GATE 2017

ENGINEERING MATHEMATICS

Handwritten Notes

Common for

AE/AG/BT/CE/CH/EC/EE/IN/ME/MN/MT/PE/PI

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This book is intended to provide basic knowledge on Engineering Mathematics to the GATE aspirants. Even though the syllabus is same, the questions appearing in different papers follows different patterns. So the GATE aspirants are advised to go through their respective paper syllabus (available in official GATE website) and previous question papers to understand the depth of the subject required to prepare for GATE exam. Most of the solved problems in this material are the questions appeared in EC/EE/ME/CE/IN/PI papers. So the remaining branches students are advised to solve the previous problems from their respective papers. EE branch students are advised to prepare *transform theory,* which is not available in this book.

This book is still under preparation. If you find any mistakes in this, please inform me through an e-mail. Don't share this book without proper citation and credits.

Thank You

All The Best

SYLLABUS – GATE 2016

1. MECHANICAL ENGINEERING – ME
2. METALLURGICAL ENGINEERING - MT
3. PRODUCTION AND INDUSTRIAL ENGINEERING - PI

Linear Algebra: Matrix algebra, systems of linear equations, eigenvalues and eigenvectors.

Calculus: Functions of single variable, limit, continuity and differentiability, mean value theorems, indeterminate forms; evaluation of definite and improper integrals; double and triple integrals; partial derivatives, total derivative, Taylor series (in one and two variables), maxima and minima, Fourier series; gradient, divergence and curl, vector identities, directional derivatives, line, surface and volume integrals, applications of Gauss, Stokes and Green's theorems.

Differential equations: First order equations (linear and nonlinear); higher order linear differential equations with constant coefficients; Euler-Cauchy equation; initial and boundary value problems; Laplace transforms; solutions of heat, wave and Laplace's equations.

Complex variables: Analytic functions; Cauchy-Riemann equations; Cauchy's integral theorem and integral formula; Taylor and Laurent series. **(Except for MT paper)**

Probability and Statistics: Definitions of probability, sampling theorems, conditional probability; mean, median, mode and standard deviation; random variables, binomial, Poisson and normal distributions.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations; integration by trapezoidal and Simpson's rules; single and multi-step methods for differential equations.

CIVIL ENGINEERING - CE

Linear Algebra: Matrix algebra; Systems of linear equations; Eigen values and Eigen vectors.

Calculus: Functions of single variable; Limit, continuity and differentiability; Mean value theorems, local maxima and minima, Taylor and Maclaurin series; Evaluation of definite and indefinite integrals, application of definite integral to obtain area and volume; Partial derivatives; Total derivative; Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Ordinary Differential Equation (ODE): First order (linear and non-linear) equations; higher order linear equations with constant coefficients; Euler-Cauchy equations; Laplace transform and its application in solving linear ODEs; initial and boundary value problems.

Partial Differential Equation (PDE): Fourier series; separation of variables; solutions of onedimensional diffusion equation; first and second order one-dimensional wave equation and two-dimensional Laplace equation.

Probability and Statistics: Definitions of probability and sampling theorems; Conditional probability; Discrete Random variables: Poisson and Binomial distributions; Continuous random variables: normal and exponential distributions; Descriptive statistics - Mean, median, mode and standard deviation; Hypothesis testing.

Numerical Methods: Accuracy and precision; error analysis. Numerical solutions of linear and non-linear algebraic equations; Least square approximation, Newton's and Lagrange polynomials, numerical differentiation, Integration by trapezoidal and Simpson's rule, single and multi-step methods for first order differential equations.

ELECTRONICS AND COMMUNICATION ENGINEERING – EC

Linear Algebra: Vector space, basis, linear dependence and independence, matrix algebra, eigen values and eigen vectors, rank, solution of linear equations – existence and uniqueness.

Calculus: Mean value theorems, theorems of integral calculus, evaluation of definite and improper integrals, partial derivatives, maxima and minima, multiple integrals, line, surface and volume integrals, Taylor series.

Differential equations: First order equations (linear and nonlinear), higher order linear differential equations, Cauchy's and Euler's equations, methods of solution using variation of parameters, complementary function and particular integral, partial differential equations, variable separable method, initial and boundary value problems.

Vector Analysis: Vectors in plane and space, vector operations, gradient, divergence and curl, Gauss's, Green's and Stoke's theorems.

Complex Analysis: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula; Taylor's and Laurent's series, residue theorem.

Probability and Statistics: Mean, median, mode and standard deviation; combinatorial probability, probability distribution functions - binomial, Poisson, exponential and normal; Joint and conditional probability; Correlation and regression analysis.

Numerical Methods: Solutions of non-linear algebraic equations, single and multi-step methods for differential equations, convergence criteria.

ELECTRICAL ENGINEERING - EE

Linear Algebra: Matrix Algebra, Systems of linear equations, Eigenvalues, Eigenvectors.

Calculus: Mean value theorems, Theorems of integral calculus, Evaluation of definite and improper integrals, Partial Derivatives, Maxima and minima, Multiple integrals, Fourier series, Vector identities, Directional derivatives, Line integral, Surface integral, Volume integral, Stokes’s theorem, Gauss’s theorem, Green’s theorem.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Method of variation of parameters, Cauchy’s equation, Euler’s equation, Initial and boundary value problems, Partial Differential Equations, Method of separation of variables.

Complex variables: Analytic functions, Cauchy’s integral theorem, Cauchy’s integral formula, Taylor series, Laurent series, Residue theorem, Solution integrals.

Probability and Statistics: Sampling theorems, Conditional probability, Mean, Median, Mode, Standard Deviation, Random variables, Discrete and Continuous distributions, Poisson distribution, Normal distribution, Binomial distribution, Correlation analysis, Regression analysis.

Numerical Methods: Solutions of nonlinear algebraic equations, Single and Multi-step methods for differential equations.

Transform Theory: Fourier Transform, Laplace Transform, z-Transform.

INSTRUMENTATION ENGINEERING - IN

Linear Algebra: Matrix algebra, systems of linear equations, Eigen values and Eigen vectors.

Calculus: Mean value theorems, theorems of integral calculus, partial derivatives, maxima and minima, multiple integrals, Fourier series, vector identities, line, surface and volume integrals, Stokes, Gauss and Green's theorems.

Differential equations: First order equation (linear and nonlinear), higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy's and Euler's equations, initial and boundary value problems, solution of partial differential equations: variable separable method.

Analysis of complex variables: Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, residue theorem, solution of integrals.

Probability and Statistics: Sampling theorems, conditional probability, mean, median, mode and standard deviation, random variables, discrete and continuous distributions: normal, Poisson and binomial distributions.

Numerical Methods: Matrix inversion, solutions of non-linear algebraic equations, iterative methods for solving differential equations, numerical integration, regression and correlation analysis.

1. PETROLEUM ENGINEERING – PE
2. CHEMICAL ENGINEERING - CH

Linear Algebra: Matrix algebra, Systems of linear equations, Eigen values and eigenvectors.

Calculus: Functions of single variable, Limit, continuity and differentiability, Taylor series, Mean value theorems, Evaluation of definite and improper integrals, Partial derivatives, Total derivative, Maxima and minima, Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Cauchy's and Euler's equations, Initial and boundary value problems, Laplace transforms, Solutions of one dimensional heat and wave equations and Laplace equation.

Complex variables: Complex number, polar form of complex number, triangle inequality.

Probability and Statistics: Definitions of probability and sampling theorems, Conditional probability, Mean, median, mode and standard deviation, Random variables, Poisson, Normal and Binomial distributions, Linear regression analysis.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations. Integration by trapezoidal and Simpson's rule. Single and multi-step methods for numerical solution of differential equations.

1. AGRICULTURE ENGINEERING – AG
2. BIOTECHNOLOGY – BT
3. MINING ENGINEERING - MN

Linear Algebra: Matrices and determinants, systems of linear equations, Eigen values and eigen vectors.

Calculus: Limit, continuity and differentiability; partial derivatives; maxima and minima; sequences and series; tests for convergence; Fourier series, Taylor series.

Vector Calculus: Gradient; divergence and curl; line; surface and volume integrals; Stokes, Gauss and Green's theorems. **(Except for BT paper)**

Differential Equations: Linear and non-linear first order Ordinary Differential Equations (ODE); Higher order linear ODEs with constant coefficients; Cauchy's and Euler's equations; Laplace transforms; Partial Differential Equations - Laplace, heat and wave equations.

Probability and Statistics: Mean, median, mode and standard deviation; random variables; Poisson, normal and binomial distributions; correlation and regression analysis; tests of significance, analysis of variance (ANOVA).

Numerical Methods: Solutions of linear and non-linear algebraic equations; numerical integration - trapezoidal and Simpson's rule; numerical solutions of ODE.

AEROSPACE ENGINEERING - AE

Linear Algebra: Vector algebra, Matrix algebra, systems of linear equations, rank of a matrix, eigenvalues and eigenvectors.

Calculus: Functions of single variable, limits, continuity and differentiability, mean value theorem, chain rule, partial derivatives, maxima and minima, gradient, divergence and curl, directional derivatives. Integration, Line, surface and volume integrals. Theorems of Stokes, Gauss and Green.

Differential Equations: First order linear and nonlinear differential equations, higher order linear ODEs with constant coefficients. Partial differential equations and separation of variables methods.

LINEAR ALGEBRA

LINEAR ALGEBRA

Matrix: A matrix is a rectangular array of numbers or functions arranged in 'm' rows and 'n' columns such that each row has same no. of elements (n) and each column has same no. of elements (m).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

a_{ij} denotes element in i^{th} row and j^{th} column.

Row matrix: Matrix having single row. Ex: $[2\ 4\ 6\ 8]$

Column matrix: Matrix having single column. Ex: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Square matrix: Matrix having equal no. of rows and columns.

Ex: $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$

* The elements along diagnol of a square matrix (a_{ij}, $i=j$) are called leading or principle diagonal elements.

* The sum of diagonal elements of a square matrix A is called Trace of A.

Diagonal matrix: A square matrix, except the leading diagonal elements are equal to zero is called diagonal matrix.

Ex: $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Scalar matrix: A diagonal matrix, whose leading diagonal elements are equal is called Scalar matrix

Ex: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Unit matrix: A diagonal matrix, whose leading diagonal elements all equal to '1' is called unit matrix.

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

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Null matrix: All the elements of matrix are zero Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Symmetic matrix: A square matrix with $a_{ij} = a_{ji}$ for all i and j.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Skew symmetric matrix: A square matrix with $a_{ij} = -a_{ji}$ for all i & j

Ex: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

* For a skew symmetrix matrix the leading diagonal elements all are equal to zero.

Matrix transpose: Intechanging of elements in rows with the corresponding elements in the columns. resulting matrix is denoted by 'A^T' or 'A''

Ex: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{m\times n} \Rightarrow A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}_{n\times m}$

* If $A = A^T$ then it is symmetric * $(A^T)^T = A$
* If $A = -A^T$ then it is skew - symmetric. * $(AB)^T = B^T . A^T$
* Every given square matrix can be expressed as sum of symmetric & skew-symmetric matrices.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew Symmetric}}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$\frac{1}{2}(A + A^T) = \underbrace{\begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix}}_{\text{Symmetric}} \qquad \frac{1}{2}(A - A^T) = \underbrace{\begin{bmatrix} 0 & -1/2 \\ +1/2 & 0 \end{bmatrix}}_{\text{Skew-Symmetric}}$$

Triangular matrix:

* A square matrix, whose elements below the leading diagonal are zero is called upper triangular matrix.

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Ex: $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ upper triangular.

* A square matrix whose elements above the leading diagonal are equal to zero is called lower triangular matrix.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ Lower triangular

Multiplication by scalar: $K\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} Ka_1 & Kb_1 & Kc_1 \\ Ka_2 & Kb_2 & Kc_2 \end{bmatrix}$

Multiplication of two matrices:

$$[A]_{m\times n} \cdot [B]_{n\times p} = [AB]_{m\times p}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix} = \begin{bmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 \end{bmatrix}$$

* Multiplication is possible only if no. of columns in first matrix is equal to no. of rows in second matrix.
* $AB \neq BA$.
* $(AB)^T = B^T \cdot A^T$

Determinant of matrix:

* For 1x1 matrix, the number itself is determinent.
* For 2x2 matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinent is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Minor: Minor of an element is the determinent obtained by deleting the row and column in which the element is present.

Ex: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow$ a_{11} minor is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

a_{23} minor is $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$

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<u>cofactor</u>: co.factor of any element 'a_{ij}' in a matrix is equal to $(-1)^{i+j} \cdot m_{ij}$. where 'm_{ij}' is minor of 'a_{ij}'.

Ex:
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then co.factor of } b_3 \text{ i.e } B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

<u>Determinant</u>: Determinant of a matrix is defined as sum of product of elements of any row or column with corresponding co.factor.

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1$$
$$= a_1 (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$
$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

Tips for GATE:

* while calculating determinant, always select a row or column with more number of elements equal to '0'

Ex:
$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 0 & 6 \end{vmatrix} = 4[6-15] = -36 \quad [\text{select } 2^{nd} \text{ row}]$$

* Try to remember co.factor signs $(-1)^{i+j}$ for each element as

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{3\times3} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}_{4\times4}$$

<u>Properties of determinants</u>:

1. Determinant remains unaltered by changing its rows into columns and columns into rows. $|A| = |A^T|$
2. If two parellel lines of determinant are interchanged, then the sign of determinant changes (same numerical value)
3. If two parellel lines are identical then det = 0
   If all the elements in a row or column are zeros then det = 0

* 4. If each element of a row or column is multiplied by same factor, then determinant also multiplied by same factor.

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$$\begin{vmatrix} a_1 & Kb_1 & c_1 \\ a_2 & Kb_2 & c_2 \\ a_3 & Kb_3 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$|KA| = K^n |A|$

5. The determinant of upper or lower triangular matrix is equal to product of leading diagonal elements.

Ex: $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1[18-0] = 18$
$= \underbrace{1 \times 3 \times 6}_{\text{diagonal elements.}}$

6. If the elements of determinant Δ are function of 'x' and if 'K' parellel lines becomes equal when $x=a$ then $(x-a)^{K-1}$ is a factor of Δ.

Ex: $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (a-b)(a-c)(b-c)$

by putting
$a=b, a=c, b=c$.

**
7. Product of determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 & a_1l_3+b_1m_3+c_1n_3 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 & a_2l_3+b_2m_3+c_2n_3 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 & a_3l_3+b_3m_3+c_3n_3 \end{vmatrix}$$

8. $|\text{co.factor matrix of } A| = |A|^{n-1}$ $n=$ order

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ co.factor matrix $= \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$|A| = 4-6 = -2$ $|\text{co.factor matrix}| = 4-6 = -2$

9. If $|A| = 0$, then A is called singular matrix.

10. If $|A| \neq 0$, then A is called non-singular matrix.

Adjoint matrix: Transpose of the co.factor matrix.

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ co.factor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$\text{Adj}[A] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

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* $A(\text{Adj } A) = (\det A) I$
* $\det(\text{Adj } A) = (\det A)^{n-1}$ $\quad n = \text{order}$

$$|\text{adj} A| = \left| |A| . A^{-1} \right|$$
$$= |A|^n |A^{-1}|$$
$$= |A|^n . |A|^{-1} = |A|^{n-1}$$

Inverse of square matrix: A matrix "B" is said to be inverse of a non-singular matrix "A" if $AB = BA = I$.

$$A^{-1} = \frac{\text{Adj}[A]}{|A|} \qquad A \cdot A^{-1} = I \qquad (\text{Adj}[A])^{-1} = \frac{A}{|A|}$$

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* $\text{adj}(\text{adj}[A]) = |A|^{n-2} . A$

we know

$A . \text{adj}[A] = |A| . I$

replace A by adj[A]

$\text{adj} A . [\text{adj}(\text{adj} A)] = |\text{Adj} A| . I$

$A^{-1} |A| [\text{adj}(\text{adj} A)] = |\text{Adj} A| . I$

$\text{adj}(\text{adj} A) = \frac{|A|^{n-1}}{|A|} . A$

$= |A|^{n-2} . A$

Properties:

* $(AB)^{-1} = B^{-1} . A^{-1}$
* If $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$
* If a non-singular matrix A is symmetric then A^{-1} is also symmetric.
* Every odd order symmetric matrix is singular i.e $|A| = 0$ $\Rightarrow A^{-1}$ does not exist for that.

Problems

1. consider matrix $X_{4\times3}$, $Y_{4\times3}$, $P_{2\times3}$ then order of $[(P.(X^TY)^{-1})P^T]^T$

$$\left[\left(P_{2\times3} \cdot (X^T_{4\times3} \cdot Y_{4\times3})^{-1}\right) . P^T_{2\times3}\right]^T = \left[\left(P_{2\times3} \cdot (X_{3\times4} \cdot Y_{4\times3})^{-1}\right) P_{3\times2}\right]^T$$
$$= \left[(P_{2\times3} \cdot (3\times3)^{-1} . P_{3\times2}\right]^T$$
$$= [P_{2\times3} . P_{3\times2}]^T = (2\times2)^T = (2\times2)$$

Tip for GATE:

Determinant calculation [for matrix with less no. of zero elements]

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, (Applicable for 3x3)

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$|A| = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

2. The following represents equation of straight line $\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$

The line passes through ?

a) (0, 0) b) (3, 4) ✓ c) (4, 3) d) (4, 4)

$$x(8) - 2(y) + 4(y-8) = 0$$

$$4x + 4 = 16 \quad \text{option b ✓}$$

3. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ then top row of A^{-1} is ?

$|A| = 1 \qquad A^{-1} = \dfrac{\text{Adj } A}{|A|} = \dfrac{(\text{co. factor})^T}{|A|}$

co. factors of 1st column $= \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

1st row of $A^{-1} = [5 \ \ -3 \ \ 1]$

4. If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 0.5 & a \\ 0 & b \end{bmatrix}$ then $a+b$?

$A \cdot A^{-1} = I \Rightarrow \begin{bmatrix} 1 & 2a - 0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$3b = 1 \Rightarrow b = 1/3 \quad \& \quad 2a = 0.1/3 \Rightarrow a = 1/60 \Rightarrow (a+b) = \frac{7}{20}$

5. If $A = (a_{ij})_{m \times n}$ such that $a_{ij} = i+j$, $\forall i, j$; then sum of the elements of A is

$$A = \begin{bmatrix} 1+1 & 1+2 & 1+3 & \cdots & 1+n \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ \vdots & \vdots & \vdots & & \vdots \\ m+1 & m+2 & m+3 & \cdots & m+n \end{bmatrix}_{m \times n}$$

$$\begin{matrix} \downarrow 1\times m & \downarrow 2\times m & \downarrow 3\times m & \cdots & \downarrow n\times m & \rightarrow n \text{ terms} \\ \frac{m(m+1)}{2} & \frac{m(m+1)}{2} & \frac{m(m+1)}{2} & \cdots & \frac{m(m+1)}{2} & \rightarrow n \text{ terms} \end{matrix}$$

$$\text{Sum of elements} = n \times \frac{m(m+1)}{2} + [m + 2m + 3m + \cdots + nm]$$

$$= \frac{mn(m+1)}{2} + m[1+2+3+\cdots+n]$$

$$= \frac{mn(m+1)}{2} + \frac{mn(n+1)}{2}$$

$$= \frac{mn}{2}(m+n+2)$$

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6. If $A=(a_{ij})_{3\times3}$, $B=(b_{ij})_{3\times3}$ such that $b_{ij}=2^{i+j}\cdot a_{ij}\ \forall\ i,j$; If $|A|=2$ then $|B|=?$ (a) 2^{10} (b) 2^{11} (c) 2^{12} ✓(d) 2^{13}

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}=2$$

$$|B|=\begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33}\end{vmatrix}=2^2\cdot2^3\cdot2^4\begin{vmatrix} a_{11} & 2a_{12} & 2^2a_{13} \\ a_{21} & 2a_{22} & 2^2a_{23} \\ a_{31} & 2a_{32} & 2^2a_{33}\end{vmatrix}$$

$$=2^9\cdot2\cdot2^2\cdot|A| = 2^{13}$$

7. If $A=(a_{ij})_{n\times n}$ such that $a_{ij}=i^2-j^2\ \forall\ i,j$. Then find sum of all the elements of A? $[a_{ij}=i^n-j^n]$

$$A=\begin{bmatrix} 0 & -3 & -8 & \cdots & (1^2-n^2) \\ 3 & 0 & -5 & \cdots & (2^2-n^2) \\ 8 & 5 & 0 & \cdots & (3^2-n^2) \\ \vdots & \vdots & \vdots & & \\ (n^2-1^2) & (n^2-2^2) & (n^2-3^2) & \cdots & 0\end{bmatrix}_{n\times n}$$

Above matrix is a skew symmetric matrix. It has all the diagonal elements equal to zero. When added up, non diagonal elements cancel out each other, resulting in final sum=0. [always]

8. The value of $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1\end{vmatrix}=?$

(or) Apply $R_2\to R_2-R_1$
$R_3\to R_3-R_1$
then find Δ with 3^{rd} column

$C_1\to C_1+C_2$ $$=\begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1\end{vmatrix}$$

$$=\underbrace{\begin{vmatrix} 1 & b & 1 \\ 1 & 1+b & 1 \\ 1 & 2b & 1\end{vmatrix}}_{\text{same lines}}+\begin{vmatrix} 2b & b & 1 \\ 2b & 1+b & 1 \\ 2b & 2b & 1\end{vmatrix}$$

$$=0+2b\underbrace{\begin{vmatrix} 1 & b & 1 \\ 1 & 1+b & 1 \\ 1 & 2b & 1\end{vmatrix}}_{\text{same lines}}$$

$=0$

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9. Find the determinant of $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ac \\ 1/c & c & ab \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} abc/a & a & bc \\ abc/b & b & ac \\ abc/c & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} bc & a & bc \\ ac & b & ac \\ ab & c & ab \end{vmatrix} = 0$$

← same lines

10. Find the value of $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

For similar type of problems

1. Apply $C_1 \to C_1 + C_2 + C_3 + C_4$
2. Apply $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$
3. Now find 'det' by taking 1st column.

$$C_1 \to C_1 + C_2 + C_3 + C_4 \quad = \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \quad = \begin{vmatrix} x+3a & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} \rightarrow \text{upper triangular matrix}$$

$$= (x+3a)(x-a)(x-a)(x-a) = (x+3a)(x-a)^3$$

11. $\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} = (x+10).x.x.x = (x+10).x^3$

12. $\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} = (1+a+b+c+d).1.1.1 = (1+a+b+c+d)$

13. If $A_{m\times n}$ and $B_{n\times p}$ are two matrices, then the no. of multiplications and additions in computing AB are ?

To find each element of AB, we require 'n' multiplications and $(n-1)$ additions.

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There are 'mp' elements in matrix $[AB]_{m\times p}$. Therfore

Total no. of multiplications required = mpn

Total no. of additions required = $mp(n-1)$

14. The no. of different $n\times n$ symmetric matrices with each element being either '0' or '1' is

For symmetric matrices, If once we fix the values of elements along & above the diagonal then no need to fix the " below the diagonal.

$$\begin{bmatrix} 1/0 & 1/0 & 1/0 & \cdots & 1/0 \\ & 1/0 & 1/0 & \cdots & 1/0 \\ & & 1/0 & \cdots & 1/0 \\ & & & \cdots & 1/0 \end{bmatrix}_{n\times n} \begin{array}{l} \rightarrow 2^n \text{ ways} \\ \rightarrow 2^{n-1} \text{ ways} \\ \rightarrow 2^{n-2} \text{ ways} \\ \rightarrow 2^1 \text{ ways} \end{array}$$

$\therefore$ Total no. of symmetric matrices $= 2^n \cdot 2^{n-1} \cdot 2^{n-2} \cdots 2^1$

$= 2^{1+2+3\cdots+n} = 2^{n(n+1)/2}$

15. If A, B, C, D be $n\times n$ matrices, each with non zero determinant. $ABCD = I$ then $B^{-1} = ?$

All are $n\times n$ matrices: $ABCD = I$

$A^{-1} \cdot ABCD = I \cdot A^{-1} \Rightarrow BCD = A^{-1}$

$B^{-1} \cdot BCD = B^{-1} A^{-1} \Rightarrow CD = B^{-1} A^{-1}$

$CDA = B^{-1} \cdot A^{-1} \cdot A \Rightarrow CDA = B^{-1}$

Shortcut:

CEB D GAF

D^{-1}

D^{-1} = GAFCEB

<u>Orthogonal</u> <u>matrix:</u> A matrix is said to be orthogonal if $A \cdot A^T = A^T \cdot A = I$ $\Rightarrow |A| = \pm 1$

$A \cdot A^T = I \Rightarrow A^{-1} \cdot A \cdot A^T = A^{-1} \cdot I$

$\Rightarrow I \cdot A^T = A^{-1} \cdot I \Rightarrow A^T = A^{-1}$

If A is orthogonal matrix then A^{-1} & A^T are also orthogonal matrices.

Ex: For matrix $M = \begin{pmatrix} 3/5 & 4/5 \\ x & 3/5 \end{pmatrix}$ such that $M^T = M^{-1}$. Find x?

Sol: $M \cdot M^T = I \Rightarrow \begin{pmatrix} 3/5 & 4/5 \\ x & 3/5 \end{pmatrix} \begin{pmatrix} 3/5 & x \\ 4/5 & 3/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow x = -\frac{4}{5}$

* Determinant of odd order skew symmetric matrix is a perfect square.

Ex: $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} = 0+16 = 16 = (4)^2$

16. If x and y are two non-zero matrices of the same order such that $xy = (0)_{m\times n}$ then

(a) $|x| \neq 0$, $|y| = 0$ b) $|x| = 0$, $|y| \neq 0$

(c) $|x| \neq 0$, $|y| \neq 0$ ✓(d) $|x| = 0$, $|y| = 0$

$xy = 0 \Rightarrow |xy| = 0 \Rightarrow |x|.|y| = 0$

then either $|x| = 0$ or $|y| = 0$ or both $|x| = |y| = 0$

(i) If $|x| = 0$, $|y| \neq 0$ y^{-1} exist

$xy = 0 \Rightarrow xy.y^{-1} = 0 \Rightarrow xI = 0$

$\Rightarrow x = 0$ [Given x is non zero]

(ii) If $|y| = 0$, $|x| \neq 0$ x^{-1} exist $\Rightarrow y = 0$ [Given y is non zero]

$\therefore$ Both $|x| = |y| = 0$

EC-08 2 Marks

17. Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ then $(AA^T)^{-1} = ?$

Given A is orthogonal $\Rightarrow A.A^T = I$

$\Rightarrow (A.A^T)^{-1} = I^{-1} = I$

18. Find the inverse of $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Given Matrix is a orthogonal $\Rightarrow A^{-1} = A^T$.

19. If a_{ij} is defined by $a_{ij} = i^2 - j^2 \ \forall i,j$ where $1 \le i, j \le n$ then find A^{-1} for $n = 5$?

Given $A = (a_{ij})_{5\times5}$ $a_{ij} = i^2 - j^2 = -(j^2 - i^2) = -a_{ji}$

$\therefore$ A is skew symmetric and odd order

$\Rightarrow |A| = 0$ & A^{-1} does not exist

Rank of a Matrix: A matrix is said to be of rank r when

(i) it has atleast one non-zero minor of order 'r'

(ii) Every minor of order higher than r vanishes.

Rank of A is denoted by $\rho(A)$. Rank of null matrix = 0

Elementary transformations of a matrix:

1. Interchange of any rows / columns
2. Multiplication of any row / column by a non zero number.
3. Addition of constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Elementary transformations do not change order or rank of the matrix.

