Theoretical Guide
Lenhadoras de Segtree

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Contents

1 Misc **1**
- 1.1 Input by file . . . 1
- 1.2 Check for overflow . . . 1

2 C++ **1**
- 2.1 Priority Queue . . . 1
- 2.2 Ordered set and multiset . . . 1

3 Number Theory **1**
- 3.1 Approximation of Number of Divisors . . . 1
- 3.2 Prime counting function - $\pi(x)$ . . . 1
- 3.3 Number of divisors . . . 2
- 3.4 Number of digits of n in base b . . . 2
- 3.5 K leading digits of n! . . . 2
- 3.6 Number of digits of n! in base b . . . 2
- 3.7 Sum of divisors . . . 2
- 3.8 Sum of digits of n in base b . . . 2
- 3.9 Fermat's Theorems . . . 2
- 3.10 Product of divisors . . . 2

4 Constants **3**

5 Combinatorics **3**
- 5.1 Burnside's Lemma . . . 3
- 5.2 Catalan Numbers . . . 3
  - 5.2.1 K-th convolution of Catalan . . . 3

6 Identities **3**

7 Geometry **3**
- 7.1 Sines Rule . . . 3
- 7.2 Cossines Rule . . . 4
- 7.3 Integer Coordinates in a Line . . . 4
- 7.4 Boundary points . . . 4
- 7.5 Pick's Theorem . . . 4
- 7.6 Perimeter . . . 4
  - 7.6.1 Circle . . . 4
- 7.7 Areas . . . 4
  - 7.7.1 Circle . . . 4
  - 7.7.2 Triangle . . . 4
  - 7.7.3 Square . . . 4
  - 7.7.4 Rectangle . . . 4
  - 7.7.5 Rhombus . . . 4
  - 7.7.6 Circular Sector . . . 5
- 7.8 Volumes . . . 5
  - 7.8.1 Sphere . . . 5
  - 7.8.2 Prism . . . 5
  - 7.8.3 Pyramid . . . 5
  - 7.8.4 Cone . . . 5
- 7.9 Triangle Existence Condition and Degenerate Triangles . . . 5
- 7.10 Shoelace Formula . . . 5
- 7.11 Distances . . . 5
  - 7.11.1 Euclidean . . . 5
  - 7.11.2 Manhattan . . . 6
- 7.12 Maximum possible manhattan distance between two points given n points . . . 6

8 Bitwise **6**
- 8.1 XOR from 1 to n . . . 6
- 8.2 Turn bit on or off . . . 6
- 8.3 Check if bit is on or off . . . 6
- 8.4 Number of bits on . . . 6
- 8.5 Count leading zeros . . . 6
- 8.6 MSB . . . 6
- 8.7 Count trailing zeros . . . 6
- 8.8 LSB . . . 6
- 8.9 NOT . . . 7

8.10 AND . . . 7
8.11 OR . . . 7
8.12 NAND . . . 7
8.13 NOR . . . 7
8.14 XOR . . . 7
8.15 XNOR . . . 7

9 Progressions **7**
9.1 Arithmetic Progression . . . 7
9.1.1 General Term . . . 7
9.1.2 Sum . . . 7
9.1.3 Sum of Second Order Arithmetic Progression . . . 8
9.2 Geometric Progression . . . 8
9.2.1 General Term . . . 8
9.2.2 Sum . . . 8
9.2.3 Infinite Sum . . . 8

10 Basic Math **8**
10.1 Divisibility Criteria . . . 8
10.1.1 2 . . . 8
10.1.2 3 . . . 8
10.1.3 4 . . . 8
10.1.4 5 . . . 8
10.1.5 6 . . . 8
10.1.6 7 . . . 8
10.1.7 8 . . . 8
10.1.8 9 . . . 8
10.2 Logarithm . . . 8
10.3 Recurring Decimal . . . 8

1 Misc

1.1 Input by file

freopen("input.txt","r",stdin);
freopen("output.txt","w",stdout);

1.2 Check for overflow

Returns false if there is no overflow and true if there is overflow. The variable v stores the result of the operation.

```
long long v;
__builtin_add_overflow(a, b, v);
cout << v;

__builtin_sub_overflow(a, b, v);
__builtin_mul_overflow(a, b, v);
```

2 C++

2.1 Priority Queue

```
template<class T> using min_priority_queue =
priority_queue<T, vector<T>, greater<T>>;
```

2.2 Ordered set and multiset

```
typedef tree<pair<ll, ll>, null_type, less<pair<ll, ll>>,
rb_tree_tag, tree_order_statistics_node_update> ordered_set;
```

To change to multiset switch equal to less_equal.

