(06 - ME): The solution of D.E $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

Sol:

$$P = 2x \qquad Q = e^{-x^2}$$

$$I.F = e^{\int P dx} = e^{x^2}$$

Solution is $y(e^{x^2}) = \int e^{-x^2} \cdot e^{x^2} dx + C$

$$y \cdot e^{x^2} = x + C$$

Given $y(0) = 1 \Rightarrow 1 \cdot e^0 = 0 + C \Rightarrow C = 1$

$$\therefore y e^{x^2} = x + 1$$

$$y = (x+1) e^{-x^2}$$

(CE-07): The solution of D.E $\frac{dy}{dx} = x^2 y$ with the condition that $y = 1$ at $x = 0$ is

Sol:

$$\frac{1}{y} dy = x^2 dx$$

I.O.B.S $\log y = \frac{x^3}{3} + C$

$$y = e^{x^3/3} \cdot e^C$$

Given $y = 1$ at $x = 0 \Rightarrow 1 = e^0 \cdot e^C \Rightarrow e^C = 1$

$\therefore$ solution is $y = e^{x^3/3}$

(CE-09): Solution of D.E $3y\frac{dy}{dx} + 2x = 0$ represents a family of

Sol:

$$3y\,dy + 2x\,dx = 0$$

I.O.B.S $3\frac{y^2}{2} + 2\frac{x^2}{2} = C$

$$\frac{x^2}{1} + \frac{y^2}{(2/3)} = C$$

Represents a family of ellipses

(ME-09): The solution of $x\frac{dy}{dx} + y = x^4$ with $y(1) = \frac{6}{5}$ is

Sol:

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = x^3$$

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(IN-10): consider the differential equation $\frac{dy}{dx}+y=e^x$ with $y(0)=1$. Then the value of $y(1)$ is

Sol: $I.F = e^{\int dx} = e^x$

Solution is $y.e^x = \int e^x.e^x dx + C$

$$ye^x = \frac{e^{2x}}{2}+C$$

Given $y(0)=1 \Rightarrow 1=\frac{1}{2}+C \Rightarrow C=\frac{1}{2}$

$$\therefore ye^x = \frac{e^{2x}}{2}+\frac{1}{2}$$

$$y=\frac{e^x}{2}+\frac{1}{2}e^{-x} = \frac{1}{2}(e^x+e^{-x})$$

$$y(1) = \frac{e^1+e^{-1}}{2}$$

(EE-11): With K as constant, the possible solution for the 1st order D.E $\frac{dy}{dx}=e^{-3x}$ is

Sol:
$$dy = e^{-3x}dx$$

I.O.B.S $\quad y = \frac{e^{-3x}}{-3}+K$

(EC-11): The solution of the D.E $\frac{dy}{dx}=Ky$, $y(0)=C$ is

Sol:
$$\frac{1}{y}dy = K dx$$

$$\log y = Kx + C_1$$

$$y = e^{Kx}.e^{C_1}$$

Given $y(0)=C \Rightarrow C=e^{C_1}$

$$\therefore y = e^{Kx}.C$$

(ME-11): Conside the D.E, $\frac{dy}{dx}=(1+y^2)x$. The general solution with constant 'c' is

Sol:
$$\frac{1}{1+y^2}dy = x\,dx$$

I.O.B.S $\quad \tan^{-1}y = \frac{x^2}{2}+C \Rightarrow y=\tan\left(\frac{x^2}{2}+C\right)$

(CE-11): The solution of D.E $\frac{dy}{dx}+\frac{y}{x}=x$ with condition that $y=1$ at $x=1$ is

Sol:

$$I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

solution is $y.x = \int x.x\,dx + C$

$$xy = \frac{x^3}{3}+C$$

Given $y=1$ at $x=1$

$$1 = \frac{1}{3}+C \Rightarrow C=\frac{2}{3}$$

$$\therefore xy = \frac{x^3}{3}+\frac{2}{3} \Rightarrow y = \frac{x^2}{3}+\frac{2}{3x}$$

EC,EE,IN-12: With initial condition $x(1)=0.5$, the solution of the D.E $t\frac{dx}{dt}+x=t$ is

Sol:

$$\frac{dx}{dt}+\frac{x}{t}=1$$

$$I.F = e^{\int \frac{1}{t}dt} = e^{\log_e t} = t$$

solution is $x.t = \int t.1\,dt + C$

$$xt = \frac{t^2}{2}+C$$

Given $x(1)=0.5 \Rightarrow 0.5(1) = \frac{1}{2}(1)+C \Rightarrow C=0$

$$\therefore xt = \frac{t^2}{2} \Rightarrow x = \frac{t}{2}$$

(CE-12): The solution of O.D.E $\frac{dy}{dx}+2y=0$ for the boundary condition, $y=5$ at $x=1$ is

Sol:

$$I.F = e^{\int P dx} = e^{2x}$$

Sol^n is $y.e^{2x} = \int 0\,dx + C$

$$y = c.e^{-2x}$$

Given $y=5$ at $x=1 \Rightarrow 5 = ce^{-2}$

$$c = \frac{5}{e^2}$$

$$y = \frac{5}{e^2}e^{-2x}$$

⑩ Q (EC-13): A system described by a linear, constant co-efficient ordinary, first ODE has an exact solution given by $y(t)$ for $t>0$, when the forcing function is $x(t)$ and the initial condition is $y(0)$. If one wishes to modify the system so that the solution becomes $-2y(t)$ for $t>0$, we need to

Sol: Solution is $y(t) = \frac{1}{I.F}\int \underbrace{x(t)}_{\text{forcing fun}}(I.F)\,dt + \underbrace{y(0)}_{\text{Initial cond}}$

To get $-2y(t)$ multiply on b.s with '-2'

$$-2y(t) = \frac{1}{I.F}\int (-2x(t))(I.F)\,dt + (-2y(0))$$

∴ Forcing function changes to $-2x(t)$

Initial condition " " $-2y(0)$

(ME-14): The general solution of the D.E $\frac{dy}{dx} = \cos(x+y)$, with 'c' as a constant is

Sol: Let $x+y=t \Rightarrow 1+\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} - 1 = \cos t$$

$$\frac{dt}{dx} = 1+\cos t = 2\cos^2\frac{t}{2}$$

$$\frac{1}{2}\sec^2\frac{t}{2}\,dt = dx$$

$$\tan\left(\frac{t}{2}\right) = x+C \Rightarrow \tan\left(\frac{x+y}{2}\right) = x+C$$

(ME-14): The solution of the initial value problem $\frac{dy}{dx} = -2xy$ $y(0)=2$ is

Sol:

$$\frac{1}{y}dy = -2x\,dx$$

$$\log y = -x^2 + C$$

$$y = e^{-x^2}\cdot e^{C}$$

Given $y(0) = 2 \Rightarrow 2 = e^{C}$

$$y = 2e^{-x^2}$$

Higher order linear differential equation:

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. The general linear differential equation of n^{th} order is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X$$

Where $P_1, P_2, \ldots, P_n$ are functions of x or constants

Linear differential equations with constant co.efficients are of the form

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + K_n y = X \rightarrow (1)$$

Where $K_1, K_2, \ldots K_n$ are constants.

Operator D:

Denote $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, $\frac{d^3}{dx^3} = D^3$, ---

from (1) $D^n y + K_1 D^{n-1} y + K_2 D^{n-2} y + \cdots + K_n y = X$

$$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y = X$$

$$f(D) \cdot y = X \rightarrow (2)$$

Where $f(D) = D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n$

i.e a polynomial in D

The solution of eq (2) is in the form

$y =$ complementing function + particular integral

$= C.F + P.I$

Complementary function: It is the solution of a differential equation of the form $f(D) y = 0$

particular solution: It is the one particular solution of the given equation

(11) Rules for finding complementing function:

In the D.E $f(D)y=0$ the equation $f(D)=0$ is called auxilary equation (A.E). Depending on the roots of A.E, we have following cases for complementing function.

Case I: When roots of the A.E are real and distinct say $m_1, m_2, m_3, \ldots, m_n$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II: When roots of the A.E are real and repeated say $m_1, m_1, m_1, m_4, m_5, \ldots, m_n$

$$C.F = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}$$

Case III: When roots of the A.E are complex and distinct say $\alpha + i\beta, m_3, m_4, \ldots, m_n$

$$C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Case IV: When roots of A.E are complex and repeated say $\alpha \pm i\beta, \alpha \pm i\beta, m_5, m_6, \ldots, m_n$

$$C.F = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Prob: Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Sol: A.E $\Rightarrow D^2 + 5D + 6 = 0$

$$(D+2)(D+3) = 0$$

$$D = -2, -3$$

C.F is $y = c_1 e^{-2x} + c_2 e^{-3x}$

Prob: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Sol: A.E $\Rightarrow D^2 + 2D + 1 = 0$

$(D+1)^2=0 \Rightarrow D=-1,-1.$

$C.F = (C_1+C_2x)e^{-x}$

prob: solve D.E $\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0$

sol: $D^2+D+1=0 \Rightarrow D=\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$

$C.F = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right]$

prob: Solve $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+3x=0$

sol: A.E $D^2+2D+3=0 \Rightarrow D=-1\pm\sqrt{2}i$

$C.F\ x=e^{-t}[C_1\cos\sqrt{2}t+C_2\sin\sqrt{2}t]$

prob: Solve $\frac{d^3y}{dx^3}+y=0$

sol: A.E $D^3+1=0 \Rightarrow (D+1)(D^2-D+1)=0$

$D=-1,\ \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$

$C.F = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right]+C_3e^{-x}$

prob: solve $\frac{d^3x}{dt^3}-6\frac{d^2x}{dt^2}+11\frac{dx}{dt}-6x=0$

sol: A.E $D^3-6D^2+11D-6=0$

$$\begin{array}{r|rrrr} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$(D-1)(D^2-5D+6)=0$

$D=1,2,3$

C.F $x=C_1e^t+C_2e^{2t}+C_3e^{3t}$

Inverse operator: An operator $\frac{1}{f(D)}$ which is operating with $f(D)y$ gives 'y'. Here $f(D)$ & $\frac{1}{f(D)}$ are called inverse operators to each other.

For the D.E $f(D).y=x$, the particular Integral given by $P.I=\frac{1}{f(D)}x$

Depending on the function 'x' we have the following cases.

Rules for particular Integration (P.I):

Case I: When $X = e^{ax}$

$$P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \left[\text{Replace D by a if } f(a) \neq 0 \right]$$

when $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(D)} e^{ax}$$

$$= x \cdot \frac{1}{f'(a)} \cdot e^{ax} \qquad \text{if } f'(a) \neq 0$$

If $f'(a) = 0$, then

$$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax} \qquad \text{if } f''(a) \neq 0$$

$\vdots$

prob: Find the P.I of $(D^2+3D+2)y = e^{2x}$

Sol:
$$P.I = \frac{1}{D^2+3D+2} e^{2x}$$

$$= \frac{1}{4+6+2} e^{2x} = \frac{1}{12} e^{2x}$$

prob: Find the P.I of $(D^2+3D+2)y = e^{-2x}$

Sol:
$$P.I = \frac{1}{D^2+3D+2} e^{-2x} \qquad f(-2) = 4-6+2 = 0$$

$$= \frac{x}{2D+3} e^{-2x} = \frac{x}{-4+3} e^{-2x}$$

$$= -xe^{-2x}$$

prob: Find the P.I of $(D^2+2D+1)y = e^{-x}$

Sol:
$$P.I = \frac{1}{D^2+2D+1} e^{-x} \qquad f(-1) = 1-2+1 = 0$$

$$= \frac{x}{2D+2} e^{-x} = \frac{x^2}{2} e^{-x} \qquad f'(-1) = -2+2 = 0$$

Note:
1. $X = \sinh x = \frac{e^x - e^{-x}}{2}$

2. $X = K = K \cdot e^{0x}$ 3. $X = a^x = e^{\log_e a^x} = e^{x \log_e a}$

Case II: When $x = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I = \frac{1}{f(D^2)}\sin(ax+b)$$

Replace D^2 by $-a^2$

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b) \quad \text{if } f(-a^2)\neq 0$$

If $f(-a^2)=0$

$$\frac{1}{f(D^2)}\sin(ax+b) = x\cdot\frac{1}{f'(-a^2)}\sin(ax+b) \quad \text{if } f'(-a^2)\neq 0$$

If $f'(-a^2)=0$

$$\frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b) \quad \text{if } f''(-a^2)\neq 0$$

$\vdots$

prob: Find the P.I of $(D^2+1)y = \sin(x+2)$

Sol:
$$P.I = \frac{1}{D^2+1}\sin(x+2) \qquad f(-1)=0$$

$$= \frac{x}{2D}\sin(x+2)$$

$$= \frac{x.D}{2D^2}\sin(x+2) = -\frac{x}{2}D\left[\sin(x+2)\right]$$

$$= -\frac{x}{2}\cos(x+2)$$

prob: Find the P.I of $(D^2+3D+2)y = \cos(2x-1)$

Sol:
$$P.I = \frac{1}{D^2+3D+2}\cos(2x-1)$$

$$= \frac{1}{-4+3D+2}\cos(2x-1)$$

$$= \frac{1}{3D-2}\cos(2x-1)$$

$$= \frac{3D+2}{(3D-2)(3D+2)}\cos(2x-1)$$

$$= \frac{3D+2}{9D^2-4}\cos(2x-1) = \frac{3D+2}{9(-4)-4}\cos(2x-1)$$

$$= \frac{-1}{40}\left[-6\sin(2x-1)+2\cos(2x-1)\right]$$

Note: 1. When $X = \sin^2 x = \frac{1-\cos 2x}{2}$

2. When $X = \sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Case III: When $X = x^n$

$$P.I = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$$

Expand $[f(D)]^{-1}$ in increasing powers of 'D' and operate on x^n.

Prob: Find the P.I of $(D^2+2D+1)y = (x^2+x)$

Sol: $P.I = \frac{1}{D^2+2D+1}(x^2+x)$

$= \frac{1}{(1+D)^2}(x^2+x) = (1+D)^{-1}[x^2+x]$

$= [1-2D+3D^2-4D^3+\ldots][x^2+x]$

$= x^2+x-2(2x+1)+3(2) = x^2-3x+4.$

Prob: Find P.I of $(D^2+\cancel{3}^{2}D+3)y = x$

Sol: $P.I = \frac{1}{D^2+2D+3} x$

$= \frac{1/3}{1+\frac{2}{3}D+D^2/3} x = \frac{1}{3}\left[1+\frac{2}{3}D+\frac{D^2}{3}\right]^{-1} x$

$= \frac{1}{3}\left[1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)+\left(\frac{2D}{3}+\frac{D^2}{3}\right)^2 - \ldots\right] x$

$= \frac{1}{3}\left[x-\frac{2}{3}\right]$

Prob: Find the P.I of $(D^2+2D)y = x$

Sol: $P.I = \frac{1}{D^2+2D} x$

$= \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x$

$= \frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{2}-\right] x$

$= \left[\frac{1}{2D}-\frac{1}{4}+\frac{D}{4}\right] x$ [high order terms vanishes]

$= \frac{1}{2}\frac{x^2}{2}-\frac{x}{4}+\frac{1}{4} = \frac{x^2-x+1}{4}$

Case IV: When $X = e^{ax}\cdot V$, V being a function of 'x'

$$P.I = \frac{1}{f(D)}(e^{ax}\cdot V) = e^{ax}\cdot\frac{1}{f(D+a)}\cdot V$$

Prob: Find P.I of $(D^2-2D+4)y = e^x \cos x$

Sol:
$$P.I = \frac{1}{D^2-2D+4}e^x\cdot\cos x$$
$$= e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x$$
$$= e^x\frac{1}{D^2+3}\cos x = e^x\cdot\frac{1}{-1+3}\cos x = \frac{e^x\cos x}{2}$$

Case V: When X is any other function of x.

$$P.I = \frac{1}{f(D)}X$$

If $f(D) = (D-m_1)(D-m_2)\cdots$, Resolve into partial fractions

$$P.I = \left[\frac{A_1}{(D-m_1)} + \frac{A_2}{(D-m_2)} + \cdots\right]X$$

$$\boxed{\text{complete solution} = C.F + P.I}$$

Equations reducible to linear equation with constant coefficients:

1. Cauchy's homogeneous linear D.E: An equation of the form
$$x^n\frac{d^ny}{dx^n} + k_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_{n-1}x\frac{dy}{dx} + k_ny = X$$
is called cauchy's linear D.E. To convert this equation into linear equation with constant co.efficients.

Put $\log x = t \Rightarrow x = e^t$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x} \Rightarrow x\cdot\frac{dy}{dx} = Dy \qquad \text{if } D = \frac{d}{dt}$$

$$x^2\frac{dy}{dx} = D(D-1)y \quad , \quad x^3\frac{dy}{dx} = D(D-1)(D-2)y \;\cdots$$

By substituting these values this equation reduces to the form $f(D)\cdot y = T$

(14) Prob: Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol: This is Cauchy's Linear D.E

Put $x = e^t \Rightarrow t = \log x$

$D(D-1)y - Dy + y = t$

$\Rightarrow [D-1]^2 y = t$

C.F: A.E $\Rightarrow (D-1)^2 = 0 \Rightarrow D = 1, 1$

$$C.F = (C_1 + C_2 t)e^t$$

P.I: $y = \frac{1}{(D-1)^2}t = (1-D)^{-2}t$

$= [1 + 2D + 3D^2 + \cdots]t = t + 2$

Complete solution $y = (C_1 + C_2 t)e^t + t + 2$

$t = \log x \Rightarrow \quad y = (C_1 + C_2 \log x)x + \log x + 2$

2. ~~Lagrange's~~ Legendre's linear D.E: An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_{n-1}(ax+b)\frac{dy}{dx} + k_n y = X$$

Put $\log_e(ax+b) = t \quad \Rightarrow ax+b = e^t$

$(ax+b)\frac{dy}{dx} = aDy$

$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y \qquad D = \frac{d}{dt}$

$\vdots$

Then the equation reduces to the form $f(D).y = T$

Initial value problem: A D.E along with the condition at only one ~~variable~~ value of the variable is called an Initial value problem.

Ex: $\frac{d^2y}{dx^2} + y = 0, \quad y(0) = 1, \quad y'(0) = -1$

Boundary value problem: A D.E along with conditions at two or more points of the independent variable is called a boundary value problem.

For I.V.P & B.V.P, the conditions are given to find the values of the arbitrary constants present in the general solution.

Previous problems

(ME-93): The differential $\frac{d^2y}{dx^2}+\frac{dy}{dx}+\underline{\sin y}=0$ is

(a) linear ✓(b) non-linear (c) homogeneous (d) of degree 2

(94): The differential equation $\frac{d^4y}{dx^4}+P\frac{d^2y}{dx^2}+Ky=0$ is

✓(a) Linear of fourth order

(b) Linear and fourth degree

(c) Non-linear of fourth order

(d) Non-homogeneous

(PI-94): Solve for y if $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=0$ with $y(0)=1$ and $y'(0)=-2$

Sol:

$$D^2+2D+1=0$$

$$(D+1)^2=0 \Rightarrow D=-1,-1$$

$\therefore$ Solution is $y=(C_1+C_2t)e^{-t} \longrightarrow (1)$

Given $y(0)=1 \Rightarrow 1=(C_1+0)e^{-0}$

$$\Rightarrow C_1=1 \longrightarrow (2)$$

$$y'=(C_1+C_2t)(-e^{-t})+e^{-t}\cdot C_2$$

Given $y'(0)=-2 \Rightarrow -2=-C_1\cdot e^0+e^0C_2$

$$\Rightarrow -2=-C_1+C_2 \Rightarrow C_2=-1$$

$\therefore$ The general solution is $y=(1-t)e^{-t}$

[RHS in Question $=0 \Rightarrow$ P.I $=0$]

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(EC-94): $y = e^{-2x}$ is a solution of D.E $y'' + y' - 2y = 0$

(a) True (b) False

Sol:

$$D^2 + D - 2 = 0$$

$$D = 1, -2$$

$$C.F = C_1 e^{x} + C_2 e^{-2x}$$

$\therefore e^{-2x}$ is a solution

(ME-94): Solve for y if $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ with $y(0) = 1$ and $y'(0) = 2$

Sol:

$$(D^2 + 2D + 1)y = 0$$

$$D = -1, -1$$

Solution is $y = (C_1 + C_2 t)e^{-t}$

Given $y(0) = 1 \Rightarrow 1 = (C_1)e^0 = C_1$

$y'(0) = 2 \Rightarrow y' = -(C_1 + C_2 t)e^{-t} + e^{-t} \cdot C_2$

$$2 = -(C_1)e^0 + e^0 C_2$$

$$2 = -C_1 + C_2 \Rightarrow C_2 = 3$$

$\therefore$ Solution is $y = (1 + 3t)e^{-t}$

(95): The D.E $y'' + (x^3 \sin x)^5 y' + y = \cos x^3$ is

(a) homogeneous (b) non linear

(c) 2nd order linear (d) non-homogeneous with constant co.efficient

(ME-95): The solution to the D.E $f''(x) + 4f'(x) + 4f(x) = 0$

Sol:

$$D^2 + 4D + 4 = 0$$

$$D = -2, -2$$

Solution is $y = (C_1 + C_2 x)e^{-2x}$

$$= C_1 e^{-2x} + C_2 x e^{-2x}$$

$f_1(x) = e^{-2x}$, $f_2(x) = x e^{-2x}$ are two solutions (as per options)

(1995): The solution of a D.E $y'' + 3y' + 2y = 0$ is

Sol:
$$D^2 + 3D + 2 = 0$$
$$D = -1, -2$$
$\therefore$ solution is $y = C_1 e^{-x} + C_2 e^{-2x}$

(1996): Solve $\frac{d^4 v}{dx^4} + 4\lambda^4 v = 1 + x + x^2$

Sol:
$$D^4 + 4\lambda^4 = 0$$
$$(D^2 + 2\lambda^2)^2 - 4D^2\lambda^2 = 0$$
$$(D^2 + 2\lambda^2 - 2D\lambda)(D^2 + 2\lambda^2 + 2D\lambda) = 0$$
$$D = -\lambda \pm \lambda i, \lambda \pm \lambda i$$

$$C.F = e^{-\lambda x}[C_1 \cos\lambda x + C_2 \sin\lambda x] + e^{\lambda x}[C_3 \cos\lambda x + C_4 \sin\lambda x]$$

$$P.I = \left(\frac{1}{D^4 + 4\lambda^4}\right)(1 + x + x^2)$$
$$= \frac{1}{4\lambda^4}\left[1 + \frac{D^4}{4\lambda^4}\right]^{-1}(1 + x + x^2)$$
$$= \frac{1}{4\lambda^4}\left[1 - \frac{D^4}{4\lambda^4}\right](1 + x + x^2) = \frac{1 + x + x^2}{4\lambda^4}$$

Complete solution is $y = C.F + P.I$

(ME-96): The particular solution for the D.E $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos x$ is

Sol:
$$P.I = \frac{1}{D^2 + 3D + 2} 5\cos x$$
$$= \frac{1}{3D + 1} 5\cos x$$
$$= \frac{3D - 1}{(3D + 1)(3D - 1)} 5\cos x = \frac{3D - 1}{9D^2 - 1} 5\cos x$$
$$= \frac{3D - 1}{-9 - 1} 5\cos x$$
$$= -\frac{1}{2}(3D - 1)\cos x = 1.5\sin x + 0.5\cos x$$

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(CE-98): solve $\frac{d^4y}{dx^4} - y = 15\cos 2x$.

Sol: $D^4 - 1 = 0 \Rightarrow D = \pm 1, \pm i$

C.F $\Rightarrow$ $y = C_1 e^{x} + C_2 e^{-x} + e^{0x}[C_3 \cos x + C_4 \sin x]$

$$P.I = \frac{1}{D^4 - 1} 15 \cdot \cos 2x$$

$$= \frac{1}{(D^2)^2 - 1} 15 \cdot \cos 2x = \frac{1}{(-4)^2 - 1} 15 \cos 2x$$

$$= \cos 2x$$

$\therefore$ complete solution $y = C.F + P.I$

(1998): The general solution of the D.E $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Sol: This is cauchy's homogeneous linear D.E

put $x = e^t \Rightarrow t = \log x$

$$\therefore \quad D(D-1) - D + 1 = 0$$

$$D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$$

solution is $y = (A + Bt)e^t$

$$= (A + B\log x)x$$

$$= Ax + Bx\log x$$

(1998): The radial displacement in a rotating disc is governed by the D.E $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{u}{x^2} = 8x$ where u is the displacement and x is the radius. If $u=0$ at $x=0$ and $u=2$ at $x=1$, calculate the displacement $x = 1/2$

Sol:

$$x^2 u'' + x u' - u = 8x^3$$

put $x = e^t \Rightarrow t = \log x$

C.F $\Rightarrow$ $D(D-1) + D - 1 = 0$

$$D^2 - 1 = 0 \Rightarrow D = \pm 1$$

$$u_c = C_1 e^t + C_2 e^{-t} = C_1 x + \frac{C_2}{x}$$

P.I $\Rightarrow$ $u_p = \frac{1}{D^2-1} 8x^3$

$= -[1-D^2]^{-1} 8x^3$

$= -[1+D^2+D^4+\dots] 8x^3$

$= -8x^3 - 48x$

Complete solution is $u = C_1 x + \frac{C_2}{x} - 8x^3 - 48x$

Given $u=0$ at $x=0$ $\Rightarrow$ $0 = $? [data wrong]

(1999): The equation $\frac{d^2y}{dx^2} + (x^2+4x)\frac{dy}{dx} + y = x^8 - 8$ is a

Sol: Non-homogeneous linear ordinary D.E

(2000): Find the solution of the D.E $\frac{d^2y}{dt^2} + \lambda^2 y = \cos(\omega t + K)$ with initial conditions $y(0)=0$, $\frac{dy(0)}{dt} = 0$. Here λ, ω, & K are constants.

