2.1) Encuentre la expresión del espectro de Fourier (forma exponencial y trigonométrica) para la señal $x(t) = |6\sin(3t + \frac{\pi}{4})|^2$ con $t \in [-\pi, \pi]$

$$x(t) = |6\sin(3t + \frac{\pi}{4})|^2 = 6^2 \sin^2(3t + \frac{\pi}{4})$$

Por identidad:

$$\sin^2(\theta) = \frac{1}{2} - \frac{\cos(2\theta)}{2}$$

se obtiene:

$$x(t) = 36\left(\frac{1}{2} - \frac{\cos(6t + \pi/2)}{2}\right) = \frac{\overset{18}{\cancel{36}}}{\cancel{2}} - 18\cos(6t + \pi/2)$$

$$x(t) = 18 - 18\cos(6t + \pi/2)$$

Ahora,

$$\cos(\theta + \pi/2) = -\sin(\theta)$$

$$x(t) = 18 + 18\sin(6t)$$

Forma trigonométrica:

$$x(t) = a_0 + \sum_{n=-N}^{N} a_n \cos(n w_0 t) + b_n \sin(n w_0 t)$$

Ya que $x(t)$ corresponde a una función seno, el seno presenta simetría impar, entonces $x(t) = -x(-t)$

Entonces,

$$a_n = 0$$

Finalmente:

$$x(t) = 18 + 18\sin(6t) = a_0 + \sum_{n=-N}^{n} b_n \sin(n w_0 t)$$

$$a_0 = c_0 = \frac{1}{t_f - t_i} \int_{t_i}^{t_f} x(t)\,dt$$

$$a_0 = \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} 18 + 18 \sin(6t)\, dt$$

$$= \frac{18}{2\pi} \int_{-\pi}^{\pi} dt + \frac{18}{2\pi} \int_{-\pi}^{\pi} \sin(6t)\, dt$$

- $\frac{18}{2\pi} \int_{-\pi}^{\pi} \sin(6t)\, dt$

$$= \frac{18}{2\pi} - \cos(6t) \Big|_{-\pi}^{\pi} = \frac{18}{2\pi} \left[- \cos(6(\pi)) + \cos(6(-\pi)) \right]$$

$$= \frac{18}{2\pi} [0] = 0$$

- $\frac{18}{2\pi} \int_{-\pi}^{\pi} dt = \frac{18}{2\pi}(\pi) - \frac{18}{2\pi} = 9 - (-9) = 18$

$$a_0 = 18$$

Ahora, para b_n =

$$b_n = \frac{2}{t_f - t_i} \int_{t_i}^{t_f} x(t) \sin(n w_0 t)\, dt$$

$$b_n = \frac{2}{\pi - (-\pi)} \int_{-\pi}^{\pi} 18 + 18 \sin(6t) \sin(n w_0 t)\, dt$$

$$b_n = \frac{2}{2\pi} \left[\underbrace{\int_{-\pi}^{\pi} 18 \sin(n w_0 t)\, dt}_{1} + \underbrace{\int_{-\pi}^{\pi} 18 \sin(6t) \sin(n w_0 t)\, dt}_{2} \right]$$

Por identidad:

$$\sin(\theta) \sin(\alpha) = \frac{\cos(\theta - \alpha) - \cos(\theta + \alpha)}{2}$$

$\sin\theta = \sin(6t)$ $\quad$ $\sin\alpha = \sin(n w_0 t)$

$$= \frac{\cos(6t - nw_0t) - \cos(6t + nw_0t)}{2} = \frac{\cos((6-nw_0)t) - \cos(6+nw_0)t)}{2}$$

$$W_0 = \frac{2\pi}{T} \qquad T = 2\pi \qquad W_0 = \frac{2\pi}{2\pi} = 1 \ (rad/s)$$

$$b_n = \frac{2}{2\pi}\left[\underbrace{\int_{-\pi}^{\pi} 18\sin(nt)\,dt}_{1} + \int_{-\pi}^{\pi} 18 \underbrace{\frac{\cos(6-n)t) - \cos(6+n)t}{2}}_{2}\,dt\right]$$

