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Acknowledgements: Text

Problem 1.

(i) Find the solution of the following 1st-order PDE:

$$u_t + tu_x = 0, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = e^x, \quad x \in \mathbb{R}$$

This equation is of the form

$$u_t + p(x,t)u_x = 0$$

whereby the general solution is of the form

$$u(x,t) = f(\phi(x,t))$$

where $\phi(x,t) = C$ is the general solution to the ODE $\dfrac{\mathrm{d}x}{\mathrm{d}t} = p(x,t)$. Given that $p(x,t) = t$ in this problem, we see that the general solution to $\phi(x,t)$ is

$$x - \frac{1}{2}t^2 = C$$

Whereby u is constant along this characteristic curve $\phi(x,t) = x - \dfrac{1}{2}t^2$.

The initial condition, given by $u(x,0) = e^x$, therefore implies that the solution to the 1st-order PDE is

$$u(x,0) = f(x) = e^x \Rightarrow u(x,t) = f\left(x - \frac{1}{2}t^2\right) = e^{x-\frac{1}{2}t^2}$$

(ii) Find the solution of the following 1st-order PDE:

$$u_t + 2u_x - u = 0, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = x^2, \quad x \in \mathbb{R}$$

We perform a linear change of variables:

$$\alpha = ax + bt$$
$$\beta = cx + dt$$

And we see that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}$$

Therefore

$$u_t + 2u_x = (b+2a)u_\alpha + (d+2c)u_\beta$$

And by choosing $b = 1$, $a = 0$, $d = -2c = -2 \Rightarrow c = 1$, we obtain $u_\alpha = u$, and therefore, we have that

$$\ln u = \alpha + f(\beta) = t + f(x-2t)$$

The initial conditions give us

$$\ln u(x,0) = f(x) = 2\ln x$$

So that our final form of the general solution is therefore

$$\begin{aligned} \ln u &= \alpha + f(\beta) = t + 2\ln(x-2t) \\ u &= (x-2t)^2 \cdot e^t \end{aligned}$$

(iii) We have the following parabolic PDE:

$$\phi_{xx} + 2\phi_{xy} + \phi_{yy} = 1$$

We know that the families of curves $\xi = c$ where c is a constant are the characteristic curves. Therefore, the change of variables is given by

$$\begin{aligned} \xi &= y - \frac{2}{2\cdot 1}x = y - x \\ \eta &= x \end{aligned}$$

And the transformed equation therefore becomes

$$a\phi_{\xi\xi} + b\phi_{\xi\eta} + c\phi_{\eta\eta} = 1$$

Where (a, b, c) are given by

$$
\begin{aligned}
a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 2(-1)(1) = -2 \\
b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C(\xi_y\eta_y) = 2(1)(-1) + 2 = 0 \\
c &= 0
\end{aligned}
$$

And therefore the canonical equation is $2\phi_{\xi\xi} + 1 = 0$. Now we solve it as follows:

$$
\begin{aligned}
\phi_{\xi\xi} &= -\frac{1}{2} \\
\phi_{\xi} &= -\frac{1}{2}\xi + c_1 + f(\eta) \\
\phi &= -\frac{1}{4}\xi^2 + (c_1 + f(\eta))\xi + g(\eta) + c_2 \\
&= -\frac{(y-x)^2}{4} + (c_1 + f(x))(y - x) + g(x) + c_2
\end{aligned}
$$

(iv) We are given the elliptic PDE

$$u_{xx} - 6u_{xy} + 12u_{yy} = 0$$

And now let us find a transofmration of independent variables to convert it to the form of the Laplace's equation $\nabla^2 f = 0$.

