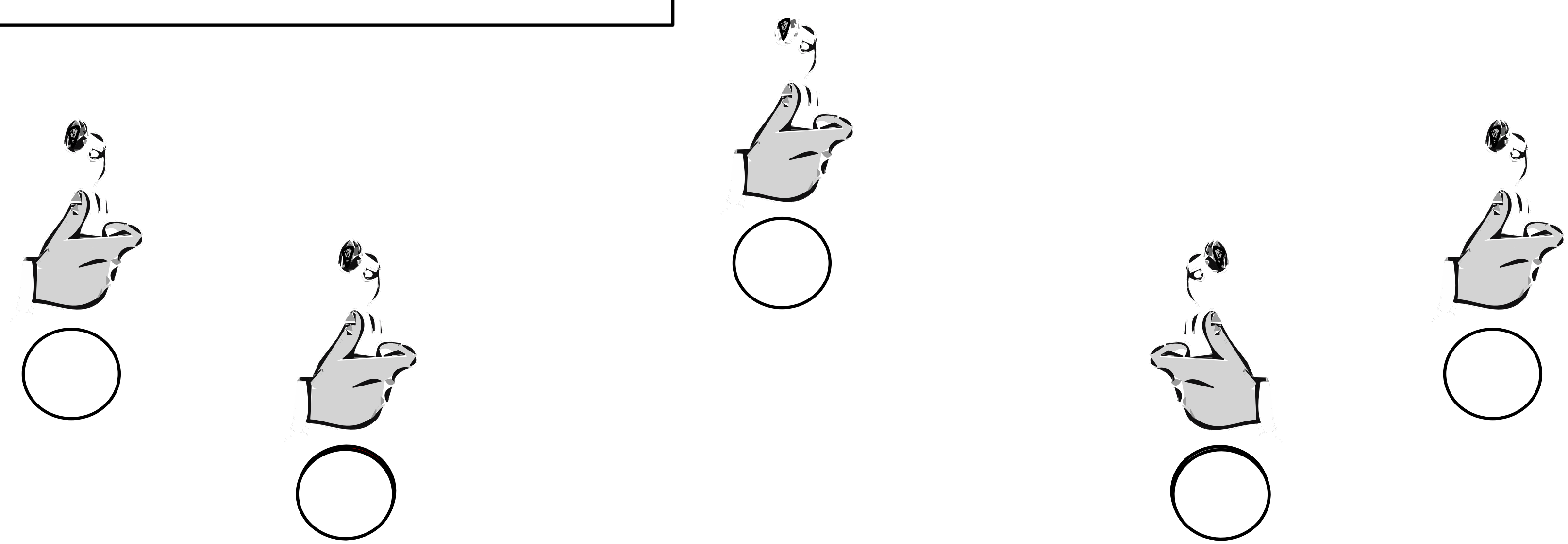

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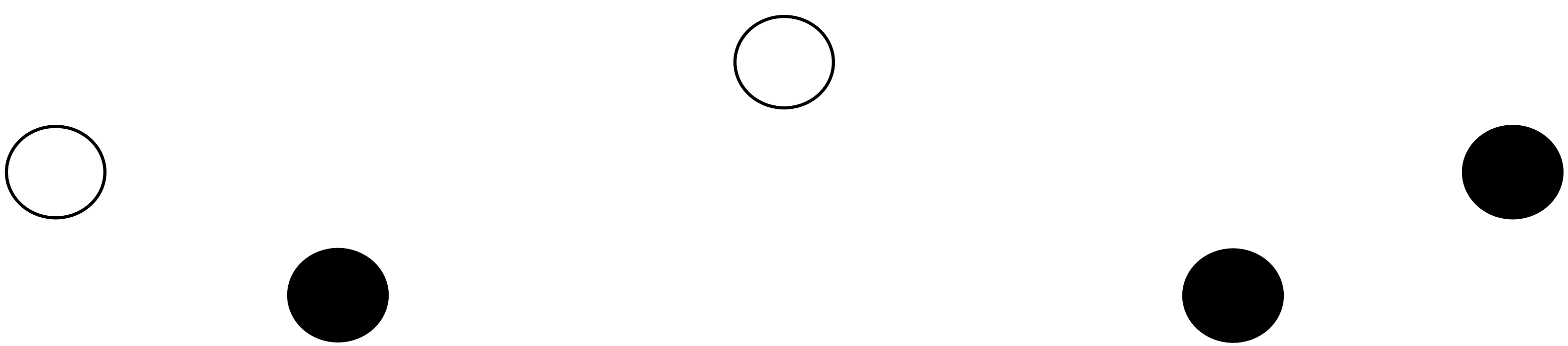

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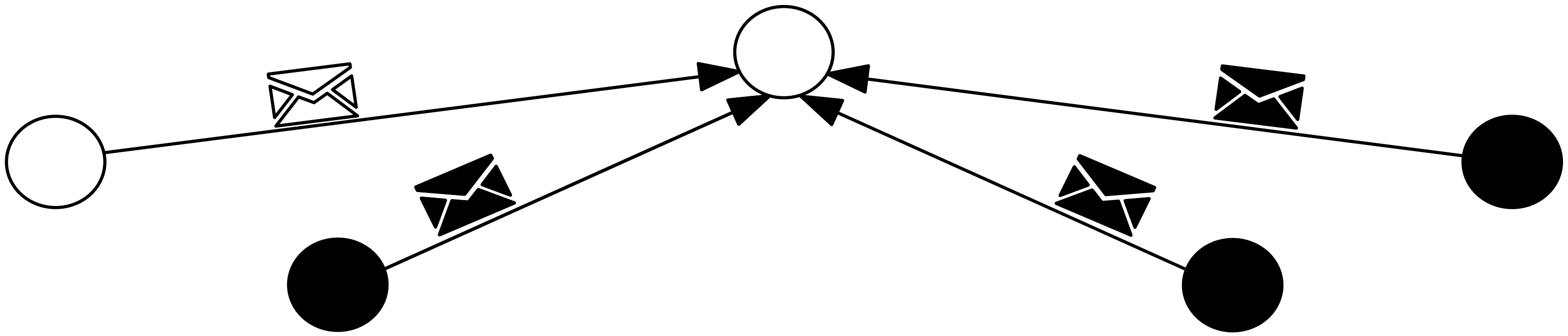

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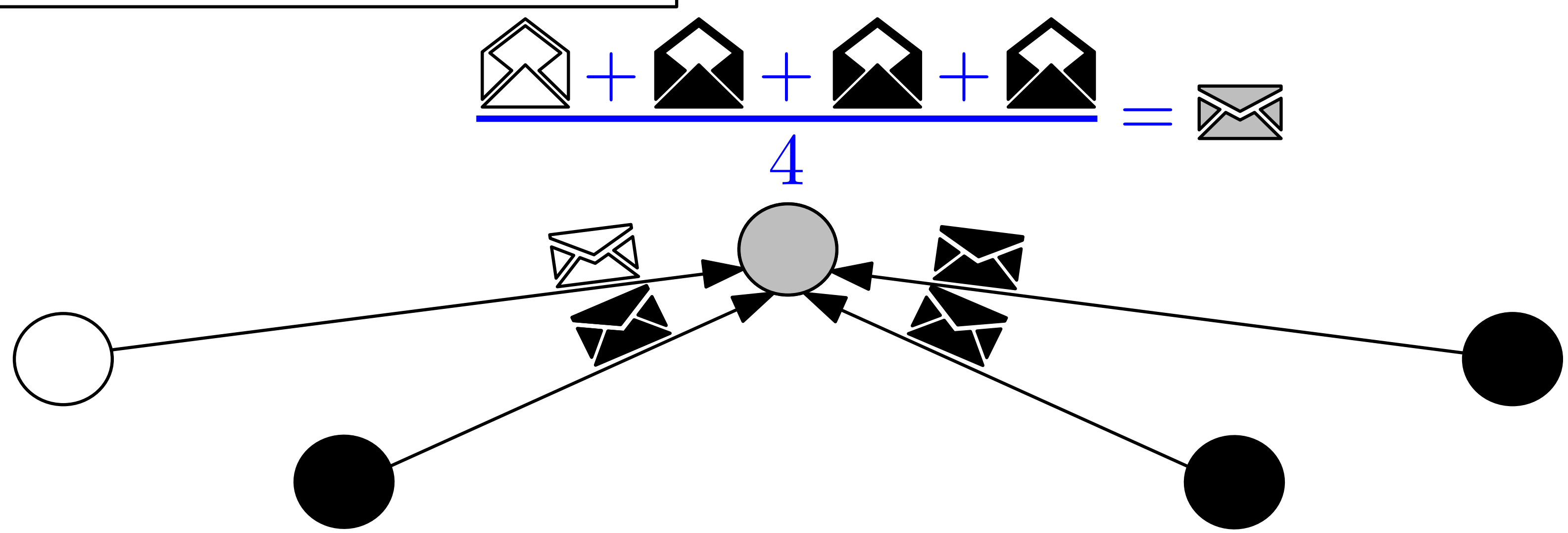