Ex: Find rank of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$$\begin{matrix} R_2 - R_1 \rightarrow R_2 \\ R_3 - 2R_1 \rightarrow R_3 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A \text{ (Say)}$$

$$|A| = 0 \quad \Rightarrow \text{rank} \neq 3$$

2^{nd} order Minor $\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix} = -8 \neq 0 \qquad \therefore \text{rank} = 2$

Gauss - Jordan Method to find Inverse:

1. write two matrices A & I side by side
2. Reduce A to I by performing same row operations on both. Then other matrix represent A^{-1}.

Reducing A to I:

1. using R_1, make 1^{st} element in $R_2, R_3 \ldots$ to zero
2. using R_2, make 2^{nd} element in $R_3, R_4 \ldots$ to zero continue like this
3. using R_n make last element in $R_{n-1}, R_{n-2} \ldots$ to zero
4. using R_{n-1} make last-1 element in $R_{n-2}, R_{n-3} \ldots$ to zero
5. Proper constant multiplication make it as I

Ex: Find inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$A = \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - 3R_3 \sim \left[\begin{array}{ccc:ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - 3R_2 \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\hookrightarrow A^{-1}$

<u>Normal form of a matrix</u>: Every non-zero matrix of rank 'r' can be reduced by sequence of elementary transformations to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ called the normal form of A.

For every matrix of rank 'r', there exist non-singular matrix P and Q such that. $PAQ = I$.

Ex: For matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ find non singular matrix P & Q such that PAQ is in normal form. Hence find the rank of A.

Sol:

$$A = IAI$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row transformation $\to$ Same transformation in pre-factor matrix

Every column transformation $\to$ Same transformation in post factor matrix

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

normal form $= \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ ∴ Rank = 2

Row echelon form: A matrix 'A' is said to be in row echelon form if

(i) Zero rows should occupy last rows, if any

(ii) The no. of zeros before a non zero element of each row is less than the no. of such zero's before the non zero element of the next row.

Ex: $\begin{bmatrix} 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* Rank of a matrix = No. of non-zero rows in row echelon form
* Non-zero rows are called linearly independent rows/vector.
* To reduce any matrix into row echelon form, we should use only row transformations.

Ex: $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$ Find K if (i) Rank(A) = 2
(ii) Rank(A) = 3

$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & K-1 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$
$R_4 \to R_3 - R_2$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K+2 & 0 \end{bmatrix}$ $\Rightarrow$ For $\rho(A) = 2 \to K = -2$
For $\rho(A) = 3 \to K \neq -2$

* $\rho(O_{n\times n}) = 0$
* $\rho(I_{n\times n}) = n$
* $\rho\{\text{adj}(I_{n\times n})\} = n$
* $\rho(A) = \rho(A^T)$
* $\rho(A \pm B) \leq \rho(A) + \rho(B)$
* $\rho(A-B) \leq \rho(A) - \rho(B)$
* $\rho(AB) \geq \rho(A) + \rho(B) - n$ if A&B are $n\times n$ matrices

** $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

* If A is an $m\times n$ matrix, then $\rho(A) \leq \min(m,n)$

* If $\rho(A_{n\times n}) = n$ then $\rho(\text{Adj}A) = n$
* If $\rho(A_{n\times n}) = n-1$ then $\rho(\text{Adj}A) = 1$
* If $\rho(A_{n\times n}) = n-2$ then $\rho(\text{Adj}.A) = 0$
* Rank of non-singular matrix ($|A| \neq 0$) of order n is 'n'
* Rank of singular matrix ($|A| = 0$) of order n is $< n$
* If $A_{m\times 1}$, $B_{1\times n}$ are two non-zero matrices then $\rho(AB) = 1$
* Nullity of A = No. of columns of $A - \rho(A)$
* Rank of diagonal matrix = no. of non-zero elements in the diagonal.

2M Prob: If $A = (a_{ij})_{m\times n}$, such that $a_{ij} = i.j \ \forall\ i,j$ then $\rho(A) = ?$

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & & & & \\ m & 2m & 3m & \cdots & mn \end{bmatrix}_{m\times n}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$\vdots$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \Rightarrow \text{Rank} = 1$$

Prob: If $X = (x_1\ x_2 \ldots x_n)^T$ is n-type non zero vector then
(i) $\rho(XX^T)$ (ii) $\rho(X^TX)$

Sol: $X_{n\times 1}$, $X^T_{1\times n}$ $\Rightarrow \rho(X.X^T) = \min\{\rho(X), \rho(X^T)\}$

$$= \min\{1, 1\}$$

$$\rho(X\cdot X^T) = 1 \quad [X \text{ is non zero}]$$

Prob: The rank of 5×6 matrix Q is 4 then which of the following statement is true

a) Q will have 4 L.I rows & 4 L.I columns
b) Q will have 4 L.I rows & 5 L.I columns
c) QQ^T is invertible d) Q^TQ is invertible.

$\rho(5\times 6) = 4$ 4 non-zero rows & columns.

$Q_{5\times 6} \cdot Q^T_{6\times 5} = (QQ^T)_{5\times 5} \to$ Invertible

$|QQ^T|_{5\times 5} \neq 0 \to \rho(QQ^T) = 5$ (Not 4)

$Q^T_{6\times 5} \cdot Q_{5\times 6} = (Q^T \cdot Q)_{6\times 6} \to$ Invertible

$|Q^T \cdot Q|_{6\times 6} \neq 0 \to \rho(Q^T \cdot Q) = 6$ (Not 4)

Linearly dependent vectors: Two vectors x_1, x_2 are lineary dependent if one vector is expressed as multiple of other vector. x_1 & x_2 ⇒ Same directional vectors

Ex: $x_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$

$k_1x_1 + k_2x_2 + \dots + k_nx_n = 0$

[for more than 2 vectors]

$x_2 = 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3x_1$ or $x_1 = \frac{1}{3}x_2$

Alteast one k, but not all equal to zero

x_1, x_2 are linearly dependent.

Linearly Independent vectors: Two vectors x_1, x_2 are Linearly independent if it is not possible to express one vector as a multiple of other. vector.

$k_1x_1 + k_2x_2 + \dots + k_nx_n = 0$
All zeros

* If Rank of A = no. of given vectors or $|A| \neq 0$ the given vectors are said to be L.I

* If Rank of A < no. of given vectors or $|A| = 0$ the given vectors are said to be L.D

* If the given vectors are linearly dependent then anyone of the vector can be expressed as linear combination of other vectors.

* The set of unit vectors of Identity matrix are L.I

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0 \Rightarrow \text{L.I vectors}$$

* The set of vectors having atleast one zero vector are L.D.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

* Dimension: = No. of linearly independent vectors. = no. of non zero rows in R.E form = no. of non zero ~~rows~~ columns in C.E form

* Basis: = Set of linearly independent vectors = set of non zero rows in R.E form = set of non zero columns in C.E form

Ex: R.E form = $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 8 \\ 0 & 0 & -5 \end{bmatrix}$

Dimension = 3

Basis = $\{(1\ -2\ 4), (0\ 3\ 8), (0\ 0\ -5)\}$

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2M

Prob: If $a_1, a_2 \ldots, a_m$ are n-dimensional vectors with $m > n$, the vectors are L.D. The matrix Q is $a_1, a_2, \ldots a_m$ as columns. The rank of $Q = ?$ (a) m (b) n (c) between m&n ✓(d) $< m$

Sol:

$$Q = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ | & | & | & & | \\ | & | & | & & | \end{bmatrix}_{n \times m} = (Q)_{n\times m}$$

$$\rho(Q_{n\times m}) \le \min(n, m)$$

$$\rho(Q_{n\times m}) \le m \begin{cases} = m & \times \text{ (L.D)} \\ < m & \checkmark \end{cases}$$

Prob: The nullity of $A = \begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix} = 1$ The value of $k = ?$

$$\text{nullity} = n(A) - \rho(A)$$

$$1 = 3 - \rho(A) \Rightarrow \rho(A) = 2 \Rightarrow |A| = 0$$

$$\begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0 \Rightarrow k = -1$$

<u>Non Homogenious system of equations:</u> A system of equation of the form

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is called non homogeneous system of linear equations in 3 variables. These can be written in matrix form as $AX = B$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

<u>Augmented matrix:</u> Write elements of matrix B in A by adding as one more column at the end. Resulting matrix is called augmented matrix and is denoted by

$$[A \mid B] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

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Solution of linear system of equations: [In Exam use calculator]

1. **Cramer's rule:**

$$x = \frac{|A_1|}{|A|} \qquad y = \frac{|A_2|}{|A|} \qquad z = \frac{|A_3|}{|A|}$$

where $A_1 = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$ $A_2 = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$ $A_3 = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$

2. **Matrix inversion Method:**

$$AX = B \Rightarrow X = A^{-1}.B$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^{-1}_{3\times3} . \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{3\times1}$$

3. **Reducing to echelon form:** Reduce the Augmented matrix to R.E form and write the equations again

Ex: Solve $x+2y-z=3$, $3x-y+2z=1$; $2x-2y+3z=2$, $x-y+z=-1$?

Sol:
$$[A\,|\,B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$$

$R_3 \to 7R_3 - 6R_2$
$R_4 \to 7R_4 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$5z = 20 \Rightarrow z = 4$
$-7y + 6z = -8 \Rightarrow y = 4$
$x + 2y - z = 3 \Rightarrow x = -1$

** **Consistency:**

1. If $\rho[A\,|\,B] \neq \rho[A]$ then the system of equations are inconsistent. i.e there is no solution.
2. If $\rho[A\,|\,B] = \rho[A] = n$ (no. of variables) then the equations are consistent and have unique solution.
3. If $\rho[A\,|\,B] = \rho[A] = r\ (<n)$ then the equations are consistent and have infinite number of solutions with $(n-r)$ independent variables.

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Prob: For what values of λ & μ the system of equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) No solution (ii) unique solution (iii) Infinite solutions.

Sol:

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(i) No solution $\lambda=3$, $\mu \neq 10$

(ii) unique solution $\lambda \neq 3$

(iii) Infinite solutions $\lambda=3$, $\mu=10$

Prob: For what values of a & b the system of eq^{n}s have $x+2y+3z=6$, $x+3y+5z=9$, $2x+5y+az=b$ have (i) No solution (ii) unique (iii) Infinite.

Sol:

$$[A|B] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{bmatrix}$$

(i) No solution $a=8$, $b \neq 15$

(ii) unique sol^n $a \neq 8$

(iii) Infinite sol^n $a=8$, $b=15$

Note: For the system $AX=B$, where $A_{m\times n}$, $X_{n\times 1}$, $B_{m\times 1}$

(i) If $m<n$ (no. of eq^{n}s < no. of variables) then there is no unique solution. It may have infinite or no solution.

(ii) If $m \geq n$ (no. of eq^{n}s $\geq$ no. of variables) then all the three conditions exists.

Prob: The system $x+2y+3z=8$ and $2x+4y+6z=10$ has

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 2 & 4 & 6 & 10 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ $\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 0 & 0 & -6 \end{array}\right]$

$\rho(A) \neq \rho(A|B)$ No Solution

Prob: If A is a 3x4 matrix and $Ax = B$ is an inconsistent system. Then the highest possible rank of A is

$$\rho(A) \le \min\{3,4\}$$

$$\rho(A) \le 3 \begin{cases} = 3 \times \text{ inconsistent} \\ = 2 \checkmark \text{ highest possible.} \end{cases}$$

Note: consider the system of equations $ax + by = e$
$cx + dy = f$

The above system of equations have

(1) No solution if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$

(2) unique solution if $\frac{a}{c} \neq \frac{b}{d}$

(3) Infinite solutions if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

1M

Prob: The system of equations $4x + 2y = 7$, $2x + y = 6$ have

$\frac{4}{2} = \frac{2}{1} \neq \frac{7}{6}$ $\therefore$ No solution

Prob: $x + 2y = 5$, $2x + 3y = 9$ have

$\frac{1}{2} \neq \frac{2}{3} \neq \frac{5}{9}$ $\therefore$ unique solution

2M

Prob: How many solutions does the following system of eq's have $-x + 5y = -1$ $x - y = 2$ $x + 3y = 3$

(a) Infinite (b) exactly 2 (c) unique sol^n d) No sol^n

Sol: $[A|B] = \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right]$

$R_3 \to R_2 + R_1$
$R_3 \to R_3 + R_1$ $\sim \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right]$

$R_3 \to R_3 - 2R_2$ $\sim \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\rho(A) = \rho(A|B) =$ No. of unknowns $= 2$

$\therefore$ unique solution exists.

Prob: For the set of eq^{n}s $x_1 + 2x_2 + x_3 + x_4 = 2$, $3x_1 + 6x_2 + 3x_3 + 3x_4 = 6$ which of the following statement is true

1) There exist only trivial solution

(2) No solution (3) unique non trivial sol^n (4) Infinite no. of non trivial sol^n.

No. of equations (2) < no. of variables (n) = 4

Infinite solutions (non trivial)

Homogenous system of equations: A system of equations of the form $a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$. These system of eq^{n}s can be written in the matrix form as $AX = 0$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These system of eq^{n}s are always consistent i.e $x=0, y=0, z=0$ always satisfy these equations.

(i) If Rank(A) = no. of variables (n) then system of equations will have zero solution or trivial solution or unique solution

(ii) If Rank(A) < n then the system of equations will have non trivial or non zero or infinite solutions in (n-r) independent variables.

Prob: The homogenous system $x+y+z=0$, $(a+1)y+(a+1)z=0$, $(a^2-1)z=0$ has two linearly independent solutions then a = ?

a) 1   b) -1   c) 0   d) 2

Sol:
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & a+1 & a+1 \\ 0 & 0 & a^2-1 \end{bmatrix}$$

$n - r = 2$

$3 - r = 2 \Rightarrow r = 1$

To make last rows = 0 $\Rightarrow a = -1$

Prob: If the homogenous system $AX=0$ has co.efficient matrix of order $n\times n$ then the condition for existance of non trivial sol^n is

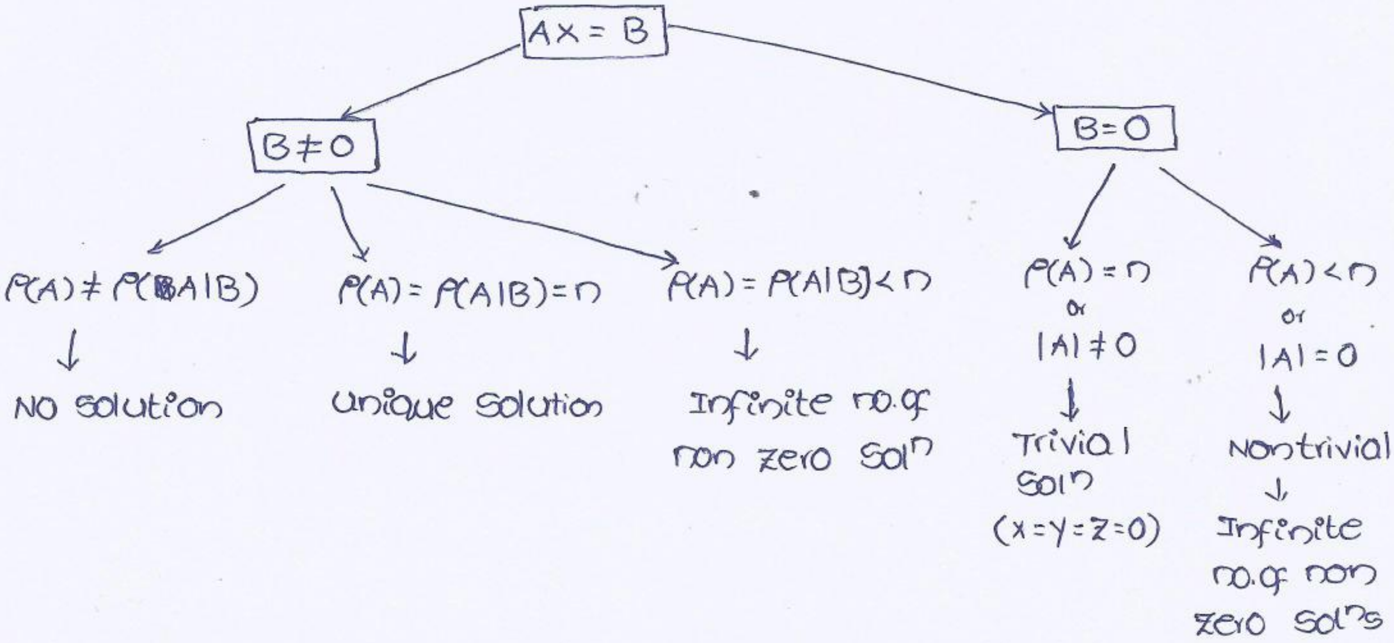
(i) $\rho(A) = n$ (ii) A is non singular (iii) $|A| \neq 0$ ✓(iv) A is singular

Note: 1. The necessary condition for existance of a non trivial solⁿ for the homogenous system $Ax=0$ is (i) A is singular (ii) $|A|=0$ (iii) $\text{Rank}[A]<n$

2. For homogenous system $A_{m\times n}\cdot X_{n\times 1}=O_{m\times 1}$

(i) If $m<n$ (no. of eqⁿs < no. of variables) always non trivial solⁿ exists

(ii) If $m\geq n$ both the conditions may exist.



**Eigen values & Eigen vectors

The roots of the characterstic eqⁿ $|A-\lambda I|=0$ are called Eigen values of matrix A.

Ex: $A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}$ $\quad |A-\lambda I|=0$

$$\begin{vmatrix}-5-\lambda & 2\\ 2 & -2-\lambda\end{vmatrix}=0 \Rightarrow (5+\lambda)(2+\lambda)-4=0$$

$$\lambda=-1,-6.$$

Ex: $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$

$$\begin{vmatrix}1-\lambda & 1 & 3\\ 1 & 5-\lambda & 1\\ 3 & 1 & 1-\lambda\end{vmatrix}=0 \Rightarrow \lambda^3-7\lambda^2+36=0$$

$$\Rightarrow \lambda=-2,6,3$$

* Eigen values are also called characterstic roots or latent roots or proper values. The set of eigen values of matrix A is called as spectrum of A.

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12

Properties:

* Sum of eigen values of A = Trace of A
* Product of eigen values of A = Det of A
* A & A^T will have same eigen values
* The eigen values of Symmetric matrices are always real.
* The " " of skew " " " " purely imaginary or zero's
* For an upper / Lower triangular matrix the eigen values are its principal diagnol elements.
* If A has eigen values $\lambda_1, \lambda_2, \lambda_3 ---$ then A^2 has eigen values $\lambda_1^2, \lambda_2^2, \lambda_3^2 ---$, A^{-1} has eigen values $1/\lambda_1$, $1/\lambda_2$, ---
* If one of eigen values of A = 0, then A^{-1} doesn't exist. Since $|A| = 0$
* Eigen values of KA are $K\lambda_1, K\lambda_2 ---$

  " " " $A - KI$ are $\lambda_1 - K, \lambda_2 - K ---$

  " " " $Adj(A)$ are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, ---$
* Eigen values of scalar / Diagonal matrix are diagonal elements.
* " " " Null matrix are all zeros
* " " " unity matrix are all 1's
* Eigen values of orthogonal matrix are unit modulus i.e ± 1

Prob: $A = \begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ Eigen values of A are

(i) $3, 3+5j, 6-j$ (ii) $-6+5j, 3-j, 3+j$ (iii) $3-j, 3+j, 5+j$ (iv) $3, -1+3j, -1-3j$. ✓

Cross check: Trace of $A = -1-1+3 = 1$

(iv) option Sum of ~~diagn~~ eigen values $= 3-1+\cancel{3j}-1-\cancel{3j} = 1$

Prob: If $(-1+\sqrt{3})$ is eigen value of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ then other two eigen values are

(i) $(-1, -\sqrt{3}, 1)$ ✓(ii) $(-1-\sqrt{3}, 2)$ (iii) $(-1+\sqrt{3}, 1)$ (iv) $(1+\sqrt{3}, -2\sqrt{3})$

Cross check: (ii) $0 = -1+\sqrt{3}-1-\sqrt{3}+2 = 0$

Eigen vectors: The eigen vector corresponding to eigen value $\lambda = \lambda_r$ of the matrix A is the solution of homogenous system $[A - \lambda_r I]x = 0$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ Eigen values are 4, −1.

$\lambda = 4$: Eigen vector is $[A - 4I]x = 0$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} x = 0$$

$$R_2 \to R_2 + R_1 \quad \sim \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $y = k \Rightarrow -3x + 2y = 0 \rightarrow x = 2k/3$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2k/3 \\ k \end{bmatrix}$ Infinite values. Ex: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\lambda = -1$: Eigen vector is $[A + I] = 0$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to 2R_2 - 3R_1 \Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $y = l \Rightarrow 2x + 2y = 0 \Rightarrow x = -l$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l \\ l \end{bmatrix}$$

Note: 1. Eigen vectors corresponding to given eigen values are always infinite in number

2. If $\lambda = \lambda_r$ is an eigen ~~vector~~ value of a matrix A then it always gives a non trivial solution for the homogenous system $[A - \lambda_r I]x = 0$

Note: $Ax = \lambda x$

$Ax = \lambda x I$ → Trivial solution $\Rightarrow |A - \lambda I| \neq 0$

$(A - \lambda I)x = 0$ → $\underbrace{\text{Infinite solution}}_{\text{Non trivial solution}} \Rightarrow |A - \lambda I| = 0$

* Eigen vectors corresponding to different eigen values of a real symmetric matrix are always orthogonal.

Ex: $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ Eigen vectors are $\underset{x_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \underset{x_2}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$

$$x_1^T \cdot x_2 = [2 \quad 1] \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= [-2 + 2] = [0]_{1\times 1}$$

$\therefore x_1, x_2$ are Orthogonal vectors

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Prob: The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigen vector of the matrix $A = \begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. The eigen value corresponding to the eigen vector is ___

Sol: $(A - \lambda I)x = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider any row $\Rightarrow$ $-2-\lambda+4+3=0 \Rightarrow \lambda = 5$

Prob: The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has an eigen value $= 3$. Sum of other two eigen values is ___

Sol: $1+0+P = 3+\lambda_2+\lambda_3 \Rightarrow \lambda_2+\lambda_3 = P-2$

Prob: Consider $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. The eigen values of A are 4 & 8. The values of x, y are

$2+y = 4+8$ $\qquad$ $\begin{vmatrix} 2 & 3 \\ x & y \end{vmatrix} = 4 \times 8$

$y = 12-2 = 10$ $\qquad$ $2(10)-3x = 32 \Rightarrow x = -4$

Prob: The eigen values of 2×2 matrix A are given by -2 & -3 resp. The eigen values of $(x+I)^{-1}.(x+5I)$

Sol: $(x+I)^{-1}.(x+5I) \Rightarrow \dfrac{1}{(\lambda+1)}.(\lambda+5)$

$$= \frac{\lambda+1+4}{\lambda+1} = 1+\frac{4}{\lambda+1}$$

$$\lambda = -2 \Rightarrow 1+\frac{4}{-2+1} = -3$$

$$\lambda = -3 \Rightarrow 1+\frac{4}{-3+1} = -1$$

Prob: The eigen vector of 3×3 matrix are $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are orthogonal. What will be the 3rd orthogonal eigen vector.

Sol: $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad$ Let $x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Here 3 vectors are orthogonal

$x_1^T x_3 = 0$ $\qquad$ $x_2^T x_3 = 0$

$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x+z = 0 \rightarrow (i)$ $\qquad$ $x-z = 0 \rightarrow (ii)$

add (i) & (ii) $\quad 2x = 0$

$$x = 0$$

put in (i) $\quad 0 + z = 0 \Rightarrow z = 0$

Zero vector can't be a eigen vector $y \neq 0 \quad \therefore y = c$ say. $(\neq 0)$

The third eigen vector is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Prob: The eigen vector of 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are given by $\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ b \end{bmatrix}$, what is $(a+b)$?

Sol: $\lambda = 1, 2$ $\qquad$ put $\lambda = 1$

$(A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 1\begin{bmatrix} 1 \\ a \end{bmatrix} \Rightarrow 1 + 2a = 1 \Rightarrow a = 0$

put $\lambda = 2 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 2\begin{bmatrix} 1 \\ b \end{bmatrix} \Rightarrow 1 + 2b = 2 \Rightarrow b = 1/2$

$$a + b = 0 + 1/2 = 1/2.$$

Prob: which of the following is an eigen vector of the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Sol: $[A - 2I]x = 0$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0 \Rightarrow \underbrace{\begin{matrix} y = 0 \\ z = 0 \end{matrix}}_{\text{fix}}$ $\quad$ Let $x = k$ $\quad \therefore$ vector $= \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Prob: Which of the following is an eigen vector of matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol: ✓a) $[1 \; 0 \; 0]^T$ $\quad$ b) $[1 \; 0 \; 1]^T$ $\quad$ c) $[0 \; 1 \; 1]^T$ $\quad$ d) $[1 \; 1 \; 1]^T$

Prob: which of the following is an eigen vector of $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$

(a) $[-1 \;\; 1]^T$ $\quad$ ✓b) $[3 \;\; -1]^T$ $\quad$ c) $[1 \;\; -1]^T$ $\quad$ d) $[-2 \;\; 1]^T$

$$\lambda = 1, -2$$

$\lambda = 1 \Rightarrow \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\qquad -x - 3y = 0$

Let $y = k \Rightarrow x = -3k$

$\therefore$ vector $= \begin{bmatrix} -3k \\ k \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ for $k = -1$

Note: $\quad |A - \lambda I| = \lambda^2 - Tr(A).\lambda + \det(A)$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $Tr(A) = a + d$

$$\det(A) = ad - bc.$$

Cayley - Hamilton theorem: Every square matrix satisfies it's own characteristic equation i.e if the characteristic equation of the n^{th} order square matrix A is

$$|A-\lambda I| = (-1)^n\lambda^n + K_1\lambda^{n-1} + \dots + K_n = 0$$

then $$(-1)^n A^n + K_1 A^{n-1} + \dots + K_n = 0$$

Prob: If $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ then $A^{-1} = ?$

Sol: $$A^2 + 7A + 6I = 0 \Rightarrow A + 7I + 6A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{6}(-A - 7I)$$

Prob: If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ then $A^8 =$

Sol: $$\lambda^2 + 0\lambda - 5 = 0 \Rightarrow \lambda^2 = 5I$$

$$A^2 = 5I \Rightarrow A^8 = 625I = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$$

Prob: If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ then $A^{-1} =$

Sol: $$\lambda = 1,1,1 \Rightarrow (\lambda-1)^3 = 0$$

$$(A-1)^3 = 0$$

$$A^3 - 1 + 3A - 3A^2 = 0$$

$$A^2 - A^{-1} + 3I - 3A = 0 \Rightarrow A^{-1} = A^2 - 3A + 3I$$

* By cayley - hamilton theorem : $A^2 - A[Tr(A)] + |A|.I = 0$
* For 3x3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ characteristic equation is

$$\lambda^3 - \lambda^2[Tr(A)] + \lambda\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\right\} - |A| = 0$$

Method to find higher powers of A:

Let $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - (-3)\lambda + 2 = 0$

$$\Rightarrow \lambda = -1, -2$$

If λ values are not repeated then $\lambda^n = a\lambda + b \rightarrow (1)$

for $\lambda = -1$ $\quad (-1)^n = -a + b \rightarrow (2)$

for $\lambda = -2$ $\quad (-2)^n = -2a + b \rightarrow (3)$

put (3) in (2) $\Rightarrow$ (-1

$\therefore b = (-1)^n + a$

$= (-1)^n + (-1)^n - (-2)^n = 2(-1)^n - (-2)^n$

put a & b in eq.(1) $\Rightarrow \lambda^n = [(-1)^n - (-2)^n]\lambda + [2(-1)^n - (-2)^n]$

by C-H theorem $A^n = [(-1)^n - (-2)^n]A + [2(-1)^n - (-2)^n]I$

Ex: put $n=9$ $\quad A^9 = 511A + 510I$

Properties:

Idompotent $A^2 = A$

Involutory $A^2 = I$

Nilpotent $A^n = 0$ n- index

Hermetian $A = \bar{A}^T = A^\theta = A^*$

skew hermetian $A = \bar{A}^T = -A^\theta = -A^*$

unitory $u \cdot u^\theta = I \Rightarrow u^{-1} = u^\theta$

prob: The matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix}$ is

$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Nilpotent

prob: The matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$ is

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow$ Involutory

prob: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

$A^2 = I \Rightarrow$ Involutory $\Rightarrow A \cdot A = I \rightarrow A \cdot A^T = I$

$A^T = A \Rightarrow$ Symmetric $\qquad \Rightarrow$ orthogonal.