The roots of the characteristic polynomial are given by

$$\lambda_{\pm} = \frac{-6 \pm \sqrt{6^2 - 48}}{2} = -3 \pm \sqrt{3^2 - 12} = -3 \pm i\sqrt{3}$$

And therefore, we see that the characteristics of the equation are:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3 \pm i\sqrt{3} \Rightarrow y = (-3 \pm i\sqrt{3})x + c$$

So we see that the two families of the characteristic curves are:

$$c_1 = y + (3 + i\sqrt{3})x, \quad c_2 = y + (3 - i\sqrt{3})x$$

And therefore the required transformation variables ξ and η are

$$
\begin{aligned}
\xi &= \frac{c_1 + c_2}{2} = y + 3x \\
\eta &= \frac{c_1 + c_2}{2i} = \sqrt{3}x
\end{aligned}
$$

So the transformed equation therefore becomes

$$a\phi_{\xi\xi} + b\phi_{\xi\eta} + c\phi_{\eta\eta} = 0$$

Where (a, b, c) are given by

$$\begin{aligned}
a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = (3)^2 - 6(3) + 12 = 3 \\
b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C(\xi_y\eta_y) = 2(1)(3)(\sqrt{3}) - 6\sqrt{3} = 0 \\
c &= 3
\end{aligned}$$

And therefore the canonical equation is $3\phi_{\xi\xi} + 3\phi_{\eta\eta} = 0 \Rightarrow \phi_{\xi\xi} + \phi_{\eta\eta} = \nabla^2\phi = 0$ which is the Laplacian in (ξ,η) coordinates.

Problem 2.

We consider the partial differential equation:

$$u_t + \alpha u_x - \beta u_{xxx} + \gamma u_{xxxx} = 0$$

along a periodic domain $[0, 2\pi]$ with boundary conditions:

$$u(0) = u(2\pi), \quad u_x(0) = u_x(2\pi), \quad u_{xx}(0) = u_{xx}(2\pi), \quad u_{xxx}(0) = u_{xxx}(2\pi)$$

with the following initial conditions:

$$u(x, 0) = \cos^2 x - \sin^2 x$$

Because the periodic is doubly periodic, $u(x, t)$ must take on the form:

$$u(x, t) = \sum_{k=-\infty}^{\infty} u_k(t) \cdot e^{ikx}$$

And now, substituting the above into the initial PDE and eliminating e^{ikx}, we get

$$\frac{\partial u_k}{\partial t} + \alpha i k u_k + \beta i k^3 u_k + \gamma k^4 u_k = 0$$
$$\frac{\partial u_k}{\partial t} + u_k(\alpha i k + \beta i k^3 + \gamma k^4) = 0$$

Doing the separation of variables, we then obtain

$$\begin{aligned}\frac{\mathrm{d}u_k}{u_k} &= -(\alpha i k + \beta i k^3 + \gamma k^4)\,\mathrm{d}t \\ \ln|u_k| &= -(\alpha i k + \beta i k^3 + \gamma k^4)t + c \\ \therefore u_k &= Ae^{-(\alpha i k + \beta i k^3 + \gamma k^4)t}\end{aligned}$$

So the final form of $u(x, t)$ is

$$u(x, t) = \sum_{k=-\infty}^{\infty} A_k e^{-(\alpha i k + \beta i k^3 + \gamma k^4)t} \cdot e^{ikx}$$

Given the initial conditions, we see that

$$\begin{aligned}u(x, t) &= \sum_{k=-\infty}^{\infty} A_k e^{ikx} = \cos^2 x - \sin^2 x \\ &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 - \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 = \frac{e^{2ix} + 2 + e^{-2ix}}{4} + \frac{e^{2ix} - 2 + e^{-2ix}}{4} \\ &= \frac{1}{2}(e^{2ix} + e^{-2ix})\end{aligned}$$

Therefore we see that $A_{-2} = A_2 = 1/2$ and all other $A_k = 0$, such that

$$u(x,t) = \frac{1}{2}\left(e^{-(2\alpha i + 8\beta i + 16\gamma)t} \cdot e^{2ix} + e^{(2\alpha i + 8\beta i - 16\gamma)t} \cdot e^{-2ix}\right)$$

Problem 3.