Sol: C.F $D^2+\lambda^2 = 0 \Rightarrow D = \pm\lambda i$

$y_c = C_1 \cos\lambda t + C_2 \sin\lambda t$

P.I $y_p = \frac{1}{D^2+\lambda^2}\cos(\omega t + K)$

$= \frac{1}{-\omega^2+\lambda^2}\cos(\omega t + K)$

$\therefore y = y_c + y_p = C_1\cos\lambda t + C_2 \sin\lambda t + \frac{1}{\lambda^2-\omega^2}\cos(\omega t + K)$

Given $y(0) = 0$

$0 = C_1 + 0 + \frac{1}{\lambda^2-\omega^2}\cdot\cos K \Rightarrow C_1 = \frac{\cos K}{\omega^2-\lambda^2}$

$\frac{dy}{dt} = -C_1 \sin\lambda t\cdot\lambda + C_2\cdot\lambda\cos\lambda t + \frac{-\omega}{\lambda^2-\omega^2}\sin(\omega t + K)$

Given $\frac{dy(0)}{dt} = 0$

$0 = 0 + C_2\lambda + \frac{\omega}{\omega^2-\lambda^2}\cdot\sin K$

$C_2 = \frac{\omega\sin K}{\lambda(\omega^2-\lambda^2)}$

$y = \frac{\cos K}{\omega^2-\lambda^2}\cos\lambda t + \frac{\omega\sin K}{\lambda(\omega^2-\lambda^2)}\sin\lambda t + \frac{1}{\lambda^2-\omega^2}\cos(\omega t + K)$

17 (CE-01): The Solution for the following D.E with boundary conditions $y(0)=2$ and $y'(1)=-3$ is, where $\frac{d^2y}{dx^2}=3x-2$

Sol: C.F $\Rightarrow$ $D^2=0$

$D=0,0$

$$y_c=(C_1+C_2x)e^0=C_1+C_2x$$

P.I $\Rightarrow$ $y_p=\frac{1}{D^2}(3x-2)$

$$=\frac{1}{D}\left[3\frac{x^2}{2}-2x+C_3\right]$$

$$=3\cdot\frac{x^3}{2\times3}-2\frac{x^2}{2}+C_3x+C_4$$

$$=\frac{x^3}{2}-x^2+C_3x+C_4$$

$$y=y_c+y_p=C_1+C_2x+C_3x+C_4-x^2+\frac{x^3}{2}$$

Given $y(0)=2$ $\Rightarrow$ $2=C_1+C_4 \rightarrow (1)$

$y'=C_2+C_3-x^2+\frac{x^3}{2} \Rightarrow -3=C_2+C_3-\frac{1}{2}$

$\Rightarrow C_2+C_3=-5/2 \rightarrow (2)$

$$\therefore\ y=2-\frac{5}{2}x+x^2+\frac{x^3}{2}$$

(2001): Solve the D.E $\frac{d^2y}{dx^2}+y=x$ with the following conditions (i) at $x=0$, $y=1$ (ii) at $x=0$, $y'=1$

Sol: C.F $D^2+1=0 \Rightarrow D=\pm i$

$$y_c=C_1\cos x+C_2\sin x$$

P.I $y_p=\frac{1}{D^2+1}x=[1+D^2]^{-1}x$

$$=[1-D^2+D^4-\dots]x$$

$$=x$$

Complete Solution $y=x+C_1\cos x+C_2\sin x$

(i) At $x=0$, $y=1$ $\Rightarrow$ $1=C_1+C_2(0) \Rightarrow C_1=1$

(ii) At $x=0$, $y'=1$ $\quad y'=1+C_1(-\sin x)+C_2\cos x$

$1=1+0+C_2 \Rightarrow C_2=0$

$$\therefore\ y=x+\cos x$$

(IN-05): The general solution of D.E $(D^2-4D+4)y=0$ is of the form

Sol: $$D^2-4D+4=0 \Rightarrow D=2,2$$

$$y=(C_1+C_2x)e^{2x}$$

(EE-05): The solution of the first order differential eqn $\dot{x}(t)=-3x(t)$, (completed)

(EE-05): For the equation $\ddot{x}(t)+3\dot{x}(t)+2x(t)=5$, then solution $x(t)$ approaches the following values as $t\to\infty$

Sol: C.F $$D^2+3D+2=0$$

$$D=-1,-2$$

$$x_c=C_1e^{-t}+C_2e^{-2t}$$

P.I $$x_p=\frac{1}{D^2+3D+2}\cdot 5$$

$$=\frac{1}{D^2+3D+2}5\cdot e^{0t}=\frac{1}{0+0+2}\cdot 5=\frac{5}{2}$$

Complete solution $x=x_c+x_p$

$$=C_1e^{-t}+C_2e^{-2t}+\frac{5}{2}$$

As $t\to\infty$ $$x=C_1e^{-\infty}+C_2e^{-2\infty}+\frac{5}{2}$$

$$=5/2$$

(CE-05): The solution $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+17y=0$; $y(0)=1$, $\left(\frac{dy}{dx}\right)_{x=\pi/4}=0$ in the range $0<x<\pi/4$ is given by

Sol: C.F $$D^2+2D+17=0 \Rightarrow D=-1\pm 4i$$

Soln is $$y=e^{-x}[C_1\cos 4x+C_2\sin 4x]$$

$$\frac{dy}{dx}=e^{-x}[-4C_1\sin 4x+4C_2\cos 4x]$$
$$+[C_1\cos 4x+C_2\sin 4x](-e^{-x})$$

Given $y(0)=1$

$$1=e^0[C_1] \Rightarrow C_1=1$$

(18)

$$0 = e^{-\pi/4}\left[-4C_1(0)+4C_2\right]+(-e^{-\pi/4})[C_1]$$

$$0 = 1-4C_2 \Rightarrow C_2 = 1/4$$

Solution is $y = e^{-x}\left[\cos 4x + \frac{1}{4}\sin 4x\right]$

(ME-05): The complete solution of O.D.E $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+qy=0$ $y = C_1e^{-x}+C_2e^{-3x}$ then P and q are

Sol:

$$m_1 = -1 \quad m_2 = -3$$

$$(D+1)(D+3)=0$$

$$D^2+4D+3=0 \Rightarrow P=4,\ q=3$$

(ME-05): (above Question). Find solution of D.E $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+(q+1)y=0$?

Sol:

$$D^2+4D+4=0 \Rightarrow (D+2)^2=0$$

$$D=-2,-2$$

$$y=(C_1+C_2x)e^{-2x}$$

(EC-05): A Solution of the D.E $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6x=0$ is given by

Sol:

$$D^2-5D+6=0 \Rightarrow D=2,3$$

$$y = C_1e^{2x}+C_2e^{3x}$$

~~(CE-06): The Solution of D.E~~ $\cancel{x^2\frac{dy}{dx}}$

(IN-06): For the initial value problem $y''+2y'+101y = (10.4)e^x$, $y(0)=1.1$ and $y(0)=-0.9$. Find solution?

Sol: C.F

$$D^2+2D+101=0$$

$$D=-1\pm 10i$$

$$y_c = e^{-x}[C_1\cos 10x + C_2\sin 10x]$$

P.I

$$y_p = \frac{1}{D^2+2D+10}(10.4e^x)$$

$$= \frac{1}{104}(10.4e^x) = 0.1e^x$$

$$y = e^{-x}[C_1\cos 10x + C_2\sin 10x] + 0.1e^{x}$$

$$y' = e^{-x}[-10C_1\sin 10x + 10C_2\cos 10x] + [C_1\cos 10x + C_2\sin 10x](-e^{-x}) + 0.1e^{x}$$

$y(0) = 0.1$ & $y'(0) = -0.9$

$\Rightarrow C_1 = 1,\ C_2 = 0$

$$\therefore y = e^{-x}\cos 10x + 0.1e^{x}$$

(EC-06): For the D.E $\frac{d^2y}{dx^2} + k^2y = 0$, the boundary conditions are (i) $y=0$ for $x=0$ and (ii) $y=0$ for $x=a$. The form of non zero solution of y are

(a) $y = \sum_m A_m \sin\left(\frac{m\pi x}{a}\right)$ (b) $y = \sum_m A_m \cos\left(\frac{m\pi x}{a}\right)$

(c) $y = \sum_m A_m x^{\pi m/a}$ (d) $y = \sum_m A_m e^{-\frac{m\pi x}{a}}$

Sol:

$$(D^2 + k^2)y = 0$$

$$D^2 + k^2 = 0 \Rightarrow D = \pm ki$$

$$y = C_1\cos(kx) + C_2\sin kx$$

Given $y=0$ at $x=0 \Rightarrow y = 0 = C_1$

$y = 0$ for $x = a \Rightarrow 0 = C_2\sin ka$

for non-zero solution $C_2 \neq 0 \Rightarrow \sin ka = 0$

$ka = n\pi \quad n \in Z$

$k = \frac{n\pi}{a},\ n \in Z$

$$\therefore y = C_2\sin\left(\frac{n\pi x}{a}\right)$$

Linear combination of the above is also a solution

(ME-06): For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the P.I is

Sol: $$\text{P.I} = \frac{3e^{2x}}{D^2+4D+3} = \frac{3e^{2x}}{4+8+3} = \frac{e^{2x}}{5}$$

(19)

(EC-07): The solution of the D.E $K^2\frac{d^2y}{dx^2} = y - y_2$ under the boundary conditions (i) $y = y_1$ at $x=0$ and (ii) $y = y_2$ at $x = \infty$ where K, y_1 and y_2 are constants is

Sol:

$$K^2D^2y - y = -y_2$$

$$D^2y - \frac{1}{K^2}y = \frac{-y_2}{K^2}$$

$$\left(D^2 - \frac{1}{K^2}\right)y = -\frac{y_2}{K^2}$$

C.F $\Rightarrow$ $D^2 - \frac{1}{K^2} = 0 \Rightarrow D = \pm\frac{1}{K}$

$$y_c = C_1e^{-x/K} + C_2e^{x/K}$$

P.I $\Rightarrow$ $y_p = \frac{1}{D^2 - \frac{1}{K^2}}\left(\frac{-y_2}{K^2}\right)e^{0x}$

$$= K^2 \cdot \frac{y_2}{K^2} = y_2$$

$\therefore$ complete solution $y = C_1e^{-x/K} + C_2e^{x/K} + y_2$

Given $y = y_1$ at $x = 0$ $\Rightarrow$ $y_1 = C_1 + C_2 + y_2$

Given $y = y_2$ at $x = \infty$ $\Rightarrow$ $y_2 = C_1(0) + C_2(\infty) + y_2$

$$C_2 = \frac{0}{\infty} = 0$$

$$C_1 = y_1 - y_2$$

$$\therefore y = (y_1 - y_2)e^{-x/K} + y_2$$

(ME-08): Given that $x'' + 3x = 0$ and $x(0) = 1$, $x'(0) = 1$ what is $x(1)$?

Sol:

$D^2 + 3 = 0 \Rightarrow D = \pm\sqrt{3}i$

$$x = C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t$$

Given $x(0) = 1$ $\Rightarrow$ $1 = C_1$

Given $x'(0) = 1$ $\Rightarrow$ $x' = -\sqrt{3}C_1\sin\sqrt{3}t + \sqrt{3}C_2\cos\sqrt{3}t$

$$1 = \sqrt{3}C_2 \Rightarrow C_2 = 1/\sqrt{3}$$

$$x = \cos\sqrt{3}t + \frac{1}{\sqrt{3}} \sin\sqrt{3}t$$

$$\therefore x(1) = \cos\sqrt{3} + \frac{1}{\sqrt{3}} \sin\sqrt{3} = 0.4096$$

(ME-08): It is given that $y'' + 2y' + y = 0$, $y(0) = 0$ & $y(1) = 0$ what is $y(0.5)$?

Sol:

$$D^2 + 2D + 1 = 0 \Rightarrow (D+1)^2 = 0$$

$$D = -1, -1$$

$$y = (C_1 + C_2 x) e^{-x}$$

Given $y(0) = 0 \Rightarrow 0 = C_1$

$y(1) = 0 \Rightarrow 0 = C_1 + C_2 e^{-1} \Rightarrow C_2 = 0$

$$y = 0 \Rightarrow y(0.5) = 0$$

(PI-08): The solutions of the D.E $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ are

Sol:

$$D^2 + 2D + 2 = 0$$

$$D = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y = e^{-x} (C_1 \cos x + C_2 \sin x)$$

(or)

$$y = C_1 e^{-(1+i)x} + C_2 e^{-(i-i)x}$$

(PI-09): The homogeneous part of the D.E $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r$ (p, q, r are constants) has real distinct roots if

Sol:

$$D^2 + PD + Q = 0$$

$$D = \frac{-P \pm \sqrt{P^2 - 4Q}}{2}$$

Roots are real and distanct if $P^2 - 4Q > 0$

(PI-09): The solution of the D.E $\frac{d^2y}{dx^2} = 0$ with boundary conditions (i) $\frac{dy}{dx} = 1$ at $x = 0$ (ii) $\frac{dy}{dx} = 1$ at $x = 1$ is

Sol:

$$D^2 = 0 \Rightarrow D = 0, 0$$

$$y = (C_1 + C_2 x)e^0 = C_1 + C_2 x$$

$$y' = 0 + C_2 = C_2$$

$y'(0) = 1 \Rightarrow 1 = C_2$

$$\therefore y = C_1 + x$$

(EE-10): For the D.E $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$ with initial conditions $x(0) = 1$ and $\left(\frac{dx}{dt}\right)_{t=0} = 0$ the solution

Sol:

$$D^2 + 6D + 8 = 0$$

$$D = -2, -4$$

$$\therefore x = C_1 e^{-2t} + C_2 e^{-4t}$$

Given $x(0) = 1 \Rightarrow 1 = C_1 + C_2 \rightarrow ①$

$x'(0) = 0 \Rightarrow x' = -2C_1 e^{-2t} - 4C_2 e^{-4t}$

$$0 = -2C_1 - 4C_2 \rightarrow ②$$

from ① & ② $C_1 = 2$, $C_2 = -1$

$$\therefore \text{solution } x = 2e^{-2t} - e^{-4t}$$

(EC-10): The function $n(x)$ satisfy the d.E $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$ where L is a constant. The boundary conditions are $n(0) = K$, $n(\infty) = 0$. The solution to this equation is

Sol:

$$D^2 - \frac{1}{L^2} = 0 \Rightarrow D = \pm\frac{1}{L}$$

$$n(x) = C_1 e^{\frac{1}{L}x} + C_2 e^{-\frac{1}{L}x}$$

Given $n(0) = K \Rightarrow K = C_1 + C_2$

$n(\infty) = 0 \Rightarrow 0 = C_1(\infty) + C_2(0)$

$$C_1 = 0$$

$$\therefore C_2 = K$$

$$\therefore n(x) = Ke^{-\frac{x}{L}}$$

(CE-10): The solution to the O.D.E $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ is

Sol:
$$D^2+D-6=0$$
$$(D+3)(D-2)=0 \Rightarrow D=-3,2$$
$$y=C_1e^{-3x}+C_2e^{2x}$$

(IN-11): Consider the D.E $y''+2y'+y=0$ with boundary conditions $y(0)=1$ & $y(1)=0$. The value of $y(2)$ is

Sol:
$$D^2+2D+1=0 \Rightarrow D=-1,-1$$
$$y=(C_1+C_2x)e^{-x}$$

Given $y(0)=1 \Rightarrow 1=C_1$

$y(1)=0 \Rightarrow 0=(C_1+C_2)e^{-1} \Rightarrow C_1+C_2=0$
$$C_2=-1$$
$$\therefore y=(1-x)e^{-x}$$
$$y(2)=(1-2)e^{-2}=-e^{-2}$$

(PI-11): The solution of D.E $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=9x+6$ with C_1 and C_2 as constants is

Sol: C.F $\Rightarrow$ $D^2+6D+9=0 \Rightarrow D=-3,-3$
$$y_c=(C_1+C_2x)e^{-3x}$$

P.I $\Rightarrow$ $y_p=\frac{1}{D^2+6D+9}(9x+6)$
$$=\frac{1}{(D+3)^2}(9x+6)=\frac{1}{9}\left[1+\frac{D}{3}\right]^{-2}(9x+6)$$
$$=\frac{1}{9}\left[1-2\frac{D}{3}+3\left(\frac{D^2}{9}\right)-\cdots\right](9x+6)$$
$$=\frac{1}{9}\left[(9x+6)-\frac{2}{3}(9)\right]=x+\frac{2}{3}-\frac{2}{3}$$
$$=x$$
$$\therefore y=y_c+y_p$$
$$=(C_1+C_2x)e^{-3x}+x$$

(21) (ME, PI-12): Consider the D.E $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$ with the boundary conditions of $y(0)=0$ and $y(1)=1$. The complete solution of D.E is

Sol: Let $x=e^t \Rightarrow t=\log x$

$$[D(D-1)+D-4]=0$$

$$D^2-4=0 \Rightarrow D=\pm 2$$

$$\therefore y=c_1e^{2t}+c_2e^{-2t}=c_1x^2+c_2x^{-2}$$

Given $y(0)=1 \Rightarrow 1=0+c_2(\infty) \Rightarrow c_2=0$

$y(1)=1 \Rightarrow 1=c_1+c_2 \Rightarrow c_1=1$

$$\therefore y=x^2$$

(IN-2012): The maximum value of the solution $y(t)$ of the D.E $\ddot{y}(t)+y(t)=0$ with initial condition $\dot{y}(0)=1$ and $y(0)=1$ for $t\geq 0$ is

Sol: $D^2+1=0 \Rightarrow D=\pm i$

$$y=c_1\cos t+c_2\sin t$$

Given $y(0)=1 \Rightarrow 1=c_1$

$\dot{y}(0)=1 \Rightarrow y'=-c_1\sin t+c_2\cos t$

$$1=c_2$$

$$\therefore y=\cos t+\sin t$$

maximum at $\dot{y}=0$

$\Rightarrow -\sin t+\cos t=0$

$\sin t=\cos t$

$\Rightarrow t=\pi/4$

$\ddot{y}=-\cos t-\sin t$

$\ddot{y}\Big|_{t=\frac{\pi}{4}}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\sqrt{2}<0$

Maximum value $y_{max}=\cos\pi/4+\sin\pi/4$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$$

(ME-13): The solution to the D.E $\frac{d^2u}{dx^2} - K\frac{du}{dx} = 0$ where K is a constant subjected to the boundary conditions $u(0)=0$ and $u(L)=U$ is

Sol:

$$\cancel{D^2-K=0 \Rightarrow D=\pm\sqrt{K}} \quad D^2 - KD = 0$$

$$\cancel{u = C_1\cos\sqrt{K}x + C_2\sin\sqrt{K}x}$$

$$D(D-K) = 0 \Rightarrow D = 0, K$$

$$\therefore u = C_1 e^0 + C_2 e^{Kx}$$

Given $u(0)=0 \Rightarrow 0 = C_1 + C_2 \rightarrow ①$

$u(L) = U \Rightarrow U = C_1 + C_2 e^{KL} \rightarrow ②$

$② - ① \Rightarrow U = C_2(e^{KL}-1) \Rightarrow C_2 = \frac{U}{e^{KL}-1}$

$$C_1 = \frac{U}{1-e^{KL}}$$

$$\therefore u(x) = \frac{U}{1-e^{KL}} + \frac{U}{e^{KL}-1}\cdot e^{Kx}$$

(EC-14): If the characteristic equation of D.E $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$ has two equal roots, then the values of α are

Sol: $D^2 + 2\alpha D + 1 = 0$

$$D = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4}}{2}$$

For equal roots $(2\alpha)^2 - 4 = 0 \Rightarrow \alpha^2 = 1$

$\alpha = \pm 1$

(EC-14): If a and b are constants, the most general solution of the D.E $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$ is

Sol: $D^2 + 2D + 1 = 0 \Rightarrow D = -1, -1$

$$x = (a + bt)e^{-t}$$

(EC-14): With initial values $y(0) = y'(0) = 1$, the solution of the D.E $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ at $x=1$ is

Sol: $(D+2)^2=0 \Rightarrow D=-2,-2$.

$$\therefore y=(c_1+c_2x)e^{-2x}$$

Given $y(0)=1 \Rightarrow 1=c_1$

$y'(0)=1 \Rightarrow y'=-c_2\,2xe^{-2x}+c_2e^{-2x}-2c_1e^{-2x}$

$$1=-2+c_2(0+1)$$

$$c_2=3$$

$$\therefore y=(1+3x)e^{-2x}$$

At $x=1$ $\quad y=4e^{-2}=0.541$

(EE-14): The solution for the D.E $\frac{d^2x}{dt^2}=-9x$, with initial conditions $x(0)=1$ and $\left.\frac{dx}{dt}\right|_{t=0}=1$ is

Sol:

$$\frac{d^2x}{dt^2}+9x=0$$

$$D^2+9=0 \Rightarrow D=\pm 3i$$

$$x=c_1\cos 3t+c_2\sin 3t$$

Given

$x(0)=1 \quad 1=c_1$

$x'(0)=1 \quad x'=-3c_1\,\cancel{\cos 3t}\,\sin 3t+3c_2\cos 3t$

$$1=3c_2 \Rightarrow c_2=1/3$$

$$\therefore x=\cos 3t+\frac{1}{3}\sin 3t$$

(EE-14): Consider the D.E $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$. Find ~~the~~ a Solution?

Sol: put $x=e^t \Rightarrow t=\log x$

$$D(D-1)+D-1=0$$

$$D^2-1=0 \Rightarrow D=\pm i$$

$$y=c_1e^{-t}+c_2e^{-t}$$

$$y=c_1x+c_2\frac{1}{x}$$

According to options $y=\frac{1}{x}$ is a solution $(c_1=0, c_2=1)$

(ME-14): If $y=f(x)$ is the solution of $\frac{d^2y}{dx^2}=0$ with the boundary conditions $y=5$ at $x=0$, and $\frac{dy}{dx}=2$ at $x=10$, $f(15)=$ ___

Sol:

$$D^2=0 \Rightarrow D=0,0$$

$$y=(C_1+C_2x)e^{0x} = C_1+C_2x$$

Given $y=5$ at $x=0 \Rightarrow 5=C_1$

$\frac{dy}{dx}=2$ at $x=10 \Rightarrow y'=C_2=2$

$$\therefore y = 5+2x = f(x)$$

$$f(15) = 5+30 = 35$$

(CE-14): Water is flowing at a steady rate through a homogeneous and saturated horizontal soil strip of 10m length. The strip is being subjected to a constant water head (H) of 5m at the begining and 1m at the end. If the governing equation of flow in the soil strip is $\frac{d^2H}{dx^2}=0$ (where x is distance along soil strip), the value of H (in m) at the middle of the strip is

Sol:

$$\frac{d^2H}{dx^2}=0 \Rightarrow D^2=0$$

$$H=(C_1+C_2x)e^{0.x}$$

$$= C_1+C_2x$$

Given $H=5m$ at $x=0$ $\quad 5=C_1$

$H=1m$ at $x=10$ $\quad 10=C_1+10C_2 \Rightarrow C_2=\frac{-2}{5}$

$$\therefore H = 5-\frac{2}{5}x$$

At middle of the strip i.e at $x=5$

$$H=5-\frac{2}{5}.5 = 5-2=3$$

Partial Differential Equations

Notation: $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ $\frac{\partial^2 z}{\partial x^2} = r$ $\frac{\partial^2 z}{\partial x . \partial y} = s$ $\frac{\partial^2 z}{\partial y^2} = t$

Formation of P.D.E: unlike the case of ODE, which arise from the elimination of arbitrary constants ; the PDE can be formed by elimination of arbitrary constants or arbitrary functions.

1. If equation involves n arbitrary constants and n independent variable then we can obtain a PDE of only first order
2. If an equation involves n arbitrary functions then we can obtain a PDE of n^{th} order
3. If the no. of arbitrary constants are more than the no. of independent variables then we will obtain a PDE of higher order.

Prob: PDE corresponding to $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $\rightarrow (1)$

Sol:
$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \qquad 2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$
$$p = \frac{x}{a^2} \qquad q = y/b^2$$

from (1) $2z = px + qy$
$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

Prob: The PDE of $z = f(x^2 - y^2)$

Sol:
$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)\, 2x = p$$
$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) = q$$
$$\frac{p}{q} = -\frac{x}{y}$$
$$py + qx = 0$$
$$\frac{\partial z}{\partial x} y + \frac{\partial z}{\partial y} x = 0$$

Prob: The PDE of $z = f(x+ay) + g(x-ay)$

$$\frac{\partial z}{\partial x} = p = f'(x+ay) + g'(x-ay)$$

$$\frac{\partial z}{\partial y} = q = f'(x+ay)\cdot a + g'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = r = f''(x+ay) + g''(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = t = f''(x+ay)\cdot a^2 + g''(x+ay)\cdot a^2$$

$$= a^2[f''(x+ay) + g''(x+ay)]$$

$$= a^2 r$$

$$\frac{\partial^2 z}{\partial y^2} - a^2 \frac{\partial z}{\partial x^2} = 0$$

Solving PDE:

First order Linear PDE: A Linear PDE of first order, commonly known as Lagrange's linear equation is of the form

$$Pp + Qq = R \qquad P, Q \text{ and } R \text{ are functions of } x, y, z.$$

This equation is called quasi-linear equation.

* When P, Q and R are independent of 'z' then it is linear equation.

To solve this equation

(i) form the subsidary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ii) Solve these simultaneous equation, which gives $u = a$ and $v = b$ as solutions.

(iii) Write the complete solution as $\phi(u,v) = 0$ (or) $u = f(v)$

Prob: Solve $\frac{y^2 z}{x} p + xzq = y^2 \Rightarrow y^2 z p + x^2 z q = y^2 x$

Sol: Subsidary equation $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$

From the first two factors $x^2 z\, dx = y^2 z\, dy$

$$x^2dx = y^2dy$$

I.O.B.S $\quad \frac{x^3}{3} = \frac{y^3}{3} + c_1 \Rightarrow x^3 - y^3 = a$ (const.)

From first & third fractions $\quad xdx = zdz$

I.O.B.S $\quad \frac{x^2}{2} = \frac{z^2}{2} + c_2$

$x^2 - z^2 = b$ (const.)

$\therefore$ complete solution is $x^3 - y^3 = f(x^2 - z^2)$

Non-linear equations of first order: Those equations in which p and q occur other than in the first degree are called non-linear PDE of first order. There are four standard forms of these equations

form I: Equations of the type $f(p,q) = 0$

It's complete solution is $z = ax + by + c$. where a and b are connected by the relation $f(a,b) = 0$.

From $f(a,b) = 0$, express $b = \phi(a)$

$\therefore$ The required solution is $z = ax + \phi(a)y + c$.

where a and c are arbitrary constants.

prob: Solve $\sqrt{p} + \sqrt{q} = 1$

sol:

$$f(p,q) = \sqrt{p} + \sqrt{q} - 1 = 0$$

$$f(a,b) = \sqrt{a} + \sqrt{b} - 1 = 0 \Rightarrow \sqrt{b} = 1 - \sqrt{a}$$

$$\Rightarrow b = (1 - \sqrt{a})^2$$

The solution is $z = ax + by + c$

$$= ax + (1 - \sqrt{a})^2 y + c$$

form II: Equations of the type $f(z,p,q) = 0$

To solve this

(i) assum $u = x + ay$ and substitute $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$ in the given equation

(ii) Solve the resulting ODE in z and u

(ii) replace u by $x+ay$

Prob: Solve $p(1+q)=qz$

sol:

$$\frac{dz}{du}\left(1+a\frac{dz}{du}\right)=az\frac{dz}{du}$$

$$1+a\frac{dz}{du}=az \Rightarrow \frac{a}{az-1}dz=du$$

$$\int\frac{a}{az-1}dz=\int du+c$$

$$\log(az-1)=u+c$$

$$=x+ay+c$$

Form III: Equations of the type $f(x,p)=g(y,q)$

Assum $f(x,p)=g(y,q)=a$

$f(x,p)=a$ $\qquad$ $g(y,q)=a$

Solve for p $\qquad$ Solve for q

$p=\phi(x)$ $\qquad$ $q=\varphi(y)$

Since $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$dz=p\,dx+q\,dy$$

$$dz=\phi(x)dx+\varphi(y)dy$$

I.O.B.S

$$z=\int\phi(x)dx+\int\varphi(y)dy+c$$

prob: Solve $p^2+q^2=x+y$

$$p^2-x=-q^2+y=a \text{ say}$$

$p^2-x=a$ $\qquad$ $-q^2+y=a$

$p=\sqrt{a+x}$ $\qquad$ $q=\sqrt{y-a}$

$$dz=\sqrt{a+x}\,dx+\sqrt{y-a}\,dy$$

$$z=\frac{2}{3}(a+x)^{3/2}+\frac{2}{3}(y-a)^{3/2}+b$$

Form IV: $z = px + qy + f(p,q)$.