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CALCULUS

CALCULUS

Basics (pre-calculus):

Logorithms: If $\log_b a = m$ then $a = b^m$

1. $\log_c(ab) = \log_c a + \log_c b$
2. $\log_c(a/b) = \log_c a - \log_c b$
3. $\log_{b^n} a^m = \frac{m}{n} \log_b a$
4. $\log_b a = \frac{1}{\log_a b} = \log_c a \cdot \log_b c = \frac{\log_c a}{\log_c b}$
5. $a^x = e^{\log_e a^x} = e^{x \log_e a}$ $\qquad \log_e \rightarrow \ln$
6. $\log_a a = 1$, $\log_a 1 = 0$, $\log_c(1/a) = -\log_c a$

Ex: $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = 2.322$

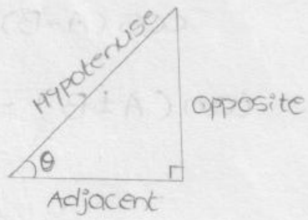
GATE IN(08): The expression $e^{-\ln x}$ for $x > 0$ is equal to

Sol: $e^{-\ln x} = e^{-\log_e x}$

$$= e^{\log_e(1/x)} = 1/x = x^{-1}$$

Trignometry:

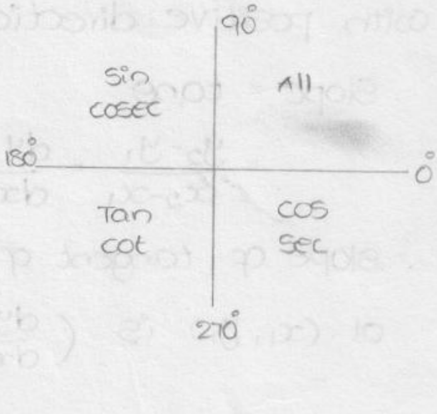
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\text{cosec}\,\theta}$$



$$\cos\theta = \frac{\text{Adjacent}}{\text{hypotenuse}} = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{\text{opposite}}{\text{Adjacent}} = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$$

	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞



(i) If the angle is $(90 \pm \theta)$, $(270 \pm \theta)$ i.e about vertical line, then co-function of initial function will apply

(ii) If the angle is $(180 \pm \theta)$, $(360 \pm \theta)$ i.e. about horizontal line then same function will apply.

(iii) The positivity or negativity of the resulting function depends on the quadrant in which initial function lies.

Ex: $\sin(270+\theta) = -\cos\theta$ $\cot(270-\theta) = \tan\theta$

$\cos(90+\theta) = -\sin\theta$ $\sec(180+\theta) = -\sec\theta$

Formulas:

1. $\sin^2\theta + \cos^2\theta = 1$; $\operatorname{cosec}^2\theta - \cot^2\theta = 1$; $1+\tan^2\theta = \sec^2\theta$.

2. $\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \overset{\sin}{\cancel{\cos}} B$

put $B=A \Rightarrow \sin 2A = 2\sin A \cos A$

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

3. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

put $B=A$ $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A = 2\cos^2 A - 1$

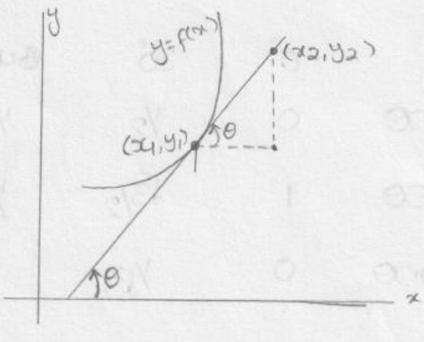
$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

4. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$

put $B=A \Rightarrow \tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Slope of a line: It is the tangent of angle made by the line with positive direction of x-axis.



Slope $= \tan\theta$

$= \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{dy}{dx}$

slope of tangent of curve $y=f(x)$ at (x_1, y_1) is $\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}$

Equation of straight line:

1. point slope form: point is (x_1, y_1) & slope m

$$y - y_1 = m(x - x_1)$$

2. Two point form:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

3. slope intercept form: slope is 'm', y-intercept is 'c'

$$y = mx + c$$

4. Intercept form: 'x' intercept is 'a', 'y' intercept is 'b'

$$\frac{x}{a} + \frac{y}{b} = 1$$

5. General equation of straight line $ax + by + c = 0$

$$y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$$

slope $= -a/b$ y-intercept $-c/b$

$$ax + by = -c$$

$$\frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1 \Rightarrow x\text{- intercept} = -c/a$$

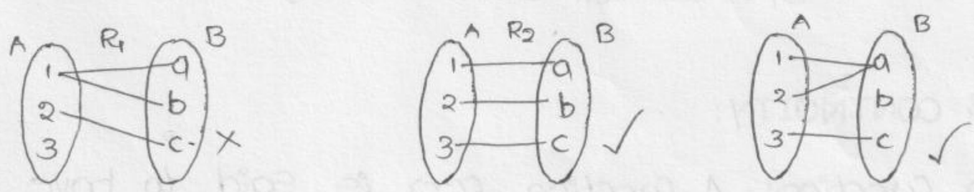
Function: A function from set A to set B is a relation from A to B satisfying the condition

"To each element in A, there exist a unique element in B"

i.e (i) Every element in A must be associated

(ii) It must be associated with a unique element in B.

Ex:



Explicit function: $z = f(x_1, x_2, \ldots x_n)$

Dependent variable (z); Independent variable ($x_1, x_2, \ldots x_n$)

Implicit function: $\phi(z, x_1, x_2, \ldots x_n) = c$

composit function: If $z = f(x, y)$ where $x = \phi(t)$ & $y = \psi(t)$ i.e

z is a function of some function.

Some special functions:

1. Even function: $f(-x) = f(x)$ Ex: x^2, $\cos x$ ...

2. odd function: $f(-x) = -f(x)$ Ex: x, $\sin x$ --

3. Modulus function:

$$f(x) = |x| = \begin{cases} x & ; x>0 \\ -x & ; x<0 \\ 0 & ; x=0 \end{cases}$$

GATE-99: $f(x) = e^{x}$ neither even nor odd

4. Step / Greatest integer function:

$$f(x) = [x] = n \in Z \quad \text{where } n \le x \le n+1$$

Ex: $[7.2] = 7$; $[7.999] = 7$; $[-1.2] = -2$

Symmetric properties of the curve: Let $f(x,y) = c$ be the eqn of a curve

(i) If $f(x,y)$ contains only even powers of x i.e $f(-x,y) = f(x,y)$ then it is symmetric about y-axis

(ii) If $f(x,y)$ contains only even powers of y i.e $f(x,-y) = f(x,y)$ then it is symmetric about x-axis

(iii) If $f(x,y) = f(y,x)$ then it is symmetric about $y=x$.

GATE (97): The curve given by the equation $x^2+y^2 = 3axy$ is

(a) symmetrical about x-axis (b) y-axis (c) line $y=x$

(d) tangential to $x=y=a/3$

Sol: $f(x,y) = f(y,x) \Rightarrow x^2+y^2 = 3axy$

$\therefore$ Symmetrical about line $y=x$.

LIMITS & CONTINUITY:

Limit of a function: A function $f(x)$ is said to have limit value 'l' as x tend to 'a' if

$$\lim_{x\to a} f(x) = l$$

Left limit: when $x<a$, $x\to a$ $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

Right limit: when $x>a$, $x\to a$ $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

A limit exists if LHL = RHL

Ex: If $f(x) = \begin{matrix} 2x+3 & \text{for } x \geq 2 \\ x+9 & \text{for } x<2 \end{matrix}$ then $\underset{x\to 2}{lt} f(x) = ?$

sol: $\underset{x\to 2^+}{lt} 2x+3 = 7 \neq \underset{x\to 2^-}{lt} x+9 = 11$

$\therefore$ Limit doesnot exist.

Indeterminate form: $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

Standard limits:

(i) $\lim\limits_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

(ii) $\underset{x\to 0}{lt} \frac{e^{mx}-1}{x} = e$

(iii) $\underset{x\to 0}{lt} \frac{a^x-1}{x} = \log_e a$

(iv) $\underset{x\to 0}{lt} [1+ax]^{1/x} = e^a$

(v) $\underset{x\to 0}{lt} [1+\frac{a}{x}]^x = e^a$

(vi) $\underset{x\to 0}{lt} \frac{\sin x}{x} = 1$ or $\underset{x\to 0}{lt} \frac{\tan x}{x} = 1$

(vii) $\underset{x\to \infty}{lt} \frac{\sin x}{x} = 0$

(viii) $\underset{x\to 0}{lt} \frac{\sin mx}{x} = m$

(ix) $\underset{x\to 0}{lt} \left[\frac{a^x+b^x}{2}\right]^{1/x} = \sqrt{ab}$

(x) $\underset{x\to 0}{lt} [\cos x + a\sin bx]^{1/x} = e^{ab}$

(xi) $\underset{x\to 0}{lt} \left[\frac{1-\cos ax}{x^2}\right] = \frac{a^2}{2}$

(xii) $\underset{x\to \infty}{lt} \frac{\log_e x}{x} = 0$

Limit properties:

1. $\underset{x\to a}{lt} [c\, f(x)] = c \underset{x\to a}{lt} f(x)$

2. $\underset{x\to a}{lt} [f(x) \pm g(x)] = \underset{x\to a}{lt} f(x) \pm \underset{x\to a}{lt} g(x)$

3. $\underset{x\to a}{lt} [f(x) \cdot g(x)] = \underset{x\to a}{lt} f(x) \cdot \underset{x\to a}{lt} g(x)$

4. $\underset{x\to a}{lt} \frac{f(x)}{g(x)} = \frac{\underset{x\to a}{lt} f(x)}{\underset{x\to a}{lt} g(x)}$ provided $\underset{x\to a}{lt} g(x) \neq 0$

5. $\underset{x\to a}{lt} [f(x)]^n = \left[\underset{x\to a}{lt} f(x)\right]^n$ where 'n' is a real number.

6. $\underset{x\to a}{lt} \sqrt[n]{f(x)} = \sqrt[n]{\underset{x\to a}{lt} f(x)}$

7. $\underset{x\to a}{lt} C = C$; C is a real number

8. If $P(x)$ is a polynomial then $\underset{x\to a}{lt} P(x) = P(a)$

(GATE-95): $\lim_{x\to 0} x\sin(1/x) = \underline{\quad}$.

sol: Let $x = t \Rightarrow 1/x = 1/t$, $t \to \infty$

$$\lim_{x\to 0} x\sin(1/x) = \lim_{t\to\infty} \frac{\sin t}{t} = 0$$

(GATE-99): $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} =$

sol: $$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1+1/n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}} = \frac{1}{\sqrt{1+0}} = 1$$

(GATE-01 IN): $\lim_{x\to\pi/4} \frac{\sin 2(x-\pi/4)}{x-\pi/4}$

sol: Let $x-\pi/4 = t \Rightarrow t\to 0$

$$\lim_{x\to\pi/4} \frac{\sin 2(x-\pi/4)}{x-\pi/4} = \lim_{t\to 0} \frac{\sin 2t}{t} = 2$$

(GATE-02 CE): $\lim_{n\to\infty} n^{1/n} =$

sol: Let $y = \lim_{n\to\infty} n^{1/n}$ Apply log on b.s

$$\log_e y = \lim_{n\to\infty} \log_e n^{1/n} = \lim_{n\to\infty} \frac{1}{n}\log_e n = 0$$

$$\log_e y = 0 \Rightarrow y = e^0 = 1$$

(GATE-03): $\lim_{x\to 0} \frac{\sin^2 x}{x} = \underline{0}$

sol: $$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \times \sin x$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \sin x$$

$$= 1 \times 0 = 0$$

(GATE-04): $\lim_{x\to 0} \frac{x^3+x^2}{2x^3-7x^2} =$

sol: $$\lim_{x\to 0} \frac{x^3+x^2}{2x^3-7x^2} = \lim_{x\to 0} \frac{x^2(x+1)}{x^2(2x-7)}$$

$$= \lim_{x\to 0} \frac{x+1}{2x-7} = \frac{1}{-7} = -\frac{1}{7}$$

(GATE-07 ME): $\lim_{x\to 0} \frac{e^x - (1+x+x^2/2)}{x^3}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

by substituting we will get

$$\lim_{x\to 0} \frac{\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\infty}{x^3} = \lim_{x\to 0}\left(\frac{1}{3!}+\frac{x}{4!}+\frac{x^2}{5!}+\cdots+\infty\right)$$

$$= \frac{1}{3!}+0 = \frac{1}{6}$$

(GATE-07 EC): $\lim_{\theta\to 0} \frac{\sin(\theta/2)}{\theta}$ is

Sol: $\lim_{x\to 0} \frac{\sin mx}{x} = m \Rightarrow \lim_{\theta\to 0} \frac{\sin(\frac{1}{2}\cdot\theta)}{\theta} = \frac{1}{2} = 0.5$

(GATE-08 CS): $\lim_{x\to\infty} \frac{x-\sin x}{x+\cos x} =$

Sol: $= \lim_{x\to\infty} \frac{1-\frac{\sin x}{x}}{1+\frac{\cos x}{x}} = \frac{1-\lim_{x\to\infty}\frac{\sin x}{x}}{1+\lim_{x\to\infty}\frac{\cos x}{x}} = \frac{1-0}{1+0} = 1$

(GATE-08 ME): $\lim_{x\to 8} \frac{x^{1/3}-2}{x-8}$

Sol: $= \lim_{x\to 8} \frac{x^{1/3}-8^{1/3}}{x-8}$ $\quad \lim_{x\to a}\frac{x^n-a^n}{x-a} = n\cdot a^{n-1}$

$= \frac{1}{3}\cdot 8^{\frac{1}{3}-1} = \frac{1}{3}\times 8^{-2/3} = \frac{1}{3}\times\frac{1}{4} = \frac{1}{12}$

(GATE-08 PI): $\lim_{x\to 0} \frac{\sin x}{e^x\cdot x} =$

Sol: $\lim_{x\to 0} \frac{\sin x}{e^x\cdot x} = \lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{1}{e^x}$

$= 1\times\frac{1}{e^0} = 1\times 1 = 1$

(GATE-10 CS): $\lim_{n\to\infty}(1-1/n)^{2n} =$

Sol: $\lim_{n\to\infty}(1-1/n)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{-1}{n}\right)^n\right]^2$

$= \left[\lim_{n\to\infty}\left(1+\frac{(-1)}{n}\right)^n\right]^2 = (e^{-1})^2 = e^{-2}$

(GATE-14 CE): $\lim_{x\to\infty} \frac{x+\sin x}{x} =$

Sol: $\lim_{x\to\infty}\frac{x}{x} + \underbrace{\lim_{x\to\infty}\frac{\sin x}{x}}_{0} = \lim_{x\to\infty} 1 + 0 = 1+0 = 1$

Pblm: $\lim_{x\to 0}\frac{|x|}{x} = ?$

Sol: LHL $= \lim_{x\to 0^-}\frac{|x|}{x} = \lim_{h\to 0}\frac{|-h|}{-h} = -1$

RHL $= \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{h\to 0}\frac{|h|}{h} = 1$

LHL $\neq$ RHL $\therefore$ limit doesn't exist.

Continuity of $f(x)$: A function $f(x)$ is said to be continous at $x=a$, when $\lim_{x\to a} f(x) = f(a)$

Ex: $f(x) = \begin{cases} 2x+3 & \text{for } x>2 \\ x+5 & \text{for } x<2 \\ 15 & \text{for } x=2 \end{cases}$

$$\lim_{x\to 2^+} f(x) = 7 = \lim_{x\to 2^-} f(x) = 7 \neq f(2) = 15$$

Limit exists but not continous.

(GATE-97): If $y=|x|$ for $x<0$ and $y=x$ for $x\geq 0$ then

✓(a) $\frac{dy}{dx}$ is discontinous at $x=0$ (b) y is discontinous at $x=0$

(c) y is not defined at $x=0$ (d) Both y & dy/dx are discontinous at $x=0$

Sol: $y=|x|$ is continous everywhere

$\frac{dy}{dx} = \frac{|x|}{x}$ is not continous at $x=0$ because $\lim_{x\to 0} \frac{|x|}{x}$ = not exist

$\therefore$ Ans: (a)

(GATE-2013 CS): which of the following function is continous at $x=3$

✓(a) $f(x) = \begin{cases} 2 & \text{if } x=3 \\ x-1 & \text{if } x>3 \\ \frac{x+3}{3} & \text{if } x<3 \end{cases}$ (b) $f(x) = \begin{cases} 4 & \text{if } x=3 \\ 8-x & \text{if } x\neq 3 \end{cases}$

(c) $f(x) = \begin{cases} x+3 & \text{if } x\leq 3 \\ x-4 & \text{if } x>3 \end{cases}$ (d) $f(x) = \frac{1}{x^3-27}$ if $x\neq 3$

Sol:

(a) $\lim_{x\to 3^+} f(x) = 3-1 = 2$ $\lim_{x\to 3^-} f(x) = \frac{3+3}{3} = 2$ $f(3) = 2$

$\therefore$ $f(x)$ is continous at $x=3$

Ans: (a)

Note: If a function is continous at $x=a$ then limit of the function also exists at $x=a$ and is equal to $f(a)$.

Differentiability: A function $f(x)$ is said to be differentiable at $x=a$ if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. and is denoted $f'(a)$

In general the derivative function is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Ex: If $f(x) = x^2$ then $f'(x) = 2x$ & $f'(a) = 2a$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} x+a = 2a$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{x^2+\Delta x^2+2\cdot x\cdot \Delta x - x^2}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x(\Delta x+2x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \Delta x + 2x = 0+2x = 2x$$

* Left hand derivative: $LHD = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h}$

Right hand derivative: $RHD = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

Necessary condition for function to be differential is

$$LHD = RHD$$

* Every differentiable function is a continous function. But every continous function is not differentiable.

Note:

(i) $|x|$ is not differentiable at $x=0$

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$LHD \neq RHD \Rightarrow |x|$ not differentiable.

(ii) $|x-a|$ is not differentiable at $x=a$

(iii) $|ax+b|$ is not differentiable at $x=-b/a$

(GATE-95): The function $f(x) = |x+1|$ on the interval $[-2,0]$

Sol: $|x+a|$ is continous everywhere.

$|x+a|$ is not differentiable at $x=-a$.

$\therefore |x+1|$ is not differentiable at $x=-1 \in [-2,0]$

Ans: continous, but not differentiable.

(GATE-07 IN): The function $f(x) = |x|^3$, at $x=0$ is = (where x is real)

(a) continous but not differentiable

(b) once differentiable but not twice

✓(c) twice " " thrice (d) thrice differentiable

Sol:

$$f(x) = |x|^3 = \begin{cases} x^3 & ; x>0 \\ -x^3 & ; x<0 \\ 0 & ; x=0 \end{cases}$$

$f(0)=0$, LHL = RHL = 0 $\Rightarrow$ continous.

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{h} = \lim_{h\to 0} \frac{|-h|^3-0}{-h} = \lim_{h\to 0} \frac{-h^3}{-h} = \lim_{h\to 0} h^2 = 0$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|^3-0}{-h} = \lim_{h\to 0} -h^2 = 0$$

LHD = RHD $\Rightarrow$ differentiable.

$$f'(x) = \begin{cases} 3x^2 & ; x>0 \\ -3x^2 & ; x<0 \\ 0 & ; x=0 \end{cases} \quad f''(x) = \begin{cases} 6x & ; x>0 \\ -6x & ; x<0 \\ 0 & ; x=0 \end{cases} \quad f'''(x) = \begin{cases} 6 & ; x>0 \\ -6 & ; x<0 \\ \text{Not diff'ble} & ; x=0 \end{cases}$$

$\therefore f(x)$ is twice differentiable but not thrice.

(GATE-10 ME): The function $y = |2-3x|$

Sol:

$y = |2-3x|$ continous $\forall x \in R$

$y = |2-3x|$ not differentiable at $x = 2/3$

$2-3x=0$
$x = 2/3$

Derivatives of Some functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	$\to nx^{n-1}$	e^x	$\to e^x$	$\sec x$	$\to \sec x \cdot \tan x$
$\sqrt{x}$	$\to \frac{1}{2\sqrt{x}}$	$\log_e^x$	$\to 1/x$	$\mathrm{cosec}\, x$	$\to -\mathrm{cosec}\, x \cdot \cot x$
$\frac{1}{x^n}$	$\to \frac{-n}{x^{n+1}}$	$\sin x$	$\to \cos x$	$\sin^{-1}x$ (or) $-\cos^{-1}x$	$\to \frac{1}{\sqrt{1-x^2}}$
		$\cos x$	$\to -\sin x$	$\tan^{-1}x$ (or) $-\cot^{-1}x$	$\to \frac{1}{1+x^2}$
$\frac{1}{\sqrt{x}}$	$\to \frac{-1}{2x\sqrt{x}}$	$\tan x$	$\to \sec^2 x$		
a^x	$\to a^x \log_e a$	$\cot x$	$\to -\mathrm{cosec}^2 x$	$\sec^{-1}x$ (or) $-\mathrm{cosec}^{-1}x$	$\to \frac{1}{x\sqrt{x^2-1}}$

Note: (1) $(uv)' = uv' + u'v$

(2) $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

(3) $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

(4) Leibniz formula: $(f.g)^n = nc_0 f^n g + nc_1 f^{n-1} g' + \dots + nc_n f \cdot g^n$

where 'n' is order of differentiation.

L'Hospital rule: If $\lim_{x \to a} \frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

Note: 1. In case of other indeterminate forms $0.\infty, 0^0, \infty^0, 1^\infty, \infty - \infty$, first we have to reduce it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use this rule

2. For the forms $0^0, \infty^0, 1^\infty$ take logarithm and then take limits

3. $0^\infty = 0$, $\infty^\infty = \infty$, $\infty . \infty = \infty$, $\infty + \infty = \infty$, $\infty^{-\infty} = 0$ are not indeterminate

Ex: $\lim_{x\to 0} \frac{\ln(1+x)}{\sin x} = \frac{0}{0}$ form

Apply L'H rule $= \lim_{x \to 0} \frac{1/(1+x)}{\cos x} = \frac{1}{1} = 1$

Ex: $\lim_{x\to 0} x^x = 0^0$ form   Let $y = x^x$

Apply logarithm first $\ln y = \ln x^x = x \ln x$

$$\lim_{x\to 0} \ln y = \lim_{x \to 0} x . \ln x$$

$$\ln\left[\lim_{x\to 0} y\right] = \lim_{x\to 0} \frac{\ln x}{1/x} = \frac{\infty}{\infty} \text{ Apply L'H}$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} = -\lim_{x\to 0} x = 0$$

$$\lim_{x\to 0} y = \lim_{x\to 0} x^x = e^0 = 1$$

(GATE-93 ME): $\lim_{x\to 0} \frac{x(e^x - 1) + 2(\cos x - 1)}{x(1-\cos x)}$ $\frac{0}{0}$

Sol: using L'H rule

$$\lim_{x \to 0} \frac{(e^x - 1) + x.e^x + 2(-\sin x)}{(1-\cos x) + x(\sin x)} \quad \frac{0}{0}$$

Again apply L'H rule.

$$= \lim_{x \to 0} \frac{e^x + e^x + xe^x - 2\cos x}{\sin x + \sin x + x\cos x} \quad \frac{0}{0}$$

Again apply L'H rule

$$= \lim_{x \to 0} \frac{e^x + e^x + e^x + xe^x + 2\sin x}{\cos x + \cos x + \cos x - x\sin x} = \frac{1+1+1+0}{1+1+1-0} = 1$$

(GATE-95 CS): $\lim_{x \to \infty} \frac{x^3 - \cos x}{x^2 + (\sin x)^2} \quad \frac{\infty}{\infty}$

Sol: Apply L' Hospital

$$\lim_{x \to \infty} \frac{3x^2 + \sin x}{2x + 2\sin x \cdot \cos x} = \lim_{x \to \infty} \frac{3x + \frac{\sin x}{x}}{2 + \frac{\sin 2x}{x}} = \frac{3 \times \infty + 0}{2 + 0} = \infty$$

(GATE-99 IN): $\lim_{x \to 0} \frac{1}{10} \cdot \frac{1 - e^{-j5x}}{1 - e^{-jx}} \quad \frac{0}{0}$

Sol: Apply L' Hospital

$$\lim_{x \to 0} \frac{1}{10} \cdot \frac{-(-5j)e^{-j5x}}{-(-j) \cdot e^{-jx}} = \frac{5j}{10j} = \frac{1}{2}$$

(GATE-99): $\lim_{x \to a} (x-a)^{x-a} \quad 0^0$

Sol: Let $y = (x-a)^{(x-a)} \Rightarrow \log y = (x-a)\log(x-a)$

$$\lim_{x \to a} \log y = \lim_{x \to a} [(x-a)\log(x-a)]$$

$$\log\left[\lim_{x \to a} y\right] = \lim_{x \to a} \frac{\log(x-a)}{1/(x-a)} \quad \frac{\infty}{\infty}$$

Apply L'H rule $= \lim_{x \to a} \frac{1/x-a}{-1/(x-a)^2} = \lim_{x \to a} -\cancel{(x-a)} = 0$

$$\lim_{x \to a} y = \lim_{x \to a} (x-a)^{(x-a)} = e^0 = 1$$

(GATE-07 PI): $\lim_{x \to \pi/4} \frac{\cos x - \sin x}{x - \pi/4} \quad \frac{0}{0}$

Sol: Apply L'H rule $= \lim_{x \to \pi/4} \frac{-\sin x - \sin x}{1} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(GATE-12 ME, PI): $\lim_{x \to 0} \left(\frac{1 - \cos x}{x^2}\right) \quad \frac{0}{0}$

Sol: Apply L'H rule $= \lim_{x \to 0} \frac{\sin x}{2x} \quad \frac{0}{0}$

Again apply L'H $= \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$

(GATE-14 ME): $\lim_{x \to 0} \frac{x - \sin x}{1 - \cos x} \quad \frac{0}{0}$

Sol: Apply L'H rule

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \quad \frac{0}{0}.$$

again apply L'H rule $= \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$

(GATE-14 ME): $\lim_{x\to 0}\left(\frac{e^{2x}-1}{\sin(4x)}\right) \quad \frac{0}{0}$

Sol: Apply L'H $= \lim_{x\to 0} \frac{2.e^{2x}}{4.\cos 4x} = \frac{2}{4} = \frac{1}{2}$

** (GATE-14 CE): $\lim_{a\to 0} \frac{x^a - 1}{a} = \frac{0}{0}$

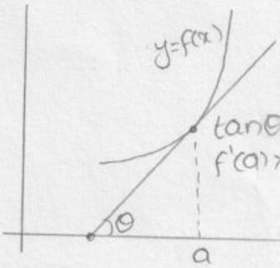
Sol: Apply L'H differentiate w.r.to a

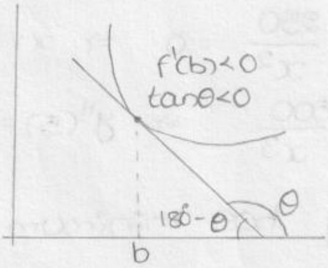
$$= \lim_{a\to 0} \frac{x^a . \log x - 0}{1} = x^0 . \log x = \log x.$$

Increasing (or) decreasing nature of function:

A function $y = f(x)$ will have increasing nature at $x = a$, if $f'(a) > 0$

A function $y = f(x)$ will have decreasing nature at $x = b$ if $f'(b) < 0$

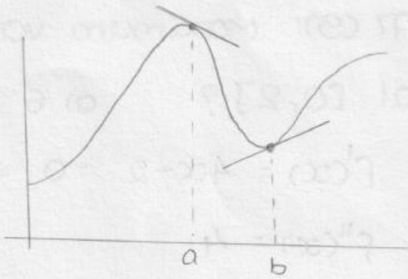




Local

Maxima and minima:

To obtain the maxima or minima



1. Find $f'(x)$ and solve the equation $f'(x) = 0$ to obtain the stationary points say $x = a, b, c$.
2. Find $f''(x)$
3. If $f''(a) < 0$, then at $x = a$. $f(x)$ will have maximum value

   If $f''(b) > 0$, then at $x = b$. $f(x)$ will have minimum value

   If $f''(c) = 0$, then at $x = c$. $f(x)$ won't have minimum/maximum. The point is called saddle point.

