The Minkowski-Weyl Theorem

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Outline

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Goal of Thesis

- Prove the Minkowski-Weyl Theorem
- Implement the proof in C++

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Outcome of Thesis

- Minkowski-Weyl Theorem is proven in line with Ziegler
- Simple implementation for command line in C++
- Pointed and full-dimensional polyhedra are explored for the purposes of verifying the implementation

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1. Goal and Outcome
   - Goal / Outcome
   - Strengths and Weaknesses
2. The Work
   - Not Original
   - Implementation
   - Pointed / Full-Dimensional Polyhedra
3. Closing Remarks
   - If I could do it again
   - Summary

Strengths

- Proofs are self-contained
- Material is “non-trivial”
- Figures and diagrams

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- Proofs missing quantification
- Backwards definitions
- Notation and concepts for matrices
- (Typesetting issues, spelling errors and minor mistakes)

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What I "Borrowed"

From Ziegler:

- Fourier Motzkin Elimination for cones
- Farkas Lemmas
- General idea for polyhedral reductions

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V/H Polyhedra/Cones

Let $U \in \mathbb{R}^{d\times l}$, $V \in \mathbb{R}^{d\times m}$, $A \in \mathbb{R}^{m\times d}$

- V-Cone:
  $\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} \equiv \text{cone}(U)$
- V-Polytope:
  $\{V\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^T\boldsymbol{\lambda} = 1\} \equiv \text{conv}(V)$
- V-Polyhedron:
  $\{U\mathbf{t} + V\boldsymbol{\lambda} \mid \mathbf{t}, \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^T\boldsymbol{\lambda} = 1\} \equiv \text{cone}(U) + \text{conv}(V)$
- H-Cone:
  $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$
- H-Polyhedron:
  $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$

Minkowski-Weyl Theorem

- General Statement:
  V-Polyhedra and H-Polyhedra are different representations of the same objects
- For Cones:
  V-Cones and H-Cones are different representations of the same objects

First, the proof is done for cones, then polyhedra are reduced to cones.

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Picture of Proof

$$
\begin{tikzcd}
\boxed{\left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \,\middle|\, [-\mathbf{b}|A] \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\}} \arrow[rr, leftrightarrow] \arrow[dd, dash, dashed] & & \boxed{\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}} \\
& \boxed{\operatorname{cone} \begin{pmatrix} 0 & I & -I \\ I & A & -A \end{pmatrix}} \arrow[dr, bend left] & \\
\boxed{\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}} \arrow[ur, bend left] & & \boxed{\operatorname{cone}(U)} \arrow[dl, bend left] \arrow[dd, dash, dashed] \\
& \boxed{\begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix} \mathbf{x} \leq \mathbf{0}} \arrow[ul, bend left] & \\
& \boxed{\operatorname{cone}(U) + \operatorname{conv}(V)} \arrow[r, leftrightarrow] & \boxed{\operatorname{cone} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ U & V \end{pmatrix}}
\end{tikzcd}
$$

Figure 1: Diagram of the proof $P_H \leftrightarrow P_V$

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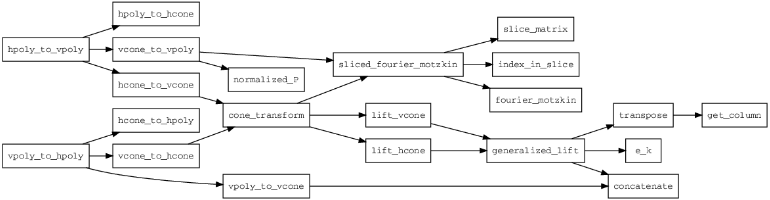
Files and Includes

file	includes
linear_algebra.h	<C++ standard library>
fourier_motzkin.h	linear_algebra.h
polyhedra.h	fourier_motzkin.h
main.cpp	polyhedra.h
test_functions.h	linear_alebra.h
test.cpp	test_functions.h, polyhedra.h

Callgraph


hpoly_to_hcone
hpoly_to_vpoly
vcone_to_vpoly
normalized_P
hcone_to_vcone
slice_matrix
sliced_fourier_motzkin
index_in_slice
fourier_motzkin
cone_transform
lift_vcone
transpose
get_column
hcone_to_hpoly
vpoly_to_hpoly
vcone_to_hcone
lift_hcone
generalized_lift
e_k
vpoly_to_vcone
concatenate

Matrix fourier_motzkin(Matrix,k)

```
  const auto z_end = partition(M.begin(), M.end(),
      [k](const Vector &v) { return v[k] == 0; });
  const auto p_end = partition(z_end, M.end(),
      [k](const Vector &v) { return v[k] > 0; });
```

Partition `M` into logical sets Z, P, N that satisfy the following:

set	range	property
Z	[`M.begin()`, `z_end`)	`it` $\in Z \Leftrightarrow$ `(*it)[k]` $= 0$
P	[`z_end`, `p_end`)	`it` $\in P \Leftrightarrow$ `(*it)[k]` > 0
N	[`p_end`, `M.end()`)	`it` $\in N \Leftrightarrow$ `(*it)[k]` < 0