It's complete solution is $z = ax + by + f(a,b)$

Prob: Solve $z = px + qy + pq$ (CE-2010)

Solution is $z = ax + by + ab$

Higher order PDE:

Homogeneous linear equations with constant co-efficients:

An equation of the form $\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x,y)$ in which k's are constants, is called a homogeneous linear partial D.E of n^{th} order with constant co-efficients. It is called homogeneous because all terms contain derivatives of same order.

Denote $\frac{\partial z}{\partial x} = D$ and $\frac{\partial z}{\partial y} = D'$ so that

$$[D^n + k_1 D^{n-1} \cdot D' + \cdots + k_n (D')^n] z = F(x,y)$$

$$f(DD')z = F(x,y)$$

Solution is $z = C.F + P.I$

Rules for C.F: For the P.D.E $f(DD')z = 0$, $f(\lambda = 0)$ is the Auxilary Equation. Where $\lambda = D/D'$.

Depending on roots of A.E, we have the following cases

(i) case-1: When the roots are different say $m_1, m_2, m_3 \ldots$

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \cdots$$

(ii) case-2: When the roots are repeated say $m, m, m \ldots$

$$C.F = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \cdots$$

Prob: Solve $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \cdot \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

Sol:

$$[D^2 - 3DD' + 2(D')^2]z = 0$$

A.E is $\lambda^2-3\lambda+2=0 \Rightarrow \lambda=1,2$

$$z=f_1(y+x)+f_2(y+2x)$$

Prob: solve $\frac{\partial^2 z}{\partial x^2}-a^2\frac{\partial^2 z}{\partial y^2}=0$

Sol

$$[D^2-a^2(D')^2]z=0$$

A.E is $\lambda^2-a^2=0 \Rightarrow \lambda=\pm a$

$$z=f_1(y+ax)+f_2(y-ax)$$

Prob: solve $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0$

Sol:

$$[D^2+2DD'+(D')^2]z=0$$

A.E is $\lambda^2+2\lambda+1=0 \Rightarrow \lambda=-1,-1$

$$z=f_1(y-x)+x f_2(y-x)$$

Rules for P.I:

From the symbolic form $P.I=\frac{1}{f(D,D')}F(x,y)$

(i) When $F(x,y)=e^{ax+by}$

$$P.I=\frac{1}{f(D,D')}e^{ax+by}$$

Put $D=a$ and $D'=b$

(ii) When $F(x,y)=\sin(mx+ny)$ (or) $\cos(mx+ny)$

$$P.I=\frac{1}{f(D^2,DD',D'^2)}\sin \text{ or } \cos(mx+ny)$$

put $D^2=-m^2$, $DD'=-mn$, $(D')^2=-n^2$

(iii) When $F(x,y)=x^m y^n$

$$P.I=\frac{1}{f(DD')}x^m y^n=[f(DD')]^{-1}x^m y^n$$

Expand in ascending powers of D'/D & operate on $x^m y^n$

(iv) When $F(x,y)=$ Any function of x & y

$$P.I=\frac{1}{f(D,D')}F(x,y)$$

(26) Resolve $\frac{1}{F(D,D')}$ into fractions and operate each fraction on $F(x,y)$ such that $\frac{1}{D-mD'}\cdot F(x,y)$.

Replace $y = c - mx$

$$\therefore \frac{1}{D-mD'}\cdot F(x,y) = \int F(x, c-mx)\,dx$$

After integration replace $c = y+mx$

Classification of PDE: A PDE of the form

$$A\cdot\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x\cdot\partial y} + C\frac{\partial^2 z}{\partial y^2} + F(x,y,z,p,q) = 0$$ is said to be

(i) elliptic if $B^2-4AC<0$

(ii) parabola if $B^2-4AC=0$

(iii) Hyperbola if $B^2-4AC>0$

Prob: $x^2\frac{\partial^2 z}{\partial x^2} + (1-y^2)\frac{\partial^2 z}{\partial y^2} = 0$ where $-\infty<x<\infty$ $x\neq 0$, $-1<y<1$ is

Sol: $B^2-4AC = -4x^2(1-y^2) < 0$

$\therefore$ ellipse

Prob: $(1+x^2)r + (5+2x^2)s + (4+x^2)t = 0$

Sol: $B^2-4AC = 9 > 0$ $\therefore$ hyperbola

Prob: $r+4s+4t=0$

Sol: $B^2-4AC = 16-16 = 0$ $\therefore$ parabola

previous problems

(IN-13): The type of partial D.E $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ is

Sol: we can write it as [ME-96 Similar]

$$1\cdot\frac{\partial^2 f}{\partial x^2} + 0\cdot\frac{\partial^2 f}{\partial x\cdot\partial t} + 0\cdot\frac{\partial^2 f}{\partial t^2} + \left(-\frac{\partial f}{\partial t}\right) = 0$$

$B^2-4AC = 0$

$\therefore$ parabolic

(ME-07): The PDE $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}=0$ has degree & order

(a) 1, 2 (b) 1, 1 (c) 2, 1 (d) 2, 2

(ME-10): The blasius equation $\frac{d^3f}{d\eta^3}+\frac{f}{2}\frac{d^2f}{d\eta^2}=0$ is a

Sol: 3^{rd} order non linear ordinary differential eqn

(ME-13): The PDE $\frac{\partial u}{\partial t}+u\cdot\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}$ is a

Sol: 2^{nd} order non linear

COMPLEX VARIABLES

Complex variables

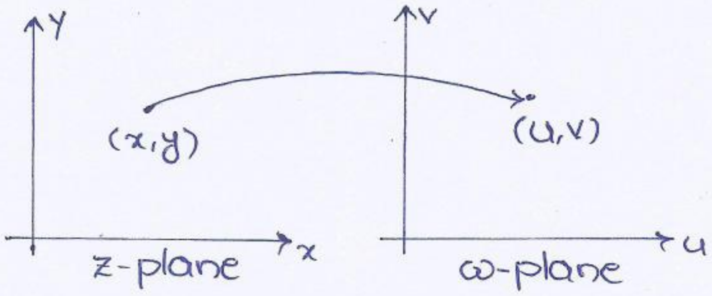
A variable of the form $z = x + iy$ where $i^2 = -1$ and x, y are real is called complex variable. By considering $x = r\cos\theta$, $y = r\sin\theta$ we have

$$z = r[\cos\theta + i\sin\theta] = re^{i\theta}$$ is called polar form or Mod amplitude form. where $r = |z| = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

We can associate a complex valued function to the complex variable z then we can write

$$\omega = f(z) = u(x,y) + i\,v(x,y)$$

To represent this complex function we require two planes namely z-plane & ω-plane. For each point (x,y) in z-plane there exist a point (u,v) in ω-plane with the function $\omega = f(z)$. The point (u,v) is called image of (x,y)



Every complex function $f(z)$ can always be expressable in the form $\omega = f(z) = u(x,y) + i\,v(x,y)$ where $u(x,y)$ is called real part of $f(z)$ and $v(x,y)$ is called imaginary part of $f(z)$.

Ex: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy$

$f(z) = e^z = e^{x+iy} = e^x\cos y + i\,e^x\sin y$

$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$

$f(z) = \log_e z = \log_e(x+iy) = \log_e(re^{i\theta})$

$= \log_e r + i\theta.\log_e e = \log_e\sqrt{x^2+y^2} + i.\tan^{-1}(y/x)$

$= \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}(y/x)$

prob: The mod amplitude form of $\frac{1+i}{1-i}$ is

sol: $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{2i}{2} = i$

$r = \sqrt{0^2+1^2} = 1 \qquad \theta = \tan^{-1}(1/0) = \pi/2$

$\therefore$ Mod amplitude form $= 1\,e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$

prob: If $\alpha + i\beta = \frac{1}{a+ib}$ then $(\alpha^2+\beta^2)(a^2+b^2) =$

sol: $\alpha + i\beta = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

$\therefore \alpha = \frac{a}{a^2+b^2} \qquad \beta = \frac{-b}{a^2+b^2}$

$\alpha^2+\beta^2 = \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{1}{(a^2+b^2)}$

12 EC, EE, IN
07 IN
(GATE 96 ME)

$(a^2+b^2)(\alpha^2+\beta^2) = \underline{1}$

prob: The value of $i^i =$

sol: $i^i = e^{\log_e i^i} = e^{i\log_e i} = e^{i\left[\frac{1}{2}\log_e 1 + i\tan^{-1}(1/0)\right]}$

$= e^{i \cdot i\pi/2} = e^{-\pi/2}$

* $\log_e i = i\pi/2 \Rightarrow i = e^{i\pi/2}$

prob: The value of $\sqrt{2i}$

sol: $\sqrt{2i} = \sqrt{2}.\sqrt{i} = \sqrt{2}.e^{\log_e\sqrt{i}} = \sqrt{2}.e^{\frac{1}{2}\log_e i}$

$= \sqrt{2}\,e^{\frac{1}{2}(i\pi/2)} = \sqrt{2}\,e^{i\pi/4} = \sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 1+i$

* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

* $e^{iy} = 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \cdots$

$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$ [Maclaurin's expansion]

$= \cos y + i\sin y$ [Euler's formula]

* $e^{2n\pi i} = \cos(2n\pi) + i\sin(2n\pi) = 1$

$\Rightarrow e^z = e^{z+2n\pi i}$, n is any integer

$\therefore e^z$ is periodic with period $2\pi i$ (GATE 97 CE)

* $e^{-iy} = \cos y - i\sin y$

* $\cos z = \frac{1}{2}(e^{iz}+e^{-iz})$ $\sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$

$\cosh z = \frac{1}{2}(e^{z}+e^{-z})$ $\sinh z = \frac{1}{2}(e^{z}-e^{-z})$

* $\cos z = \frac{1}{2}(e^{iz}+e^{-iz}) = \frac{1}{2}[e^{i(x+iy)}+e^{-i(x+iy)}]$

$= \frac{1}{2}[\cos x(e^{-y}+e^{y}) + i\sin x(e^{-y}-e^{y})]$

$= \cos x.\cosh y - i\sin x.\sinh y$

* $\sin z = \sin x.\cosh y + i\cos x.\sinh y$

* $\sin(iz) = \frac{1}{2i}[e^{i(iz)}-e^{-i(iz)}] = \frac{1}{2i}[e^{-z}-e^{z}]$

$= \frac{i}{2}[e^{z}-e^{-z}] = i\sinh z$

$\cos(iz) = \frac{1}{2}[e^{i(iz)}+e^{-i(iz)}] = \frac{1}{2}[e^{-z}+e^{z}] = \cosh z$

* $\sinh z = \frac{1}{i}\sin(iz) = -i\sin(i(x+iy))$

$= i\sin(y-ix) = \sinh x.\cos y + i\cosh x.\sin y$

$\cosh z = \cosh x.\cos y + i\sinh x.\sin y$

* In general $\ln z = \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x) + i2n\pi$. The principal value is obtained when $n=0$

prob: Find the general and principle values of $\log(1+\sqrt{3}i)$

sol: $|z| = \sqrt{1+3} = 2$ $Arg(z) = \tan^{-1}(\sqrt{3}) = \pi/3$

$\log(1+\sqrt{3}i) = \log 2 + i(\frac{\pi}{3}+2n\pi)$, n any integer

principal value $\mathrm{Log}(1+\sqrt{3}i) = \log 2 + i\pi/3$ $(n=0)$

prob: Find general and principal values of $(-i)^i$

sol: $(-i)^i = e^{i\log(-i)} = e^{i[\log 1 + i(-\frac{\pi}{2}+2n\pi)]}$

$= e^{\frac{\pi}{2}-2n\pi}$

principal value $= e^{\pi/2}$

Properties of Moduli:

- $|z_1+z_2| \leq |z_1|+|z_2|$

  $|z_1-z_2| \leq |z_1|+|z_2|$

  $|z_1+z_2| \geq |z_1|-|z_2|$

  $|z_1-z_2| \geq |z_1|-|z_2|$

- $|z_1 z_2 z_3 \ldots z_n| = |z_1|.|z_2|.\ldots.|z_n|$

  $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(GATE 94 IN): The real part of the complex number $z = x+iy$ is given by (a) $Re(z) = z - z^*$ (b) $Re(z) = \frac{z-z^*}{2}$ ✓(c) $= \frac{z+z^*}{2}$ (d) $= z+\overset{*}{z}$

Sol: $z = x+iy \Rightarrow z^* = x - iy$

$$Re(z) = x = \frac{z+z^*}{2}$$

(GATE 97 IN): The complex number $z = x+iy$ which satisfy the equation $|z+1| = 1$ lie on

(a) Circle with $(1,0)$ as center and radius 1

✓(b) circle with $(-1,0)$ as center and radius 1

(c) y-axis (d) x-axis

Sol: $|z+1| = 1 \Rightarrow |x+iy+1| = 1$

$$(x+1)^2 + y^2 = 1$$

Equation of circle with (x_0, y_0) as center and r as radius is $(x-x_0)^2 + (y-y_0)^2 = r^2$

$\therefore$ $|z+1| = 1$ is a circle with center $(-1,0)$ and radius $= 1$

(GATE 2005 CE): Which one of the following is not true for the complex numbers z_1 and z_2?

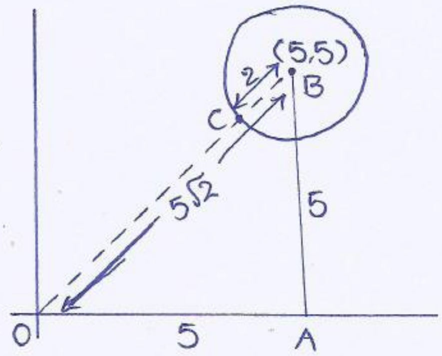
(a) $\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$ (b) $|z_1+z_2| \le |z_1| + |z_2|$ ✓(c) $|z_1+z_2| \le ||z_1| - |z_2||$

(d) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

$$|z_1+z_2| \geq |z_1| - |z_2|$$

(GATE 2005 IN): Consider the circle $|z-5-5i| = 2$ in the complex number plane (x,y) with $z = x+iy$. The minimum ~~number~~ distance from origin to circle is?

Sol:



$OB^2 = OA^2 + AB^2 = 50$

$OB = 5\sqrt{2}$

$OC = OB - CB = 5\sqrt{2} - 2$

(GATE-06 EC): For the function of a complex variable $\omega = \ln z$ (where $\omega = u + iv$ and $z = x + iy$) the u = constant lines get mapped in the z-plane as

Sol: $\omega = \ln z = \ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x)$

$$u = \frac{1}{2}\ln(x^2+y^2) = C \quad \text{(constant)}$$

$$\ln(x^2+y^2) = 2C$$

$$x^2+y^2 = e^{2C} \Rightarrow x^2+y^2 = (e^C)^2$$

Represent concentric circles with radius e^C.

(GATE-07 PI): If a complex number $z = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ then z^4 is

Sol: $z = \frac{\sqrt{3}}{2} + i\frac{1}{2} \Rightarrow iz = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \omega$ say

[ω is a root of $x^3 = 1$]

$$i^4 z^4 = \omega^4 = \omega^3 . \omega \quad [\omega^3 = 1]$$

$$z^4 = \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(GATE-08 PI): The value of the expression $\frac{-5+i10}{3+4i}$

Sol: $\frac{(-5+i10)}{(3+4i)} \times \frac{(3-4i)}{(3-4i)} = \frac{25+50i}{25} = 1+2i$

(GATE-08 EC): The equation $\sin(z) = 10$ has

(a) no real or complex solution (b) Exactly two distinct complex sol^s

(c) a unique solution ✓(d) an infinite number of complex sol^ns

Sol: $\sin z = 10 \Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 10$

$$\Rightarrow (e^{iz}) - \frac{1}{e^{iz}} = 20i$$

$$\Rightarrow (e^{iz})^2 - 20i(e^{iz}) - 1 = 0$$

$$e^{iz} = \frac{-(-20i) \pm \sqrt{-400+4}}{2} = 10i \pm 3\sqrt{11}\,i = i[10 \pm 3\sqrt{11}]$$

$$iz = \log_e i[10 \pm 3\sqrt{11}]$$

$$= \log_e i + \log_e(10 \pm 3\sqrt{11})$$

$$iz = \log 1 + i\left(\frac{\pi}{2} \pm 2n\pi\right) + \log(10 \pm 3\sqrt{11})$$

$$z = \frac{\pi}{2} \pm 2n\pi - i\log(10 \pm 3\sqrt{11})$$

$\therefore$ z has infinite no of complex sol^ns

(GATE-2013 EE): square roots of $-i$, where $i=\sqrt{-1}$ are

(a) $i, -i$ ✓(b) $\cos(-\pi/4)+i\sin(-\pi/4)$, $\cos(3\pi/4)+i\sin(3\pi/4)$

(c) $\cos(\pi/4)+i\sin(3\pi/4)$, $\cos(3\pi/4)+i\sin(\pi/4)$

(d) $\cos(3\pi/4)+i\sin(-3\pi/4)$, $\cos(-3\pi/4)+i\sin(3\pi/4)$

Sol: $i=e^{i\pi/2} \Rightarrow -i=\frac{1}{i}=e^{-i\pi/2}$

$\sqrt{-i}=(-i)^{1/2}=e^{-i\pi/4}$

and $-i=i^3=e^{i3\pi/2} \Rightarrow \sqrt{-i}=e^{i3\pi/4}$

$\therefore$ square roots of $-i$ are $e^{-i\pi/4}$, $e^{i3\pi/4}$

(GATE-14 EE): All the values of the multi valued complex function 1^i, where $i=\sqrt{-1}$ are.

(a) purely imaginary ✓(b) real & non negative (c) on the unit circle

(d) equal in real and imaginary parts.

Sol: $1^i=(\cos 2n\pi+i\sin 2n\pi)^i$

$=(e^{i2n\pi})^i=e^{-2n\pi}=\frac{1}{e^{2n\pi}}$ It is real & non-negative.

(GATE-14 ME): The argument of complex number $\frac{1+i}{1-i}$

Sol: $\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{2i}{2}=i=0+1.i$

Argument $=\tan^{-1}(1/0)=\tan^{-1}(\infty)=\pi/2$

Analytic function: A function $f(z)$ is said to be analytic in a region R of z-plane if the derivative of $f(z)$ exists at each and every point in that region.

Necessary and sufficient condition: For a function $f(z)=u+iv$ to be analytic in the region 'R'

(i) Partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ must exist in 'R'

(ii) They should satisfy $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

(or)

$$\underbrace{u_x=v_y, \quad u_y=-v_x}_{\text{Cauchy Reiman or C-R equations}}$$

prob: $f(z) = z^2$.

$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u = x^2 - y^2 \qquad v = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

$u_x = v_y \qquad u_y = -v_x$

$\therefore f(z) = z^2$ is analytic throughout the z-plane.

prob: $f(z) = \bar{z}$

$\bar{z} = x - iy \Rightarrow u = x \quad v = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y$

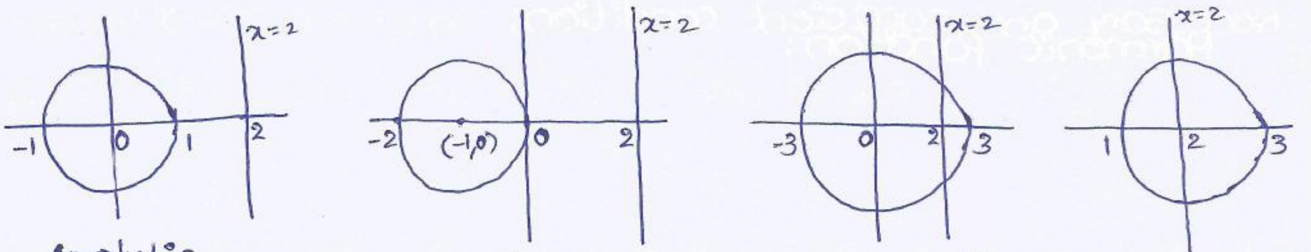
$\therefore f(z) = \bar{z}$ is not analytic throughout the z-plane.

Note: 1. Any function which involves $\bar{z}$ is always not analytic throughout the z-plane.

2. Every polynomial in z is always analytic throughout the z-plane.

Ex: $f(x) = u + iv$ suppose if $u_x = \frac{1}{x-2}$, $u_y = 1$, $v_x = -1$, $v_y = \frac{1}{x-2}$

This function is analytic in the regions which does not contain $x = 2$ like $|z| = 1$, $|z+1| = 1$ and this function is not analytic in the regions which contains $x = 2$ like $|z| = 3$, $|z-2| = 1$

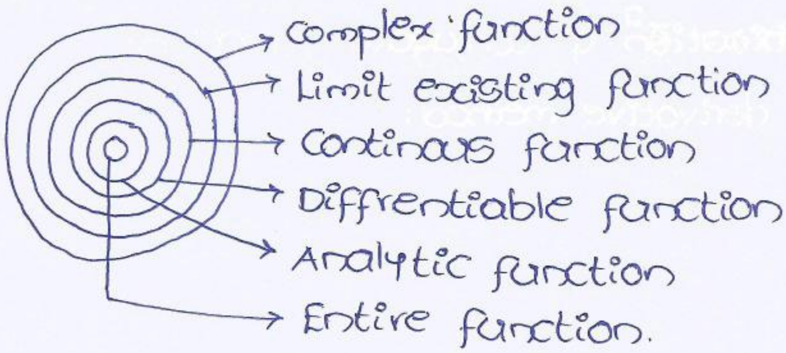


Analytic   Analytic   Not analytic   Not analytic

Entire function (or) Holomorphic (or) Regular function:

A function which is analytic in the entire z-plane is called entire function.

Ex: Every polynomial in z.



C-R equations in polar form:

$$f(z) = u + iv \Rightarrow f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

P.d.w.r. to 'r' $\quad f'(re^{i\theta}) \times e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \rightarrow (1)$

P.d.w.r. to 'θ' $\quad f'(re^{i\theta}) \times r \times i \times e^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \rightarrow (2)$

from (1) & (2) $\quad ri\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$$-r.\frac{\partial v}{\partial r} + i.r.\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \theta} + i.\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r.\frac{\partial v}{\partial r} \qquad \frac{\partial v}{\partial \theta} = r.\frac{\partial u}{\partial r}$$

$$\Rightarrow \quad v_r = -\frac{1}{r}u_\theta \qquad u_r = \frac{1}{r}v_\theta$$

Properties of analytic function: If $f(z) = u + iv$ is analytic function then

1. $u(x,y) = c_1$ & $v(x,y) = c_2$ are orthogonal to eachother

   Ex: $f(z) = z^2 = x^2 - y^2 + i\,2xy$

   $x^2 - y^2 = c_1$ & $2xy = c_2$ are $\perp$'ar to each other.

2. If $u(x,y)$ is a harmonic function then $v(x,y)$ is also a harmonic function.

   Harmonic function: A function $H(x,y)$ which satisfies the laplace equation $\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$ is called harmonic function.

$$u_x = v_y \qquad u_y = -v_x$$

$$u_{xx} = v_{xy} \qquad u_{yy} = -v_{yx}$$

$u_{xx} + u_{yy} = 0$

lly $\quad v_{xx} + v_{yy} = 0$

If $f(z) = u + iv$ is analytic then u & v satisfies laplace equation

⑤

Determination of Conjugate function:

Total derivative method: Let the real part of $u(x,y)$ of an analytic function $f(z) = u + iv$ be given. Then to find $v(x,y)$ we proceed as follows. The total derivative of v is given by

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

By using C-R eq^{n}s $\quad dv = -\frac{\partial u}{\partial y} \cdot dx + \frac{\partial u}{\partial x} \cdot dy$

$$v = \int(-u_y)\,dx + \int u_x \cdot dy + C$$

** y const $\qquad$ *** terms independent of 'x'.

Similarly, If $v(x,y)$ is given. Then to find $u(x,y)$.

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy = v_y\,dx - v_x\,dy \quad [\text{using C-R}$$

$$u = \int v_y\,dx - \int v_x\,dy + C$$

Milne - Thompson method:

1. If $u(x,y)$ is given

   Take $f'(z) = u_x - i u_y$

   Replace x by z and y by '0' in $f'(z)$

   Then integrate $f'(z)$ with respect to z.

2. If $v(x,y)$ is given

   Take $f'(z) = v_y + i v_x$

   Replace x by z and y by '0' in $f'(z)$

   Then integrate $f'(z)$ with respect to z.

(GATE-05 PI): The function $\omega = u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ is not analytic at the point

✓(a) $(0,0)$ $\qquad$ (b) $(0,1)$ $\qquad$ (c) $(1,0)$ $\qquad$ (d) $(2,\infty)$

Sol: $\omega = \log(x+iy)$ $\qquad$ log is not defined at origin

(GATE-07 CE): potential function ϕ is given as $\phi = x^2 - y^2$. what will be the stream function ψ with the condition $\psi = 0$ at $x=0,\ y=0$?

