1 Minkowski-Weyl Theorem

Definition 1 (non-negative linear combination)**.** *Let* $U \in \mathbb{R}^{d\times p}$*,* $\mathbf{t} \in \mathbb{R}^p, \mathbf{t} \geq \mathbf{0}$*, then* $\sum_{1\leq j\leq p} t_j U^j = U\mathbf{t}$ *is called a* ***non-negative linear combination*** *of* U*.*

Definition 2 (V-Cone)**.** *Let* $U \in \mathbb{R}^{d\times p}$*. The set of all non-negative linear combinations of* U *is denoted* $\operatorname{cone}(U)$*. Such a set is called a* ***V-Cone****.*

Definition 3 (convex combination)**.** *Let* $V \in \mathbb{R}^{d\times n}$*,* $\boldsymbol{\lambda} \in \mathbb{R}^n, \boldsymbol{\lambda} \geq \mathbf{0}, \sum_{1\leq j\leq n} \lambda_j = 1$*, then* $\sum_{1\leq j\leq n} \lambda_j V^j$ *is called a* ***convex combination*** *of* V*. The set of all convex combinations of* V *is denoted* $\operatorname{conv}(V)$*.*

Definition 4 (V-Polyhedron)**.** *Let* $V \in \mathbb{R}^{d\times n}$*,* $U \in \mathbb{R}^{d\times p}$*. Then the set*

$$\{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \operatorname{cone}(U), \mathbf{y} \in \operatorname{conv}(V)\}$$

is called a V-Polyhedron.

Definition 5 (H-Polyhedron)**.** *Let* $A \in \mathbb{R}^{m\times d}$*,* $\mathbf{b} \in \mathbb{R}^m$*. Then the set*

$$\left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; A\mathbf{x} \leq \mathbf{b}\right\}$$

is called an H-Polyhedron.

Definition 6 (H-Cone)**.** *Let* $A \in \mathbb{R}^{m\times d}$*. Then the set*

$$\left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; A\mathbf{x} \leq \mathbf{0}\right\}$$

is called an H-Cone.

The following theorem is the basic result to be proved in this thesis, which states that V-Polyhedra and H-Polyhedra are two different representations of the same objects.

Theorem 1 (Minkowski-Weyl Theorem)**.** *Every V-Polyhedron is an H-Polyhedron, and every H-Polyhedron is a V-Polyhedron.*

The proof proceeds by first showing that V-Cones are representable as H-Cones, and H-Cones are representable as V-Cones. Then it is shown that the case of polyhedra can be reduced to cones.

Theorem 2 (Minkowski-Weyl Theorem for Cones)**.** *Every V-Cone is an H-Cone, and every H-Cone is a V-Cone.*

A simple but useful property of cones is that they are closed under addition and positive scaling.

Proposition 1. *Let* C *be either an H-Cone or a V-Cone, for each* i $\mathbf{x}^i \in C$*, and* $c_i \geq 0$*. Then:*

$$\sum_i c_i \mathbf{x}^i \in C$$

Proof. First we prove Proposition 1 for H-Cones, then for V-Cones. If, for each i, $A\mathbf{x}^i \leq \mathbf{0}$, then $A(c_i\mathbf{x}^i) = t_i A\mathbf{x}^i \leq \mathbf{0}$, and

$$A\left(\sum_i c_i\mathbf{x}^i\right) = \sum_i A(c_i\mathbf{x}^i) = \sum_i c_i A\mathbf{x}^i \leq \sum_i \mathbf{0} \leq \mathbf{0}$$

So, $\sum_i c_i\mathbf{x}^i \in C$ when C is an H-Cone. Next, suppose that $C = \text{cone}(U)$, and for each i, $\exists \mathbf{t}_i \geq \mathbf{0} : \mathbf{x}^i = U\mathbf{t}_i$. Then $c_i\mathbf{t}_i \geq \mathbf{0}$, and $\sum_i c_i\mathbf{t}_i \geq \mathbf{0}$. Therefore

$$\sum_i c_i\mathbf{x}^i = \sum_i c_i U\mathbf{t}_i = \sum_i U(c_i\mathbf{t}_i) = U\left(\sum_i c_i\mathbf{t}_i\right)$$

So, $\sum_i c_i\mathbf{x}^i \in C$ when C is a V-Cone. □

This proposition will be used in the following way: if we wish to show that $\sum_i c_i\mathbf{x}^i$ in a member of some cone C, it suffices to show that, for each i, $c_i \geq 0$ and $\mathbf{x}^i \in C$.