Matrix fourier_motzkin(Matrix,k)

```
  for (auto p_it = z_end; p_it != p_end; ++p_it) {
    for (auto n_it = p_end; n_it != M.end(); ++n_it) {
      result.push_back(
        (*p_it)[k]*(*n_it) - (*n_it)[k]*(*p_it));
    }
  }
```

This function creates the sets which correspond to:

$$\left\{ B_i^k B_j - B_j^k B_i \mid i \in P, j \in N \right\}, \quad \left\{ Y_k^i Y^j - Y_k^j Y^i \mid i \in P, j \in N \right\}$$

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Testing Methods

- Suppose we have an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, know that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(V)$, and would like to test if $\text{cone}(V) = \text{cone}(V')$

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\begin{aligned}
AV' \leq \mathbf{0} &\Rightarrow \text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \\
V \subseteq V' &\Rightarrow \text{cone}(V) \subseteq \text{cone}(V') \\
\text{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} &\Rightarrow AV' \leq \mathbf{0} \\
\text{cone}(V) \subseteq \text{cone}(V') &\overset{?}{\Rightarrow} V \subseteq V'
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- The last item would create an equivalence
  - Must relax "$\subseteq$" (vectors vs rays)
  - Requires notion of "essentially unique" representation

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Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
  - Pointed Polyhedra have at least one vertex
  - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have “essentially unique” sets of generators / contraints
- These “essentially unique” sets make it easy to test for equivalence
- The characterizations are similar to “linear independence”

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- These “essentially unique” sets are “minimal”
- A set V is called *minimal* for $\text{cone}(V)$ if

$$(\forall \mathbf{v} \in V) \;\; \text{cone}(V \setminus \{\mathbf{v}\}) \subsetneq \text{cone}(V)$$

- A set A is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ if

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Pointed V-Cones

- The following statements are equivalent.
  1. $\text{cone}(V)$ is pointed.
  2. $\mathbf{t} \geq \mathbf{0},\ [V\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}]$
- Suppose that $\text{cone}(V)$ is pointed. The following two statements are equivalent:
  1. V is minimal
  2. $\mathbf{t} \geq \mathbf{0},\ \mathbf{v} = V\mathbf{e}_i,\ [\mathbf{v} = V\mathbf{t} \Rightarrow \mathbf{t} = \mathbf{e}_i]$
- “No generator can be non-trivially generated”

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Farkas Lemma

1. Farkas Lemma: Let $U \in \mathbb{R}^{d \times n}$. Precisely one of the following is true:

$$(\exists \mathbf{t} \geq \mathbf{0}) : \mathbf{x} = U\mathbf{t}$$

$$(\exists \mathbf{y}) : U^T\mathbf{y} \leq 0, \ \mathbf{y}^T\mathbf{x} > 0$$

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Full-Dimensional H-Cones

- A set V of vectors is *full-dimensional* if

$$\forall \mathbf{y} \neq \mathbf{0}, \forall c \in \mathbb{R}, \exists \mathbf{v} \in V : \ \mathbf{y}^T \mathbf{v} \neq c$$

- The following two statements are equivalent:
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is full dimensional and A is minimal
  2. $\text{cone}(A^T)$ is pointed and A is minimal
- The Farkas Lemma can be used to prove the following:

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \text{cone}(A^T) = \text{cone}(A'^T)$$

($\text{cone}(A^T)$ is what I call the "dual cone")

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Cones and Polyhedra

- General polyhedra are decomposed into a "characteristic-cone" and polytope
- Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \leq b\} = \text{cone}(U) + \text{conv}(V)$, and let $\mathbf{r}$ be a vector. The following are equivalent:
  1. $A\mathbf{r} \leq \mathbf{0}$
  2. $(\forall \mathbf{x} \in P)(\forall \alpha > 0)\ \mathbf{x} + \alpha\mathbf{r} \in P$
  3. $\mathbf{r} \in \text{cone}(U)$
- $2 \Rightarrow 3$ requires the Farkas Lemma
- Note that (2) in the proof above is independent of A and U.

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Pointed V-Polyhedra

- Minimality for the set $\mathrm{conv}(V)$ – a polytope – is given by the vertex set
- In $P = \mathrm{cone}(U) + \mathrm{conv}(V)$ we need U to be minimal as for V-Cones
- Complication: a vertex of $\mathrm{conv}(V)$ may not be a vertex of P

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- If $\mathbf{v}$ is a vertex of $\mathrm{cone}(U) + \mathrm{conv}(V)$, then $[\mathbf{v} = U\mathbf{t} + V\lambda] \Rightarrow \mathbf{t} = \mathbf{0}$ (I call $\mathbf{v}$ here *U-free*)
- If $\mathbf{v}$ is a vertex of $\mathrm{cone}(U) + \mathrm{conv}(V)$, then $\mathbf{v}$ is a vertex of $\mathrm{conv}(V)$
- Let $P = \mathrm{cone}(U) + \mathrm{cone}(V)$ be pointed. The following are equivalent
  1. (U, V) is minimal for P
  2. U is minimal for $\mathrm{cone}(U)$, V is the vertex set of P
  3. U is minimal for $\mathrm{cone}(U)$, V is the vertex set of $\mathrm{conv}(V)$, and V is U-free

