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BACHELOR THESIS

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Minkowski-Weyl Theorem

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Dedication.

Title: Minkowski-Weyl Theorem

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Abstract: The Minkowski-Weyl Theorem is proven for polyhedra by first showing the proof for cones, then the reductions from polyhedra to cones. The proof follows Ziegler [1], and uses Fourier-Motzkin elimination. A C++ implementation is given for the enumeration algorithm suggested by the proof, as well a means of testing the implementation against some special polyhedra. The Farkas Lemma is then proven and used to prove the validity of the testing methods.

Keywords: Minkowski-Weyl Theorem polyhedra Fourier-Motzkin C++

Contents

Introduction

Polyhedra are fundamental mathematical objects. Two ways of describing polyhedra are:

1. A finite intersection of half-spaces

2. The *Minkowski-Sum* of the *convex-hull* of a finite set of rays and a finite set of points

The Minkowski-Weyl Theorem is a fundamental result in the theory of polyhedra. It states that both means of representation are equivalent. The proof given here is algorithmic in nature, using a technique known as *Fourier-Motzkin elimination.* The algorithm implied by the proof is then implemented in C++.

1. Minkowski-Weyl Theorem

1.1   Polyhedra

Definition 1.1.1 (Non-negative Linear Combination)**.** Let $U \in \mathbb{R}^{d \times p}, \mathbf{t} \in \mathbb{R}^p, \mathbf{t} \geq \mathbf{0}$, then $\sum_{1 \leq j \leq p} t_j U^j = U\mathbf{t}$ is called a *non-negative linear combination of U*.

Definition 1.1.2 (V-Cone)**.** Let $U \in \mathbb{R}^{d \times p}$. The set of all non-negative linear combinations of U is denoted cone(U). Such a set is called a *V-Cone.*

Definition 1.1.3 (Convex Combination)**.** Let $V \in \mathbb{R}^{d \times n}, \boldsymbol{\lambda} \in \mathbb{R}^n, \boldsymbol{\lambda} \geq \mathbf{0}, \sum_{1 \leq j \leq n} \lambda_j = 1$, then $\sum_{1 \leq j \leq n} \lambda_j V^j$ is called a *convex combination* of V. The set of all convex combinations of V is denoted conv(V).

Definition 1.1.4 (V-Polyhedron)**.** Let $V \in \mathbb{R}^{d \times n}$, $U \in \mathbb{R}^{d \times p}$. Then the set

$$\{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \text{cone}(U), \, \mathbf{y} \in \text{conv}(V)\}$$

is called a *V-Polyhedron.*

Note: Given two sets P and Q, the set $P + Q = \{p + q \mid p \in p, \, q \in Q\}$ is called the *Minkowski Sum* of P and Q. Therefore, we will write a V-Polyhedron as cone$(U) + $conv$(V)$ for some U and V.

Definition 1.1.5 (H-Polyhedron)**.** Let $A \in \mathbb{R}^{m \times d}$, $\mathbf{b} \in \mathbb{R}^m$. Then the set

$$\left\{\mathbf{x} \in \mathbb{R}^d \,\middle|\, A\mathbf{x} \leq \mathbf{b}\right\}$$

is called an *H-Polyhedron.*

Definition 1.1.6 (H-Cone)**.** Let $A \in \mathbb{R}^{m \times d}$. Then the set

$$\left\{\mathbf{x} \in \mathbb{R}^d \,\middle|\, A\mathbf{x} \leq \mathbf{0}\right\}$$

is called an *H-Cone.*

A simple but useful property of cones is that they are closed under addition and positive scaling.

Proposition 1.1.1 (Closure Property of Cones)**.** *Let C be either an H-Cone or a V-Cone, for each i $\mathbf{x}^i \in C$, and $c_i \geq 0$. Then:*

$$\sum_i c_i \mathbf{x}^i \in C$$

Proof. First we prove Proposition 1.1.1 for H-Cones, then for V-Cones. If, for each i, $A\mathbf{x}^i \leq \mathbf{0}$, then $A(c_i\mathbf{x}^i) = t_i A\mathbf{x}^i \leq \mathbf{0}$, and

$$A\left(\sum_i c_i\mathbf{x}^i\right) = \sum_i A(c_i\mathbf{x}^i) = \sum_i c_i A\mathbf{x}^i \leq \sum_i \mathbf{0} \leq \mathbf{0}$$

So, $\sum_i c_i\mathbf{x}^i \in C$ when C is an H-Cone. Next, suppose that $C = \text{cone}(U)$, and for each i, $\exists \mathbf{t}_i \geq \mathbf{0} : \mathbf{x}^i = U\mathbf{t}_i$. Then $c_i\mathbf{t}_i \geq \mathbf{0}$, and $\sum_i c_i\mathbf{t}_i \geq \mathbf{0}$. Therefore

$$\sum_i c_i\mathbf{x}^i = \sum_i c_i U\mathbf{t}_i = \sum_i U(c_i\mathbf{t}_i) = U\left(\sum_i c_i\mathbf{t}_i\right)$$

So, $\sum_i c_i\mathbf{x}^i \in C$ when C is a V-Cone. $\qquad\square$

This proposition will be used in the following way: if we wish to show that $\sum_i c_i\mathbf{x}^i$ in a member of some cone C, it suffices to show that, for each i, $c_i \geq 0$ and $\mathbf{x}^i \in C$.

1.2 Minkowski-Weyl Theorem

The following theorem is the basic result to be proved in this thesis, which states that V-Polyhedra and H-Polyhedra are two different representations of the same objects.

Theorem 1 (Minkowski-Weyl Theorem). *Every V-Polyhedron is an H-Polyhedron, and every H-Polyhedron is a V-Polyhedron.*

The proof proceeds by first showing that V-Cones are representable as H-Cones, and H-Cones are representable as V-Cones. Then it is shown that the case of polyhedra can be reduced to cones.

Theorem 2 (Minkowski-Weyl Theorem for Cones). *Every V-Cone is an H-Cone, and every H-Cone is a V-Cone.*

---

2. Proof of the Minkowski-Weyl Theorem

2.1 Every V-Cone is an H-Cone

Definition 2.1.1 (Coordinate Projection). Let I be the identity matrix. Then the matrix I' formed by deleting some rows from I is called a **coordinate-projection**.

The proof rests on the following two lemmas:

Lemma 2.1.1 (Lifting a V-Cone). *Every V-Cone is a coordinate-projection of an H-Cone.*

Lemma 2.1.2 (Projecting an H-Cone). *Every coordinate-projection of an H-Cone is an H-Cone.*

Proof. Given Lemma 2.1.1 and Lemma 2.1.2, the proof follows simply. Given a V-Cone, we use Lifting a V-Cone to get a description involving coordinate-projection of an H-Cone. Then we can apply Projecting an H-Cone in order to get an H-Cone. $\qquad\square$

Proof of Lifting a V-Cone. We prove that every V-Cone is a coordinate-projection of an H-Cone, by giving an explicit formula. Let $U \in \mathbb{R}^{d \times p}$, and observe that

$$\text{cone}(U) = \{U\mathbf{t} \mid \mathbf{t} \in \mathbb{R}^p, \mathbf{t} \geq \mathbf{0}\} = \left\{\mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \in \mathbb{R}^p)\, \mathbf{x} = U\mathbf{t}, \mathbf{t} \geq \mathbf{0}\right\}$$

We will collect $\mathbf{t}$ and $\mathbf{x}$ on the left side of the inequality, treating $\mathbf{t}$ as a variable and expressing its contraints as linear inequalities, then project away the coordinates corresponding to $\mathbf{t}$. The following expression takes one step:

$$\mathbf{t} \geq \mathbf{0} \Leftrightarrow -I\mathbf{t} \leq \mathbf{0} \tag{2.1}$$

Using the equality: $a = 0 \Leftrightarrow a \leq 0 \wedge -a \leq 0$, and block matrix notation, we take the second step.

$$\mathbf{x} = U\mathbf{t} \Leftrightarrow \mathbf{x} - U\mathbf{t} = \mathbf{0} \Leftrightarrow \begin{pmatrix} I & -U \\ -I & U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \tag{2.2}$$

Comparing (2.1) and (2.2), we define a matrix tranform:

Transform 1 (V-Cone Lift).

$$\text{T}_{\text{V}}(U) = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix}$$

So we define $A' = T_V(U)$, then we can rewrite $\mathrm{cone}(U)$:

$$\mathrm{cone}(U) = \left\{\mathbf{x} \in \mathbb{R}^d \,\middle|\, A'\begin{pmatrix}\mathbf{x}\\\mathbf{t}\end{pmatrix} \leq \mathbf{0}\right\}$$

Let $\Pi \in \{0,1\}^{d \times (d+p)}$ be the identity matrix in $\mathbb{R}^{(d+p)\times(d+p)}$, but with the last p-rows deleted. Then Π is a coordinate projection, and the above expression can be written:

$$\mathrm{cone}(U) = \Pi\left(\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0}\right\}\right) \tag{2.3}$$

This is a coordinate projection of an H-Cone, and Lifting a V-Cone is shown. $\square$

To prove Projecting an H-Cone, we use two separate propositions.

Proposition 2.1.1 (Projecting Null Columns)**.** *Let $B \in \mathbb{R}^{m'\times(d+p)}$, B' be B with the last p columns deleted, and Π the identity matrix with the last p rows deleted (i.e. $B' = \Pi B$). Furthermore, suppose that the last p columns of B are $\mathbf{0}$. Then*

$$\Pi\left(\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid B\mathbf{y} \leq \mathbf{0}\right\}\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid B'\mathbf{x} \leq \mathbf{0}\right\}$$

Proof. Recall that $B\mathbf{y} \leq \mathbf{0}$ means that $(\forall i)\,\langle B_i, \mathbf{y}\rangle \leq 0$. Because the last p columns of B are $\mathbf{0}$, any row B_i of B can be written $(B_i', \mathbf{0})$, with $\mathbf{0} \in \mathbb{R}^p$. We can also rewrite $\mathbf{y} \in \mathbb{R}^{d+p}$ as $(\mathbf{x}, \mathbf{w})$ with $\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^p$, so that $\mathbf{x} = \Pi(\mathbf{y})$. Then

$$\langle B, \mathbf{y}\rangle = \langle(B_i', \mathbf{0}), (\mathbf{x}, \mathbf{w})\rangle = \langle B_i', \mathbf{x}\rangle = \langle B_i', \Pi(\mathbf{y})\rangle$$

It follows that

$$\langle B_i, \mathbf{y}\rangle \leq 0 \Leftrightarrow \langle B_i', \Pi(\mathbf{y})\rangle \leq 0$$

Since B_i is an arbitrary row of B, the proposition is shown. $\square$

In order to use the above proposition, we need a matrix with columns which are $\mathbf{0}$. The next proposition shows us how to obtain such a matrix from another, while maintaining certain properties.

Proposition 2.1.2 (Fourier Motzkin Elimination for H-Cones)**.** *Let $B \in \mathbb{R}^{m_1 \times (d+p)}$, $1 \leq k \leq (d+p)$, and $\mathbf{x} = \sum_{i \neq k} x_i \mathbf{e}_i$. Then there exists a matrix $B' \in \mathbb{R}^{m_2 \times (d+p)}$ with the following properties:*

1. *Every row of B' is a postive linear combination of rows of B.*

2. *m_2 is finite.*

3. *The k-th column of B' is $\mathbf{0}$.*

4. *$(\exists t)B(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$*

Proof. Partition the rows of B as follows:

$$P = i \mid B_i^k > 0$$
$$N = j \mid B_j^k < 0$$
$$Z = l \mid B_l^k = 0$$

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Then let B' be a matrix with rows of the following forms:

$$
\begin{aligned}
C_l &= B_l & &| \, l \in Z \\
C_{ij} &= B_i^k B_j - B_j^k B_i & &| \, i \in P, j \in N
\end{aligned}
$$

1 and *2* are clear. *3* is satisfied for rows indexed by Z by definition That it holds for the other rows, observe:

$$
\langle C_{ij}, \mathbf{e}_k \rangle = \left\langle B_i^k B_j - B_j^k B_i, \mathbf{e}_k \right\rangle = B_i^k B_j^k - B_j^k B_i^k = 0
$$

The right direction of *4* is shown in the following calculations. Because $B_l^k = 0$:

$$
\langle B_l, \mathbf{x} + t\mathbf{e}_k \rangle = \langle B_l, \mathbf{x} \rangle + t B_l^k = \langle B_l, \mathbf{x} \rangle = \langle C_l, \mathbf{x} \rangle
$$

So $\langle B_l, \mathbf{x} + t\mathbf{e}_k \rangle \le 0 \Rightarrow \langle C_l, \mathbf{x} \rangle \le 0$. For rows indexed by P, N, because $B_i^k B_j^k - B_j^k B_i^k = 0$ we have:

$$
\left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \right\rangle = \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \right\rangle
$$

Because B_i^k and $-B_j^k$ are non-negative:

$$
\langle B_i, \mathbf{x} + t\mathbf{e}_k \rangle \le 0, \ \langle B_j, \mathbf{x} + t\mathbf{e}_k \rangle \le 0 \Rightarrow \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \right\rangle \le 0
$$

Therefore $\left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \right\rangle \le 0$, and the right implication is shown.