Absolute maxima/minima: If $f(x)$ defined in interval $[a, b]$ then

Absolute minimum value $= \min\{f(a), f(b), \text{all local minimum values}\}$

Absolute maximum value $= \max\{f(a), f(b), \text{all local maximum values}\}$

If stationary points are out of given interval then don't consider them

Ex: The function $f(x) = x^3 - 9x^2 + 24x - 12$ has

Sol: $f'(x) = 3x^2 - 18x + 24 = 0$ $\qquad f''(x) = 6x - 18$

$x = 2, 4$ $\qquad f''(2) = -6 < 0$

$f''(4) = 6 > 0$

$\therefore f(x)$ has Max. at $x = 2$, Min at $x = 4$.

(GATE-07 EC)

Ex: The maximum value of $f(x) = x^2 - x - 2$ in the interval $[-4, 4]$ is

✓(a) 18 (b) 10 (c) -2.25 (d) indeterminate

Sol: $f'(x) = 2x - 1 = 0 \Rightarrow x = 1/2$

$f''(x) = 2$ can't find

$f(-4) = 18$ $\quad f(4) = 10$ $\quad f(1/2) = -2.25$

$\therefore f(x)$ has maximum at $x = -4$.

(GATE-94): The function $y = x^2 + \frac{250}{x}$ at $x = 5$ attains

Sol: $y' = 2x - \frac{250}{x^2} = 0 \Rightarrow x = 5$

$y'' = 2 + \frac{500}{x^3} \Rightarrow y''(5) = 2 + 4 = 6 > 0$

$\therefore$ y has minimum at $x = 6$

(GATE-97 CS): Maximum value of function $f(x) = 2x^2 - 2x + 6$ in the interval $[0, 2]$? a) 6 ✓b) 10 c) 12 d) 5.5

$\downarrow$ $f(1/2)$

Sol: $f'(x) = 4x - 2 = 0 \Rightarrow x = 1/2$

$f''(x) = 4$

$f(0) = 0 + 6 = 6$ $\qquad f(2) = 8 - 4 + 6 = 10$

$\therefore$ Maximum value $= 10$ in $[0, 2]$

(GATE-99 CE): Number of inflection points for the curve $y = 2x^4 + x$

Sol: $\frac{d^2y}{dx} = 0 \Rightarrow 8 \times 3x^2 = 0$

$\Rightarrow x = 0 \Rightarrow y = 0$

$\therefore$ (0,0) is the only inflection point.

(GATE-05 EE): For the function $f(x) = x^2 e^{-x}$, the max. occurs when $x = ?$

Sol: $f'(x) = e^{-x}[-x^2 + 2x] = 0 \Rightarrow x = 0, 2$ $\qquad f''(0) = 2 > 0$

$f''(x) = -e^{-x}[-x^2 + 2x] + e^{-x}[-2x + 2]$ $\qquad f''(2) = -2e^{-2} < 0$

$\therefore f(x)$ has max. at $x = 2$

(GATE-07 EE): The function $f(x) = (x^2-4)^2$ has (x is real number)

(a) only one minimum (b) only two minima (c) Three minima

(d) Three maxima.

Sol: $f'(x) = 2(x^2-4)\cdot 2x = 0 \Rightarrow x = \pm 2, 0$

$f''(x) = 4(3x^2-4)$

$f''(0) = -16 < 0$ $\quad f''(2) = 32 > 0$ $\quad f''(-2) = 32 > 0$

$\therefore f(x)$ has minimum at $x = \pm 2$

(GATE-08 CS): The number of distinct extrema for the curve $3x^4 - 16x^3 + 24x^2 + 37$ is

Sol: $f'(x) = 12x^3 - 48x^2 + 48x = 0 \Rightarrow x = 0, 2, 2$

$f''(x) = 36x^2 - 96x + 48$

$f''(0) = 48 > 0$ $\quad f''(2) = 0$ No extrema

$\therefore f(x)$ has only one extrema (minimum) at $x = 0$

(GATE-08 EC): For real values of 'x', the minimum value of function $f(x) = e^x + e^{-x}$ is

Sol: $f(x) = e^x + e^{-x}$ where $x \in R$

$f'(x) = e^x - e^{-x} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = 0$

$f''(x) = e^x + e^{-x} \Rightarrow f''(0) = 1 + 1 = 2 > 0$

$\therefore f(x)$ has Minimum at $x = 0$, Minimum value $f(0) = 2$

(GATE-10 EC): If $e^y = x^{1/x}$ then y has a

Sol: $e^y = x^{1/x} \Rightarrow y = \log x^{1/x} = \dfrac{\log x}{x}$

$y' = \dfrac{1-\log x}{x^2} = 0 \Rightarrow x = e$

$y'' = \dfrac{x^2(-1/x) - (1-\log x)x}{x^4}$

$y''(e) = -e/e^4 < 0 \Rightarrow y$ has maximum at $x = e$

(GATE-12 EC, EE, IN): The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$ Similar Ques. in GATE-14 EC

Sol: $f'(x) = 3x^2 - 18x + 24 = 0 \Rightarrow x = 2, 4$

$f''(x) = 6x - 18$ $\quad f''(2) = -6 < 0$ $\quad f'(4) = 6 > 0$

$\hookrightarrow$ local maximum at $x = 2$

Local maximum = $f(2) = 25$

In the interval $[1,6]$ $\quad f(1) = 21 \quad f(6) = 41$

$\therefore$ Absolute maximum = $\max\{f(1), f(6), \text{local max}\}$

$= \max\{21, 41, 25\} = 41$

(GATE-14 EC): For $0 \le t < \infty$ the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

Sol: $f'(t) = (-e^{-t} + 4e^{-2t}) = 0 \Rightarrow e^{-t} = 1/4$

$\Rightarrow t = \log_e 4$

$f''(t) = (e^{-t} - 8e^{-2t})$

$f''(\log_e 4) = e^{-\log_e 4} - 8e^{-2\log_e 4} = \frac{1}{4} - 8\cdot\frac{1}{16} = -1/4 < 0$

$\therefore f(t)$ has maximum value at $t = \log_e 4$

(GATE-14 EC): The max value of function $f(x) = \ln(1+x) - x$ (where $x > -1$) occurs at $x =$

Sol: $f'(x) = \frac{1}{1+x} - 1 = 0 \Rightarrow x = 0$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1 < 0$

$\therefore f(x)$ has maximum at $x = 0$

(GATE-14 EE): The max. value of the function $f(x) = xe^{-x}$ in the interval $(0, \infty)$ is

Sol: $f'(x) = (-x\cdot e^{-x} + e^{-x}) = 0 \Rightarrow x = 1$

At $x = 1$, $f''(x) < 0$

$\therefore$ Maximum exists at $x = 1$ and $f(1) = 1\cdot e^{-1} = 1/e$

(GATE-14 EE): Minimum of the real valued function $f(x) = (x-1)^{2/3}$ occurs at $x =$

Sol: $f(x) = \left[(x-1)^{1/3}\right]^2 \quad \therefore f(x) \ge 0$ always

The minimum value of $f(x)$, $f(x) = 0$

$(x-1)^{2/3} = 0 \Rightarrow x = 1$

(GATE-14 EE): The minimum value of $f(x) = x^3 - 3x^2 - 24x + 100$ in interval $[-3, 3]$ is

Sol: $f'(x) = 3x^2 - 6x - 24 = 0 \Rightarrow x = -2, 4$

but $x = 4 \notin [-3, 3]$

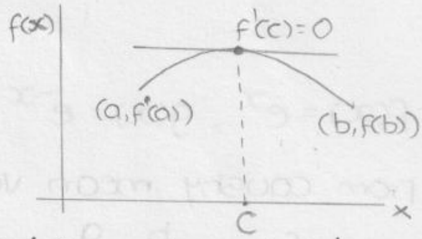
$f''(x) = 6x - 6 \Rightarrow f''(-2) = -18 < 0$ We. get maximum

$\therefore$ Absolute minimum = $\min\{f(-3), f(3)\}$ $\quad f(-3) = 118 \quad f(3) = 28$

$= \min\{118, 28\} = 28$

MEAN VALUE THEOREMS:

Rolle's theorem: If $f(x)$ is a function defined in an interval $[a, b]$ such that

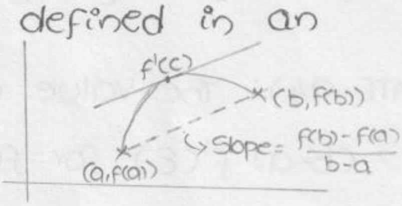


(1) $f(x)$ is continous in $[a,b]$

(2) $f'(x)$ exists in (a,b)

(3) $f(a) = f(b)$ then there exist $c \in (a,b)$ such that $f'(c) = 0$.

Langranges theorem: If $f(x)$ is a function defined in an interval $[a,b]$ such that



(1) $f(x)$ is continous in $[a,b]$

(2) $f'(x)$ is exists in (a,b) then there exist $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \min_{a \le x \le b} f(x) \le \frac{f(b)-f(a)}{b-a} \le \max_{a \le x \le b} f'(x)$$

Cauchy's mean value theorem: If $f(x)$, $g(x)$ are two functions defined in an interval such that

(1) $f(x)$, $g(x)$ are continous in $[a,b]$

(2) $f'(x)$, $g'(x)$ exists in (a,b)

(3) $g'(x) \neq 0 \ \forall\ x \in (a,b)$ then there exist $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

prob: The value 'c' of mean value theorem for $f(x) = x^2 - 5x + 6$ in the interval $[2,3]$ is

Sol: $f(2) = 4 - 10 + 6 = 0$

$f(3) = 9 - 15 + 6 = 0$

From rolle's theorem $\exists\ c \in (2,3)$ such that $f'(c) = 0$

$\Rightarrow 2c - 5 = 0 \Rightarrow c = 5/2$

prob: $f(x) = x^3 - 4x^2 + 4x$ in the interval $[0, 2]$ is

$f(0) = 0 \qquad f(2) = 8 - 16 + 8 = 0$

$\therefore$ From rolle's theorem $\exists\ c \in (0,2)$ such that $f'(c) = 0$

$3c^2 - 8c + 4 = 0 \Rightarrow c = 2,\ 2/3 \qquad c = 2 \notin (0,2)$

$c = 2/3 \in (0,2)$

prob: $f(x) = 1 + x^2$ in $[1,2]$ is

$f(1) = 2 \qquad f(2) = 5 \qquad f(a) \neq f(b)$

from langranges theorem $f'(c) = \frac{5-2}{1} = 3$

$2c = 3 \Rightarrow c = 3/2 \in (1,2)$

prob: $f(x) = e^x,\ g(x) = e^{-x}$ in $[a,b]$

sol: From caushy mean value theorem

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} \Rightarrow -e^{2c} = -e^{a+b}$$

$$\Rightarrow 2c = a+b \Rightarrow c = \frac{a+b}{2} \in (a,b)$$

(GATE-94): The value of $\mathcal{E}$ in the mean value theorem of $f(b) - f(a) = (b-a) f'(\mathcal{E})$ for $f(x) = Ax^2 + Bx + C$ in (a,b) is

sol: $f'(\mathcal{E}) = \frac{f(b) - f(a)}{b-a}$

$$2A\mathcal{E} + B = \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b-a}$$

$$2A\mathcal{E} + B = A(b+a) + B$$

By comparing

$$2\mathcal{E} = a+b \Rightarrow \mathcal{E} = \frac{a+b}{2}$$ **

(GATE-95): If $f(0) = 2$ and $f'(x) = \frac{1}{5-x^2}$, then the lower and upper bounds of $f(1)$ estimated by the mean value theorem are

sol: Let $f(x)$ be defined in $[0,1]$ by lagrange's M.V.T

$\exists\, c \in (0,1)$ such that $f'(c) = \frac{f(1) - f(0)}{1-0}$

$$\frac{1}{5-c^2} = \frac{f(1) - 2}{1}$$

We know that $\text{Min}\{f'(x)\} < f'(c) < \text{Max}\{f'(x)\} \qquad c \in (0,1)$

$$\frac{1}{5} < f(1) - 2 < \frac{1}{4} \Rightarrow 2.2 < f(1) < 2.25$$

TAYLOR SERIES: The Taylor series of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Maclaurin's Series: put $a = 0$

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Important expansions:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

4. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

5. $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

6. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

Prob: In the taylor series of $f(x) = e^x$ about $x = 2$ the co.eff of $(x-2)^4$ is

Sol: $= \frac{1}{4!} f^{IV}(2) = \frac{1}{4!} e^x \Big|_{x=2} = \frac{e^2}{24}$

Prob: In the taylor series of $f(x) = e^x + \sin x$ about $x - \pi$ the co.eff of $(x-\pi)^2$ is

Sol: $= \frac{1}{2!} f''(\pi) = \frac{1}{2!}\left[e^x - \sin x\right]\Big|_{x=\pi} = \frac{e^\pi}{2}$

Prob: The linear approximation for e^{-x} around $x = 2$ is

Sol: $f(x) = f(2) + (x-2) f'(2)$

$= e^{-2} + (x-2)(-e^{-2}) = e^{-2}(3-x)$

(GATE-00 CE): The Taylor series expansion of $\sin x$ about $x = \pi/6$ is

Sol: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$

$f(x) = \sin x$ and $a = \pi/6 \Rightarrow \sin(\pi/6) = 1/2 = f(a)$

$f'(x) = \cos x \Rightarrow f'(a) = \cos \pi/6 = \sqrt{3}/2$

$f''(x) = -\sin x \Rightarrow f''(a) = -\sin \pi/6 = -1/2$

$\sin x = \frac{1}{2} + (x - \pi/6)\sqrt{3}/2 + \frac{(x-\pi/6)^2}{2!}\left(-\frac{1}{2}\right) + \cdots$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 + \cdots$

(GATE-01 CE): Limit of the following series as x approaches $\pi/2$ is $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x$

Sol: $\lim_{x \to \pi/2} \sin x = \sin \pi/2 = 1$

(GATE-09 EC): The Taylor series expansion of $\frac{\sin x}{x-\pi}$ at $x=\pi$ is given by

sol: $$\frac{\sin x}{x-\pi} = \frac{1}{x-\pi}\left[\sin\pi + (x-\pi)(\cos\pi) + \frac{(x-\pi)^2}{2!}(-\sin\pi) + \cdots\right]$$

$$= \frac{1}{x-\pi}\left[0 + (x-\pi)(-1) + \frac{(x-\pi)^2}{2!}(0) + \frac{(x-\pi)^3}{3!} - \cdots\right]$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!}$$

(or)

put $x-\pi = t \Rightarrow x = \pi + t$

$$f = \frac{\sin(\pi+t)}{t} = -\frac{\sin t}{t} \quad \text{about } t=0$$

$$= -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right] = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \cdots$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} \cdots$$

(GATE-14 EC): The Taylor series expansion of $3\sin x + 2\cos x$ is

sol: $$3\sin x + 2\cos x = 3\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] + 2\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= 2 + 3x - x^2 - \frac{x^3}{2} + \cdots$$

PARTIAL DIFFERENTIATION:

If $f(x,y)$ is a function of two variables (x,y) then

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Ex: $f(x) = x^3 + 3xy^2$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 \qquad \frac{\partial f}{\partial y} = 3x(2y) = 6xy$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial f^2}{\partial y^2} = 6x$$

$$\frac{\partial^2 f}{\partial y . \partial x} = 6y \qquad \frac{\partial^2 f}{\partial x . \partial y} = 6y$$

Euler's theorem: If $f(x,y)$ is a homogenous function of degree 'n' then

$$x \cdot \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x . \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Homogeneous function: A function $f(x,y)$ is said to be a homogeneous function of degree 'n' if

$f(x,y) = x^n \cdot \phi(y/x)$ (or) $y^n \psi(x/y)$.

Ex: 1. $f(x,y) = x^3 + 3xy^2 = x^3\left[1+3\left(y/x\right)^2\right]$ deg = 3

2. $f(x,y) = \log_e y - \log_e x = x^0 \log_e(y/x)$ deg = 0

3. $f(x,y) = y\sin(y/x) = y\sin\left(\frac{1}{x/y}\right)$ deg = 1

4. $f(x,y) = \sin\left(\frac{x^4+y^4}{x-y}\right)$ Not homogeneous

Note: If $u(x,y) = f(x,y) + g(x,y) + h(x,y)$, where f, g & h are homogenous function with degree m, n & p respectively then

(a) $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = mf + ng + ph$

(b) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g + p(p-1)h$

Note: If F(u) is homogenous function in two variables x & y with degree 'n', then

(a) $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = F(u)$.

(b) $x^2\cdot\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = F(u)[F'(u)-1]$

Total differentiation: If $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$, then the total derivative of 'z' w.r.to 't' is

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

Total differential co.efficient $dz = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy$

Note: 1. If $f(x,y) = C$ is an implicit function then $\frac{dy}{dx} = \frac{-f_x}{f_y}$

(2) If $z = f(x,y)$ where $x = \phi(u,v)$ & $y = \psi(u,v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

pblm: If $z = e^x\sin y$, where $x = \log t$ & $y = t^2$ then $\frac{dz}{dt} = ?$

Sol: $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$ $x = \log t$, $t = e^x$

$= e^x\cdot\sin y\cdot\frac{1}{t} + e^x\cos y\cdot 2t$

$= \frac{e^x}{t}(\sin y + 2t^2\cos y) = \sin y + 2y\cos y$

prob: The total derivative of x^2y w.r.to 'x', where x & y are connected by the relation $x^2+xy+y^2=1$ is

sol: Let $u=x^2y \Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

$$= 2xy + \cancel{y^2}\, x^2\, dy/dx$$

Given relation $x^2+xy+y^2=1 = f(x,y)$ say

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{2x+y}{x+2y}$$

$$\therefore \frac{du}{dx} = 2xy + x^2\left(-\frac{2x+y}{x+2y}\right) = 2xy - x^2\left(\frac{2x+y}{x+2y}\right)$$

prob: If $u = f(x+cy) + g(x-cy)$ then $\frac{u_{xx}}{u_{yy}} =$

sol: Let $r = x+cy$ $\quad s = x-cy$

$$\frac{\partial r}{\partial x} = 1 \quad \frac{\partial r}{\partial y} = c \quad \frac{\partial s}{\partial x} = 1 \quad \frac{\partial s}{\partial y} = -c$$

$$u = f(r) + g(s) \Rightarrow u_x = f'(r)\cdot\frac{\partial r}{\partial x} + g'(s)\cdot\frac{\partial s}{\partial x} = f'(r) + g'(s)$$

$$u_{xx} = f''(r) + g''(s)$$

$$u_y = f'(r)\cdot\frac{\partial r}{\partial y} + g'(s)\cdot\frac{\partial s}{\partial y} = f'(r)\cdot c + g'(s)(-c)$$

$$\therefore u_{yy} = f''(r)c^2 + g''(s)\cdot c^2$$

$$\frac{u_{xx}}{u_{yy}} = \frac{1}{c^2} = c^{-2}$$

prob: If $\mu = \frac{x^2y}{x^{5/2}+y^{5/2}}$ then $x^2\mu_{xx} + 2xy\,\mu_{xy} + y^2\mu_{yy} =$

sol:

$$\mu = \frac{x^2y}{x^{5/2}+y^{5/2}} = \frac{x^2y}{x^2[x^{1/2}+y^{1/2}]} = \frac{y}{x^{1/2}+y^{1/2}}$$

$$= \frac{y}{y^{1/2}\left[\left(\frac{x}{y}\right)^{1/2}+1\right]} = y^{1/2}\cdot\frac{1}{\left[1+(x/y)^{1/2}\right]}$$

$$\therefore n = 1/2$$

$$x^2\mu_{xx} + 2xy\,\mu_{xy} + y^2\mu_{yy} = n(n-1)\mu = \frac{1}{2}\left(\frac{1}{2}-1\right)\mu$$

$$= -\frac{1}{4}\mu.$$

prob: If $u = \operatorname{cosec}^{-1}\left[\frac{x^{1/4}-y^{1/4}}{x^{1/5}+y^{1/5}}\right]$ then $xu_x + yu_y =$

sol: $\operatorname{cosec} u = \frac{x^{1/4}-y^{1/4}}{x^{1/5}+y^{1/5}}$ $\qquad n = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

$$xu_x + yu_y = n\frac{f(u)}{f'(u)} = \frac{1}{20}\times\frac{\operatorname{cosec} u}{-\operatorname{cosec} u\cdot\cot u} = -\frac{1}{20}\tan u.$$

(GATE-08 ME): Let $f = y^x$ what is $\frac{\partial^2 f}{\partial x \cdot \partial y}$ at $x=2,\ y=1$?

Sol: $f = y^x \Rightarrow \frac{\partial f}{\partial y} = x \cdot y^{x-1}$ (x const)

$$\frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} = y^{x-1} + x \cdot y^{x-1} \cdot \log y \cdot (1) \qquad (y \text{ const})$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\substack{x=2 \\ y=1}} = y^{x-1} + x \cdot y^{x-1} \log y = 1^1 + 2 \cdot 1^1 \log 1 = 1$$

Maxima & Minima for function of two variables:

Let $z = f(x,y)$

consider $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \cdot \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Method: (i) Find p, q, r, s, t.

(ii) Equate p & q to zero for obtaining the stationary points

(iii) At each stationary point find r, s, t.

(a) If $rt - s^2 > 0$, $r > 0$ then the function $f(x,y)$ has a minima at the stationary point

(b) If $rt - s^2 > 0$, $r < 0$ then $f(x,y)$ has a maximum at the stationary point

(c) If $rt - s^2 < 0$, then $f(x,y)$ has no extreme at that point and is known as saddle point.

Constrained Maxima & minima:

Lagranges method of undetermined multipliers:

Let $f(x,y,z)$ & $\phi(x,y,z) = c$ then $F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z) = 0$

$$\left.\begin{aligned} F_x &= 0 \quad \text{i.e} \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \\ F_y &= 0 \quad \text{i.e} \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \\ F_z &= 0 \quad \text{i.e} \quad \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \text{Lagrange's eq}^{n}\text{s}$$

λ - lagranges multiplier.

(GATE-93 ME): The function $f(x,y) = x^2 y - 3xy + 2y + x$ has

Sol: $\frac{\partial f}{\partial x} = 2xy - 3y + 1 = 0$ $\qquad$ $\frac{\partial f}{\partial y} = x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

at $x=1 \Rightarrow 2(1)y-3y+1=0 \Rightarrow y=1 \Rightarrow (1,1)$

at $x=2 \Rightarrow 2(2)y-3y+1=0 \Rightarrow y=-1 \Rightarrow (2,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 2y \Rightarrow \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} = 2 \qquad \frac{\partial f^2}{\partial x \cdot \partial y} = 2x-3$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

$$\left[\frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2}\right] - \left[\frac{\partial^2 f}{\partial x \cdot \partial y}\right]^2 < 0 \quad \text{at both } (1,1) \;\&\; (2,-1)$$

$\therefore$ NO extremum

(GATE-02) : The function $f(x,y) = 2x^2+2xy-y^3$ has stationary points

Sol:
$$\frac{\partial f}{\partial x} = 4x+2y=0 \qquad \frac{\partial f}{\partial y} = 2x-3y^2=0$$

$$\Rightarrow 2x=-y \qquad \Rightarrow -y-3y^2=0 \Rightarrow y=0, -1/3$$

$$\therefore x = 0, 1/6$$

$\therefore (0,0), (1/6, -1/3)$ are the stationary points

(GATE-07 PI): For the function $f(x,y) = x^2-y^2$ defined on R^2, the point $(0,0)$ is

Sol:
$$\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x=0 \qquad \frac{\partial f}{\partial y} = \cancel{0}-2y=0 \Rightarrow y=0$$

$$(x,y) = (0,0)$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = -2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \cdot \partial y}\right)^2 = 2(-2) = -4 < 0$$

$\therefore$ Neither maxima nor minima

prob: If $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$ then $x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$

Sol:
$$e^u = \cancel{\log} \frac{x^4+y^4}{x+y} = P \text{ say} \qquad n=3$$

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = nP \Rightarrow x \cdot e^u \cdot \frac{\partial u}{\partial x} + y \cdot e^u \cdot \frac{\partial u}{\partial y} = ne^u$$

$$\Rightarrow x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 3$$

Prob: The minimum value of $x^2+y^2+z^2$ where $x+y+z=1$ is

Sol:
$$f = x^2+y^2+z^2 \qquad \phi = x+y+z-1 \qquad x+y+z=1$$

$$F = f+\lambda\phi \qquad x = 1/3 \quad y = 1/3 \quad z = 1/3$$

$$\left.\begin{aligned} f_x &= 2x+\lambda=0 \Rightarrow \lambda=-2x \\ f_y &= 2y+\lambda=0 \Rightarrow \lambda=-2y \\ f_z &= 2z+\lambda=0 \Rightarrow \lambda=-2z \end{aligned}\right\} \Rightarrow x=y=z$$

$$f_{min} = x^2+x^2+x^2 = 3x^2 = 3(1/3) = 1$$

Integration: If $\frac{d}{dx} f(x) = F(x)$ then $\int F(x) = f(x) + C$

$f(x)$	$\int f(x)$	$f(x)$	$\int f(x)$
x^n	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$ (or) $-\cos^{-1}x$
$1/x$	$\log_e x$	$\frac{1}{1+x^2}$	$\tan^{-1}x$ (or) $-\cot^{-1}x$
$1/x^2$	$-1/x$	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}x$ (or) $-\operatorname{cosec}^{-1}x$
$\sqrt{x}$	$\frac{2}{3}x\sqrt{x}$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(x/a)$
$1/\sqrt{x}$	$2\sqrt{x}$	$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}(x/a)$
e^x	e^x	$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}(x/a)$
a^x	$a^x/\log_e a$		
$\sin x$	$-\cos x$		
$\cos x$	$\sin x$		
$\sec^2 x$	$\tan x$		
$\operatorname{cosec}^2 x$	$-\cot x$		
$\sec x . \tan x$	$\sec x$		
$\operatorname{cosec} x . \cot x$	$-\operatorname{cosec} x$		

$* \int \frac{f'(x)}{f(x)} dx = \log_e |f(x)|$ *

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\log_e \cos x = \log_e |\sec x|$

