The Minkowski-Weyl Theorem

Nathan Chappell[1]

[1]Charles University
Faculty of Mathematics and Physics

Defense of Bachelor's Thesis, 2019

Outline

Outline

Goal of Thesis

- Prove the Minkowski-Weyl Theorem
- Implement the proof in C++

Goal of Thesis

- Prove the Minkowski-Weyl Theorem
- Implement the proof in C++

Outcome of Thesis

- Minkowski-Weyl Theorem is proven in line with Ziegler
- Simple implementation for command line in C++
- Pointed and full-dimensional polyhedra are explored for the purposes of verifying the implementation

Outcome of Thesis

- Minkowski-Weyl Theorem is proven in line with Ziegler
- Simple implementation for command line in C++
- Pointed and full-dimensional polyhedra are explored for the purposes of verifying the implementation

Outcome of Thesis

- Minkowski-Weyl Theorem is proven in line with Ziegler
- Simple implementation for command line in C++
- Pointed and full-dimensional polyhedra are explored for the purposes of verifying the implementation

Outline

Strengths

- Proofs are self-contained
- Material is "non-trivial"
- Figures and diagrams

Strengths

- Proofs are self-contained
- Material is "non-trivial"
- Figures and diagrams

Strengths

- Proofs are self-contained
- Material is "non-trivial"
- Figures and diagrams

Weaknesses

- Proofs missing quantification
- Backwards definitions
- Notation and concepts for matrices
- (Typesetting issues, spelling errors and minor mistakes)

Weaknesses

- Proofs missing quantification
- Backwards definitions
- Notation and concepts for matrices
- (Typesetting issues, spelling errors and minor mistakes)

Weaknesses

- Proofs missing quantification
- Backwards definitions
- Notation and concepts for matrices
- (Typesetting issues, spelling errors and minor mistakes)

Weaknesses

- Proofs missing quantification
- Backwards definitions
- Notation and concepts for matrices
- (Typesetting issues, spelling errors and minor mistakes)

Goal and Outcome
The Work
Closing Remarks
Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Outline

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

What I "Borrowed"

From Ziegler:

- Fourier Motzkin Elimination for cones
- Farkas Lemmas
- General idea for polyhedral reductions

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

What I "Borrowed"

From Ziegler:

- Fourier Motzkin Elimination for cones
- Farkas Lemmas
- General idea for polyhedral reductions

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

What I "Borrowed"

From Ziegler:

- Fourier Motzkin Elimination for cones
- Farkas Lemmas
- General idea for polyhedral reductions

Goal and Outcome
The Work
Closing Remarks
Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

V/H Polyhedra/Cones

Let $U \in \mathbb{R}^{d \times l}$, $V \in \mathbb{R}^{d \times m}$, $A \in \mathbb{R}^{m \times d}$

- V-Cone:
  $\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} \equiv \text{cone}(U)$

- V-Polytope:
  $\{V\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^T\boldsymbol{\lambda} = 1\} \equiv \text{conv}(V)$

- V-Polyhedron:
  $\{U\mathbf{t} + V\boldsymbol{\lambda} \mid \mathbf{t}, \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^T\boldsymbol{\lambda} = 1\} \equiv \text{cone}(U) + \text{conv}(V)$

- H-Cone:
  $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$

- H-Polyhedron:
  $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Minkowski-Weyl Theorem

- General Statement:
  V-Polyhedra and H-Polyhedra are different representations of the same objects

- For Cones:
  V-Cones and H-Cones are different representations of the same objects

First, the proof is done for cones, then polyhedra are reduced to cones.

Minkowski-Weyl Theorem

- General Statement:
  V-Polyhedra and H-Polyhedra are different representations of the same objects
- For Cones:
  V-Cones and H-Cones are different representations of the same objects

First, the proof is done for cones, then polyhedra are reduced to cones.