2 Every V-Cone is an H-Cone

Definition 7 (Coordinate Projection)**.** *Let I be the identity matrix. Then the matrix I' formed by deleting some rows from I is called a **coordinate-projection**.*

The proof rests on the following two propostions:

(V1) Every V-Cone is a coordinate-projections of an H-Cone.

(V2) Every coordinate-projection of an H-Cone is an H-Cone.

Proof. Given (V1) and (V2), the proof follows simply. Given a V-Cone, we use (V1), to get a description involving coordinate-projection of an H-Cone. Then we can apply (V2) in order to get an H-Cone. □

Proof of (V1). We prove that every V-Cone is a coordinate-projection of an H-Cone, by giving an explicit formula. Let $U \in \mathbb{R}^{d\times p}$, and observe that

$$\text{cone}(U) = \{U\mathbf{t} \mid \mathbf{t} \in \mathbb{R}^p,\, \mathbf{t} \geq \mathbf{0}\} = \left\{\mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \in \mathbb{R}^p)\, \mathbf{x} = U\mathbf{t},\, \mathbf{t} \geq \mathbf{0}\right\}$$

We will collect $\mathbf{t}$ and $\mathbf{x}$ on the left side of the inequality, treating $\mathbf{t}$ as a variable and expressing its contraints as linear inequalities, then project away the coordinates corresponding to $\mathbf{t}$. The following expression takes one step:

$$\mathbf{t} \geq \mathbf{0} \Leftrightarrow -I\mathbf{t} \leq \mathbf{0} \tag{1}$$

And using the equality: $a = 0 \Leftrightarrow a \leq 0 \wedge -a \leq 0$, and block matrix notation, we take the second step.

$$\mathbf{x} = U\mathbf{t} \Leftrightarrow \mathbf{x} - U\mathbf{t} = \mathbf{0} \Leftrightarrow \begin{pmatrix} I & -U \\ -I & U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \tag{2}$$

Comparing (1) and (2), we define a new matrix $A' \in \mathbb{R}^{(p+2d)\times(d+p)}$:

$$A' = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix}$$

then we can rewrite cone(U):

$$\text{cone}(U) = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; A' \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \right\}$$

Let $\Pi \in \{0,1\}^{d\times(d+p)}$ be the identity matrix in $\mathbb{R}^{(d+p)\times(d+p)}$, but with the last p-rows deleted. Then Π is a coordinate projection, and the above expression can be written:

$$\text{cone}(U) = \Pi\left(\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0}\right\}\right) \tag{3}$$

This is a coordinate projection of an H-Cone, and (V1) is shown. □

To prove (V2), we use two separate propositions.

Proposition 2. *Let $B \in \mathbb{R}^{m'\times(d+p)}$, B' be B with the last p columns deleted, and Π the identity matrix with the last p rows deleted (i.e. $B' = \Pi B$). Furthermore, suppose that the last p columns of B are $\mathbf{0}$. Then*

$$\Pi\left(\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid B\mathbf{y} \leq \mathbf{0}\right\}\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid B'\mathbf{x} \leq \mathbf{0}\right\}$$