$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log_e |\sin x|$

$\int \sec x \, dx = \int \frac{\sec x . (\sec x + \tan x)}{\sec x + \tan x} dx = \log_e |\sec x + \tan x|$

$\int \operatorname{cosec} x \, dx = \log_e |\operatorname{cosec} x - \cot x|$

Integration by parts:

ILATE

$\int uv \, dx = u \int v \, dx - \int u' \left(\int v \, dx \right) dx$

$= uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Ex: $\int \log_e x \, dx = \log_e x \int dx - \int \frac{1}{x} \cdot x \, dx = x(\log_e x - 1)$

$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x = e^x(x^2 - 2x + 2)$

$\int x e^x dx = x e^x - e^x = e^x(x-1)$

$$\int \frac{1}{1-\cos x}\,dx = \int \frac{(1+\cos x)}{(1-\cos x)(1+\cos x)}\,dx$$

$$= \int \frac{(1+\cos x)}{1-\cos^2 x}\,dx = \int \frac{1+\cos x}{\sin^2 x}\,dx$$

$$= \int \operatorname{cosec}^2 x\,dx + \int \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\,dx$$

$$= -\cot x + \int \cot x \cdot \operatorname{cosec} x\,dx = -\cot x - \operatorname{cosec} x$$

prob: $\int \frac{x^4}{x^2+1}\,dx = \int \frac{x^4-1+1}{x^2+1}\,dx$

$$= \int \frac{(x^2+1)(x^2-1)+1}{x^2+1}\,dx = \int (x^2-1)\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^3}{3} - x + \tan^{-1} x$$

prob: $\int \frac{\sin^8 x}{\cos^{10} x}\,dx = \int \tan^8 x \cdot \sec^2 x\,dx$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int t^8\,dt = \frac{t^9}{9} = \frac{\tan^9 x}{9}$$

prob: $\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{1-\cos^2 x}{1+\cos x}\,dx$

$$= \int \frac{(1-\cos x).(1+\cos x)}{1+\cos x}\,dx = \int (1-\cos x)\,dx$$

$$= x - \sin x$$

prob: $\int \frac{x^2-a^2}{x^2+a^2}\,dx = \int \frac{x^2+a^2-2a^2}{x^2+a^2}\,dx$

$$= \int dx - 2a^2 \int \frac{1}{x^2+a^2}\,dx = x - 2a^2 \cdot \frac{1}{a} \cdot \tan^{-1}(x/a)$$

$$= x - 2a\tan^{-1}(x/a)$$

prob: $\int \frac{x}{\sqrt{1-x^2}}\,dx =$

Let $1-x^2 = t \Rightarrow -2x\,dx = dt \Rightarrow x\,dx = \frac{-dt}{2}$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{t}}\cdot\left(\frac{-dt}{2}\right) = \frac{-1}{2}\int \frac{1}{\sqrt{t}}\,dt$$

$$= -\frac{1}{2}\cdot 2\sqrt{t} = -\sqrt{t} = -\sqrt{1-x^2}$$

prob: $\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$

Let $x+1 = t \Rightarrow dx = dt$

$$= \int \frac{1}{t^2+1}\,dt = \tan^{-1} t = \tan^{-1}(1+x)$$

DEFINITE INTEGRALS: If $\int f(x)\,dx = F(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

Note: $$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)\,dx\right] = f(v)\cdot\frac{dv}{dx} - f(u)\cdot\frac{du}{dx}$$

properties of definite integral:

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) If $c \in (a,b)$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(iii) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(iv) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

(v) $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ If } f(x) \text{ is even} \\ 0 & ; \text{ If } f(x) \text{ is odd} \end{cases}$$

(vi) $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ If } f(2a-x) = f(x) \\ 0 & ; \text{ If } f(2a-x) = -f(x) \end{cases}$$

(vii) $\int_0^a x\cdot f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$; If $f(a-x) = f(x)$

(viii) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$; If $f(x+a) = f(x)$

(ix) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left[\frac{n-1}{n}\times\frac{n-3}{n-2}\times\cdots\times\frac{2}{3} \text{ (or) } \frac{1}{2}\right]K$$

where $K = \begin{cases} 1 & ; \text{ if } n \text{ is } \cancel{\text{even}} \text{ odd} \\ \pi/2 & ; \text{ if } n \text{ is } \cancel{\text{odd}} \text{ even} \end{cases}$

(x) $$\int_0^{\pi/2} \sin^m x\cdot\cos^n x\,dx = \left\{\frac{[(m-1)(m-3)\cdots 2 \text{ (or) } 1][(n-1)(n-3)\cdots 2 \text{ (or) } 1]}{[(m+n)(m+n-2)(m+n-4)\cdots]}\right\}\cdot K$$

where $K = \begin{cases} \pi/2 & ; \text{ when } m \,\&\, n \text{ are even} \\ 1 & ; \text{ Otherwise} \end{cases}$

Prob: $\int_{-2\pi}^{2\pi} \cos^6 x\,dx$

Sol: $f(x) = \cos^6 x$ is a even function $\int_{-2\pi}^{2\pi} \cos^6 x\,dx = 2\int_0^{2\pi} \cos^6 x\,dx$

$f(2\pi - x) = [\cos(2\pi - x)]^6 = \cos^6 x$

$$\int_{-2\pi}^{2\pi} \cos^6 x\,dx = 2\int_0^{2\pi} \cos^6 x\,dx = 2\cdot 2\int_0^{\pi} \cos^6 x\,dx$$

$$= 4 \cdot 2 \int_0^{\pi/2} \cos^6 x \, dx \qquad \cos^6(\pi - x) = [-\cos x]^6 = \cos^6 x$$

$$= 8 \times \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{4}$$

prob: $\int_1^3 x\sqrt{x^2-1}\, dx$

Let $x^2 - 1 = t^2$ $\quad x = 1 \Rightarrow t = 0$

$2x\,dx = 2t\,dt$ $\quad x = 3 \Rightarrow 3^2 - 1 = t^2$

$x\,dx = t\,dt$ $\quad t = \sqrt{8}$

$$= \int_0^{\sqrt{8}} t \cdot t\,dt = \int_0^{\sqrt{8}} t^2\,dt$$

$$= \frac{t^3}{3}\Big|_0^{\sqrt{8}} = \frac{(\sqrt{8})^3}{3} = \frac{8\sqrt{8}}{3}$$

prob: $\int_0^1 x \cdot \frac{\sin^{-1}x}{\sqrt{1-x^2}}\, dx$

Let $\sin^{-1} x = t \Rightarrow x = \sin t$ $\quad x = 0 \Rightarrow t = \sin^{-1} 0 = 0$

$dx = \cos t\, dt$ $\quad x = 1 \Rightarrow t = \sin^{-1} 1 = \pi/2$

$$= \int_0^{\pi/2} \sin t \cdot \frac{t}{\cos t} \cdot \cos t\, dt$$

$$= \int_0^{\pi/2} t \sin t\, dt = \left[t(-\cos t) - 1(-\sin t)\right]_0^{\pi/2}$$

$$= \left[-t\cos t + \sin t\right]_0^{\pi/2} = [1 - 0] = 1$$

prob: $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx = \int_0^1 \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}\, dx$

$$= \int_0^1 \frac{(1-x)}{\sqrt{1-x^2}}\, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx - \frac{1}{2}\int_0^1 \frac{2x}{\sqrt{1-x^2}}\, dx$$

$$= \left[\sin^{-1} x + \sqrt{1-x^2}\right]_0^1 = \sin^{-1} 1 - \sqrt{1-0} = \frac{\pi}{2} - 1$$

prob: $\int_0^2 |1-x|\, dx = \int_0^1 (1-x)\, dx - \int_1^2 (1-x)\, dx$

$(1-x) > 0$ for $0 < x < 1$

$(1-x) < 0$ for $1 < x < 2$

$$= \left[x - \frac{x^2}{2}\right]_0^1 - \left[x - \frac{x^2}{2}\right]_1^2$$

$$= \left[1 - \frac{1}{2}\right] - \left[2 - \frac{4}{2} - 1 + \frac{1}{2}\right] = 1$$

prob: $\int_{-\pi}^{\pi} (|x| + \sin x)\, dx = \int_{-\pi}^{\pi} |x|\, dx + \int_{-\pi}^{\pi} \sin x\, dx$

$|x|$ even

$\sin x$ odd

$$= 2\int_0^{\pi} |x|\, dx + 0$$

$$= 2\left(x^2/2\right)_0^{\pi} = \pi^2$$

prob: $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\, dx$

Sol: Let $f(x) = \tan x \Rightarrow f(0 + \pi/2 - x) = \tan\left(\frac{\pi}{2} - x\right) = \cot x$

$$\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\, dx = \int_0^{\pi/2} \frac{f(x)}{f(x) + f(0 + \pi/2 - x)} = \frac{\pi/2 - 0}{2}$$

$$= \pi/4$$

prob: $\int_0^n [x]\, dx =$

sol: $I = \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \dots$

$= (x]_1^2 + 2(x)_2^3 + 3(x)_3^4 + \dots$

$= (2-1) + 2(3-1) + 3(4-3) + 4(5-4) + \dots$

$= 1+2+3+4\dots = \frac{n(n+1)}{2}$

prob: $\int_0^{\pi/2} \log(\tan x)\, dx$ is

sol: $\int_0^{\pi/2} \log(\tan x)\, dx = I$ say $\qquad \int_0^a f(x)dx = \int_0^a f(a-x)\, dx$

$I = \int_0^{\pi/2} \log[\tan(\pi/2 - x)]\, dx = \int_0^{\pi/2} \log(\cot x)\, dx$

$I + I = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\, dx$

$2I = \int_0^{\pi/2} \log(\tan x \cdot \cot x)\, dx \qquad \tan x \cdot \cot x = 1$

$2I = \int_0^{\pi/2} \log(1)\, dx = 0 \Rightarrow I = 0$

prob: $\int_{-\pi}^{\pi} \log \frac{(1+\sin x)}{(1-\sin x)}\, dx = I$

$f(-x) = \log\left(\frac{1-\sin x}{1+\sin x}\right) = -\log\left(\frac{1+\sin x}{1-\sin x}\right) = -f(x)$

odd function $\therefore I = 0$

(GATE-97): If $\phi(x) = \int_0^{x^2} \sqrt{t}\, dt$ then $\frac{d\phi}{dx} =$

sol: $\phi = \frac{2}{3} t\sqrt{t}\Big|_0^{x^2} = \frac{2}{3} x^2\sqrt{x^2} - 0 = \frac{2}{3} x^3$

$\frac{d\phi}{dx} = \frac{2}{3} \cdot 3 \cdot x^2 = 2x^2$

(GATE-01): The value of the integral $I = \int_0^{\pi/4} \cos^2 x\, dx$

sol: $I = \int_0^{\pi/4} \cos^2 x\, dx = \int_0^{\pi/4} \left[\frac{1+\cos 2x}{2}\right] dx$

$= \frac{x}{2}\Big|_0^{\pi/4} + \frac{\sin 2x}{4}\Big|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4} \cdot \sin(\pi/2)$

$= \frac{\pi}{8} + \frac{1}{4}$

(GATE-02): $\int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{1+\cos x}\, dx = I$

$f(-x) = -f(x) \Rightarrow$ odd function $\therefore I = 0$

(GATE-05 IN): $\int_{-1}^{1} \frac{1}{x^2}dx$

$= 2\int_0^1 \frac{1}{x^2}dx$

$= 2\cdot\left(-\frac{1}{x}\right)_0^1 = \infty$

Sol: $= \int_{-1}^{0} \frac{1}{x^2}dx + \int_0^{1}\frac{1}{x^2}dx = \left(\frac{x^{-1}}{-1}\right)_{-1}^{0} + \left(\frac{x^{-1}}{-1}\right)_0^{1}$

$= (\infty - 1) + (1 - \infty) = \infty$

(GATE-08 PI): $\int_{-\pi/2}^{\pi/2} (x\cos x)dx$

Sol: $f(x) = x\cos x \Rightarrow f(-x) = -x\cos(-x) = -x\cdot\cos x = -f(x)$

$\Rightarrow$ odd function $\quad \therefore I = 0$

(GATE-10 EE): $P = \int_0^1 xe^x dx$

Sol: $\int_0^1 xe^x dx = \left[xe^x - e^x\right]_0^1 = (e-e) + (e^0 - 0) = 1$

(GATE-11 CS): $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}dx = I$

Sol: $I = \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}dx = \int_0^{\pi/2} e^{2ix}dx = \left[\frac{e^{2ix}}{2i}\right]_{\pi/2}^{0}$

$= \frac{e^{i\pi}}{2\cancel{\pi}i} - \frac{1}{2\cancel{\pi}i} = \frac{\cos\pi + i(0)}{2\cancel{\pi}i} - \frac{1}{2\cancel{\pi}i} = -\frac{1}{2i} - \frac{1}{2i} = \frac{-2}{2i}$

$= -\frac{1}{i} = i$

(GATE-13 ME): $\int_1^e \sqrt{x}\cdot\ln(x)\cdot dx$

$\int u\,v = uv - \int v\,du$

$u = \ln x$

Sol: $= \left[\left(\ln x \cdot \frac{x^{3/2}}{3/2}\right)_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^{3/2}}{3/2}dx\right]$

$= \frac{2}{3}\cdot e^{3/2} - 0 - \int_1^e \frac{2}{3}\cdot x^{1/2}dx$

$= \frac{2}{3}e^{3/2} - \frac{2}{3}\left(\frac{x^{3/2}}{3/2}\right)_1^e = \frac{2}{3}e^{3/2} - \frac{4}{9}e^{3/2} + \frac{4}{9} = \frac{2}{9}e^{3/2} + \frac{4}{9}$

(GATE-13 CE): $\int_0^{\pi/6} \cos^4 3\theta\cdot\sin^3 6\theta\cdot d\theta$

Sol: Let $3\theta = t \Rightarrow \theta = t/3 \Rightarrow d\theta = dt/3$

$\theta = 0 \Rightarrow t = 0$

$\theta = \pi/6 \Rightarrow t = \pi/2$

$I = \int_0^{\pi/2} \cos^4 t\cdot\sin^3 2t\cdot dt/3$

$= \frac{1}{3}\int_0^{\pi/2} \cos^4 t\cdot(2\sin t\cdot\cos t)^3 dt$

$= \frac{8}{3}\int_0^{\pi/2} \cos^7 t\cdot\sin^3 t\,dt = \frac{8}{3}\left[\frac{(6\times4\times2)(2)}{10\times8\times6\times4\times2}\right] = 1/15$

(GATE-14 ME): $\int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2 + \cos(x-1)}dx$

Sol: $f(2-x) = \frac{(2-x-1)^2\sin(2-x-1)}{(2-x-1)^2 + \cos(2-x-1)} = \frac{(x-1)^2(-\sin(x-1))}{(x-1)^2 + \cos(x-1)} = -f(x)$

According to $\int_0^{2a} f(x)\,dx = 0$ if $f(2a-x) = -f(x)$

$\therefore I = 0$

Improper Integrals:

First Kind: $\int_a^b f(x)\,dx$ if $a=-\infty$ (or) $b=\infty$ (or) both

i.e $\int_{-\infty}^{b} f(x)\,dx$ or $\int_a^\infty f(x)\,dx$ (or) $\int_{-\infty}^{\infty} f(x)\,dx$

Second Kind: $\int_a^b f(x)\,dx$ if a & b are finite but $f(x)$ is infinite for some $x \in [a,b]$

Ex: $\int_{-1}^{1} \log(1+x)\,dx$; $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$; $\int_0^3 \frac{1}{x^2-5x+4}\,dx$

convergence of an improper integral:

(i) If $\int_a^b f(x)\,dx$ = finite, then it is a convergent improper integral

(ii) If $\int_a^b f(x)\,dx$ = Infinite, then it is a divergent improper integral

prob: $\int_0^\infty \frac{1}{a^2+x^2}\,dx$

sol:
$$I = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^\infty = \frac{1}{a}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$
$$= \frac{1}{a}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2a}$$ finite (Convergent)

prob: $\int_{-\infty}^{0} e^{ax}\cos px\,dx$

sol:
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$
$$I = \left[\frac{e^{ax}}{a^2+p^2}(a\cos px + p\sin px)\right]_{-\infty}^{0} = \frac{a}{a^2+p^2} - 0 = \frac{a}{a^2+p^2}$$ finite (Convergent)

prob: $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$

sol:
$$I = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}}\,dx$$
$$= \underbrace{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx}_{\text{even}} + \underbrace{\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx}_{\text{odd}}$$
$$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + 0$$
$$= 2\sin^{-1}x\Big|_0^1 = 2\sin^{-1}1 = 2\cdot\pi/2 = \pi$$ finite

comparision test: For 1st kind of improper integrals

(a) Let $0 \le f(x) \le g(x)$ then

(i) $\int_a^b f(x)dx$ converges if $\int_a^b g(x)$ is convergent.

(ii) $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)$ is divergent.

(b) Limit form: Let $f(x)$ & $g(x)$ be two positive functions such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$ (non-zero finite) then $\int_a^b f(x)dx$ & $\int_a^b g(x)dx$ both are converge or diverge together.

prob: $\int_1^\infty e^{-x^2}dx$

sol: $e^{x^2} \ge e^x \;\forall\; x \ge 1 \Rightarrow e^{-x^2} \le e^{-x} \;\forall\; x \le 1$

$\int_1^\infty e^{-x}dx = -e^{-x}\Big|_1^\infty = 1 \;\therefore$ convergent

$\therefore \int_1^\infty e^{-x^2}dx$ is also convergent.

prob: $\int_2^\infty \frac{1}{\log x}dx$

sol: $\log x < x \;\forall\; x \ge 2 \Rightarrow \frac{1}{\log x} > \frac{1}{x} \;\forall\; x \ge 2$

$\int_2^\infty \frac{1}{x}dx = \log x\Big|_2^\infty =$ infinite $\Rightarrow$ divergent

$\therefore \int_2^\infty \frac{1}{\log x}dx$ is also divergent.

Comparison test: For 2nd kind of improper integrals

Limit form: Let $f(x)$ & $g(x)$ be two positive functions suchthat

(i) 'a' is a point of discontinuity and $\lim_{x\to a^+} \frac{f(x)}{g(x)} = l_1$ (non zero finite)

(ii) 'b' is a point of discontinuity and $\lim_{x\to b^-} \frac{f(x)}{g(x)} = l_2$ (non zero finite)

then $\int_a^b f(x)dx$ & $\int_a^b g(x)dx$ both converge (or) diverge together.

prob: $\int_1^2 \frac{\sqrt{x}}{\log x}dx$

$\frac{1}{\log x} > \frac{1}{x} \;\forall\; x > 1 \Rightarrow \frac{\sqrt{x}}{\log x} > \frac{1}{\sqrt{x}} \;\forall\; x > 1$

$\int_1^2 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_1^2 = 2\sqrt{2} - 2 =$ finite $\Rightarrow$ convergent

$\int_1^2 \frac{\sqrt{x}}{\log x} dx$ may / may not be convergent

2nd Method: $f(x) = \frac{\sqrt{x}}{\log x}$ Let $g(x) = \frac{1}{x \log x}$

$\lim_{x \to 1^+} \frac{f(x)}{g(x)} = 1\sqrt{1} = 1$

Let $\log x = t$ $\quad x=1 \Rightarrow t=0$; $x=2 \Rightarrow t = \log 2$

$\frac{1}{x} dx = dt$

$$\int_1^2 \frac{1}{x \log x} dx = \int_0^{\log 2} \frac{1}{t} \cdot dt$$

$$= \log t \Big|_0^{\log 2} = \text{infinite (diverge)}$$

$\therefore \int_1^2 \frac{\sqrt{x}}{\log x} dx$ is also divergent.

(GATE-00): $\lim_{a \to \infty} \int_1^a x^{-4} dx$

(a) diverges (b) converges to $1/3$ c) converges to $-1/a^3$ d) converges to 0

Sol: $\lim_{a \to \infty} \int_1^a x^{-4} dx = \lim_{a \to \infty} \left[\frac{x^{-3}}{-3}\right]_1^a = \lim_{a \to \infty} \left[\frac{a^{-3}}{-3} - \frac{1}{-3}\right]$

$$= \lim_{a \to \infty} -\frac{1}{3a^3} + \frac{1}{3} = 0 + \frac{1}{3} = 1/3$$

(GATE-05): $\int_1^\infty x^{-3} dx$

$$= \frac{x^{-2}}{-2}\Big|_1^\infty = -\frac{1}{2}\left[\frac{1}{\infty} - 1\right] = 1/2$$

(GATE-08 ME): Which of the following integrals is unbounded?

(a) $\int_0^{\pi/4} \tan x\, dx$ (b) $\int_0^\infty \frac{1}{1+x^2} dx$ (c) $\int_0^\infty x e^{-x} dx$ (d) $\int_0^1 \frac{1}{1-x} dx$

Sol: (a) $\int_0^{\pi/4} \tan x\, dx = \log \sec x\Big]_0^{\pi/4} = \log\sqrt{2} - \log 1 = \log\sqrt{2}$

(b) $\int_0^\infty \frac{1}{1+x^2} dx = \tan^{-1} x\Big|_0^\infty = \tan^{-1}\infty - \tan^{-1} 0 = \pi/2 - 0 = \pi/2$

(c) $\int_0^\infty x e^{-x} dx = \left[x \cdot \frac{e^{-x}}{-1} - \frac{e^{-x}}{(-1)^2}\right]_0^\infty = 0 - 0 + \frac{1}{(-1)^2} = 1$

(d) $\int_0^1 \frac{1}{1-x} dx = \left[\frac{\log(1-x)}{-1}\right]_0^1 = \log 0 + \log 1 = \infty + 0 = \infty$ $\Rightarrow$ unbounded

(GATE-10 ME): $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$ $\Rightarrow$ even function

$$= 2\int_0^\infty \frac{1}{1+x^2} dx = 2 \cdot \pi/2 = \pi$$

Gamma function: $\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1} dx \quad (n>0)$.

Note: (i) $\Gamma 1 = 1$

(ii) $\Gamma 1/2 = \sqrt{\pi}$

(iii) $\Gamma(n+1) = n\Gamma n \quad \forall\, n>0$

(iv) $\Gamma(n+1) = n! \quad \forall\, n \in Z^+$

(v) $\int_0^{\infty} e^{-ax} \cdot x^{n-1} dx = \frac{\Gamma n}{a^n}$

prob: $\int_0^{\infty} e^{-x^2} dx =$

Sol: Let $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{1}{2} t^{-1/2} dt$

$$I = \int_0^{\infty} e^{-t} \cdot \frac{1}{2} t^{1/2} dt = \frac{1}{2}\int_0^{\infty} e^{-t} \cdot t^{1/2-1} dt = \frac{1}{2}\Gamma 1/2 = \frac{\sqrt{\pi}}{2}$$

prob: $\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

prob: $\int_0^1 (x \log x)^4 dx =$

Sol: Let $\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$x=0 \Rightarrow t=-\infty$
$x=1 \Rightarrow t=0$

$$I = \int_{-\infty}^{0} [e^{-t}(-t)]^4 (-e^{-t} dt) = \int_0^{\infty} e^{-5t} \cdot t^{5-1} dt$$

$$= \frac{\Gamma 5}{5^5} = \frac{4!}{5^5}$$

$\log_a 0 = \begin{cases} \infty \ ; & a<1 \\ -\infty \ ; & a>1 \end{cases}$

Beta function: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad (m>0,\ n>0)$

Note: (i) $\beta(m,n) = \beta(n,m)$

(ii) $\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$

(iii) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{n+m}} dx$

(iv) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$

i.e $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$(p>-1/2,\ q>-1/2)$

prob: $\int_0^2 x^7 (16-x^4)^{10} dx =$

Soln: Let $x^4 = 16t \Rightarrow 4x^3 dx = 16\,dt \Rightarrow x^3 dx = 4\,dt$

$x=0 \Rightarrow t=0$
$x=2 \Rightarrow 16=16t$
$t=1$

$$I = \int_0^1 16t(16-16t)^{10} 4\,dt = 16^{11} \cdot 4\int_0^1 t(1-t)^{10} dt$$

$$= 4 \times 16^{11} \times \beta(2,11)$$

$$= 4 \times 16^{11} \times \frac{\Gamma 2 \cdot \Gamma 11}{\Gamma 13} = 4 \times 16^{11} \times \frac{1 \times 10!}{12!}$$

prob: $\int_0^\infty \left(\frac{x}{1+x^2}\right)^3 dx$

sol: Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$I = \int_0^{\pi/2}\left(\frac{\tan\theta}{\sec^2\theta}\right)^3 \sec^2\theta\, d\theta = \int_0^{\pi/2}\frac{\tan^3\theta}{\sec^4\theta}\cdot d\theta$$

$$= \int_0^{\pi/2}\sin^3\theta\cdot\cos\theta\, d\theta = \frac{1}{2}\beta\left(\frac{3+1}{2}, \frac{1+1}{2}\right) = \frac{1}{2}\beta(2,1)$$

$$= \frac{1}{2}\times\frac{\Gamma 2\cdot\Gamma 1}{\Gamma 3} = \frac{1}{2}\times\frac{1\times 1}{2} = \frac{1}{4}$$

(GATE-94 ME): $\int_0^\infty e^{-y^3}\cdot y^{1/2}\, dy$

sol: put $y^3 = t \Rightarrow 3y^2\, dy = dt$

$\Rightarrow t^{1/3} = y$ $\qquad y^{1/2}dy = \frac{1}{3}y^{-3/2}dt = \frac{1}{3}t^{-1/2}dt$

$$\int_0^\infty e^{-t}\cdot\frac{1}{3}t^{-1/2}dt = \frac{1}{3}\int_0^\infty e^{-t}\cdot t^{\frac{1}{2}-1}dt = \frac{1}{3}\Gamma(1/2) = \frac{\sqrt{\pi}}{3}$$

(GATE-10 PI): $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2/2}dx$ even function

sol: put $x^2/2 = t \Rightarrow x\, dx = dt$

$\Rightarrow \sqrt{2t}\cdot dx = dt \Rightarrow dx = \frac{1}{\sqrt{2}}t^{-1/2}dt$

$$I = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-t}\frac{1}{\sqrt{2}}t^{-1/2}dt = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-t}\cdot t^{\frac{1}{2}-1}dt$$

$$= \frac{1}{\sqrt{\pi}}\cdot\Gamma(1/2) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

MULTIPLE INTEGRALS:

double integral: An integral of the form $\int_c^d \boxed{\int_a^b f(x)dx}\, dy$ is called double integral. To evaluate this integral we have to follow the following steps.