Minkowski-Weyl Theorem

- General Statement:
  V-Polyhedra and H-Polyhedra are different representations of the same objects

- For Cones:
  V-Cones and H-Cones are different representations of the same objects

First, the proof is done for cones, then polyhedra are reduced to cones.
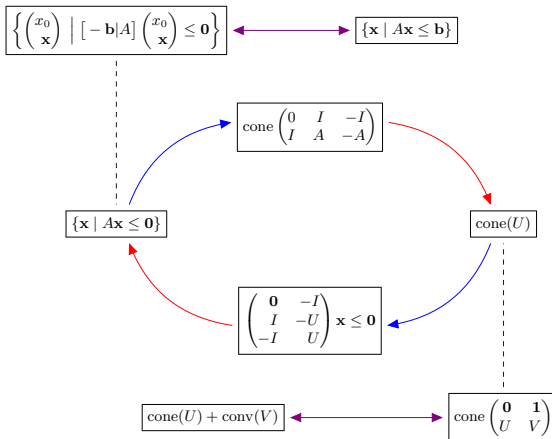
Picture of Proof

Figure 1: Diagram of the proof $P_H \leftrightarrow P_V$

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Outline

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Files and Includes

file	includes
`linear_algebra.h`	`<C++ standard library>`
`fourier_motzkin.h`	`linear_algebra.h`
`polyhedra.h`	`fourier_motzkin.h`
`main.cpp`	`polyhedra.h`
`test_functions.h`	`linear_alebra.h`
`test.cpp`	`test_functions.h, polyhedra.h`

Callgraph
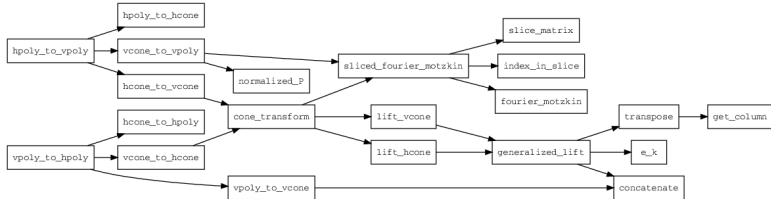
Matrix fourier_motzkin(Matrix,k)

```cpp
23    const auto z_end = partition(M.begin(), M.end(),
24        [k](const Vector &v) { return v[k] == 0; });
25    const auto p_end = partition(z_end, M.end(),
26        [k](const Vector &v) { return v[k] > 0; });
```

Partition M into logical sets Z, P, N that satisfy the following:

set	range	property
Z	[M.begin(), z_end)	it $\in Z \Leftrightarrow$ (*it)[k] $= 0$
P	[z_end, p_end)	it $\in P \Leftrightarrow$ (*it)[k] > 0
N	[p_end, M.end())	it $\in N \Leftrightarrow$ (*it)[k] < 0

Matrix fourier_motzkin(Matrix,k)

```
30    for (auto p_it = z_end; p_it != p_end; ++p_it) {
31      for (auto n_it = p_end; n_it != M.end(); ++n_it) {
32        result.push_back(
33          (*p_it)[k]*(*n_it) - (*n_it)[k]*(*p_it));
34      }
35    }
```

This function creates the sets which correspond to:
$$\left\{ B_i^k B_j - B_j^k B_i \mid i \in P, j \in N \right\}, \quad \left\{ Y_k^i Y^j - Y_k^j Y^i \mid i \in P, j \in N \right\}$$

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Outline

Testing Methods

- Suppose we have an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, know that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(V)$, and would like to test if $\text{cone}(V) = \text{cone}(V')$

$$
\begin{aligned}
AV' \leq \mathbf{0} &\Rightarrow \text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \\
V \subseteq V' &\Rightarrow \text{cone}(V) \subseteq \text{cone}(V') \\
\text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} &\Rightarrow AV' \leq \mathbf{0} \\
\text{cone}(V) \subseteq \text{cone}(V') &\stackrel{?}{\Rightarrow} V \subseteq V'
\end{aligned}
$$

- The last item would create an equivalence
  - Must relax "$\subseteq$" (vectors vs rays)
  - Requires notion of "essentially unique" representation

Nathan Chappell   The Minkowski-Weyl Theorem

Testing Methods

- Suppose we have an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, know that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(V)$, and would like to test if $\text{cone}(V) = \text{cone}(V')$

$$AV' \leq \mathbf{0} \qquad\qquad \Rightarrow \text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$
$$V \subseteq V' \qquad\qquad \Rightarrow \text{cone}(V) \subseteq \text{cone}(V')$$
$$\text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \Rightarrow AV' \leq \mathbf{0}$$
$$\text{cone}(V) \subseteq \text{cone}(V') \qquad \overset{?}{\Rightarrow} V \subseteq V'$$