(i) The solution to a unit impulse at the origin in an infinite domain has the form

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right)$$

We wish to verify that for $t > 0$, the above solution is indeed a solution of the heat conduction equation $u_t = \kappa u_{xx}$.

$$\begin{aligned}
u_t &= -\frac{1}{2t}\frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right) + \frac{x^2}{4kt^2}\frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right) \\
&= \left(-\frac{1}{2t} + \frac{x^2}{4\kappa t}\right) u \\
u_x &= -\frac{x}{2\kappa t} u \\
\therefore u_{xx} &= -\frac{1}{2\kappa t} u + \left(\frac{x}{2\kappa t}\right)^2 u = \left(-\frac{1}{2\kappa t} + \frac{x^2}{4\kappa^2 t}\right) u = \frac{u_t}{\kappa}
\end{aligned}$$

(ii) We next perform a change of independent variables $(x,t) \to (\xi, t')$, where

$$\xi = x - Ut, \quad t' = t$$

Therefore, we see that

$$\begin{aligned}
u_t &= u_{t'} t'_t + u_\xi \xi_t = u_{t'} - U u_\xi \\
u_x &= u_{t'} t'_x + u_\xi \xi_x = u_\xi \\
u_{xx} &= u_{t't'} t'^2_x + 2u_{t'\xi}\xi_x t'_x + u_{\xi\xi}\xi_x^2 + u_{t'} t'_{xx} + u_\xi \xi_{xx} \\
&= 0 + 0 + u_{\xi\xi} + 0 + 0 = u_{\xi\xi}
\end{aligned}$$

So the linear convection-diffusion equation is transformed into

$$u_t + U u_x = \kappa u_{xx} \Rightarrow u_{t'} - U u_\xi + U u_\xi = u_{t'} = u_{\xi\xi}$$

The solution of which is

$$\begin{aligned}
u(\xi, t') &= \frac{1}{\sqrt{4\pi\kappa t'}} \exp\left(-\frac{\xi^2}{4\kappa t'}\right) \\
\therefore u(x - Ut, t) &= \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x-Ut)^2}{4\kappa t}\right)
\end{aligned}$$

(iii) We consider discretized forms of the components of the linear convection-diffusion equation as follows:

$$
\begin{aligned}
u_t &= \frac{\partial u}{\partial t} = \frac{P_i^{n+1} - P_i^n}{\tau} \\
u_x &= \frac{\partial u}{\partial x} = \frac{P_{i+1}^n - P_{i-1}^n}{2\Delta x} = \frac{P_{i+1}^n - P_{i-1}^n}{2/M} = \frac{M(P_{i+1}^n - P_{i-1}^n)}{2} \\
u_{xx} &= M^2(P_{i+1} + P_{i_1} - 2P_i)
\end{aligned}
$$

Such that we convert the discrete Markov process to:

$$
\begin{gathered}
P_i^{n+1} - P_i^n + \left(p - \frac{1}{2}\right)(P_{i+1}^n - P_{i-1}^n) = \frac{1}{2}(P_{i+1} + P_{i_1} - 2P_i) \\
\tau u_t + \frac{2p-1}{M} u_x = \frac{2}{M^2} u_{xx} \Rightarrow u_t + \frac{2p-1}{M\tau} = \frac{1}{2M^2\tau} u_{xx}
\end{gathered}
$$

Comparing against the actual linear convection-diffusion equation, and noting that $t = N\tau$, we have that

$$
\begin{aligned}
U = \frac{2p-1}{M\tau} = \frac{2p-1}{Mt/N} &\Rightarrow Ut = \frac{(2p-1)N}{M} \\
\kappa = \frac{1}{2M^2\tau} &\to \kappa t = \frac{N}{2M^2}
\end{aligned}
$$

