Bounds with Imperfect Instruments: An Extension of Nevo and Rosen

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1 Introduction

The use of instrumental variables (IV) estimators has answered many important questions and greatly improved the credibility of results in applied microeconomics. Some important economic and policy questions remain unanswered on account of an absence of a truly exogenous IV in the setting of the question. Nevo and Rosen (2012) (NR, henceforth) address estimation with what they term "Imperfect Instrumental Variables" (IIV): instruments that are correlated with an endogenous variable but that are not entirely exogenous. They show that it is possible, under the assumption that the instrument is no more endogenous than the endogenous variable itself, to improve upon one of the two previously known bounds. This is done through the use of an additional IV, which is simply a weighted sum of the endogenous regressor and the main instrumental variable.

Here, without making any additional assumptions, we present new bounds that often improve upon the bound that NR does not address. We do so by leveraging a result from the Statistics literature which provides additional information about the relationship between the assumed sets of parameter values in each of the cases that they examine. The rest of the paper proceeds as follows: Section 2 summarizes the NR result, Section 3 explains our method and the new bounds that we arrive at, Section 4 presents results from a simulation and Section 5 concludes.

2 Result from Nevo and Rosen

Assume the following data generating process:

$$y_i = x_i\beta + u_i$$

Then an IIV exists when, for some z, $E[xu] \neq 0$, $E[zu] \neq 0$, $E[xz] \neq 0$. It is widely known that assumptions about the direction of correlations between x, z, and u describe the direction of the bias for both β_{OLS} and the standard β_{IV}^{Z}, thus providing bounds. NR show that a new set of bounds can be obtained when Assumption 1 holds:

Assumption 1 *The instrument and the endogenous variable are correlated with the unobservables in the same direction:* $\rho_{zu}\rho_{xu} > 0$.

NR propose a new instrument:

$$V(\lambda) = \sigma_x z - \lambda\sigma_z x$$

They show that for $\lambda = \lambda^* = \rho_{zu}/\rho_{xu}$, $V(\lambda)$ is both relevant and valid. While $V(\lambda^*)$ is not feasible, as it is a function of unknown parameters ρ_{zu} and ρ_{xu}, NR show that $V(\lambda = 1)$ improves upon bounds on the true value of β.A value of $\lambda = 1$ corresponds to the most conservative case of Assumption 2:

Assumption 2 *The instrument is no more endogenous than the endogenous variable:* $\rho_{zu}^2 \leq \rho_{xu}^2$.

Using the above, NR arrive at the following bounds.

Table 1: Bounds for an IIV Under the Assumptions 1 and 2

	$\rho_{xz} \leq 0$	$\rho_{xz} \geq 0$
Case 1: $\rho_{xu}, \rho_{zu} < 0$	$\beta_{OLS} < \beta_{IV}^{V(1)} < \beta < \beta_{IV}^{Z}$	Case 3: $\beta > \max\{\beta_{IV}^{V(1)}, \beta_{IV}^{Z}\} > \beta_{OLS}$
Case 2: $\rho_{xu}, \rho_{zu} > 0$	$\beta_{IV}^{Z} < \beta < \beta_{IV}^{V(1)} < \beta_{OLS}$	Case 4: $\beta < \min\{\beta_{IV}^{V(1)}, \beta_{IV}^{Z}\} < \beta_{OLS}$

3 Tighter Bounds Leveraging an Implicit Assumption

In this section, we use a result from the Statistics literature to tighten the bounds presented in NR. We first discuss Transitivity in Correlation, the concept that we apply here. Following this we examine bounds in the two sided bounds cases in Table 2 (cases 1 and 2).

3.1 Transitivity in Correlation

In general, if two variables, A and C, are each positively correlated with a third variable, B, it is possible that A and C could be correlated positively, negatively, or not at all with each other. However, Langford et al. (2001) show that, in some circumstances, correlations can be transitive. They prove that a sufficient condition for positive correlation between A and C, when $\rho_{AB}\rho_{BC} > 0$, can be stated as follows: $\rho_{AB}^2 + \rho_{BC}^2 > 1 \implies \rho_{AC} > 0$. Later work by Lipovetsky (2002) shows that a sufficient condition for negative correlation between A and C, when $\rho_{AB}\rho_{BC} < 0$, can be stated as follows: $\rho_{AB}^2 + \rho_{BC}^2 > 1 \implies \rho_{AC} < 0$. We combine these two results to define Theorem 1:[1].

Theorem 1 *A sufficient condition for the product of three related correlations to be positive ($\rho_{AB}\rho_{BC}\rho_{AC} > 0$) is for the sum of the squares of at least one pair of correlations to be greater than one:* $\rho_{AB}^2 + \rho_{BC}^2 > 1 \implies \rho_{AC} > 0$.