Now suppose that $B'\mathbf{x} \le \mathbf{0}$. The task is to find a t so that $B\mathbf{x} \le \mathbf{0}$. Because rows indexed by Z have $B_l^k = 0$, $B'(\mathbf{x} + t\mathbf{e}_k) \le \mathbf{0} \Rightarrow B\mathbf{x} \le \mathbf{0}$. So the task is to find a t so that the inequality holds for rows indexed by P and N. Observe

$$
\begin{aligned}
\forall i \in P, \forall j \in N \quad & \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \right\rangle \le \ 0 & &\Leftrightarrow \\
\forall i \in P, \forall j \in N \quad & \left\langle B_i^k B_j, \mathbf{x} \right\rangle \le \left\langle B_j^k B_i, \mathbf{x} \right\rangle & &\Leftrightarrow \\
\forall i \in P, \forall j \in N \quad & \left\langle B_i / B_i^k, \mathbf{x} \right\rangle \le \left\langle B_j / B_j^k, \mathbf{x} \right\rangle & &\Leftrightarrow \\
& \max_{i \in P} \left\langle B_i / B_i^k, \mathbf{x} \right\rangle \le \min_{j \in N} \left\langle B_j / B_j^k, \mathbf{x} \right\rangle
\end{aligned}
$$

Note that the third inequality changes directions because $B_j^k < 0$. Now we choose t to lie in this last interval, and show that we can use it to satisfy all of the constraints given by B. So, we have a t such that

$$
\max_{i \in P} \left\langle B_i / B_i^k, \mathbf{x} \right\rangle \le t \le \min_{j \in N} \left\langle B_j / B_j^k, \mathbf{x} \right\rangle
$$

In particular,

$$
(\forall j \in N) \quad \left\langle B_j / B_j^k, \mathbf{x} \right\rangle \ge t \Rightarrow \langle B_j, \mathbf{x} \rangle - B_j^k t \le 0
$$

Again, the inequality changes directions because $B_j^k < 0$. Now consider a row B_j from B:

$$
\langle B_j, \mathbf{x} - t\mathbf{e}_k \rangle = B_j \mathbf{x} - B_j^k t \le 0
$$

Similarly,

$$(\forall i \in P) \quad t \geq B_i / B_i^k \mathbf{x} \Rightarrow 0 \geq B_i \mathbf{x} - B_i^k t$$

Now consider a row B_i from B:

$$\langle B_i, \mathbf{x} - t\mathbf{e}_k \rangle = B_i \mathbf{x} - B_i^k t \leq 0$$

So, we've demonstrated that $\mathbf{x} - t\mathbf{e}_k$ satisfies all the constraints from B, and the left implication is shown. So *4* holds. □

Remark 1 (Fourier Motzkin Matrix)**.** Fourier Motzkin Elimination for H-Cones highlights the properties of the matrix B'. Upon close inspection, we can create a Matrix Y such that $B' = YB$, and every element of Y is non-negative. Create the following set of row vectors Y

$$\mathbf{e}_l \qquad\qquad | \; l \in Z$$
$$B_i^k \mathbf{e}_j - B_j^k \mathbf{e}_i \; | \; i \in P, \; j \in N$$

Since the basis vectors simply select rows during matrix multiplication, it is clear that

$$B' = YB$$

Proof of Projecting an H-Cone. Here we prove the case that the coordinate projection is onto the first d of $d + p$ coordinates. Let $\left\{ \mathbf{y} \in \mathbb{R}^{d+p} : A'\mathbf{y} \leq \mathbf{0} \right\}$ be the H-Cone we need to project, and Π the coordinate-projection we need to apply (the identity matrix with the last p columns deleted). For each $1 \leq k \leq p$ we can use Fourier Motzkin Elimination for H-Cones in an incremental manner, starting with A'.

> let $B_0 := A'$
> for $1 \leq k \leq p$
>   let $B_k :=$ result of proposition 2 applied to B_{k-1}, $\mathbf{e}_{d+k}$
> endfor
> return B_p

Consider the resulting B. Property *2* holds throughout, so B is finite. After each iteration, property *3* holds for k, so the k-th column is $\mathbf{0}$. Since each iteration only results from non-negative combinations of the result of the previous iteration (property *1*), once a column is $\mathbf{0}$ it remains so. Therefore, at the end of the process, the last p columns of B are all $\mathbf{0}$. Then, by Projecting Null Columns, we can apply Π to B by simply deleting the last p columns of B. Denote this resulting matrix A. We still need to check that

$$\Pi \left\{ \mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0} \right\} = \left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\} \tag{2.4}$$

This follows from the following:

$$A'\mathbf{y} \leq \mathbf{0} \Rightarrow A(\Pi(\mathbf{y})) \leq \mathbf{0} \tag{2.5}$$
$$A\mathbf{x} \leq \mathbf{0} \Rightarrow (\exists t_1) \ldots (\exists t_p) A'(\mathbf{x} + t_1 \mathbf{e}_{d+1} + \cdots + t_p \mathbf{e}_{d+p}) \leq \mathbf{0} \tag{2.6}$$

The key observation of this verification utilizes property 4 of Fourier Motzkin Elimination for H-Cones:

$$(\exists t)B(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$$

In what follows, let $\mathbf{x} = \sum_{1 \leq j \leq d} x_j \mathbf{e}_j$. The above property is applied sequentially to the sets B_k as follows:

$$
\begin{aligned}
(\exists t_p)(\exists t_{p-1})\dots(\exists t_1) \quad & B_0(\mathbf{x} + t_1\mathbf{e}_p + t_2\mathbf{e}_{p-1} + \cdots + t_p\mathbf{e}_d) \leq \mathbf{0} \Leftrightarrow \\
(\exists t_p)\dots(\exists t_2) \quad & B_1(\mathbf{x} + t_2\mathbf{e}_{d+2} + \cdots + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} \qquad \Leftrightarrow \\
& \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
(\exists t_p) \quad & B_{p-1}(\mathbf{x} + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} \qquad\qquad\qquad \Leftrightarrow \\
& B_p\mathbf{x} \leq \mathbf{0}
\end{aligned}
$$

Because $A' = B_0$, and A is B_p with the last p columns deleted, (2.5) and (2.6) hold, therefore (2.4) holds, and the proof of Projecting an H-Cone is complete, and we've shown that a coordinate projection of an H-Cone is again an H-Cone. $\quad\square$

With Lifting a V-Cone and Projecting an H-Cone proven, we are now certain that any V-Cone is also an H-Cone.

2.2 Every H-Cone is a V-Cone

Definition 2.2.1 (Coordinate Hyperplane). A set of the form

$$\left\{\mathbf{x} \in \mathbb{R}^{d+m} \mid \langle \mathbf{x}, \mathbf{e}_k \rangle = 0\right\} = \left\{\mathbf{x} \in \mathbb{R}^{d+m} \mid x_k = 0\right\}$$

is called a *coordinate-hyperplane.*

This is how coordinate hyperplanes will be used. We consider a V-Cone intersected with some coordinate hyperplanes, and write it in the following way:

$$\left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}\right\} \tag{2.7}$$

If we suppose that $U' \subset \mathbb{R}^{d+m}$, and Π is the identity matrix with the last m rows deleted, then this is just a convenient way of writing:

$$\Pi\Big(\operatorname{cone}(U') \cap \{x_{d+1} = 0\} \cap \cdots \cap \{x_{d+m} = 0\} \Big) \tag{2.8}$$

The proof rests on the following three propostions:

Lemma 2.2.1 (Lifting an H-Cone). *Every H-Cone is a coordinate-projection of a V-Cone intersected with some coordinate hyperplanes.*

Lemma 2.2.2 (Intersecting a V-Cone). *Every V-Cone intersected with a coordinate-hyperplane is a V-Cone.*

Lemma 2.2.3 (Projecting a V-Cone). *Every coordinate-projection of a V-Cone is an V-Cone.*

Proof. Given lemma 2.2.1, lemma 2.2.2, and lemma 2.2.3, the proof follows simply. Given an H-Cone, we use Lifting an H-Cone to get a description involving the coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. We apply Intersecting a V-Cone as many times as necessary to elimintate the intersections, then we can apply Projecting a V-Cone in order to get a V-Cone. $\quad\square$

Proof of Lifting an H-Cone. Let $A \in \mathbb{R}^{m \times d}$, we now show that the H-Cone

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$$

can be written as the projection of a V-Cone intersected with some hyperplanes. We use the following transform.

Transform 2 (V-Cone Lift).

$$\mathrm{T_H}(A) = \begin{pmatrix} \mathbf{0} & I & -I \\ I & A & -A \end{pmatrix}$$

Define

$$U' = \mathrm{T_H}(A) = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}, \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}, 1 \leq j \leq d, 1 \leq i \leq m \right\}$$

We then claim:

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\} = \left\{ \mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t} \right\} \qquad (2.9)$$

First, considering (2.7) and (2.8), observe that this is a coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. Next, we note that

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$$

We can write this as a sum with all positive coefficients if we split up the x_j as follows:

$$x_j^+ = \begin{cases} x_j & x_j \geq 0 \\ 0 & x_j < 0 \end{cases} \qquad x_j^- = \begin{cases} 0 & x_j \geq 0 \\ -x_j & x_j < 0 \end{cases}$$

Then we have

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} \qquad (2.10)$$

where $x_j^+, x_j^- \geq 0$. Also observe that

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$$

This can also be written

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \qquad (2.11)$$

(2.10) and (2.11) together show

$$A\mathbf{x} \leq \mathbf{0} \Rightarrow (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

Conversely, suppose

$$(\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

We would like to show that $A\mathbf{x} \leq \mathbf{0}$. Let x_j^+, x_j^-, w_i take the values of $\mathbf{t}$ that are coefficients of $\begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$, $\begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}$, and $\begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}$ respectively, and denote $x_j = x_j^+ - x_j^-$. Then we have

$$\begin{aligned}
\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} &= \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\
&= \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}
\end{aligned}$$

where $\mathbf{w} \geq \mathbf{0}$. By (2.11) we have $A\mathbf{x} \leq \mathbf{0}$. So (2.9) holds. $\qquad \square$

The proof of Intersecting a V-Cone relies upon the following proposition.

Proposition 2.2.1 (Fourier Motzkin Elimination for V-Cones)**.** *Let* $Y \in \mathbb{R}^{(d+m) \times n_1}$, $1 \leq k \leq m$, *and* $\mathbf{x}$ *satisfy* $x_k = 0$. *Then there exists a matrix* $Y' \in \mathbb{R}^{(d+m) \times n_2}$ *with the following properties:*

1. *Every column of* Y' *is a postive linear combination of columns of* Y.

2. n_2 *is finite.*

3. *The k-th row of* Y' *is* $\mathbf{0}$.

4. $(\exists \mathbf{t} \geq \mathbf{0})\mathbf{x} = Y\mathbf{t} \Leftrightarrow (\exists \mathbf{t}' \geq \mathbf{0})\mathbf{x} = Y'\mathbf{t}'$

Proof. We partition the columns of Y:

$$\begin{aligned}
P &= i \mid Y_k^i > 0 \\
N &= j \mid Y_k^j < 0 \\
Z &= l \mid Y_k^l = 0
\end{aligned}$$

We then define Y':

$$Y' = \left\{ Y^l \mid l \in Z \right\} \cup \left\{ Y_k^i Y^j - Y_k^j Y^i \mid i \in P, j \in N \right\}$$

1 and *2* are clear. *3* can be seen from:

$$\begin{aligned}
\left\langle Y'^l, \mathbf{e}^k \right\rangle &= 0 \\
\left\langle Y'^{ij}, \mathbf{e}^k \right\rangle &= \left\langle Y_k^i Y^j - Y_k^j Y^i, \mathbf{e}^k \right\rangle = Y_k^i Y_k^j - Y_k^j Y_k^i = 0 \qquad (2.12)
\end{aligned}$$

Before moving on to the proof, we first note how we may write our vectors.

$$Y\mathbf{t} = \sum_{l \in Z} t_k Y^k + \sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j$$

$$Y'\mathbf{t} = \sum_{l \in Z} t_k Y^k + \sum_{\substack{i \in P \\ j \in N}} t_{ij}(Y_k^i Y^j - Y_k^j Y^i)$$

Then, by Closure Property of Cones (cone closure), to show that the proposition is true, we need only show that, given any $t_i, t_j \geq 0$ ($t_{ij} \geq 0$), there exists $t_{ij} \geq 0$ ($t_i, t_j \geq 0$) such that

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} t_{ij}(Y_k^i Y^j - Y_k^j Y^i) \tag{2.13}$$

Proposition 2.2.2 (Sum Mixture). *Suppose that*

$$\sum_{i \in P} t_i Y_{d+1}^i + \sum_{j \in N} t_j Y_{d+1}^j = 0 \qquad Y_k^j < 0 < Y_k^i$$

Then the following holds

$$(t_i, t_j \geq 0) \Rightarrow (\exists t_{ij} \geq 0) \qquad \text{such that (2.13) holds}$$
$$(t_{ij} \geq 0) \Rightarrow (\exists t_i, t_j \geq 0) \qquad \text{such that (2.13) holds}$$

Sum Mixture. First note that if all $t_i = 0, t_j = 0$, then choosing $t_{ij} = 0$ satisfies (2.13), likewise if all $t_{ij} = 0$, then $t_i = 0, t_j = 0$ satisfies (2.13). So suppose that some $t_i \neq 0, t_j \neq 0, t_{ij} \neq 0$.

The right hand side of (2.13) can be written

$$\sum_{j \in N} \left(\sum_{i \in P} t_{ij} Y_k^i \right) Y^j + \sum_{i \in P} \left(-\sum_{j \in N} t_{ij} Y_k^j \right) Y^i$$

This means, given $t_{ij} \geq 0$, we can choose $t_j = \sum_{i \in P} t_{ij} Y_k^i$, and $t_i = -\sum_{j \in N} t_{ij} Y_k^j$, both of which are greater than 0.