- The last item would create an equivalence
  - Must relax "$\subseteq$" (vectors vs rays)
  - Requires notion of "essentially unique" representation

Nathan Chappell   The Minkowski-Weyl Theorem

Testing Methods

- Suppose we have an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, know that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(V)$, and would like to test if $\text{cone}(V) = \text{cone}(V')$

$$AV' \leq \mathbf{0} \qquad\qquad \Rightarrow \text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$
$$V \subseteq V' \qquad\qquad \Rightarrow \text{cone}(V) \subseteq \text{cone}(V')$$
$$\text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \Rightarrow AV' \leq \mathbf{0}$$
$$\text{cone}(V) \subseteq \text{cone}(V') \qquad \overset{?}{\Rightarrow} V \subseteq V'$$

- The last item would create an equivalence
  - Must relax "$\subseteq$" (vectors vs rays)
  - Requires notion of "essentially unique" representation

Nathan Chappell   The Minkowski-Weyl Theorem

Goal and Outcome   Not Original
The Work   Implementation
Closing Remarks   Pointed / Full-Dimensional Polyhedra

Testing Methods

- Suppose we have an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, know that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \mathrm{cone}(V)$, and would like to test if $\mathrm{cone}(V) = \mathrm{cone}(V')$

$$
\begin{aligned}
AV' \leq \mathbf{0} &\Rightarrow \mathrm{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \\
V \subseteq V' &\Rightarrow \mathrm{cone}(V) \subseteq \mathrm{cone}(V') \\
\mathrm{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} &\Rightarrow AV' \leq \mathbf{0} \\
\mathrm{cone}(V) \subseteq \mathrm{cone}(V') &\overset{?}{\Rightarrow} V \subseteq V'
\end{aligned}
$$

- The last item would create an equivalence
  - Must relax "$\subseteq$" (vectors vs rays)
  - Requires notion of "essentially unique" representation

Nathan Chappell   The Minkowski-Weyl Theorem

Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
    - Pointed Polyhedra have at least one vertex
    - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have "essentially unique" sets of generators / contraints
- These "essentially unique" sets make it easy to test for equivalence
- The characterizations are similar to "linear independence"

Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
  - Pointed Polyhedra have at least one vertex
  - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have "essentially unique" sets of generators / contraints
- These "essentially unique" sets make it easy to test for equivalence
- The characterizations are similar to "linear independence"

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
  - Pointed Polyhedra have at least one vertex
  - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have "essentially unique" sets of generators / contraints
- These "essentially unique" sets make it easy to test for equivalence
- The characterizations are similar to "linear independence"

Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
    - Pointed Polyhedra have at least one vertex
    - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have "essentially unique" sets of generators / contraints
- These "essentially unique" sets make it easy to test for equivalence
- The characterizations are similar to "linear independence"

Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
    - Pointed Polyhedra have at least one vertex
    - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have "essentially unique" sets of generators / contraints
- These "essentially unique" sets make it easy to test for equivalence
- The characterizations are similar to "linear independence"

Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
  - Pointed Polyhedra have at least one vertex
  - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have "essentially unique" sets of generators / contraints
- These "essentially unique" sets make it easy to test for equivalence
- The characterizations are similar to "linear independence"

Pointed/ Full-Dimensional Polyhedra

- These "essentially unique" sets are "minimal"
- A set V is called *minimal* for cone(V) if

$$(\forall \mathbf{v} \in V) \ \text{cone}(V \setminus \{\mathbf{v}\}) \subsetneq \text{cone}(V)$$

- A set A is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ if

$$(\forall A_i \in A) \ \{\mathbf{x} \mid A \setminus \{A_i\} \ \mathbf{x} \leq \mathbf{0}\} \supsetneq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$

- The definitions are similar for general polyhedra

Pointed/ Full-Dimensional Polyhedra

- These "essentially unique" sets are "minimal"
- A set V is called *minimal* for $\operatorname{cone}(V)$ if

$$(\forall \mathbf{v} \in V) \ \operatorname{cone}(V \setminus \{\mathbf{v}\}) \subsetneq \operatorname{cone}(V)$$

- A set A is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ if

$$(\forall A_i \in A) \ \{\mathbf{x} \mid A \setminus \{A_i\} \, \mathbf{x} \leq \mathbf{0}\} \supsetneq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$