Sol: Given $\phi(x,y) = x^2 - y^2$

Take $d\psi = \psi_x\,dx + \psi_y \cdot dy$

$$= -\phi_y \cdot dx + \phi_x \cdot dy$$

$$= -(-2y)\,dx + 2x\cdot(dy)$$

$$\psi = 2xy + K$$

At $x=0, y=0 \Rightarrow \psi = 2xy + K = 0 \Rightarrow K = 0$

$\therefore$ stream function is $\psi(x,y) = 2xy$

(GATE - 09 ME): An analytic function of a complex variable $z=x+iy$ is expressed as $f(z) = u(x,y) + iv(x,y)$ where $i=\sqrt{-1}$. If $u = xy$ then expression for v should be

Sol: Given $u = xy$

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy$$

$$dv = -x dx + y dy$$

$$v = -\frac{x^2}{2} + \frac{y^2}{2} + K = \frac{y^2 - x^2}{2} + K$$

(GATE - 2010 PI): If $f(x+iy) = x^3 - 3xy^2 + i\phi(x,y)$ where $i=\sqrt{-1}$ and $f(x+iy)$ is an analytic function then $\phi(x,y)$ is

Sol: Given $\varphi(x,y) = x^3 - 3xy^2$

$$d\phi = \phi_x dx + \phi_y dy = -\varphi_y dx + \varphi_x dy$$

$$d\phi = -(0 - 6xy)\,dx + (3x^2 - 3y^2)\,dy$$

$$\phi = \frac{6x^2y}{2} + \left(-3\cdot\frac{y^3}{3}\right) + K$$

$$= 3x^2y - y^3 + K$$

(GATE - 11 CE): For an analytic function $f(x+iy) = u(x,y) + iv(x,y)$, u is given by $u = 3x^2 - 3y^2$. The expression for v, considering K to be constant is

Sol: Given $u = 3x^2 - 3y^2$

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy$$

$$dv = -(0 - 6y)\,dx + 6x\,dy$$

$$v = 6xy + K$$

(GATE - 14 EC): The real part of an analytic function $f(z)$ where $z = x+iy$ is given by $e^{-y}\cos x$. The imaginary part of $f(z)$ is

Sol: Given $u = e^{-y}\cos x$

⑥

$$dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy$$

$$= -u_y dx + u_x \cdot dy$$

$$= e^{-y}\cos x\, dx - e^{-y}\sin x\, dy$$

$$v = e^{-y}\sin x + C$$

(GATE-14 ME): An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x,y) + i\,v(x,y)$, where $i = \sqrt{-1}$. If $u(x,y) = 2xy$ then $v(x,y)$ must be

Sol: Given $u = 2xy$

$$dv = v_x dx + v_y dy = -u_y dx + u_x dy$$

$$= -2x\,dx + 2y\,dy$$

$$v = -x^2 + y^2 + k$$

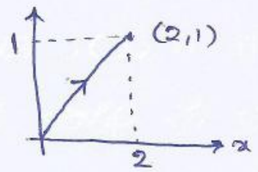
(GATE-14 ME): $u(x,y) = x^2 - y^2$. $v(x,y) = ?$

Sol: $dv = -u_y dx + u_x dy = 2y\,dx + 2x\,dy$

$$v = 2xy + C$$

Complex integration: An integral of the form $\int_C f(z)\,dz$ where $dz = dx + i\,dy$ is called a complex integral. This integral value depends not only on $f(z)$ but also on the curve 'C'.

Prob: The value of $\int_0^{2+i} \bar{z}\, dz$ along (i) $y = x/2$ (ii) The real axis to 2 and then vertically to $2+i$.

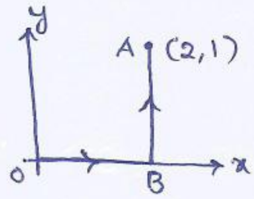


Sol: (i) Along $y = x/2 \Rightarrow dy = \frac{dx}{2}$

$$\int_0^{2+i}(x - iy)(dx + i\,dy) = \int_0^2\left(x - i\frac{x}{2}\right)\left(dx + i\frac{dx}{2}\right)$$ (or convert to y)

$$= \left(1 - \tfrac{i}{2}\right)\left(1 + \tfrac{i}{2}\right)\int_0^2 x\cdot dx$$

$$= \left(1 + \tfrac{1}{4}\right)\left(\tfrac{x^2}{2}\right)_0^2 = 5/2$$



(ii) Along OB: $y = 0 \Rightarrow dy = 0$

x varies from 0 to 2

$$\int_{OB} \bar{z}\, dz = \int_0^2 x\, dx = \left(\frac{x^2}{2}\right)_0^2 = 2$$

Along BA: $x = 2 \Rightarrow dx = 0$, y varies from 0 to 1

$$\int_{BA} \bar{z}\, dz = \int_0^1 (2 - iy)(i\,dy) = i(2 - i/2) = 2i + 1/2$$

$$\therefore \int_{OBA} \bar{z}\, dz = 2 + \frac{1}{2} + 2i = \frac{5}{2} + 2i$$

prob: The value of $\int_{1-i}^{2+i} x\, dz$ along $x = t+1,\ y = 2t^2 - 1$ is

$x = t+1 \Rightarrow dx = dt$ $\qquad x = 1 \Rightarrow t = 0$

$y = 2t^2 - 1 \Rightarrow dy = 4t\, dt$ $\qquad x = 2 \Rightarrow t = 1$

$$\therefore \int_0^1 (t+1)(dt + i\,4t\,dt) = \int_0^1 (t+1)(1 + i4t)dt$$

$$= \frac{t^2}{2} + 4i\frac{t^3}{3} + t + 4i\frac{t^2}{2} = \frac{3}{2} + \frac{10}{3}i$$

Caushy's Integral theorem:

If $f(z)$ is analytic inside and on a closed curve 'C' then $\oint_C f(z)dz = 0$

Ex: $\oint_{|z|=1} (z^2 + 3z + 5)\, dz = 0$ [polynomial in z is analytic everywhere]

$\oint_{|z|=1} \frac{z^2+3z+5}{z-2}\, dz = 0$ [Analytic inside $|z|=1$. but not at $z=2$]

$\oint_{|z|=3} \frac{z^2+3z+5}{z-2}\, dz$ C.I.T is not applicable.

Caushy's integral formula:

If $f(z)$ is analytic inside and on a closed curve 'C' and $z=a$ is a point inside 'C' then

$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

Ex: $\oint_{|z|=3} \frac{z^2+3z+5}{z-2}\, dz = 2\pi i\, f(2) = 2\pi i\,[4+6+5]$

$= 30\pi i$

Generalised caushy's integral formula:

If $f(z)$ is analytic inside and on a closed curve 'C' and $z=a$ is a point inside 'C' then

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = 2\pi i\, \frac{f^n(a)}{n!}$$

Ex: $\oint_{|z|=3} \frac{z^2+3z+5}{(z-2)^2}\, dz = 2\pi i\, \frac{f'(2)}{1!} = 2\pi i\,[4+3]$

$= 14\pi i$

Generalized caushy's integral formula: If $f(z)$ is analytic inside and on a closed curve 'c' and $z=a$ is a point inside 'c' then

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = 2\pi i\,\frac{f^{n}(a)}{n!}$$

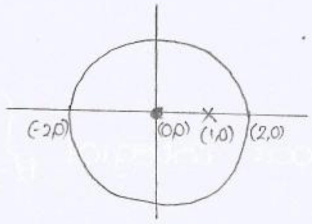
Ex: $\displaystyle\oint_{|z|=3} \frac{z^2+3z+5}{(z-2)^2}\,dz = 2\pi i\,\frac{f'(2)}{1!} = 14\pi i$

Procedure to follow:

1. Draw the given curve 'c'.
2. Factorize the denominator of given function and mark the roots
   (i) If all the roots are out side of curve 'c' then according to cauchy's integral theorem given integral $= 0$
   (ii) If some roots are inside 'c' then
   (a) write down numerator and the factors of denominator which corre-sponds to roots outside the curve 'c' (if any) as $f(z)$
   (b) Depending on inside roots whether it is a simple root or multiple root apply caushy's integral formula or generalized caushy's integral formula.

prob: $\displaystyle\oint \frac{z^2-z+1}{z-1}\,dz$ where c is (i) $|z|=2$ (ii) $|z|=\frac{1}{2}$

(i) $C: |z|=2$



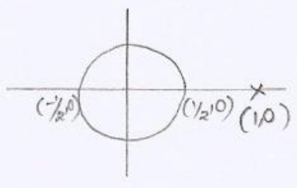
Root: $z=1$
$\Rightarrow (1,0)$

$z=1$ lies inside given $C: |z|=2$

$\therefore$ Let $f(z)=z^2-z+1$ (only numerator)

$$\oint \frac{z^2-z+1}{z-1}\,dz = 2\pi i\, f(1) \quad (z=1 \text{ is a simple root})$$
$$= 2\pi i\,[1-1+1] = 2\pi i$$

(ii) $C: |z|=\frac{1}{2}$

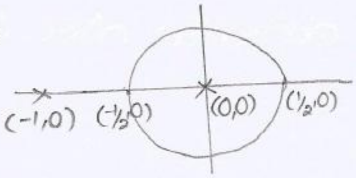


$z=1$ lies outside of $C: |z|=\frac{1}{2}$

$$\therefore \oint \frac{z^2-z+1}{z-1}\,dz = 0$$

prob: $\oint_C \frac{2z+1}{z^2+z}\,dz$ where C is $|z| = 1/2$.

Sol:



Denominator: $z^2+z = z(z+1)$

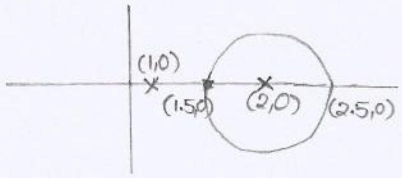
Roots: $z=0$ $\quad$ $z=-1$

$z=0$ lies inside and $z=-1$ lies outside of 'c'

$$\therefore f(z) = \frac{2z+1}{z+1}$$

$$\therefore \oint_C \frac{2z+1}{z^2+z}\,dz = \oint_C \frac{f(z)}{z}\,dz = 2\pi i \cdot f(0) = 2\pi i\left[\frac{1}{1}\right] = 2\pi i$$

prob: $\oint_C \frac{z}{(z-1)(z-2)^2}\,dz$ where C is $|z-2| = 1/2$



Denominator : $(z-1)(z-2)^2$

Roots : $z=1$ $\quad$ $z=2$
Simple $\quad$ multiple

$z=1$ lies outside and $z=2$ lies inside

$$\therefore f(z) = \frac{z}{z-1}$$

$$\oint_C \frac{z}{(z-1)(z-2)^2}\,dz = \oint_C \frac{f(z)}{(z-2)^2}\,dz = 2\pi i\,\frac{f'(2)}{1!}$$

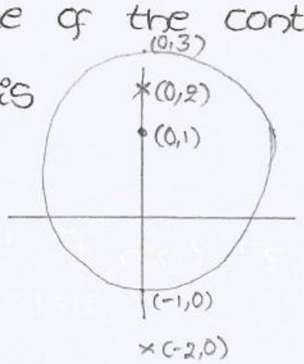
$$= 2\pi i\left[\frac{(z-1)-z}{(z-1)^2}\right]_{z=2} = -2\pi i$$

prob: $\oint_C \frac{2z+1}{(2z-1)^2}\,dz$ where C is $|z| = 1$

$$= \frac{1}{4}\oint_C \frac{2z+1}{(z-1/2)^2} = \frac{1}{4}\cdot 2\pi i\,\frac{f'(1/2)}{1!} \qquad f(z) = 2z+1$$

$$= \frac{1}{4}\cdot 2\pi i \times 2 = \pi i$$

(GATE-2006 EC): The value of the contour integral $\int_{|z-j|=2} \frac{1}{z^2+4}\,dz$ in the positive sense is



Sol: $C: |z-j| = 2$

$z^2+4 = (z+2i)(z-2i)$

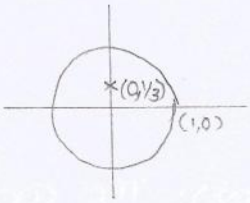
$z=-2i$ $\quad$ $z=2i$
outside $\quad$ inside

$$\Rightarrow f(z) = \frac{1}{(z+2i)}$$

$$\therefore \int_{|z-j|=2} \frac{1}{z^2+4}\,dz = \int_{|z-j|=2} \frac{f(z)}{(z-2i)}\,dz = 2\pi i\,f(2i)$$

$$= 2\pi i \cdot \frac{1}{4i} = \pi/2$$

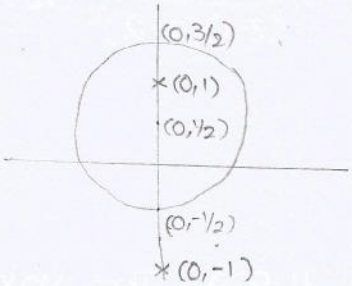
(GATE 06 CE): Using cauchy's integral theorem, the value of the integral (integration being taken in contour clockwise direction) $\oint_C \frac{z^3-6}{3z-i}\,dz$ is where C is $|z|=1$



Sol: $\oint_C \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\oint_C \frac{z^3-6}{z-i/3}\,dz$

$$= \frac{1}{3}\cdot 2\pi i\cdot\left[\left(\tfrac{i}{3}\right)^3-6\right] = \frac{2\pi}{81}-4\pi i$$

(GATE EC 07): The value of $\oint_C \frac{1}{z^2+1}\,dz$ where C is the contour $|z-i/2|=1$ is



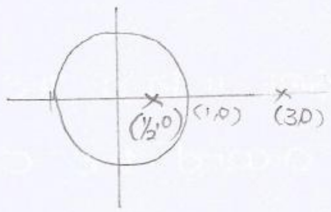
Sol: $\oint_C \frac{1}{z^2+1}\,dz = \oint_C \frac{1/z+i}{z-i}\,dz = 2\pi i\cdot\frac{1}{2i} = \pi$

(GATE ME 08): The integral $\oint f(z)\,dz$ evaluated around the unit circle on the complex plane for $f(z)=\frac{\cos z}{z}$

Sol: $C: |z|=1$, $z=0$ is inside 'C'

$$\therefore \oint_C \frac{\cos z}{z}\,dz = 2\pi i\cos(0) = 2\pi i$$

(GATE CE 09): The value of integral $\oint_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}\,dz$ where c is a closed curve given by $|z|=1$ is

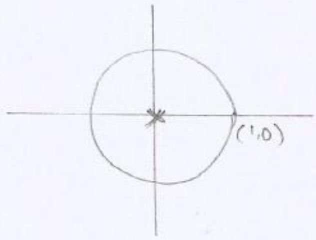


Sol: $\oint_C \frac{\cos(2\pi z)}{(2z-1)(z-3)}\,dz = \frac{1}{2}\oint_C \frac{\cos(2\pi z)}{(z-1/2)(z-3)}\,dz$

$$= \frac{1}{2}\oint_C \frac{\cos(2\pi z)/(z-3)}{(z-1/2)}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i\cdot\frac{\cos(\pi)}{(1/2-3)} = \frac{2\pi i}{5}$$

(GATE-09 EC): If $f(z)=C_0+C_1z^{-1}$ then $\oint_{|z|=1}\frac{1+f(z)}{z}\,dz$ is



Sol: $f(z)=C_0+C_1z^{-1}=C_0+\frac{C_1}{z}$

$$\oint_{|z|=1}\frac{1+C_0+\frac{C_1}{z}}{z}\,dz = \oint_{|z|=1}\frac{z+zC_0+C_1}{z^2}\,dz$$

$$= \frac{2\pi i}{1!}\left[1+C_0\right]_{z=0} = 2\pi i(1+C_0)$$

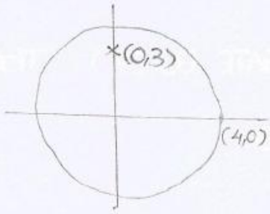
(GATE-09 IN): The value of $\oint_C \frac{\sin z}{z} dz$, where the contour of the integration is a simple closed curve around the origin is

Sol: $z=0$ is inside the curve surrounds it.

$$\therefore \oint_C \frac{\sin z}{z} dz = 2\pi i \sin(0) = 0$$

(GATE-10 IN): The contour C in the adjoining figure is described by $x^2+y^2=16$. Then the value of $\oint_C \frac{z^2+8}{(0.5)z-(1.5)j} dz$

Sol: Given $x^2+y^2=16 \Rightarrow |z|=4$

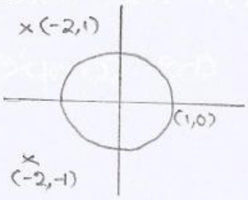


$$\oint_C \frac{z^2+8}{\frac{1}{2}z-\frac{3}{2}j} dz = 2\oint_C \frac{z^2+8}{z-3j} dz$$

$$= 2\left[2\pi j\left((3j)^2+8\right)\right] \quad (z=3j \text{ lies inside } |z|=4)$$

$$= -4\pi j$$

(GATE-11 EC): The value of integral $\oint_C \frac{-3z+4}{z^2+4z+5} dz$, when C is the circle $|z|=1$ is given by



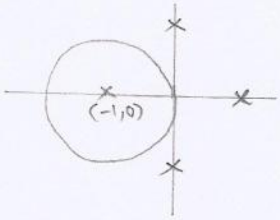
Sol: $z^2+4z+5 = (z+2-i)(z+2+i)$

$$z=-2+i \qquad z=-2-i$$

Both roots lies outside $C: |z|=1$

$$\therefore \oint_C \frac{-3z+4}{z^2+4z+5} dz = 0$$

(GATE-11 PI): The value of $\oint_C \frac{z^2}{z^4-1} dz$ using cauchy's integral around the circle $|z+1|=1$ where $z=x+iy$ is



Sol: $\oint_C \frac{z^2}{z^4-1} dz = \oint_C \frac{z^2}{(z^2-1)(z^2+1)} dz$

$$= \frac{1}{2}\oint_C \left[\frac{1}{z^2-1} + \frac{1}{z^2+1}\right] dz$$

$$= \frac{1}{2}\oint_C \frac{1}{z^2-1} dz + \frac{1}{2}\oint_C \frac{1}{z^2+1} dz$$

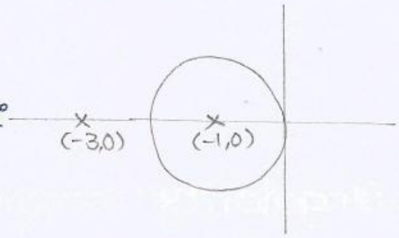
$$= \frac{1}{2}\oint_C \frac{1}{(z-1)(z+1)} dz + \frac{1}{2}\oint_C \frac{1}{(z+i)(z-i)} dz$$

Only $z=-1$ lies inside the circle $|z+1|=1$

$$\oint_C \frac{z^2}{z^4-1} dz = \frac{1}{2}\cdot 2\pi i\left[\frac{1}{-1-1}\right] + \cancel{\tfrac{1}{2}}\,\cancel{2\pi i}\ 0$$

$$= -\frac{\pi i}{2}$$

(GATE - 12 EC, EE, IN): Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in the z-plane such that $|z+1| = 1$, the value of $\frac{1}{2\pi j} \oint_C f(z)\,dz$ is



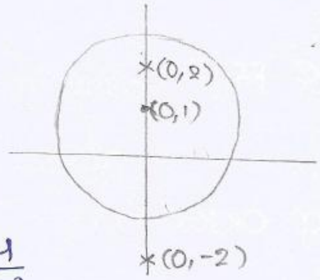
Sol: $z = -1$ lies inside & $z = -3$ lies outside of given $C: |z+1| = 1$

$$\frac{1}{2\pi j}\oint_C f(z)\,dz = \frac{1}{2\pi j}\oint_C \frac{1}{z+1}\,dz - \frac{1}{2\pi j}\oint_C \frac{2}{z+3}\,dz$$

$$= \frac{1}{2\pi j}\cdot 2\pi j\,(1)_{z=-1} - \frac{1}{2\pi j}(0)$$

$$= 1.$$

(GATE - 13 EE): $\oint \frac{z^2-4}{z^2+4}\,dz$ evaluated anticlockwise around the circular $|z-i| = 2$, where $i = \sqrt{-1}$ is



Sol: $z^2 + 4 = (z+2i)(z-2i)$

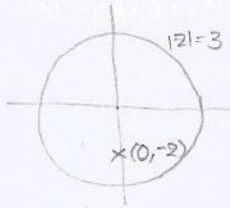
$z = -2i \qquad z = 2i$

$z = -2i$ lies outside $\Rightarrow f(z) = \frac{z^2-4}{z+2i}$

$$\oint \frac{z^2-4}{z^2+4}\,dz = \oint \frac{f(z)}{z-2i}\,dz$$

$$= 2\pi i \cdot f(2i) = -4\pi$$

(GATE - 14 EC): C is a closed path in the z-plane given by $|z| = 3$. The value of the integral $\oint_C \frac{z^2 - z + 4j}{z+2j}\,dz$ is

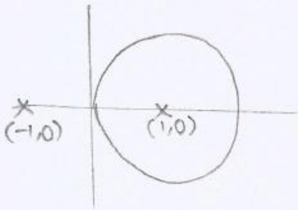


Sol: $z = -2j$ lies inside $|z| = 3$

$$\oint_C \frac{z^2-z+4j}{z+2j}\,dz = 2\pi j\left[(-2j)^2 - (-2j) + 4j\right]$$

$$= -4\pi(3+2j)$$

(GATE - 14 EE): Integration of the complex function $f(z) = \frac{z^2}{z^2-1}$, in the counterclockwise direction, around $|z-1| = 1$, is



Sol: $z^2 - 1 = (z+1)(z-1)$

$z = -1$ lies outside of $C: |z-1| = 1$

$\therefore F(z) = \frac{z^2}{(z+1)}$

$$\oint_C f(z) = \oint_C \frac{F(z)}{(z-1)} = 2\pi i\,F(1) = 2\pi i \cdot \frac{1}{2} = \pi i$$

Residue theorem: If $f(z)$ is analytic inside and on a closed curve 'c' except at a finite number of singularities inside 'c' then

$$\oint_c f(z)\,dz = 2\pi i\,[\text{Sum of residues over the singularities}]$$

Singularity: A point where $f(z)$ fails to be analytic is called a singularity of $f(z)$

Ex: $\dfrac{1}{(z-3)(z+7)}$ then $z=3,\ z=-7$ are singularities of $f(z)$

Singularities are of 4-types:

1. Simple pole
2. pole of order 'n'
3. Essential singularity
4. Removable singularity.

Simple pole: $f(z) = \dfrac{1}{(z-3)(z+7)}$. Here $z=3,-7$ are simple poles of $f(z)$

pole of order 'n': $f(z) = \dfrac{1}{(z+2)^2(z-5)}$. Here $z=-2$ is a pole of order '2' and $z=5$ is a pole of order '1' or simple pole.

Essential singularity: $f(z) = \sin\left(\dfrac{1}{z-1}\right) = \dfrac{1}{z-1} - \dfrac{1}{(z-1)^3 3!} + \dfrac{1}{(z-1)^5 5!} - \cdots$

Here $z=1$ is essential singularity of $f(z)$.

Removable singularity: $f(z) = \dfrac{z^2-3z+2}{(z-1)(z+2)} = \dfrac{z-2}{z+2}$

Here $z=1$ is a removable singularity. and $z=-2$ is a simple pole.

Calculation of residue:

1. If $z=a$ is a simple pole of $f(z)$ then

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} (z-a)\,f(z)$$

2. If $z=a$ is a pole of order 'n' of $f(z)$ then

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left\{(z-a)^n f(z)\right\}$$

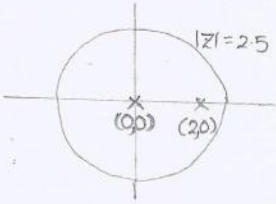
* Residue method is used to find $\oint f(z)\,dz$ when there are 2 or more poles of $f(z)$ exists inside the region 'c'.

Procedure to find $\oint_C f(z)\,dz$ using residue method:

1. Remove removable singularities of $f(z)$ if any.
2. Draw the region 'c' and mark the poles of $f(z)$
3. Residue corresponds to poles outside 'c' is $=0$
4. Find the residues corresponds to poles inside 'c' using above formula.

$$\oint_C f(z)\,dz = 2\pi i\,[\text{Sum of residues}]$$

prob: $\oint_C \frac{z^2+1}{z^2-2z}\,dz$ where $C: |z| = 2.5$



$$\oint_C \frac{z^2+1}{z^2-2z}\,dz = \oint_C \frac{z^2+1}{z(z-2)}\,dz$$

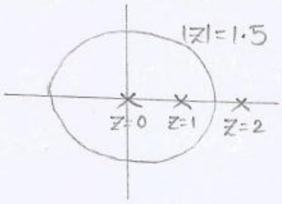
$z=0,\ z=2$ are simple poles & lies inside $|z| = 2.5$

$$\operatorname{Res} f(z)\big|_{z=0} = \lim_{z\to 0} \cancel{z}\cdot\frac{z^2+1}{\cancel{z}(z-2)} = \frac{0+1}{-2} = -\tfrac{1}{2}$$

$$\operatorname{Res} f(z)\big|_{z=2} = \lim_{z\to 2} \cancel{(z-2)}\,\frac{z^2+1}{z\cancel{(z-2)}} = \frac{4+1}{2} = \tfrac{5}{2}$$

$$\oint_C f(z)\,dz = 2\pi i\left[-\frac{1}{2}+\frac{5}{2}\right] = 4\pi i$$

prob: $\oint_C \frac{1-2z}{z(z-1)(z-2)}\,dz$ where 'c' is $|z| = 1.5$



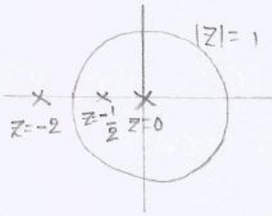
sol: $z=0,1$ lies inside $|z| = 1.5$ & are simple poles

$$\operatorname{Res} f(z)\big|_{z=0} = \lim_{z\to 0} \cancel{z}\cdot\frac{1-2z}{\cancel{z}(z-1)(z-2)} = \frac{1}{2}$$

$$\operatorname{Res} f(z)\big|_{z=1} = \lim_{z\to 1} \cancel{(z-1)}\cdot\frac{1-2z}{z\cancel{(z-1)}(z-2)} = \frac{-1}{-1} = 1$$

$$\therefore \oint_C f(z)\,dz = 2\pi i\left[\frac{1}{2}+1\right] = 3\pi i$$

prob: $\oint_C \frac{2z-1}{z(z+2)(2z+1)}\,dz$ where 'c' is $|z|=1$



sol:

$$\operatorname{Res} f(z)\big|_{z=0} = \lim_{z\to 0} \cancel{z}\cdot\frac{2z-1}{\cancel{z}(z+2)(2z+1)} = -\tfrac{1}{2}$$

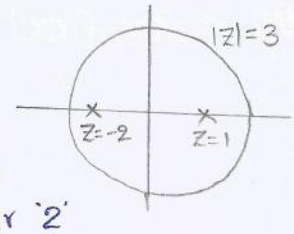
$z=0,\ z=-\tfrac{1}{2}$ lies inside

** $$\operatorname{Res} f(z)\big|_{z=-1/2} = \lim_{z\to -1/2} \cancel{(z+\tfrac{1}{2})}\cdot\frac{2z-1}{z(z+2)\cdot 2\cancel{(z+\tfrac{1}{2})}}$$

$$= \tfrac{4}{3}$$

$$\oint_C f(z)\,dz = 2\pi i\left(-\tfrac{1}{2}+\tfrac{4}{3}\right) = \frac{10\pi i}{6}$$

prob: $\oint_C \frac{z^2}{(z-1)^2(z+2)} dz$ where 'c' is $|z|=3$



Sol: Both $z=1$, $z=-2$ lies inside $|z|=3$

$z=-2$ is a simple pole & $z=1$ is pole of order '2'

$$\text{Res } f(z)\Big|_{z=1} = \underset{z\to 1}{lt} \frac{1}{1!} \frac{d}{dz}\left\{(z-1)^2 \cdot \frac{z^2}{(z-1)^2(z+2)}\right\}$$

$$= \underset{z\to 1}{lt} \frac{d}{dz}\left\{\frac{z^2}{z+2}\right\}$$

$$= \underset{z\to 1}{lt} \frac{(z+2)2z - z^2}{(z+2)^2} = \frac{6-1}{9} = 5/9$$

$$\text{Res } f(z)\Big|_{z=-2} = \underset{z\to -2}{lt} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$$\oint_C f(z)\,dz = 2\pi i\left[\frac{5}{9}+\frac{4}{9}\right] = 2\pi i$$