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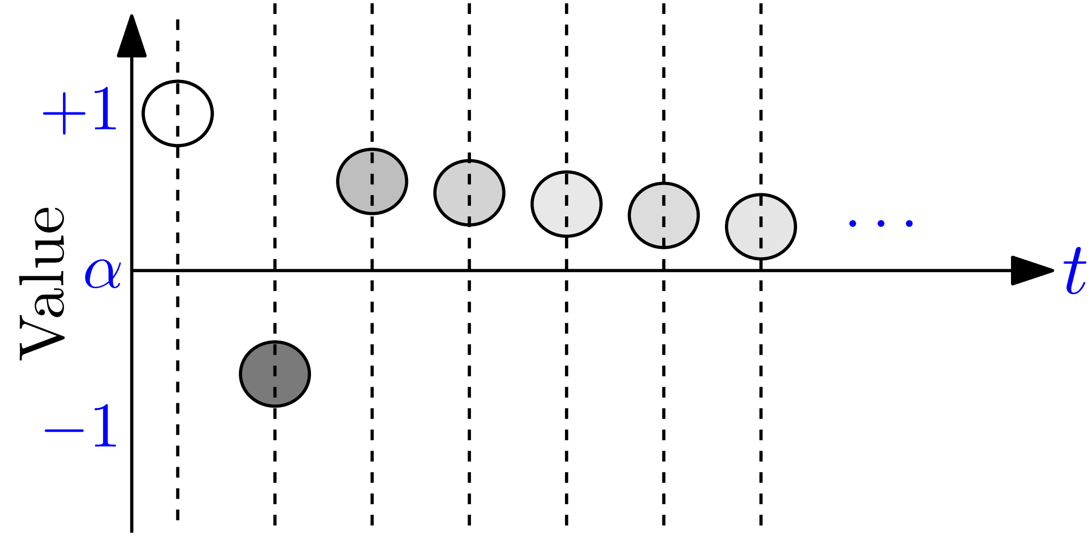

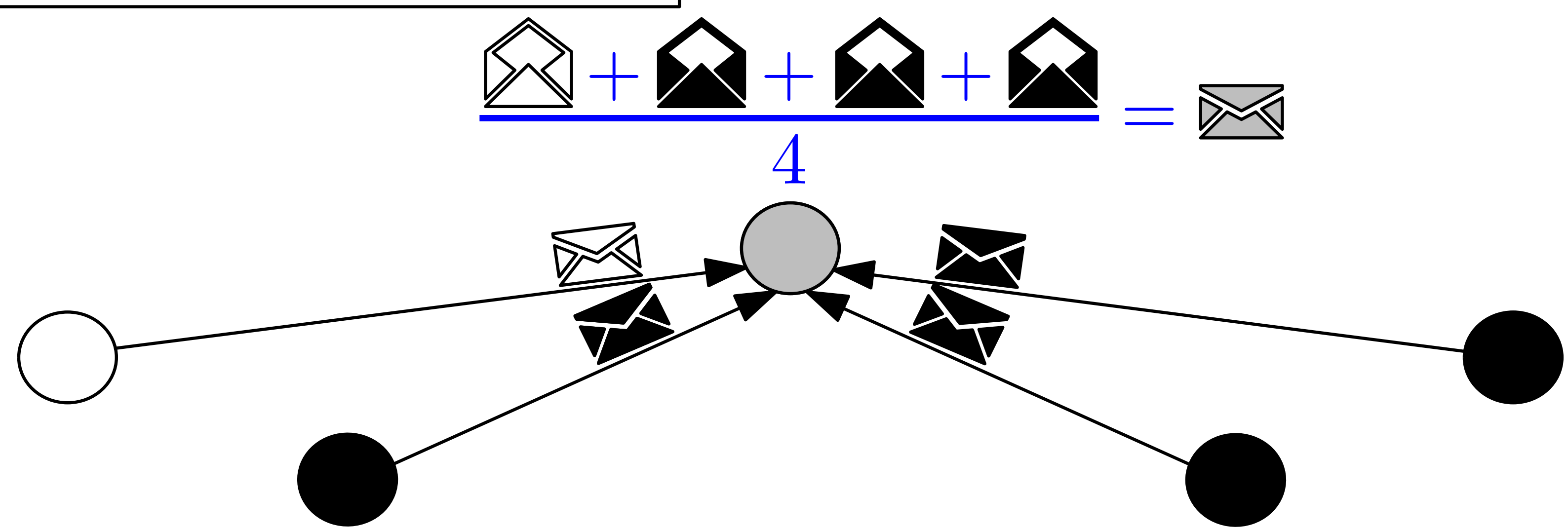

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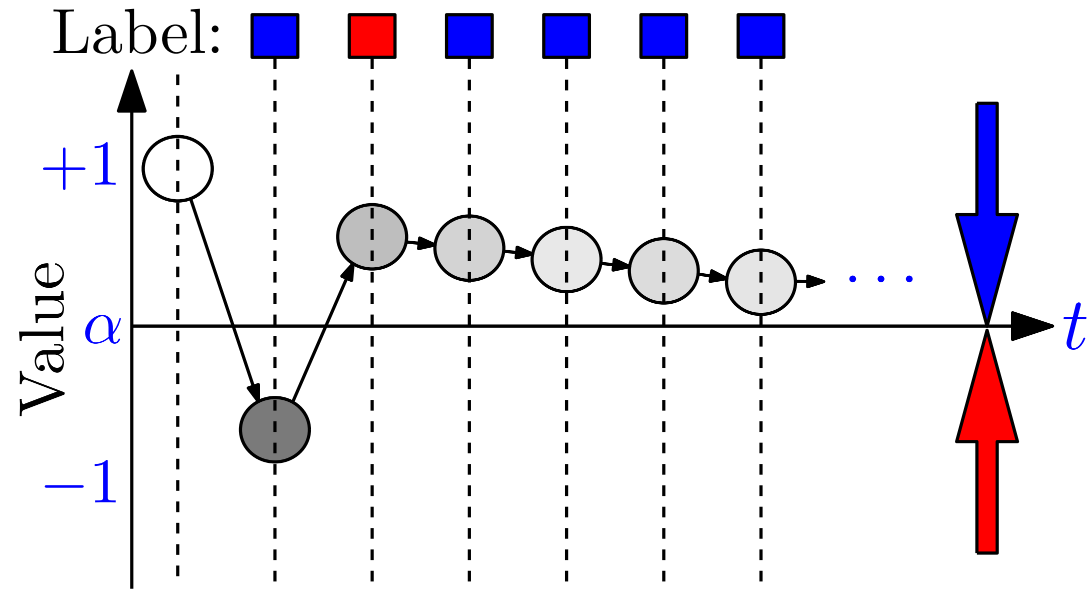

