Difference of nth powers identity

We will prove the following identity for all $x, c \in \mathbb{R}, n \in \mathbb{N}$:
$x^n - c^n = (x-c)\sum_{i=1}^{n} x^{n-i}c^{i-1}$

Proof. By induction on n.
If $n = 1$, we have $x - c = (x-c)\sum_{i=1}^{1} x^{1-i}c^{i-1} = (x-c)(1)$

Now for some fixed natural number, k, assume:
$x^{k-1} - c^{k-1} = (x-c)\sum_{i=1}^{k-1} x^{k-1-i}c^{i-1}$

Multiply both sides of the equation by $x + c$:
$(x+c)(x^{k-1} - c^{k-1}) = (x+c)(x-c)\sum_{i=1}^{k-1} x^{k-1-i}c^{i-1}$

Distribute on the left side:
$x^k - c^k - xc^{k-1} + x^{k-1}c = (x+c)(x-c)\sum_{i=1}^{k-1} x^{k-1-i}c^{i-1}$

Now distribute the $x - c$ term on the right side:
$... = (x+c)(\sum_{i=1}^{k-1} x^{k-i}c^{i-1} - \sum_{i=1}^{k-1} x^{k-1-i}c^{i})$

And distribute the $x + c$ term on the right side:
$... = \sum_{i=1}^{k-1} x^{k+1-i}c^{i-1} - \sum_{i=1}^{k-1} x^{k-i}c^{i} + \sum_{i=1}^{k-1} x^{k-i}c^{i} - \sum_{i=1}^{k-1} x^{k-1-i}c^{i+1}$

Now extract an xc from the third sum and a c^2 from the fourth sum:
$... = \sum_{i=1}^{k-1} x^{k+1-i}c^{i-1} - \sum_{i=1}^{k-1} x^{k-i}c^{i} + (xc)(\sum_{i=1}^{k-1} x^{k-1-i}c^{i-1}) - (c^2)(\sum_{i=1}^{k-1} x^{k-1-i}c^{i-1})$

The third and fourth sums are now identical, so we can combine them:
$... = \sum_{i=1}^{k-1} x^{k+1-i}c^{i-1} - \sum_{i=1}^{k-1} x^{k-i}c^{i} + c(x-c)(\sum_{i=1}^{k-1} x^{k-1-i}c^{i-1})$

Which, by the induction hypothesis, is equivalent to:
$... = \sum_{i=1}^{k-1} x^{k+1-i}c^{i-1} - \sum_{i=1}^{k-1} x^{k-i}c^{i} + c(x^{k-1} - c^{k-1})$

Now we bring the left side of the equation back:
$x^k - c^k - xc^{k-1} + x^{k-1}c = \sum_{i=1}^{k-1} x^{k+1-i}c^{i-1} - \sum_{i=1}^{k-1} x^{k-i}c^{i} + c(x^{k-1} - c^{k-1})$

And move some terms over to the right, to isolate the $x^k - c^k$:
$x^k - c^k = xc^{k-1} - x^{k-1}c + \sum_{i=1}^{k-1} x^{k+1-i}c^{i-1} - \sum_{i=1}^{k-1} x^{k-i}c^{i} + c(x^{k-1} - c^{k-1})$

Now we turn our attention to the two remaining sums.
For concision and clarity, we make the following designations:
Let $\alpha = \sum_{i=1}^{k-1} x^{k+1-i}c^{i-1}$
Let $\beta = \sum_{i=1}^{k-1} x^{k-i}c^{i}$

So that we have:
$x^k - c^k = xc^{k-1} - x^{k-1}c + \alpha - \beta + c(x^{k-1} - c^{k-1})$

We also introduce two new terms:
Let $\gamma = \sum_{i=1}^{k} x^{k+1-i} c^{i-1}$
Let $\delta = \sum_{i=1}^{k} x^{k-i} c^{i}$

Observe that $\alpha = \gamma - xc^{k-1}$ and $\beta = \delta - c^k$.

We substitute these results into our equation:
$x^k - c^k = xc^{k-1} - x^{k-1}c + \gamma - xc^{k-1} - \delta - c^k + c(x^{k-1} - c^{k-1})$

Simplifying and rearranging terms:
$x^k - c^k = \gamma - \delta - x^{k-1}c - c^k + c(x^{k-1} - c^{k-1})$
$x^k - c^k = \gamma - \delta - c(x^{k-1} - c^{k-1}) + c(x^{k-1} - c^{k-1})$
$x^k - c^k = \gamma - \delta$

We replace γ and δ with their definitions:
$x^k - c^k = \sum_{i=1}^{k} x^{k+1-i} c^{i-1} - \sum_{i=1}^{k} x^{k-i} c^{i}$

And extract an x and c from the sums to make them identical:
$x^k - c^k = (x) \sum_{i=1}^{k} x^{k-i} c^{i-1} - (c) \sum_{i=1}^{k} x^{k-i} c^{i-1}$

A final combining of like terms concludes the proof by induction:
$x^k - c^k = (x - c) \sum_{i=1}^{k} x^{k-i} c^{i-1}$

$\square$