- The definitions are similar for general polyhedra

Pointed/ Full-Dimensional Polyhedra

- These "essentially unique" sets are "minimal"
- A set V is called *minimal* for $\text{cone}(V)$ if

$$(\forall \mathbf{v} \in V) \; \text{cone}(V \setminus \{\mathbf{v}\}) \subsetneq \text{cone}(V)$$

- A set A is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ if

$$(\forall A_i \in A) \; \{\mathbf{x} \mid A \setminus \{A_i\} \, \mathbf{x} \leq \mathbf{0}\} \supsetneq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$

- The definitions are similar for general polyhedra

Goal and Outcome   Not Original
The Work   Implementation
Closing Remarks   Pointed / Full-Dimensional Polyhedra

Pointed/ Full-Dimensional Polyhedra

- These "essentially unique" sets are "minimal"
- A set V is called *minimal* for $\mathrm{cone}(V)$ if

$$(\forall \mathbf{v} \in V) \ \mathrm{cone}(V \setminus \{\mathbf{v}\}) \subsetneq \mathrm{cone}(V)$$

- A set A is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ if

$$(\forall A_i \in A) \ \{\mathbf{x} \mid A \setminus \{A_i\} \, \mathbf{x} \leq \mathbf{0}\} \supsetneq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$

- The definitions are similar for general polyhedra

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Pointed V-Cones

- The following statements are equivalent.
  1. cone(V) is pointed.
  2. $t \geq 0$, $[Vt = 0 \Rightarrow t = 0]$
- Suppose that cone(V) is pointed. The following two statements are equivalent:
  1. V is minimal
  2. $t \geq 0$, $[Vt = Ve_i \Rightarrow t = e_i]$
- "No generator can be non-trivially generated"

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Pointed V-Cones

- The following statements are equivalent.
  1. $\text{cone}(V)$ is pointed.
  2. $\mathbf{t} \geq \mathbf{0}$, $[V\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}]$
- Suppose that $\text{cone}(V)$ is pointed. The following two statements are equivalent:
  1. V is minimal
  2. $\mathbf{t} \geq \mathbf{0}$, $[V\mathbf{t} = V\mathbf{e}_i \Rightarrow \mathbf{t} = \mathbf{e}_i]$
- "No generator can be non-trivially generated"

Goal and Outcome

The Work

Closing Remarks

Not Original

Implementation

Pointed / Full-Dimensional Polyhedra

Pointed V-Cones

- The following statements are equivalent.
    1. $\mathrm{cone}(V)$ is pointed.
    2. $\mathbf{t} \geq \mathbf{0}$, $[V\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}]$
- Suppose that $\mathrm{cone}(V)$ is pointed. The following two statements are equivalent:
    1. V is minimal
    2. $\mathbf{t} \geq \mathbf{0}$, $[V\mathbf{t} = V\mathbf{e}_i \Rightarrow \mathbf{t} = \mathbf{e}_i]$
- "No generator can be non-trivially generated"

Farkas Lemma

1. Farkas Lemma: Let $U \in \mathbb{R}^{d \times n}$. Precisely one of the following is true:

$$(\exists \mathbf{t} \geq \mathbf{0}) : \mathbf{x} = U\mathbf{t}$$
$$(\exists \mathbf{y}) : U^T\mathbf{y} \leq 0, \ \mathbf{y}^T\mathbf{x} > 0$$

2. i.e. a point is contained in a cone or can be separated from it with a hyperplane

Nathan Chappell   The Minkowski-Weyl Theorem

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Farkas Lemma

1. Farkas Lemma: Let $U \in \mathbb{R}^{d \times n}$. Precisely one of the following is true:

$$(\exists \mathbf{t} \geq \mathbf{0}) : \mathbf{x} = U\mathbf{t}$$
$$(\exists \mathbf{y}) : U^T \mathbf{y} \leq 0, \ \mathbf{y}^T \mathbf{x} > 0$$

2. i.e. a point is contained in a cone or can be separated from it with a hyperplane

Full-Dimensional H-Cones

- A set *V* of vectors is *full-dimensional* if

$$\forall \mathbf{y} \neq \mathbf{0}, \forall c \in \mathbb{R}, \exists \mathbf{v} \in V : \ \mathbf{y}^T \mathbf{v} \neq c$$