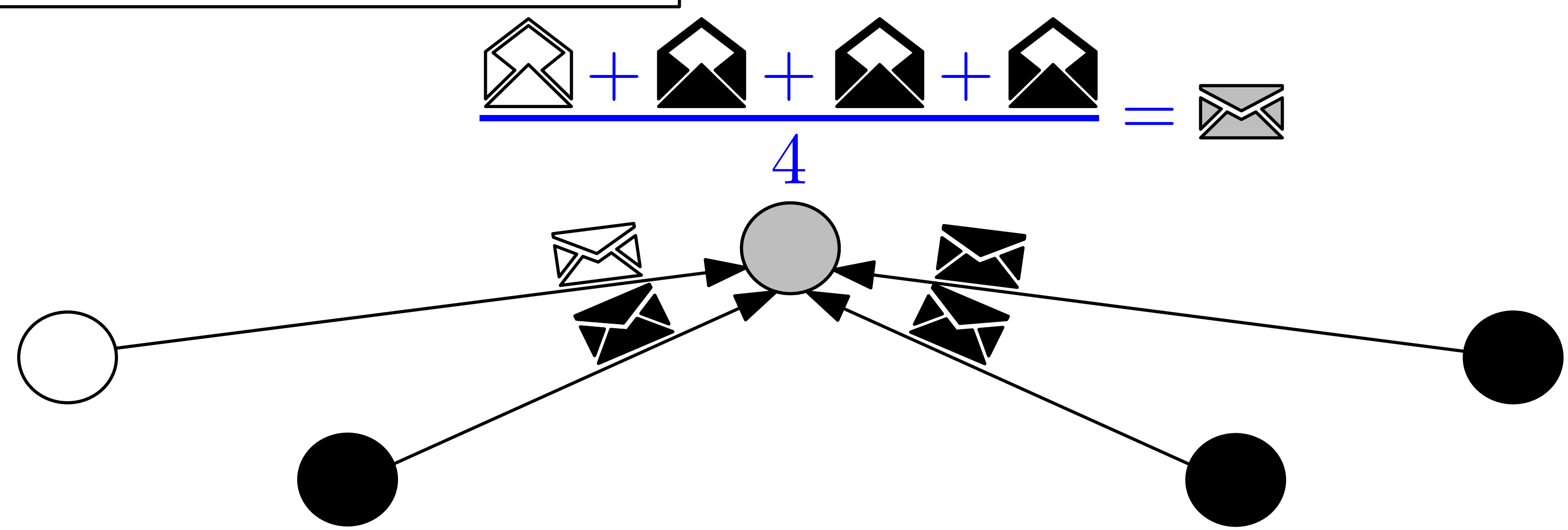

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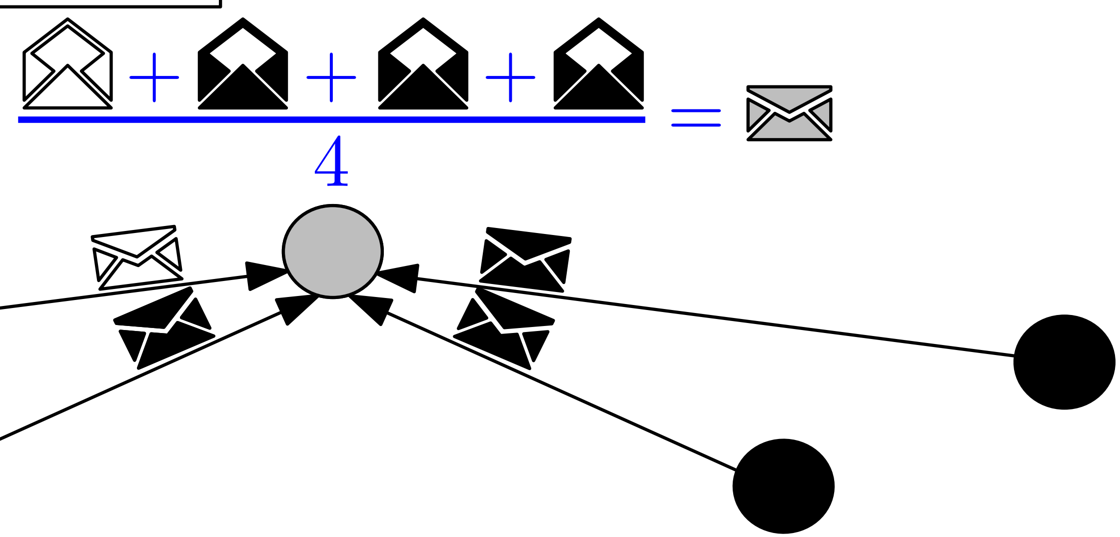
Well studied process [Shah '09]:

- Converges to (weighted) global average of initial values,
- Convergence time = mixing time of G,
- Important applications in fault-tolerant self-stabilizing consensus.



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Averaging is a **linear** dynamics

$$\mathbf{x}^{(t)} = \begin{pmatrix} \circ \\ \bullet \\ \circ \\ \bullet \\ \bullet \end{pmatrix}$$

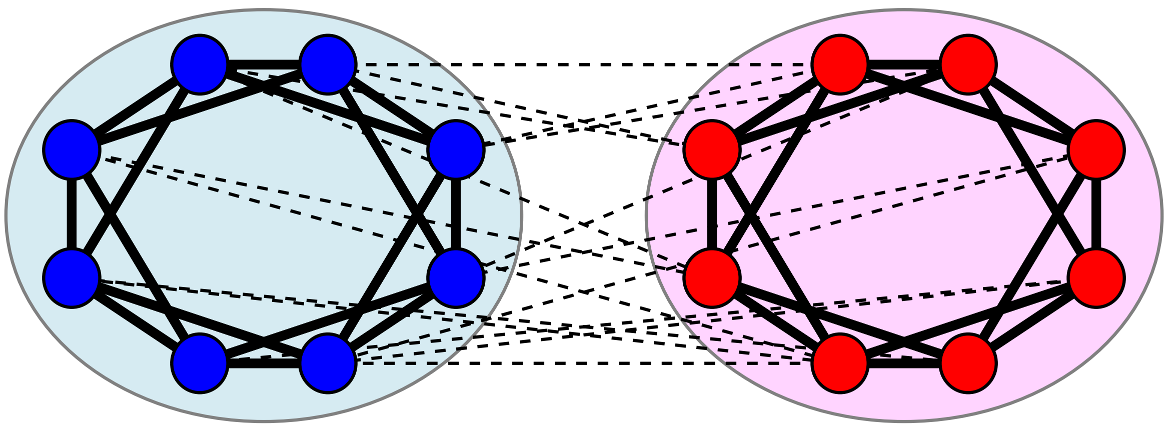
$$\boxed{\mathbf{x}^{(t)} = P \cdot \mathbf{x}^{(t-1)} = P^t \cdot \mathbf{x}^{(0)}}$$

P transition matrix of random walk

Toy Case: Regular Stochastic Block Model

Regular SBM (RSBM) [Brito et al. SODA'16]. A graph $G = (V_1 \dot{\bigcup} V_2, E)$ s.t.

- $|V_1| = |V_2|$,
- $G\big|_{V_1}, G\big|_{V_2} \sim$ random a-regular graphs
- $G\big|_{E(V_1,V_2)} \sim$ random b-regular bipartite graph.



4-regular 4-regular

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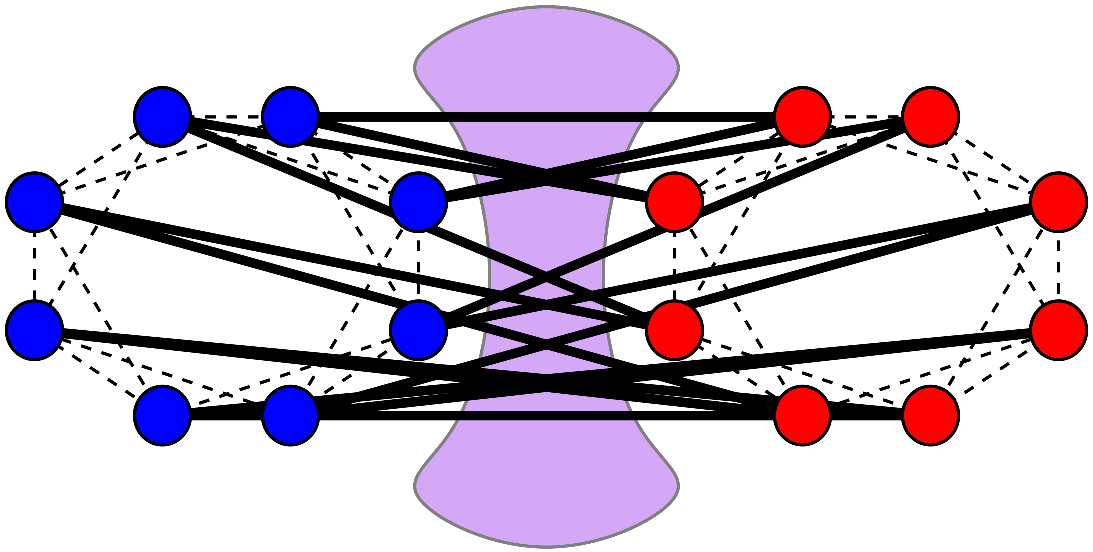

