

# Theoretical Guide

## Lenhadoras de Segtree

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## 1 Number Theory

$$(a + b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$

$$(a - b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$

$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$

$$a^b \bmod m = (a \bmod m)^b \bmod m$$

$$a \equiv b \pmod{m} \iff (b - a) | m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$

$$\text{lcm}(a, b) \times \gcd(a, b) = a \times b$$

$$\text{lcm}(a, b) = \frac{a \times b}{\gcd(a, b)} = \frac{a}{\gcd(a, b)} \times b$$

$$\gcd(a, b) = b ? \gcd(b, a \% b) : a$$

### 1.1 Approximation of Number of Divisors

The number of divisors of  $n$  is about  $\sqrt[3]{n}$ .

$n$	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

### 1.2 Fermat's Theorems

Let  $P$  be a prime number and  $a$  an integer, then:

$$a^p \equiv a \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

**Lemma:** Let  $p$  be a prime number and  $a$  and  $b$  integers, then:

$$(a + b)^p \equiv a^p + b^p \pmod{p}$$

**Lemma:** Let  $p$  be a prime number and  $a$  an integer. The inverse of  $a$  modulo  $p$  is  $a^{p-2}$ :

$$a^{-1} \equiv a^{p-2} \pmod{p}$$

### 1.3 K leading digits of n!

A similar idea can be used to calculate the first digits of exponentiation.

$$\log_{10} n! = \log_{10}(1 \times 2 \times 3 \times \dots \times n) = \log_b 1 + \log_{10} 2 + \log_{10} 3 + \dots + \log_{10} n$$

Decimal part:

$$q = \log_{10} n! - (\text{int}) \log_{10} n!$$

Leading digits:

$$b = \text{pow}(10, q)$$

```
// Shift decimal point k-1 times
for ( int i = 0; i < k - 1; i++ ) {
    b *= 10;
}
```

$$\text{leadingdigits} = \lfloor b \rfloor$$

## 1.4 Sum of divisors

Given the prime factorization

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3}$$

The sum of divisors of n is

$$\phi(n) = \frac{p_1^{e_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{e_2+1} - 1}{p_2 - 1} \cdot \frac{p_3^{e_3+1} - 1}{p_3 - 1}$$

## 1.5 Prime counting function - $\pi(x)$

Expected to have  $\frac{x}{\log x}$  primes within  $[1, x]$ . The prime counting function is asymptotic to  $\frac{x}{\log x}$ , by the prime number theorem.

x	10	10 <sup>2</sup>	10 <sup>3</sup>	10 <sup>4</sup>	10 <sup>5</sup>	10 <sup>6</sup>	10 <sup>7</sup>	10 <sup>8</sup>
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

## 1.6 Number of digits of n in base b

If

$$\sqrt[k]{n} < b$$

then n has k or less digits when written in base b.

## 1.7 Product of divisors

Given the prime factorization

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3}$$

The product of divisors of n is

$$p(n) = n^{d(n)/2}$$

$$p(n) = (p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3})^{(e_1+1) \cdot (e_2+1) \cdot (e_3+1)/2}$$

$$p(n) = p_1^{e_1 \cdot (e_1+1)(e_2+1) \cdot (e_3+1)/2} \cdot p_2^{e_2 \cdot (e_1+1)(e_2+1) \cdot (e_3+1)/2} \cdot p_3^{e_3 \cdot (e_1+1)(e_2+1) \cdot (e_3+1)/2}$$

For any  $e_i$ , it is guaranteed that either  $e_i$  or  $e_i + 1$  will be divisible by 2.

When calculating the exponent  $e_1 \cdot (e_1 + 1)(e_2 + 1) \cdot (e_3 + 1)/2$ , get it % MOD - 1, from Fermat's Theorem.

## 1.8 Number of divisors

Given the prime factorization

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3}$$

The number of divisors of n is

$$d(n) = (e_1 + 1) * (e_2 + 1) * (e_3 + 1)$$

## 1.9 Sum of digits of n in base b

$$f(n, b) = \begin{cases} n & n < b \\ f\left(n, \left\lfloor \frac{n}{b} \right\rfloor\right) + (n \bmod b) & n \geq b \end{cases}$$

## 1.10 Number of digits of n! in base b

$$\lfloor \log_b n! \rfloor + 1 = \lfloor \log_b (1 \times 2 \times 3 \times \dots \times n) \rfloor + 1 = \lfloor \log_b 1 + \log_b 2 + \log_b 3 + \dots + \log_b n \rfloor + 1$$

## 2 C++

### 2.1 Priority Queue

```
template<class T> using min_priority_queue =
priority_queue<T, vector<T>, greater<T>>;
```

### 2.2 Ordered set and multiset

```
typedef tree<pair<ll, ll>, null_type, less<pair<ll, ll>>,
rb_tree_tag, tree_order_statistics_node_update> ordered_set;
```

To change to multiset switch equal to less\_equal.