- The following two statements are equivalent:
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is full dimensional and *A* is minimal
  2. $\mathrm{cone}(A^T)$ is pointed and *A* is minimal

- The Farkas Lemma can be used to prove the following:

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \mathrm{cone}(A^T) = \mathrm{cone}(A'^T)$$

($\mathrm{cone}(A^T)$ is what I call the "dual cone")

Full-Dimensional H-Cones

- A set V of vectors is *full-dimensional* if

$$\forall \mathbf{y} \neq \mathbf{0}, \forall c \in \mathbb{R}, \exists \mathbf{v} \in V : \ \mathbf{y}^T \mathbf{v} \neq c$$

- The following two statements are equivalent:
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is full dimensional and A is minimal
  2. $\text{cone}(A^T)$ is pointed and A is minimal

- The Farkas Lemma can be used to prove the following:

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \text{cone}(A^T) = \text{cone}(A'^T)$$

($\text{cone}(A^T)$ is what I call the "dual cone")

Full-Dimensional H-Cones

- A set V of vectors is *full-dimensional* if

$$\forall \mathbf{y} \neq \mathbf{0}, \forall c \in \mathbb{R}, \exists \mathbf{v} \in V : \ \mathbf{y}^T \mathbf{v} \neq c$$

- The following two statements are equivalent:
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is full dimensional and A is minimal
  2. $\text{cone}(A^T)$ is pointed and A is minimal

- The Farkas Lemma can be used to prove the following:

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \text{cone}(A^T) = \text{cone}(A'^T)$$

($\text{cone}(A^T)$ is what I call the "dual cone")

Cones and Polyhedra

- General polyhedra are decomposed into a "characteristic-cone" and polytope
- Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \leq b\} = \text{cone}(U) + \text{conv}(V)$, and let $\mathbf{r}$ be a vector. The following are equivalent:

  1. $A\mathbf{r} \leq \mathbf{0}$
  2. $(\forall \mathbf{x} \in P)(\forall \alpha > 0)\ \mathbf{x} + \alpha\mathbf{r} \in P$
  3. $\mathbf{r} \in \text{cone}(U)$

- $2 \Rightarrow 3$ requires the Farkas Lemma
- Note that (2) in the proof above is independent of A and U.

Cones and Polyhedra

- General polyhedra are decomposed into a "characteristic-cone" and polytope
- Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \leq b\} = \text{cone}(U) + \text{conv}(V)$, and let $\mathbf{r}$ be a vector. The following are equivalent:
  1. $A\mathbf{r} \leq \mathbf{0}$
  2. $(\forall \mathbf{x} \in P)(\forall \alpha > 0)\ \mathbf{x} + \alpha\mathbf{r} \in P$
  3. $\mathbf{r} \in \text{cone}(U)$
- $2 \Rightarrow 3$ requires the Farkas Lemma
- Note that (2) in the proof above is independent of A and U.

Cones and Polyhedra

- General polyhedra are decomposed into a "characteristic-cone" and polytope
- Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \leq b\} = \text{cone}(U) + \text{conv}(V)$, and let $\mathbf{r}$ be a vector. The following are equivalent:
  1. $A\mathbf{r} \leq \mathbf{0}$
  2. $(\forall \mathbf{x} \in P)(\forall \alpha > 0)\ \mathbf{x} + \alpha\mathbf{r} \in P$
  3. $\mathbf{r} \in \text{cone}(U)$
- $2 \Rightarrow 3$ requires the Farkas Lemma
- Note that (2) in the proof above is independent of A and U.

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Cones and Polyhedra

- General polyhedra are decomposed into a "characteristic-cone" and polytope
- Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \leq b\} = \text{cone}(U) + \text{conv}(V)$, and let $\mathbf{r}$ be a vector. The following are equivalent:
  1. $A\mathbf{r} \leq \mathbf{0}$
  2. $(\forall \mathbf{x} \in P)(\forall \alpha > 0)\ \mathbf{x} + \alpha\mathbf{r} \in P$
  3. $\mathbf{r} \in \text{cone}(U)$
- $2 \Rightarrow 3$ requires the Farkas Lemma
- Note that (2) in the proof above is independent of A and U.

