BME 556: HW6

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Velocity profiles, Flow rate, and Wall shear stress can be calculated for pulsatile flow by separating the steady and oscillatory profiles:

Steady

Velocity For steady flow in a tube we can make the following assumptions:

1. $v_z = f(r)$
2. $v_r = v_\theta = 0$
3. Constant viscosity (newtonian), and density
4. $p = f(z)$
5. $g_z = 0$

Under these assumptions, the z component of the cylindrical Navier-Stokes equations can be reduced and solved with the following boundary conditions:

1. $\left.\dfrac{dv_z}{dr}\right|_{r=0} \neq \infty$
2. $v_z(r = a) = 0$

$$\begin{aligned}
0 &= -\frac{dp}{dz} + \mu\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right) \\
\frac{r}{\mu}\frac{dp}{dz} &= \frac{d}{dr}\left(r\frac{dv_z}{dr}\right) \\
\frac{dv_z}{dr} &= \frac{r}{2\mu}\frac{dp}{dz} + \frac{c}{r} \rightarrow \text{BC 1:} \quad c = 0 \\
v_z &= \frac{r^2}{4\mu}\frac{dp}{dz} + c, \ \text{BC 2:} \quad v_z(r = a) = 0 \\
c &= -\frac{a^2}{4\mu}\frac{dp}{dz}
\end{aligned}$$

Flow Rate Given the above velocity profile, we can calculate the flow rate Q with:

$$\begin{aligned}
Q &= \int_0^a v_z(r) 2\pi r dr \\
&= \int_0^a \frac{a^2}{4\mu}\frac{dp}{dz}\left(1-\left(\frac{r}{a}\right)^2\right) 2\pi r dr \\
&= \frac{2\pi a^2}{4\mu}\frac{dp}{dz}\int_0^a r - \frac{r^3}{a^2} dr \\
&= \frac{2\pi a^2}{4\mu}\frac{dp}{dz}\left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^a \\
&= \frac{2\pi a^2}{4\mu}\frac{dp}{dz}\left(\frac{a^2}{2} - \frac{a^4}{4a^2}\right) \\
&= \frac{2\pi a^4}{8\mu}\frac{dp}{dz}(1-1/2) \\
&= \frac{\pi a^4}{8\mu}\frac{dp}{dz}
\end{aligned}$$

Shear Stress We can similarly calculate shear stress from the velocity profile:

$$\begin{aligned}
\tau_{rz} &= 2\mu D_{rz} = 2\mu * \frac{1}{2}\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{dz}\right) \\
&= \mu\left(\frac{\partial v_z(r,t)}{\partial r}\right) \\
&= \frac{a^2}{4}\frac{dp}{dz}\frac{d}{dr}\left(1 - \frac{r^2}{a^2}\right) \\
&= \frac{a^2}{4}\frac{dp}{dz}\left(-\frac{2r}{a^2}\right) \\
\tau_{rz}\big|_{r=a} &= \frac{a^2}{4}\frac{dp}{dz}\left(-\frac{2a}{a^2}\right) \\
&= -\frac{a}{2}\frac{dp}{dz}
\end{aligned}$$

Oscillatory

Velocity For unsteady flow in a tube, we can make the following assumptions:

1. $v_z = f(r,t)$
2. $v_r = v_\theta = 0$
3. Constant viscosity (newtonian), and density
4. $p = f(z,t)$
5. $g_z = 0$

These assumptions give us the following simplified z component of the cylindrical Navier-Stokes equations:

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right]$$

If we assume the pressure gradient is a sum of sine or cosine waves and represent this as a sum of exponentials: $\frac{\partial p(z,t)}{\partial z} = p_s + p_o \sum_{n=1}^{N} (\cos n\omega t + i \sin n\omega t) = p_s + p_o \sum_{n=1}^{N} e^{in\omega t}$, where n is the harmonic of the wave. Then we can solve the governing equation separately for each harmonic and sum the terms:

$$\rho \frac{\partial v_z}{\partial t} = -p_o e^{in\omega t} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right]$$

Postulating that the velocity profile $v_z(r,t)$ will follow a similar profile as the pressure gradient allows us to separate the r and t components: $v_z(r,t) = V(r)e^{in\omega t}$. Using this separation of variables we have that:

- $\frac{\partial v_z}{\partial t} = in\omega V(r) e^{in\omega t}$
- $\frac{\partial v_z}{\partial r} = \frac{dV(r)}{dr} e^{in\omega t}$
- $\frac{d}{dr} r \frac{dV(r)}{r} = \frac{dV(r)}{dr} + r \frac{d^2 V(r)}{dr}$

$$\rho in\omega V(r) e^{in\omega t} = -p_o e^{in\omega t} + \mu \left[\frac{dV(r)}{dr} + r \frac{d^2 V(r)}{dr} \right] e^{in\omega t}$$

$$\frac{p_0}{\mu} = \frac{d^2 V(r)}{dr^2} + \frac{1}{r} \frac{dV(r)}{dr} - \frac{\rho ni\omega V(r)}{\mu}$$