## 3 Bitwise

### 3.1 NOT

$A$	$X$
0	1
1	0

### 3.2 AND

$A$	$B$	$X$
0	0	0
0	1	0
1	0	0
1	1	1

### 3.3 OR

$A$	$B$	$X$
0	0	0
0	1	1
1	0	1
1	1	1

### 3.4 NAND

$A$	$B$	$X$
0	0	1
0	1	1
1	0	1
1	1	0

### 3.5 NOR

$A$	$B$	$X$
0	0	1
0	1	0
1	0	0
1	1	0

### 3.6 XOR

$A$	$B$	$X$
0	0	0
0	1	1
1	0	1
1	1	0

### 3.7 XNOR

$A$	$B$	$X$
0	0	1
0	1	0
1	0	0
1	1	1

### 3.8 XOR from 1 to n

$$f(n) = \begin{cases} n & n \equiv 0 \pmod{4} \\ 1 & n \equiv 1 \pmod{4} \\ n+1 & n \equiv 2 \pmod{4} \\ 0 & n \equiv 3 \pmod{4} \end{cases}$$

### 3.9 Number of bits on

`__builtin_popcount(x)`

### 3.10 Count leading zeros

```
__builtin_clz(z)
__builtin_clzll(z)
```

### 3.11 MSB

```
32 - _builtin_clz(x)
64 - __builtin_clzll(x)
```

### 3.12 Count trailing zeros

```
__builtin_ctz(x)
__builtin_ctzll(x)
```

### 3.13 LSB

```
__builtin_ffs(X)
```

### 3.14 Turn bit on or off

Turn on bit  $i$   $x |= (1 << i)$   
 Turn off bit  $i$   $x \&= \sim(1 << i)$

### 3.15 Check if bit is on or off

Check if bit is on  $x \& (1 << i)$   
 Check if bit is off  $!(x \& (1 << i))$

## 4 Constants

```
LLINF = 0x3f3f3f3f3f3f3f3fLL
```

```
PI = acos(-1)
```

## 5 Progressions

### 5.1 Geometric Progression

#### 5.1.1 General Term

$$a_1 q^{n-1}$$

#### 5.1.2 Sum

$$\frac{a_1(q^n - 1)}{q - 1}$$

#### 5.1.3 Infinite Sum

$$-1 < q < 1$$

$$\frac{a_1}{1 - q}$$

### 5.2 Arithmetic Progression

#### 5.2.1 General Term

$$a_1 + (n - 1)r$$

#### 5.2.2 Sum

$$\frac{(a_1 + a_n)n}{2}$$

#### 5.2.3 Sum of Second Order Arithmetic Progression

$a_1$  is the first element of the original progression,  $b_1$  is the first element of the derived progression,  $n$  is the number of elements of the original progression and  $r$  is the ratio of the derived progression

$$a_1 n + \frac{(b_1 n(n - 1))}{2} + \frac{r n(n - 1)(n - 2)}{6}$$

## 6 Basic Math

### 6.1 Logarithm

$$\log_b mn = \log_b m + \log_b n \quad \log_b \frac{m}{n} = \log_b m - \log_b n \quad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \quad \log_b n = \log_a n \log_b a \quad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \quad \log_b a = \frac{1}{\log_a b} \quad \log_b a \log_a c = \log_b c$$

$$\log_b 1 = 0 \quad \log_b b = 1$$

### 6.2 Recurring Decimal

To find whether a fraction in its most simple form is a recurring decimal, find the prime factors of the denominator. If there are any prime factors other than 2 and 5 then the fraction is a recurring decimal.

### 6.3 Divisibility Criteria

#### 6.3.1 2

The last digit is either 0, 2, 4, 6 or 8

#### 6.3.2 3

The sum of the digits is also divisible by 3

#### 6.3.3 4

The last two digits form a number that is divisible by 4

#### 6.3.4 5

The last digit is either 0 or 5

#### 6.3.5 6

It has to be divisible by both 2 and 3

#### 6.3.6 7

The subtraction of the number formed without the last digit and the last digit times 2 is also divisible by 7

#### 6.3.7 8

The last three digits form a number that is divisible by 8

#### 6.3.8 9

The sum of the digits is also divisible by 9

## 7 Combinatorics

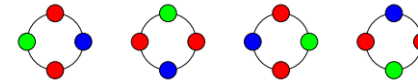
### 7.1 Burnside's Lemma

Burnside's lemma can be used to count the number of combinations so that only one representative is counted for each group of symmetric combinations. For example, if we have a necklace with different colored pearls and we want to know how many different combinations we can make.