Pointed V-Polyhedra

- Minimality for the set $\text{conv}(V)$ – a polytope – is given by the vertex set

- In $P = \text{cone}(U) + \text{conv}(V)$ we need U to be minimal as for V-Cones

- Complication: a vertex of $\text{conv}(V)$ may not be a vertex of P

Goal and Outcome
The Work
Closing Remarks

Not Original
Implementation
Pointed / Full-Dimensional Polyhedra

Pointed V-Polyhedra

- Minimality for the set $\text{conv}(V)$ – a polytope – is given by the vertex set
- In $P = \text{cone}(U) + \text{conv}(V)$ we need U to be minimal as for V-Cones
- Complication: a vertex of $\text{conv}(V)$ may not be a vertex of P

Pointed V-Polyhedra

- Minimality for the set $\mathrm{conv}(V)$ – a polytope – is given by the vertex set
- In $P = \mathrm{cone}(U) + \mathrm{conv}(V)$ we need U to be minimal as for V-Cones
- Complication: a vertex of $\mathrm{conv}(V)$ may not be a vertex of P

Pointed V-Polyhedra

- If **v** is a vertex of $\text{cone}(U) + \text{conv}(V)$, then
  $[\mathbf{v} = U\mathbf{t} + V\lambda] \Rightarrow \mathbf{t} = \mathbf{0}$
  (I call **v** here *U-free*)

- If **v** is a vertex of $\text{cone}(U) + \text{conv}(V)$, then **v** is a vertex of $\text{conv}(V)$

- Let $P = \text{cone}(U) + \text{cone}(V)$ be pointed. The following are equivalent

  1. (U, V) is minimal for P
  2. U is minimal for $\text{cone}(U)$, V is the vertex set of P
  3. U is minimal for $\text{cone}(U)$, V is the vertex set of $\text{conv}(V)$, and V is *U*-free

Pointed V-Polyhedra

- If **v** is a vertex of $\mathrm{cone}(U) + \mathrm{conv}(V)$, then
  $[\mathbf{v} = U\mathbf{t} + V\lambda] \Rightarrow \mathbf{t} = \mathbf{0}$
  (I call **v** here *U-free*)

- If **v** is a vertex of $\mathrm{cone}(U) + \mathrm{conv}(V)$, then **v** is a vertex of $\mathrm{conv}(V)$

- Let $P = \mathrm{cone}(U) + \mathrm{cone}(V)$ be pointed. The following are equivalent
    1. (U, V) is minimal for P
    2. U is minimal for $\mathrm{cone}(U)$, V is the vertex set of P
    3. U is minimal for $\mathrm{cone}(U)$, V is the vertex set of $\mathrm{conv}(V)$, and V is U-free

Pointed V-Polyhedra

- If **v** is a vertex of $\text{cone}(U) + \text{conv}(V)$, then
  $[\mathbf{v} = U\mathbf{t} + V\boldsymbol{\lambda}] \Rightarrow \mathbf{t} = \mathbf{0}$
  (I call **v** here *U-free*)

- If **v** is a vertex of $\text{cone}(U) + \text{conv}(V)$, then **v** is a vertex of $\text{conv}(V)$

- Let $P = \text{cone}(U) + \text{cone}(V)$ be pointed. The following are equivalent
  1. (U, V) is minimal for P
  2. U is minimal for $\text{cone}(U)$, V is the vertex set of P
  3. U is minimal for $\text{cone}(U)$, V is the vertex set of $\text{conv}(V)$, and V is *U-*free

Farkas Lemma 2

- We need another form of the Farkas Lemma:

$$(\exists \mathbf{t} \geq \mathbf{0}) \mathbf{t}^T A = \mathbf{y}, \, \mathbf{t}^T \mathbf{b} \leq c \Leftrightarrow \begin{cases} (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{y}^T \mathbf{x} \leq 0 \text{ and} \\ (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{b} \Rightarrow \mathbf{y}^T \mathbf{x} \leq c \end{cases}$$

- Basically, a constraint is valid for a polyhedron if and only if it is a non-negative combination of rows of constraints (plus some change)