(GATE 08 EC): The residue of the function $f(z) = \frac{1}{(z+2)^2(z-2)^2}$ at $z=2$

Sol: $$\text{Res } f(z)\Big|_{z=2} = \frac{1}{1!} \underset{z\to 2}{lt} \frac{d}{dz}\left\{(z-2)^2 \cdot \frac{1}{(z+2)^2(z-2)^2}\right\}$$

$$= \frac{1}{1!} \underset{z\to 2}{lt} \frac{d}{dz}\left\{\frac{1}{(z+2)^2}\right\}$$

$$= \underset{z\to 2}{lt} \frac{-2}{(z+2)^3} = -1/32$$

(GATE 08 EE): Given $x(z) = \frac{z}{(z-a)^2}$ with $|z|>a$, the residue of $x(z)\,z^{n-1}$ at $z=a$ for $n\geq 0$ will be

Sol: Let $f(z) = x(z) \cdot z^{n-1} = \frac{z^n}{(z-a)^2}$

$z=a$ is a pole of order '2'

$$\text{Res } f(z)\Big|_{z=a} = \underset{z\to a}{lt}\left[\frac{d}{dz}\left\{(z-a)^2 \cdot \frac{z^n}{(z-a)^2}\right\}\right]$$

$$= \underset{z\to a}{lt}\left[\frac{d}{dz} \cdot z^n\right]$$

$$= \underset{z\to a}{lt}\; n z^{n-1} = n a^{n-1}$$

(GATE-10 EC): The residues of a complex function $x(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles

Sol: All the poles $z=0,1,2$ are simple poles

$\text{Res } x(z)\Big|_{z=0} = 1/2$ $\qquad$ $\text{Res } x(z)\Big|_{z=1} = 1$ $\qquad$ $\text{Res } x(z)\Big|_{z=2} = -3/2$

Taylor series of $f(z)$: Taylor series of $f(z)$ about $z=a$ is given by $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$

* Consist of only positive powers of $(z-a)$

Laurent's series: Laurent's series of $f(z)$ about $z=a$ is

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n (z-a)^n$$

* It consists of both positive & negative powers of $(z-a)$

prob: write $\frac{1}{(z-1)(z+3)}$ in powers of $(z-1)$

sol:
$$\frac{1}{(z-1)(z+3)} = \frac{1}{(z-1)(z-1+4)}$$
$$= \frac{1}{4(z-1)\left(1+\frac{z-1}{4}\right)}$$
$$= \frac{1}{4(z-1)}\left[1+\left(\frac{z-1}{4}\right)\right]^{-1}$$
$$= \frac{1}{4(z-1)}\left[1-\left(\frac{z-1}{4}\right)+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3+\cdots\right]$$
$$= \frac{1}{4(z-1)} - \frac{1}{4^2} + \frac{1}{4^3}(z-1) - \frac{1}{4^4}(z-2)^2 + \cdots$$

Residue: In laurent's series expansion of $f(z)$ about $z=a$ ~~if there is only one term with negative power of (z-a) then z=a is~~ the co.efficient of $(z-a)^{-1}$ is called the "residue"

* In laurent's series expansion of $f(z)$ about $z=a$

1. If there is only one term with negative power of $(z-a)$ then $z=a$ is called a Simple pole.

3. If there are infinite number of terms with negative power of $(z-a)$ then $z=a$ is called an essential singular

2. If there are 'n' terms with negative power of $(z-a)$ then $z=a$ is called pole of order 'n'

4. If there are no terms with negative power of $(z-a)$ then $z=a$ is called Removable Singularity

prob: $\frac{1}{(z-1)(z+3)} = \boxed{\frac{1}{4}}(z-1)^{-1} - \frac{1}{4^2} + \frac{1}{4^3}(z-1) - \frac{1}{4^4}(z-1)^2 + \cdots$

Residue $= \left.\frac{1}{(z+3)}\right|_{z=1} = \frac{1}{4}$

prob: Expand $\frac{1}{(z-1)^2(z+3)}$ interms of $(z-1)$ & find residue at $z=1$

sol: $\frac{1}{(z-1)^2(z+3)} = \frac{1}{4(z-1)^2}\left[1-\left(\frac{z-1}{4}\right)+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3+\cdots\right]$

$= \frac{1}{4}(z-1)^{-2} - \boxed{\frac{1}{4^2}}(z-1)^{-1} + \frac{1}{4^3} - \frac{1}{4^4}(z-1) + \cdots$

Residue$|_{z=1} = \lim_{z\to 1} \frac{d}{dz}\left\{\frac{1}{z+3}\right\}$

$= \lim_{z\to 1} \frac{-1}{(z+3)^2} = -\frac{1}{4^2}$

prob: $\oint_{|z|=3} (z-2)\cos\left(\frac{1}{z-2}\right)dz$

sol: $(z-2)\cos\left(\frac{1}{z-2}\right) = (z-2)\left[1 - \frac{1}{2!}\frac{1}{(z-2)^2} + \frac{1}{4!}\cdot\frac{1}{(z-2)^4} - \cdots\right]$

$= (z-2) - \boxed{\frac{1}{2!}}\cdot\frac{1}{(z-2)} + \frac{1}{4!}\cdot\frac{1}{(z-2)^3} - \cdots$

$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{2}\right) = -\pi i$

prob: $\oint_C z^2 e^{1/z}dz$ where C is $|z|=1$

$z^2e^{1/z} = z^2\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2\cdot\frac{1}{2!} + \cdots\right]$

$= z^2 + z + \frac{1}{2!} + \boxed{\frac{1}{3!}}\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^2} + \cdots$

$\oint_C f(z)dz = 2\pi i\left(\frac{1}{3!}\right) = \pi i/3$

prob: $\oint_C \frac{e^{-z}}{z^2}dz$ where C is $|z|=1$

caushy integral formula: $2\pi i\frac{f'(0)}{1!} = 2\pi i\frac{(-e^{-z})_{z=0}}{1} = -2\pi i$

caushy residue theorem: $\text{Res} f(z)|_{z=0} = \lim_{z\to 0}\frac{d}{dz}(e^{-z}) = \lim_{z\to 0} -e^{-z} = -1$

$\oint_C f(z)\,dz = 2\pi i(-1) = -2\pi i$

Laurentz series: $\frac{e^{-z}}{z^2} = \frac{1}{z^2}\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right]$

$\oint f(z)dz = 2\pi i(-1) = -2\pi i$

(GATE-07 IN): For the function $\frac{\sin z}{z^3}$ of a complex variable z, the point z=0 is

Sol: $\frac{\sin z}{z^3} = \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$

$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$

Highest negative power is $\frac{1}{z^2}$. $\therefore$ $z=0$ is pole of order '2'

(GATE-11 IN): The contour integral $\int_C e^{1/z}\,dz$ with C as the counte clockwise unit circle in the z-plane is equal to

Sol: $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$

$\text{Res } f(z)\big|_{z=0} = 1$

$\therefore \oint_C e^{1/z}\,dz = 2\pi i(1) = 2\pi i = 2\pi\sqrt{-1}$

PROBABILITY AND STATISTICS

PROBABILITY & STATISTICS

Random experiment: An ~~outcom~~ experiment whose output cannot be predicted but it is one among the several possibilities.

Ex: Tossing coin, rolling dyes .

Event: The outcome of the random experiment is called event. and the chance of this event is $P(E) = \dfrac{\text{favourable cases}}{\text{Total no. of cases}}$

and $0 \leq P(E) \leq 1$

permutations & combinations:

Factorial: Let n be positive integer, $n! = n \times (n-1) \times (n-2) \times \dots 3 \times 2 \times$

and $0! = 1$

permutations: The different arrangements of a given number of things by taking some or all at a time

Ex: All permutations made with letters a, b, c by taking two at a time are (ab, ba, ac, ca, bc, cb)

Ex: All permutations made with letters a, b, c by taking all at a time are (abc, acb, bac, bca, cab, cba)

Number of permutations: Number of permutations of n things, taken 'r' at a time is given by

$$nP_r = n.(n-1).(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Ex: $6P_2 = 6 \times 5 = 30$ $\quad 7P_3 = 7 \times 6 \times 5 = 210$

* Number of all permutations of n things, taken all at a time $= n!$

* If there are 'n' subjects of which P_1 are alike of one kind; P_2 are alike of another kind; and so on P_r are alike of r^{th} kind such that $(P_1 + P_2 + P_3 + \dots + P_r) = n$

No. of permutations of these 'n' objects $= \dfrac{n!}{P_1!\, P_2! \dots P_r!}$

Combinations: Each of different groups / selections which can be formed by taking some or all of a number of objects.

Ex: select 2 out of 3 boys A, B, C. Then possible selections are AB, BC, CA. (AB and BA represent same selection)

Number of combinations: The number of combinations of n things, taken 'r' at a time is $nC_r = \frac{n!}{r!\,(n-r)!}$

Note: $nC_n = 1$ and $nC_0 = 1$, $nC_r = nC_{n-r}$, $nC_r \cdot r! = nP_r$

Ex: $11C_4 = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$

$16C_{13} = 16C_{16-13} = 16C_3 = \frac{16\times15\times14}{3\times2\times1} = 560.$

Ex: The no. of shake hands out of 10 friends if each shakehand with other $= 10C_2$ $(A \rightleftharpoons B)$

Ex: The no. of e-mails out of 10 friends if each giving e-mail to other $= 10P_2$ $(A \to B,\ A \leftarrow B)$

* Selection of r things out of n things = no. of selections in which one particular thing is already there + no. of selections in which that particular thing is not there.

$$nC_r = (n-1)C_{(r-1)} + (n-1)C_r$$

Ex: $6C_3 = 5C_2 + 5C_3 = 20$

Circular arrangements: The no. of ways to arrange 'n' things in a circular order is given by $(n-1)!$

Ex: The no. of ways to arrange a, b, c in a circular order is 2 ways they are

In case of circle there is no starting & ending points. once we fix one of the place, as * then the remaining n-1 places can be arranged themselves like in a row in $(n-1)!$ ways.

Arrangements

	In a row					In a circular order	
	6 boys & 6 Girls	6 boys & 5 girls	6 boys & 4 girls	6A's & 6B's	6A's & 6 Girls	6 boys & 6 girls	6 boys & 5 girls
1. Total ways	$12!$	$11!$	$10!$	$\frac{12!}{6!\,6!}$	$\frac{12!}{6!}$	$11!$	$10!$
2. All girls (B's) must be together	$7!\,6!$	$7!\,5!$	$7!\,4!$	$\frac{7!}{6!} = 7$	$\frac{7!}{6!} \times 6! = 7!$	$6!\,6!$	$6!\,5!$
3. No two girls (B's) must be together	$6!\,[{}^7C_6 \times 6!]$	$6!\,[{}^7C_5 \times 5!]$	$6!\,[{}^7C_4 \times 4!]$	$\frac{6!}{6!} \times {}^7C_6$	$\frac{6!}{6!} \times {}^7C_6 \times 6!$	$5!\,6!$	$5!\,[{}^6C_5 \times 5!]$
4. All girls (B's) not be together	$12! - 7!\,6!$	$11! - 7!\,5!$	$10! - 7!\,4!$	$\frac{12!}{6!\,6!} - 7$	$\frac{12!}{6!} - 7!$	$11! - 6!\,6!$	$10! - 6!\,5!$
5. No two boys (A's) & no two girls (B's) must be together	$6!\,6! + 6!\,6!$	$6!\,5!$	0	$1\times1 + 1\times1$	$1\times6! + 1\times6!$	$5!\,6!$	0

Note: If difference b/w no. of boys & girls is 2 or more then no 2 boys & no 2 girls sit together in a row is not possible.

Note: If difference b/w no. of boys & girls is 1 or more then no 2 boys & no 2 girls sit together in a circle is not possible

Elementary event: outcome of a random experiment

Spample space: set of all possible outcomes, denoted by S.

Ex: Tossing a coin $S = \{H, T\}$

Rolling a die $S = \{1, 2, 3, 4, 5, 6\}$

The event represented by set 'S' is a sure event so, it always occurs.

Impossible event: If there is no outcome of a experiment it is called impossible event and denoted by ϕ.

Equally likely event: If all the elementary events of 'S' have the same chance of occuring, then the events are said to be equally likely events.

Mutually exclusive Events: Two events cannot occur together

Ex: Let 2 dice are throwed.

E_1: sum of the numbers on the dice ≤ 4 $= \{2, 3, 4\}$

E_2: sum of the numbers on the dice > 9 $= \{10, 11, 12\}$

sample space $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$$E_1 \cap E_2 = \phi$$

Mutually exhaustive events:

If $E_1 \cup E_2 \cup E_3 \ldots \cup E_m = S$ then the events are exhaustive.

* If both $E_1 \cup E_2 \cup E_3 \ldots \cup E_m = S$ and $E_i \cap E_j = \phi$ $i \neq j$ occur then the events are called mutually exclusive & exhaustive.

Combination of events:

$E_1 \cup E_2$ = E_1 or E_2 or both occur

$E_1 \cap E_2$ = E_1 and E_2 both occur

E_1^c or $\bar{E}_1$ = set of elements of S which do not belong to E_1

$E_1 - E_2$ = Event E_1 but not E_2

* $A \cap \bar{A} = \phi$, A and $\bar{A}$ are mutually exclusive.

Probability of an event:

$$P(A) = P = \frac{\text{number of elementary events favourable to } A}{\text{Total number of equally likely elementary events}}$$

* $0 \le P(A) \le 1$
* $P(\phi) = 0$ and $P(S) = 1$
* Law of addition : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

* If A & B are mutually exclusive then $A \cap B = \phi$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B)$$

* $P(\text{not } A) = P(\bar{A}) = 1 - P(A)$

Tossing coins: 1 coin $\rightarrow$ 2 ways $\rightarrow \{T, H\}$

2 coins $\rightarrow 2^2$ ways $\rightarrow \{TT, TH, HT, HH\}$

⋮

n coins $\rightarrow 2^n$ coins

Prob: Two coins are tossed together what is the probability of getting

(i) Same face : $\frac{2}{2^2}$ $\{HH, TT\}$

(ii) More heads than tails: $\frac{1}{2^2}$ $\{HH\}$

Prob: If 3 coins tossed

(i) $\frac{2}{2^3}$ $\{HHH, TTT\}$

(ii) $\frac{3C_2 + 3C_3}{2^3}$ $\{HHT, HTH, THH, HHH\}$

H	T	
0	3 ×	
1	2 ×	
2	1 ✓	$\rightarrow 3C_2$
3	0 ✓	$\rightarrow 3C_3$

Prob: If 4 coins tossed

(i) $\frac{2}{2^4}$ $\{HHHH, TTTT\}$

(ii) $\frac{4C_3 + 4C_4}{2^4}$

H	T	
0	4	
1	3	
2	2	
3	1 ✓	$\rightarrow 4C_3$
4	0 ✓	$\rightarrow 4C_4$

prob: A coin is tossed untill it shows the same face in two consecutive throws. what is the probability of success by tossing a coin not more than 4 times.

Sol: Success at 2^{nd} time HH , TT $\rightarrow$ 2 ways

" at 3^{rd} time HTT, THH $\rightarrow$ 2 ways

" at 4^{th} time THTT, HTHH $\rightarrow$ 2 ways

$$\text{Probability} = \frac{2}{2^2} + \frac{2}{2^3} + \frac{2}{2^4} = \frac{7}{8}$$

prob: Not more than 3 times

Sol:
$$\text{probability} = \frac{2}{2^2} + \frac{2}{2^3} = \frac{3}{4}$$

prob: A&B Simultaneously toss a coin till one of them gets head. If A starts the game then what is the probability for

(i) B to win (ii) A to win

Sol: (i) A started $\rightarrow$ B win

TH TTTH TTTTTH

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1/4}{1-1/4} = \frac{1}{3}$$

$a + ar + ar^2 + \cdots = \frac{a}{1-r}$ if $|r|<1$

(ii) A started $\rightarrow$ A win

H TTH TTTTH

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{1/2}{1-1/4} = 2/3$$

prob: A coin is tossed untill which shows head. what is the probability of success by tossing a coin an odd no. of times

Sol: H TTH TTTTH

$$1/2 + 1/2^3 + 1/2^5 + \cdots = \frac{1/2}{1-1/4} = 2/3$$

prob: An even no. of times

Sol: TH TTTH

$$1/2^2 + 1/2^4 + 1/2^6 + \cdots = \frac{1/4}{1-1/4} = 1/3$$

4

Rolling dice: 1 die - 6 ways $\{1, 2, 3, 4, 5, 6\}$

2 dice - 6^2 ways $\{11, 12, 13, 14, 15, 16, 21, 22, 23 \dots\}$

3 dice - 6^3 ways

Prob: Two dice are rolled together. what is the probability of getting (i) Same face $6/6^2 = 1/6$ $\{11, 22, 33, 44, 55, 66\}$

(ii) Sum as 10 $3/6^2$ $\{64, 46, 55\}$

(iii) Sum morethan 10 $3/6^2$ $\{56, 65, 66\}$

(iv) First number more than second num. $15/6^2$

Note: When two dice are rolled then $A = B \rightarrow 6$ ways

$A > B \rightarrow 15$ ways

$B > A \rightarrow 15$ ways.

prob: 3 dice rolled

(i) same face $\frac{6}{6^3} = 1/36$

(ii) Sum as morethan 16 $= \frac{3+1}{6^3}$ $\{(6,6,5), (5,6,6), (6,5,6), (6,6,6)\}$

prob: 4 dice rolled

(i) Same face $= \frac{6}{6^4}$

(ii) Sum as 20 $= \frac{35}{6^4}$

5 5 5 5 $\rightarrow$ 1 way

5 5 6 4 $\rightarrow \frac{4!}{2!} = 12$

6 6 5 3 $\rightarrow \frac{4!}{2!} = 12$

6 6 4 4 $\rightarrow \frac{4!}{2!\,2!} = 6$

6 6 6 2 $\rightarrow \frac{4!}{3!} = 4$

35 ways

(iii) Sum as 21 $= \frac{20}{6^4}$

6 5 5 5 $\rightarrow$ 4

6 6 5 4 $\rightarrow$ 12

6 6 6 3 $\rightarrow$ 4

20

Prob: A & B simultaneously roll dice till one of them gets the number '4'. If A starts the game, what is the probability for

(i) B to win

$$\underbrace{\frac{5}{6}}_{\text{A lost}} \times \underbrace{\frac{1}{6}}_{\text{B won}} + \underbrace{\frac{5}{6}}_{\text{A lost}} \cdot \underbrace{\frac{5}{6}}_{\text{B lost}} \cdot \underbrace{\frac{5}{6}}_{\text{A lost}} \cdot \underbrace{\frac{1}{6}}_{\text{B won}} + \dots = \frac{5/36}{1 - \frac{25}{36}} = \frac{5}{11}$$

(ii) A to win $1-5/11 = 6/11$

prob: There are two biased dice of which 1st dice shows an even number twice as frequently as odd number, 2nd dice shows 5, thrice as frequently as any other number. If these dice are rolled together what is the probabity of getting

(i) Sum as 10 (ii) Sum as morethan 10

	dice1	dice2
1	k	l
2	$2k$	l
3	k	l
4	$2k$	l
5	k	$3l$
6	$2k$	l
	$9k=1$	$8l$
	$k=1/9$	$l=1/8$

(i) Sum as 10

(5,5), (6,4), (4,6)

$$\frac{1}{9}\times\frac{3}{8}+\frac{2}{9}\times 1/8+\frac{2}{9}\times 1/8$$

(ii) Sum more than 10

(5,6), (6,5), (6,6)

$$\frac{1}{9}\times\frac{1}{8}+\frac{2}{9}\times\frac{3}{8}+\frac{2}{9}\times 1/8 =$$

Pack of cards:

52
- 26 black
  - 13 spades ♤
    - Honours (4): A, J, Q, K — $\underbrace{\text{A, J, Q, K}}_{\text{face cards}}$
    - Number (9): 2 to 10
  - 13 clubs ♧
- 26 Red
  - 13 diamonds ◇
  - 13 hearts ♡

Suits

prob: Two cards are drawn at random from a pack of cards both must belong to same suit & both must belong to different suits.

sol: (i) $\frac{4C_1 \cdot 13C_2}{52C_2}$ (ii) $\frac{4C_2 \cdot 13C_1 \cdot 13C_1}{52C_2}$

prob: Three cards

sol: (i) $\frac{4C_1 \cdot 13C_3}{52C_3}$ (ii) $\frac{4C_3 \cdot 13C_1 \cdot 13C_1 \cdot 13C_1}{52C_3}$

prob: Four cards

sol: (i) $\frac{4C_1 \cdot 13C_4}{52C_4}$ (ii) $\frac{4C_4 \cdot 13C_1 \cdot 13C_1 \cdot 13C_1 \cdot 13C_1}{52C_4}$

5.

Prob: A card is drawn at random from a pack of cards. What is the probability that it is (i) King (ii) King or red (iii) King or Queen (iv) King or face card (v) King and Queen.

Sol: (i) $4/52$

(ii) $P(K \cup R) = P(K) + P(R) - P(K \cap R)$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

(iii) $P(K \cup Q) = P(K) + P(Q)$ K, Q are mutually exclusive

$$= \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$$

(iv) $P(K \cup F) = P(K) + P(F) - P(K \cap F)$

$$= \frac{4}{52} + \frac{12}{52} - \frac{4}{52} = 12/52$$

(v) $P(K \cap Q) = 0$

Prob: A card is drawn from a pack of 52 cards. Find the probability that (i) neither diamond nor a face (ii) neither 10 nor a King comes.

Sol: (i) $P(D^c \cap F^c) = 1 - P(D \cup F) = 1 - [P(D) + P(F) - P(D \cap F)]$

$$= 1 - \left[\frac{13}{52} + \frac{12}{52} - \frac{3}{52}\right]$$

(ii) $P(10^c \cap K^c) = 1 - P(10 \cup K) = 1 - [P(10) + P(K) - P(10 \cap K)]$

$$= 1 - \left[\frac{4}{52} + \frac{4}{52} - 0\right] = \frac{44}{52}$$

Leap year concept:

Leap year	Non leap year
366 days	365 days
52 weeks + 2 days	52 weeks + 1 day
↓ SM, MT, TW, WT, TF, FS, SS	↓ S, M, T, W, T, F, S.

Prob: The probability for a leap year selected at random to have (i) 53 Sundays (ii) 52 Sundays is

Sol: (i) $2/7$ (ii) $5/7$.

prob: probability for non-leap year to have (i) 53 sundays (ii) 52 sundays.

sol: (i) $1/7$ (ii) $6/7$

Multiplication theorem / conditional probability:

$$P(A\cap B) = P(A)\cdot P(B/A) = P(B)\cdot P(A/B)$$

$\hookrightarrow$ conditional probability

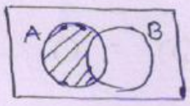
$$P(A\cap B\cap C) = P(A)\cdot P(B/A)\cdot P(C/A\cap B)$$

Independent events: Two events are said to be independent if $P(A\cap B) = P(A).\ P(B)$

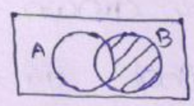
6. $P(A^c\cup B^c) = P((A\cap B)^c)$
$= 1 - P(A\cap B)$

* If A & B are two events then

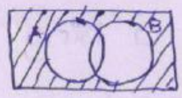
1. $P(A) + P(A^c) = 1$
2. $P(A\cap B^c) = P(A) - P(A\cap B)$ only 'A'



3. $P(A^c\cap B) = P(B) - P(A\cap B)$ only 'B'



4. $P(A^c\cap B^c) = P((A\cup B)^c) = 1 - P(A\cup B)$ Neither A nor B



5. $P(A^c/B) = \dfrac{P(A^c\cap B)}{P(B)} = \dfrac{P(B) - P(A\cap B)}{P(B)}$

$$= 1 - \frac{P(A\cap B)}{P(B)} = 1 - P(A/B)$$

prob: Two cards are drawn one after the other without replacement what is the probability that 1st is King and 2nd is Queen

sol: $\dfrac{4}{52}\times\dfrac{4}{51}$

prob: 1st card is known to be King, what is the probability for 2nd card is Queen

$4/51$

prob: If it is with replacement then what is the probability that 1st is King and 2nd is queen is

$\dfrac{4}{52}\times\dfrac{4}{52}$

Total probability theorem: Let $B_1, B_2, \ldots B_n$ be a set of mutually exclusive and exhaustive events of the sample space S with $P(B_K) \neq 0$, $K = 1, 2, \ldots n$ and let A be any event of S. Then

$$P(A) = \sum_{i=1}^{n} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$$

Prob: A factory has four independent units A, B, C, D which produce 40%, 30%, 20%, 10% of identical items respectively. The percentage of defective items produced by these units are 2%, 1%, 0.5%, 0.25% respectively. If an item is selected at random, find probabity that the item is defective.

Sol: M : defective item

$$P(M) = P(A) \cdot P(M/A) + P(B) \cdot P(M/B) + P(C) \cdot P(M/C) + P(D) \cdot P(M/D)$$

$$= 0.4\,(0.02) + 0.3\,(0.01) + 0.2\,(0.5) + 0.1\,(0.25) = 0.01225$$

Prob: Box A contains 6 red and 4 blue balls, box B contains 3 red and 7 blue balls. Two balls are drawn at random from box A and placed in box B. Now a ball is drawn at random from Box B. Find the probability that it is blue ball.

Sol:

Balls taken from A	probability	Box B contains	probability of drawing blue ball
2 red	$\frac{6C_2}{10C_2} = 1/3$	5 red, 7 blue	$7C_1/12C_1 = 7/12$
1 red, 1 blue	$\frac{6C_1 \cdot 4C_1}{10C_2} = \frac{8}{15}$	4 red, 8 blue	$\frac{8C_1}{12C_1} = 2/3$
2 blue	$\frac{4C_2}{10C_2} = \frac{2}{15}$	3 red, 9 blue	$\frac{9C_1}{12C_1} = 3/4$

$$P(E) = P(B_1) \cdot P(E/B_1) + P(B_2)\, P(E/B_2) + P(B_3) \cdot P(E/B_3)$$

$$= \frac{1}{3} \cdot \frac{7}{12} + \frac{8}{15} \cdot \frac{2}{3} + \frac{2}{15} \cdot \frac{3}{4} = 0.65$$

** **Baye's theorem:** (probability of causes)

$$P(B_K/A) = \frac{P(B_K) \cdot P(A/B_K)}{\sum_{K=1}^{n} P(B_K) \cdot P(A/B_K)}$$

Taking ball from a bag / box:

prob: There are 2 bags of which 1st bag contains 5 red & 4 green balls, 2nd bag contain 6 red & 7 green balls. A ball is drawn at random from one of the bags

(i) what is the probability that it is a red ball

$$\frac{1}{2} \times \frac{5}{9} + \frac{1}{2} \times \frac{6}{13}$$

(ii) It is found to be red. what is the probability that it is obtained from 2nd bag

$$= \frac{\frac{1}{2} \times \frac{6}{13}}{\frac{1}{2} \times \frac{5}{9} + \frac{1}{2} \times \frac{6}{13}} =$$

(iii) If chance of selection of 1st bag is 1/3 & 2nd bag is 2/3 then

$$= \frac{2/3 \times 6/13}{1/3 \times 5/9 + 2/3 \times 6/13}$$

prob: A bag A contains 8 white and 4 black balls. A second bag B contains 5 white and 6 black balls. One ball is drawn at random from bag A and is placed in Bag B. Now, a ball is drawn at random from bag B. It is found that this ball is white. Find the probability that black ball has been transfered from bag A.