Now suppose we have been given $t_i \geq 0, t_j \geq 0$. First observe:

$$0 = \sum_{i \in P} t_i Y_k^i + \sum_{j \in N} t_j Y_k^j \Rightarrow \sum_{i \in P} t_i Y_k^i = -\sum_{j \in N} t_j Y_k^j$$

Denote the value in this equality as σ, and note that $\sigma > 0$. Then

$$\sum_{i \in P} t_i Y^i = \frac{-\sum_{j \in N} t_j Y_k^j}{\sigma} \sum_{i \in P} t_i Y^i = \sum_{\substack{i \in P \\ j \in N}} -\frac{t_i t_j}{\sigma} Y_k^j Y^i$$

$$\sum_{j \in N} t_j Y^j = \frac{\sum_{i \in P} t_i Y_k^i}{\sigma} \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma} Y_k^i Y^j$$

Combining these results, we have

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma}(Y_k^i Y^j - Y_k^j Y^i)$$

$\square$

Finally, we can conclude that, given $\mathbf{t} \geq \mathbf{0}$, if $Y\mathbf{t}$ has a 0 in the final coordinate, then we can write it as $Y'\mathbf{t}'$ where $\mathbf{t}' \geq \mathbf{0}$, and any non-negative linear combination of vectors from Y' can be written as a non-negative linear combination of vetors from Y, and will necessarily have the k-th coordinate be 0 by property *3*. So property *4* holds. □

Proof of Intersecting a V-Cone. In Fourier Motzkin Elimination for V-Cones, the assumption that $x_k = 0$ in property *4* creates the set $\text{cone}(Y) \cap \{\mathbf{x} \mid x_k = 0\}$. This set, by property *4*, is $\text{cone}(Y')$. □

Proof of Projecting a V-Cone. We shall prove that the coordinate-projection of a V-Cone is again a V-Cone. Let Π be the relevant projection, then we have:

$$\Pi\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U\mathbf{t}) \mid \mathbf{t} \geq \mathbf{0}\} = \{(\Pi U)\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\}$$

The last equality follows from associativity of matrix multiplication. Therefore,

$$\Pi\big(\text{cone}(U)\big) = \text{cone}\big(\Pi U\big)$$

□

Having shown that H-Cones are V-Cones, we complete the proof of the Minkowski-Weyl Theorem for cones is complete.

2.3 Reducing Polyhedra to Cones

Definition 2.3.1 (Hyperplane)**.** Let $\mathbf{y} \in \mathbb{R}^d$, $c \in \mathbb{R}$. Then a set of the form

$$\Big\{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{y}, \mathbf{x} \rangle = c\Big\}$$

is called a *hyperplane.*

2.3.1 H-Polyhedra $\leftrightarrow$ H-Cones

Proposition 2.3.1. *Every H-Polyhedron can be written as an H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$, and any H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$ is an H-Polyhedron.*

Proof. We begin by re-writing the expression:

$$A\mathbf{x} \leq \mathbf{b} \Leftrightarrow -\mathbf{b} + A\mathbf{x} \leq \mathbf{0} \Leftrightarrow \big[-\mathbf{b}|A\big]\begin{pmatrix}1\\\mathbf{x}\end{pmatrix} \leq \mathbf{0}$$

Note that

$$\Big\{\mathbf{x} \mid \big[-\mathbf{b}|A\big]\begin{pmatrix}1\\\mathbf{x}\end{pmatrix} \leq \mathbf{0}\Big\} = \Big\{\begin{pmatrix}x_0\\\mathbf{x}\end{pmatrix} \mid \big[-\mathbf{b}|A\big]\begin{pmatrix}x_0\\\mathbf{x}\end{pmatrix} \leq \mathbf{0}\Big\} \cap \{\mathbf{x} \mid x_0 = 1\}$$

It follows that

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \Big\{\mathbf{x} \mid \big[-\mathbf{b}|A\big]\mathbf{x} \leq \mathbf{0}\Big\} \cap \{\mathbf{x} \mid x_0 = 1\}$$

□

2.3.2 V-Polyhedra ↔ V-Cone

Proposition 2.3.2 (V-Polyhedron → V-Cone)**.**

$$\text{cone}(U) + \text{conv}(V) = \text{cone}\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ U & V \end{pmatrix} \cap \{\mathbf{x} \mid x_0 = 1\}$$

Proof. For the value 1 to appear in the first coordinate, a convex combination of the vectors from $(\mathbf{1}, V)$ must be taken. After that, any non-negative combination of $(\mathbf{0}, U)$ added to this vector won't affect the 1 in the first coordinate. □

This shows that a V-Polyhedron may be written as an intersection of a V-Cone and the hyperplane $\{\mathbf{x} \mid x_0 = 1\}$. It is more difficult to show that, given a V-Cone, that you can intersect it with the hyperplane $\{\mathbf{x} \mid x_0 = 1\}$ and get a V-Polyhedron out of it.

Proposition 2.3.3 (V-Cone → V-Polyhedron)**.** *Let Π be the identity matrix with the first row deleted. Then, for any set $C_U = \text{cone}(U)$ there are sets W and V such that*

$$\text{cone}(V) \cap \{\mathbf{x} \mid x_0 = 1\} = \text{cone}(W) + \text{conv}(V)$$

Proof. We partition U into the sets:

$$P = i \mid U_0^i > 0$$
$$N = j \mid U_0^j < 0$$
$$Z = l \mid U_0^l = 0$$

And define two new sets:

$$W = \left\{ U^l \mid l \in Z \right\} \cup \left\{ U_0^i U^j - U_0^j U^i \mid i \in P, j \in N \right\}$$
$$V = \left\{ U^i / U_0^i \mid i \in P \right\}$$

Then I claim that

$$C_U = \text{cone}(W) + \text{conv}(V)$$

Say $\mathbf{x} \in \text{cone}(W)$, and $\mathbf{y} \in \text{conv}(V)$. Then $\mathbf{x}$ can be written

$$\mathbf{x} = \sum_{l \in Z} t_l U^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (U_0^i U^j - U_0^j U^i)$$

$$= \sum_{l \in Z} t_l U^l + \sum_{j \in N} \left(\sum_{i \in P} t_{ij} U_0^i \right) U^j + \sum_{i \in P} \left(\sum_{j \in N} -t_{ij} U_0^j \right) U^i$$

So $\mathbf{x} \in C_U$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{x} \rangle = \sum_{l \in Z} t_l U_0^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (U_0^i U_0^j - U_0^j U_0^i) = 0$$

So $x_0 = 0$. Similarly, $\mathbf{y}$ can be written:

$$\mathbf{y} = \sum_{i \in P} \lambda_i U^i / U_0^i, \quad \sum_{i \in P} \lambda_i = 1$$

So $\mathbf{y} \in C_U$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{y} \rangle = \sum_{i \in P} \lambda_i U_0^i / U_0^i = \sum_{i \in P} \lambda_i = 1$$

So $y_0 = 1$ and $x_0 + y_0 = 1$. It follows that $\mathbf{x} + \mathbf{y} \in C_U$.

Next, suppose that $\mathbf{z} \in C_U$, then $\mathbf{z}$ can be written

$$\mathbf{z} = \sum_{l \in Z} t_l U^l + \sum_{i \in P} t_i U^i + \sum_{j \in N} t_j U^j$$

It will be convenient to use shorter notation for these sums. Define the following:

$$\boldsymbol{\sigma}_Z = \sum_{l \in Z} t_l U^l, \quad \sigma_l = \sum_{l \in Z} t_l U_0^l = 0$$

$$\boldsymbol{\sigma}_P = \sum_{i \in P} t_i U^i, \quad \sigma_i = \sum_{i \in P} t_i U_0^i$$

$$\boldsymbol{\sigma}_N = \sum_{j \in N} t_j U^j, \quad \sigma_j = \sum_{j \in N} t_j U_0^j$$

Then it holds that

$$\langle \mathbf{e}_0, \mathbf{z} \rangle = \sigma_l + \sigma_i + \sigma_j = \sigma_i + \sigma_j = 1 \quad \Rightarrow \quad -\sigma_j / \sigma_i = 1 - 1/\sigma_i$$

$$\boldsymbol{\sigma}_P = \boldsymbol{\sigma}_P / \sigma_i + (1 - 1/\sigma_i)\boldsymbol{\sigma}_P = \boldsymbol{\sigma}_P / \sigma_i - (\sigma_j / \sigma_i)\boldsymbol{\sigma}_P$$

Using the new notation, we can rewrite $\mathbf{z}$:

$$\mathbf{z} = \boldsymbol{\sigma}_Z + \boldsymbol{\sigma}_P + \boldsymbol{\sigma}_N = \boldsymbol{\sigma}_Z + \frac{\boldsymbol{\sigma}_P}{\sigma_i} - \frac{\sigma_j}{\sigma_i}\boldsymbol{\sigma}_P + \frac{\sigma_i}{\sigma_i}\boldsymbol{\sigma}_N = \boldsymbol{\sigma}_Z + \frac{\boldsymbol{\sigma}_P}{\sigma_i} + \frac{\sigma_i \boldsymbol{\sigma}_N - \sigma_j \boldsymbol{\sigma}_P}{\sigma_i}$$

Using 'Closure Property of Cones' on page 3, we need only show that

1. $\boldsymbol{\sigma}_Z \in \text{cone}(W)$

2. $(\sigma_i \boldsymbol{\sigma}_N - \sigma_j \boldsymbol{\sigma}_P) \in \text{cone}(W)$

3. $\boldsymbol{\sigma}_P / \sigma_i \in \text{conv}(V)$

Since each $U^l : l \in Z$ is in C_V, (1) holds. We also have:

$$\sigma_i \boldsymbol{\sigma}_N - \sigma_j \boldsymbol{\sigma}_P = \sum_{i \in P} t_i \sum_{j \in N} t_j U_0^i U^j - \sum_{j \in N} t_j \sum_{i \in P} t_i U_0^j U^i = \sum_{\substack{i \in P \\ j \in N}} t_i t_j (U_0^i U^j - U_0^j U^i)$$

So (2) holds. Finally,

$$\boldsymbol{\sigma}_P / \sigma_i = \sum_{i \in P} t_i U^i / \sigma_i = \sum_{i \in P} (t_i U_0^i / \sigma_i)(U^i / U_0^i)$$

Since $\sum_{i \in P}(t_i U_0^i / \sigma_i) = \sigma_i / \sigma_i = 1$, it follows that $\boldsymbol{\sigma}_P / \sigma_i \in \text{conv}(V)$. $\qquad \square$

2.4 Picture of the Proof

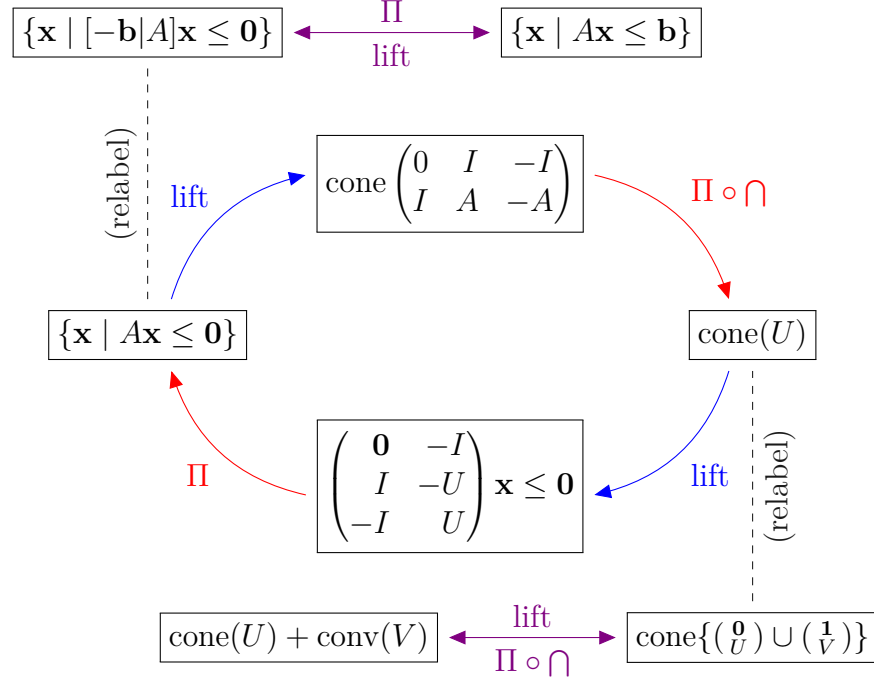Here we show a diagram that represent the proof of the Minkowski-Weyl Theorem.

Figure 2.1: Diagram of the proof $P_H \leftrightarrow P_V$

Figure 2.1 shows the flow from an H-Polyhedron to a V-Polyhedron and back. There are arrows for transformations back and forth from polyhedra to cones, arrows to show the transformation between cones and intermediate representation, and arrows to show where Fourier Motzkin elimination is applied to reduce these intermediate represenations to standard cones. V-Cones are lifted to H-Cones which need to be projected (Π), and H-Cones are lifted to V-Cones which need to be intersected and projected ($\Pi \circ \bigcap$).