2-regular bipartite

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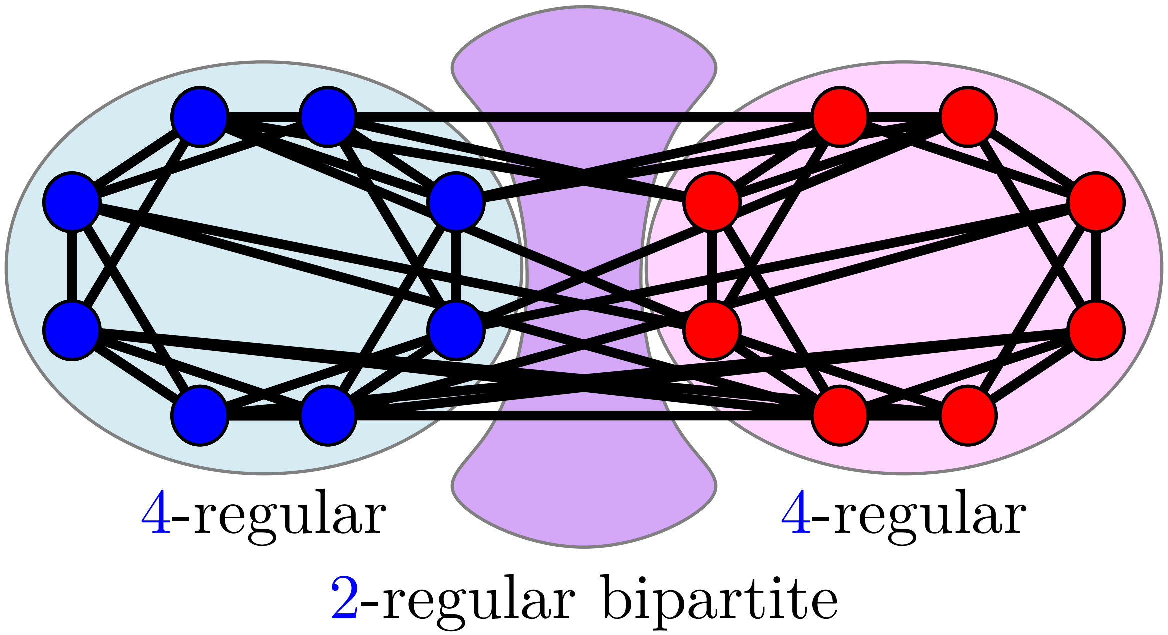

Analysis on Regular SBM

$P \longrightarrow$ symmetric $\Longrightarrow$ orthonormal eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$ and real eigenvalues $\lambda_1, ..., \lambda_n$.

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Regular SBM $\Longrightarrow$ $P \frac{1}{\sqrt{n}} \chi = \left(\frac{a-b}{a+b}\right) \cdot \frac{1}{\sqrt{n}} \chi$

$$\frac{1}{a+b} \left(\begin{array}{c:c} \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots a \text{ “1”s} \cdots & \cdots b \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \hdashline \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots b \text{ “1”s} \cdots & \cdots a \text{ “1”s} \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array} \right) \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = \frac{a-b}{a+b} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

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$\mathbf{v}_1 = \frac{1}{\sqrt{n}} \mathbf{1}$ with (largest) eigenvalue 1

Regular SBM $\implies$ $P \frac{1}{\sqrt{n}} \chi = (\frac{a-b}{a+b}) \cdot \frac{1}{\sqrt{n}} \chi$

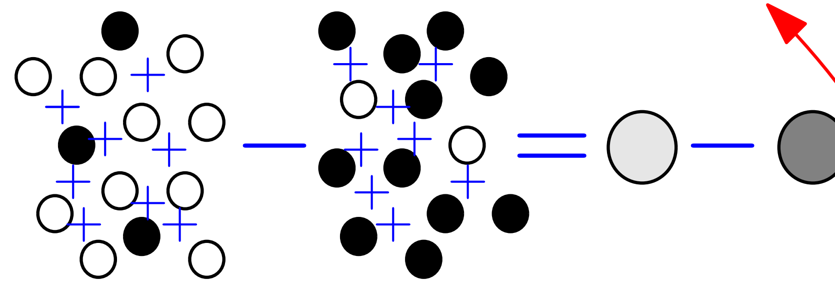
W.h.p. $\max\{\lambda_3, |\lambda_n|\}(1+\delta) < \frac{a-b}{a+b} = \lambda_2$, then

$$\mathbf{x}^{(t)} = \frac{1}{n}(\mathbf{1}^\intercal \mathbf{x}^{(0)})\mathbf{1} + \left(\frac{a-b}{a+b}\right)^t \frac{1}{n}(\chi^\intercal \mathbf{x}^{(0)})\chi + \mathbf{e}^{(t)}$$

with $\|\mathbf{e}^{(t)}\| \le (\max\{\lambda_3, |\lambda_n|\})^t \sqrt{n}$

Analysis on Regular SBM

$\frac{1}{n}\sum_{u\in V_1}\mathbf{x}^{(0)}(u) - \frac{1}{n}\sum_{u\in V_2}\mathbf{x}^{(0)}(u)$



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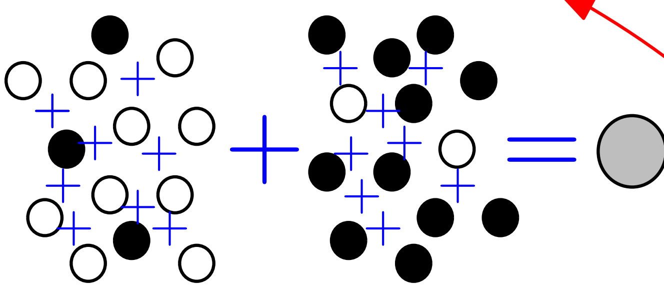

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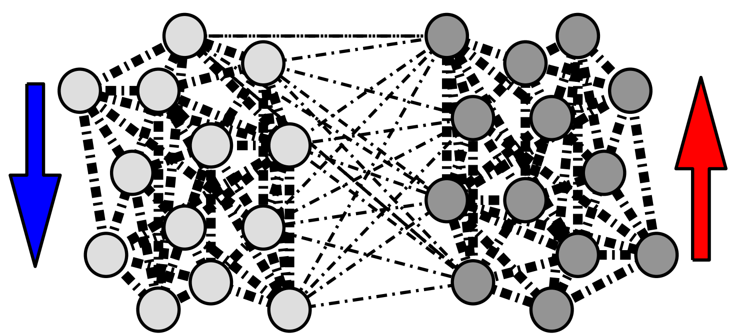

$$\boxed{\text{sign}(\mathbf{x}^{(t)}(u) - \mathbf{x}^{(t-1)}(u)) \propto \text{sign}(\chi(u))}$$

Roadmap

- Intro to Computational Dynamics
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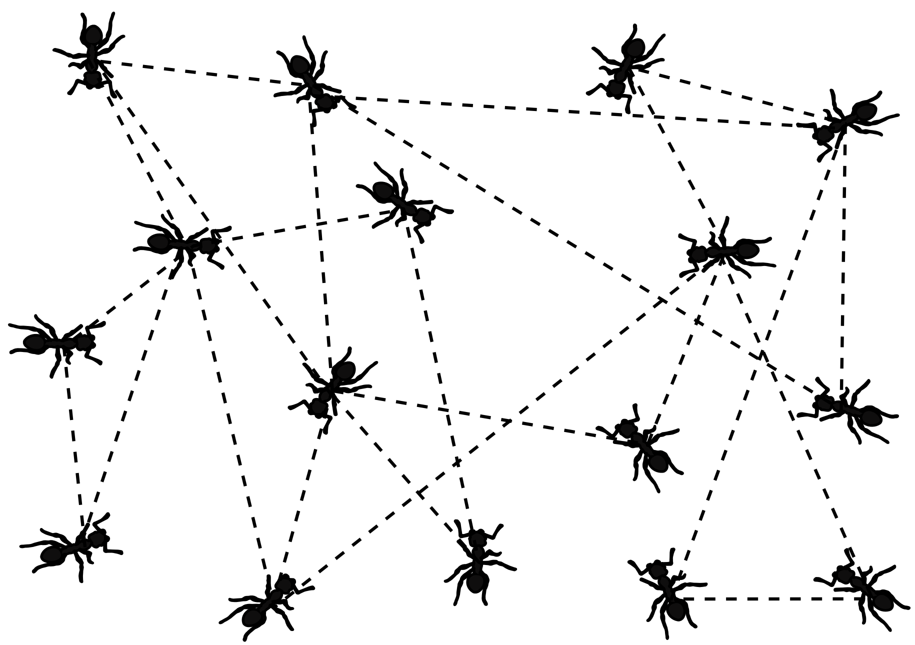