3 Number Theory

$$(a+b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$
$$(a-b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$
$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$
$$a^b \bmod m = (a \bmod m)^b \bmod m$$
$$a \equiv b \pmod m \iff (b-a)|m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$
$$\operatorname{lcm}(a,b) \times \gcd(a,b) = a \times b$$
$$\operatorname{lcm}(a,b) = \frac{a \times b}{\gcd(a,b)} = \frac{a}{\gcd(a,b)} \times b$$
$$gcd(a,b) = b?gcd(b, a\%b) : a$$

3.1 Approximation of Number of Divisors

The number of divisors of n is about $\sqrt[3]{n}$.

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

3.2 Prime counting function - $\pi(x)$

Expected to have $\frac{x}{\log x}$ primes within $[1, x]$. The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

3.3 Number of divisors

Given the prime factorization

$$n = p_1^{e_1}.p_2^{e_2}.p_3^{e_3}$$

The number of divisors of n is

$$d(n) = (e1 + 1) * (e2 + 1) * (e3 + 1)$$

3.4 Number of digits of n in base b

If

$$\sqrt[k]{n} < b$$

then n has k or less digits when written in base b.

3.5 K leading digits of n!

A similar idea can be used to calculate the first digits of exponentiation.

$$\log_{10} n! = \log_{10}(1 \times 2 \times 3 \times ... \times n) = \log_b 1 + \log_{10} 2 + \log_{10} 3 + ... + \log_{10} n$$

Decimal part:

$$q = \log_{10} n! - (int) \log_{10} n!$$

Leading digits:

$$b = pow(10, q)$$

```
// Shift decimal point k-1 times
for ( int i = 0; i < k - 1; i++ ) {
    b *= 10;
}
```

$$leadingdigits = \lfloor b \rfloor$$

3.6 Number of digits of n! in base b

$\lfloor \log_b n! \rfloor + 1 = \lfloor \log_b(1 \times 2 \times 3 \times ... \times n) \rfloor + 1 = \lfloor \log_b 1 + \log_b 2 + \log_b 3 + ... + \log_b n \rfloor + 1$

3.7 Sum of divisors

Given the prime factorization

$$n = p_1^{e_1}.p_2^{e_2}.p_3^{e_3}$$

The sum of divisors of n is

$$\phi(n) = \frac{p_1^{e_1+1} - 1}{p_1 - 1}.\frac{p_2^{e_2+1} - 1}{p_2 - 1}.\frac{p_3^{e_3+1} - 1}{p_3 - 1}$$

3.8 Sum of digits of n in base b

$$f(n, b) = \begin{cases} n & n < b \\ f\left(n, \left\lfloor \frac{n}{b} \right\rfloor + (n \mod b)\right) & n \geq b \end{cases}$$

3.9 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a + b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

3.10 Product of divisors

Given the prime factorization

$$n = p_1^{e_1}.p_2^{e_2}.p_3^{e_3}$$

The product of divisors of n is

$$p(n) = n^{d(n)/2}$$

$$p(n) = (p_1^{e_1}.p_2^{e_2}.p_3^{e_3})^{(e1+1).(e2+1).(e3+1)/2}$$

$$p(n) = p_1^{e_1.(e_1+1)(e2+1).(e3+1)/2}.p_2^{e_2.(e_1+1)(e2+1).(e3+1)/2}.p_3^{e_3.(e_1+1)(e2+1).(e3+1)/2}$$

For any e_i, it is guaranteed that either e_i or $e_i + 1$ will be divisible by 2.

When calculating the exponent $e_1.(e_1+1)(e2+1).(e3+1)/2$, get it % MOD - 1, from Fermat's Theorem.

4 Constants

```
LLINF = 0x3f3f3f3f3f3f3f3fLL
```

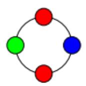
```
PI = acos(-1)
```

5 Combinatorics

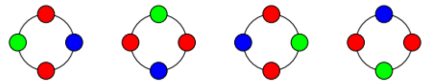
5.1 Burnside's Lemma

Burnside's lemma can be used to count the number of combinations so that only one representative is counted for each group of symmetric combinations. For example, if we have a necklace with different colored pearls and we want to know how many different combinations we can make.

For example, if we have a necklace colored like this:



This variations are the same if we consider that we can rotate the necklace:



When the number of steps is k, the number of necklaces that remain the same are:

$$m^{gcd(k,n)}$$

The number of diferent combinations for m colors and a necklace of size n is

$$\sum_{i=0}^{n-1} \frac{m^{gcd(i,n)}}{n}$$

So a necklace of length 4 with 3 colors has

$$\frac{3^4 + 3 + 3^2 + 3}{4} = 24$$

5.2 Catalan Numbers

The Catalan number C_n equals the number of valid parenthesis expressions that consist of n left parentheses and n right parentheses.