Sol:

B_1: Transfer white ball

$$P(B_1) = \frac{8}{12} = \frac{2}{3}$$

Now bag B contains 6 white and 6 black balls

B_2: Transfer black ball

$$P(B_2) = \frac{4}{12} = \frac{1}{3}$$

Now bag B contains 5 white & 7 black balls

E: Drawing white ball from Bag B

$$P(E/B_1) = \frac{6}{12} = \frac{1}{2} \qquad P(E/B_2) = \frac{5}{12}$$

probability that black ball is transfered

$$P(B_2/E) = \frac{P(B_2) \cdot P(E/B_2)}{P(E)} = \frac{5/36}{17/36}$$

$$= \frac{5}{17}$$

Prob: The probability for a man to know the answer for a question in an exam is $1/3$. If he don't know the answer, then he guess the answer. The probability for the guessed answer to be correct is $1/4$. He answered one question and is found to be correct. What is probability that he knows the answer.

Sol: E_1 : Knows answer $\quad P(E_1) = 1/3$

E_2 : guess answer (Don't know answer) $\quad P(E_2) = 1 - 1/3 = 2/3$

A : Answer is correct

$P(A/E_1)$: Probability for known answer to be correct (Not given)

** Assume $P(A/E_1) = 1$

$P(A/E_2)$: probability for guessed answer to be correct $= 1/4$

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

$$= \frac{1/3 \cdot 1}{1/3 \cdot 1 + 2/3 \cdot 1/4} = 2/3$$

Venn diagrams:

Prob: In a town 50% people read newspaper A, 40% read B, 20% read both A and B. A person is selected at random what is the probability that for him to read

(i) Atleast one = only one + Both

$$= \left(\frac{30}{100} + \frac{20}{100}\right) + \frac{20}{100} = \frac{70}{100}$$

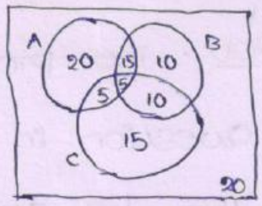
(ii) Atmost one = No one + only one

$$= \frac{30}{100} + \left(\frac{30}{100} + \frac{20}{100}\right) = \frac{80}{100}$$

(iii) Exactly one $= \frac{30}{100} + \frac{20}{100} = \frac{50}{100}$

(iv) None $= \frac{30}{100}$

Prob: In a town 45% read A, 40% read B, 35% read C, 20% read A&B 15% read B&C, 10% read A&C, 5% read A&B&C. A person selected at random. What is the prob. that



(i) Atleast one $= 80/100$

(ii) At most one $= \frac{20}{100} + \frac{20}{100} + \frac{10}{100} + \frac{15}{100} = \frac{65}{100}$

(iii) Exactly one $= \frac{20}{100} + \frac{10}{100} + \frac{15}{100} = \frac{45}{100}$

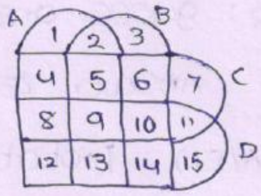
(iv) None $= \frac{20}{100}$

(v) Atleast two = two + Three $= \frac{5}{100} + \frac{15}{100} + \frac{10}{100} + \frac{5}{100} = \frac{35}{100}$

(vi) Atmost two = None + One + two $= \frac{20}{100} + \frac{20}{100} + \frac{15}{100} + \frac{10}{100} + \frac{15}{100} + \frac{10}{100} + \frac{5}{100} = \frac{95}{100}$

(vii) Exactly two $= \frac{30}{100}$

Note: If 4 variables then venn diagram



Atleast one to happen:

prob: A room has 3 bulb holders. A bag contains 15 bulbs of which 5 are fused. 3 bulbs selected at random to fit into holders. What is the probability that room get lighted

sol: Room get lighted = Atleast one light should glow

= 1 − No light should glow

$= 1 - \frac{5C_3}{15C_3}$

prob: The probability for A to solve a problem is $1/3$. that of B to solv is $1/7$. What is the probability that problem will be solved

sol: problem solved = Atleast one should solve

= 1 − No one should solve

$= 1 - \frac{2}{3} \cdot \frac{2}{7} = \frac{21-4}{21} = 17/21$

prob: The probability for a man to hit a target is $1/3$. What is the probability for him to hit the target atleast once out of 5 chances.

sol: probability ~~for~~ to hit atleast once = 1 − Not hitting in 5 chances

$= 1 - \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = 1 - \left(\frac{2}{3}\right)^5$

Miscellaneous problems:

1. prob: A number is Selected at random from 1^{st} 200 natural numbers. Find the probability that it is divible by 6 or 8.

Sol: $\frac{33}{200} + \frac{25}{250} - \frac{8}{200} = \frac{1}{4}$

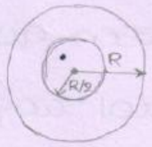
6)200(33
198
2

8)200(25
200
0

2|8,6
4,3
L.C.M = 24

24)200(8
192
8

2. prob: A point is selected at random inside circle. what is the probability that the point is closer to the center than to it's circumference.



Sol: For the point is to be closer to center, it should inside the circle with radius R/2

$\therefore$ probability $= \frac{\pi(R/2)^2}{\pi R^2} = \frac{1}{4}$

3. Prob: In a certain college 4% men & 1% women are taller than 1.8m, furthermore 60% of Students are women. A student is selected at random & is taller than 1.8m. What is the probability that the Student is a women?

Sol: E_1 : Student is men

E_2 : Student is women

A : Taller than 1.8m

$$P(E_2/A) = \frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$$

$$= \frac{\frac{60}{100} \times \frac{1}{100}}{\frac{40}{100} \times \frac{4}{100} + \frac{60}{100} \times \frac{1}{100}} = 3/11$$

4. Prob: A coin is weighted so that $P(H) = 2/3$, $P(T) = 1/3$ is tossed. If head appears then a number is selected at random from 1 to 9. If tail appears a number is " " " " 1 to 5. Find the probabity of getting an even number.

$$= \underbrace{\frac{2}{3}}_{\to P(H)} \times \underbrace{\frac{4}{9}}_{\to \text{Even number from } 1-9} + \underbrace{\frac{1}{3}}_{\to P(T)} \times \underbrace{\frac{2}{5}}_{\to \text{Even number from } 1-5}$$

5. prob: A determinant is selected at random from set of all determinants of order 2×2 whose entries are either zero or one. only. what is the probability that the choosen determinant is

positive?

Sol: $\begin{bmatrix} - & - \\ - & - \end{bmatrix} = 2\times2\times2\times2 = 16$ ways.

$\underbrace{ad}_{\text{Equal to '1'}} - \underbrace{bc}_{\text{Equal to '0'}} > 0$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 3$

only possibility

probability = $3/16$

6.Prob: A quadratic eqn of the form $ax^2+bx+c=0$ is selected at random from all possibles whose co-eff's a, b, c are distinct and are selected from the set $\{1,2,3,4,6,8,9\}$. What is the probability that choosen quatratic equation to have equal roots?

Sol: Total ways = $7P_3$

Equal roots $\Rightarrow b^2 = 4ac$ $\begin{cases} 6^2 = 4.1.9 \\ 6^2 = 4.9.1 \end{cases}$

$$\text{probability} = \frac{2}{7P_3}$$

Note: If $P(E) = m/n$

odds in favour of E is $m : (n-m)$

odds against of E is $(n-m) : m$

7. Prob: The odds are 2 to 5 in favour of A to solve a problem & 3 to 7 in against B to solve the problem. What is the probability for problem to be solved?

Sol: $m : n-m = 2:5$ $\qquad$ $n-m : m = 3:7$

$m = 2, \ n = 7$ $\qquad$ $m = 7, \ n = 10$

$P(A) = 2/7$ $\qquad$ $P(B) = 7/10$

probability for problem to be solved $= 1 - 5/7 \cdot 3/7 = 55/70$

RANDOM VARIABLE:

A variable whose value cannot be predicted but it is one among the several possibilities is called a Random variable (R.V)

Ex: No. of Heads turns up when tossing a coin 3 times is a R.V. Here R.V takes values 0, 1, 2, 3

We can distribute probability for R.V as below.

x	0	1	2	3
$P(x=x_i)$	$1/8$	$3/8$	$3/8$	$1/8$

1-Dimensional R.V: Assigning a real value to each outcome of an experiment is called 1 Dimensional R.V. The corresponding data is known as univariate data.

2-Dimensional R.V: connecting 2 outcomes at a time to the one real value provided that, those outcomes are drawn from same sample space. The corresponding data is known as bi-variate data.

Discrete Random variable: If a random variable x takes a finite number or countably infinite number of values, then x is called a discrete R.V. We denote the possible values taken by x as $x_1, x_2, \ldots x_n$.

Let $P(x=x_i)=P_i$ then P_i is called the probability function if it satisfies the following conditions

(i) $P_i \geq 0$ for all i

(ii) $\sum_i P_i = 1$

The collection of pairs (x_i, P_i), $i=1,2,3\ldots$

x	x_1	x_2	x_3	$\ldots$
$P(x=x_i)$	P_1	P_2	P_3	$\ldots$

is called the probability distribution or discrete probability distribution or probability mass function

Cumulative distribution function: The function $F(x)$ defined by $F(x) = P(X \le x)$

$= \sum_{x_i \le x} p(x_i)$ is called cumulative distribution function.

We can write $p(x_i) = P(X = x_i) = F(x_i) - F(x_{i-1})$

properties: 1. $0 \le F(x) \le 1$

2. If $x_1 < x_2$ then $F(x_1) \le F(x_2)$

3. $P(a \le X \le b) = F(b) - F(a)$

Expectation: The expectation of a discrete R.V 'x' is defined as

$$E(X) = \sum_{i=0}^{n} x_i\, p(x_i)$$

Mean $\mu_x = E(X) = \sum_{i=0}^{n} x_i\, p(x_i) = \sum_i x_i p_i$

Variance $\sigma_x^2 = E[(x - \mu_x)^2] = \sum_i (x_i - \mu_x)^2 p_i$

$= E(X^2) - [E(X)]^2$

$= \sum_i x_i^2 p_i - \mu_x^2$

prob: A R.V X has following probability distribution

x	0	1	2	3	4
$P(x)$	C	$2C$	$2C$	C^2	$5C^2$

Find C? Evaluate $P(X<3)$, $P(0<X<4)$. Determine distribution function of X. Find the mean and variance of X.

sol: $\sum_{x=0}^{4} P(x) = 1 \Rightarrow C + 2C + 3C + C^2 + 5C^2 = 1$

$\Rightarrow 6C^2 + 5C - 1 = 0$

$\Rightarrow C = 1/6, -1$

not possible, because $P(x) \ge 0$

$\therefore C = 1/6$

$P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$= C + 2C + 2C = 5C = 5/6$

$P(0<X<4) = P(X=1) + P(X=2) + P(X=3)$

$= 2C + 2C + C^2 = 25/36$

probability distribution and distribution function are

x	0	1	2	3	4
$p(x)$	$1/6$	$2/6$	$2/6$	$1/36$	$5/36$
$F(x)$	$1/6$	$3/6$	$5/6$	$31/36$	1

Mean $\mu_x = \Sigma x_i p_i = 0 + \frac{2}{6} + \frac{4}{6} + \frac{3}{36} + \frac{20}{36} = \frac{59}{36}$

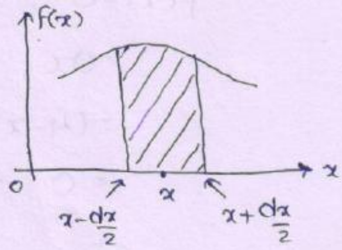
variance $\sigma_x^2 = E(x^2) - [E(x)]^2$

$$= \left[0\left(\tfrac{1}{6}\right) + 1\left(\tfrac{2}{6}\right) + 4\left(\tfrac{2}{6}\right) + 9\left(\tfrac{1}{36}\right) + 16\left(\tfrac{5}{36}\right)\right] - \left(\tfrac{59}{36}\right)^2 = 1.4529$$

<u>Continous</u> <u>Random</u> <u>variable</u>: A R.V X is said to be continous R.V if it takes all possible values in a given interval I_x.

<u>probability</u> <u>density</u> <u>function</u>: consider a small interval $(x - \frac{dx}{2}, x + \frac{dx}{2})$ of length dx about the point 'x'. Let $f(x)$ be a continous function so that $f(x)dx$ represents probability that X falls in this interval. That is

$$P\left(x - \frac{dx}{2} \le X \le x + \frac{dx}{2}\right) = f(x)\,dx$$



The function $f(x)dx$ represents area under the curve $f(x)$. The function $f(x)$ is called probability density function(pdf) of X.

properties: (i) $f(x) \ge 0$ for all $x \in I_x$

(ii) $\int_{I_x} f(x)\,dx = 1$

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Now $P(X = a) = P(a \le x \le a) = \int_a^a f(x)\,dx = 0$

$$\therefore P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$$

Mean $= \int_{I_x} x f(x)\,dx = \mu_x$

variance $= \sigma_x^2 = \int_{I_x} (x - \mu_x)^2 \cdot f(x)\,dx$

cumulative density function (cdf):

$$F(x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

For the interval $I_x = [a, b]$

$$\text{Mean} = \mu = \int_a^b x f(x)\,dx$$

$$\text{variance} = \sigma^2 = \int_a^b x^2 f(x)\,dx - (\mu)^2$$

Properties: (i) $0 \le F(x) \le 1$, $-\infty < x < \infty$

(ii) $F(x)$ is a non decreasing function i.e if $x_1 < x_2$ then $F(x_1) \le F(x_2)$

(iii) $F(\infty) = 1$ and $F(-\infty) = 0$

(iv) $F(x)$ may have countable number of discontinuties

(v) $f(x) = F'(x)$ at all points where $F(x)$ is differentiable.

prob: If the density function of a continous R.V x is given by

$f(x) = 0$, $x < 0$
$= ax$, $0 \le x \le 2$
$= (4-x)a$, $2 \le x \le 4$
$= 0$, $x > 4$

(i) Find a?
(ii) Find cdf of X
(iii) Find $P(X > 2.5)$

sol: (i) $\int_0^4 f(x)\,dx = 1$

$$\int_0^2 ax\,dx + \int_2^4 a(4-x)\,dx = 1 \Rightarrow 2a + 2a = 1 \Rightarrow a = 1/4$$

(ii) $x < 0$: $F(x) = 0$

$0 \le x \le 2$: $F(x) = \int_0^x \frac{x}{4}\,dx = \frac{x^2}{8}$

$2 \le x \le 4$: $F(x) = \int_0^2 \frac{x}{4}\,dx + \int_2^x \frac{1}{4}(4-x)\,dx = \frac{1}{8}(8x - x^2 - 8)$

$x > 4$: $F(x) = 1$

(iii) $P(X > 2.5) = \int_{2.5}^4 \frac{1}{4}(4-x)\,dx$

$$= \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_{2.5}^4 = 9/32$$

prob: If the probability density function of a continous R.V is given by $f(x) = e^{x}$, $0 \le x < \infty$. Find mean and variance?

sol: Mean $= \mu = \int_0^\infty x \cdot e^{-x} dx = -[(x+1)e^{-x}]_0^\infty = 1$

Variance $= \int_0^\infty x^2 e^{-x} dx - (\mu)^2$

$= (-x^2 e^{-x})_0^\infty + 2\int_0^\infty x e^{-x} dx - 1 = 2 - 1 = 1$

properties of Expectation:

1. If X is a R.V and 'a' is a constant, then $E(ax) = aE(x)$
2. If X & Y are R.V's then $E(X+Y) = E(X) + E(Y)$

$E(X-Y) = E(X) - E(Y)$

$E(X.Y) = E(X) \cdot E(Y/X) = E(Y) \cdot E(X/Y)$

3. If X & Y are independent R.V's $E(X.Y) = E(X) \cdot E(Y)$
4. If $Y = aX + b$ where a and b are constants. Then $E(Y) = aE(X) + b$
5. $E(\text{constant}) = \text{constant}$

properties of variance:

1. If X is a R.V and 'a' is a constant

$V(aX) = a^2 V(X)$

$V(-Y) = (-1)^2 V(Y) = V(Y)$

2. If X and Y are independent R.V's $V(x \pm y) = V(x) + V(y)$
3. If a and b are constant and X and Y are independent R.V's

$V(aX - bY) = a^2 V(X) + b^2 V(Y)$

$V\left(\frac{X}{a} - \frac{Y}{b}\right) = \frac{1}{a^2} V(X) + \frac{1}{b^2} V(Y)$

4. If $Y = aX + b \Rightarrow V(Y) = V(aX + b)$

$= a^2 V(X) + 0$ $[\because V(\text{const}) = 0]$

$= a^2 V(X)$

5. If X & Y two R.V's (dependent R.V's)

$V(X+Y) = V(X) + V(Y) + 2\,\underbrace{COV(X,Y)}_{\text{Covariance of }(X,Y)}$

$COV(X,Y) = E(X.Y) - E(X).E(Y)$

$\rightarrow COV(X,X) = E(X^2) - (E(X))^2 = V(X)$

$\rightarrow COV(a,b) = \cancel{COV(ab) = COV}$

$= E(ab) - E(a).E(b) = ab - ab = 0$

Note: The mean and variance for the sum of the numbers on the dice is $E(x) = \frac{7n}{2}$ $V(x) = \frac{35}{12}n$ where 'n' is number of dice rolled.

prob: 3 unbiased dice are thrown. Find the mean and variance for the sum of the numbers on them?

$$E(x) = \frac{7n}{2} = \frac{7\times3}{2} = \frac{21}{2}$$

$$V(x) = \frac{35}{12}n = \frac{35\times3}{12} = 35/4$$

prob: Two unbiased dice are rolled. Find the expectation for sum 7 on them?

sol: x is assuming only 7. $\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$$E(\text{Sum }7) = 7.P(7) = 7\times\frac{6}{36} = 7/6$$

prob: A player tosses 3 coins. He wins 500 rupees if '3' heads occur. 300 rupees if 2 heads occur, 100 rupees if only 1 head occur. On the otherhand he looses 1500 rs if 3 tails occur. Find value of the game?

sol: x: No. of heads possibility

x	3	2	1	0
P(x)	1/8	3/8	3/8	1/8

VALUE OF GAME = GAIN - LOSE = Gain × it's prob - Lose × it's prob

$$E(x) = 500\times\frac{1}{8} + 300\times 3/8 + 100\times 3/8 - 1500\times 1/8$$

$$= \frac{200}{8} = 25$$

Note: If Game is said to be fair, the expected value of game is said to be 0. (No loss, No gain)

prob: If 'x' is a continous R.V, and $f(x) = Kx^2e^{-x}$, $0<x<\infty$

(i) Find value of K (ii) Mean and variance.

sol:

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} Kx^2e^{-x}dx = 1$$

Gamma function $\Gamma n = \int_0^{\infty} e^{-x}.x^{n-1}dx$; $n>0$

$$\Gamma n = (n-1)\Gamma(n-1) = (n-1)!$$ $[\Gamma 1 = 1, \Gamma 0 = \text{Not defined}]$

$K\Gamma 3 = 1 \Rightarrow K \cdot 2! = 1 \Rightarrow K = 1/2$

(ii) Mean $E(x) = \int_0^\infty x \cdot f(x)\,dx$

$= \frac{1}{2}\int_0^\infty x \cdot x^2 e^{-x}\,dx = \frac{1}{2}\int_0^\infty x^3 e^{-x}\,dx$

$= \frac{1}{2}\Gamma(4) = \frac{1}{2}\times 3! = 3.$

variance $E(x^2)-[E(x)]^2 = \frac{1}{2}\int_0^\infty x^2 \cdot x^2 e^{-x}\,dx - (3)^2$

$= \frac{1}{2}\int_0^\infty x^4 e^{-x}\,dx - 9$

$= \frac{1}{2}\Gamma 5 - 9 = \frac{1}{2}\times 4! - 9 = 3$

Prob: If $f(x) = |x| \quad -1 < x < 1$. Find variance?

Sol: $E(x) = \int_{-1}^{1} x\, f(x)\,dx = \int_{-1}^{1} x\,|x|\,dx$

odd fun $\times$ ~~odd~~ even fun = odd fun

$= -\int_{-1}^{0} x^2\,dx + \int_0^1 x^2\,dx = 0$

$\int_{-a}^{a}$ odd fun $dx = 0$

$E(x^2) = \int_{-1}^{1} x^2 f(x)\,dx = \int_{-1}^{1} x^2 |x|\,dx$

even $\times$ even = even

$\int_{-a}^{a}$ even $= 2\int_0^a$ even

$= 2\times\int_0^1 x^3\,dx$

$= 2\times\left(\frac{x^4}{4}\right)_0^1 = 2\times 1/4 = 1/2$

variance $= E(x^2) - [E(x)]^2 = 1/2$

2-Dimensional Random variable:

Joint probability function or mass function of (x, y):

If (x, y) be a 2D-R.V then

$P(x = x_i, y = y_i) = P(x = x_i \cap y = y_i) = P(x_i, y_j) = P_{ij}$

P_{ij} is called joint probability function of (x, y)

(i) $P_{ij} \geq 0$, for all i, j

(ii) $\sum_i \sum_j P_{ij} = 1$

Marginal probababability function:

Marginal prob. function of x, $P(x = x_i) = \sum_{j=1}^{n} P(x_i, y_j)$

Marginal prob. function of y; $P(y = y_j) = \sum_{i=1}^{m} P(x_i, y_j)$

conditional probability function:

conditional probability function of x given $y = y_j$ is

$$P(x = x_i / y = y_j) = \frac{P(x = x_i \cap y = y_j)}{P(y = y_j)} = \frac{P_{ij}}{P(y = y_j)}$$

conditional probability function of y given $x = x_i$ is

$$P(y = y_j / x = x_i) = \frac{P[y = y_j \cap x = x_i]}{P(x = x_i)} = \frac{P_{ij}}{P(x = x_i)}$$

Two R.V'S x and y are said to be independent, if

$$P(x = x_i,\ y = y_j) = P(x = x_i) \cdot P(y = y_j)$$

Cumulative distribution function:

The CDF of (x, y) is defined by $F(x, y) = P(x \le x \text{ and } y \le y)$

$$= \sum_i \sum_j P_{ij},\ y_j \le y,\ x_i \le x.$$

properties:

(i) $F(-\infty, y) = 0$, $F(x, -\infty) = 0$, $F(-\infty, \infty) = 1$

(ii) $F(x, y)$ is a monotonic non decreasing function.

(iii) $P(a < x < b,\ y \le y) = F(b, y) - F(a, y)$

(iv) $P(x \le x,\ c < y < d) = F(x, d) - F(x, c)$

(v) At all points of continuity of $f(x, y)$, $\frac{\partial^2 F}{\partial x \cdot \partial y} = f(x, y)$

continous:

Joint probability density function:

If $D \in R$ then $P((x, y) \in D) = \iint_D f(x, y)\, dx\, dy$

$$P(a \le x \le b,\ c \le y \le d) = \int_c^d \int_a^b f(x, y)\, dx \cdot dy$$

properties: (i) $f(x, y) \ge 0$

(ii) $\iint_R f(x, y)\, dx\, dy = 1$

* cdf of (x, y) is $F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\, dx \cdot dy$

* Marginal density of x, $f_x(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

* Marginal density of y, $f_y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$

Marginal distribution functions are $F_X(x) = \int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx$

$$F_Y(y) = \int_{-\infty}^{y}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy$$

$$\therefore f_X(x) = \frac{dF_X(x)}{dx} \qquad f_Y(y) = \frac{dF_Y(y)}{dy}$$

$\Rightarrow P(a \le x \le b) = \int_a^b f_X(x)\,dx$

$P(c \le y \le d) = \int_c^d f_Y(y)\,dy$.

$\Rightarrow$ conditional density of x given y = $f(x/y) = \dfrac{f(x,y)}{f_Y(y)}$

conditional density of y given x = $f(y/x) = \dfrac{f(x,y)}{f_X(x)}$

$\Rightarrow$ Two continous random variables are independent if $f(x,y) = f_X(x)f_Y(y)$.

Prob: The Joint pdf of the R.V x and y is given by $f(x,y) = K(xy+y^2)$ $0 \le x \le 1$, $0 \le y \le 2$. Find (i) $P(>1)$ (ii) $P(x>1/2, y<1)$ (iii) $P(x+y \le 1)$

Sol: (i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = 1$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2} K(xy+y^2)\,dy\,dx = 1$$

$$\Rightarrow K\int_{x=0}^{1}\left[x\cdot\left(\frac{y^2}{2}\right)_0^2 + \left(\frac{y^3}{3}\right)_0^2\right]dx = 1 \qquad \Rightarrow K = 3/11$$

$$P(y>1) = \frac{3}{11}\int_{x=0}^{1}\int_{y=1}^{2}(xy+y^2)\,dy\,dx = \frac{111}{132}$$

(ii) $P(x>1/2, y<1) = \dfrac{3}{11}\displaystyle\int_{x=1/2}^{1}\int_{y=0}^{1}(xy+y^2)\,dy\,dx = \frac{17}{176}$

(iii) $P(x+y \le 1) = \dfrac{3}{11}\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x}(xy+y^2)\,dy\,dx$

$$= \frac{3}{11}\int_{x=0}^{1}\left[x\left[\frac{y^2}{2}\right]_0^{1-x} + \left[\frac{y^3}{3}\right]_0^{1-x}\right]dx = \frac{3}{88}$$

Functions of Random variables:

Let x be a R.V defined on the sample space S and let g be a function suchthat $y = g(x)$ is also a R.V defined on S. Then pdf of y is given by $f_Y(y) = \left|\dfrac{dx}{dy}\right| f_X(x)$

Prob: A continous R.V x has pdf $f(x) = x/2 \quad 0<x<2$
$= 0 \quad$ elsewhere.

Find the pdf of $Y = 3x+2$

Sol: $Y = 3x+2 \Rightarrow x = \frac{y-2}{3}$ $\quad 0<x<2 \Rightarrow 2<y<8$

$$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x)$$

$$f_Y(y) = \frac{1}{3}\cdot\frac{x}{2} = \frac{1}{18}(y-2) \qquad 2<y<8$$

↳ function of y

Transformation of 2-D R.V's:

U, V defined by transformations $u = u(x,y)$ and $v = v(x,y)$ then joint pdf $g_{UV}(u,v)$ of transformed variables U and V is given by

$$g_{UV}(u,v) = |J|\, f_{XY}(x,y)$$

where $|J| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$ = Jacobian

prob: The pdf of R.V's (X,Y) is given by $f(x,y) = \frac{1}{4}e^{-\left(\frac{x+y}{2}\right)}$, $x>0$ $y>0$
$= 0$ elsewhere

Find the distribution of $(X-Y)/4$.