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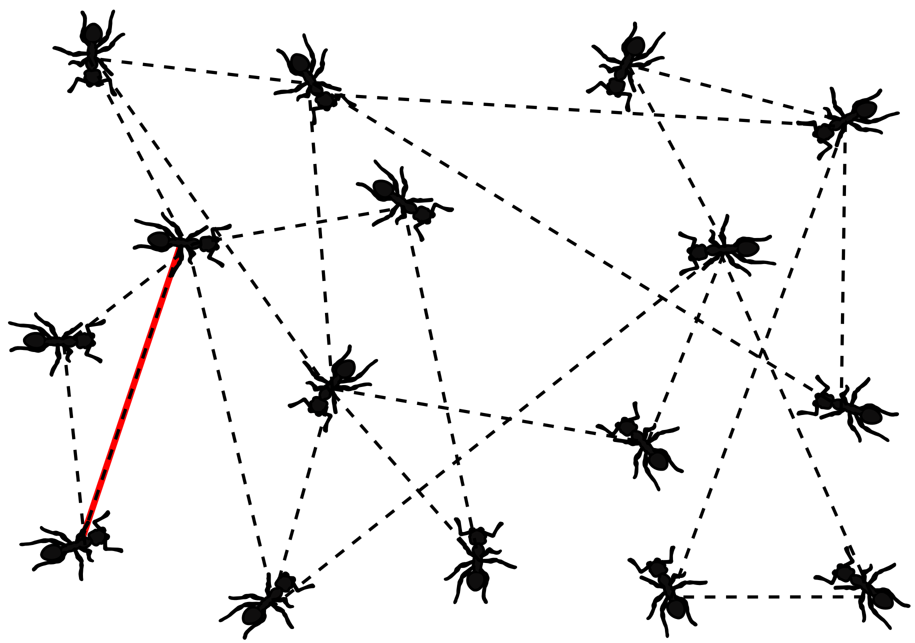

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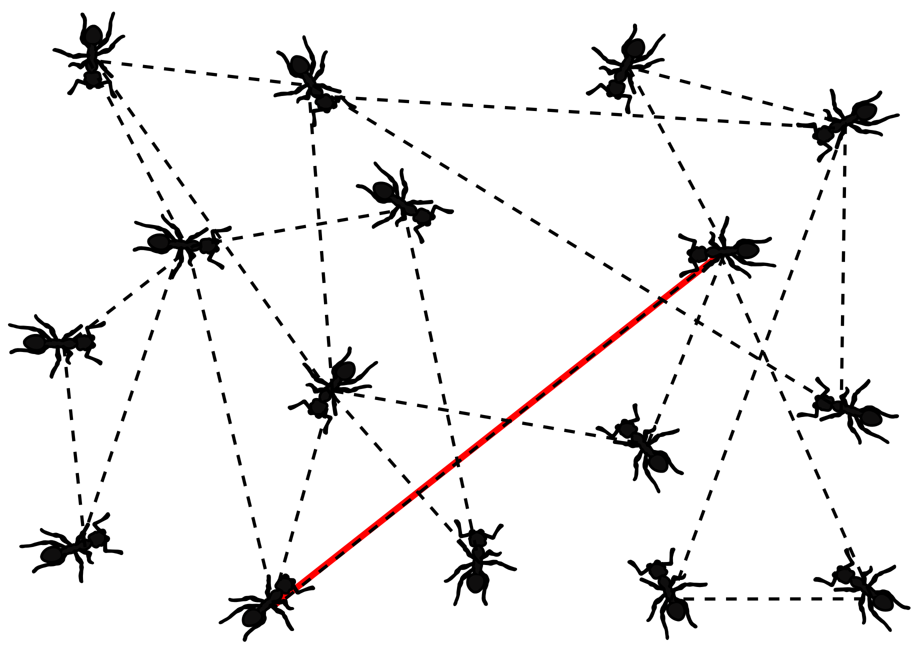

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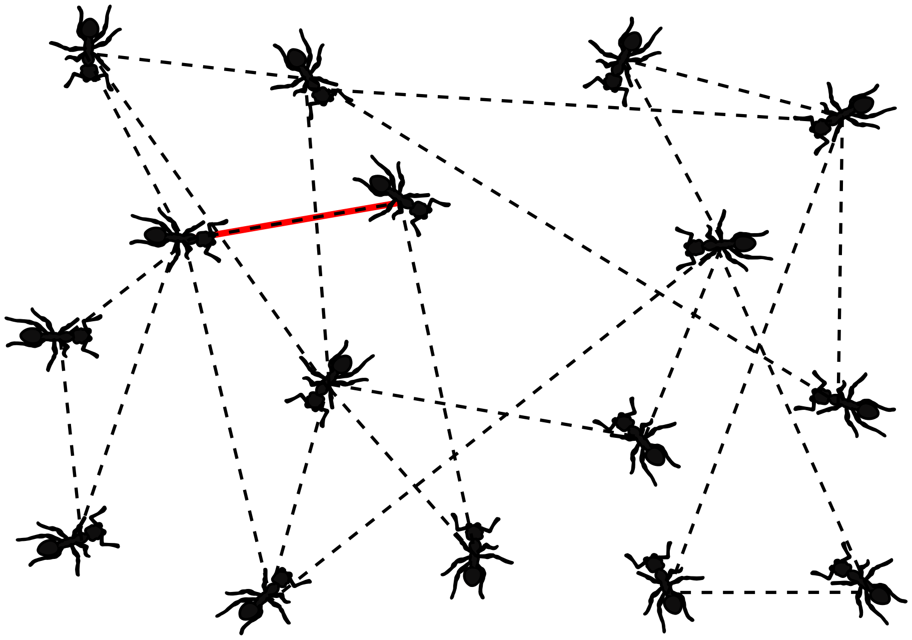

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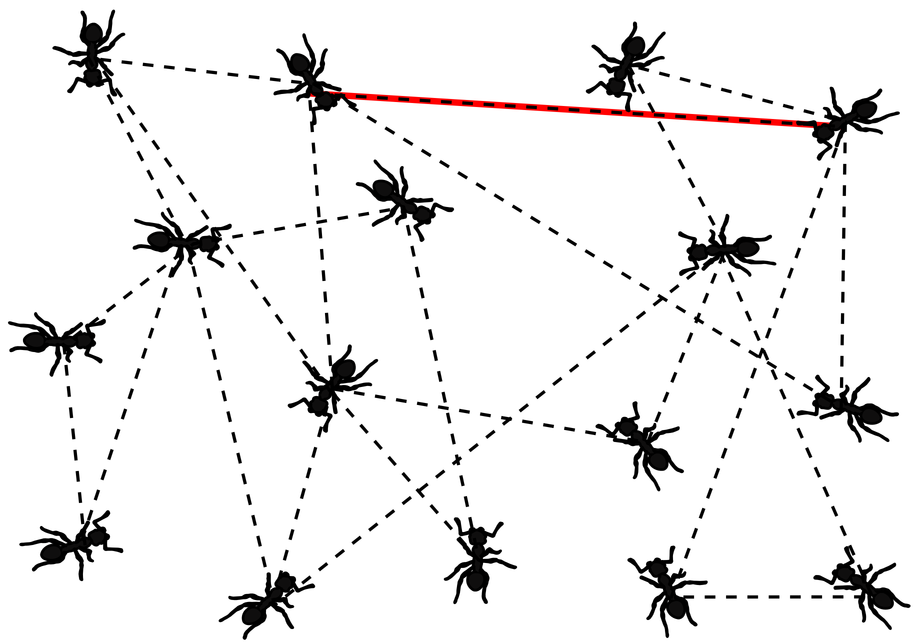
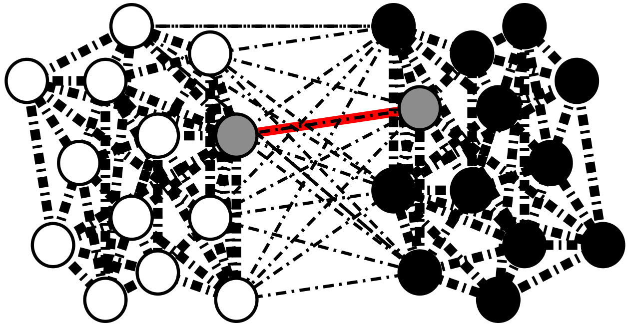


Sparsification of the Averaging Dynamics

!!!: The *variance* of picking a random edge breaks the monotonicity and seems to prevent concentration.