3. C++ Implementation

The above transformations have been implemented in C++. Program `main.cpp` takes one argument specifying the type of input object. It reads the description of the object from standard input, and writes the result of the implied transformation to standard output. If no arguments are supplied, then a `usage` message is given. The `usage` message, which also contains the input format for the objects, is:

```
usage: ./main input_type

The input object is read on stdin, and the result of the
transform to sent to stdout.  input_type determines the type
of input and output:

    -vc  # transforms a vcone into an hcone
    -vp  # transforms a vpolyhedron into an hpolyhedron
    -hc  # transforms an hcone into a vcone
    -hp  # transforms an hpolyhedron into a vpolyhedron

input format is as follows:
  hcone := dimension   ws (vector ws)*
  vcone := dimension   ws (vector ws)*
  hpoly := dimension+1 ws (vector ws constraint ws)*
  vpoly := dimension   ws ('U' | 'V') ws vpoly_vecs*

  ws         := whitespace, as would be read by "cin >> ws;"
  dimension  := a positive integer.  For hpoly, add one to
                the dimension of the space (this extra
                dimension is for the constraint)
  vector     := (dimension) doubles separated by whitespace
  constraint := a double (the value b_i in <A_i,x> <= b_i)
  'V' | 'U'  := the literal character 'U' or 'V'
  vpoly_vecs := (['U'] ws vector) | (['V'] ws vector)

VPOLY ONLY:

  vpoly contains two matrices:
    U - contains the rays of the vpolyhedron
    V - contains the points of the vpolyhedron

  On input, enter 'U' or 'V' to indicate which matrix should
  receive the vectors that follow.  You can switch back and
  forth as you like, but either 'U' or 'V' must be entered
  before starting to input vectors.

EXAMPLES:
```

```
$ ./main -vc <<< "2 1 0"

OUTPUT:

2
-0 -1
0 1
0 0
-1 0

$ ./main -hc <<< "2 1 0 0 1"

OUTPUT:

2
-1 0
0 0
0 0
0 -1

$ ./main -vp <<< "2 U 1 0 V 0 0 1 1"

OUTPUT:

3
0 0 -0
0 0 -0
0 1 1
0 0 1
-1 1 -0
-1 0 -0
0 0 -0
0 -1 -0
$ ./main -hp <<< "3 0 -1 0 0 1 1 -1 1 0"

OUTPUT:

2
U
1 0
0 0
0 0
0 0
V
0 0
1 1
```

The files pertaining to the implementation will be discussed in the following sections, but here is a table showing the include dependencies followed by a short

summary of the files.

file	includes
linear_algebra.h	
fourier_motzkin.h	linear_algebra.h
polyhedra.h	fourier_motzkin.h
main.cpp	polyhedra.h
test_functions.h	linear_alebra.h
test.cpp	test_functions.h, polyhedra.h

Here is a very brief summary of the files mentioned in the above table, more details are given in sequent sections.

- `linear_algebra.h`
  Types `Vector` and `Matrix`, and some basic functionality for them

- `fourier_motzkin.h`
  Fourier Motzkin elimination, Minkowski-Weyl Theorem for cones

- `polyhedra.{cpp,h}`
  Transforms between polytopes and polyhedra, Minkowski-Weyl Theorem

- `test_functions.h`
  Types and functions for testing the algorithms

- `test.cpp`
  Test cases for the algorithms and the functions from `test_functions.h`

3.1 Code

The relevant code will be displayed with commentary below. Some of the code relating to C++ specific technicalities and I/O is ommitted.

3.2 `linear_algebra.h`

The types `Vector` and `Vectors` are used in the representation of polyhedra. The `std::valarray` template is used because it has built-in vector-space operations (sum and scaling). `std::vector`, is used as a container of `Vectors`, however other containers could be used.

```
10  using Vector = std::valarray<double>;
11  using Vectors = std::vector<Vector>;
```

The `class` `Matrix` implements a subset of what a *C++ Container* should. It is the primary type for representing polyhedra, and directly represents Cones, as well as H-Polyhedra. The interface is designed to enforce the following invariant:

$$(\forall v \in \texttt{vectors})\, \texttt{v.size() == d}$$

The *factory* function `read_Matrix` is provided to read a `Matrix` from an `istream`. It is necessary because the value of `d` can't be known before reading some of the stream.

```cpp
13  class Matrix {
14  // invariant: d >= 0
15  // invariant: (forall valid i) vectors[i].size() == d
16  public:
17    const size_t d; // size of all Vectors
18  private:
19    Vectors vectors;
20  public:
21    // needed for back_insert_iterator
22    using value_type = Vector;
23
24    Matrix(size_t d);
25    Matrix(std::initializer_list<Vector>&&);
26    bool check() const; // checks each Vector has size d
27
28    //defaults don't work because of const member
29    Matrix(const Matrix&);
30    Matrix(Matrix&&);
31    Matrix &operator=(const Matrix&);
32    Matrix &operator=(Matrix&&);
33    Matrix &operator=(std::initializer_list<Vector>&&);
34
35    static Matrix read_Matrix(std::istream&);
36
37    Vectors::iterator        begin();
38    Vectors::iterator        end();
39    Vectors::const_iterator begin() const;
40    Vectors::const_iterator end()   const;
41
42    bool    empty() const;
43    size_t  size()  const;
44    Vector& back();
45
46    Vector& add_Vector();
47    void push_back(const Vector &v);
48    void push_back(Vector &&v);
49  };
```

The struct VPoly gather two Matrixs needed to represent a V-Polyhedron. The Matrix U corresponds to the rays that generate the cone, and the Matrix V corresponds to the points, i.e.

$$\text{vpoly} = \text{cone}(\text{vpoly.U}) + \text{conv}(\text{vpoly.V})$$

```cpp
51  struct VPoly {
52    const size_t d;
53    Matrix U; // rays
54    Matrix V; // points
55
56    VPoly(size_t d) : d{d}, U{d}, V{d} {}
57    VPoly(std::initializer_list<Vector>&&,
58          std::initializer_list<Vector>&&);
```

```
59    bool check() const;
60
61    static VPoly read_VPoly(std::istream&);
62 };
```

The `class input_error` is thrown to indicate an invalid input to the program, and provide some clue as to why it failed. Here are two command line examples:

```
$ ./main -vc <<< "0"
terminate called after throwing an instance of 'input_error'
  what():  bad d: 0
Aborted (core dumped)
$ ./main -vc <<< "2 1"
error reading matrix, vector 1
terminate called after throwing an instance of 'input_error'
  what():  failed to read vector: istream failed
Aborted (core dumped)
```

```
64 class input_error : public std::runtime_error {
65 public:
66    input_error(const char*s);
67    input_error(const std::string &s);
68 };
```

`operator>>` and `operator<<` implement the input format described in `usage.txt`.

```
70 std::istream& operator >>(std::istream&, Vector&);
71 std::istream& operator >>(std::istream&, Matrix&);
72 std::istream& operator >>(std::istream&, VPoly&);
```

```
74 std::ostream& operator <<(std::ostream& o, const Vector&);
75 std::ostream& operator <<(std::ostream& o, const Matrix&);
76 std::ostream& operator <<(std::ostream& o, const VPoly&);
```

`usage()` outputs the usage message shown above.

```
78 int usage();
```

3.3 `linear_algebra.cpp`

`e_k` creates the canonical basis `Vector` $\mathbf{e}_k \in \mathbb{R}^d$.

```
232 Vector e_k(size_t d, size_t k) {
233    Vector result(d);
234    result[k] = 1;
235    return result;
236 }
```

`concatentate` takes the `Vectors` `l` $\in \mathbb{R}^{\texttt{l.size()}}$ and `r` $\in \mathbb{R}^{\texttt{r.size()}}$ and `return`s the `Vector` `(l,r)` $\in \mathbb{R}^{\texttt{l.size() + r.size()}}$

```
239  Vector concatenate(const Vector &l, const Vector &r) {
240    Vector result(l.size() + r.size());
241    copy(begin(l), end(l), begin(result));
242    copy(begin(r), end(r), next(begin(result), l.size()));
243    return result;
244  }
```

get_column returns the k-th column of the Matrix M. Note that while a Matrix may logically represent either a collection of row or column Vectors, get_column is only used in the function transpose, where this distinction is unimportant.

```
249  Vector get_column(const Matrix &M, size_t k) {
250    if (!(0 <= k && k < M.d)) {
251      throw std::out_of_range("k < 0 || M.d <= k");
252    }
253    Vector result(M.size());
254    size_t result_row{0};
255    for (auto &&row : M) {
256      result[result_row++] = row[k];
257    }
258    return result;
259  }
```

transpose returns the transpose of Matrix M.

```
262  Matrix transpose(const Matrix &M) {
263    if (M.empty()) {
264      return M;
265    }
266    Matrix result{M.size()};
267    // for every column of M,
268    for (size_t k = 0; k < M.d; ++k) {
269      result.push_back(get_column(M,k));
270    }
271    return result;
272  }
```

A slice object can be used to conveniently obtain a subset of a valarray. slice_matrix returns the Matrix obtained by applying the slice s to each Vector of the Matrix.

```
275  Matrix slice_matrix(const Matrix &M, const std::slice &s) {
276    Matrix result{s.size()};
277    transform(M.begin(), M.end(), back_inserter(result),
278      [s](const Vector &v) { return v[s]; });
279    return result;
280  }
```

3.4 fourier_motzkin.cpp

A slice object is determined by three fields: start, size, and stride, and implicitly represents all indices of the form:

$$\sum_{0 \le k < \texttt{size}} \texttt{start} + k \cdot \texttt{stride}$$

Therefore:

$$i \in \texttt{slice} \Leftrightarrow i - \texttt{start} \equiv 0 \mod (\texttt{stride}), \quad \texttt{start} \leq i \leq \texttt{start} + \texttt{stride} \cdot \texttt{size}$$

```
11  bool index_in_slice(size_t index, const slice &s) {
12    return ((index - s.start()) % s.stride() == 0) &&
13           s.start() <= index &&
14           index <= s.start() + s.stride()*(s.size()-1);
15  }
```

`fourier_motzkin` takes a `Matrix M` and a coordinate `k` and creates the set which either corresponds to a projection of an H-Cone (without actually doing the projection), or the intersection of a V-Cone with a coordinate-hyperplane.

```
20  Matrix fourier_motzkin(Matrix M, size_t k) {
21    Matrix result{M.d};
22    // Partition into Z,P,N
23    const auto z_end = partition(M.begin(), M.end(),
24        [k](const Vector &v) { return v[k] == 0; });
25    const auto p_end = partition(z_end, M.end(),
26        [k](const Vector &v) { return v[k] > 0; });
27    // Move Z to result
28    move(M.begin(), z_end, back_inserter(result));
29    // convolute vectors from P,N
30    for (auto p_it = z_end; p_it != p_end; ++p_it) {
31      for (auto n_it = p_end; n_it != M.end(); ++n_it) {
32        result.push_back(
33          (*p_it)[k]*(*n_it) - (*n_it)[k]*(*p_it));
34      }
35    }
36    return result;
37  }
```

The lines:

```
23    const auto z_end = partition(M.begin(), M.end(),
24        [k](const Vector &v) { return v[k] == 0; });
25    const auto p_end = partition(z_end, M.end(),
26        [k](const Vector &v) { return v[k] > 0; });
```

Partition `M` into logical sets Z, P, N that satisfy the following:

set	range	property
Z	[`M.begin()`, `z_end`)	it $\in Z \Leftrightarrow$ (*it)[k] $= 0$
P	[`z_end`, `p_end`)	it $\in P \Leftrightarrow$ (*it)[k] > 0
N	[`p_end`, `M.end()`)	it $\in N \Leftrightarrow$ (*it)[k] < 0

The line:

```
28    move(M.begin(), z_end, back_inserter(result));
```

Moves Z into the result. The lines:

```
30    for (auto p_it = z_end; p_it != p_end; ++p_it) {
31      for (auto n_it = p_end; n_it != M.end(); ++n_it) {
32        result.push_back(
```

```
33          (*p_it)[k]*(*n_it) - (*n_it)[k]*(*p_it));
34      }
35    }
```

convolute the vectors in the way described in 'Fourier Motzkin Elimination for H-Cones' on page 6 and 'Fourier Motzkin Elimination for V-Cones' on page 11 (concerning projecting an H-Cone and intersecting a V-Cone with a coordinate-hyperplane), and push them into the result `Matrix`. In particular, it creates the sets which correspond to

$$\left\{B_i^k B_j - B_j^k B_i \mid i \in P,\, j \in N\right\}, \quad \left\{Y_k^i Y^j - Y_k^j Y^i \mid i \in P,\, j \in N\right\}$$

`sliced_fourier_motzkin` applies `fourier_motzkin` to `Matrix M` for each $k \notin$ `s`, then slices the resulting `Matrix` using `slice_matrix` and `s`. This is the realization of the algorithms indicated by the proofs of either direction of the Minkowski-Weyl Theorem for cones.

```
40  Matrix sliced_fourier_motzkin(Matrix M, const slice &s) {
41    for (size_t k = 0; k < M.d; ++k) {
42      if (!index_in_slice(k,s)) {
43        M = fourier_motzkin(M, k);
44      }
45    }
46    return slice_matrix(M, s);
47  }
```

When transforming an H-Cone to a V-Cone, it first must be written as a V-Cone of a new matrix, then it is intersected with coordinate-hyperplanes and projected. Similarly, when a V-Cone is transformed into an H-Cone, it must be written as and H-Cone of a new matrix then projected with coordinate-projections. The transformations are described in (**??**) and (**??**), and summarized here:

$$A \to \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix} \quad U \to \begin{pmatrix} \mathbf{0} & I & -I \\ I & A & -A \end{pmatrix}$$