Proof. Recall that $B\mathbf{y} \leq \mathbf{0}$ means that $(\forall i)\, \langle B_i, \mathbf{y}\rangle \leq 0$. By the way we've defined B, any row B_i of B can be written $(B_i', \mathbf{0})$, with $\mathbf{0} \in \mathbb{R}^p$. Rewriting $\mathbf{y} \in \mathbb{R}^{d+p}$ as $(\mathbf{x}, \mathbf{w})$ with $\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^p$, so that $\mathbf{x} = \Pi(\mathbf{y})$. Then

$$\langle B, \mathbf{y}\rangle = \langle (B_i', \mathbf{0}), (\mathbf{x}, \mathbf{w})\rangle = \langle B_i', \mathbf{x}\rangle = \langle B_i', \Pi(\mathbf{y})\rangle$$

It follows that

$$\langle B_i, \mathbf{y}\rangle \leq 0 \Leftrightarrow \langle B_i', \Pi(\mathbf{y})\rangle \leq 0$$

Since B_i is an arbitrary row of B, the proposition is shown. □

In order to use the above proposition, we need a matrix with $\mathbf{0}$ columns. The next proposition shows us how to do so, one column at a time.

Proposition 3. *Let $B \in \mathbb{R}^{m_1\times(d+p)}$, $1 \leq k \leq p$, and $\mathbf{x} = \sum_{i\neq k} x_i \mathbf{e}_i$. Then there exists a matrix $B' \in \mathbb{R}^{m_2\times(d+p)}$ with the following properties:*

1. *Every row of B' is a postive linear combination of rows of B.*
2. *m_2 is finite.*
3. *The k-th column of B' is $\mathbf{0}$.*

4. $(\exists t)B(\mathbf{x}+t\mathbf{e}_k)\leq \mathbf{0} \Leftrightarrow B'\mathbf{x}\leq \mathbf{0}$

Proof. Partition the rows of B as follows:

$$
\begin{aligned}
P &= i \mid B_i^k > 0\\
N &= j \mid B_j^k < 0\\
Z &= l \mid B_l^k = 0
\end{aligned}
$$

Then let B' be a matrix with rows of the following forms:

$$
\begin{aligned}
C_l &= B_l && \mid l \in Z\\
C_{ij} &= B_i^k B_j - B_j^k B_i && \mid i \in P, j \in N
\end{aligned}
$$

1 and *2* are clear. *3* can be seen from:

$$
\begin{aligned}
&\langle C_l, \mathbf{e}_k\rangle = 0\\
&\langle C_{ij}, \mathbf{e}_k\rangle = \left\langle B_i^k B_j - B_j^k B_i, \mathbf{e}_k\right\rangle = B_i^k B_j^k - B_j^k B_i^k = 0 \qquad (4)
\end{aligned}
$$

The right direction of *4* is shown in the following calculations. Because $B_l^k = 0$:

$$\langle B_l, \mathbf{x}+t\mathbf{e}_k\rangle = \langle B_l, \mathbf{x}\rangle + tB_l^k = \langle B_l, \mathbf{x}\rangle = \langle C_l, \mathbf{x}\rangle$$

So:

$$\langle B_l, \mathbf{x}+t\mathbf{e}_k\rangle \leq 0 \Rightarrow \langle C_l, \mathbf{x}\rangle \leq 0$$

For rows indexed by P, N, we observe (13), and have:

$$\left\langle B_i^k B_j - B_j^k B_i, \mathbf{x}+t\mathbf{e}_k\right\rangle = \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x}\right\rangle$$

Now, we use property *1*:

$$\langle B_i, \mathbf{x}+t\mathbf{e}_k\rangle \leq 0,\ \langle B_j, \mathbf{x}+t\mathbf{e}_k\rangle \leq 0 \Rightarrow \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x}+t\mathbf{e}_k\right\rangle \leq 0$$