Para 1)

$$\frac{2}{2\pi}\int_{-\pi}^{\pi} 18\sin(nt)\,dt = \frac{9}{\pi}\int_{-\pi}^{\pi}\sin(nt)\,dt = \frac{-18}{n\pi}\cos(nt)\Big|_{-\pi}^{\pi}$$

$$= \frac{-18}{n\pi}\left[\cancel{\cos(n\pi) - \cos(-n\pi)}^{\,0}\right] = 0$$

Para 2)

$$\frac{2}{2\pi}\int_{-\pi}^{\pi} 18\,\frac{\cos((6-n)t) - \cos((6+n)t)}{2}\,dt = \frac{18}{2\pi}\int_{-\pi}^{\pi}\cos((6-n)t) - \cos((6+n)t)\,dt$$

$$= \frac{18}{2\pi}\left[\int_{-\pi}^{\pi}\cos((6-n)t)\,dt - \int_{-\pi}^{\pi}\cos((6+n)t)\,dt\right]$$

$$= \frac{18}{2\pi}\left[\frac{\sin((6-n)t)}{6-n}\Big|_{-\pi}^{\pi} - \frac{\sin((6+n)t)}{6+n}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{18}{2\pi}\left[\left[\frac{\sin((6-n)\pi)}{6-n} - \frac{\sin((6-n)-\pi)}{6+n}\right] - \left[\frac{\sin((6+n)\pi)}{6+n} - \frac{\sin((6+n)-\pi)}{6+n}\right]\right]$$

$$= \frac{18}{2\pi}\left[\left[\frac{\sin((6-n)\pi - \sin((6-n)-\pi)}{6-n}\right] - \left[\frac{\sin((6+n)\pi) - \sin((6+n)-\pi)}{6+n}\right]\right]$$

$$= 18\,\frac{\sin((6-n)\pi) - \sin((6-n)-\pi))}{2\pi(6-n)} - 18\,\frac{\sin((6+n)\pi) - \sin((6+n)\cdot\pi)}{2\pi(6+n)}$$

Para $n \neq 6$, $b = n$. No obstante, para $n=6$ debemos calcular el límite y aproximar la indeterminación $\frac{0}{0}$.

$$b_6 = 18 \lim_{n \to 6} \frac{\frac{d}{dn} \sin((6-n)\pi) - \sin((6-n)-\pi)}{\frac{d}{dn} 2\pi(6-n)}$$

$$b_6 = 18 \lim_{n \to 6} \frac{\cos(6-n)\pi)(-\pi) - \cos(-(6-n)\pi)\pi)}{-2\pi}$$

$$b_6 = 18 \frac{\cos(0)(-\pi) - \cos(0)\pi}{-2\pi} = 18 \frac{(-2\pi)}{-2\pi} = 18$$

$$b_6 = 18 \quad y \quad b_{-6} = -18$$

Por tanto,

$$a_n = \begin{cases} 18 & n=0 \\ 0 & \forall n \setminus \{0\} \end{cases}$$

$$b_n = \begin{cases} 18 & n=6 \\ -18 & n=-6 \\ 0 & \forall n \setminus \{6,-6\} \end{cases}$$

Forma exponencial:

$$C_0 = a_0 = 18$$

y $$C_n = \frac{a_n - j\, b_n}{2}$$

$$C_6 = \frac{0 - j18}{2} = \frac{-j18}{2} = -j9 \qquad C_{n-6} = j9$$

Se obtiene:

$$C_n = \begin{cases} 18 & n=0 \\ -9j & n=6 \\ 9j & n=-6 \\ 0 & \forall n \setminus \{0,6,-6\} \end{cases}$$