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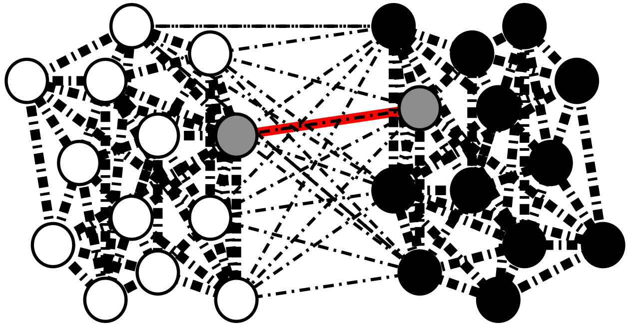

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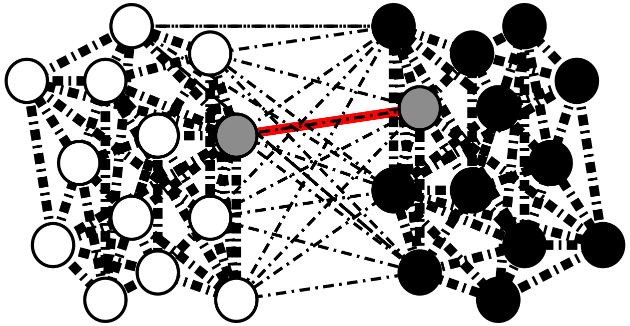

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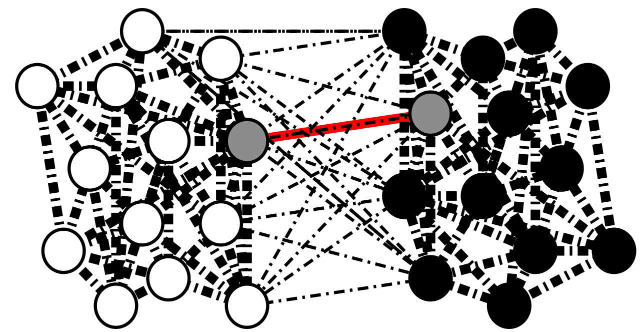

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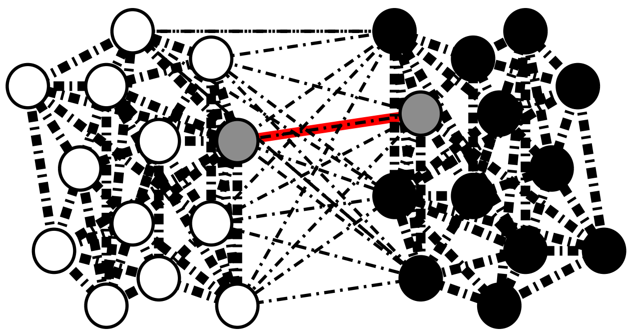

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Problem: no concentration tools for matrix *products*
(e.g. no logarithm for noncommutative matrices)

Community Sensitive Labeling

CSL(m,T):

- At the outset
$$\mathbf{x}_u^{(0)} \sim \mathrm{Unif}(\{-1,+1\}^m).$$
- In each round, the endpoints of the random edge choose a random index $j \in [m]$ and set
$$\mathbf{x}_u(j) = \mathbf{x}_v(j) = \frac{\mathbf{x}_u(j) + \mathbf{x}_v(j)}{\mathbf{2}};$$ (cfr [Boyd et al. '06]).
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Thm. $G = (V_1 \dot{\bigcup} V_2, E)$ regular SBM s.t. $d\epsilon^4 \gg b \log^2 n$, then CSL($m$,$T$) with $m = \Theta(\epsilon^{-1} \log n)$ and $T = \Theta(\log n)$ labels all nodes but a set U with size $|U| \leq \sqrt{\epsilon} n$, in such a way that

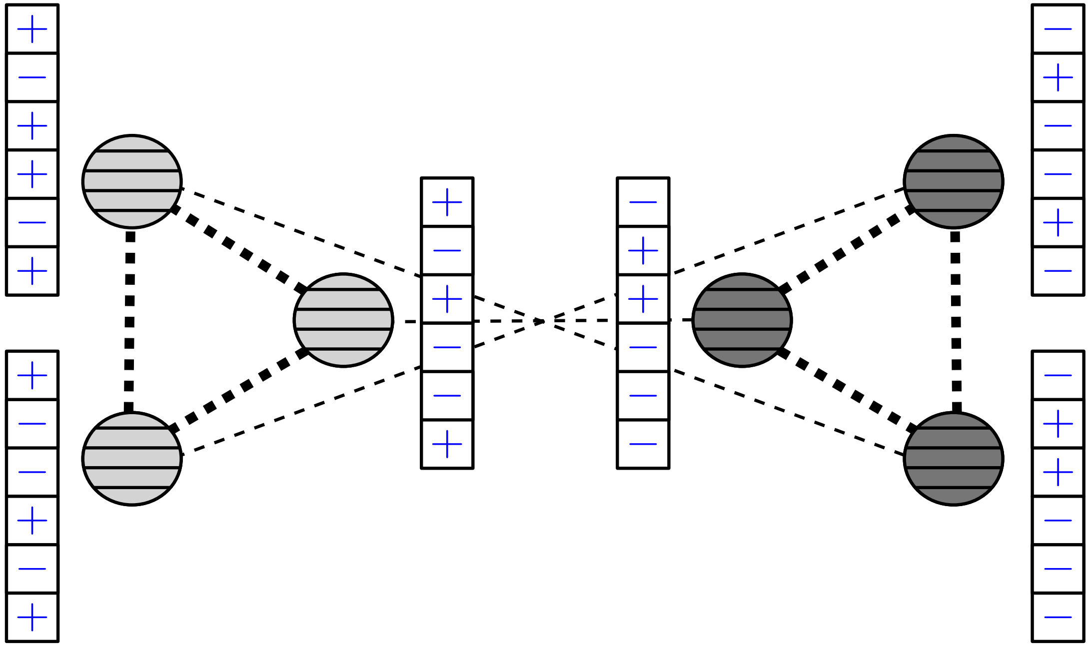
- the labels of nodes in the same community agree on at least $5/6$ entries, and
- the labels of nodes in different communities differ in more than $1/6$ entries.

Community Sensitive Labeling

Example:

> 2 different labels
$\Longrightarrow$ foes!

≤ 2 different labels
$\Longrightarrow$ friends!



$\boxed{\textbf{Warning}\text{: not a dynamics!}}$

Analysis 1/3

Proof Ingredient 1. We are done if, for any fixed component j, all *lucky* nodes $u \notin U$ are such that

$$\Pr\left(h_u = \mathbf{sgn}\Big(\sum_{v \in V(u)} \mathbf{x}_v\Big)\right) \geq \frac{99}{100}.$$

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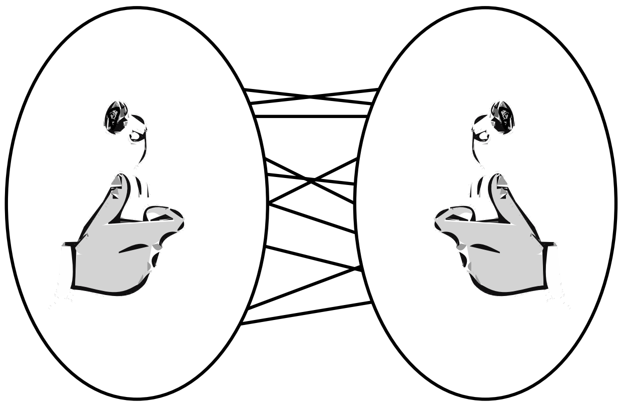

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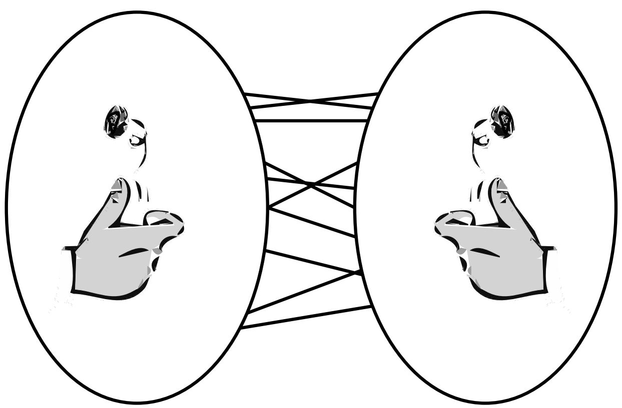

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sign of $\mathbf{x}_u$ at (local) time T

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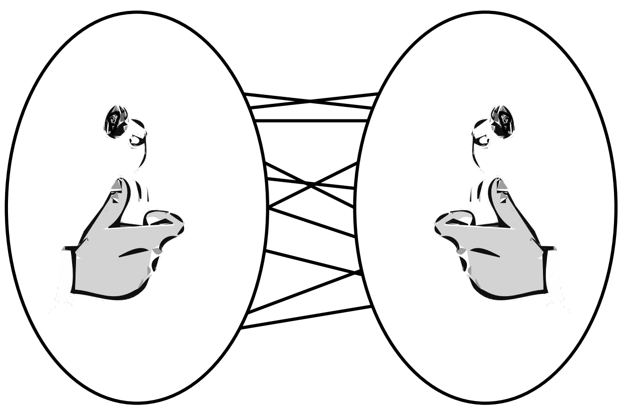

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Problem: bound $|U| = \#$*unlucky* nodes (i.e. $h_u := \mathbf{sgn}(\mathbf{x}_u^{(T)})$ is wrong with small prob.).