Therefore

$$\left\langle B_i^k B_j - B_j^k B_i, \mathbf{x}\right\rangle \leq 0$$

Now suppose that $B'\mathbf{x}\leq \mathbf{0}$. The task is to find a t so that $B\mathbf{x}\leq \mathbf{0}$. Looking at (13), any choice of t we make will be okay for rows indexed by Z. So the task is to find a t so that the inequality holds for rows indexed by P and N. Observe

$$
\begin{aligned}
&\forall i \in P, \forall j \in N \quad \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x}\right\rangle \leq 0 \Leftrightarrow\\
&\forall i \in P, \forall j \in N \quad \left\langle B_i^k B_j, \mathbf{x}\right\rangle \leq \left\langle B_j^k B_i, \mathbf{x}\right\rangle \Leftrightarrow\\
&\forall i \in P, \forall j \in N \quad \left\langle B_j / B_j^k, \mathbf{x}\right\rangle \geq \left\langle B_i / B_i^k, \mathbf{x}\right\rangle \Leftrightarrow\\
&\min_{j\in N} \left\langle B_j / B_j^k, \mathbf{x}\right\rangle \geq \max_{i\in P} \left\langle B_i / B_i^k, \mathbf{x}\right\rangle
\end{aligned}
$$

Note that the third inequality changes directions because $B_j^k < 0$. Now we choose t to lie in this last interval, and show that we can use it to satisfy all of the constraints given by B. So, we have a t such that

$$\min_{j \in N} \left\langle B_j / B_j^k, \mathbf{x} \right\rangle \geq t \geq \max_{i \in P} \left\langle B_i / B_i^k, \mathbf{x} \right\rangle$$

In particular,

$$\begin{aligned} &(\forall j \in N) \quad \left\langle B_j / B_j^k, \mathbf{x} \right\rangle \quad \geq t \Rightarrow \\ &(\forall j \in N) \quad \langle B_j, \mathbf{x} \rangle - B_j^k t \leq 0 \end{aligned}$$

Again, the inequality changes directions because $B_j^k < 0$. Now consider a row B_j from B:

$$\langle B_j, \mathbf{x} - t\mathbf{e}_k \rangle = B_j \mathbf{x} - B_j^k t \leq 0$$

Similarly,

$$\begin{aligned} &(\forall i \in P) \quad t \geq B_i / B_i^k \mathbf{x} \quad \Rightarrow \\ &(\forall i \in P) \quad 0 \geq B_i \mathbf{x} - B_i^k t \end{aligned}$$

Now consider a row B_i from B:

$$\langle B_i, \mathbf{x} - t\mathbf{e}_k \rangle = B_i \mathbf{x} - B_i^k t \leq 0$$

So, we've demonstrated that $\mathbf{x} - t\mathbf{e}_k$ satisfies all the constraints from B, and the left implication is shown. So *4* holds. □

Now to prove:

(V2) Every coordinate-projection of an H-Cone is an H-Cone.

proof of (V2). Here we prove the case that the coordinate projection is onto the first d of $d+p$ coordinates. Let $\left\{\mathbf{y} \in \mathbb{R}^{d+p} : A'\mathbf{y} \leq \mathbf{0}\right\}$ be the H-Cone we need to project, and Π the coordinate-projection we need to apply (the identity matrix with the last p columns deleted). For each $1 \leq k \leq p$ we can use proposition 3 in an incremental manner, starting with A'.

let $B_0 := A'$
for $1 \leq k \leq p$
  let $B_k :=$ result of proposition 2 applied to B_{k-1}, $\mathbf{e}_{d+k}$
endfor
return B_p

Consider the resulting B. Property *2* holds throughout, so B is finite. After each iteration, property *3* holds for k, so the k-th column is $\mathbf{0}$. Since each iteration only results from non-negative combinations of the result of the previous iteration (property *1*), once a column is $\mathbf{0}$ it remains so. Therefore, at the end of the process, the last p columns of B are all $\mathbf{0}$. Then, by proposition 2, we can apply Π to B by simply deleting the last p columns of B. Denote this resulting matrix A. We still need to check:

$$A'\mathbf{y} \leq \mathbf{0} \Leftrightarrow A(\Pi(\mathbf{y})) \leq \mathbf{0} \tag{5}$$

$$(\exists t_1)\dots(\exists t_p)A'(\mathbf{x} + t_1\mathbf{e}_{d+1} + \cdots + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} \Leftrightarrow A\mathbf{x} \leq \mathbf{0} \tag{6}$$