Let us now transform x and t to x_n and t_n such that

$$
x_n = x + c_x, \quad t_n = t + c_t
$$

Such that when $t = 0$)

$$
\begin{aligned}
u(x_n, 0) &= \frac{1}{\sqrt{4\pi\kappa c_t}} \exp\left(-\frac{(x + c_x - Uc_t)^2}{4\kappa c_t}\right) \\
&= \frac{1}{\sqrt{2\pi(0.05)^2}} \exp\left(-\frac{(x - 0.2)^2}{2(0.05)^2}\right)
\end{aligned}
$$

So, we see that c_t and c_x are:

$$
\begin{aligned}
c_t &= \frac{2(0.05)^2}{4\kappa} = \frac{1}{800\kappa} \\
c_x &= -0.2 + Uc_t
\end{aligned}
$$

And therefore, the final equation therefore becomes

$$
\begin{aligned}
u(x_n, t_n) &= \frac{1}{\sqrt{4\pi\kappa t_n}} \exp\left(-\frac{(x_n - Ut_n)^2}{4\kappa t_n}\right) \\
&= \frac{1}{\sqrt{4\pi\kappa(t + c_t)}} \exp\left(-\frac{(x + c_x - U(t + c_t))^2}{4\kappa(t + c_t)}\right) \\
&= \frac{1}{\sqrt{4\pi\kappa(t + c_t)}} \exp\left(-\frac{(x - 0.2 - Ut)^2}{4\kappa(t + c_t)}\right) \\
&= \frac{1}{\sqrt{4\pi(\kappa t + 1/800)}} \exp\left(-\frac{(x - 0.2 - Ut)^2}{4(\kappa t + 1/800)}\right)
\end{aligned}
$$

So we have the analytical solution for $t = 0$ and $t = N\tau$

$$
\begin{aligned}
u(x, 0) &= \frac{20}{\sqrt{2\pi}} e^{-200(x-0.2)^2} \\
u(x, N\tau) &= \frac{1}{\sqrt{4\pi(N/2M^2 + 1/800)}} \exp\left(-\frac{(x - 0.2 - (2p-1)N/M)^2}{4(N/2M^2 + 1/800)}\right)
\end{aligned}
$$

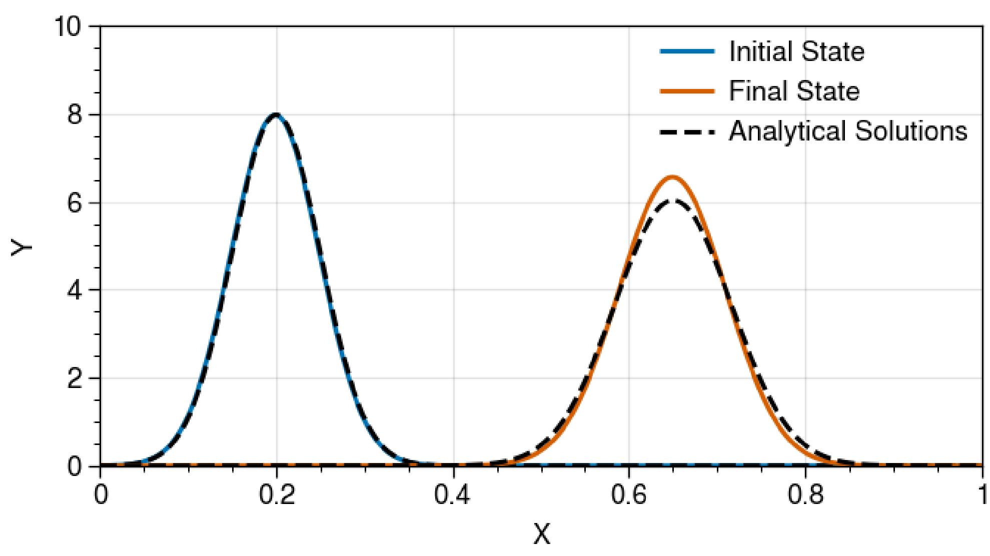

Figure 1: Plots of our analytic solutions (black-dashed), and initial (blue) and final (red) numerical solutions to the linear convection-diffusion equation. Here $M = 400$, $N = 300$ and $p = 0.8$.