Since the sum of any two of the squared correlations being greater than one guarantees a positive product of the three correlations ($\rho_{AB}\rho_{BC}\rho_{AC} > 0$), then, by transposition, if the product of the three correlations is negative ($\rho_{AB}\rho_{BC}\rho_{AC} < 0$), it must be the case that none of the pairs of the correlations sum to more than one. We define the following corollary:

[1] Lipovetsky and Conklin (2004) for a similar for a similar combined expression of the two above findings.

Corollary 1 *The necessary conditions for the product of three related correlations to be negative ($\rho_{AB}\rho_{BC}\rho_{AC} < 0$) is for the sum of the squares of each of the pairwise correlations to be less than one, i.e.,*

$$\rho^2_{AB} + \rho^2_{BC} \leq 1$$
$$\rho^2_{AB} + \rho^2_{AC} \leq 1$$
$$\rho^2_{AC} + \rho^2_{BC} \leq 1$$

Note that for Cases 1 and 2 in Table 1, the product of the three ρ terms is negative. Thus the above corollary gives a general definition of the necessary condition for two-sided bounds. This approach is not applicable to Cases 3 and 4, as in these cases $\rho_{AB}\rho_{BC}\rho_{AC} > 0$. Below we use this result to obtain a new set of two-sided bounds.

We are the first to apply the above result to the econometrics literature. To our knowledge, the only other paper using this result in a similar context is Mauro (1990). This paper, published in the Psychology literature, concerns omitted variables bias. The paper presents an equivalent condition to that used by Langford et al. (2001) and then uses it to provide a range of possible estimates in an empirical exercise. However, no explicit bounding formula for β is provided.

3.2 Tightening the Bounds

The assumptions already placed upon the correlations in Cases 1 and 2 imply that $\rho_{xu}\rho_{zu}\rho_{xz} < 0$, thus Corollary 1 applies:

$$\rho^2_{xu} + \rho^2_{xz} < 1 \tag{1}$$

We solve Equation 1 for ρ^2_{xu}, substitute for the definition of ρ_{xu} and take the square root of each side:

$$\begin{aligned} \rho^2_{xu} &< 1 - \rho^2_{xz} \\ \rho_{xu} = \frac{\sigma_{xu}}{\sigma_x \sigma_u} &\in \pm\sqrt{1 - \rho^2_{xz}} \end{aligned}$$

We then multiply each side by $\frac{\sigma_u}{\sigma_x}$ and substitute the definition of bias to obtain the following.[2]

$$(\beta_{OLS} - \beta) \in \pm\frac{\sigma_u}{\sigma_x}\sqrt{1 - \rho^2_{xz}}$$

[2]For the simple linear model ($y_i = x_i\beta + u_i$), the well known probability limit of the bias in β_{OLS} is $\frac{\sigma_{xu}}{\sigma^2_x}$.

Finally, shifting the set by $-\beta_{OLS}$ and then transforming it by a negative we have two new bounds:

$$
\begin{aligned}
-\beta &\in \pm\frac{\sigma_u}{\sigma_x}\sqrt{1-\rho_{xz}^2}-\beta_{OLS} \\
\beta &\in \beta_{OLS}\pm\frac{\sigma_u}{\sigma_x}\sqrt{1-\rho_{xz}^2} \qquad (2) \\
\beta &\in (\beta_L, \beta_U) \qquad (3)
\end{aligned}
$$

We now provide some intuition about these usefulness of these new bounds. In Case 1 (see Table 1), β_L is clearly dominated by both β_{OLS} (the pre-existing lower bound), as well as by $\beta_{IV}^{V(1)}$ (NR's new tighter lower bound). However, β_H may dominate the pre-existing upper bound of β_{IV}^Z. As ρ_{xz} approaches 0, β_H approaches a maximum bias of $\beta_{OLS}+\frac{\sigma_u}{\sigma_x}$. Ironically, this largest possible value of β_H may also provide the most likely improvement upon β_{IV}^Z: while the bias in β_H is always finite, the bias in β_{IV}^Z, $\frac{\rho_{zu}}{\rho_{xz}}\frac{\sigma_u}{\sigma_x}$, approaches infinity. In short, weak instruments provide the worst case scenario for β_H and β_{IV}^Z, but the worst case scenario for β_{IV}^Z is more problematic. Similar logic applies to β_L in Case 2. However, it should be clear that our bounds do not always dominate the preexisting bounds.

4 Conclusion and Future Work

IV estimation is an important tool in applied microeconomics. Recent work has explored the bounding of estimates when IV is biased. In this paper, we introduce an under used tool to this literature transitivity in correlations. Without making any further assumptions, we present a new set of bounds that may improve estimation, especially when instruments are weak. Future work will explore application of these bounds. One important issue is the choice of an estimator of σ_u, which is itself biased, though it may be bounded. We will also report results from a simulation exercise.

References

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