$$W \xrightarrow{h} S \xrightarrow{g} T$$

1. $h : W \to S$

$\mathrm{Hom}(h, T) : \mathrm{Hom}(S, T) \to \mathrm{Hom}(W, T)$

$\mathrm{Hom}(h, T)(g) \longmapsto g \circ h$

$$\mathrm{Hom}(h, T)(g)(w) = g(h(w))$$

why $|T| \geq 2$?

Show: h is surjective iff

$\mathrm{Hom}(h, T)$ is injective

$g \circ h = \mathrm{Hom}(h, T)(g)$

$$\begin{tikzcd} W \arrow[rr, dashed] \arrow[dr, "h"'] & & T \\ & S \arrow[ur, "g"'] & \end{tikzcd}$$

Assume h surjective and that $g_1 \neq g_2 : S \to T$. To show $g_1 \circ h \neq g_2 \circ h$.

$\exists x \in S \ni g_1(x) \neq g_2(x)$

Say $g_1(x) = t_1$ and $g_2(x) = t_2$

$t_1 \neq t_2$

Since h is surjective, $\exists w \in W \ni$ $h(w) = x$. Since

$\mathrm{Hom}(h,T)(g_1) = g_1 \circ h = t_1$

$\mathrm{Hom}(h,T)(g_2) = g_2 \circ h = t_2$

we see that $g_1 \circ h \neq g_2 \circ h$, so $\mathrm{Hom}(h,T)$ is injective.

Aside: if we had a function $\varphi : W \to T$ (so $\varphi \in \mathrm{Hom}(W,T)$), then is surjectivity of $h : W \to S$ enough to induce a function $g : S \to T$?

$w \qquad \varphi(w)$

$$\begin{tikzcd}
W \arrow[r, "\varphi"] \arrow[d, "h"'] & T \\
S \arrow[ur, dashed, "g?"'] &
\end{tikzcd}$$

x

The idea: can $g(x)$ be defined by this data for all $x \in S$?

You might think so: since h is onto, $\exists\, w \in W \ni h(w) = x$. So define $g(x) = \varphi(w)$.

BUT NO! there is a huge gap in this argument!

<u>Problem is:</u> g is not well-defined, because x might have multiple pre-images!

If $h(w_1) = x = h(w_2)$ with $w_1 \neq w_2$, we don't know that $\varphi(w_1) = \varphi(w_2)$. So what then is the value of $g(x)$? $\varphi(w_1)$ or $\varphi(w_2)$?

[If, however, φ is constant on all h-pre-images of x (all $x \in S$), then this <u>would</u> define g.]