For $\Omega_n = a\sqrt{\frac{\rho n\omega}{\mu}}$, this becomes:

$$\frac{d^2 V(r)}{dr^2} + \frac{1}{r} \frac{dV(r)}{dr} - \frac{i\Omega_n^2 V(r)}{a^2} = \frac{p_0}{\mu}$$

If we do a change of variables $\zeta = \Lambda \frac{r}{a}$, then $\frac{d}{dr} = \frac{d}{d\zeta} \frac{d\zeta}{dr} = \frac{d}{d\zeta} \frac{\Lambda}{a}$, and $\frac{d^2}{dr^2} = \frac{d^2}{d\zeta^2} \frac{\Lambda^2}{a^2}$. If $\Lambda = \left(\frac{i-1}{\sqrt{2}} \right) \Omega$, then we get:

$$\frac{\Lambda^2}{a^2} \frac{d^2 V(\zeta)}{d\zeta^2} + \frac{\Lambda}{a\zeta} \frac{\Lambda}{a} \frac{dV(\zeta)}{d\zeta} - \frac{i\Omega_n^2 V(\zeta)}{a^2} = \frac{p_0}{\mu}$$

$$\frac{-i\Omega_n^2}{a^2} \frac{d^2 V(\zeta)}{d\zeta^2} + \frac{-i\Omega^2}{a^2} \frac{1}{\zeta} \frac{dV(\zeta)}{d\zeta} - \frac{i\Omega^2 V(\zeta)}{a^2} = \frac{p_0}{\mu}$$

If we divide by $\frac{-i\Omega_n^2}{a^2}$, then separate the solution into homogenous and particular solutions:

$$V = V_h + V_p$$

$$0 = \frac{d^2 V(\zeta)}{d\zeta^2} + \frac{1}{\zeta} \frac{dV(\zeta)}{d\zeta} + V(\zeta) \text{ } (V_h)$$

$$0 = \zeta^2 \frac{d^2 V(\zeta)}{d\zeta^2} + \zeta \frac{dV(\zeta)}{d\zeta} + (\zeta^2 - \alpha^2) V(\zeta), \text{ Where } \alpha = 0$$

This last equation is known as Bessel's differential equation, and has a solution of the form: $V_h(\zeta) = A_n J_0(\zeta) + B_n Y_0(\zeta)$, where J_0 is the 0th order Bessel function of the 1st kind, and Y_0 is the 0th order Bessel function of the second kind. However, $Y_0(0) = -\infty$, so for the solution to be finite at the center, $B_n = 0$.

$$\begin{aligned} V(\zeta) &= V_h(\zeta) + V_p(\zeta) \\ &= A_n J_0(\zeta) + \frac{i p_o a^2}{\mu \Omega^2} \end{aligned}$$

Our next boundary condition is that at the wall, the velocity must be 0. At the wall, $\zeta = \Lambda_n \frac{a}{a}$ $V(\Lambda_n) = 0$. If we plug in this boundary condition and then substitute $V(\zeta)$ back into $v_z = V(\zeta)e^{in\omega t}$, we get:

$$\begin{aligned} 0 &= A_n J_0(\Lambda_n) + \frac{i p_o a^2}{\mu \Omega_n^2} \\ A_n &= -\frac{\frac{i p_o a^2}{\mu \Omega_n^2}}{J_0(\Lambda_n)} \\ V(\zeta) &= \frac{i p_o a^2}{\mu \Omega_n^2}\left(1 - \frac{J_0(\zeta)}{J_0(\Lambda_n)}\right) \\ v_z^n(\zeta, t) &= \frac{i p_o a^2}{\mu \Omega_n^2}\left(1 - \frac{J_0(\zeta)}{J_0(\Lambda_n)}\right) e^{in\omega t} \end{aligned}$$

Flow Rate

$$\begin{aligned} Q_n(t) &= \int_0^a 2\pi r v_z^n(r,t) dr \\ &= \frac{2\pi p_o a^2}{\mu \Omega_n^2} e^{in\omega t} \int_0^a r\left(1 - \frac{J_0(\zeta)}{J_o(\Lambda)}\right) dr \end{aligned}$$