Sol: transformations $u = \frac{1}{4}(x-y)$, $v = y$ $\qquad y>0 \Rightarrow v>0$

$\Rightarrow x = 4u+v$, $y = v$ $\qquad x>0 \Rightarrow 4u+v>0$

$v>-4u$ if $u<0$

$v>0$ if $u>0$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} = 4$$

$$g_U(u,v) = |J|\, f_{XY}(x,y)$$

$$= \frac{4}{4}\cdot e^{-(4u+2v)/2} = e^{-(2u+v)}, \quad -\infty<u<\infty$$

for $u<0$: $f_U(u) = \int_{-4u}^{\infty} e^{-(2u+v)}\,dv = e^{-2u}(e^{4u}) = e^{2u}$

for $v>0$: $f_U(u) = \int_0^{\infty} e^{-(2u+v)}\,dv = e^{-2u}$

$\therefore f_U(u) = e^{-2|u|} \quad -\infty<u<\infty$

*** **PROBABILITY DISTRIBUTIONS:**

Binomial or bernoullies distribution: A R.V X whose probability distribution is given by the formula $P(X=r) = nC_r\, p^r q^{n-r}$ then X is called binomial R.V. Here n denotes no. of times experiment is conducted.

p = probability of success

q = probability of failure $= 1-p$

For Binomial distribution

Mean $E(x) = np$ $\qquad E[x^2] = n(n-1)p^2 + np$

$var(x) = np(p-1) = npq$

central moment (μ_k): $\quad \mu_k = E\{x - E(x)\}^k$

For the binomial distribution $B(n,p)$, we have $x=r$, $E(x) = np$, $q = 1-p$

$$\mu_k = \sum_{r=0}^{n} (r-np)^k \, {}^nC_r \, p^r q^{n-r}$$

$$\mu_{k+1} = pq\left[\frac{d\mu_k}{dp} + nk\,\mu_{k-1}\right]$$

$\rightarrow \mu_0 = 1,\ \mu_1 = 0 \Rightarrow \mu_2 = npq$ = variance

$\mu_3 = npq(q-p)$ = 3^{rd} central moment

Skewness: $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ Skewness defined interms of 'μ_3'

$\Gamma = \sqrt{\beta_1}$ Γ is also a measure of skewness.

Note: If $\mu_3 = 0 \Rightarrow \beta_1 = 0$ Then curve is symmetry

If $\mu_3 = -ve \Rightarrow$ ~~β~~ Then curve is negatively skewed.

If $\mu_3 = +ve \Rightarrow$ Then curve is positively skewed.

* Let x be a binomial variate. Then the moment generating function (MGF) of x is defined by

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^{n} e^{tr}\,{}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} {}^nC_r\,(pe^t)^r q^{n-r}$$

$$= (q + pe^t)^n$$

Ex: When a coin is tossed 3 times. x denotes the R.V no. of heads turned up. Then here $p = q = 1/2$, $n=3$ and

$P(x=0) = 3C_0 (1/2)^0 (1/2)^3 = 1/8$

$P(x=1) = 3C_1 (1/2)^1 (1/2)^2 = 3/8$ ---

x	0	1	2	3
$p(x)$	1/8	3/8	3/8	1/8

Mean $= \Sigma x_i p_i = \frac{3+6+3}{8} = 3/2 = np$

* $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(1-2p)^2}{npq}$ $\qquad \Gamma_1 = \frac{1-2p}{\sqrt{npq}}$

If $p = 1/2 \longrightarrow$ Symmetry

$p < 1/2 \longrightarrow$ positively skewed $\qquad p > 1/2 \rightarrow$ Negatively skewed.

prob: For a binomial distribution, mean is 10 & variance is 5. How many times the experiment is conducted?

sol: $\frac{npq}{np} = \frac{5}{10} \Rightarrow q = 1/2 \Rightarrow p = 1/2$

$\Rightarrow n = 20$

prob: Find the probability of getting 9 exactly 2 in 3 times with a pair of dice.

sol: $n = 3, \quad x = 2$ $\quad \{(5,4),(4,5),(6,3),(3,6)\}$

$P(\text{sum}=9) = \frac{4}{36} = 1/9 \qquad q = 1 - 1/9 = 8/9$

required prob. $= nC_x\, p^x q^{n-x}$

$= 3C_2 (1/9)^2 (8/9)^1 = \frac{8}{243}$

prob: The probability of man hitting the target is 1/3.

(i) If he fires 5 times, what is the probability of his hitting the target atleast twice

(ii) How many times must he fire so that probability of his hitting the target atleast once is more than 90%.

sol: $n = 5, \; p = 1/3, \; q = 2/3$

(i) $P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(x=1) + P(x=0)]$

$= 1 - \left[(2/3)^5 + 5C_1 (1/3)(2/3)^4\right] = \frac{131}{243}$

(ii) $P(x \geq 1) > 90\% \Rightarrow 1 - P(x=0) > 0.9$

$\Rightarrow P(x=0) < 0.1$

$\Rightarrow q^n < 0.1 \Rightarrow (2/3)^n < 0.1 \Rightarrow n =$

prob: 2 dice are rolled 120 times. Find the average no. of times in which the number on the first die exceeds no. on 2nd.

sol: $n = 120 \qquad p = 15/36$

$A = B \rightarrow 6$ ways
$A > B \rightarrow 15$ "
$A < B \rightarrow 15$ "

Average = Mean = $np = 120 \times \frac{15}{36} = 50$

prob: If x is a R.V and $E(x) = 4$, $V(x) = 4/3$

Find ~~V(x) = 4/3~~ (i) $P(x \leq 2)$ (ii) Comment on β_1

sol: $E(x) = np = 4 \qquad V(x) = npq = 4/3$

$\frac{npq}{np} = \frac{4/3}{4} \Rightarrow q = 1/3$, $P = 2/3$.

$n = 4 \times 3/2 = 6$

$P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$

$= \left(\frac{1}{3}\right)^6 + 6C_1\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + 6C_2\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 = \frac{73}{729}$

(ii) Since $P = \frac{2}{3} > 1/2$ Negatively skewed.

<u>poisson's</u> <u>distribution</u>: A R.V 'x' whose probability distribution is given by $P(x=r) = \frac{e^{-\lambda}.\lambda^r}{r!}$ Then x is called poisson R.V and the distribution is called poisson distribution.

$$P(x; \lambda > 0) = P(r) = \begin{cases} \frac{e^{-\lambda}.\lambda^r}{r!}, & \lambda > 0, \ 0 \le r < \infty \\ 0 & \text{otherwise} \end{cases}$$

* Mean = variance = $\lambda = np$
* It is used when observations are HIGH and SUCCESS probability is low. i.e $n \to \infty$ and $P \to 0$
* $E(x) = \text{mean} = \lambda$

  $V(x) = \mu_2 = \lambda$

  $\mu_3 = \lambda$

  $\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$

  Since $\lambda > 0$, poisson distribution is always <u>positively skewed</u>.
* MGF $M_x(t) = E[e^{t\lambda}] = e^{\lambda(e^t - 1)}$

<u>Prob</u>: A product is supposed to contain 2% defective items. What is the probability in a sample of 50 items to contain (i) Exactly 2 (ii) less than 2 defective items.

<u>sol</u>: using Bi-nomial $dist^n$

$P = 2/100 = 1/50$ $n = 50$

$P(x=2) = 50C_2(1/50)^2(49/50)^{48}$

Not possible

using poisson's $dist^n$

$\lambda = np = 50 \times 1/50 = 1$

$P(x=2) = \frac{e^{-1}.1^2}{2!} = 1/2e$

$P(x<2) = 1/e + 1/e = 2/e$

<u>prob</u>: A telephone switch board receives 20 calls on an average during an hour. Find the probability for a period of 5 mins (i) NO calls (ii) Exactly 3 (iii) Atleast 2 calls received.

sol: For 60 min $\longrightarrow$ 20 calls (avg)

5 min $\longrightarrow \frac{20}{60} \times 5 = 1.65$ calls (avg) $= \lambda$

(i) $P(x=0) = \dfrac{e^{-1.65}(1.65)}{0!} = e^{-1.65}$

(ii) $P(x=3) = \dfrac{e^{-1.65}(1.65)^3}{3!}$

(iii) $P(X \geq 2) = 1 - P(X<2)$

$= 1 - [P(X=0) + P(X=1)] = \left[1 - \left[e^{-1.65} + \dfrac{1.65e^{-1.65}}{1!}\right]\right]$

prob: For a Poisson distribution with R.V as x, $P(x=1) = P(x=2)$

Then $P(x=4) =$

sol:

$e^{-\lambda}.\lambda^1 = \dfrac{e^{-\lambda}.\lambda^2}{2} \Rightarrow \lambda = 2$

$P(x=4) = \dfrac{e^{-2}.2^4}{4!} = \dfrac{4}{6e^2}$

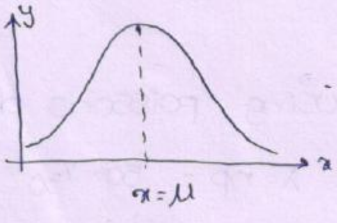
Normal distribution (Gaussian distribution):

A continuous R.V 'x' whose probability density function is given by

$$N(x;\mu,\sigma^2) = f(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < x < \infty,\ -\infty < \mu < \infty,\ 0 < \sigma < \infty \\ 0, & \text{Otherwise} \end{cases}$$

The mean of normal distribution is μ & variance is σ^2 and also $\int_{-\infty}^{\infty} f(x)dx = 1$.

By taking values of x & $f(x)$ in xy plane we can get the graph of normal dist^n as below.



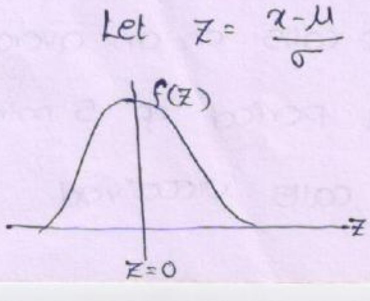
To calculate $P(a<x<b)$ for the normal dist^n we have to calculate

$$\int_a^b f(x)\,dx = \int_a^b e^{-\alpha x^2} dx$$

Which cannot be evaluated analytically.

Standard Normal distribution:

Let $Z = \dfrac{x-\mu}{\sigma} \Rightarrow f(Z) = \dfrac{1}{\sqrt{2\pi}} e^{-Z^2/2} \quad -3 \leq Z \leq 3$



Mean $= 0$, variance $= 1$

prob: The avg. height of a student in a class is 175cm with a standard deviation of 10cm. A student is selected at random from that class. What is the prob. for him to have height b/w (i) B/w 170 to 180cm (ii) more than 180.. Given $P(0<Z<0.5) = 0.1915$

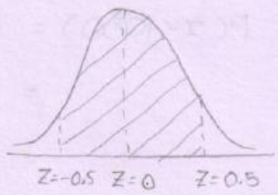
Sol: $\mu = 175\text{cm}$ $\sigma = 10\text{cm}$

(i) $P(170 < x < 180)$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-175}{10}$$

At $x = 170 \Rightarrow Z = -5/10 = -0.5$

At $x = 180 \Rightarrow Z = 5/10 = 0.5$

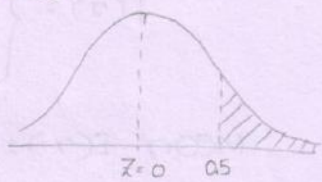


$P(170<x<180) = P(-0.5<Z<0.5)$

$= 2\times P(0<Z<0.5) = 2\times 0.1915 = 0.383$

(ii) $P(x>180) = P(Z>0.5)$

$= \frac{1}{2} - P(0<Z<0.5) = \frac{1}{2} - 0.1915 = 0.3185$

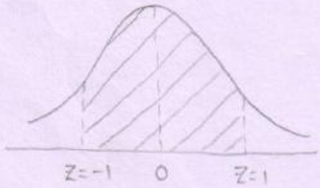


prob: The resistance of resistors is observed to be normal with mean resistance of 100Ω, SD of 2Ω, what % of resistors will have resistance (i) b/w 98Ω to 102Ω (ii) more than 102Ω. given $P(-\infty<Z<1) = 0.8413$

Sol: $\mu = 100$ $\sigma = 2$

(i) $P(98<x<102)$

$$Z = \frac{x-100}{2} \qquad \begin{array}{l} x=98 \Rightarrow Z=-1 \\ x=102 \Rightarrow Z=1 \end{array}$$



$P(98<x<102) = P(-1<Z<1) = 2\times P(0<Z<1)$

$= 2\times[0.8413-0.5] = 0.6826 = 68.26\%$

(ii) $P(x>102) = P(Z>1)$

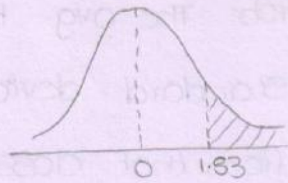
$= 1-0.8413 = 0.1587 = 15.87\%$

prob: The avg. life of an electric ~~field~~ bulb of a company is 2040 hrs with a S.D of 60 hrs. A bulb of this company is purchased. what is the prob. that the bulb is likely to burn for (i) More than 2150 hrs (ii) Less than 1950 hrs. Given $P(0<Z<1.83) = 0.4664$ $P(0<Z<1.5) < 0.4332$.

sol: $\mu = 2040 \quad \sigma = 60$

(i) $P(x > 2150)$

$$z = \frac{x - 2040}{60} \qquad x = 2150 \Rightarrow z = \frac{110}{60} = {}^{11}/_{6} = 1.83$$

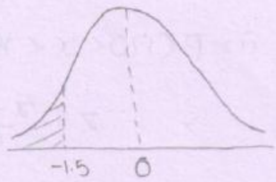


$P(x > 2150) = P(z > 1.83)$

$= 0.5 - P(0 < z < 1.83) = 0.5 - 0.4664 = 0.0336$

(ii) $P(x < 1950)$

$$z = \frac{1950 - 2040}{60} = -1.5$$



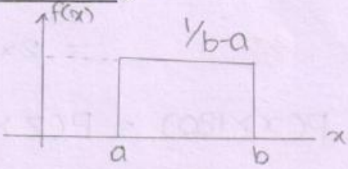
$P(x < 1950) = P(z < -1.5)$

$= 0.5 - P(-1.5 < z < 0)$

$= 0.5 - P(0 < z < 1.5) = 0.5 - 0.4332 = 0.0668$

<u>uniform</u> <u>distribution</u> (<u>Rectangular</u> <u>distribution</u>):

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ & a < b \\ 0 & \text{otherwise} \end{cases}$$



Mean $E(x) = \frac{a+b}{2}$

variance $V(x) = \frac{(b-a)^2}{12}$

* If x is uniform R.V in the interval $-a < x < a$ and it's density function is

$$f(x) = \frac{1}{2a} \qquad \text{Mean} = 0$$
$$\text{variance} = a^2/3$$

STATISTICS: collection of data, analysis of data & interpretation of data.

Types of data: (i) Grouped and ungrouped

(ii) Closed and open.

Grouped data: If data is in the form of class intervals and frequency then the data is known as grouped data or distributing the frequencies to their corresponding class intervals, then the data is known as frequency distribution.

ungrouped data: If data contains only observations, without any class intervals then the data is known as ungrouped data. or raw data.

closed data: If the class intervals are in a continous form without any discontinuity, then the data is known as closed data otherwise open data.

Mean (Average):

$$\bar{x}_{UGD} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\bar{x}_{GD} = \frac{\sum_{i=1}^{n} f_i x_i}{N}$$

n: no. of observations

N: Sum of frequencies

x: mid point, $\frac{UL+LL}{2}$

F: Frequencies

Median:

1. If 'n' is odd, the middle observation is the median
2. If 'n' is even, average between middle observations

provided that

(i) Data is rearranged either in increasing / decreasing order

(ii) No. of observations above the middle is equal to the number of observations below.

$$Md = l + \frac{\left(\frac{N}{2} - m\right) \times c}{f}$$

l = lower limit for Ideal class

f = frequency of Ideal class

c = class length

m = cumulative frequency

Mode: The most frequently repeated observation is known as mode.

For grouped data $M_0 = l + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) C$ $\quad \Delta_1 = f - f_1$, $\Delta_2 = f - f_2$

Relation: Mode = 3×median − 2×mode

prob: Raw data 2, 4, 2, 3, 5, 2, 6, 5, 2, 7

sol: $\text{Mean} = \frac{\text{Total Sum of observations}}{\text{No. of observations}} = \frac{38}{10} = 3.8$

Ascending order 2, 2, 2, 2, 3, 4, 5, 5, 6, 7

$\text{median} = \frac{3+4}{2} = \frac{7}{2} = 3.5$

Mode = Most repeated value = 2

prob: Grouped data:

C.I	0-10	10-20	20-30	30-40	40-50	50-60
frequency	3	4	12	24	13	4

sol:

C.I	freq f_i	Mid x_i	cum.freq
0-10	3	5	3
10-20	4	15	7
20-30	12 f_1	25	19 m
30-40	24 f	35	43
40-50	13 f_2	45	56
50-60	4	55	60 N
	60		

Ideal class

l

x_i = Mid value of class

$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{2020}{60} = 33.66$

$\text{Median} = l + \frac{\left(\frac{N}{2} - m\right)}{f} \times C$

$= 30 + \frac{(30-19)}{24} \times 10 = 34.58$

$\text{Mode} = l + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times C$ $\quad \Delta_1 = f - f_1 = 12$, $\Delta_2 = f - f_2 = 11$

$= 30 + \frac{12}{12+11} \times 10 = 35.21$.

Range: Maximum − minimum

Greatest value − Least value.

Standard deviation: $\sqrt{\text{variance}} = (S.D)$

$\text{variance} = (S.D)^2 = \sigma_x^2$

$$\sigma_x^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - (\bar{x})^2$$

* variance is the sum of squares of deviation from mean.

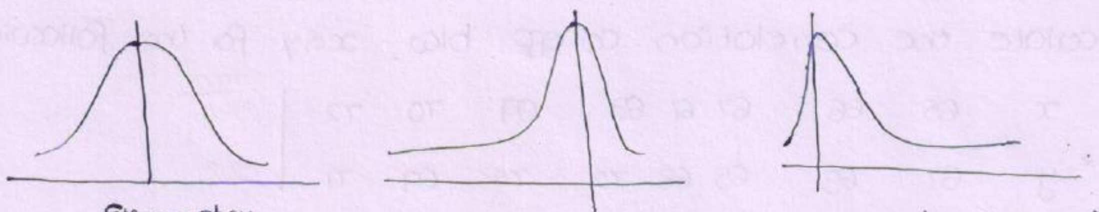
For grouped data

$$\sigma_x^2 = \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i\right)^2$$

$$\sigma_x^2 = \frac{1}{N}\sum f_i (x_i - \bar{x})^2$$

co.efficient of variation $C.V = \frac{\text{standard deviation}}{\text{Mean}} \times 100$

Skewness: "Lack of symmetry."



Symmetry   Negatively skewed   positively skewed

pearson's co.efficient of skewness:

$$S_{KP} = \frac{\text{Mean} - \text{Mode}}{\sigma} = \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

practical limit of SKP $-3 \le S_{KP} \le 3$

For symmetry $S_{KP} = 0$

Symmetry	Negative skewness	positive skewness
Mode = Median = Mean	Mode > Median > Mean	Mode < Median < Mean

CORRELATION & REGRESSION:

The degree of relation b/w the two variables is known as correlation.

positive correlation: If the changes in the both the variables are in the same direction (increasing or decreasing) Then those variables are known as positively correlated variables.

Negative correlation: If the changes in the one variable is affecting the changes of other variables in reverse direction, then those variables are known as negatively correlated.

correlation co.efficient b/w x, y is given by $r = \frac{\mathrm{cov}(x,y)}{\sigma_x \cdot \sigma_y}$

$$r = \frac{\frac{1}{n}\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2}\sqrt{\frac{1}{n}\sum (y_i - \bar{y})^2}} = \frac{\sum XY}{\sqrt{\sum X^2}\sqrt{\sum Y^2}}$$

$X = x_i - \bar{x}$

$Y = y_i - \bar{y}$

* $-1 \le r \le 1$
* If $0 < r \le 1$ then x & y are positively correlated
* If $-1 \le r < 0$ then x & y are negatively correlated
* If $r = 0$ then there is no correlation b/w x & y
* If x & y are independent R.V's then co.variance = 0 i.e $\text{cov}(x,y) = 0$

$\Rightarrow r(x,y) = 0$. But converse is not true.

prob: calculate the correlation co.eff b/w x & y for the following data.

									Total
x	65	66	67	67	68	69	70	72	544
y	67	68	65	68	72	72	69	71	552

sol: $\bar{x} = \frac{544}{8} = 68$ $\bar{y} = \frac{552}{8} = 69$

									Σ
$X = x - \bar{x}$	-3	-2	-1	-1	0	1	2	4	
$Y = y - \bar{y}$	-2	-1	-4	-1	3	3	0	2	
XY	6	2	4	1	0	3	0	8	24
X^2	9	4	1	1	0	1	4	16	36
Y^2	9	1	16	1	9	9	0	4	44

$$r = \frac{24}{\sqrt{36}\sqrt{44}} = 0.6$$ $\therefore$ x & y are positively correlated.

<u>Regression</u>: The linear relationship b/w 2D random variable is known as regression.

Lines of regression: y on x $(y - \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$

x on y $(x - \bar{x}) = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$

Regression co.efficient y on x, $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

Regression co.efficient x on y, $b_{xy} = r\frac{\sigma_x}{\sigma_y}$

$$b_{xy} \cdot b_{yx} = r^2$$

Correlation co.eff. $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$ (Geometric mean)

* Both regression co.eff. must have same sign.

i.e if both +ve $\Rightarrow$ r is +ve

if both -ve $\Rightarrow$ r is -ve

* Two regression lines are passing through the point $(\bar{x}, \bar{y})$

* If $b_{xy} = b_{xy} \Rightarrow r\frac{\sigma_x}{\sigma_y} = r\frac{\sigma_y}{\sigma_x} \Rightarrow \sigma_y^2 = \sigma_x^2$

$\therefore$ variances are also equal.

* Angle b/w regression lines $\theta = \tan^{-1}\left(\frac{1-r^2}{|r|} \cdot \frac{\sigma_x \cdot \sigma_y}{(\sigma_x)^2 + (\sigma_y)^2} \right)$

If $r = 0 \Rightarrow \theta = \pi/2 \Rightarrow \perp'$or

$r = 1 \Rightarrow \theta = 0$ or $\pi \Rightarrow$ coinside (perfect correlation)

Prob: The regression equation are $3x + 2y = 1$; $2x + 4y = 0$

(i) Find r? (ii) $\bar{x}, \bar{y}$

Sol: which ever the co.eff (magnitude) in the expression is higher then it is the DEPENDENT VARIABLE.

x on y

$3x + 2y = 1$

$x = \frac{1}{3} - \frac{2}{3}y$

$b_{xy} = -2/3$

y on x

$2x + 4y = 0$

$y = -\frac{2}{4}x$

$b_{yx} = -1/2$

$r = \sqrt{b_{xy} \cdot b_{yx}}$

$= \sqrt{-2/3 \times -1/2}$

$= -1/\sqrt{3}$ (both -ve)

(ii) Means $\bar{x}, \bar{y}$ also ~~pass through~~ on the lines.

$$\begin{array}{r} 6\bar{x} + 4\bar{y} = 2 \\ 2\bar{x} \pm 4\bar{y} = 0 \\ \hline 4\bar{x} = 2 \end{array} \Rightarrow \bar{x} = 1/2 \qquad \bar{y} = -1/4$$

Note: $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$$= \frac{\text{cov}(x,y)}{\sigma_x \cdot \sigma_y} \cdot \frac{\sigma_y}{\sigma_x} = \frac{\text{cov}(x,y)}{\sigma_x^2} = \frac{\text{cov}(x,y)}{\text{var}(x)}$$

similarly $b_{xy} = \frac{\text{cov}(x,y)}{\text{var}(y)}$

NUMERICAL METHODS

NUMERICAL METHODS

To find solutions to

1. Algebraic and Transcendental equations
2. System of linear equations
3. Integration of a function
4. Differential equations.

① Solution of Algebraic and transcendental equations:

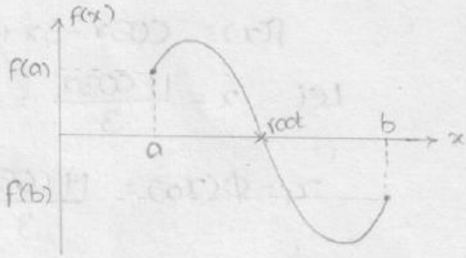
Algebraic equations: polynomials of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$$

Transcendental equations: combination of polynomial, exponential and trigonometric functions.

Ex: $\tan x = x$, $xe^x - 1 = 0$, $\cos x - xe^x = 0$

Intermediate value theorem: If $f(x)$ is a continous function defined on $[a,b]$ and $f(a)$, $f(b)$ are having opposite signs, then there exist atleast one root of $f(x)$ in $[a,b]$



Ex: $f(x) = x^3 + x - 1$

$f(0) = -1$ (-ve)

$f(1) = 1+1-1$ (+ve)

$\therefore$ one root in the interval $[0,1]$

Iterative Method (Successive approximation):

1. Express $f(x) = 0$ as $x = \phi(x)$
2. choose an initial approximation x_0 to the root in the given interval
3. Find the next approximations using $x_1 = \phi(x_0)$
   $x_2 = \phi(x_1)$
   $x_3 = \phi(x_2)$ ---
4. Stop the iteration if two consecutive iterations gives the same approximation.