Analysis 2/3

Proof Ingredient 2. W.h.p. T happens in (global) time $\Theta(n \log n)$.

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Next idea

$\Longrightarrow$ if for any $t = \Theta(n \log n)$ we prove $\approx \epsilon^2 n$ nodes u are *bad*, namely

$$(\mathbf{x}_u^{(t)} - \sum_{v \in V(u)} \mathbf{x}_v^{(0)})^2 > \frac{\epsilon^2}{n}$$

then we can bound the *unlucky nodes* by bounding a *spreading process*:

- At time $10n \log n$, $\approx \epsilon^2 n$ nodes are *bad/unlucky*, and
- at each following round, a good node become bad **iff** we pick a *cross edge* or an *edge touching a bad node*.

Analysis 3/3: Second Moment Analysis

Proof Ingredient 3. If $\sum_u (\mathbf{x}_u^{(10n\log n)} - \sum_{v \in V(u)} \mathbf{x}_v^{(0)})^2$ is small *(Ingredient 2)*, it remains small for $\mathcal{O}(n \log n)$ rounds.

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Use Markov ineq. on

$$\mathbf{E}\Big[\sum_u (\mathbf{x}_u^{(t)} - \sum_{v\in V(u)} \mathbf{x}_v^{(0)})^2\Big]$$

$$= \mathbb{E}\left[\|\mathbf{x}^{(t)} - \pi_{\mathbf{v}_{1,2}}(\mathbf{x}^{(0)})\|^2\right]$$

$$= \mathbf{E}\Big[\|\pi_{\mathbf{v}_{\geq 2}}(\mathbf{x}_u^{(t)}) - \pi_{\mathbf{v}_2}(\mathbf{x}_u^{(0)})\|^2\Big]$$

$$\leq \mathbf{E}\Big[\|\prod P^{(i)} \pi_{\mathbf{v}_2}(\mathbf{x}_u^{(t)}) - \pi_{\mathbf{v}_2}(\mathbf{x}_u^{(0)})\|^2\Big]$$

$$+ \mathbf{E}\Big[\|\prod P^{(i)} \pi_{\mathbf{v}_{\geq 3}}(\mathbf{x}_u^{(0)})\|^2\Big].$$

$\pi_{\mathbf{v}_i}(\mathbf{x})$ projection on i-th eigenspace

$P^{(i)}$ matrix of averaging at time i

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$$+ \mathbf{E}\left[\|\prod P^{(i)} \pi_{\mathbf{v}_{\geq 3}}(\mathbf{x}_u^{(0)})\|^2\right]. \longleftarrow \text{Need double recurrence}$$

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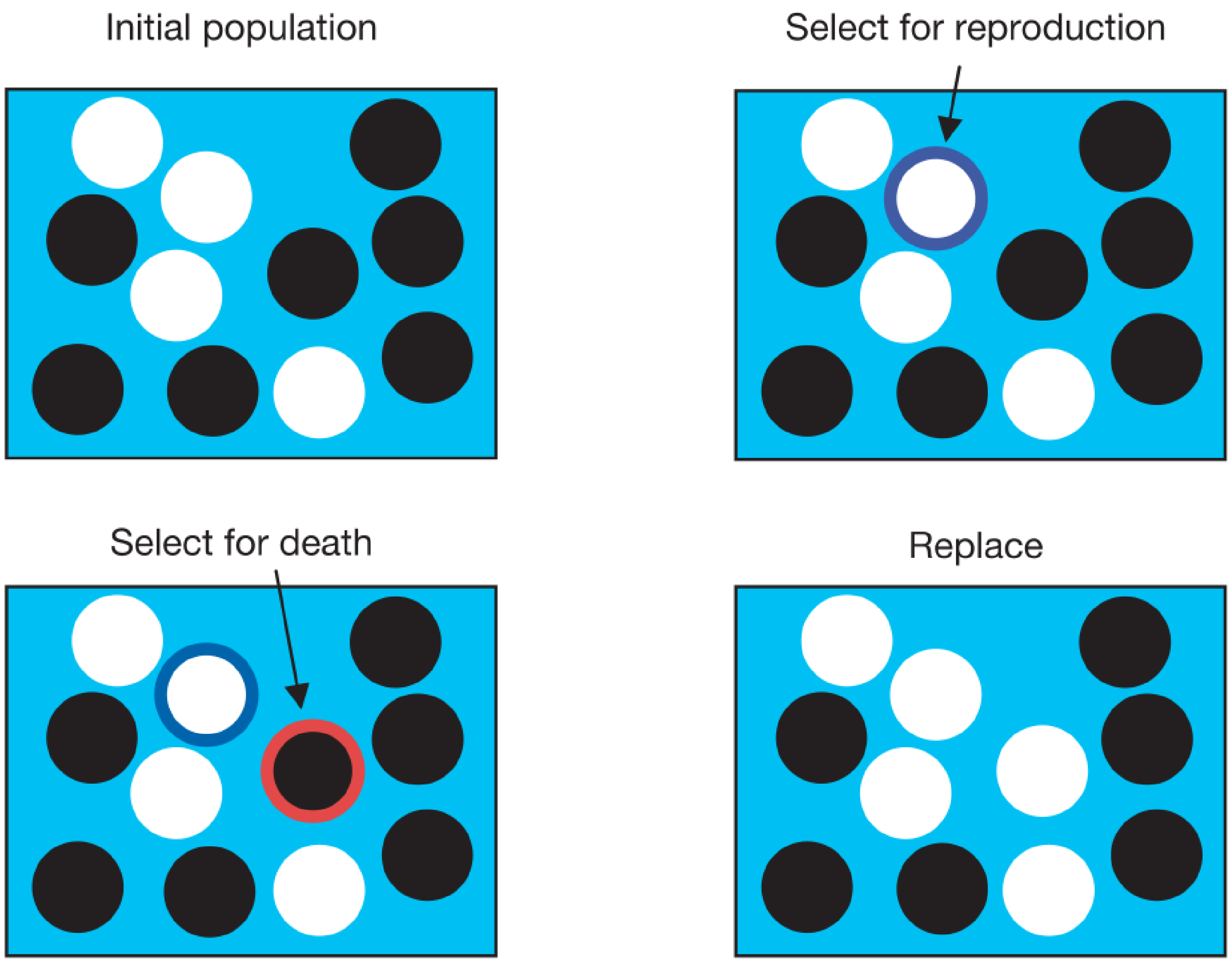
Evolutionary Dynamics on Graphs

[Lieberman, Hauert & Nowak, Nature '05]:

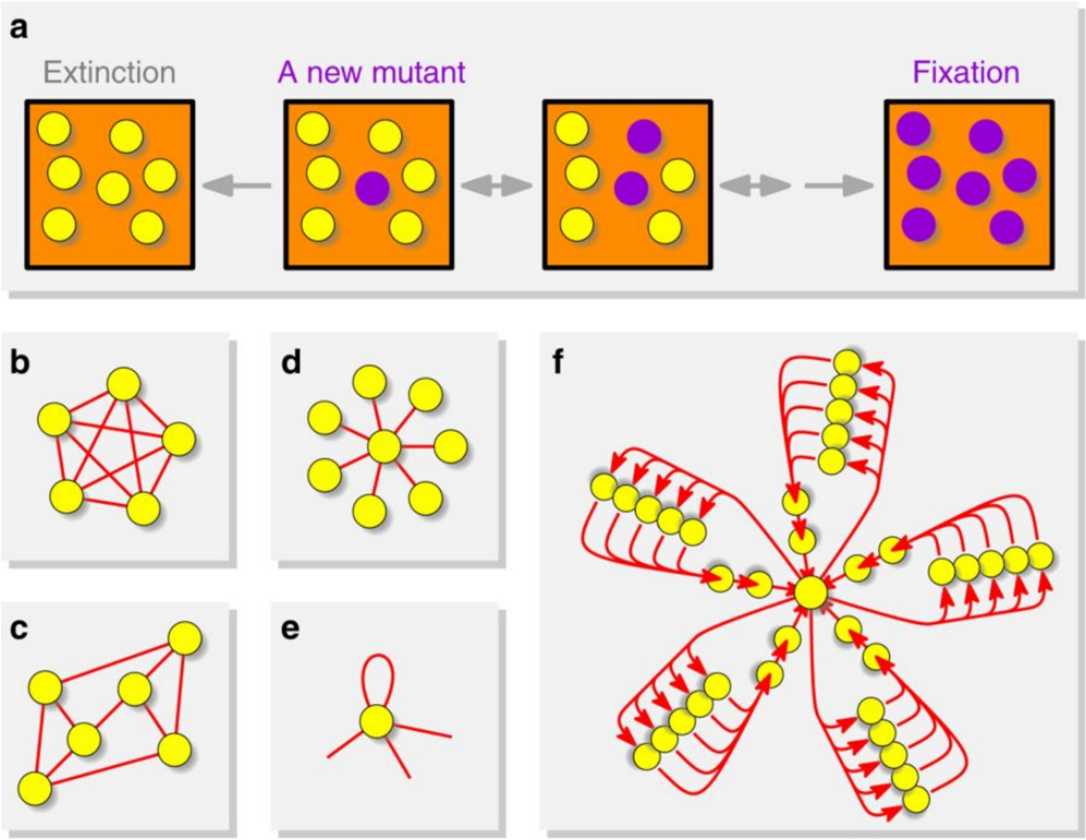


A node is selected randomly according to its fitness and it replaces a random neighbor

Picture from Lieberman et al.

The Moran Process and Fixation Probability

[Giakkoupis '16, Galanis et al. J. ACM '17, Goldberg et al.x2 '18, Pavlogiannis et al. Comm. Bio. '18]:
Probability that a mutant with fitness r conquers a population with fitness 1 on a family of graphs $\{G_n\}_n$.
Are there families G_n with probability $1 - o_n(1)$?



Picture from Pavlogiannis et al.

The Speed of Speciation

The Moran process doesn't provide an explanation for *speciation*

"What is needed now is a shift in focus to identifying more general rules and patterns in the dynamics of speciation. The crucial step in achieving this goal is the development of simple and general dynamical models that can be studied not only numerically but analytically as well. [...] Speciation is expected to be triggered by changes in the environment. Once genetic changes underlying speciation start, they go to completion very rapidly."

[Gavrilets, Evolution '03]

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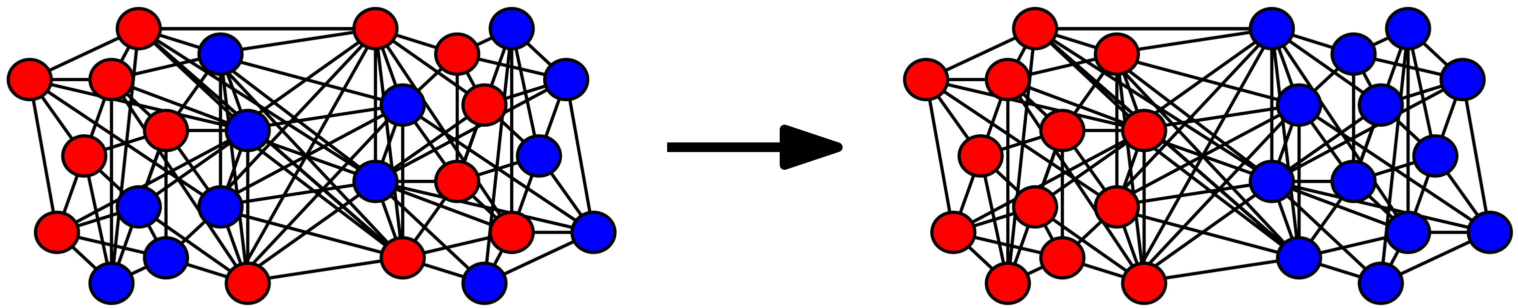
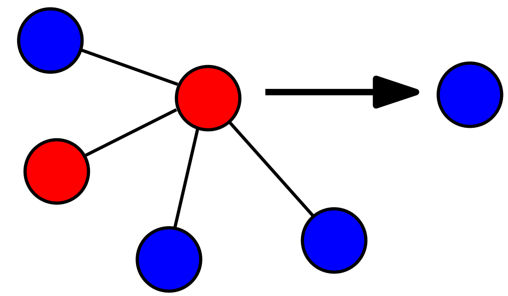
Problem: A simple evolutionary-graph-theoretic proof of principle for speciation.



y-Degree Majority Dynamics

Node gets color x with probability

$$\left(\frac{\#\text{neighbors with col. } x}{\text{degree}}\right)^y$$



$y = 1 \implies$ Voter Dynamics (Moran Process)

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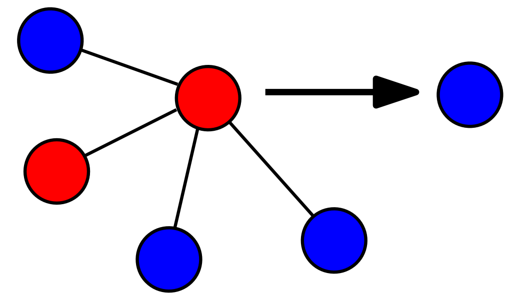

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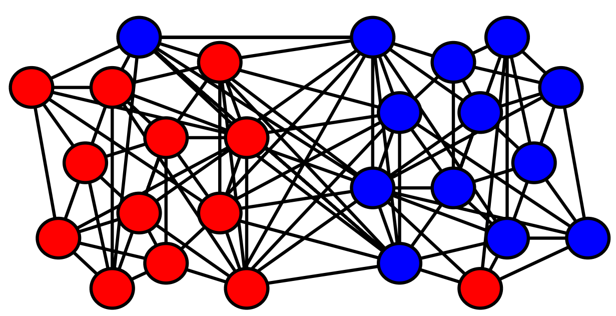
[Cooper et al.x3, ICALP'14, DISC'15, DISC'17]: 2-Choice Dynamics can be related to the *spectral structure* of the graph!

$$\sum_{x\in V}\left(\frac{B(x)}{d}\right)^2 = \|P\mathbf{1}_B\|_2^2 \le \frac{B^2}{n} + \lambda^2 B.$$

$B(x)$ blue neighbors of x, P trans. matrix of graph, $\mathbf{1}_B$ indicator vector of blue-col. nodes, B overall number of blue-col. nodes, λ second-largest eigenvalue of P

Metastability of 2-Choices Dynamics

Theorem [Cruciani, N., Scornavacca, AAAI'19].
G d-regular graph divided in 2 *clusters*, where cut is a b-regular bipartite graph. Each node initially blue or red u.a.r. If $b/d = \mathcal{O}(1/\sqrt{n})$ and spectral radius of clusters is $\mathcal{O}(n^{-\frac{1}{4}})$, then with prob. $\Omega(1)$, after $\mathcal{O}(\log n)$ *time*, clusters are *almost-monochromatic*, with *different colors*, and remains so for $n^{\Omega(1)}$ *time* w.h.p.



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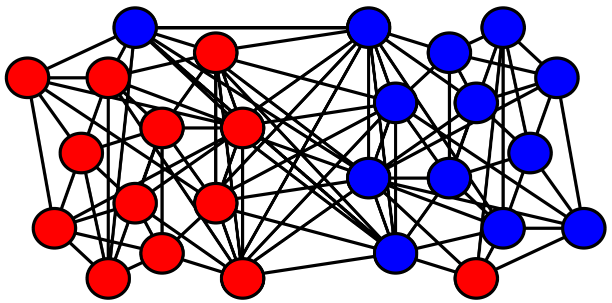

Corollary: LPA. First analytical result on a sparse Label Propagation Algorithm (class of clustering heuristics).

Thank You!