We can substitute $r = \frac{a}{\Lambda}\zeta$ and $dr = \frac{a}{\Lambda}d\zeta$. We can solve this integral by using the identity: $\int_0^a x J_0(x) dx = a J_1(a)$

$$\begin{aligned} &= \frac{2\pi p_o a^2}{\mu \Omega_n^2} e^{in\omega t} \frac{a^2}{\Lambda_n^2 J_0(\Lambda_n)} \int_0^{\Lambda_n} \zeta\left(J_0(\Lambda_n) - J_0(\zeta)\right) d\zeta \\ &= \frac{2\pi p_o a^2}{\mu \Omega_n^2} e^{in\omega t} \frac{a^2}{\Lambda_n^2 J_0(\Lambda_n)} \left(\int_0^{\Lambda_n} \zeta J_0(\Lambda_n) - \int_0^{\Lambda_n} \zeta J_0(\zeta) d\zeta\right) \\ &= \frac{2\pi p_o a^2}{\mu \Omega_n^2} e^{in\omega t} \frac{a^2}{\Lambda_n^2 J_0(\Lambda_n)} \left(\frac{\Lambda_n^2}{2} J_0(\Lambda_n) - \Lambda_n J_1(\Lambda_n)\right) \\ &= \frac{2\pi p_o a^2}{\mu \Omega_n^2} e^{in\omega t} \frac{a^2}{2}\left(1 - \frac{2 J_1(\Lambda_n)}{\Lambda_n J_0(\Lambda_n)}\right) \\ &= \frac{\pi p_o a^4}{\mu \Omega_n^2}\left(1 - \frac{2 J_1(\Lambda_n)}{\Lambda_n J_0(\Lambda_n)}\right) e^{in\omega t} \end{aligned}$$

Shear Stress We can use the constitutive relation $\tau_{rz} = 2\mu\frac{1}{2}\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)$, and the identity $\frac{dJ_0(x)}{dx} = -J_1(x)$:

$$\begin{aligned}
\tau_n(t) &= -\mu \left(\frac{\partial v_n(r,t)}{\partial r}\right)_{r=a} e^{in\omega t} \\
&= -\frac{ip_o a^2}{\Omega_n^2}\left[\frac{d}{dr}\left(1 - \frac{J_0(\zeta)}{J_0(\Lambda_n)}\right)\right]_{r=a} e^{in\omega t} \\
&= -\frac{ip_o a^2}{\Omega_n^2}\frac{\Lambda_n}{a}\left[\frac{d}{d\zeta}\left(1 - \frac{J_0(\zeta)}{J_0(\Lambda_n)}\right)\right]_{\zeta=\Lambda_n} e^{in\omega t} \\
&= -\frac{ip_o a^2}{\Omega_n^2}\frac{\Lambda_n}{a}\left[\left(\frac{J_1(\zeta)}{J_0(\Lambda_n)}\right)\right]_{\zeta=\Lambda_n} e^{in\omega t} \\
&= -\frac{ip_o a^2}{\Omega_n^2}\frac{\Lambda_n}{a}\left(\frac{J_1(\Lambda_n)}{J_0(\Lambda_n)}\right) e^{in\omega t} \\
&= -\frac{p_o a}{\Lambda_n}\left(\frac{J_1(\Lambda_n)}{J_0(\Lambda_n)}\right) e^{in\omega t}
\end{aligned}$$

The last simplification was made since $\frac{i\Lambda}{\Omega^2} = \frac{1}{\Lambda}$

Pulsatile

Velocity Flow is driven by the following combined pressure gradient:

$$\frac{dp}{dz} = -p_s - p_o \sum_{n=1}^{N} C_n e^{i(n\omega t + \phi_n)}, \text{ Where } \omega = \frac{2\pi}{T}$$

Combining our steady and oscillatory velocity profiles, we get:

$$v_{z,o}(r,t) = \frac{p_s a^2}{4\mu}\left(1 - \left(\frac{r}{a}\right)^2\right) + p_o \sum_{n=1}^{N} \frac{iC_n a^2}{\mu\Omega_n^2}\left(1 - \frac{J_0(\zeta)}{J_0(\Lambda_n)}\right) e^{i(\omega nt - \phi_n)}$$

Flow Rate Similarly, combining the steady and oscillatory portions of flow rate, we get:

$$Q(t) = \frac{\pi a^4 p_s}{8\mu} + \sum_{n=1}^{N} \frac{\pi p_n a^4}{\mu\Omega_n^2}\left(1 - \frac{2J_1(\Lambda_n)}{\Lambda_n J_0(\Lambda_n)}\right) e^{in\omega t}$$

Wall shear stress And finally, for wall shear stress we can similarly combine our steady and oscillatory profiles.

$$\tau(t) = -\frac{ap_s}{2} + \sum_{n=1}^{N} -\frac{p_n a}{\Lambda_n}\left(\frac{J_1(\Lambda_n)}{J_0(\Lambda_n)}\right) e^{in\omega t}$$