Note that the tranformation of U can be written:

$$U \to \begin{pmatrix} \mathbf{0} & I \\ I & A \\ -I & -A \end{pmatrix}^T$$

Remembering that a `Matrix` is either a collection of row *or* column `Vector`s, it is not surprising that these two transformations can be written as one function of a `Matrix` and some coefficients. In `generalized_lift`, the coefficients are given as an `array<`double`, 5> C`, so the overall transformation can be illustrated as:

$$\text{Matrix M} \to \begin{pmatrix} \mathbf{0} & \text{C[0]}I \\ \text{C[1]}I & \text{C[2]M} \\ \text{C[3]}I & \text{C[4]M} \end{pmatrix}$$

where `Matrix M` is a collection of row `Vector`s, or

$$\text{Matrix M} \to \begin{pmatrix} \mathbf{0} & \text{C[1]}I & \text{C[3]}I \\ \text{C[0]}I & \text{C[2]M} & \text{C[4]M} \end{pmatrix}$$

where `Matrix M` is a collection of column `Vector`s.

```
64  Matrix generalized_lift(const Matrix &cone,
65                          const array<double,5> &C) {
66    const size_t d = cone.d;
67    const size_t n = cone.size();
68    Matrix result{d+n};
69    Matrix cone_t = transpose(cone);
70    // |0  C[0]*I|   |0      |
71    //               |C[0]*I|
72    for (size_t i = 0; i < n; ++i) {
73      result.add_Vector()[d+i] = C[0];
74    }
75    size_t k = 0;
76    // |C[1]*I C[2]*U|   |C[1]*I|
77    //                   |C[2]*A|
78    for (auto &&row_t : cone_t) {
79      result.push_back(
80        concatenate(C[1]*e_k(d,k++), C[2]*row_t));
81    }
82    k = 0;
83    // |C[3]*I C[4]*U|   |C[3]*I|
84    //                   |C[4]*A|
85    for (auto &&row_t : cone_t) {
86      result.push_back(
87        concatenate(C[3]*e_k(d,k++), C[4]*row_t));
88    }
89    return result;
90  }
```

`lift_vcone` and `lift_hcone` implement the appropriate transformation using `generalized_lift` and providing the appropriate coefficients in `array<double, 5> C`.

```
98   Matrix lift_vcone(const Matrix &vcone) {
99     return generalized_lift(vcone, {-1,1,-1,-1,1});
100  }
```

```
107  Matrix lift_hcone(const Matrix &hcone) {
108    return generalized_lift(hcone, {1,1,1,-1,-1});
109  }
```

`cone_transform` consolidates the logic of the V-Cone $\rightarrow$ H-Cone and H-Cone $\rightarrow$ V-Cone transformations by accepting a `Matrix cone` and a `Lift`.

```
112  Matrix cone_transform(const Matrix &cone, LiftSelector lift) {
113    if (cone.empty()) {
114      throw logic_error{"empty cone for transform"};
115    }
116    switch (lift) {
117      case LiftSelector::lift_vcone: {
118        return sliced_fourier_motzkin(
119          lift_vcone(cone), slice(0, cone.d, 1));
120      } break;
121      case LiftSelector::lift_hcone: {
122        return sliced_fourier_motzkin(
```

```
123          lift_hcone(cone), slice(0, cone.d, 1));
124      } break;
125      default: {
126        throw std::logic_error{"invalid LiftSelector"};
127      }
128    }
129 }
```

vcone_to_hcone and hcone_to_vcone specialize cone_transform by providing the appropriate Lift.

```
131 Matrix vcone_to_hcone(Matrix vcone) {
132    return cone_transform(vcone,LiftSelector::lift_vcone);
133 }
```

```
135 Matrix hcone_to_vcone(Matrix hcone) {
136    return cone_transform(hcone,LiftSelector::lift_hcone);
137 }
```

3.5 `polyhedra.cpp`

hpoly_to_hcone and hcone_to_hpoly implement the Matrix transforms:

$$\texttt{hpoly\_to\_hcone} : (A|b) \rightarrow (-b|A), \quad \texttt{hcone\_to\_hpoly} : (-b|A) \rightarrow (A|b)$$

These very simple transforms are done with the cshift function, which "circularly shifts" the elements of a Vector (provided as part of the interface to valarray).

```
13 Matrix hpoly_to_hcone(Matrix hpoly) {
14    transform(hpoly.begin(), hpoly.end(), hpoly.begin(),
15        [](Vector v) {
16          v[v.size()-1] *= -1;
17          return v.cshift(-1);
18        });
19    return hpoly;
20 }
```

```
24 Matrix hcone_to_hpoly(Matrix hcone) {
25    transform(hcone.begin(), hcone.end(), hcone.begin(),
26        [](Vector v) {
27          v[0] *= -1;
28          return v.cshift(1);
29        });
30    return hcone;
31 }
```

vpoly_to_vcone implements the VPoly transform:

$$\texttt{vpoly} \rightarrow \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \texttt{vpoly.U} & \texttt{vpoly.V} \end{pmatrix}$$

```
36  Matrix vpoly_to_vcone(VPoly vpoly) {
37    //requires increase in dimension
38    Matrix result{vpoly.d+1};
39    for (auto &&u : vpoly.U) {
40      result.push_back(concatenate({0},u));
41    }
42    for (auto &&v : vpoly.V) {
43      result.push_back(concatenate({1},v));
44    }
45    return result;
46  }
```

normalized_P calculates the V in (**??**). Let Π be the identity matrix with the 0-th row deleted, and $P = \{\mathbf{u} \in U : u_0 > 0\}$. then this is the result of:

$$\Pi(\text{cone}(P) \cap \{x_0 = 1\})$$

```
50  Matrix normalized_P(const Matrix &U) {
51    if (U.d <= 1) {
52      throw std::logic_error{"can't normalize U!"};
53    }
54    Matrix result{U.d-1};
55    std::slice s{1,result.d,1};
56    for (auto &&v : U) {
57      // select the vectors with positive 0-th coordinate
58      if (v[0] <= 0) { continue; }
59      // normalize the selected vectors,
60      result.push_back(v[0] == 1 ? v[s] : (v / v[0])[s]);
61    }
62    return result;
63  }
```

vcone_to_vpoly implements the full tranformation in (**??**).

```
67  VPoly vcone_to_vpoly(Matrix vcone) {
68    VPoly result{vcone.d-1};
69    result.U = sliced_fourier_motzkin(
70                vcone, slice(1,vcone.d-1,1));
71    result.V = normalized_P(vcone);
72    return result;
73  }
```

hpoly_to_vpoly and vpoly_to_hpoly implement the complete transformations promised by the file.

```
77  VPoly hpoly_to_vpoly(Matrix hpoly) {
78    return vcone_to_vpoly(
79            hcone_to_vcone(
80              hpoly_to_hcone(move(hpoly))));
81  }
```
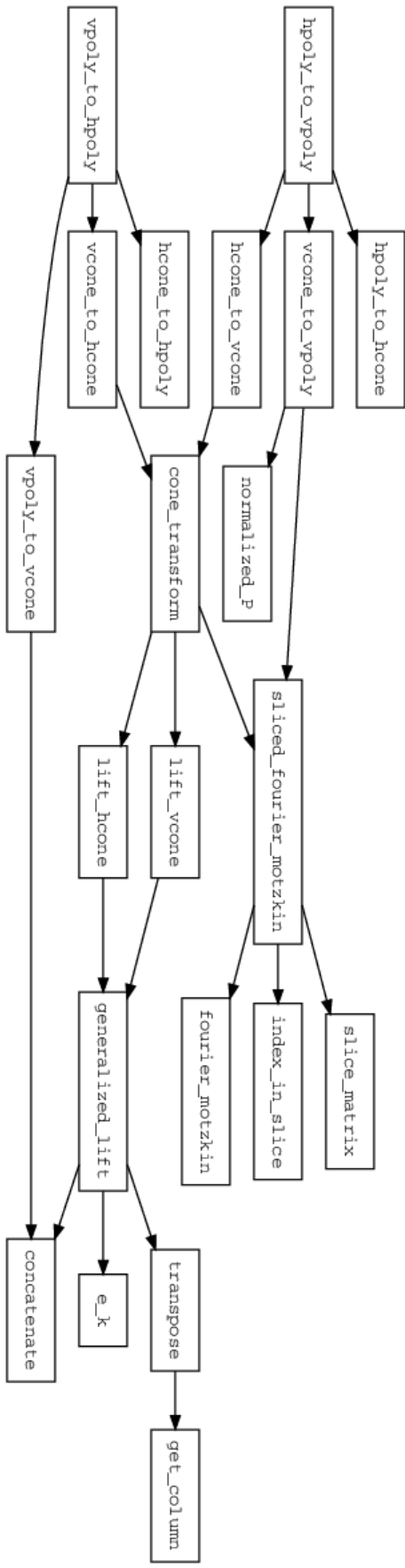
```
83  Matrix vpoly_to_hpoly(VPoly vpoly) {
84    return hcone_to_hpoly(
85            vcone_to_hcone(
```

```
86              vpoly_to_vcone(move(vpoly))));
87 }
```

3.6   Picture of the Program

In the following diagram, the nodes represent functions, and the edges can be read as "calls." Such a diagram is known as a "callgraph," and is only intended to give an overview of the program.

4. Testing

In the next sections, the methods used for testing the program described above will be discussed.

Notation: Let $AU \leq \mathbf{b}$ be shorthand for $(\forall \mathbf{u} \in U)A\mathbf{u} \leq \mathbf{b}$.

4.1 Testing H-Cone $\to$ V-Cone

Suppose we have an H-Cone $C_A = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, and would like to test if a V-Cone $C_{V'} = \mathrm{cone}(V')$ represents the same set. It's easy to check that

$$AV' \leq \mathbf{0} \Rightarrow C_{V'} \subseteq C_A$$

It's not clear what to do to check if $C_A \subseteq C_{V'}$. Suppose we had a set V, and we knew that $C_A = \mathrm{cone}(V)$, and that $C_A = C_{V'} \Rightarrow V \subseteq V'$. Then we'd have the following situation:

$$
\begin{aligned}
AV' \leq \mathbf{0} &\Rightarrow C_{V'} \subseteq C_A \\
V \subseteq V' &\Rightarrow C_A \subseteq C_{V'} \\
C_{V'} = C_A &\Rightarrow V \subseteq V' \\
C_{V'} = C_A &\Rightarrow AV' \leq \mathbf{0}
\end{aligned}
$$

The problem is now to come up with such a set V. We will need to relax the requirements on V a little bit, but not in a way that reduces its utility. The set is desribed in the next proposition, but first we introduce the notion of equivalence vectors:

Definition 4.1.1 (vector equivalence)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, and suppose that $\mathbf{u}/\|\mathbf{u}\| = \mathbf{v}/\|\mathbf{v}\|$. Then say that $\mathbf{u}, \mathbf{v}$ are equivalent, and write:

$$\mathbf{u} \simeq \mathbf{v}$$

Definition 4.1.2 (Extreme)**.** Let $V \in \mathbb{R}^{d \times n}$, if no member of V is a positive linear combination of other elements of V then V is called *extreme*.

Proposition 4.1.1. *Let $V \in \mathbb{R}^{d \times n}$ be extreme, and $C = \mathrm{cone}(V)$. Then*

$$C = \mathrm{cone}(V') \Rightarrow (\forall \mathbf{v} \in V)(\exists \mathbf{v}' \in V') : \mathbf{v} \simeq \mathbf{v}'$$

Proof. Let $\mathbf{v} \in V$, so that $\mathbf{v} = V\mathbf{e}_k$. Since $C = \mathrm{cone}(V')$, there exists a matrix A with all non-negative entries such that $V' = VA$. There is also a non-negative b such that $\mathbf{v} = V'b$. Then $\mathbf{v} = (VA)b = V(Ab)$. Since A and b contain only non-negative entries, so does Ab. Since $\mathbf{v}$ is not a non-negative combination of other vectors from V, Ab must be the basis vector $\mathbf{e}_k$. Then if $i \neq k$, $\mathbf{e}_i^T(Ab) = 0$, or $(\mathbf{e}_i^T A)b = \sum_j A_i^j b_j = 0$. Since $A_i^j, b_j \geq 0$, we have:

$$
\begin{aligned}
(\forall i \neq k) \quad A_i^j > 0 &\Rightarrow b_j = 0 \\
(\forall i \neq k) \quad b_j > 0 &\Rightarrow A_i^j = 0
\end{aligned}
$$

Furthermore, we have $\langle A_k, b \rangle = 1$, so for some l, $A_i^l, b_l > 0$. Then,

$$(\forall i \neq k) \quad A_i^l = 0$$

Now let $\mathbf{b}' = \mathbf{e}_l / A_k^l$. Then it immediately follows that $\langle A_k, \mathbf{b}' \rangle = 1$. Also,

$$(\forall i \neq k) \quad A_i^l = 0 \quad \Rightarrow \quad \langle A_i, b' \rangle = A_i^l / A_k^l = 0$$

We conclude that $A\mathbf{b}' = \mathbf{e}_k = A\mathbf{b}$, and that $\mathbf{v} = V(A\mathbf{b}) = V(A\mathbf{b}') = (VA)\mathbf{b}' = U(\mathbf{e}_l / A_k^l)$. If $\mathbf{v}' = U\mathbf{e}_l$, that is $\mathbf{v}'$ is the l-th vector of U, then $\mathbf{v} = \mathbf{v}' / A_k^l$, or

$$\mathbf{v} / \|\mathbf{v}\| = (\mathbf{v}'/A_k^l) / \left\|\mathbf{v}'/A_k^l\right\| = \mathbf{v}' / \|\mathbf{v}'\|$$

So $\mathbf{v} \simeq \mathbf{v}'$. $\qquad\square$

Let's denote $(\forall \mathbf{v} \in V)(\exists \mathbf{v}' \in V') : \mathbf{v} \simeq \mathbf{v}'$ as $V \sqsubseteq V'$. Then, considering the discussion before the proposition, we have the following result.

Equivalence Criteria 1 (H-Cone $\rightarrow$ V-Cone). *Say $V \in \mathbb{R}^{d \times n}$ is extreme, and let $C_A = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \mathrm{cone}(V)$. Then*

$$C_A = \mathrm{cone}(V') \;\Leftrightarrow\; AV' \leq \mathbf{0}, V \sqsubseteq V'$$

Test 1 (H-Cone $\rightarrow$ V-Cone). We now have a method for testing the program. First, we hand-craft an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ based on some extreme set V, then run our program to get a set V', with the alleged property that $\mathrm{cone}(V') = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. If we confirm Equivalence Criteria 1, then our program has succeeded.