Then, using (5) and (6), it is easy to see that:

$$\Pi\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0}\right\} = \left\{\mathbf{x} \in \mathbb{R}^{d} \mid A\mathbf{x} \leq \mathbf{0}\right\} \tag{7}$$

The key observation of this verification utilizes property *4* of proposition 3:

$$(\exists t)B(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$$

In what follows, let $\mathbf{x} = \sum_{1 \leq j \leq d} x_j\mathbf{e}_j$. The above property is applied sequentially to the sets B_k as follows:

$$\begin{array}{rll}
(\exists t_p)(\exists t_{p-1})\dots(\exists t_1) & B_0(\mathbf{x} + t_1\mathbf{e}_p + t_2\mathbf{e}_{p-1} + \cdots + t_p\mathbf{e}_d) \leq \mathbf{0} & \Leftrightarrow \\
(\exists t_p)\dots(\exists t_2) & B_1(\mathbf{x} + t_2\mathbf{e}_{d+2} + \cdots + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow \\
\vdots & \vdots & \vdots \\
(\exists t_p) & B_{p-1}(\mathbf{x} + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow \\
 & B_p\mathbf{x} \leq \mathbf{0} & \Leftrightarrow
\end{array}$$

Because $A' = B_0$, and A is B_p with the last p columns deleted, (5) and (6) hold, therefore (7) holds, and the proof of (V2) is complete, and we've shown that a coordinate projection of an H-Cone is again an H-Cone. $\square$

With (V1) and (V2) proven, we are now certain that any V-Cone is also an H-Cone.

3 Every H-Cone is a V-Cone

Definition 8 (Coordinate Hyperplane)**.** *A set of the form*

$$\left\{\mathbf{x} \in \mathbb{R}^{d+m} \mid \langle \mathbf{x}, \mathbf{e}_k \rangle = 0\right\} = \left\{\mathbf{x} \in \mathbb{R}^{d+m} \mid x_k = 0\right\}$$

is called a coordinate-hyperplane.

We will use coordinate-hyperplanes in the following way. We consider a V-Cone intersected with some coordinate hyperplanes, and write it in the following way:

$$\left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}\right\} \tag{8}$$

If we suppose that $U' \subset \mathbb{R}^{d+m}$, and Π is the identity matrix with the last m rows deleted, then this is just a convenient way of writing:

$$\Pi\big(\,\mathrm{cone}(U') \cap \{x_{d+1} = 0\} \cap \cdots \cap \{x_{d+m} = 0\}\,\big) \tag{9}$$

The proof rests on the following three propostions:

H1 Every H-Cone is a coordinate-projection of a V-Cone intersected with some coordinate hyperplanes.

H2 Every V-Cone intersected with a coordinate-hyperplane is a V-Cone

H3 Every coordinate-projection of a V-Cone is an V-Cone.

Proof. Given *H1*, *H2*, and *H3*, the proof follows simply. Given an H-Cone, we use *H1* to get a description involving the coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. We apply *H2* as many times as necessary to elimintate the intersections, then we can apply *H3* in order to get a V-Cone. □

Proof of H1. Let $A \in \mathbb{R}^{m \times d}$, we now show that the H-Cone

$$\big\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0}\big\}$$

can be written as the projection of a V-Cone intersected with some hyperplanes. Define U':

$$U' = \left\{ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}, 1 \leq j \leq d,\, 1 \leq i \leq m \right\}$$

Then we claim:

$$\big\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0}\big\} = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t} \right\} \tag{10}$$

First, considering (8) and (9), observe that this is a coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. Next, we note that

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$$

We can write this as a sum with all positive coefficients if we split up the x_j as follows:

$$x_j^+ = \begin{cases} x_j & x_j \geq 0 \\ 0 & x_j < 0 \end{cases} \qquad x_j^- = \begin{cases} 0 & x_j \geq 0 \\ -x_j & x_j < 0 \end{cases}$$