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El error relativo se calcula:

$$E_x[\%] = \left[1 - \frac{1}{P_x}\sum_{n=-N}^{N} |c_n|^2\right] 100\%$$

Para P_x =

$$P_x = \frac{1}{T}\int_{t_i}^{t_f} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |18 + 18\sin(6t)|^2 dt$$

$$P_x = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} 18^2 + 2(18)18\sin(6t) + 18^2\sin^2(6t)\, dt\right]$$

$$P_x = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} 18^2\, dt - \int_{-\pi}^{\pi} 648\sin(6t)\,dt + 18^2\int_{-\pi}^{\pi}\left(\frac{1}{2} - \frac{\cos(12t)}{2}\, dt\right)\right]$$

$$P_x = \frac{1}{2\pi}\left[324t\Big|_{-\pi}^{\pi} - \frac{648}{6}\cos(6t)\Big|_{-\pi}^{\pi} + 324\left[\frac{1}{2}t\Big|_{-\pi}^{\pi} - \frac{\sin(12t)}{24}\Big|_{-\pi}^{\pi}\right]\right]$$

$$P_x = \frac{1}{2\pi}\Big[\big[324(\pi) - 324(-\pi)\big] - 108\,\big[\cos(6(\pi)) - \cos(6(-\pi))\big]^{0} +$$

$$324\left[\left[\frac{1}{2}(\pi) - \frac{1}{2}(-\pi)\right] - \left[\frac{\sin(12\pi)}{24} - \frac{\sin(12(-\pi))}{24}\right]^{0}\right]\Big]$$

$$P_x = \frac{1}{2\pi}\left[648\pi + 0 + 324\pi\right] = \frac{972\pi}{2\pi} = 486$$

$$P_x = 486$$

$$E_x = \left[1 - \frac{(-9)^2 + (18)^2 + (9^2)}{486}\right] \times 100\% = 0\%.$$

2-2) Sea la señal portadora $c(t) = A_c \cos(2\pi F_c t)$, con $A_c, F_c \in \mathbb{R}$, y la señal mensaje $m(t) \in \mathbb{R}$. Encuentre el espectro en la frecuencia de la señal modula en amplitud (AM), $y(t) = \left(1 + \frac{m(t)}{A_c}\right) c(t)$.

La transformada de Fourier de la señal modulada se puede encontrar como:

$$Y(w) = F\{y(t)\} = F\left\{\left(1 + \frac{m(t)}{A_c}\right) c(t)\right\}$$

$$= F\left\{\left(1 + \frac{m(t)}{A_c}\right)\right\} + F\{c(t)\} = F\{c(t)\} + \frac{1}{A_c} F\{m(t)\, c(t)\}$$

Utilizando las tablas de Fourier:

$$c(w) = F\{c(t)\} = F\{A_c \cos(2\pi F_c t)\}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$c(w) = A_c\, F\left\{\frac{e^{j2\pi F_c t} + e^{-j2\pi F_c t}}{2}\right\}$$

$$A_c\left[F\left\{\frac{e^{j2\pi F_c t}}{2}\right\} + F\left\{\frac{e^{-j2\pi F_c t}}{2}\right\}\right]$$

Decimos que:

- $F\{e^{\pm j w_0 t}\} = 2\pi\, \delta(w \mp w_0)$

$$\frac{A_c}{2}\left[2\pi\, \delta(w - 2\pi F_c) + 2\pi\, \delta(w + 2\pi F_c)\right]$$

$$A_c \pi\, \delta(w - 2\pi F_c) + A_c \pi\, \delta(w + 2\pi F_c)$$

$$c(w) = A_c \pi\, \delta\left((w - 2\pi F_c) + (w + 2\pi F_c)\right)$$

- $F\left\{\frac{m(t)\,(A_c \cos(2\pi F_c t))}{A_c}\right\} = F\{\cos(2\pi F_c t)\, m(t)\}$

$$F\left\{\frac{m(t)\, e^{j2\pi F_c t}}{2}\right\} + F\left\{\frac{m(t)\, e^{-j2\pi F_c t}}{2}\right\}$$

$$\frac{M(\omega - 2\pi F_c t)}{2} + \frac{M(\omega + 2\pi F_c t)}{2}$$

$$= \frac{1}{2} M [(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t)]$$

$$\boxed{Y(\omega) = A_c \pi \delta [(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t)] + \frac{1}{2} M [(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t)]}$$