For example, if we have a necklace colored like this:



This variations are the same if we consider that we can rotate the necklace:



When the number of steps is  $k$ , the number of necklaces that remain the same are:

$$m^{\gcd(k,n)}$$

The number of diferent combinations for  $m$  colors and a necklace of size  $n$  is

$$\sum_{i=0}^{n-1} \frac{m^{\gcd(i,n)}}{n}$$

So a necklace of length 4 with 3 colors has

$$\frac{3^4 + 3 + 3^2 + 3}{4} = 24$$

## 8 Misc

### 8.1 Check for overflow

Returns false if there is no overflow and true if there is overflow. The variable `v` stores the result of the operation.

```
long long v;
__builtin_add_overflow(a, b, v);
cout << v;
```

```
__builtin_sub_overflow(a, b, v);
__builtin_mul_overflow(a, b, v);
```

### 8.2 Input by file

```
freopen("input.txt", "r", stdin);
freopen("output.txt", "w", stdout);
```

## 9 Geometry

### 9.1 Triangle Existence Condition

$$a + b \geq c$$

$$a + c \geq b$$

$$b + c \geq a$$

### 9.2 Distances

#### 9.2.1 Euclidean

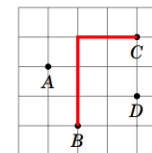
$$d(p, q) = \sqrt{(q.x - p.x)^2 + (q.y - p.y)^2}$$

#### 9.2.2 Manhattan

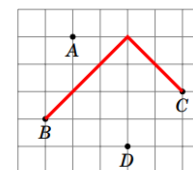
$$|p.x - q.x| + |p.y - q.y|$$

### 9.3 Maximum possible manhattan distance between two points given n points

Given  $n$  points, for instance:



Rotate all coordinates  $45^\circ$  so that  $(x, y)$  becomes  $(x + y, y - x)$ , so,  $p$  becomes  $p'$  and  $q$  becomes  $q'$ .



The maximum manhattan distance is obtaining by choosing the two points that maximize:

$$\max(|p'.x - q'.x|, |p'.y - q'.y|)$$

### 9.4 Sines Rule

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

### 9.5 Cossines Rule

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

### 9.6 Pick's Theorem

$$A = a + \frac{b}{2} - 1$$

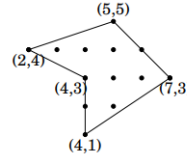
where  $A$  is the area of the polygon,  $a$  is the number of integer points inside the polygon and  $b$  is the number of integer points in the boundary of the polygon (not counting the vertexes).

## 9.7 Boundary points

The number of integer points in the boundary of a polygon is:

$$B = v + b$$

where  $v$  is the number of vertices (integer points as well) and  $b$  is the number of integer points situated between two vertices, like in the following figure:



$b$  can be calculated for every line connecting two points (including the line between the last and the first point) as follows:

$$boundary\_points(p, q) = \begin{cases} |p.y - q.y| - 1 & p.x = q.x \\ |p.x - q.x| - 1 & p.y = q.y \\ gcd(|p.x - q.x|, |p.y - q.y|) - 1 & \text{otherwise} \end{cases}$$

## 9.8 Perimeter

### 9.8.1 Circle

$$2\pi r$$

## 9.9 Areas

### 9.9.1 Circle

$$\pi r^2$$

### 9.9.2 Triangle

$$\frac{b * h}{2}$$

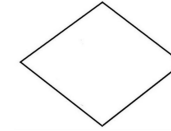
### 9.9.3 Square

$$l^2$$

### 9.9.4 Rectangle

$$hr$$

### 9.9.5 Rhombus



$D$  is the biggest diagonal and  $d$  is the smallest diagonal

$$A = \frac{1}{2} * D * d$$

## 9.10 Volumes

### 9.10.1 Sphere

$$\frac{4}{3}\pi r^3$$

### 9.10.2 Prism

$$V = bh$$

### 9.10.3 Pyramid

$$\frac{bh}{3}$$

### 9.10.4 Cone

$$\frac{\pi r^2 h}{3}$$

## 9.11 Shoelace Formula

Calculates the area of a polygon.

$$A = \frac{1}{2} \left| \sum_{i=1}^{n-1} (p_i \times p_{i+1}) \right| = \frac{1}{2} \left| \sum_{i=1}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) \right|$$

Where the points  $p_1, p_n, \dots$  are in adjacent order and the first and last vertex is the same, that is,  $p_1 = p_n$



## 10 Identities

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$
$$\sum_{i=1}^n \frac{1}{i} \approx \log n \quad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$