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Catalan numbers are also related to trees:

- there are C_n binary trees of n nodes
- there are C_{n-1} rooted trees of n nodes.

5.2.1 K-th convolution of Catalan

Finds the count of balanced parentheses sequences consisting of n+k pairs of parentheses where the first k symbols are open brackets.

$$C_k = \frac{k+1}{n+k+1}\binom{2n+k}{n}$$

6 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log n \qquad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

7 Geometry

7.1 Sines Rule

$$\frac{a}{sin(\alpha)} = \frac{b}{sin(\beta)} = \frac{c}{sin(\gamma)}$$

7.2 Cossines Rule

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

7.3 Integer Coordinates in a Line

Given the line segment from (x1, x2) to (y1, y2), the number of points situated in this line segment for which both x and y are integers is given by:

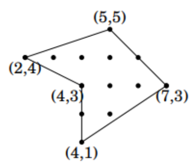
$$gcd(|x1 - x2|, |y1 - y2|) + 1$$

7.4 Boundary points

The number of integer points in the boundary of a polygon is:

$$B = v + b$$

where v is the number of vertices (integer points as well) and b is the number of integer points situated between two vertices, like in the following figure:



b can be calculated for every line connecting two points (including the line between the last and the first point) as follows:

$$boundary\_points(p,q) = \begin{cases} |p.y - q.y| - 1 & \text{p.x = q.x} \\ |p.x - q.x| - 1 & \text{p.y = q.y} \\ gcd(|p.x - q.x|, |p.y - q.y|) - 1 & \end{cases}$$

7.5 Pick's Theorem

$$A = a + \frac{b}{2} - 1$$

where A is the area of the polygon, a is the number of integer points inside the polygon and b is the number of integer points in the boundary of the polygon (not counting the vertexes).

7.6 Perimeter

7.6.1 Circle

$$2\pi r$$

7.7 Areas

7.7.1 Circle

$$\pi r^2$$

7.7.2 Triangle

$$\frac{b * h}{2}$$

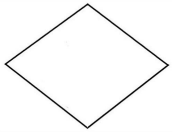
7.7.3 Square

$$l^2$$

7.7.4 Rectangle

$$hr$$

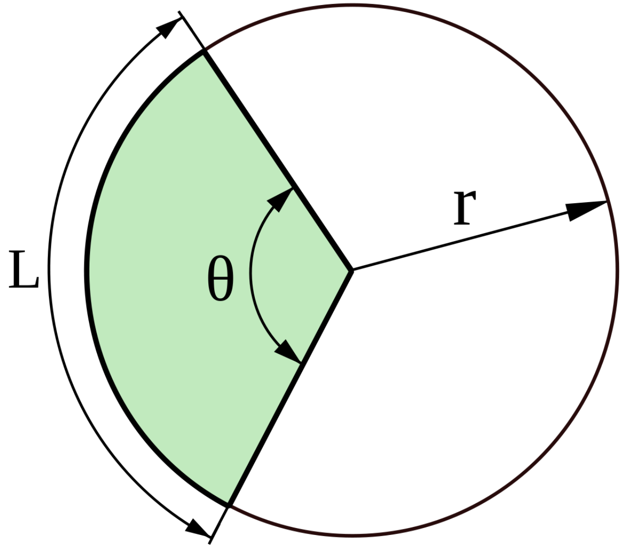
7.7.5 Rhombus



D is the biggest diagonal and d is the smallest diagonal

$$A = \frac{1}{2} * D * d$$

7.7.6 Circular Sector



$$A = \frac{l * d}{2}$$

For α in radians:

$$A = \frac{r^2 * \alpha}{2}$$

For θ in degrees:

$$A = \frac{\theta * \pi * r^2}{360°}$$

7.8 Volumes

7.8.1 Sphere

$$\frac{4}{3}\pi r^3$$

7.8.2 Prism

$$V = bh$$

7.8.3 Pyramid

$$\frac{bh}{3}$$

7.8.4 Cone

$$\frac{\pi r^2 h}{3}$$

7.9 Triangle Existence Condition and Degenerate Triangles

$$a + b \geq c$$
$$a + c \geq b$$
$$b + c \geq a$$

If it's a requirement that the triangle isn't degenerate (all of its vertices are collinear), then the existence condition is:

$$a + b > c$$
$$a + c > b$$
$$b + c > a$$

7.10 Shoelace Formula

Calculates the area of a polygon.