1. First we evaluate the inner box & then the outer box.
2. while integrating w.r.to one variable we treat the other variable as constant
3. If the limits are constant limits then we need to follow the order in which dx & dy are present.
4. If the limits are variable limits then we need to follow the order in which the limits are present. and they are limits for that variable which is not present in the limits.

prob: $\int_{x=0}^{1}\int_{y=0}^{x^2}(x+y)\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=0}^{x^2} x\cdot dy + \int_{y=0}^{x^2} y\,dy\right]dx$

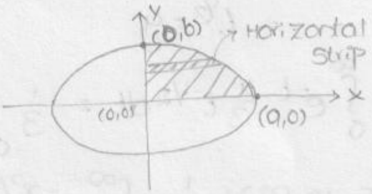
$$= \int_{x=0}^{1}\left[x\cdot(y)_0^{x^2} + \left(\frac{y^2}{2}\right)_0^{x^2}\right]dx$$

$$= \int_{x=0}^{1}\left[x^3 + \frac{x^4}{2}\right]dx = \left[\frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \frac{1}{4} + \frac{1}{10}$$

prob: $\int_0^4\int_0^{x^2} e^{y/x}\cdot dy\cdot dx = \int_0^4\left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$

$$= \int_0^4 [xe^x - x]\,dx = (xe^x - e^x)_0^4 - \left(\frac{x^2}{2}\right)_0^4 = 3e^4 - 7$$

prob: The value of $\iint_R xy\,dx\,dy$. where R is a region in the +ve quadrant at the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is



Sol: Consider horizontal strip

$x = 0$ to $x = \frac{a}{b}\sqrt{b^2 - y^2}$

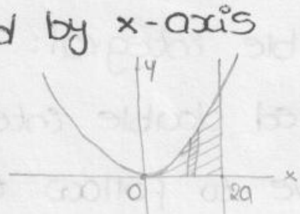
$y = 0$ to $y = b$

$$\iint_R xy\,dx\,dy = \int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy\cdot dx\cdot dy$$

$$= \int_{y=0}^{b} y\left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy = \int_{y=0}^{b}\frac{y}{2}\left[\frac{a^2}{b^2}(b^2-y^2) - 0^2\right]dy$$

$$= +\frac{a^2}{2}\int_{y=0}^{b}(y - y^3)\,dy = +\frac{a^2}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^b = +\frac{a^2}{2}\left[\frac{b^2}{4}\right] = \frac{a^2b^2}{8}$$

prob: $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis $x = 2a$, $x^2 = 4ay$ $(a>0)$



Sol: Consider vertical strip

$y = 0$ to $y = x^2/4a$, $x = 0$ to $2a$

$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dx\cdot dy$$

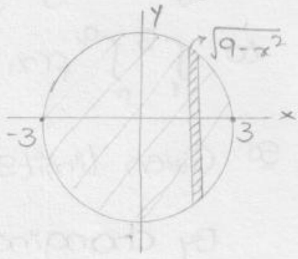
$$= \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx = \int_{x=0}^{2a} x\cdot\frac{x^4}{32a^2}\,dx$$

$$= \frac{1}{32a^2}\int_{x=0}^{2a} x^5\,dx = \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{1}{32a^2}\cdot\frac{64a^6}{6} = \frac{a^4}{3}$$

Note: $\iint_R dx\,dy$ always represents area bounded by R

$\iiint_V dx\,dy\,dz$ always represents volume bounded by V.

Prob: $\iint_R dx\,dy$ where R is region bounded by the circle $x^2+y^2=9$



Sol: Consider vertical strip

$$y=-\sqrt{9-x^2} \text{ to } \sqrt{9-x^2}, \quad x=-3 \text{ to } 3$$

$$\iint_R dx\,dy = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy \cdot dx$$

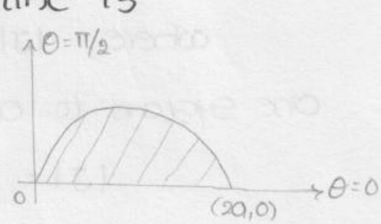
$$= \int_{-3}^{3} (y)_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} 2\sqrt{9-x^2}\,dx = 2\cdot 2\int_0^3 \sqrt{9-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 = 9\pi$$

Prob: The value of $\iint_R r^2\sin\theta\,dr\,d\theta$, where 'r' is the region bounded by the semi circle $r=2a\cos\theta$ above initial line is

Sol: $r=0$ to $2a\cos\theta$, $\theta=0$ to $\pi/2$



$$\iint_R r^2\sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\sin\theta\int_{r=0}^{2a\cos\theta} r^2\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\sin\theta\left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{8a^3\cos^3\theta\cdot\sin\theta}{3}\,d\theta$$

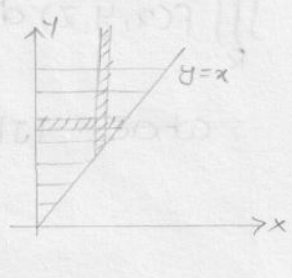
Let $\cos\theta = t \Rightarrow -\sin\theta\,d\theta = dt$ | $\theta=0 \Rightarrow t=1$, $\theta=\pi/2 \Rightarrow t=0$

$$= \int_1^0 \frac{8a^3}{3}t^3(-dt) = \frac{8a^3}{3}\int_0^1 t^3\,dt$$

$$= \frac{8a^3}{3}\left(\frac{t^4}{4}\right)_0^1 = \frac{8a^3}{3}\cdot\frac{1}{4} = 2a^3/3$$

Change of Order of integration:

Prob: $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$



Sol: Given limits are $y=x$ to $y=\infty$
$x=0$ to $x=\infty$

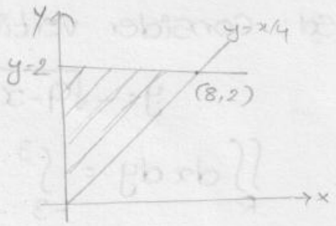
Horizontal strip $y=0$ to $y=\infty$, $x=0$ to y

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = \int_{y=0}^{\infty}\left[\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\right]dy = \int_0^\infty e^{-y}\,dy$$

$$= -e^{-y} = [0+1] = 1$$

(GATE 05)

prob: By changing the order of integration $\int_0^8 \int_{x/4}^2 f(x,y)\,dy\,dx$ leads to $\int_r^s \int_p^q f(x,y)\,dx\,dy$ then q is



sol: Given limits $y = x/4$ to 2, $x = 0$ to 8

By changing limits, $y = 0$ to 2, $x = 0$ to $4y$

$$\therefore q = 4y$$

prob: Evaluate $\int_0^2 \int_0^x \int_0^{\sqrt{x+y}} z\,dz\,dy\,dx$

sol: $I = \int_0^2 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\,dx = \int_0^2 \left[\frac{(x+y)^2}{4}\right]_0^x dx = 2$

Change of variables: In a double integral, if $x = f(u,v)$ & $y = g(u,v)$ then $\iint_R \phi(x,y) = \iint_R \phi(f,g)\,|J|\,du\,dv = \iint_R \phi(u,v)\,|J|\,du\,dv$

where, $|J| \to$ Jacobian of transformation used to transform one system to other.

$$|J| = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

cartesian form $(x,y) \longrightarrow$ polar form (r,θ):

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\Rightarrow \iint_R \phi(x,y)\,dx\,dy = \iint_R \varphi(r,\theta)\,r\,dr\,d\theta$$

→ In a triple integral if $x = f(u,v,\omega)$, $y = g(u,v,\omega)$, $z = h(u,v,\omega)$ then $\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R \varphi(u,v,\omega)\,|J|\,du\,dv\,d\omega$

$$\text{where } |J| = J\left(\frac{x,y,z}{u,v,\omega}\right) = \frac{\partial(x,y,z)}{\partial(u,v,\omega)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \omega} \end{vmatrix}$$

cartesian (x,y,z) to cylindrical polar form (r,θ,z):

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\iiint_R \phi(x,y,z)\,dx\,dy\,dz = \iiint_R \varphi(r,\theta,z)\, r\,dr\,d\theta\,dz$$

Cartesian to spherical polar form:
(r,θ,ϕ)

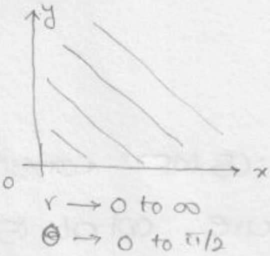
$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$|J| = \begin{vmatrix} \sin\theta\cdot\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\cdot\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\iiint_R \phi(x,y,z)\,dx\,dy\,dz = \iiint_R \varphi(r,\theta,\phi)\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

(IN 07)

prob: $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$



$r \to 0$ to ∞

$\theta \to 0$ to $\pi/2$

sol: Let $x = r\cos\theta$, $y = r\sin\theta$ then $|J| = r$

& $x^2+y^2 = r^2$

$$\int_0^\infty\int_0^{\pi/2} e^{-r^2}\cdot r\cdot dr\,d\theta$$

Let $r^2 = t \Rightarrow r\,dr = dt/2$

$$= \int_0^{\pi/2}\int_0^\infty e^{-t}\cdot dt/2\cdot d\theta = \int_0^{\pi/2}\frac{1}{2}\left[-e^{-t}\right]_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \pi/4$$

(GATE 05)

prob: By a change of variable $x(u,v) = uv$ & $y(u,v) = v/u$ in double integration, the integrand $f(x,y)$ changes to $f(uv, v/u)\,\phi(u,v)$ then $\phi(u,v)$ is

sol: $\phi(u,v) = |J| = \begin{vmatrix} v & u \\ -\frac{v}{u^2} & 1/u \end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}$

(GATE-95): By reversing the order of integration $\int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$ may be represented as

(a) $\int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$ (b) $\int_0^2\int_y^{\sqrt{y}} f(x,y)\,dx\,dy$ ✓(c) $\int_0^4\int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$ (d) $\int_{x^2}^{2x}\int_0^2 f(x,y)\,dy\,dx$

sol: $y = x^2$ to $y = 2x$ $\qquad$ $x=0$ to $x=2$

$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$

$x = \sqrt{y} \qquad x = y/2 \qquad y = 0^2 = 0 \qquad y = 2^2 = 4$

$\qquad\qquad\qquad\qquad = 2(0) = 0 \qquad = 2(2) = 4$

$$\therefore \int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx = \int_{y=0}^{4}\int_{x=y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

(GATE-04): The volume of an object expressed in spherical co-ordinate is given by $v = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\phi \, dr\, d\phi\, d\theta$. The value of the integral

Sol: $v = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{r^3}{3}\right]_0^1 \sin\phi \, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi}(-\cos\phi)_0^{\pi/3} d\theta$

$= \frac{1}{3}\times\frac{1}{2}\times 2\pi = \pi/3$

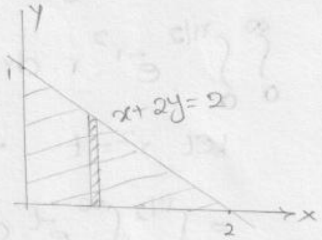
(GATE-08 CS): $\int_0^3\int_0^x (6-x-y)\,dx\,dy$

Sol: $\int_0^3 [6y - xy - y^2/2]_0^x \, dx = \int_0^3 (6x - x^2 - x^2/2)\,dx$

$= \int_0^3 (6x - 3x^2/2)\,dx = 6\left(\frac{x^2}{2}\right)_0^3 - \frac{3}{2}\left(\frac{x^3}{3}\right)_0^3$

$= 6\times\frac{9}{2} - \frac{3}{2}\times\frac{27}{3} = 27 - 13.5 = 13.5$

(GATE-08 ME): consider shaded triangular region P shown in the figure. what is $\iint_P xy\,dx\,dy$?



Sol: consider vertical strip

$x + 2y = 2$
$\Downarrow$
$y = \frac{2-x}{2}$

$y = 0$ to $\frac{2-x}{2}$

$x = 0$ to 2

$I = \int_0^2\int_0^{\frac{2-x}{2}} xy \cdot dy\,dx = \int_0^2 x\left(\frac{y^2}{2}\right)_0^{\frac{2-x}{2}} dx$

$= \int_0^2 \frac{x}{2}\left(\frac{2-x}{2}\right)^2 dx = 1/6$

(GATE-14 EC): The volume under the surface $z(x,y) = x+y$ and above the triangle in the xy plane defined by $\{0 \le y \le x$ and $0 \le x \le 12\}$ is

Sol: volume $= \int_0^{12}\int_0^x z\,dy\,dx = \int_0^{12}\int_0^x (x+y)\,dy\,dx$

$= \int_0^{12}(xy + y^2/2)_0^x\,dx = \int_0^{12}(x^2 + x^2/2)\,dx$

$= \frac{3}{2}\left(x^3/3\right)_0^{12} = 864$

(GATE-14EE): To evaluate the double integral $\int_0^8\int_{y/2}^{(y/2+1)}\left(\frac{2x-y}{2}\right)dx\,dy$ we make the substitution $u = \left(\frac{2x-y}{2}\right)$ and $v = y/2$. The integral will reduce to

Sol: $v = y/2 \Rightarrow dv = \frac{1}{2}dy \Rightarrow dy = 2dv$

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$u = \left(\frac{2x-y}{2}\right) \Rightarrow du = dx$.

As $x: \frac{y}{2} \to \frac{y}{2}+1 \Rightarrow u: 0 \to 1$ and $y: 0 \to 8 \Rightarrow v: 0 \to 4$

$\therefore$ integral ~~becomes~~ become $\int_0^4 \int_0^1 2u\,du\,dv$

(GATE-14 ME): $\int_0^2 \int_0^x e^{x+y}\,dy\,dx$

sol: $\int_0^2 \int_0^x e^{x+y}\,dy\,dx = \int_0^2 e^x (e^y)_0^x\,dx = \int_0^2 e^x(e^x - 1)\,dx$

$= \int_0^2 (e^{2x} - e^x)\,dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^2 = \frac{(e^2-1)^2}{2}$

Lengths, Areas & volumes:

1. Length of an arc of a curve $y = f(x)$ between the lines $x = x_1$ & $x = x_2$ is $l = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\,dx$

2. Length of an arc of a curve $x = f(t)$ & $y = g(t)$ between $t = t_1$ to $t = t_2$ is $l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + (dy/dt)^2}$

3. Area of the region bounded by the curve $y = f(x)$ & $y = g(x)$ between $x = x_1$ & $x = x_2$ is $A = \int_{x_1}^{x_2} [g(x) - f(x)]\,dx = \int_{x_1}^{x_2} \int_{f(x)}^{g(x)} dy\,dx$

4. The volume generated by revolving $y = f(x)$ between $x = x_1$ & $x = x_2$ about x-axis is

$$V = \int_{x_1}^{x_2} \pi y^2\,dx$$

About y-axis is $V = \int_{y_1}^{y_2} \pi x^2\,dy$

5. Inpolar form: (i) about initial line (i.e $\theta = 0$) $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta\,d\theta$

(ii) about line $\theta = \pi/2$ $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta\,d\theta$

(GATE 08 ME)

prob: The length of the curve $y = \frac{2}{3}x^{3/2}$ b/w $x = 0$ & 1 is

sol: $\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2} x^{\frac{3}{2}-1} = x^{1/2}$

$l = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = 1.22$

prob: The length of curve $x = \cos^3\theta$ $y = \sin^3\theta$ b/w $\theta = 0$ & $\pi/2$ is

sol: $l = \int_0^{\pi/2} \sqrt{(3\cos^2\theta(-\sin\theta))^2 + (3\sin^2\theta \cdot \cos\theta)^2}\,d\theta = \int_0^{\pi/2} 3\sin\theta\cos\theta\,d\theta$

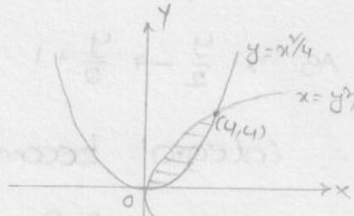
$= 3/2$.

(GATE-09 ME)

Prob: The area between the curve $y^2 = 4x$ & $x^2 = 4y$ is

Sol: $$A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{1}{4} \cdot \frac{x^3}{3}\right]_0^4$$

$$= \left[\frac{4}{3} \times 8 - \frac{1}{4} \times \frac{64}{3}\right] = 16/3$$

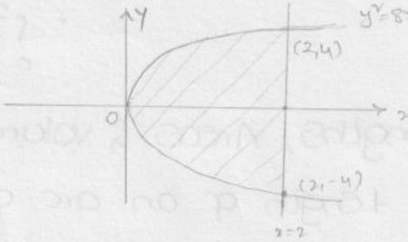


(GATE-94)

Prob: The volume generated by revolving the area bounded by parabola $y^2 = 8x$ and st. line $x = 2$ about y-axis is

Sol: $$V = \int_{y_1}^{y_2} \pi x^2 \, dy = \int_{-4}^{4} \pi \cdot \frac{y^4}{64} dy$$

$$= \frac{\pi}{64}\left[\frac{y^5}{5}\right]_{-4}^{4} = \frac{32\pi}{5}$$



Total volume $= \int_{-4}^{4} \pi x^2 dy = \int_{-4}^{4} \pi (2)^2 dy = 32\pi$

Remaining volume $= 32\pi - 32\pi/5 = 128\pi/5$

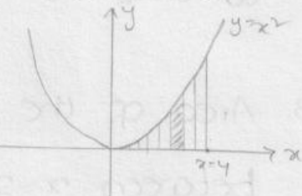
(GATE-97): Area bounded by the curve $y = x^2$ and the lines $x = 4$ and $y = 0$ is given by

Sol: Consider vertical strip $y = 0$ to $y = x^2$

$x = 0$ to $x = 4$

$$\int_{x=0}^{4}\int_{y=0}^{x^2} dx\,dy = \int_{x=0}^{4} (y)_0^{x^2} dx$$

$$= \int_0^4 x^2 dx = \left.\frac{x^3}{3}\right|_0^4 = \frac{4^3}{3} = 64/3$$

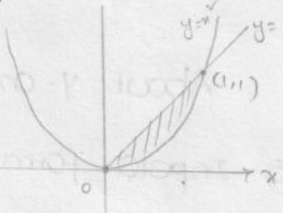


(GATE-12 ME, PI)

(GATE-04): The area enclosed b/w the parabola $y = x^2$ and the straight line $y = x$ is

Sol: $$\text{Area} = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = 1/6$$



(GATE-10 ME): The parabolic arc $y = \sqrt{x}$ $1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is

Sol: $$\text{volume} = \int_{x_1}^{x_2} \pi y^2 dx = \int_1^2 \pi (\sqrt{x})^2 dx = \pi\left(\frac{x^2}{2}\right)_1^2 = 3\pi/2$$

VECTOR CALCULUS

Vector calculus

vector differentials	vector integrals
1. Gradiant	1. Line integral
2. Divergence	2. Surface integral
3. curl	3. volume integral

Scalar function: A Scalar function $f(x,y,z)$ is a function defined at each point in a certain domain D in space. Its value is real and depends on the point $p(x,y,z)$ in space, but not on any particular co.ordinate system being used.

Ex: $x^2+y^2+z^2=c$

vector function: A function $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ defined at each point $p \in D$ is called a vector function.

Ex: $\bar{v} = 3x^2yz\,\bar{i} + 3xy^2z\,\bar{j} + 3xyz^2\,\bar{k}$

vector differential operator: It is denoted by '∇' (del)

In 2-dimension $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y}$

In 3-dimension $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad \text{(Scalar)}$$

* The equation $\nabla^2\phi = 0$ is called laplace equation. Any function which satisfies laplace eq^n is called Harmonic function

vector operations: $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$$\bar{a} \pm \bar{b} = (a_1 \pm b_1)\bar{i} + (a_2 \pm b_2)\bar{j} + (a_3 \pm b_3)\bar{k}$$

$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$ Scalar product**

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{vector product}^{**}$$

→ Magnitude of $\bar{a}$ ⇒ $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

→ unit vector along the direction of $\bar{a} = \dfrac{\bar{a}}{|\bar{a}|}$

→ Angle between two vectors $\bar{a}$ & $\bar{b}$ ⇒ $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$

→ $[\bar{a}\ \ \bar{b}\ \ \bar{c}] = \bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

→ $\bar{r} = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$ → position vector

$\dfrac{d\bar{r}}{dt}$ → velocity / Tangent vector

$\dfrac{d^2\bar{r}}{dt}$ → Accelaration vector

unit tangent vector $= \dfrac{(d\bar{r}/dt)}{|d\bar{r}/dt|}$

Gradient of a scalar function: Let $\phi(x, y, z)$ is a scalar function then

$$\text{grad}\,\phi = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

* Gradient of a scalar point function is a vector function
* Gradient gives normal to the surface
* unit normal $= \dfrac{\nabla\phi}{|\nabla\phi|}$

prob: Gradient of $\phi = 3x^2y - y^3z^2$ at $(1, -1, 2)$ is?

sol: $\nabla\phi = \bar{i}[6xy] + \bar{j}[3x^2 - 3y^2z^2] + \bar{k}[-2y^3z]$

$\nabla\phi\big|_{(1,-1,2)} = -6\bar{i} - 9\bar{j} + 4\bar{k}$ (Normal)

unit normal $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{-6\bar{i} - 9\bar{j} + 4\bar{k}}{\sqrt{36+81+16}} = \dfrac{1}{\sqrt{133}}(-6\bar{i} - 9\bar{j} + 4\bar{k})$

②

prob: $U = x+y+z$, $V = x^2+y^2+z^2$, $W = xy+yz+zx$. Find $[\nabla U\ \nabla V\ \nabla W]$

Sol: $\nabla U = \bar{i}+\bar{j}+\bar{k}$

$\nabla V = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$\nabla W = (y+z)\bar{i} + (x+z)\bar{j} + (x+y)\bar{k}$

$$[\nabla U\ \nabla V\ \nabla W] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 0$$

★ Angle between two curves is the angle between their tangents at the common point

★ Angle between two surfaces is the angle between their normals at the common point

prob: Find angle between two surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at $(2,-1,+2)$

Sol: $\nabla\phi_1 = 2x\bar{i}+2y\bar{j}+2z\bar{k}$ $\qquad$ $\nabla\phi_2 = 2x\bar{i}+2y\bar{j}-\bar{k}$

$= 4\bar{i}-2\bar{j}+4\bar{k}$ $\qquad$ $= 4\bar{i}-2\bar{j}-\bar{k}$

$\cos\theta$ = Angle b/w normals $= \dfrac{16+4-4}{\sqrt{36}\cdot\sqrt{21}} = \dfrac{8}{3\sqrt{21}}$

properties of Gradient:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(C_1 f + C_2 g) = C_1\nabla f + C_2\nabla g$; C_1, C_2 arbitrary constants.
3. $\nabla(fg) = f\cdot\nabla g + g\nabla f$
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\cdot\nabla f - f\nabla g}{g^2}$, $g\neq 0$

→ Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$

$r^2 = x^2+y^2+z^2$

$2r\,\partial r = 2x\,\partial x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

1. $$\text{Grad } r = \nabla r = \frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}$$

$$= \frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{r} = \frac{\bar{r}}{r}$$

2. $$\text{Grad}(1/r) = \nabla(1/r)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\bar{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\bar{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\bar{k}$$

$$= -\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\bar{i} - \frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\bar{j} - \frac{1}{r^2}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= -\frac{1}{r^2}\cdot\frac{x}{r}\bar{i} - \frac{1}{r^2}\cdot\frac{y}{r}\bar{j} - \frac{1}{r^2}\cdot\frac{z}{r}\cdot\bar{k} = -\frac{\bar{r}}{r^3}$$

3. $$\text{Grad}(r^2) = \nabla r^2$$

$$= \frac{\partial}{\partial x}(r^2)\bar{i} + \frac{\partial}{\partial y}(r^2)\bar{j} + \frac{\partial}{\partial z}(r^2)\bar{k}$$

$$= 2r\cdot\frac{\partial r}{\partial x}\bar{i} + 2r\cdot\frac{\partial r}{\partial y}\bar{j} + 2r\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= 2r\cdot\frac{x}{r}\bar{i} + 2\cdot r\cdot\frac{x}{r}\bar{j} + 2r\cdot\frac{x}{r}\bar{k} = 2\bar{r}$$

4. $$\text{Grad}(\log r) = \nabla(\log r)$$

$$= \frac{\partial}{\partial x}(\log r)\bar{i} + \frac{\partial}{\partial y}(\log r)\bar{j} + \frac{\partial}{\partial z}(\log r)\bar{k}$$

$$= \frac{1}{r}\cdot\frac{\partial r}{\partial x}\bar{i} + \frac{1}{r}\cdot\frac{\partial r}{\partial y}\bar{j} + \frac{1}{r}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= \frac{1}{r}\cdot\frac{x}{r}\bar{i} + \frac{1}{r}\cdot\frac{y}{r}\bar{j} + \frac{1}{r}\cdot\frac{z}{r}\cdot\bar{k} = \frac{\bar{r}}{r^2}$$

5. $$\text{Grad}(r^n) = \nabla(r^n)$$

$$= \frac{\partial}{\partial x}(r^n)\bar{i} + \frac{\partial}{\partial y}(r^n)\bar{j} + \frac{\partial}{\partial z}(r^n)\bar{k}$$

$$= nr^{n-1}\cdot\frac{\partial r}{\partial x}\bar{i} + nr^{n-1}\cdot\frac{\partial r}{\partial y}\cdot\bar{j} + nr^{n-1}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= nr^{n-1}\cdot\frac{x}{r}\bar{i} + nr^{n-1}\cdot\frac{y}{r}\bar{j} + nr^{n-1}\cdot\frac{z}{r}\bar{k} = nr^{n-2}\cdot\bar{r}$$

6. $$\text{Grad}(e^{r^2}) = \nabla(e^{r^2})$$

$$= e^{r^2}\cdot 2r\left[\frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}\right]$$

$$= e^{r^2}\cdot 2r\left[\frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}\right] = 2e^{r^2}\bar{r}$$

③

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Directional derivative: D.D of the surface ϕ in the direction of $\bar{r}$ is given by

$$D.D = \nabla\phi \cdot \hat{n} \quad \text{where } \hat{n} = \frac{\bar{r}}{|\bar{r}|}$$

→ DD is maximum in the direction of $\nabla\phi$ (normal)

→ The maximum value of DD is $|\nabla\phi|$

Prob: Find the DD of $\phi = 2xy + z^2$ in the direction of $\bar{r} = \bar{i} + 2\bar{j} + 2\bar{k}$ at the point $(1, -1, 3)$

Sol:
$$\nabla\phi = \bar{i}(2y) + \bar{j}(2x) + \bar{k}(2z)$$

$$\nabla\phi|_{(1,-1,3)} = -2\bar{i} + 2\bar{j} + 6\bar{k}$$

$$\bar{r} = \bar{i} + 2\bar{j} + 2\bar{k} \Rightarrow \hat{n} = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{9}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$DD = \nabla\phi \cdot \hat{n} = \frac{-2 + 4 + 12}{3} = 14/3$$

Prob: Find D.D of $\phi = xyz^2 + xz$ at the point $(1,1,1)$ in the direction of normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

Sol:
$$\nabla\phi = (yz^2 + z)\bar{i} + (xz^2)\bar{j} + (2xyz + x)\bar{k}$$

$$\nabla\phi|_{(1,1,1)} = 2\bar{i} + \bar{j} + 3\bar{k}$$

$$\nabla S = (3y^2)\bar{i} + (6xy + 1)\bar{j} + (-1)\bar{k}$$

$$\nabla S|_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{3\bar{i} + \bar{j} - \bar{k}}{\sqrt{11}}$$

$$DD = \nabla\phi \cdot \hat{n} = \frac{6 + 1 - 3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Divergence of vector: Let $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ be a vector point function then

$$\operatorname{div}(\bar{v}) = \nabla \cdot \bar{v}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(v_1\bar{i} + v_2\bar{j} + v_3\bar{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

* Divergence of vector point function is a scalar point function.

* Divergence gives rate at which fluid flows out of a unit volume
* If divergence of $\bar{V} = 0$ then $\bar{V}$ is called solenoidal vector.