Nathan Chappell   The Minkowski-Weyl Theorem

Farkas Lemma 2

- We need another form of the Farkas Lemma:

$$(\exists \mathbf{t} \geq \mathbf{0})\mathbf{t}^T A = \mathbf{y}, \, \mathbf{t}^T \mathbf{b} \leq c \Leftrightarrow \begin{cases} (\forall \mathbf{x})A\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{y}^T \mathbf{x} \leq 0 \textbf{ and} \\ (\forall \mathbf{x})A\mathbf{x} \leq \mathbf{b} \Rightarrow \mathbf{y}^T \mathbf{x} \leq c \end{cases}$$

- Basically, a constraint is valid for a polyhedron if and only if it is a non-negative combination of rows of constraints (plus some change)

Goal and Outcome   Not Original
The Work   Implementation
Closing Remarks   Pointed / Full-Dimensional Polyhedra

Full-Dimensional H-Polyhedra

- The following are equivalent
  1. $\{\mathbf{x} \mid A\mathbf{x} \le \mathbf{b}\}$ is full-dimensional
  2. $\mathbf{y} \ge \mathbf{0}$, $\mathbf{y}^T A = \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T \mathbf{b} > \mathbf{0}$

- $\mathbf{y}^T A = \mathbf{0}$, $\mathbf{y}^T \mathbf{b} > \mathbf{0}$ occurs when two of the bounding hyperplanes are parallel

- If $\{\mathbf{x} \mid A\mathbf{x} \le \mathbf{b}\}$ is full-dimensional, the following are equivalent

  1. $(A, \mathbf{b})$ is minimal
  2. $\mathbf{y} \ge \mathbf{0}, \mathbf{y} \ne \mathbf{e}_i, \mathbf{y}^T A = A_i \Rightarrow \mathbf{y}^T \mathbf{b} > b_i$

- "A non-trivially generated row generates a trivial constraint"

Full-Dimensional H-Polyhedra

- The following are equivalent
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional
  2. $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y}^T A = \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T \mathbf{b} > \mathbf{0}$

- $\mathbf{y}^T A = \mathbf{0}$, $\mathbf{y}^T \mathbf{b} > \mathbf{0}$ occurs when two of the bounding hyperplanes are parallel

- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional, the following are equivalent

  1. $(A, \mathbf{b})$ is minimal
  2. $\mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{e}_i, \mathbf{y}^T A = A_i \Rightarrow \mathbf{y}^T \mathbf{b} > b_i$

- "A non-trivially generated row generates a trivial constraint"

Full-Dimensional H-Polyhedra

- The following are equivalent
    1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional
    2. $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y}^T A = \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T \mathbf{b} > \mathbf{0}$
- $\mathbf{y}^T A = \mathbf{0}$, $\mathbf{y}^T \mathbf{b} > \mathbf{0}$ occurs when two of the bounding hyperplanes are parallel
- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional, the following are equivalent
    1. $(A, \mathbf{b})$ is minimal
    2. $\mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{e}_i, \mathbf{y}^T A = A_i \Rightarrow \mathbf{y}^T \mathbf{b} > b_i$
- "A non-trivially generated row generates a trivial constraint"

Nathan Chappell   The Minkowski-Weyl Theorem

Full-Dimensional H-Polyhedra

- The following are equivalent
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional
  2. $\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A = \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T \mathbf{b} > \mathbf{0}$
- $\mathbf{y}^T A = \mathbf{0}, \mathbf{y}^T \mathbf{b} > \mathbf{0}$ occurs when two of the bounding hyperplanes are parallel
- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional, the following are equivalent
  1. $(A, \mathbf{b})$ is minimal
  2. $\mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{e}_i, \mathbf{y}^T A = A_i \Rightarrow \mathbf{y}^T \mathbf{b} > b_i$
- "A non-trivially generated row generates a trivial constraint"

Dual Homogenization Cone

- For H-Polyhedra switching to the dual setting is slightly more complicated

- The following two statement are equivalent:

  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$

  2. $\text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} = \text{cone} \begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix}$

- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is minimal and full-dimensional, then either

  1. $\text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ is minimal and pointed, or

  2. $\text{cone} \begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix}$ is minimal and pointed, and

     $\text{cone} \begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix} = \text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$

Nathan Chappell   The Minkowski-Weyl Theorem

Dual Homogenization Cone

- For H-Polyhedra switching to the dual setting is slightly more complicated

- The following two statement are equivalent:

  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$

  2. $\text{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} = \text{cone}\begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix}$

- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is minimal and full-dimensional, then either