4.1.1 Farkas Lemma

The procedure for the other direction is *almost* identical, but there is a slight catch. Call a set of row vectors **extreme** if no row is a non-negative combination of the other. We would be able to use Equivalence Criteria 1 to say something similar about an extreme set of row vectors A, if we could say that:

Theorem 3 (Dual Cone).

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \mathrm{cone}(A^T) = \mathrm{cone}(A'^T)$$

To prove this proposition, we use the Farkas Lemma:

Proposition 4.1.2 (The Farkas Lemma). *Let $U \in \mathbb{R}^{d \times n}$. Precisely one of the following is true:*

$$(\exists \mathbf{t} \geq \mathbf{0}) : \mathbf{x} = U\mathbf{t}$$

$$(\exists \mathbf{y}) : U^T \mathbf{y} \leq 0, \ \langle \mathbf{x}, \mathbf{y} \rangle > 0$$

Proof. That both can't be true can be seen by:

$$\mathbf{x} = U\mathbf{t} \quad \Rightarrow \quad \mathbf{y}^T\mathbf{x} = \mathbf{y}^T U\mathbf{t} \quad \Rightarrow \quad 0 \neq 0$$

To see that at least one is true we must reconsider the process of converting a V-Cone to an H-Cone. First, from cone(U) we create the following matrix:

$$A = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix}$$

By the way A is constructed,

$$(\exists \mathbf{t}) : A \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{t} \geq \mathbf{0})\ \mathbf{x} = U\mathbf{t} \tag{4.1}$$

In the proof of the transformation, we use 'Fourier Motzkin Elimination for H-Cones' on page 6 to transform that matrix A. The 'Fourier Motzkin Matrix' on page 8 promises a sequence of matrices $Y_{d+1}, \ldots, Y_{d+n}$ with certain properties. Let $Y = (Y_{d+n})(Y_{d+(n-1)}) \ldots (Y_{d+1})$, then it can be said of Y:

1. Every element of Y is non-negative.

2. Y is finite.

3. The last n columns of YA are all $\mathbf{0}$.

4. $(\exists t_{d+1}, \ldots, t_{d+n}) A(\mathbf{x} + \sum_{i=d+1}^{d+n} t_i \mathbf{e}_i) \leq \mathbf{0} \Leftrightarrow (YA)\mathbf{x} \leq \mathbf{0}$

Note that here $\mathbf{x} \in \mathbb{R}^{d+n}$. A has three blocks of rows, which can be labeled with Z, P, N in a fairly obvious way. Then, Y can be broken up into three blocks of columns, so that

$$Y = (Y_Z\ Y_P\ Y_N)$$

Where each of $Y_Z, Y_P, Y_N \geq \mathbf{0}$. Consolidating what is known about A and Y,

$$YA = (Y_Z\ Y_P\ Y_N) \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix} = (Y'\ \mathbf{0})$$

Here, we have let $Y' = Y_P - Y_N$. Then it follows that

$$\mathbf{0} = -Y_Z - Y_P(U) + Y_N(U) = -Y_Z - Y'(U) \quad \Rightarrow \quad Y_Z = -Y'U \quad \Rightarrow \quad Y'U \leq \mathbf{0}$$

Then it holds that, for any row $\mathbf{y}' \in Y'$:

$$\mathbf{y}'U \leq \mathbf{0} \tag{4.2}$$

It is also true that

$$(YA) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} = (Y'\ \mathbf{0}) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} = Y'\mathbf{x}$$

We also have

$$(\exists \mathbf{t}) : A \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \Leftrightarrow (YA) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \Leftrightarrow Y'\mathbf{x} \leq \mathbf{0} \tag{4.3}$$

Note that here $\mathbf{x} \in \mathbb{R}^d$. So, if given some $\mathbf{x}$, the left side of (4.3) is not satisfied, then neither is the right, and there must be some row $\mathbf{y}' \in Y'$ such that the following holds:

$$\langle \mathbf{y}', \mathbf{x} \rangle > 0 \tag{4.4}$$

Then we conclude that, if the right side of (4.1) fails, then there is a vector $\mathbf{y}' \in Y'$ satisfying (4.2) and (4.4). $\qquad \square$

4.2   Testing V-Cone $\rightarrow$ H-Cone

Now we can prove Dual Cone:

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \text{cone}(A^T) = \text{cone}(A'^T)$$

Proof. First suppose that $\text{cone}(A^T) = \text{cone}(A'^T)$. Then there exists a non-negative matrix B such that $A'^T = A^T B$. Then $A\mathbf{x} \leq \mathbf{0} \Rightarrow B^T A \mathbf{x} \leq \mathbf{0} \Rightarrow A'\mathbf{x} \leq \mathbf{0}$. Precisely the same reasoning shows that $A'\mathbf{x} \leq \mathbf{0} \Rightarrow A\mathbf{x} \leq \mathbf{0}$, and we conclude that $\text{cone}(A^T) = \text{cone}(A'^T) \Rightarrow \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$.

Next suppose that $\text{cone}(A^T) \neq \text{cone}(A'^T)$, that is, let $\mathbf{z} \in \text{cone}(A), \mathbf{z} \notin \text{cone}(A')$. We must show that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$. By the Farkas Lemma, we have a $\mathbf{y}$ such that $\langle \mathbf{y}, \mathbf{z} \rangle > 0$, $A'\mathbf{y} \leq \mathbf{0}$. Clearly this means that $\mathbf{y} \in \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$. Since $\mathbf{z} \in \text{cone}(A)$, there is some $(\mathbf{t} \geq \mathbf{0}) : \mathbf{z}^T = \mathbf{t}^T A$. Then if $A\mathbf{y} \leq \mathbf{0}$, we would have $\langle \mathbf{y}, \mathbf{z} \rangle = \mathbf{t}^T A \mathbf{y} \leq 0 < \langle \mathbf{y}, \mathbf{z} \rangle$, a contradiction. So we conclude that $\mathbf{y} \notin \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. $\qquad \square$

Now, as before, we suppose that we have a set V and A, where A is extreme, and we know that $C = \text{cone}(V) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. We have the following situation:

$$
\begin{aligned}
C &= \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Rightarrow A \sqsubseteq A' \\
C &= \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Rightarrow A'V \leq \mathbf{0} \\
A &\sqsubseteq A' && \Rightarrow \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \subseteq C \\
A'V &\leq \mathbf{0} && \Rightarrow C \subseteq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}
\end{aligned}
$$

So we have the following.

Equivalence Criteria 2 (V-Cone $\rightarrow$ H-Cone)**.** *Let $A \in \mathbb{R}^{m \times d}$ be extreme, and $C_V = \text{cone}(V) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. Then*

$$C_V = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow A'V \leq \mathbf{0}, \ A \sqsubseteq A'$$

Test 2 (V-Cone $\rightarrow$ H-Cone)**.** We now have a method for testing the program. First, we hand-craft an V-Cone cone V based on some extreme set A, then run our program to get a set A', with the alleged property that $\text{cone}(V') = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. If we confirm Equivalence Criteria 2, then our program has succeeded.

4.3   Testing H-Polyhedron $\to$ V-Polyhedron

Say we have an H-Polyhedron $P_{A,\mathbf{b}} = \{\mathbf{x} \mid A\mathbf{x} \le \mathbf{b}\}$, and wish to check that our program correctly calculates a V' and U' such that $P_{A,\mathbf{b}} = \mathrm{cone}(U') + \mathrm{conv}(V')$. Again, we shall use the notion of extremity and show that under certain circumstances we can use extreme sets to demonstrate the validity of our algorithm. In this case, the definition of extreme is a little more complicated, but it asserts that a set U is extreme as before, and that no element of another set V can be expressed as a non-trival sum of a convex combination of V and a non-negative linear combination of members of U.

Definition 4.3.1 (Extreme Pair)**.** A pair of sets $U \in \mathbb{R}^{d \times n}, V \in \mathbb{R}^{d \times p}$ is called an *extreme pair* if for any $\mathbf{u} = U\mathbf{e}_k, \mathbf{v} = V\mathbf{e}_l$ the following is true:

$$\mathbf{t} \ge \mathbf{0}, \ \mathbf{u} = U\mathbf{t} \Rightarrow \mathbf{t} = \mathbf{e}_k$$
$$\mathbf{t} \ge \mathbf{0}, \boldsymbol{\lambda} \ge \mathbf{0}, \langle \boldsymbol{\lambda}, \mathbf{1} \rangle = 1, \mathbf{v} = U\mathbf{t} + V\boldsymbol{\lambda} \Rightarrow \mathbf{t} = \mathbf{0}, \boldsymbol{\lambda} = \mathbf{e}_l$$

Let us now consider the set U in the expression $\{\mathbf{x} \mid A\mathbf{x} \le b\} = \mathrm{cone}(U) + \mathrm{conv}(V)$.

Proposition 4.3.1 (Characterstic Cone)**.** *Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \le b\} = \mathrm{cone}(U) + \mathrm{conv}(V)$. Then*

$$\mathrm{cone}(U) = \{\mathbf{x} \mid A\mathbf{x} \le \mathbf{0}\}$$

Proof. We show that the following three statements are equivalent:

1. $A\mathbf{r} \le \mathbf{0}$

2. $(\forall \mathbf{x} \in P)(\forall \alpha > 0) \ \mathbf{x} + \alpha \mathbf{r} \in P$

3. $\mathbf{r} \in \mathrm{cone}(U)$

$(1 \Rightarrow 2)$. $\mathbf{x} \in P$ means that $A\mathbf{x} \le \mathbf{b}$, and $A\mathbf{r} \le \mathbf{0}$ means that $A(\mathbf{x} + \alpha \mathbf{r}) \le A\mathbf{x} \le \mathbf{b}$.
$(\neg 1 \Rightarrow \neg 2)$. Suppose $\langle A_i, \mathbf{r} \rangle > 0$, then let $\alpha > (b_i - \langle A_i, \mathbf{x} \rangle)/\langle A_i, \mathbf{r} \rangle$. We have:

$$\langle A_i, \mathbf{x} + \alpha \mathbf{r} \rangle > \langle A_i, \mathbf{x} \rangle + \frac{b_i \langle A_i, \mathbf{r} \rangle - \langle A_i, \mathbf{x} \rangle \langle A_i, \mathbf{r} \rangle}{\langle A_i, \mathbf{r} \rangle} = b_i$$

$(3 \Rightarrow 2)$. This is essentially the definition of $\mathrm{cone}(U) + \mathrm{conv}(V)$.
$(2 \Rightarrow 3)$. Now for the real work. Suppose that (2) holds, but $\mathbf{r} \notin \mathrm{cone}(U)$. Then by the Farkas Lemma, we have a $\mathbf{y}$ that satisfies $(\forall \mathbf{r} \in U) \langle \mathbf{r}, \mathbf{y} \rangle \le 0, \ \langle \mathbf{y}, \mathbf{r} \rangle > 0$. From (2) we construct a sequence: $(\mathbf{x}_n) = \mathbf{v} + n \cdot \mathbf{r}$. Then it is clear that the sequence $\langle \mathbf{y}, \mathbf{x}_n \rangle \to \infty$. It is also clear that $(\forall n) \mathbf{x}_n \in P$. We now need the following:

- A linear, real-valued function on the set $\mathrm{conv}(V)$ achieves its maximal value at some $\bar{\mathbf{v}} \in V$.

To see this is true, suppose that the linear function is given by $\langle \mathbf{y}, \cdot \rangle$, and that $\bar{\mathbf{v}}$ is an element of V such that $(\forall \mathbf{v} \in V)\ \langle \mathbf{y}, \bar{\mathbf{v}} \rangle \geq \langle \mathbf{y}, \mathbf{v} \rangle$. Then, for any $\mathbf{r} \in \text{conv}(V)$, $\mathbf{r} = \sum_{\mathbf{v} \in V} \lambda_v \mathbf{v}$ where $\sum \lambda_v = 1 \Rightarrow \lambda_v \leq 1$, and it follows

$$\langle \mathbf{y}, \mathbf{r} \rangle = \left\langle \mathbf{y}, \sum_{v \in V} \lambda_v v \right\rangle = \sum_{v \in V} \lambda_v \langle \mathbf{y}, \mathbf{v} \rangle \leq \sum_{v \in V} \lambda_v \langle \mathbf{y}, \bar{\mathbf{v}} \rangle = \langle \mathbf{y}, \bar{\mathbf{v}} \rangle$$

Now consider the maximum value of the function $\langle \mathbf{y}, \cdot \rangle$ on P. Since any element of P can be written $\mathbf{r} + \mathbf{v} \mid \mathbf{r} \in \text{cone}(U)$, $\mathbf{v} \in \text{conv}(V)$, and $(\forall \mathbf{r} \in U)\ \langle \mathbf{y}, \mathbf{r} \rangle \leq 0$, we can find the maximum value on $\text{conv}(V)$. However, $\langle \mathbf{y}, \cdot \rangle$ achievs its maximal value on $\text{conv}(V)$ at some $\bar{\mathbf{v}} \in V$, which is a contradiction with the fact that $\langle \mathbf{y}, \mathbf{x}_n \rangle \to \infty$, so we conclude that $\mathbf{r} \in \text{cone}(U)$. $\qquad \square$

Remark 2 (Characteristic Cone). Note that (2) in the proof above is independent of A and U. This means that the cone of a polyhedron is independent of its representation, i.e. if $\text{cone}(U) + \text{conv}(V) = \text{cone}(U') + \text{conv}(V')$, then $\text{cone}(U) = \text{cone}(U')$, while it is not necessarily true that $\text{conv}(V) = \text{conv}(V')$. Similarly, if $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$, the it holds that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$.