Curl of vector: Let $\bar{V} = V_1\bar{i} + V_2\bar{j} + V_3\bar{k}$ be a vector point function then

$$\text{curl}(\bar{V}) = \nabla \times \bar{V} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

* curl of vector point function is again a vector point function
* curl gives rotation of the fluid
* If $\text{curl}(\bar{V}) = \bar{0}$ then $\bar{V}$ is called irrotational vector.
* If $\bar{V}$ is linear velocity then angular velocity $\bar{\omega} = \frac{1}{2}[\text{curl}\,\bar{V}]$

prob: If $\bar{V} = x^2yz\,\bar{i} + xy^2z\,\bar{j} + xyz^2\,\bar{k}$. Find div $\bar{V}$ & curl $\bar{V}$ at $(1,-1,1)$

Sol: $\text{div}\,\bar{V} = \nabla\cdot\bar{V} = 2xyz + 2xyz + 2xyz = 6xyz\big|_{(1,-1,1)} = -6$

$$\text{curl}\,\bar{V} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = \bar{i}[xz^2 - xy^2] - \bar{j}\,[yz^2 - x^2y] + \bar{k}\,[y^2z - x^2z]$$

$$= \bar{0}$$

Properties: (vector Identity)

GATE 96 1. $\text{curl}\,(\text{grad}\,\phi) = \bar{0}$

2. $\text{div}\,(\text{curl}\,\bar{V}) = 0$

3. $\text{div}\,(\nabla\phi) = \nabla^2\phi$

4. $\text{div}\,[\phi\bar{V}] = \phi\,\text{div}(\bar{V}) + \nabla\phi\cdot\bar{V}$

GATE-95 5. $\text{curl}\,[\phi\bar{V}] = \phi\,\text{curl}(\bar{V}) + \nabla\phi\times\bar{V}$

GATE-05 6. $\text{curl}\,[\text{curl}\,\bar{V}] = \nabla(\text{div}\,\bar{V}) - \nabla^2\bar{V}$

Prob: If $\bar{F} = \nabla(2x^3y^2z^4)$; Find div $\bar{F}$; curl $\bar{F}$

Sol: $\text{div}\,\bar{F} = \text{div}\,[\nabla(2x^3y^2z^4)]$

$= \nabla^2(2x^3y^2z^4)$ from (3)

$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$\text{curl}\,\bar{F} = \text{curl}\,[\nabla(2x^3y^2z^4)]$

$= \bar{0}$ from (1)

prob: Find a, b, c of $\bar{F} = (x+2y+az)\bar{i} + (bx-3y+z)\bar{j} + (4x+cy+2z)\bar{k}$

If $\bar{F}$ is irrotational.

Sol: $$\text{curl}(\bar{F}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y+z & 4x+cy+2z \end{vmatrix}$$

$$= \bar{i}\,[c-1] - \bar{j}\,(4-a) + \bar{k}\,(b-2)$$

If $\bar{F}$ is irrotational then $\text{curl}(\bar{F}) = \bar{0} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$$\therefore\ a = 4,\ b = 2,\ c = 1$$

prob: Find 'a' if $\bar{F} = 2xy\,\bar{i} + 3x^2y\,\bar{j} - 3ayz\,\bar{k}$ is solenoidal at $(1,2,3)$.

Sol: $$\text{div}(\bar{F}) = (2y + 3x^2 - 3ay)\Big|_{(1,2,3)} = 0$$

$$4 + 3 - 6a = 0$$

$$a = 7/6$$

(GATE-93): If the linear velocity $\bar{v}$ is given by $\bar{v} = x^2y\,\bar{i} + xyz\,\bar{j} - yz^2\,\bar{k}$ then the angular velocity $\bar{w}$ at the point $(1,1,-1)$ is

Sol: $$\bar{w} = \frac{1}{2}\,\text{curl}\,\bar{v} = \frac{1}{2}\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -yz^2 \end{vmatrix}$$

$$= \frac{1}{2}\left[\bar{i}\,(-z^2 - xy) - \bar{j}\,(0-0) + \bar{k}\,(yz - x^2)\right]$$

At $(1,1,-1)$ $$= \frac{1}{2}\left[-2\bar{i} - 2\bar{k}\right] = -(\bar{i} + \bar{k})$$

(GATE-94): The directional derivative of $f(x,y) = 2x^2 + 3y^2 + z^2$ at point $P(2,1,3)$ in the direction of the vector $a = \bar{i} - 2\bar{k}$ is

Sol: $$\nabla f = 4x\,\bar{i} + 6y\,\bar{j} + 2z\,\bar{k}$$

$$\nabla f\big|_{(2,1,3)} = 8\bar{i} + 6\bar{j} + 6\bar{k}$$

$$\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{\sqrt{5}}(\bar{i} - 2\bar{k})$$

$$D.D = \nabla f \cdot \hat{n} = \frac{8-12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

(GATE-95): The derivative of $f(x,y)$ at point $(1,2)$ in the direction of vector $\bar{i}+\bar{j}$ is $2\sqrt{2}$ and in the direction of the vector $-2\bar{j}$ is -3. Then the derivative of $f(x,y)$ in the direction $-\bar{i}-2\bar{j}$ is

Sol: Let $\bar{a} = \bar{i}+\bar{j}$

$$(\nabla f)\cdot\frac{\bar{a}}{|\bar{a}|} = 2\sqrt{2} \quad \Rightarrow \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right)\cdot\left(\frac{\bar{i}+\bar{j}}{\sqrt{2}}\right) = 2\sqrt{2}$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4 \longrightarrow ①$$

Let $\bar{b} = -2\bar{j}$

$$(\nabla f)\cdot\frac{\bar{b}}{|\bar{b}|} = -3 \quad \Rightarrow \frac{-2\,\partial f/\partial y}{2} = -3 \quad \Rightarrow \frac{\partial f}{\partial y} = 3 \longrightarrow ②$$

$$\text{from ①} \quad \frac{\partial f}{\partial x} = 1$$

$\therefore$ D.D in the direction $-\bar{i}-2\bar{j}$ is $\left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right)\cdot\left(\frac{-\bar{i}-2\bar{j}}{\sqrt{5}}\right)$

$$= (\bar{i}+3\bar{j})\cdot\left(\frac{-\bar{i}-2\bar{j}}{\sqrt{5}}\right) = \frac{-7}{\sqrt{5}}$$

(GATE 96): The directional derivative of the function $f(x,y,z) = x+y$ at the point $P(1,1,0)$ along the direction $\bar{i}+\bar{j}$ is

Sol: $$D.D = \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right)_{(1,1,0)}\cdot\left(\frac{\bar{i}+\bar{j}}{\sqrt{2}}\right)$$

$$= \frac{1+1}{\sqrt{2}} = \sqrt{2}$$

(GATE 99): For the function $\phi = ax^2y - y^3$ to represent the velocity potential of an ideal fluid, $\nabla^2\phi$ should be equal to zero. In that case, the value of 'a' has to be

Sol: $$\phi = ax^2y - y^3$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x} + \frac{\partial^2\phi}{\partial y} + \frac{\partial^2\phi}{\partial z} = 0$$

$$ay - 3y = 0 \quad \Rightarrow a = 3$$

(GATE - 02): The directional derivative of the following function at (1, 2) in the direction of $(4\bar{i} + 3\bar{j})$ is : $f(x) = x^2 + y^2$

Sol: $DD = (\nabla f)_{(1,2)} \cdot \hat{n}$

$$= (2x\,\bar{i} + 2y\,\bar{j})_{(1,2)} \cdot \frac{4\bar{i} + 3\bar{j}}{\sqrt{25}} = \frac{8+12}{5} = 4$$

(GATE - 03): The vector feild $F = x\bar{i} - y\bar{j}$ is

(a) divergence free, but not irrotational

(b) irrotational, but not divergence free

✓(c) divergence free and irrotational

(d) neither divergence free nor irrotational

Sol: $\operatorname{div} \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 1 - 1 = 0$ (divergent free)

$$\operatorname{curl} \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = \bar{0} \quad \text{Irrotational}$$

(GATE-05 EE): For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, the magnitude of the gradient at the point (1, 3) is

Sol: $\operatorname{grad} u = (\nabla u)_{(1,3)} = (x\bar{i} + \frac{2}{3}y\bar{j})_{(1,3)}$

$$= \bar{i} + 2\bar{j}$$

$$|\nabla u| = \sqrt{1+4} = \sqrt{5}$$

(GATE - 05 IN): A scalar field is given by $f = x^{2/3} + y^{2/3}$, where x and y are cartesian co.ordinates. The derivative of 'f' along the line $y = x$ directed away from the origin at point (8,8) is

Sol: unit vector along $y = x$ is

$$\hat{a} = \cos\frac{\pi}{4}\,\bar{i} + \sin\frac{\pi}{4}\,\bar{j} = \frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}}\bar{j}$$

$$(\nabla f)_{(8,8)} = \left(\frac{2}{3}x^{-1/3}\,\bar{i} + \frac{2}{3}y^{-1/3}\,\bar{j}\right)_{(8,8)}$$

$$= \tfrac{1}{3}\bar{i} + \tfrac{1}{3}\bar{j}$$

$$D.D = (\nabla f) \cdot \bar{a} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} = \sqrt{2}/3$$

(GATE-06 CE): The D.D of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $P(1,2,3)$ in the direction of the vector $\bar{a} = \bar{i} - 2\bar{k}$ is

Sol: $(\nabla f)_{(1,2,3)} = (4x\bar{i} + 6y\bar{j} + 2z\bar{k})_{(1,2,3)}$

$$= 4\bar{i} + 12\bar{j} + 6\bar{k}$$

$$DD = (\nabla f) \cdot \frac{\bar{a}}{|\bar{a}|} = (4\bar{i} + 12\bar{j} + 6\bar{k}) \cdot \left(\frac{\bar{i} - 2\bar{k}}{\sqrt{5}}\right)$$

$$= \frac{4-12}{\sqrt{5}} = -8/\sqrt{5}$$

(GATE-07 CE): The velocity vector is given as $\bar{v} = 5xy\bar{i} + 2y^2\bar{j} + 3yz^2\bar{k}$. The divergence of this velocity vector at $(1,1,1)$ is

Sol: $\operatorname{div} \bar{v} = 5y + 4y + 6yz$

At $(1,1,1)$ $\operatorname{div} \bar{v} = 5 + 4 + 6 = 15$

(GATE-07 PI): The angle between two planar vector $\bar{a} = \frac{\sqrt{3}}{2}\bar{i} + \frac{1}{2}\bar{j}$ and $\bar{b} = -\frac{\sqrt{3}}{2}\bar{i} + \frac{1}{2}\bar{j}$ is

Sol: $$\cos\theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} = \frac{-3/4 + 1/4}{\sqrt{\frac{3}{4} + \frac{1}{4}} \cdot \sqrt{\frac{3}{4} + 1/4}} = -1/2$$

$$\theta = \cos^{-1}(-1/2) = 120^\circ$$

(GATE-07 EE): Divergence of the vector field $V(x,y,z) = -(x\cos xy + y)\bar{i} + (y\cos xy)\bar{j} + [(\sin z^2) + x^2 + y^2]\bar{k}$ is

Sol: $$\operatorname{div} \bar{v} = \frac{\partial}{\partial x}[-(x\cos xy + y)] + \frac{\partial}{\partial y}(y\cos xy) + \frac{\partial}{\partial z}[(\sin z^2) + x^2 + y^2]$$

$$= 2z\cos z^2$$

(GATE-08 ME): The divergence of the vector field $(x-y)\bar{i} + (y-x)\bar{j} + (x+y+z)\bar{k}$ is

Sol: $\operatorname{div} \bar{v} = 1 + 1 + 1 = 3$

(GATE-08 ME): The D.D of scalar function $f(x,y,z) = x^2 + 2y^2 + z$ at the point $P = (1,1,2)$ in the direction of vector $\bar{a} = 3\bar{i} - 4\bar{j}$ is

Sol: $$DD = (2x\bar{i} + 4y\bar{j} + \bar{k})_{(1,1,2)} \cdot \left(\frac{3\bar{i} - 4\bar{j}}{5}\right)$$

$$= \frac{6-16}{5} = -2$$

⑥

(GATE-09 CE): For a scalar function $f(x,y,z) = x^2+3y^2+2z^2$ the gradient at the point $P(1,2,-1)$ is

Sol: $\text{Grad } f = 2x\bar{i} + 6y\bar{j} + 4z\bar{k}$

At $(1,2,-1)$ $\nabla f = 2\bar{i} + 12\bar{j} - 4\bar{k}$

(GATE-09 CE): For a scalar function $f(x,y,z) = x^2+3y^2+2z^2$ the directional derivative at $P(1,2,-1)$ in the direction of $\bar{i}-\bar{j}+2\bar{k}$ is

Sol: $(\nabla f).\hat{n} = \frac{2-12-8}{\sqrt{6}} = \frac{-18}{\sqrt{6}} = -3\sqrt{6}$

(GATE-09 IN): A sphere of unit radius is centered at the origin. The unit normal at the point (x,y,z) on the surface of the sphere is the vector

✓a) (x,y,z) b) $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ c) $\left(\frac{x}{\sqrt{3}},\frac{y}{\sqrt{3}},\frac{z}{\sqrt{3}}\right)$ d) $\left(\frac{x}{\sqrt{2}},\frac{y}{\sqrt{2}},\frac{z}{\sqrt{2}}\right)$

Sol: Equation of sphere $x^2+y^2+z^2=1$

Normal to the surface $= \nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$$\text{unit normal} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{x\bar{i}+y\bar{j}+z\bar{k}}{\underbrace{\sqrt{x^2+y^2+z^2}}_{=1}} = x\bar{i}+y\bar{j}+z\bar{k}$$

(GATE-09 ME): The divergence of vector field. $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at point $(1,1,1)$ is equal to

Sol: $\text{div } \bar{F} = 3z + 2x - 2yz$

$\text{div } \bar{F}|_{(1,1,1)} = 3+2-2 = 3$

(GATE-10 EE): Divergence of the 3-dimensional radial vector field $\bar{r}$ is

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\text{div } \bar{r} = 1+1+1 = 3$

(GATE-11 CE): If $\bar{a}$ and $\bar{b}$ are two arbitrary vectors with magnitudes a and b respectively. Then $|\bar{a}\times\bar{b}|^2$ will be equal to

Sol: $$|\bar{a}\times\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 \sin^2(\bar{a},\bar{b})$$
$$= a^2b^2[1-\cos^2(\bar{a},\bar{b})]$$
$$= a^2b^2\left[1-\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|^2|\bar{b}|^2}\right] = a^2b^2\left[\frac{a^2b^2-\bar{a}\cdot\bar{b}}{a^2b^2}\right]$$
$$= a^2b^2-\bar{a}\cdot\bar{b}$$

(GATE-11 PI): If $A(0,4,3)$, $B(0,0,0)$, and $C(3,0,4)$ are three points defined in x,y,z co-ordinate system then which one of the following vectors is perpendicular to both $\bar{AB}$ & $\bar{BC}$ vectors

a) $16\bar{i}+9\bar{j}-12\bar{k}$ b) $16\bar{i}-9\bar{j}+12\bar{k}$ c) $16\bar{i}-9\bar{j}-12\bar{k}$ d) $16\bar{i}+9\bar{j}+12\bar{k}$

Sol: $$\bar{BA} = \bar{OA}-\bar{OB} = 4\bar{j}+3\bar{k}$$
$$\bar{BC} = \bar{OC}-\bar{OB} = 3\bar{i}+4\bar{k}$$

vector perpendicular to $\bar{BA}$ & $\bar{BC}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 4 & 3 \\ 3 & 0 & 4 \end{vmatrix} = \bar{BA}\times\bar{BC}$

$$= 16\bar{i}+9\bar{j}-12\bar{k}$$

(GATE-11 PI): If $T(x,y,z) = x^2+y^2+2z^2$ defines the temperature at any location (x,y,z) then the magnitude of the temperature gradient at point $P(1,1,1)$ is

Sol: $$\nabla T = 2x\bar{i}+2y\bar{j}+4z\bar{k}$$
$$(\nabla T)_{(1,1,1)} = 2\bar{i}+2\bar{j}+4\bar{k} \Rightarrow |\nabla T| = \sqrt{4+4+16} = 2\sqrt{6}$$

(GATE-11 EE): The two vectors $[1\ 1\ 1]$ and $[1\ a\ a^2]$ where $a = \frac{-1}{2}+j\frac{\sqrt{3}}{2}$ are

✓a) orthogonal b) orthonormal c) parellel d) collinear

Sol: Let $$\bar{p} = [1\ 1\ 1] = \bar{i}+\bar{j}+\bar{k}$$
$$\bar{q} = [1\ a\ a^2] = \bar{i}+\left(\frac{-1}{2}+j\frac{\sqrt{3}}{2}\right)\bar{j}+\left(\frac{-1}{2}-j\frac{\sqrt{3}}{2}\right)\bar{k}$$
$$\bar{p}\cdot\bar{q} = 1-\frac{1}{2}+j\frac{\sqrt{3}}{2}-\frac{1}{2}-j\frac{\sqrt{3}}{2} = 0$$

Cube roots of unity are $1, a, a^2$

$1+a+a^2=0$

Both vectors are orthogonal

⑦

(GATE-12 EC, EE, IN): The direction of vector A is radially outward from the origin, with $|A| = Kr^n$ where $r^2 = x^2+y^2+z^2$ and K is Cons

The value of n for which $\nabla \cdot \bar{A} = 0$

Sol: Given $|\bar{A}| = Kr^n$

unit radial vector in the direction of A is $\frac{\bar{r}}{r}$

$$\therefore \text{ vector } \bar{A} = Kr^n \cdot \frac{\bar{r}}{r} = Kr^{n-1}\bar{r}$$

Given $\nabla \cdot \bar{A} = 0$

$$K \nabla \cdot (r^{n-1}\bar{r}) = 0$$

$$\nabla \cdot [r^{n-1}x\bar{i} + r^{n-1}y\bar{j} + r^{n-1}z\bar{k}] = 0$$

$$\frac{\partial}{\partial x}(r^{n-1} \cdot x) + \frac{\partial}{\partial y}(r^{n-1}y) + \frac{\partial}{\partial y}(r^{n-1}z) = 0$$

$$\left[r^{n-1} + x(n-1)r^{n-2} \cdot \frac{\partial r}{\partial x}\right] + \left[r^{n-1} + y(n-1)r^{n-2} \cdot \frac{\partial r}{\partial y}\right] + [\cdots] = 0$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore 3r^{n-1} + (n-1)r^{n-3}[x^2+y^2+z^2] = 0$$

$$3r^{n-1} + (n-1)r^{n-3} \cdot r^2 = 0$$

$$3r^{n-1} + (n-1)r^{n-1} = 0$$

$$3+n-1 = 0$$

$$n = -2$$

(GATE-12 ME, PI): For the spherical surface $x^2+y^2+z^2=1$, the unit outward normal vector at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ is given by

Sol: $$(\nabla\phi)_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)} = (2x\bar{i} + 2y\bar{j} + 2z\bar{k})_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)}$$

$$= \sqrt{2}\bar{i} + \sqrt{2}\bar{j} \qquad \text{(Normal)}$$

$$\text{unit normal} = \frac{\sqrt{2}\bar{i} + \sqrt{2}\bar{j}}{\sqrt{2+2}} = \frac{\bar{i}}{\sqrt{2}} + \frac{\bar{j}}{\sqrt{2}}$$

(GATE-13 EC): The divergence of the vector field $\bar{A} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$ is

Sol: $$\text{div } \bar{A} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

(GATE-13 EE): The curl of gradient of scalar field defined by

$V = 2x^2y + 3y^2z + 4z^2x$ is

Sol: $\text{curl}(\text{grad } V) = \bar{0}$

(GATE-14 ECE): If $\bar{r} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$ and $|\bar{r}| = r$, then $\text{div}(r^2\nabla(\ln r)) =$

Sol: ** $\nabla[f(r)] = f'(r) \cdot \frac{\bar{r}}{r}$

$$\nabla[\log r] = \frac{1}{r} \cdot \frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$$

$$\text{div}[r^2(\nabla(\log r))] = \text{div}(\bar{r}) = 1+1+1 = 3$$

(GATE-14 EC): The magnitude of the gradient for the function

$f(x,y,z) = x^2 + 3y^2 + z^3$ at the point $(1,1,1)$ is

Sol: $\nabla f = 2x\bar{i} + 6y\bar{j} + 3z^2\bar{k}$

$$(\nabla f)_{(1,1,1)} = 2\bar{i} + 6\bar{j} + 3\bar{k}$$

$$|\nabla f| = \sqrt{4+36+9} = 7$$

(GATE-14 EE): Let $\nabla\cdot(fV) = x^2y + y^2z + z^2x$, where f and V are scalar and vector feilds respectively. If $V = y\bar{i} + z\bar{j} + x\bar{k}$ then $V\cdot(\nabla f)$ is

Sol: $\nabla\cdot(fV) = f(\nabla\cdot V) + V\cdot(\nabla f)$

$$V = y\bar{i} + z\bar{j} + x\bar{k} \Rightarrow \nabla\cdot V = 0+0+0 = 0$$

$$\therefore \nabla\cdot(fV) = V\cdot(\nabla f) = x^2y + y^2z + z^2x$$

(GATE-14 CE): A particle moves along a curve whose parametric equations are : $x = t^3 + 2t$, $y = -3e^{-2t}$, $z = 2\sin 5t$, where x, y & z show variations of distance covered by particle (in cm) with time t (in s). The magnitude of the accelaration of particle (in cm/s²) at $t=0$ is

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = (t^3+2t)\bar{i} + (-3e^{-2t})\bar{j} + (2\sin 5t)\bar{k}$

$$\text{velocity} = \frac{d\bar{r}}{dt} = (3t^2+2)\bar{i} + (6e^{-2t})\bar{j} + 10\cos 5t\,\bar{k}$$

$$\text{Acceleration} = \frac{d^2\bar{r}}{dt^2} = 6t\,\bar{i} - 12e^{-2t}\,\bar{j} - 50\sin 5t\,\bar{k}$$

At $t=0$ $\frac{d^2r}{dt^2} = -12\bar{j}$

$$\therefore \text{Magnitude} = |-12\bar{j}| = 12$$

Line integrals: Integration along a curve is called line integral and is denoted by $\int_C \bar{F}\cdot d\bar{r}$

$$\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int_C F_1 dx + \int_C F_2 dy + \int_C F_3 dz$$

prob: $\bar{F} = (5xy - 6x^2)\bar{i} + (4y - 6x)\bar{j}$ then $\int_C \bar{F}\cdot d\bar{r}$ along the curve $y = x^3$: $(1,1) \to (2,8)$

sol: $\int_C \bar{F}\cdot d\bar{r} = \int_C (5xy - 6x^2)dx + \int_C (4y - 6x)\,dy$

$$y = x^3 \Rightarrow dy = 3x^2 dx$$

$$= \int_1^2 (5x^4 - 6x^2)dx + \int_1^2 (4x^3 - 6x)\,3x^2 dx$$

$$= 76.5$$

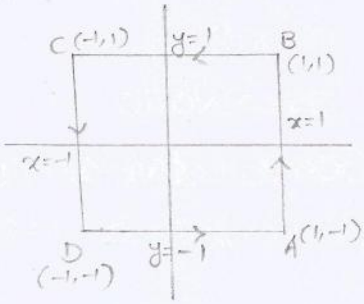
prob: $\bar{F} = x^2\bar{i} + xy\bar{j}$. Find $\int_C \bar{F}\cdot d\bar{r}$ along $y = x$: $(0,0) \to (1,1)$

sol: $\int_C \bar{F}\cdot d\bar{r} = \int_C x^2 dx + \int_C xy\,dy$

$$y = x \Rightarrow dy = dx$$

$$= \int_0^1 x^2 dx + \int_0^1 x^2 dx = 2\cdot \frac{x^3}{3}\Big|_0^1 = 2/3$$

prob: $\oint_C (x^2 + xy)\,dx + \oint (x^2 + y^2)\,dy$ where 'c' is a square bounded by $x = \pm 1$, $y = \pm 1$.



sol: Along AB : $x = 1 \Rightarrow dx = 0$

$$\int_{-1}^{1} (1 + y^2)dy = \left[y + y^3/3\right]_{-1}^{1} = 8/3$$

Along BC: $y = 1 \Rightarrow dy = 0$

$$\int_1^{-1} (x^2 + x)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^{-1} = -2/3$$

Along CD : $x = -1 \Rightarrow dx = 0$

$$\int_1^{-1} (1 + y^2)dy = -8/3$$

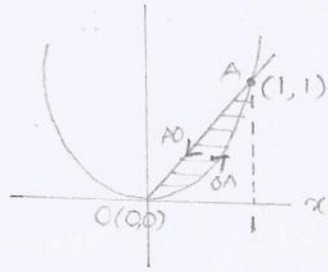
Along DA : $y=-1$

$$\int_{-1}^{1}(x^2-x)dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1} = 2/3$$

$$\therefore \oint_C (x^2+xy)dx+(x^2+y^2)dy = \frac{8}{3}+\frac{2}{3}-\frac{8}{3}-\frac{2}{3} = 0$$

prob: $\oint_C (xy+y^2)dx + x^2dy$: 'C' is bounded by $y=x$: $y=x^2$

Sol: Along OA: $y=x^2 \Rightarrow dy = 2x \cdot dx$



$$\int_{OA}(x^3+x^2)dx + \int_{OA} 2x^3 dx = \int_0^1 (x^4+3x^3)dx$$
$$= 19/20$$

Along AO: $y=x \Rightarrow dy = dx$

$$\int_{AO}(x^2+x^2)dx + \int_{AO} x^2dx = 3\int_1^0 x^2dx = x^3\Big|_1^0 = -1$$

$$\therefore \oint_C (xy+y^2)dx + x^2dy = \frac{19}{20}-1 = -1/20$$

Greens theorem: Let M&N are functions of x, y defined in a region with a simple closed curve 'c' then

$$\oint_C Mdx+Ndy = \iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

[Relation b/w line & double integration]

Stokes theorem: Let 'S' be a <u>open</u> surface bounded by a simple closed curve 'c' then GATE 13EC

GATE 09EC 13EC $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,dS$

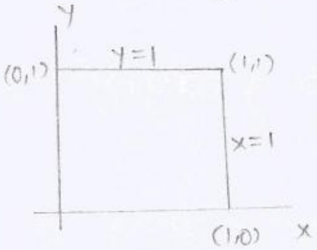
GATE 05EE [Relation b/w line & surface integration]

Gauss-divergence theorem: Let 'S' be a <u>closed</u> surface then

02EE $\oiint_S \bar{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\bar{F}\,dv$

[Relation b/w surface & volume integration]

(GATE-05): value of the integral $\oint_c xy\,dy - y^2 dx$, where c is the square cut from the first quadrant by the line $x=1$ and $y=1$ will be (use Green's theorem to change the line integral into double integral)



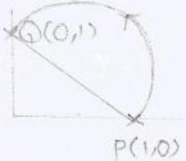
Sol: $\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$

Here $M = -y^2$ $N = +xy$

$$\oint_c xy\,dy - y^2 dx = \int_{x=0}^{1} \int_{y=0}^{1} (y + 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \frac{3}{2} x\,dx = 3/2$$

(GATE-08 EE): Consider points P and Q in xy-plane with P(1,0) and Q(0,1). The line integral $2\int_P^Q (x dx + y dy)$ along the semi circle with the line segment PQ as diameter



Sol: $M = 2x$ $N = 2y$

$\frac{\partial M}{\partial y} = 0$ $\frac{\partial N}{\partial x} = 0$

$\therefore$ From greens theorem $2\oint_P^Q (x\,dx + y\,dy) = 0$

(GATE-08 PI): If $\bar{r}$ is position vector of any point on a closed surface S that encloses the volume V then $\iint_S \bar{r} \cdot d\bar{s}$ is equal to

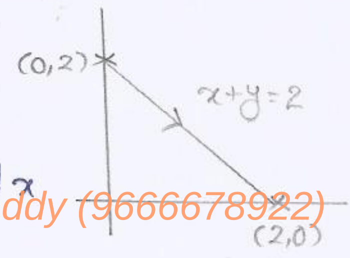
Sol: From gauss-divergence theorem

$$\iint_S \bar{r} \cdot d\bar{s} = \iiint_V (\nabla \cdot \bar{r})\,dv \qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$= \iiint_V (1+1+1)\,dv = 3V$$

(GATE-09 EE): $F(x,y) = (x^2+xy)\,\hat{a}_x + (y^2+xy)\,\hat{a}_y$. Its line integral over the straight line from $(x,y) = (0,2)$ to $(2,0)$ evaluates to