  1. $\text{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ is minimal and pointed, or

  2. $\text{cone}\begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix}$ is minimal and pointed, and

     $\text{cone}\begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix} = \text{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$

Dual Homogenization Cone

- For H-Polyhedra switching to the dual setting is slightly more complicated
- The following two statement are equivalent:

  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$
  2. $\mathrm{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} = \mathrm{cone}\begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix}$

- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is minimal and full-dimensional, then either

  1. $\mathrm{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ is minimal and pointed, or
  2. $\mathrm{cone}\begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix}$ is minimal and pointed, and
     $\mathrm{cone}\begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix} = \mathrm{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$

Pointed H-Polyhedra and Full-Dimensional V-Polyhedra

- **t** is a non-negative vector, $V \neq \emptyset$, and abbreviate linear-independent as LI. $\bar{V}$ denotes $\{\mathbf{v} - \mathbf{v}' \mid \mathbf{v}, \mathbf{v}' \in V\}$.

	Pointed	Full-Dimensional
$\mathrm{cone}(U)$	$U\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$	d LI vectors in U
$\mathrm{cone}(U) + \mathrm{conv}(V)$	$U\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$	d LI vectors in $U \cup \bar{V}$
$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$	d LI row vectors in A	$\mathbf{t}^T A = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$
$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$	d LI row vectors in A	$\mathbf{t}^T A = \mathbf{0} \Rightarrow \mathbf{t}^T \mathbf{b} > 0$

Goal and Outcome

The Work

Closing Remarks

Not Original

Implementation

Pointed / Full-Dimensional Polyhedra

Not much to see here

- The proofs are more tedious than enlightening
- The implementation is straightforward

Outline

Better MWT for Cones

- H-Cone $\rightarrow$ V-Cone. Use same transform and "tensor-notation"
- Farkas Lemma using the following facts:
    1. An intersection of closed sets is closed
    2. A projection of a closed subset of $\mathbb{R}^n$ is closed
    3. (Heine-Borel) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded
    4. A continuous function on a compact subset of $\mathbb{R}^n$ achieves its maximum
    5. Linear functions on $\mathbb{R}^n$ are continuous
    6. The Cauchy-Shwartz inequality
- Better dual cone:
    $$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(U) \Leftrightarrow \text{cone}(A^T) = \{\mathbf{x} \mid U^T\mathbf{x} \leq \mathbf{0}\}$$

Nathan Chappell   The Minkowski-Weyl Theorem

Tensor-like notation

First, we write $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \left\{ \mathbf{x} \,\middle|\, \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \text{cone} \begin{pmatrix} I & -I & \mathbf{0} \\ A & -A & I \end{pmatrix} \right\}$

Now we need to do our intersections:

- Let $\mathbf{u} \in \text{cone}(U)$ with $u_l = 0$
- Let $\mathbf{p}^i t_i$ sum over elements of U with positive l-th elements
- $\mathbf{u} = \mathbf{p}^i t_i + \mathbf{n}^j t_j + \mathbf{z}^k t_k$
- $u_l = 0 \Rightarrow p_l^i t_i + n_l^j t_j = 0 \ldots \quad \sigma := p_l^i t_i = -n_l^j t_j$
- $\mathbf{u} = \frac{\mathbf{n}^j t_j p_l^i t_i - \mathbf{p}^i t_i n_l^j t_j}{\sigma} + \mathbf{z}^k t_k = (\mathbf{n}^j p_l^i - \mathbf{p}^i n_l^j)\frac{t_i t_j}{\sigma} + \mathbf{z}^k t_k$

Outline

Nathan Chappell   The Minkowski-Weyl Theorem

Summary

- Finite generation
- The Farkas Lemma (separation)
- Pointed and Full-Dimensional polyhedra are "non-degenerate"

Summary

- Finite generation
- The Farkas Lemma (separation)
- Pointed and Full-Dimensional polyhedra are "non-degenerate"

Summary

- Finite generation
- The Farkas Lemma (separation)
- Pointed and Full-Dimensional polyhedra are "non-degenerate"

Further Considerations

- Better algorithms (Dual-Description)
- Other interpretations...
    - ...Systems of logical deduction:
      Hahn-Banach vs Incompleteness
    - ...Systems where "lift and drop" creates a dual
      representation