** The sequence of approximations converges to the root if $|\phi'(x)| < 1$

Ex: $x^3+x-1=0 : [a,b]=[0,1]$

$x=1-x^3$	$x=\sqrt[3]{1-x}$	$x(x^2+1)=1$
$\lvert\phi'(x)\rvert=\lvert 1-3x^2\rvert$	$\lvert\phi'(x)\rvert=\left\lvert\frac{1}{3}(1-x)^{-2/3}(-1)\right\rvert$	$x=\frac{1}{x^2+1}$
$=3x^2$	$\lvert\phi'(x)\rvert=\frac{1/3}{(1-x)^{2/3}}$	$\lvert\phi'(x)\rvert=\frac{-1}{(1+x^2)^2}\cdot 2x$
$\lvert\phi'(0)\rvert=0<1$	$\lvert\phi'(0)\rvert=1/3<1$	$\lvert\phi'(0)\rvert=0<1$
$\lvert\phi'(1)\rvert=3>1$	$\lvert\phi'(1)\rvert=\infty$	$\lvert\phi'(1)\rvert=1/2<1$
×	×	✓

$\therefore$ Let $\phi(x)=\frac{1}{1+x^2}$ and Let $x_0=0$

$x_1=\phi(x_0)=\frac{1}{1+0^2}=1$

using calci $\frac{1}{1+(Ans)^2}$ press "="

$x_2=\phi(x_1)=\frac{1}{1+1^2}=0.5$

$x_3=\phi(x_2)=\frac{1}{1+(0.5)^2}=0.8$

$x_4=\phi(x_3)=\frac{1}{1+(0.8)^2}=0.6097$

$\vdots$

$x_{18}=0.6822$

$x_{19}=0.6824$

$x_{20}=0.6822$

$x_{21}=0.6824$

$x_{22}=0.6822$

Ex: $\cos x=3x-1$

$f(x)=\cos x-3x+1 \quad : [0,1]$

Let $x=\frac{1+\cos x}{3}=\phi(x)$ & Let $x_0=0$

$x_1=\phi(x_0)=\frac{1+\cos 0}{3}=0.6666$

using calci $\frac{1+\cos(Ans)}{3}$ press "="

$\vdots$

$x_5=0.6071, \quad x_6=0.6071$

Bi-Section Method: Suppose that a root of $f(x)=0$ lies in the interval $I_0=(a_0,b_0)$ i.e $f(a_0)\times f(b_0)<0$ (opposite signs)

1. Bi-sect the interval to obtain $c_1=\frac{a_0+b_0}{2}$
2. Root lies in the interval $I_1=(a_0,c_1)$ if $f(a_0)\times f(c_1)<0$
   Root lies in the interval $I_2=(c_1,b_0)$ otherwise.
3. Repeat above procedure for 'n' times, finally the root is given by Midpoint of the interval

Ex: $x^3+x-1 \quad : [0,1]$

$c_1=\frac{0+1}{2}=0.5$

$f(c_1)=(0.5)^3+(0.5)-1=-0.37$ (-ve)

$f(a_0)\times f(c_1)=(-1)\times(-0.37)=0.37>0$

$\therefore$ New Interval $=(c_1,b_0)$
$=(0.5,1)$

(2)

$C_2 = \frac{0.5+1}{2} = 0.75$

$f(0.5) \times f(0.75) = -0.064 < 0$ $\therefore$ New interval $= [0.5, 0.75]$

$\vdots$

$C_{12} = 0.68225$ $\therefore$ New interval $= [0.6822, 0.6825]$

Regula falsi Method / Method of false position:

1. Assume two initial approximations x_0, x_1 such that $f(x_0) \times f(x_1) < 0$
2. Find next approximation using $x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}$
3. Root lies in the interval (x_0, x_2) if $f(x_0) \times f(x_2) < 1$
   Root lies in the interval (x_2, x_1) otherwise.
4. Continue the process untill desired accuracy is achieved.

If $f(x)$ in the interval $[a, b]$, for simplicity select $x_0 = a$, $x_1 = b$

then $x_{n+1} = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

Ex: $x^3 + x - 1 = 0 : [0 \ 1]$

$x_0 = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = 0.5$ $x_1 = \frac{0.5 f(1) - 1 f(0.5)}{f(1) - f(0.5)} = 0.6363$

$f(x_0) = -0.37$ (-ve) $\vdots$

New interval $= [0.5, 1]$

Another way: Fix one end of interval (say b) $f(b) > 0$ and apply iteration formula to find new value for other end of the interval. After desired iterations, other end of interval itself gives the root.

Secant method / Chord Method:

1. It is same as regula falsi method. The only difference is, while selecting initial approximation, we don't consider the condition $f(x_0) \times f(x_1) < 0$
2. So, this method may or may not converge
3. If it converge, it converges faster than regula falsi method.

****

Newton raphson method: The iterative formula in Newton raphson method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

→ Intially assume x_0

→ Newton raphson method fails if $f'(x) = 0$

→ If $f'(x)$ is very large then root can be find very rapidly

(99 CS) → condition for the convergence of newton raphson method is $|f(x). f''(x)| < |f'(x)|^2$

Ex: $x^3 + x - 1 = 0$: $[0, 1]$

sol: $f(x) = x^3 + x - 1 \Rightarrow f'(x) = 3x^2 + 1$

Let $x_0 = 0.5$

$$x_1 = 0.5 - \frac{(0.5)^3 + (0.5) - 1}{3(0.5)^2 + 1} = 0.714$$

$x_2 = 0.6831$ $\quad x_3 = 0.6823$ $\quad x_4 = 0.6823$

(95 CS)

prob: Find iterative formula for $\sqrt{N}$ using Newton Raphson method?

sol $x = \sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0$

$$f(x_n) = x_n^2 - N \Rightarrow f'(x_n) = 2x_n$$

From N.R Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n} = \frac{1}{2}\left[\frac{N}{x_n} + x_n\right]$$

(96-CS)

prob: Iterative formula for $\sqrt[3]{N}$

$$x = \sqrt[3]{N} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$$

$$f(x_n) = x_n^3 - N \Rightarrow f'(x_n) = 3x_n^2$$

From N.R Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

(3)

(GATE-95): Let $f(x) = x - \cos x$. using NR method find x_{n+1} using x_n

Sol: $f(x) = x - \cos x \Rightarrow f'(x) = 1 + \sin x$

$$x_{n+1} = x_n - \frac{[x_n - \cos x_n]}{1 + \sin x_n}$$

(GATE-97 CS): N.R method is used to find the root of the equation $x^2 - 2 = 0$. If the iterations are started from -1, then the iteration will

a) converges to -1 b) converges to $\sqrt{2}$ ✓ c) converges to $-\sqrt{2}$ d) not converge

Sol: from N.R method $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n}$

Iteration started from $-1 \Rightarrow x_0 = -1$

$\therefore x_1 = \dfrac{x_0^2 + 2}{2x_0} = -1.5$ $\qquad x_3 = \dfrac{x_2^2 + 2}{2x_2} = -1.4141$

$x_2 = \dfrac{x_1^2 + 2}{2x_1} = -1.4166$ $\qquad x_4 = \dfrac{x_3^2 + 2}{2x_3} = -1.4141$

$\therefore$ Iterations converge to $-\sqrt{2}$

(GATE-05 CE): Given $a > 0$, we wish to calculate its reciprocal value $\frac{1}{a}$ by using NR method for $f(x) = 0$. For $a = 7$ and starting with $x_0 = 0.2$ the first two iterations will be

Sol: $x = \frac{1}{a} \Rightarrow x - \frac{1}{a} = 0$ (or) $\frac{1}{x} - a = 0$

$f(x) = \frac{1}{x} - a \Rightarrow f'(x) = -\frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - a\right)}{\left(-\frac{1}{x_n^2}\right)} = 2x_n - ax_n^2$$

Given $a = 7$, $x_0 = 0.2$

$x_1 = 2(0.2) - 7(0.2)^2 = 0.12$

$x_2 = 2(0.12) - 7(0.12)^2 = 0.1392$

(GATE-05 ME): Starting from $x_0 = 1$, one step of NR method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value x_1 as

$f'(x) = 3x^2 + 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1) + 3(1) - 7}{3(1) + 3} = 1 + \frac{1}{2} = 1.5$$

(GATE-05 PI): The real root of the equation $xe^x = 2$ is evaluated using NR method. If the first approximation of the value of x is 0.8679, the 2nd approximation of x correct to 3 decimal places is

Sol: $f(x) = xe^x - 2 \Rightarrow f'(x) = xe^x + e^x$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{(0.8679)e^{0.8679} - 2}{(0.8679)e^{0.8679} + e^{0.8679}}$$

$$= 0.853$$

(GATE-07 CE): The following equation need to be numerically solved using NR method $x^3 + 4x - 9 = 0$. The iterative equation for this purpose is

Sol: $f(x) = x^3 + 4x - 9 \Rightarrow f'(x) = 3x^2 + 4$

$$x_{k+1} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2 + 4)}$$

$$= \frac{3x_k^3 + 4x_k - x_k^3 - 4x_k + 9}{3x_k^2 + 4} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$$

(GATE-07 EC): The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using NR method. If $x = 2$ taken as the initial approx. of the solution then the next approx. using this method will be

Sol: $f'(x) = 3x^2 - 2x + 4$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8-4+8+4}{12-4+4} = 4/3$$

(GATE-08 EE): Equation $e^x - 1 = 0$ is required to be solved using NR method with an initial guess $x_0 = -1$. Then after one step of NR method estimate x_1 of the solution will be given

Sol: $f(x) = e^x - 1 \Rightarrow f'(x) = e^x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{e^{-1} - 1}{e^{-1}} = 0.71828$$

(GATE-08 EC): The recursion relation to solve $x = e^{-x}$ using NR method is

Sol: $f(x) = x - e^{-x} \Rightarrow f'(x) = 1 + e^{-x}$

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{(1 + x_n)e^{-x_n}}{(1 + e^{-x_n})}$$

(GATE-10 CS): NR method is used to compute a root of the eq^n $x^2-13=0$ with 3.5 as the initial value. The approximation after one iteration is

Sol: $x^2-13=0 \Rightarrow x=\sqrt{13}$ $\quad x_0=3.5$

$$x_1=\frac{1}{2}\left[x_0+\frac{13}{x_0}\right]=\frac{1}{2}\left[3.5+\frac{13}{3.5}\right]=3.607$$

(GATE-11 EC): A numerical solution of the equation $f(x)=x+\sqrt{x}-3=0$ can be obtained using NR method. If the starting value is $x=2$ for the iteration then the value of x that is to be used in next step is

$x_0=2$

Sol: $f(x)=x+\sqrt{x}-3 \Rightarrow f'(x)=1+\frac{1}{2\sqrt{x}}$

$$x_1=2-\frac{2+\sqrt{2}-3}{1+1/2\sqrt{2}}=1.6939$$

(GATE-13 EE): When the NR method is applied to solve the equation $f(x)=x^3+2x-1=0$, the solution at the end of 1st iteration with the initial value as $x_0=1.2$ is

Sol: $f'(x)=3x^2+2$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1.2-\frac{(1.2)^3+2(1.2)-1}{3(1.2)^2+2}=0.705$$

(GATE-14 EE): The function $f(x)=e^x-1$ is to be solved using NR method. If the initial value of x_0 is taken 1.0, then the absolute error observed at 2nd iteration is

Sol: $f'(x)=e^x$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1-\frac{(e-1)}{e}=1/e$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=\frac{1}{e}-\frac{(e^{1/e}-1)}{e^{1/e}}=0.06$$

(GATE-14 ME): The real root of the equation $5x-2\cos x=0$ (upto two decimal places) is Ans: 0.54

Sol: $f'(x)=5+2\sin x$

Let $x_0=1$

By NR method $x_1=1-\frac{5-2\cos 1}{5+2\sin 1}=0.5631$

$x_2=0.5426$

$x_3=0.5426$

(GATE-14 PI): If the equation $\sin x = x^2$ is solved by Newton Raphson's method with the initial guess of $x=1$, then the value of x after 2 iterations would be

Sol: $f(x) = x^2 - \sin x \Rightarrow f'(x) = 2x - \cos x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \left(\frac{1-\sin 1}{2-\cos 1}\right) = 0.8915$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \simeq 0.73$$

(GATE-14 CS): In NR method, an initial guess of $x_0 = 2$ is made the sequence $x_0, x_1, x_2 \ldots$ is obtained for the function $0.75x^3 - 2x^2 - 2x + 4 = 0$. Consider the statements

(I) $x_3 = 0$ (II) The method converge to solution in a finite number of iterations. Which of the following is true?

a) only I (b) only II (c) Both I and II (d) Neither I nor II

Sol: $f(x) = 0.75x^3 - 2x^2 - 2x + 4$

$f'(x) = 2.25x^2 - 4x - 2$ $\quad x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2, \quad x_3 = 0, \quad x_4 = 2, \quad x_5 = 0 \ldots$$

$\therefore$ Iterations will not converge.

System of Non linear equations: NR method can be extend to find the roots of system of non linear equations. consider $f(x,y) = 0$, $g(x,y) = 0$.

The approximation of roots $x_{k+1} = x_k + \Delta x$ $\quad y_{k+1} = y_k + \Delta y$ $\quad k = 0,1,2\ldots$

$$\text{where } \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -J_k^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}$$

J is called Jacobian matrix $\quad J_k = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x_k, y_k)}$

→ A necessary and sufficient condition for convergence is $\rho(J_k^{-1}) < 1$. where ρ denotes the spectral radius (largest eigen value in magnitude.

Method	Order convergence	
Bi section	Linear convergence	1
Regula falsi	Linear convergence	1
Secant method	Super linear convergence	1.62
Newton Raphson	Quadratic convergence	2

Note: Let us consider nth degree polynomial $f(x)=0$

(i) No. of positive real roots of $f(x) \le$ No. of sign changes in $f(x)=0$

(ii) No. of negative real roots of $f(x) \le$ No. of sign changes in $f(-x)=0$

(iii) No. of complex roots $= n-$ (no. of +ve + no. of -ve roots)

(GATE-07 IN): The polynomial $P(x)=x^5+x+2$ has

(i) all real roots (ii) 3 real & 2 complex (iii) 1 real & 4 complex (iv) all complex

Sol: $P(x)=0$ doesn't have any sign changes. No. of positive roots $=0$

$P(-x)=-x^5-x+2$. one sign change. No. of negative roots $=1$

$\therefore$ complex roots $=5-1=4$

Note: If $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ are n roots of equations $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_{n-1}x+a_n=0$

(i) $\sum_{i=1}^{n} \alpha_i = \dfrac{-a_1}{a_0}$ (ii) $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \dots \alpha_n = (-1)^n \dfrac{a_n}{a_0}$

(GATE-08 IN): It is known that two roots of the non-linear equation $x^3-6x^2+11x-6=0$ are 1 and 3. The third root will be

Sol: $\alpha_1+\alpha_2+\alpha_3 = \dfrac{-a_1}{a_0} = 6$ (or) $\alpha_1\alpha_2\alpha_3 = (-1)^3 \cdot \dfrac{-6}{1}$

$4+\alpha_3 = 6$ $\qquad$ $3 \cdot \alpha_3 = 6$

$\alpha_3 = 2$ $\qquad$ $\alpha_3 = 6/3 = 2$

② Solution to system of linear equations:

(i) Gauss elimination method: discussed in "Matrix algebra" topic.

(ii) LU decomposition (Method of factorization)

or

DO-little method:

Let us consider system of linear eq's in matrix form $AX=B$

Decompose A into $A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore$ The system of equations $LUX = B$

Let $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solve to find y_1, y_2, y_3.

from $UX = Y \Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Solve to find x_1, x_2, x_3.

CROUT'S method: The Lower and upper triangular matrix in Crout's method are

$$A = LU$$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

3. Solution to Integration of function:

The area bounded by curve $f(x)$ is denoted by $\int_a^b f(x)\,dx$. Divide $[a, b]$ into "n" equal sub intervals ~~very~~ where length of each interval is "h" (step size)

$a = x_0$

$x_1 = x_0 + h$

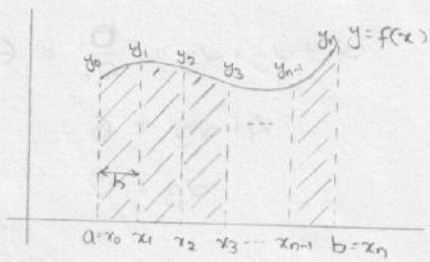
$x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 3h$

$\vdots$

$x_n = x_0 + nh \Rightarrow b = a + nh$

$$\Rightarrow h = \frac{b-a}{n}$$



(i) Trapezoidal rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

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(ii) Simpson's $\frac{1}{3}$ rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots+y_{n-2})+4(y_1+y_3+y_5+\cdots+y_{n-1})\right]$$

(iii) Simpson $\frac{3}{8}$th rule:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+\cdots+y_{n-3})+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})\right]$$

* In trapezoidal rule error is of order h^2
* In Simpson's $\frac{1}{3}$rd rule error is of order h^4
* In Simpson's $\frac{3}{8}$th rule error is of order h^5
* Simpson's $\frac{1}{3}$rd rule is applicable if number of intervals are EVEN
* Simpson's $\frac{3}{8}$th rule is applicable if number of intervals are multiples of 3.
* Trapezoidal rule is applicable for any number of intervals.
* Simpson's rule for integration gives exact results when $f(x)$ is a polynomial of degree '2'.
* Trapezoidal rule for integration gives exact results when $f(x)$ is a polynomial of degree ≤ 1 (0 or 1)

(GATE-07 ME): $\int_0^{2\pi} \sin x\,dx$ is evaluated by T.R rule with eight equal intervals. (with 5 significant digits)

Sol: $n = 8 \qquad h = 2\pi/8 = \pi/4$

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$y = \sin x$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0

T.R rule $\int_0^{2\pi} \sin x \cdot dx = \frac{\pi/4}{2}\left[(0+0)+2\left(\frac{1}{\sqrt{2}}+1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}-1-\frac{1}{\sqrt{2}}\right)\right]$

$= 0.00000$

(GATE-10 ME):

x	0	60	120	180	240	300	360
y	0	1068	-323	0	323	-355	0

Evaluate $\int_0^{2\pi} y\,dx$ using Simpson's rule.

a) 542 b) 995 c) 1444 d) 1986

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x values are in degrees. and limits of integration are interms of π $\Rightarrow$ $h = 60^\circ = \pi/3$

By Simson's 1/3 rule $\int_0^{2\pi} y\,dx = \frac{h}{3}\left[(y_0+y_6)+2(y_3)+4(y_1+y_2+y_4+y_5)\right]$

$= 995$

(GATE-11 ME): The integral $\int_1^3 \frac{1}{x}\,dx$ when evaluated using 1/3rd rule on two equal intervals each of length 1.

Sol: $h = \frac{b-a}{n} = \frac{3-1}{2} = 1$

x	1	2	3
$y = 1/x$	1	1/2	1/3
	y_0	y_1	y_2

By Simpson's 1/3rd rule

$$\int_1^3 \frac{1}{x}\,dx = \frac{1}{3}\left[(1+1/3)+2(0)+4(1/2)\right] = 1.111$$

Note: Error in Trapezoidal rule $= -\left(\frac{b-a}{12}\right)h^2 \max[f''(x)]$

Error in Simpson's $\frac{1}{3}$ rule $= -\frac{(b-a)}{180}\cdot h^4 \max[f^{IV}(x)]$

Error in Simpson's $\frac{3}{8}$ rule $= -\frac{3}{80}h^5 \max[f^{IV}(x)]$

Note: If Error tolerance (ϵ) is prescribed then the no. of sub-intervals required to achieve that accuracy are given by

In trapezoidal rule : $N^2 \geq \frac{(b-a)^3}{12\epsilon}\cdot \max[f''(x)]$

In Simpson's 1/3 rule : $N^4 \geq \frac{(b-a)^5}{2880\epsilon} \max[f^{IV}(x)]$

Prob: The minimum number of equal length subintervals needed to approximate $\int_1^2 xe^x\,dx$ to an accuracy of atleast $\frac{1}{3}\times10^{-6}$ using Trapezoidal rule.

✓(a) 1000.e (b) 1000 (c) 100e (d) 100

Sol: $f(x) = xe^x \Rightarrow f'(x) = e^x + xe^x$

$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$

e^x is increasing and x is increasing in [1,2]

$\therefore \max[f''(x)] = (2e^x + xe^x)\Big|_{x=2} = 4e^2$

Accuracy $\geq \frac{1}{3}\times10^{-6} \Rightarrow \epsilon \leq \frac{1}{3}\times10^{-6}$

$\therefore N^2 \geq \frac{(b-a)^3}{12\epsilon}\cdot\max[f''(x)] \Rightarrow N^2 \geq \frac{(1)^3}{12\times\frac{1}{3}\times10^{-6}}\times 4e^2 \Rightarrow N \geq 1000e$

Solution of ordinary differential equations:

Single step method: If we use only the values $y_n, f(x_n, y_n)$ to obtain the solution value y_{n+1} at the next nodal point $x = x_{n+1}$ then the method is called explicit single step method.

$$y_{n+1} = y_n + \phi(x_n, y_n, h)$$

If y_{n+1} and $f(x_{n+1}, y_{n+1})$ are also used, then the method is called implicit S.S method.

$$y_{n+1} = y_n + \phi(x_n, x_{n+1}, y_n, y_{n+1}, h)$$

1. Taylor series method:

$$y(x) = y(x_n) + (x - x_n) y'(x_n) + \cdots + \frac{(x - x_n)^P}{P!} y^{(P)}(x_n) + \cdots$$

Substitute $x = x_{n+1} \Rightarrow x_{n+1} - x_n = h$

$$y(x_{n+1}) = y(x_n) + h \cdot y'(x_n) + \cdots + \frac{h^P}{P!} y^{(P)}(x_n) + \cdots$$

2. Euler's method (RK first order method): If $dy/dx = f(x, y)$ & $y(x_0) = y_0$ then

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$
$$y_2 = y_1 + h \cdot f(x_1, y_1)$$
$$\vdots$$

3. Euler's backward method:

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

The non linear equation for y_{n+1} can be solved by NR method.

4. Modified Euler method / Euler - cauchy method / Heun's method / RK 2nd order method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$
$$= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + h f_n)\right]$$

where $f_n = f(x_n, y_n)$

Let $K_1 = h f(x_n, y_n)$

$K_2 = h f(x_{n+1}, y_n + K_1)$

$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$

also denoted as

$y_1^P = y_0 + h f(x_0, y_0)$

$y_1^C = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^P)\right]$

also called predictor - corrector formula or Improved Euler cauchy formula.

5. Runge's method (RK method of order 3):

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$$

$$K' = hf(x_0 + h,\ y_0 + K_1)$$

$$K_3 = hf(x_0 + h,\ y_0 + K')$$

$$K = \frac{1}{6}(K_1 + 4K_2 + K_3)$$

$$y_1 = y_0 + K$$

6. Runge-Kutta method (RK method of order 4):

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$$

$$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3)$$

$$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$y_1 = y_0 + \Delta y$$

(GATE-93): Given the differential equation $y' = x - y$ with initial condition $y(0) = 0$. The value of $y(0.1)$ calculated numerically upto the third place of decimal by the 2nd order RK method with step size $h = 0.1$ is

Sol: Given $y' = x - y \Rightarrow \frac{dy}{dx} = y' = f(x, y)$

Given $y(0) = 0 \Rightarrow x_0 = 0,\ y_0 = 0$ $\quad h = 0.1$

From RK 2nd order method

$$K_1 = h\,f(x_0, y_0) \qquad x_1 = x_0 + h$$

$$= 0.1\,[x_0 - y_0] = 0 \qquad = 0 + 0.1 = 0.1$$

$$K_2 = hf(x_1,\ y_0 + K_1)$$

$$= (0.1)f(0.1, 0) = (0.1)[0.1 - 0] = 0.01$$

$$y(0.1) = y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$= 0 + \frac{1}{2}(0 + 0.01) = \frac{1}{200} = 0.005$$

(GATE-10 EC): Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with initial condition $y(0)=0$. Using Euler's first order method with a step size of 0.1 then the value of $y(0.3)$ is

Sol: $f(x,y) = \frac{dy}{dx} = y+x \qquad h = 0.1$

$y(0) = 0 \Rightarrow x_0 = 0, \; y_0 = 0$

$\therefore x_1 = x_0 + h = 0.1 \qquad x_2 = x_0 + 2h = 0.2 \qquad x_3 = x_0 + 3h = 0.3$

$y_1 = y_0 + h\,f(x_0, y_0) = y_0 + h[x_0 + y_0] = 0$

$y_2 = y_1 + h\,f(x_1, y_1) = 0 + 0.1[0.1 + 0] = 0.01$

$y_3 = y_2 + h\,f(x_2, y_2) = 0.01 + 0.1[0.2 + 0.01] = 0.031$

$\therefore y(0.3) = 0.031$

(GATE-13 IN): While numerically solving the D.E $\frac{dy}{dx} + 2xy^2 = 0$, $y(0)=1$ using Euler's predictor corrector (improved Euler cauchy) method with step size of 0.2, the value of y after the first step is

Sol: $\frac{dy}{dx} = -2xy^2 = f(x,y) \qquad f(0,1) = 0$

$y_1^P = y_0 + h\,f(x_0, y_0) = 1 + 0.2(0) = 1$

$y_1^C = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^P)\right] = 1 + \frac{0.2}{2}(0 - 0.4) = 0.96$

(GATE-14 ME): Consider an O.D.E $\frac{dx}{dt} = 4t + 4$. If $x = x_0$ at $t=0$, the increment in x calculated using RK 4^{th} order multi-step method with a step size of $\Delta t = 0.2$ is

Sol: $\frac{dx}{dt} = 4t + 4 = f(t,x) \qquad$ at $t=0 \quad x = x_0 = 0$

$h = 0.2$

By RK fourth order method

$k_1 = h\,f(t_0, x_0) = 0.2 \times 4 = 0.8 \qquad k_2 = h\,f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right) = 0.88$

$k_3 = h\,f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right) = 0.88 \qquad k_4 = h\,f(t_0 + h, x_0 + k_3) = 0.96$

$k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] = 0.88$

$x_1 = x_0 + k = 0 + 0.88 = 0.88$

Multistep methods: [NO question in GATE till now]

1. Adams- Bashforth methods
2. Adams- moultons methods
3. Milne- Simsons methods

CURVE FITTING:

1. $y = ax + b$

Normal equations are $\Sigma y = a\Sigma x + nb \rightarrow (1)$

$\Sigma xy = a\Sigma x^2 + b\Sigma x \rightarrow (2)$

solving 1 & 2 we get a, b

2. $y = ax^2 + bx + c$

Normal equations are $\Sigma y = a\Sigma x^2 + b\Sigma x + c$

$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$

$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$

3. $y = a x^b \Rightarrow \log y = \log a + b \log x$

$Y = A + bX$ where $X = \log x$, $A = \log a$, $Y = \log y$

Normal equations are $\left.\begin{array}{l} \Sigma Y = nA + b\Sigma X \\ \Sigma XY = A\Sigma X + b\Sigma X^2 \end{array}\right\}$ Solving gives A, b

$a = \text{antilog}(A) = 10^A$

4. $y = a \cdot b^x \Rightarrow \log y = \log a + x \log b$

$Y = A + xB$ where $Y = \log y$, $A = \log a$, $B = \log b$

Normal equations are $\left.\begin{array}{l} \Sigma Y = B\Sigma x + nA \\ \Sigma xY = B\Sigma x^2 + A\Sigma x \end{array}\right\}$ Solving gives A & B

$a = 10^A$; $b = 10^B$

(GATE-08): Three values of x and y are to be fitted in a straight line in the form $y = a + bx$ by the method of least squares. Given $\Sigma x = 6$, $\Sigma y = 21$, $\Sigma x^2 = 14$, $\Sigma xy = 46$, the values of a & b are respectively.

Sol: $n = 3$ $\quad \Sigma y = na + b\Sigma x \Rightarrow 21 = 3a + 6b$

$\Sigma xy = a\Sigma x + b\Sigma x^2 \Rightarrow 46 = 6a + 14b$

Solving

$a = 3, b = 2$