Proposition 4.3.2. *A convex combination of convex combinations is another convex combination*

Proof. Let Λ represent a collection of convex combinations, that is, $\mathbf{1}^T \Lambda = \mathbf{1}^T$, and let $\boldsymbol{\lambda} \geq \mathbf{0}$, $\mathbf{1}^T \boldsymbol{\lambda} = 1$ be a convex combinator. Then $\Lambda \boldsymbol{\lambda} = \boldsymbol{\lambda}'$ where $\boldsymbol{\lambda}' \geq \mathbf{0}$, $\mathbf{1}^T \boldsymbol{\lambda}' = 1$. That $\boldsymbol{\lambda}' \geq \mathbf{0}$ is clear, then just note that $\mathbf{1}^T \boldsymbol{\lambda}' = \mathbf{1}^T \Lambda \boldsymbol{\lambda} = \mathbf{1}^T \boldsymbol{\lambda} = 1$. $\qquad \square$

Proposition 4.3.3 (Minkowski Sums). *The following two statements hold*

1. $A \subseteq B, C \subseteq D \Rightarrow A + C \subseteq B + D$

2. $P + \text{cone}(U) + \text{cone}(U) = P + \text{cone}(U)$

Proof. (1) $a \in A \Rightarrow a \in B$, $c \in C \Rightarrow c \in D$. Taken together, $a + c \in B + D$. (2) $\mathbf{t}, \mathbf{t}' \geq \mathbf{0} \Rightarrow p + U\mathbf{t} + U\mathbf{t}' = p + U(\mathbf{t} + \mathbf{t}') = p + U\mathbf{t}''$, $\mathbf{t}'' \geq \mathbf{0}$. $\qquad \square$

Equivalence Criteria 3. *Suppose that there is an extreme pair U, V such that $P_{A,\mathbf{b}} = \{\mathbf{x} \mid A\mathbf{x} \leq b\} = \text{cone}(U) + \text{conv}(V)$. Then the following are equivalent:*

1. $P_{A,\mathbf{b}} = \text{cone}(U') + \text{conv}(V')$

2. $U \sqsubseteq U'$, $V \subseteq V'$, $AU' \leq \mathbf{0}$, $AV' \leq \mathbf{b}$

Proof. $(2 \Rightarrow 1)$. There's not too much to say about this direction, it's mostly just collecting some straightforward observations and results.

(a) $U \sqsubseteq U' \Rightarrow \text{cone}(U) \subseteq \text{cone}(U')$

(b) $V \subseteq V' \Rightarrow \text{conv}(V) \subseteq \text{conv}(V')$

(c) (a) + (b) $\Rightarrow P_{A,\mathbf{b}} \subseteq \text{cone}(U') + \text{conv}(V')$

(d) $AU' \leq \mathbf{0} \Rightarrow \text{cone}(U') \subseteq \text{cone}(U)$

(e) $AV' \leq \mathbf{b} \Rightarrow \text{conv}(V') \subseteq P_{A,\mathbf{b}}$

(f) (d) + (e) $\Rightarrow \text{cone}(U') + \text{conv}(V') \subseteq P_{A,\mathbf{b}} + \text{cone}(U) = P_{A,\mathbf{b}}$

- (c) + (f) $\Rightarrow (2 \Rightarrow 1)$

(a) and (b) are clear, (c) uses Proposition 4.3.3, (d) requires Characteristic Cone, (e) is clear, and (f) uses Proposition 4.3.3.

$(1 \Rightarrow 2)$. This direction is a little more interesting. First we observe:

$$\text{cone}(U) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(U') \Rightarrow U \sqsubseteq U'$$

The equalities follow from Characteristic Cone, and the implication follows from Equivalence Criteria 1. Note that the extremeness of U and the Farkas lemma are both used here. Since we know that $\text{cone}(U) = \text{cone}(U')$, we also know that $\text{cone}(U') + \text{conv}(V') = \text{cone}(U) + \text{conv}(V')$. Next, we consider V and exploit its extremeness. Since $P_{A,\mathbf{b}} = \text{cone}(U) + \text{conv}(V')$, each $\mathbf{v}' \in V'$ can be written $U\mathbf{t} + V\boldsymbol{\lambda}$, where $\mathbf{t} \geq \mathbf{0}$ and $\boldsymbol{\lambda}$ is a convex combinator. We combine these into matrices T and Λ, so $V' = UT + V\Lambda$. But it is also true that every $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = U\mathbf{t} + V'\boldsymbol{\lambda} = U\mathbf{t} + (UT + V\Lambda)\boldsymbol{\lambda} = U\mathbf{t}' + V\boldsymbol{\lambda}'$$

Where $\mathbf{t}' \geq \mathbf{0}$, and $\boldsymbol{\lambda}'$ is a convex combinator. Because U, V is an extreme pair, we have that $\mathbf{t}' = \mathbf{0}$, and $\boldsymbol{\lambda} = \mathbf{e}_k$ for some k. Because U is extreme, it does not contain $\mathbf{0}$, and so $\mathbf{t} = \mathbf{0}$. This puts us at $\mathbf{v} = V'\boldsymbol{\lambda} = V\Lambda\boldsymbol{\lambda} = V\boldsymbol{\lambda}'$, and $\boldsymbol{\lambda}' = \mathbf{e}_k$. In order that $\boldsymbol{\lambda}' = \mathbf{e}_k$, for every column of Λ corresponding to a positive entry in $\boldsymbol{\lambda}$, only one row may contain a positive entry, and that entry must be 1. Then instead of $\boldsymbol{\lambda}$, use instead $\mathbf{e}_l$ where $\Lambda_k^l = 1$. Then $\Lambda\boldsymbol{\lambda} = \Lambda\mathbf{e}_l$, so $V\Lambda\boldsymbol{\lambda} = V\Lambda\mathbf{e}_l = V'\mathbf{e}_l = \mathbf{v}'$ where $\mathbf{v}' \in V'$. Then $\mathbf{v} \in V'$.

That $AV' \leq \mathbf{b}$ is obvious, and that $AU' \leq \mathbf{0}$ is mentioned in the remarks after Remark 2. $\qquad\square$

Test 3 (H-Polyhedron → V-Polyhedron)**.** We now have a method for testing the program. First, we hand-craft an H-Polyhedron $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ based on some extreme pair (U, V), then run our program to get the pair (U', V'), with the alleged property that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \text{cone}(U') + \text{conv}(V')$. If we confirm Equivalence Criteria 3, then our program has succeeded.

4.4 Testing V-Polyhedron → H-Polyhedron

Now we suppose we have a V-Polyhedron $P_{U,V} = \text{cone}(U) + \text{conv}(V)$, and would like to test the program which returns a matrix-vector pair $A', \mathbf{b}'$ where supposedly $P_{U,V} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. Again, we will start off with a pair $A, \mathbf{b}$ where we know that $P_{U,V} = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$, where $A, \mathbf{b}$ satisfy some nice properties, and use those properties to test if $P_{U,V} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. In order to demonstrate these properties, the Farkas Lemma will be used, but in different forms. We want to use Equivalence Criteria 2, but first we have to check:

Proposition 4.4.1. *The following statements are equivalent:*

1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$

2. $\left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \mid \begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b} & A \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \mid \begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b}' & A' \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\}$

Proof. $(2 \Rightarrow 1)$. Just set $x_0 = 1$, and move $\mathbf{b}, \mathbf{b}'$ to the right side of the inequalities. $(\neg 2 \Rightarrow \neg 1)$. Suppose that:

$$\begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b} & A \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0}, \quad \begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b}' & A' \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \not\leq \mathbf{0}$$

Observe that, by the way these sets are constructed, $x_0 \geq 0$. If $x_0 = 0$, then we have $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$, which, by the remark following Remark 2 means that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. If $x_0 > 0$, then we have:

$$A\mathbf{x} \leq x_0\mathbf{b}, \ A'\mathbf{x} \not\leq x_0\mathbf{b}' \Rightarrow A(\mathbf{x}/x_0) \leq \mathbf{b}, \ A'(\mathbf{x}/x_0) \not\leq \mathbf{b}'$$

So $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. $\qquad\square$

Now, combining the results of 'Characteristic Cone' on page 35 and proposition 4.4.1, we have the following result:

Proposition 4.4.2. *The following two statement are equivalent:*

1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$

2. $\text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} = \text{cone} \begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix}$

4.4.1 Extreme H-Polyhedra Pairs

Definition 4.4.1 (Extreme Pair)**.** A pair $A, \mathbf{b}$ is called **extreme** if $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is non-empty, and

$$\mathbf{t} \geq \mathbf{0}, \ \mathbf{t}^T(A, b) = (A_i, b_i) \Rightarrow [\mathbf{t} \neq \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{e}_i]$$

Of course, we want to say that if $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ where $(A, \mathbf{b})$ is an extreme pair, and $P = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$, then, by Proposition 4.4.2, $(A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')$. The catch is that the cones in Proposition 4.4.2 have a strange form. What we want to be true turns out to be so, but before we can prove this fact, we need a property of extreme pairs, which requires a new form of the Farkas Lemma.

4.4.2 Farkas Lemma: Round 2

Let us restate the conclusion of the Farkas Lemma:

$$\exists \mathbf{t} \geq \mathbf{0} \mid U\mathbf{t} = \mathbf{x} \Leftrightarrow \neg \exists \mathbf{y} \mid \mathbf{y}^T U \leq \mathbf{0}, \ \mathbf{y}^T \mathbf{x} > 0$$

If we let $U = (A, -A, I)$, and we get a new form:

$$\exists \mathbf{t} \geq \mathbf{0} \mid (A, -A, I)\mathbf{t} = \mathbf{x} \Leftrightarrow \neg \exists \mathbf{y} \mid \mathbf{y}^T(A, -A, I) \leq \mathbf{0}, \ \mathbf{y}^T \mathbf{x} > 0$$

Then breaking apart $\mathbf{t} = (\mathbf{t}_P, \mathbf{t}_N, \mathbf{t}_I)$, we have

$$(A, -A, I) \begin{pmatrix} \mathbf{t}_P \\ \mathbf{t}_N \\ \mathbf{t}_I \end{pmatrix} = \mathbf{x} \Rightarrow A(\mathbf{t}_P - \mathbf{t}_N) = \mathbf{x} - \mathbf{t}_I$$

If we let $\mathbf{t}_P - \mathbf{t}_N = \mathbf{z}$, then $\mathbf{z}$ is no longer constrained by $\mathbf{t} \geq \mathbf{0}$, and we have that $A\mathbf{z} \leq \mathbf{x}$. Since $\mathbf{y}^T A \leq \mathbf{0}$ and $-\mathbf{y}^T A \leq \mathbf{0}$, it must be that $\mathbf{y}^T A = \mathbf{0}$. Combining these results, relabeling $\mathbf{x}$ as $\mathbf{b}$ and $\mathbf{z}$ as $\mathbf{x}$, and $\mathbf{y}$ as $-\mathbf{y}$, we see a new form of Farkas Lemma:

Theorem 4 (Farkas Lemma 2)**.**

$$\exists \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b} \;\Leftrightarrow\; \neg \exists \mathbf{y} \geq \mathbf{0} \mid \mathbf{y}^T A = \mathbf{0},\, \mathbf{y}^T \mathbf{b} < 0$$

This form tells us that an H-Polyhedron is non-empty, or we can create an inequality that is impossible to solve from it's matrix. The next form shows a similar result, but this time it is for a specific $\mathbf{x}$, not implying the emptyness of the Polyhedra, just the constraints' failure to be satisfied at a specific point.

We can now prove some useful properties of extreme pairs.

Proposition 4.4.3. *Let $A, \mathbf{b}$ be an extreme pair. Then the following holds:*

$$\mathbf{t} \geq \mathbf{0},\, \mathbf{t}^T A = \mathbf{0} \Rightarrow [\mathbf{t} \neq \mathbf{0} \Rightarrow \langle \mathbf{t}, \mathbf{b} \rangle > 0]$$
$$\mathbf{t} \geq \mathbf{0},\, \mathbf{t}^T A = A_i \Rightarrow [\mathbf{t} \neq \mathbf{e}_i \Rightarrow \langle \mathbf{t}, \mathbf{b} \rangle > b_i]$$

Proof. Suppose that $\mathbf{t}^T A = \mathbf{0}$, $\mathbf{t} \neq \mathbf{0}$. Next suppose that $\langle \mathbf{t}, \mathbf{b} \rangle = 0$. Then $(\mathbf{t} + \mathbf{e}_i)(A, b) = (A_i, b_i)$, but $\mathbf{t} + \mathbf{e}_i \neq \mathbf{e}_i$, a contradiction. Next suppose that $\langle \mathbf{t}, \mathbf{b} \rangle < 0$. Then we have that $\exists \mathbf{t} \geq \mathbf{0}$, $\mathbf{t}^T A = \mathbf{0}$, $\mathbf{t}^T \mathbf{b} < 0$, which by Farkas Lemma 2 means that $\{\mathbf{x} \mid A\mathbf{x} \leq b\}$ is empty, a contradiction. So the first property is proven.

Now suppose that $\mathbf{t}^T A = A_i$, $\mathbf{t} \neq \mathbf{e}_i$. Then we know that $\mathbf{t}^T \mathbf{b} \neq b_i$, but if $\langle \mathbf{t}, \mathbf{b} \rangle < 0$, we'd have that $(\mathbf{t} - \mathbf{e}_i)^T A = \mathbf{0}$, $(\mathbf{t} - \mathbf{e}_i)^T \mathbf{b} < 0$, contradicting the first property. $\qquad\square$

We are now prepared to prove the following proposition.