Sol: Given straight line is $x+y=2$



$$y = 2-x \Rightarrow dy = -dx$$

$$\int \bar{F} \cdot d\bar{r} = \int_0^2 [x^2 + x(2-x)]dx - \int_0^2 [(2-x)^2 + x(2-x)]dx$$

$= 0$

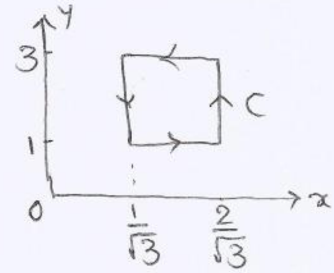
(GATE-09 PI): The line integral of the vector function $\bar{F} = 2x\hat{i} + x^2\hat{j}$ along the x-axis from $x=1$ to $x=2$ is

Sol: x-axis $\Rightarrow y=0 \Rightarrow dy=0$

$$\int \bar{F}\cdot d\bar{r} = \int_1^2 2x\,dx + \int_1^2 x^2 dy$$

$$= \int_1^2 2x\,dx = 4-1 = 3$$

(GATE-10 EC): If $\bar{A} = xy\,\hat{a}_x + x^2\hat{a}_y$ then $\oint \bar{A}\cdot d\bar{r}$ over the path shown in the figure is



Sol: Apply Green's theorem

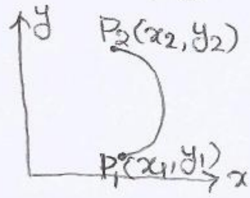
$M = xy \qquad N = x^2$

$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$

$$\oint \bar{A}\cdot d\bar{r} = \int_{x=1/\sqrt{3}}^{2/\sqrt{3}} \int_{y=1}^{3} (2x - x)\,dy\,dx$$

$$= \int_{x=1/\sqrt{3}}^{2/\sqrt{3}} x\cdot(y)_1^3\,dx = 2\cdot\left.\frac{x^2}{2}\right|_{1/\sqrt{3}}^{2/\sqrt{3}} = \frac{4}{3} - \frac{1}{3} = 1$$

(GATE-11 PI): The line integral $\int_{P_1}^{P_2} (y\,dx + x\,dy)$ from $P(x_1, y_1)$ to $P_2(x_2, y_2)$ along the semi circle P_1P_2 as shown in figure



Sol: $\int_{P_1}^{P_2} y\,dx + x\,dy = \int_{P_1}^{P_2} d(xy)$

$$= (xy)_{P_1}^{P_2} = x_2 y_2 - x_1 y_1$$

(GATE-11 EC): consider a closed surface 'S' surrounding a volume V. If $\bar{r}$ is the position vector of a point inside S with $\hat{n}$ the unit normal on S, the value of $\oiint 5\bar{r}\cdot\hat{n}\,ds$ is

Sol: From gauss-divergence theorem

$$\oiint \bar{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\bar{F}\,dv$$

$$\Rightarrow \oiint 5\bar{r}\cdot\hat{n}\,ds = 5\iiint \text{div}\,\bar{r}\,dv \qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$= 5\iiint 3\,dv$$

$$= 15V$$

(GATE 13 ME): The following surface integral is to be evaluated over a sphere for the given steady velocity vector field $F = x\bar{i} + y\bar{j} + z\bar{k}$ defined w.r.to a cartesian co-ordinate system having i, j, k as unit base vectors. $\iint_S \frac{1}{4}(F \cdot n)\,dA$ where S is sphere, $x^2+y^2+z^2=1$ and n is the outward unit normal vector to the sphere. The value of surface integral is

Sol: $$\iint_S \frac{1}{4} \bar{F}\cdot\hat{n}\,dA = \frac{1}{4}\iiint_V \nabla\cdot\bar{F}\,dv$$

$$= \frac{1}{4}\iiint 3\,dv = \frac{3}{4}(\text{volume of sphere})$$

$$= \frac{3}{4}\times\frac{4}{3}\pi r^3 = \pi r^3 = \pi$$

$x^2+y^2+z^2=1$
$\Rightarrow r=1$

(GATE-13 EE): Given a vector field $\bar{F} = y^2x\,\hat{a}_x - yz\,\hat{a}_y - x^2\,\hat{a}_z$, the line integral $\int \bar{F}\cdot dl$ evaluated along a segment on the x-axis from $x=1$ to $x=2$ is

Sol: x-axis $\Rightarrow$ $y=0$, $z=0$

$dy=0$, $dz=0$

$$\therefore \int \bar{F}\cdot dl = \int_1^2 y^2x\,dx = 0 \quad [\because y=0]$$

(GATE-14 EE): The line integral of function $F = yz\,\bar{i}$ in the counter clockwise direction, along the circle $x^2+y^2=1$ at $z=1$ is

Sol: $$\int_C \bar{F}\cdot d\bar{l} = \int_C yz\,dx$$

$$= \int (yz\,dx - 0\,dy)$$

Apply greens theorem $M = yz$ $N=0$

$\frac{\partial M}{\partial y} = z$ $\frac{\partial N}{\partial x} = 0$

at $z=1$ $\frac{\partial M}{\partial y} = 1$

$$= \iint_S (0-1)\,dx\,dy = -\iint_S dx\,dy = -S = -\pi r^2 \quad [\because r=1]$$

$$= -\pi$$

(GATE-14 ME): The integral $\oint_C (y\,dx - x\,dy)$ is evaluated along the circle $x^2+y^2 = 1/4$ traversed in counter clockwise direction. The integral is equal to

Sol: Apply green's theorem

$$\oint_C (y\,dx - x\,dy) = \iint_R (-1-1)\,dx\,dy$$

$= -2$ (Area of given circle) $\qquad x^2+y^2=(1/2)^2$

$= -2 \cdot \pi \left(\frac{1}{2}\right)^2 = -\pi/2 \qquad r = 1/2$

DIFFERENTIAL EQUATIONS

Differential Equations

Differential Equation: Equations involving differential co.efficients are called differential equations.

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{\partial y}{\partial x} \cdots$$ are called differential co.efficients

Ordinary differential equation: The differential equation in which all the differential co.efficient have reference to a single independent variable.

Ex:
1. $\frac{d^2y}{dt^2} + n^2x = 0$
2. $y = x\frac{dy}{dx} + \frac{x}{dy/dx}$
3. $e^x dx + e^y dy = 0$

Partial differential equation: The differential equation in which there are two or more independent variables and partial differential ~~di~~ coefficients with respect to any of them.

Ex:
1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$
2. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

Order of D.E: It is the order of the highest differential co.efficient that occurs in the equation.

Degree of D.E: It is the power of highest differential co.efficient provided the equation is free from fractional powers.

Ex:		order	Degree	
1.	$\left[1+\left(dy/dx\right)^2\right]^{3/2} = \left(d^2y/dx^2\right)$	2	2	[PI-05]
2.	$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0$	2	1	
3.	$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$	2	1	
4.	$\left(\frac{dy}{dx}\right)^{3/2} = \left(\frac{d^2y}{dx^2}\right)^{2/3}$	2	4	

Formation of D.E: Differential equations are formed by the elemination of orbitrary constants / functions from a relation in the variables and constants. From D.E involving 'n' arbitrary constants we get the D.E of nth order.

Prob: Form the D.E by eliminating the arbitrary constants present in $y = ax + bx^2$

$$y = ax + bx^2 \longrightarrow (1)$$

d.w.r.to x $\quad y' = a + 2bx \longrightarrow (2)$

d.w.r.to 'x' $\quad y'' = 2b$

$$b = y''/2 \longrightarrow (3)$$

from (2) & (3) $\quad y' = a + xy''$

$$a = y' - xy'' \longrightarrow (4)$$

from (1) & (4) $\quad y = (y' - xy'')x + \frac{y''}{2}x^2$

$$x^2y'' + 2xy' - 2x^2y'' = 2y$$

$$x^2y'' - 2xy' + 2y = 0$$

$$x^2\frac{d^2y}{dx} - 2x\frac{dy}{dx} + 2y = 0$$

prob: $\quad y = a + bx$

$$y = a + bx \longrightarrow (1)$$

d.w.r.to x $\quad y' = b \longrightarrow (2)$

d.w.r.to x $\quad y'' = 0$

$$\frac{d^2y}{dx^2} = 0$$

prob: $\quad y = a\cos x + b\sin x \longrightarrow (1)$

$$y' = -a\sin x + b\cos x \longrightarrow (2)$$

$$y'' = -a\cos x - b\sin x \longrightarrow (3)$$

from (1) & (3) $\quad y'' = -y$

$$y'' + y = 0$$

$$\frac{d^2y}{dx^2} + y = 0$$

prob: $\quad y = ae^{x} + be^{2x}$

$$y' = ae^{x} + 2be^{2x} = y + be^{2x} \rightarrow (1)$$

$$y'' = y' + 2be^{2x}$$

$$be^{2x} = \frac{y'' - y'}{2} \rightarrow (2)$$

from (1) & (2)

$$y' = y + \frac{y'' - y'}{2}$$

$$3y' = 2y + y'' \Rightarrow y'' + 2y - 3y' = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

prob: $\quad (x-a)^2 + (y-b)^2 = 1 \rightarrow (1)$

D.w.r. to 'x' $\quad 2(x-a) + 2(y-b)y' = 0 \rightarrow (2)$

$$(x-a) + (y-b)y' = 0$$

D.w.r. to 'x' $\quad 1 + (y-b)y'' + (y')^2 = 0$

$$y - b = -\frac{[1+(y')^2]}{y''} \rightarrow (3)$$

from (2) & (3) $\quad x - a = \frac{[1+(y')^2]\, y'}{y''} \rightarrow (4)$

from (1) & (4) $\quad \left[\frac{(1+(y')^2)\,y'}{y''}\right]^2 + \left[-\frac{(1+(y')^2)}{y''}\right]^2 = 1$

$$[1+(y')^2]^2\,[(y')^2+1] = (y'')^2$$

$$[1+(y')^2]^3 = (y'')^2$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left[\frac{d^2y}{dx}\right]^2$$

Solution of D.E:

An Equation in terms of the variables which satisfies the given differential equation is called Solution of D.E

Ex: For the D.E $\frac{d^2y}{dx^2}+y=0$, the solution is

$$y = a\cos x + b\sin x$$

also $y = 2\cos x$, $\cos x + \sin x$, $\sin 3x$ are also solutions.

The solution of D.E is of two types

General solution: The solution of D.E which contains as many arbitrary constants as that of the order of D.E is called the general solution.

Particular solution: A solution obtained from ~~particular~~ general solution by giving particular values to arbitrary constants are called particular solutions.

D.E of 1st order and 1st degree

The general form of D.E of 1st order and 1st degree is $\frac{dy}{dx} = f(x,y)$. Depending on the function $f(x,y)$ these equations are divided into 4-types

1. Variable seperable form D.E
2. Homogeneous D.E
3. Linear D.E
4. Exact D.E

③

1. variable seperable form: If in an equation it is possible to collect all functions of 'x' and 'dx' on one side and all the functions of 'y' and 'dy' on the other side, then the variables are said to be seperable. Thus the general form of such an equation is

$$\varphi(y)\,dy = \phi(x)\,dx$$

I.O.B.S $\int \varphi(y)\,dy = \int Q(x)\,dx + c$

Prob: $\frac{dy}{dx} = \frac{y}{x}$

$$\frac{1}{y}dy = \frac{1}{x}dx$$

I.O.B.S $\log_e y = \log_e x + C$ Let $C = \log_e K$

$$= \log_e x + \log_e K$$

$$\log_e y = \log_e xK$$

$y = xK$ is the solution.

Prob: $\frac{dy}{dx} = 1 + x + y + xy$

$$\frac{dy}{dx} = 1 + x + y(1+x) = (1+x)(1+y)$$

$$\frac{1}{1+y}dy = (1+x)\,dx$$

I.O.B.S $\log_e(1+y) = x + \frac{x^2}{2} + c$

Prob: $y - x\frac{dy}{dx} = a\left[y^2 + \frac{dy}{dx}\right]$

$$y - ay^2 = (x+a)\frac{dy}{dx}$$

$$\frac{1}{y-ay^2}dy = \frac{1}{x+a}dx$$

$$\frac{1}{y(1-ay)}dy = \frac{1}{x+a}dx$$

$$\left[\frac{a}{1-ay}+\frac{1}{y}\right]dy=\frac{1}{x+a}dx$$

I.O.B.S $\quad -\log_e|1-ay|+\log_e|y| = \log_e|x+a|+c \qquad c=\log_e k$

$$\log_e \frac{|y|}{|1-ay|} = \log_e |x+a|k$$

$$\therefore \quad \frac{|y|}{|1-ay|} = |x+a|k$$

$$\frac{y}{(1-ay)} = (x+a)k$$

Prob: $(x+y+1)^2 \frac{dy}{dx} = 1 \longrightarrow (1)$

put $\quad x+y+1 = t \quad$ say

$$1+\frac{dy}{dx} = \frac{dt}{dx}$$

from (1) $\quad t^2\left(\frac{dt}{dx}-1\right)=1$

$$\frac{dt}{dx} = \frac{1+t^2}{t^2} \Rightarrow \frac{t^2}{1+t^2}dt = dx$$

$$\Rightarrow \left[1-\frac{1}{1+t^2}\right]dt = dx$$

I.O.B.S

$$t-\tan^{-1}t = x+c$$

$$(x+y+1)-\tan^{-1}(x+y+1) = x+c$$

Prob: $\frac{dy}{dx} = e^{3x-2y}+x^2e^{-2y}$

$$\frac{dy}{dx} = e^{3x}\cdot e^{-2y}+x^2\cdot e^{-2y}$$

$$=(x^2+e^{3x})\cdot e^{-2y}$$

$$e^{2y}dy = (x^2+e^{3x})dx$$

I.O.B.S $\quad \frac{e^{2y}}{2} = \frac{x^3}{3}+\frac{e^{3x}}{3}+c$

$$3e^{2y} = 2(e^{3x}+x^3)+6c$$

(4)

2. Homogeneous D.E: D.E's of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$

Where f_1, f_2 are homogeneous functions of same degree are called homogeneous D.E's

To solve these equations we take substitution

$$y = vx \quad \text{so that} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

(or)

$$x = vy \quad \text{so that} \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Later, seperate the variables and integrate.

Prob: Solve $(x^2-y^2)dx - xy\,dy = 0$

$$\frac{dy}{dx} = \frac{x^2-y^2}{xy} \quad \text{Homogeneous in } x, \& \ y$$

put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x^2 v} = \frac{1-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$

$$\frac{v}{1-2v^2}\,dv = \frac{dx}{x}$$

I.O.B.S

$$-\frac{1}{4}\int \frac{4v}{1-2v^2}\,dv = \int \frac{1}{x}dx + c$$

$$-\frac{1}{4}\log(1-2v^2) = \log x + c$$

$$4\log x + \log(1-2v^2) = -4c$$

$$x^4 \cdot (1-2v^2) = e^{-4c}$$

$$x^4\left(1 - 2\frac{y^2}{x^2}\right) = K$$

$$x^2(x^2-2y^2) = K$$

Prob: Solve $(e^{x/y}+1)dx + e^{x/y}(1-x/y)dy = 0$

$$\frac{dx}{dy} = \frac{-e^{x/y}\left(1-\frac{x}{y}\right)}{e^{x/y}+1}$$ is a homogeneous eqn

put $x = \vartheta y \Rightarrow \frac{dx}{dy} = \vartheta + y\frac{d\vartheta}{dy}$

$$\vartheta + y\frac{d\vartheta}{dy} = \frac{-e^{\vartheta}(1-\vartheta)}{e^{\vartheta}+1}$$

$$y\frac{d\vartheta}{dy} = -\frac{e^{\vartheta}(1-\vartheta)}{e^{\vartheta}+1} - \vartheta = -\frac{\vartheta+e^{\vartheta}}{1+e^{\vartheta}}$$

$$-\frac{dy}{y} = \frac{1+e^{\vartheta}}{\vartheta+e^{\vartheta}}d\vartheta$$

I.O.B.S $\quad -\log y = \log_e(\vartheta+e^{\vartheta}) + c$

$$\log y + \log(\vartheta+e^{\vartheta}) = -c$$

$$y(\vartheta+e^{\vartheta}) = e^{-c} = k \text{ say}$$

$$y\left(\frac{x}{y}+e^{x/y}\right) = k$$

$$x + y\,e^{x/y} = k$$

Prob: Solve $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx - x\sec^2(y/x)dy = 0$

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)\cos^2\left(\frac{y}{x}\right)$$

put $y = \vartheta x \Rightarrow \frac{dy}{dx} = \vartheta + x\frac{d\vartheta}{dx}$

$$\vartheta + x\frac{d\vartheta}{dx} = (\vartheta\sec^2\vartheta - \tan\vartheta)\cos^2\vartheta$$

$$x\frac{d\vartheta}{dx} = -\tan\vartheta\cdot\cos^2\vartheta = \frac{-\tan\vartheta}{\sec^2\vartheta}$$

$$\frac{\sec^2\vartheta}{\tan\vartheta}d\vartheta = -\frac{1}{x}dx$$

$$\log(\tan\vartheta) = -\log x + c$$

$$x\tan\vartheta = k$$

$$x\tan(y/x) = k$$

5

Equations reducible to homogeneous form:

The equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ can be reduced to homogeneous form as follows

Case I: when $\frac{a}{a'} \neq \frac{b}{b'}$

put $x = X+h \qquad y = Y+k$

$dx = dX \qquad dy = dY$

$$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')} \rightarrow (1)$$

Solve $\left.\begin{array}{l} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{array}\right\}$ To get h & k

So that (1) becomes

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$$ is a homogenous in X & Y

Put $Y = vX$ or $X = vY$

solve it

Case II: when $\frac{a}{a'} = \frac{b}{b'}$

Let $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ say

$$\therefore \frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c'} \rightarrow (1)$$

put $ax+by = t$

D.w.r.to 'x' $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

from (1) $\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t+c}{mt+c'}$

Separate variables 'x' and 't'

solve it.

3. Linear D.E: A D.E is said to be linear if the dependant variable and it's differential co.efficients occur only in the first degree and not multiplied together.

Prob: Which of the following is a linear D.E?

(a) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=0$ (b) $\frac{d^2y}{dx^2}+y\frac{dy}{dx}+y=0$

(c) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y^2=0$ ✓(d) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

The Standard form of a 1st order linear D.E (Leibnitz linear Equation) is (CE-97)

$$\frac{dy}{dx}+Py=Q$$ where P, Q are function of 'x' or 'constants'

It's solution is $y(I.F)=\int Q(I.F)dx+C$

where $I.F=e^{\int P dx}$

Prob: Solve $(x+1)\frac{dy}{dx}-y=e^{3x}(1+x)^2$

$$\frac{dy}{dx}-\frac{1}{(x+1)}y=e^{3x}(1+x)$$

$$P=\frac{-1}{x+1} \qquad Q=e^{3x}(1+x)$$

$$I.F=e^{\int\frac{-1}{x+1}dx}=e^{-\log_e(1+x)}=\frac{1}{1+x}$$

Solution is $y\cdot\frac{1}{x+1}=\int\frac{e^{3x}(1+x)}{1+x}dx+C$

$$\frac{y}{x+1}=\frac{e^{3x}}{3}+C$$

Prob: Solve $y(\log y)dx+(x-\log y)dy=0$

$$\frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y}$$

$$P=\frac{1/y}{\log y} \qquad Q=\frac{1}{y}$$

$$I.F = e^{\int \frac{1/y}{\log y} dy} = e^{\log(\log y)} = \log y$$

solution is ~~y x~~. $x \log y = \int \frac{1}{y} \log y \, dy + c$

$$x \log y = \frac{1}{2} (\log y)^2 + c$$

$$x = \frac{1}{2} \log y + c \cdot (\log y)^{-1}$$

Prob: $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \tan x \cdot \sec^2 x$$

$$I.F = e^{\int \sec^2 x \, dx} = e^{\tan x}$$

Solution is $y(e^{\tan x}) = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx + C$

$$= \int t e^t dt + C \qquad \tan x = t$$

$$\sec^2 dx = dt$$

$$= te^t - e^t + C$$

$$y(e^{\tan x}) = e^{\tan x}(\tan x - 1) + C$$

Bernoulli's equation:

The equation $\frac{dy}{dx} + Py = Qy^n$ where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called bernoulli's equation.

Divide b.s by $y^n \Rightarrow y^{-n} \frac{dy}{dx} + Py^{1-n} = Q \rightarrow (1)$

Let $y^{1-n} = z \Rightarrow (1-n) y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$

$$y^{-n} \frac{dy}{dx} = \frac{1}{(1-n)} \frac{dz}{dx}$$

from (1) $\frac{1}{(1-n)} \frac{dz}{dx} + Pz = Q$

$$\frac{dz}{dx} + P(1-n) z = Q(1-n) \qquad \text{(CE-05)}$$

which is a Linear D.E in 'z'

Prob: Solve $x\frac{dy}{dx}+y=x^3y^6$

$$y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2 \rightarrow (1)$$

let $y^{-5}=z \Rightarrow -5y^{-6}\frac{dy}{dz}=\frac{dz}{dx}$

from(1) $-\frac{1}{5}\frac{dz}{dx}+\frac{z}{x}=x^2$

$$\frac{dz}{dx}-\frac{5}{x}\cdot z=-5x^2 \quad \text{Leibnitz eqn}$$

$$I.F=e^{-\int\frac{5}{x}dx}=e^{-5\log x}=x^{-5}$$

Solution is $z\cdot e^{-5}=\int(-5x^2)\cdot x^{-5}dx+C$

$$z\cdot e^{-5}=-5\cdot\frac{x^{-3+1}}{-3+1}+C$$

$$y^{-5}\cdot e^{-5}=-5\frac{x^{-2}}{-2}+C$$

4. Exact D.E: D.E'S which can be expressable as an exact differential of some function of x, y are called exact D.E

Ex: $ydx+xdy=0 \Rightarrow d(xy)=0$

$$\Rightarrow xy=C$$

$$\frac{1}{y}dx-\frac{x}{y^2}dy=0 \Rightarrow d\left(\frac{x}{y}\right)=0$$

$$\Rightarrow \frac{x}{y}=C$$

In general, for an equation of the form $Mdx+Ndy=0$ to be exact, the required condition is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. If it is satisfied then the solution of the equation is

$$\int_{y\ \text{const}} M\,dx+\int(\text{terms of } N \text{ independent of } x)\,dy=C$$

⑦ prob: Solve $y\sin 2x\,dx - (1+y^2+\cos^2 x)dy = 0$

$$M = y\sin 2x \qquad N = -(1+y^2+\cos^2 x)$$

$$\frac{\partial M}{\partial y} = \sin 2x \qquad \frac{\partial N}{\partial x} = 2\cos x\cdot\sin x = \sin 2x$$

Solution is $\int\limits_{y\ \text{const}} y\sin 2x + \int -(1+y^2)dy + c$

$$-y\frac{\cos 2x}{2} - y - \frac{y^3}{3} = c$$

prob: $(x^2-4xy-2y^2)dx + (y^2-4xy-2x^2)dy = 0$

$$\frac{\partial M}{\partial y} = -4x-4y \qquad \frac{\partial N}{\partial y} = -4x-4y$$

Solution is $\int\limits_{y\ \text{const}} (x^2-4xy-2y^2)\,dx + \int y^2 dy = 0$

$$\frac{x^3}{3} - 4y\cdot\frac{x^2}{2} - 2y^2x + \frac{y^3}{3} = c$$

$$x^3 - 6x^2y - 6xy^2 + y^3 = k$$

prob: $(2xy+y-\tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

$$\frac{\partial M}{\partial y} = 2x - \sec^2 y + 1 \qquad \frac{\partial N}{\partial x} = 2x - \tan^2 y$$

$$= 2x - \tan^2 y$$

Solution is $\int\limits_{y\ \text{const}} (2xy+y-\tan y)dx + \int \sec^2 y\,dx = 0$

$$2y\cdot\frac{x^2}{2} + xy - x\cdot\tan y + \tan y = c$$

Previous problems

prob: (GATE 94): The necessary & sufficient conditions for the D.E of the form $M(x,y)dx + N(x,y)dy$ to be exact is

Sol: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

CE-99: If C is a constant then the solution of $\frac{dy}{dx} = 1+y^2$ is

Sol:
$$\frac{1}{1+y^2}dy = dx$$

I.O.B.S $\tan^{-1}y = x + C$

$$y = \tan(x+C)$$

IN-08: Particular soln $y = \tan(x+3)$

ME-03: The solution of D.E $\frac{dy}{dx} + y^2 = 0$ is

Sol:
$$\frac{1}{y^2}dy = -dx$$

I.O.B.S $-\frac{1}{y} = -x + C$

$$\frac{1}{y} = x - C \Rightarrow y = \frac{1}{x+K} \quad (K = -C)$$

CE-04: Bio transformation of an organic compound having concentration (x) can be modeled using an ordinary differential eqn $\frac{dx}{dt} + Kx^2 = 0$, where K is the reaction constant. If $x = a$ at $t = 0$ then solution of the eqn is

Sol:
$$\frac{dx}{dt} = -Kx^2$$
$$-\frac{dx}{x^2} = K\,dt$$

I.O.B.S $+\frac{1}{x} = +Kt + C \quad \rightarrow (1)$

Given $x = a$ at $t = 0 \Rightarrow \frac{1}{a} = C$

$\therefore$ soln is $\frac{1}{x} = Kt + \frac{1}{a}$

EE-05: The solution of 1st order D.E $\dot{x}(t) = -3x(t)$, $x(0) = x_0$ is

EC-08: $x(0) = 2$

Sol:
$$\frac{dx}{dt} = -3x$$
$$\frac{1}{x}dx = -3\,dt$$

I.O.B.S $\log x = -3t + C$

$$x = e^{-3t}e^{C} \quad \rightarrow (1)$$

Given $x(0) = x_0$

from (1) $x_0 = e^0 \cdot e^C$

$e^C = x_0$

$\therefore$ Solution is $x = e^{-3t} \cdot x_0$

$$I.F = e^{\int \frac{1}{x} dx} = e^{\log_e x} = x$$

Solution is $y \cdot x = \int x^3 \cdot x \, dx + c$

$$y \cdot x = \frac{x^5}{5} + c$$

Given $y(1) = \frac{6}{5} \Rightarrow \frac{6}{5} = \frac{1}{5} + c \Rightarrow c = 1$

$\therefore$ Solution is $xy = \frac{x^5}{5} + 1$

$$y = \frac{x^4}{5} + \frac{1}{x}$$

PI-10: The solution of the D.E $\frac{dy}{dx} - y^2 = 1$ satisfying the condition $y(0) = 1$ is

Sol:

$$\frac{dy}{dx} = 1 + y^2$$

$$\int \frac{1}{1+y^2} dy = \int dx + c$$

$$\tan^{-1} y = x + c$$

Given $y(0) = 1 \Rightarrow \tan^{-1}(1) = 0 + c \Rightarrow c = \pi/4$

$\therefore$ Solution is $\tan^{-1} y = x + \pi/4$

$$y = \tan(x + \pi/4)$$

PI-10: Which of the following D.E has a solution given by the function $y = 5\sin(3x + \pi/3)$

Sol:

$$y = 5\sin 3x \cdot \cos \pi/3 + 5\cos 3x \cdot \sin \pi/3$$

$$= \underbrace{5\cos \pi/3}_{a} \cdot \sin 3x + \underbrace{5\sin \pi/3}_{b} \cos 3x$$

$y = a\sin 3x + b\cos 3x$ a&b are arbitrary constants

$$y' = 3a\cos 3x - 3b\sin 3x$$

$$y'' = -9a\sin 3x - 9b\cos 3x$$

$$= -9y$$

$$y'' + 9y = 0$$