Then we have

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} \tag{11}$$

where $x_j^+, x_j^- \geq 0$. Also observe that

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$$

This can also be written

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \tag{12}$$

(11) and (12) together show

$$A\mathbf{x} \leq \mathbf{0} \Rightarrow (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

Conversely, suppose

$$(\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

We would like to show that $A\mathbf{x} \leq \mathbf{0}$. Let x_j^+, x_j^-, w_i take the values of $\mathbf{t}$ that are coefficients of $\begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$, $\begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}$, and $\begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}$ respectively, and denote $x_j = x_j^+ - x_j^-$. Then we have

$$\begin{aligned}
\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} &= \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\
&= \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}
\end{aligned}$$

where $\mathbf{w} \geq \mathbf{0}$. By (12) we have $A\mathbf{x} \leq \mathbf{0}$. So (10) holds. □

The proof of *H2* relies upon the following proposition.

Proposition 4. *Let $Y \in \mathbb{R}^{(d+m) \times n_1}$, $1 \leq k \leq m$, and $\mathbf{x}$ satisfy $x_k = 0$. Then there exists a matrix $Y' \in \mathbb{R}^{(d+m) \times n_2}$ with the following properties:*

1. *Every column of Y' is a postive linear combination of rows of B.*
2. *n_2 is finite.*
3. *The k-th row of Y' is $\mathbf{0}$.*
4. *$(\exists \mathbf{t} \geq \mathbf{0})\mathbf{x} = Y\mathbf{t} \Leftrightarrow (\exists \mathbf{t}' \geq \mathbf{0})\mathbf{x} = Y'\mathbf{t}'$*

Recall that Y^i is the i-th column of Y, and Y_k^i is the element of Y in the i-th column and k-th row.

Proof. We partition the columns of Y:

$$P = i \ \ | \ Y_k^i > 0$$
$$N = j \ \ | \ Y_k^j < 0$$
$$Z = l \ \ | \ Y_k^l = 0$$

We then define Y':

$$Y' = \left\{Y^l \mid l \in Z\right\} \cup \left\{Y_k^i Y^j - Y_k^j Y^i \mid i \in P, \, j \in N\right\}$$

1 and *2* are clear. *3* can be seen from:

$$\left\langle Y'^l, \mathbf{e}^k \right\rangle = 0$$
$$\left\langle Y'^{ij}, \mathbf{e}^k \right\rangle = \left\langle Y_k^i Y^j - Y_k^j Y^i, \mathbf{e}^k \right\rangle = Y_k^i Y_k^j - Y_k^j Y_k^i = 0 \qquad (13)$$

Before moving on to the proof, we first note how to write our vectors.

$$Y\mathbf{t} = \sum_{l \in Z} t_k Y^k + \sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j$$
$$Y'\mathbf{t} = \sum_{l \in Z} t_k Y^k + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (Y_k^i Y^j - Y_k^j Y^i)$$

Then, by proposition 1, to show that the proposition is true, we need only show that, given any $t_i, t_j \geq 0$ ($t_{ij} \geq 0$), there exists $t_{ij} \geq 0$ ($t_i, t_j \geq 0$) such that

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} t_{ij} (Y_k^i Y^j - Y_k^j Y^i) \qquad (14)$$

Proposition 5. *Suppose that*

$$\sum_{i \in P} t_i Y_{d+1}^i + \sum_{j \in N} t_j Y_{d+1}^j = 0 \qquad Y_k^j < 0 < Y_k^i$$

Then the following holds

$$(t_i, t_j \geq 0) \Rightarrow (\exists t_{ij} \geq 0) \quad \text{such that (14) holds}$$
$$(t_{ij} \geq 0) \Rightarrow (\exists t_i, t_j \geq 0) \text{ such that (14) holds}$$

Proof. First note that if all $t_i = 0, t_j = 0$, then choosing $t_{ij} = 0$ satisfies (14), likewise if all $t_{ij} = 0$, then $t_i = 0, t_j = 0$ satisfies (14). So suppose that some $t_i \neq 0, t_j \neq 0, t_{ij} \neq 0$.