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Pointed V-Polyhedra

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Farkas Lemma 2

- We need another form of the Farkas Lemma:

$$(\exists \mathbf{t} \geq \mathbf{0})\mathbf{t}^T A = \mathbf{y},\ \mathbf{t}^T\mathbf{b} \leq c \Leftrightarrow \begin{cases} (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{y}^T\mathbf{x} \leq 0 \textbf{ and} \\ (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{b} \Rightarrow \mathbf{y}^T\mathbf{x} \leq c \end{cases}$$

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Full-Dimensional H-Polyhedra

- The following are equivalent
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional
  2. $\mathbf{y} \geq \mathbf{0},\ \mathbf{y}^T A = \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T \mathbf{b} > \mathbf{0}$
- $\mathbf{y}^T A = \mathbf{0},\ \mathbf{y}^T \mathbf{b} > \mathbf{0}$ occurs when two of the bounding hyperplanes are parallel
- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full-dimensional, the following are equivalent
  1. $(A, \mathbf{b})$ is minimal
  2. $\mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{e}_i, \mathbf{y}^T A = A_i \Rightarrow \mathbf{y}^T \mathbf{b} > b_i$
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Dual Homogenization Cone

- For H-Polyhedra switching to the dual setting is slightly more complicated
- The following two statement are equivalent:
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$
  2. $\text{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} = \text{cone}\begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix}$
- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is minimal and full-dimensional, then either
  1. $\text{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ is minimal and pointed, or
  2. $\text{cone}\begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix}$ is minimal and pointed, and
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Pointed H-Polyhedra and Full-Dimensional V-Polyhedra

- $\mathbf{t}$ is a non-negative vector, $V \neq \emptyset$, and abbreviate linear-independent as LI. $\bar{V}$ denotes $\{\mathbf{v} - \mathbf{v}' \mid \mathbf{v}, \mathbf{v}' \in V\}$.

	Pointed	Full-Dimensional
$\text{cone}(U)$	$U\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$	d LI vectors in U
$\text{cone}(U) + \text{conv}(V)$	$U\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$	d LI vectors in $U \cup \bar{V}$
$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$	d LI row vectors in A	$\mathbf{t}^T A = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$
$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$	d LI row vectors in A	$\mathbf{t}^T A = \mathbf{0} \Rightarrow \mathbf{t}^T\mathbf{b} > 0$

Not much to see here

- The proofs are more tedious than enlightening
- The implementation is straightforward

Outline

1. Goal and Outcome
   - Goal / Outcome
   - Strengths and Weaknesses
2. The Work
   - Not Original
   - Implementation
   - Pointed / Full-Dimensional Polyhedra
3. Closing Remarks
   - If I could do it again
   - Summary

Better MWT for Cones

- H-Cone $\to$ V-Cone. Use same transform and "tensor-notation"
- Farkas Lemma using the following facts:
  1. An intersection of closed sets is closed
  2. A projection of a closed subset of $\mathbb{R}^n$ is closed
  3. (Heine-Borel) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded
  4. A continuous function on a compact subset of $\mathbb{R}^n$ achieves its maximum
  5. Linear functions on $\mathbb{R}^n$ are continuous
  6. The Cauchy-Shwartz inequality
- Better dual cone:
  $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(U) \Leftrightarrow \text{cone}(A^T) = \{\mathbf{x} \mid U^T\mathbf{x} \leq \mathbf{0}\}$

Tensor-like notation

First, we write $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \left\{\mathbf{x} \,\middle|\, \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \text{cone}\begin{pmatrix} I & -I & \mathbf{0} \\ A & -A & I \end{pmatrix}\right\}$

Now we need to do our intersections:

- Let $\mathbf{u} \in \text{cone}(U)$ with $u_l = 0$
- Let $\mathbf{p}^i t_i$ sum over elements of U with positive l-th elements
- $\mathbf{u} = \mathbf{p}^i t_i + \mathbf{n}^j t_j + \mathbf{z}^k t_k$
- $u_l = 0 \Rightarrow p_l^i t_i + n_l^j t_j = 0 \ldots \quad \sigma := p_l^i t_i = -n_l^j t_j$
- $\mathbf{u} = \frac{\mathbf{n}^j t_j p_l^i t_i - \mathbf{p}^i t_i n_l^j t_j}{\sigma} + \mathbf{z}^k t_k = (\mathbf{n}^j p_l^i - \mathbf{p}^i n_l^j)\frac{t_i t_j}{\sigma} + \mathbf{z}^k t_k$

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Summary

- Finite generation
- The Farkas Lemma (separation)
- Pointed and Full-Dimensional polyhedra are "non-degenerate"

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Further Considerations

- Better algorithms (Dual-Description)
- Other interpretations...
  - ...Systems of logical deduction:
    Hahn-Banach vs Incompleteness
  - ...Systems where "lift and drop" creates a dual representation