Proposition 4.4.4. *Suppose that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$, where $(A, \mathbf{b})$ is an extreme pair. Then $(A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')$.*

Proof. It suffices to show that $\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ is extreme, for then:

$$\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} \sqsubseteq \begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix} \Rightarrow \begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix} \sqsubseteq \begin{pmatrix} -\mathbf{b}'^T \\ A'^T \end{pmatrix} \Rightarrow (A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')$$

So, suppose that $(\mathbf{t}, t) \geq \mathbf{0}$, $\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}(\mathbf{t}, t) = \begin{pmatrix} -b_i \\ A_i^T \end{pmatrix}$. We must show that $\mathbf{t} = \mathbf{e}_i$, $t = 0$. Suppose that it's not. Then we have $\mathbf{t}^T A = A_i$, and by Proposition 4.4.3 $\langle \mathbf{t}, -\mathbf{b} \rangle < b_i$. Since $-t \leq 0$, $\langle \mathbf{t}, -\mathbf{b} \rangle - t < b_i$, a contradiction. So we have shown that $\mathbf{t} = \mathbf{e}_i$, in which case $t = 0$, and the proposition follows. $\qquad\square$

Suppose that we have a (U, V) and $(A, \mathbf{b})$ such that $P_{UV} = \text{cone}(U) + \text{conv}(V) = \{\mathbf{x} \mid A\mathbf{x} \le \mathbf{b}\}$, and $(A, \mathbf{b})$ is an extreme pair. We run our program and get a new pair $(A', \mathbf{b}')$. Denote $P_{A', \mathbf{b}'} := \{\mathbf{x} \mid A'\mathbf{x} \le \mathbf{b}'\}$. We would like to verify that $P_{UV} = P_{A', \mathbf{b}'}$. We have the following:

$$A'U \le \mathbf{0},\ A'V \le \mathbf{b}' \Rightarrow P_{UV} \subseteq P_{A', \mathbf{b}'}$$
$$(A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')\quad \Rightarrow P_{A', \mathbf{b}'} \subseteq P_{UV}$$
$$P_{UV} = P_{A', \mathbf{b}'}\qquad\quad \Rightarrow A'U \le \mathbf{0},\ A'V \le \mathbf{b}'$$
$$P_{UV} = P_{A', \mathbf{b}'}\qquad\quad \Rightarrow (A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')$$

The last line uses Proposition 4.4.4. So we conclude:

Equivalence Criteria 4. *Suppose that there is an extreme pair* $(A, \mathbf{b})$ *such that* $P_{UV} = \text{cone}(U) + \text{conv}(V) = \{\mathbf{x} \mid A\mathbf{x} \le b\}$. *Then the following are equivalent:*

$$P_{UV} = \{\mathbf{x} \mid A'\mathbf{x} \le \mathbf{b}'\} \Leftrightarrow A'U \le \mathbf{0},\ A'V \le \mathbf{b}',\ (A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')$$

Test 4 (V-Polyhedron $\to$ H-Polyhedron)**.** We now have a method for testing the program. First, we hand-craft a V-Polyhedron $\text{cone}(U) + \text{conv}(V)$ based on some extreme pair $(A, \mathbf{b})$, then run our program to get the pair $(A', \mathbf{b}')$, with the alleged property that $\text{cone}(U) + \text{conv}(V) = \{\mathbf{x} \mid A'\mathbf{x} \le \mathbf{b}'\}$. If we confirm Equivalence Criteria 4, then our program has succeeded.

4.5 `test_functions.h`

The following types are defined for running tests of the different algorithms. They are expected to be given a descriptive name, the object on which the test will be run, and a `key` with which the result of the test will be compared. The `key` object is one of the extreme objects described above.

```
7   struct hcone_test_case {
8     std::string name;
9     Matrix hcone; // vectors for H or V cone
10    Matrix key;   // minimal generating set
11
12    bool run_test() const;
13  };

15  struct vcone_test_case {
16    std::string name;
17    Matrix vcone; // vectors for H or V cone
18    Matrix key;   // minimal generating set
19
20    bool run_test() const;
21  };

23  struct hpoly_test_case {
24    std::string name;
25    Matrix hpoly; // vectors for H-Polyhedron
26    VPoly key;    // minimal generating set
```

```
27
28    bool run_test() const;
29  };
```

```
31  struct vpoly_test_case {
32    std::string name;
33    VPoly  vpoly; // vectors for V-Polyhedron
34    Matrix key;   // minimal generating set
35
36    bool run_test() const;
37  };
```

4.6 `test_functions.cpp`

The dot-product and norm (in terms of dot product).

```
28  double operator*(const Vector &l, const Vector &r) {
29    if (l.size() > r.size()) {
30      throw runtime_error{"inner product: l > r"};
31    }
32    return inner_product(begin(l), end(l), begin(r), 0.);
33  }
```

```
35  double norm(const Vector &v) {
36    return sqrt(v*v);
37  }
```

`approximately_zero` is used during tests to avoid issues involving floating point rounding errors. For example, `1/6.0 * 2.5 - 5/12.0 == 0` will give `false`, while `approximately_zero(1/6.0 * 2.5 - 5/12.0)` will return `true`. Test cases are used where intermediate calculations don't depend on such high accuracy, and these discrepencies can be ignored.

`approximately_zero(c) ==` `true` is be denoted $c \approx 0$.

```
39  bool approximately_zero(double d) {
40    const double error = .000001;
41    bool result = abs(d) < error;
42    if (d != 0 && result) {
43      ostringstream oss;
44      oss << scientific << d;
45      log("approximately_zero " + oss.str(), 1);
46    }
47    return result;
48  }
```

Tests $c < 0 \lor c \approx 0$.

```
50  bool approximately_lt_zero(double d) {
51    return d < 0 || approximately_zero(d);
52  }
```

Tests $||\mathbf{v}|| \approx 0$. This is be denoted $\mathbf{v} \approx \mathbf{0}$.

```cpp
55  bool approximately_zero(const Vector &v) {
56    return approximately_zero(norm(v));
57  }
```

Tests $\mathbf{u}/\|\mathbf{u}\| - \mathbf{v}/\|\mathbf{v}\| \approx \mathbf{0}$. This is be denoted $\mathbf{u} \simeq \mathbf{v}$.

```cpp
59  bool is_equivalent(const Vector &l, const Vector &r) {
60    if (l.size() != r.size()) return false;
61    if (norm(l) == 0 || norm(r) == 0) {
62      return norm(l) == 0 && norm(r) == 0;
63    }
64    return approximately_zero(l / norm(l) - r / norm(r));
65  }
```

Tests $\mathbf{u} - \mathbf{v} \approx \mathbf{0}$. This is be denoted $\mathbf{u} \approx \mathbf{v}$.

```cpp
67  bool is_equal(const Vector &l, const Vector &r) {
68    if (l.size() != r.size()) return false;
69    return approximately_zero(l - r);
70  }
```

Tests $(\exists \mathbf{u} \in U) \mid \mathbf{v} \simeq \mathbf{u}$.

```cpp
72  bool has_equivalent_member(const Matrix &M,
73                            const Vector &v) {
74    if (!any_of(M.begin(), M.end(),
75      [&](const Vector &u) {
76        return is_equivalent(u,v); })) {
77      ostringstream oss;
78      oss << dashes
79          << " no equivalent member found for:\n"
80          << v << endl;
81      log(oss.str(),1);
82      return false;
83    }
84    return true;
85  }
```

Tests $(\exists \mathbf{u} \in U) \mid \mathbf{v} \approx \mathbf{u}$.

```cpp
87  bool has_equal_member(const Matrix &M,
88                        const Vector &v) {
89    if (!any_of(M.begin(), M.end(),
90      [&](const Vector &u) { return
91        is_equal(u,v); })) {
92      ostringstream oss;
93      oss << dashes
94          << " no equal member found for:\n"
95          << v << endl;
96      log(oss.str(),1);
97      return false;
98    }
99    return true;
100 }
```

Tests $(\forall v \in V)(\exists \mathbf{u} \in U) \mid \mathbf{v} \simeq \mathbf{u}$. This is be denoted $V \sqsubseteq U$.

```
103  bool subset_mod_eq(const Matrix &generators,
104                     const Matrix &vcone) {
105    return all_of(generators.begin(), generators.end(),
106      [&](const Vector &g) {
107        return has_equivalent_member(vcone, g); });
108  }
```

Tests $(\forall v \in V)(\exists \mathbf{u} \in U) \mid \mathbf{v} \approx \mathbf{u}$. This is be denoted $V \subseteq U$.

```
111  bool subset(const Matrix &generators,
112              const Matrix &vcone) {
113    return all_of(generators.begin(), generators.end(),
114      [&](const Vector &g) {
115        return has_equal_member(vcone, g); });
116  }
```

Given a `Vector constraint` and `Vector ray`, tests if
`approximately_lt_zero(ray * constraint)`. Note that if the constraint is of the
form $\langle A_i, \mathbf{v} \rangle \leq b$ for some value b, then this tests $\langle A_i, \mathtt{ray} \rangle \leq \mathbf{0}$.

```
120  bool ray_satisfied(const Vector &constraint,
121                     const Vector &ray) {
122    if (constraint.size() != ray.size() &&
123        constraint.size()-1 != ray.size()) {
124      throw runtime_error{"bad ray vs constraint"};
125    }
126    double ip = ray * constraint;
127    if (!(approximately_lt_zero(ip))) {
128      ostringstream oss;
129      oss << dashes << " ray not satisfied!\n"
130          << "ray: " << ray
131          << "\nconstraint: " << constraint
132          << "\n ray * constraint = " << ip << endl;
133      log(oss.str(), 1);
134      return false;
135    }
136    return true;
137  }
```

Test $A\mathbf{v} \leq \mathbf{0}$

```
139  bool ray_satisfied(const Matrix &constraints,
140                     const Vector &ray) {
141    return all_of(constraints.begin(), constraints.end(),
142      [&](const Vector &cv) {
143        return ray_satisfied(cv, ray); });
144  }
```

Test $AV \leq \mathbf{0}$

```
146  bool rays_satisfied(const Matrix &constraints,
147                      const Matrix &rays) {
148    return all_of(rays.begin(), rays.end(),
149      [&](const Vector &ray) {
150        return ray_satisfied(constraints, ray); });
151  }
```

Test $\langle A_i, \mathbf{v} \rangle \le b_i$

```
154  bool vec_satisfied(const Vector &constraint,
155                     const Vector &vec) {
156    size_t cback_i = constraint.size()-1;
157    if (cback_i != vec.size()) {
158      throw runtime_error{"bad vec vs constraint"};
159    }
160    double ip = vec * constraint;
161    double c_val = constraint[cback_i];
162    if (!(approximately_lt_zero(ip - c_val))) {
163      ostringstream oss;
164      oss << dashes << " vec not satisfied!\n"
165          << "vec: " << vec
166          << "\nconstraint: " << constraint
167          << "\n vec * constraint = " << ip << endl;
168      log(oss.str(), 1);
169      return false;
170    }
171    return true;
172  }
```

Test $A\mathbf{v} \le \mathbf{b}$

```
174  bool vec_satisfied(const Matrix &constraints,
175                     const Vector &vec) {
176    return all_of(constraints.begin(), constraints.end(),
177      [&](const Vector &cv) {
178        return vec_satisfied(cv, vec); });
179  }
```

Test $AV \le \mathbf{b}$

```
181  bool vecs_satisfied(const Matrix &constraints,
182                      const Matrix &vecs) {
183    return all_of(vecs.begin(), vecs.end(),
184      [&](const Vector &vec) {
185        return vec_satisfied(constraints, vec); });
186  }
```

Given an H-Cone $C = \{\mathbf{x} \mid A\mathbf{x} \le \mathbf{0}\} = \text{cone}(U)$ where U is extreme, and a `Matrix` U', determines if $C = \text{cone}(U')$. Similarly, given a V-Cone $C = \text{cone}(U) = \{\mathbf{x} \mid A\mathbf{x} \le \mathbf{0}\}$ where A is extreme, and a `Matrix` A', determines if $C = \{\mathbf{x} \mid A'\mathbf{x} \le \mathbf{0}\}$.

```
190  bool equivalent_cone_rep(const Matrix &cone,
191                           const Matrix &key,
192                           const Matrix &alt_rep) {
193    return rays_satisfied (cone, alt_rep) &&
194           subset_mod_eq  (key, alt_rep);
195  }
```

Given an H-Polytope $P = \{\mathbf{x} \mid A\mathbf{x} \le \mathbf{b}\} = \text{cone}(U) + \text{conv}(V)$ where U and V are extreme, and a pair (U', V'), determines if $P = \text{cone}(U') + \text{conv}(V')$.

```
197  bool equivalent_hpoly_rep(const Matrix &hpoly,
```

```
198                             const VPoly  &key,
199                             const VPoly  &vpoly) {
200   return rays_satisfied (hpoly, vpoly.U) &&
201          vecs_satisfied (hpoly, vpoly.V) &&
202          subset_mod_eq  (key.U, vpoly.U) &&
203          subset         (key.V, vpoly.V);
204 }
```

Given a V-Polytope $P = \text{cone}(U) + \text{conv}(V) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ where A is extreme, and a `Matrix` $(A', \mathbf{b}')$, determines if $P = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$.

```
206 bool equivalent_vpoly_rep (const VPoly  &vpoly,
207                            const Matrix &key,
208                            const Matrix &hpoly) {
209   return rays_satisfied (hpoly, vpoly.U) &&
210          vecs_satisfied (hpoly, vpoly.V) &&
211          subset_mod_eq  (key, hpoly);
212 }
```

Bibliography

[1] ZIEGLER, Gunter. *Lectures on Polytopes*. Springer-Verlag, New York, 1995. ISBN 0-387-94329-3.