$$A = \frac{1}{2}|\sum_{i=1}^{n-1}(p_i \times p_{i+1}) = \frac{1}{2}|\sum_{i=1}^{n-1}(x_i y_{i+1} - x_{i+1} y_i)|$$

Where the points $p_1, pn, \ldots$ are in adjecent order and the first and last vertex is the same, that is, $p_1 = pn$

7.11 Distances

7.11.1 Euclidean

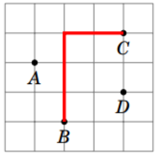
$$d(p,q) = \sqrt{(q.x - p.x)^2 + (q.y - p.y)^2}$$

7.11.2 Manhattan

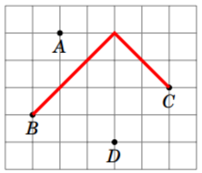
$$|p.x - q.x| + |p.y - q.y|$$

7.12 Maximum possible manhattan distance between two points given n points

Given n points, for instance:



Rotate all coordinates 45^o do that (x, y) becomes $(x+y, y-x)$, so, p becomes p' and q becomes q'.



The maximum manhattan distance is obtaining by choosing the two points that maximize:

$$max(|p'.x - q'.x|, |p'.y - q'.y|)$$

8 Bitwise

8.1 XOR from 1 to n

$$f(n) = \begin{cases} n & n \equiv 0 \pmod 4 \\ 1 & n \equiv 1 \pmod 4 \\ n+1 & n \equiv 2 \pmod 4 \\ 0 & n \equiv 3 \pmod 4 \end{cases}$$

8.2 Turn bit on or off

Turn on bit i x |= (1 << i)
Turn off bit i x &= ~(1 << i)

8.3 Check if bit is on or off

Check if bit is on x & (1 << i)
Check if bit is off !(x & (1 << i))

8.4 Number of bits on

```
__builtin_popcount(x)
```

8.5 Count leading zeros

```
__builtin_clz(z)
__builtin_clzll(z)
```

8.6 MSB

```
32 - _builtin_clz(x)
64 - __bultin_clzll(x)
```

8.7 Count trailing zeros

```
__builtin_ctz(x)
__builtin_ctzll(x)
```

8.8 LSB

```
__builtin_ffs(X)
```

8.9 NOT

A	X
0	1
1	0

8.10 AND

A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

8.11 OR

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

8.12 NAND

A	B	X
0	0	1
0	1	1
1	0	1
1	1	0

8.13 NOR

A	B	X
0	0	1
0	1	0
1	0	0
1	1	0

8.14 XOR

A	B	X
0	0	0
0	1	1
1	0	1
1	1	0

8.15 XNOR

A	B	X
0	0	1
0	1	0
1	0	0
1	1	1

9 Progressions

9.1 Arithmetic Progression

9.1.1 General Term

$$a_1 + (n-1)r$$

9.1.2 Sum

$$\frac{(a_1 + a_n)n}{2}$$

9.1.3 Sum of Second Order Arithmetic Progression

a_1 is the first element of the original progression, b_1 is the first element of the derived progression, n is the number of elements of the original progression and r is the ratio of the derived progression

$$a_1 n + \frac{(b_1 n(n-1)}{2} + \frac{rn(n-1)(n-2)}{6}$$

9.2 Geometric Progression

9.2.1 General Term

$$a_1 q^{n-1}$$

9.2.2 Sum

$$\frac{a_1(q^n - 1)}{q - 1}$$

9.2.3 Infinite Sum

$$-1 < q < 1$$

$$\frac{a_1}{1-q}$$

10 Basic Math

10.1 Divisibility Criteria

10.1.1 2

The last digit is either 0, 2, 4, 6 or 8

10.1.2 3

The sum of the digits is also divisible by 3

10.1.3 4

The last two digits form a number that is divisble by 4

10.1.4 5

The last digit is either 0 or 5

10.1.5 6

It has to be divisible by both 2 and 3

10.1.6 7

The subtraction of the number formed without the last digit and the last digit times 2 is also divisible by 7

10.1.7 8

The last three digits form a number that is divisble by 8

10.1.8 9

The sum of the digits is also divisible by 9

10.2 Logarithm

$$\log_b mn = \log_b m + \log_b n \qquad log_b \frac{m}{n} = \log_b m - \log_n n \qquad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \qquad \log_b n = \log_a n \log_b a \qquad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b a = \frac{1}{\log_a b} \qquad \log_b a \ \log_a c = \log_b c$$

$$\log_b 1 = 0 \qquad \log_b b = 1$$

10.3 Recurring Decimal

To find whether a fraction in its most simple form is a recurring decimal, find the prime factors of the denominator. If there are any prime factors other than 2 and 5 then the fraction is a recurring decimal.