The right hand side of (14) can be written

$$\sum_{j \in N} \left(\sum_{i \in P} t_{ij} Y_k^i \right) Y^j + \sum_{i \in P} \left(- \sum_{j \in N} t_{ij} Y_k^j \right) Y^i$$

This means, given $t_{ij} \geq 0$, we can choose $t_j = \sum_{i \in P} t_{ij} Y_k^i$, and $t_i = -\sum_{j \in N} t_{ij} Y_k^j$, both of which are greater than 0.

Now suppose we have been given $t_i \geq 0, t_j \geq 0$. First observe:

$$0 = \sum_{i \in P} t_i Y_k^i + \sum_{j \in N} t_j Y_k^j \Rightarrow \sum_{i \in P} t_i Y_k^i = -\sum_{j \in N} t_j Y_k^j$$

Denote the value in this equality as σ, and note that $\sigma > 0$. Then

$$\sum_{i \in P} t_i Y^i = \frac{-\sum_{j \in N} t_j Y_k^j}{\sigma} \sum_{i \in P} t_i Y^i = \sum_{\substack{i \in P \\ j \in N}} -\frac{t_i t_j}{\sigma} Y_k^j Y^i$$

$$\sum_{j \in N} t_j Y^j = \frac{\sum_{i \in P} t_i Y_k^i}{\sigma} \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma} Y_k^i Y^j$$

Combining these results, we have

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma} (Y_k^i Y^j - Y_k^j Y^i)$$

□

Finally, we can conclude that, given $\mathbf{t} \geq \mathbf{0}$, if $Y\mathbf{t}$ has a 0 in the final coordinate, then we can write it as $Y'\mathbf{t}'$ where $\mathbf{t}' \geq \mathbf{0}$, and any non-negative linear combination of vectors from Y' can be written as a non-negative linear combination of vetors from Y, and will necessarily have the k-th coordinate be 0 by property *3*. So property *4* holds. □

Proof of H2. In proposition 4, the assumption that $x_k = 0$ in property *4* creates the set $\text{cone}(Y) \cap \{\mathbf{x} \mid x_k = 0\}$. This set, by property *4*, is $\text{cone}(Y')$. □

Proof of H3. We shall prove that the coordinate-projection of a V-Cone is again a V-Cone. Let Π be the relevant projection, then we have:

$$\Pi \{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U\mathbf{t}) \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U)\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\}$$

The last equality follows from associativity of matrix multiplication. Therefore,

$$\Pi\big(\text{cone}(U)\big) = \text{cone}\left(\Pi(U)\right)$$

□

4 Reducing Polyhedra to Cones

Definition 9 (Hyperplane)**.** *Let* $\mathbf{y} \in \mathbb{R}^d$*,* $c \in \mathbb{R}$*. Then a set of the form*

$$\left\{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{y}, \mathbf{x} \rangle = c\right\}$$

is called a hyperplane.

4.1 H-Polyhedra $\leftrightarrow$ H-Cones

We show that an H-Polyhedron can be represented as the projection of an H-Cone intersected with a hyperplane. We begin by re-writing the expression:

$$A\mathbf{x} \le \mathbf{b} \Leftrightarrow -\mathbf{b} + A\mathbf{x} \le \mathbf{0} \Leftrightarrow \big[-\mathbf{b}|A\big]\begin{pmatrix}1\\ \mathbf{x}\end{pmatrix} \le \mathbf{0} \tag{15}$$

Proposition 6. *Every H-Polyhedron can be written as an H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$, and any H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$ is an H-Polyhedron.*

Proof. We extend (15):

$$\mathbf{x} \in \big\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \le \mathbf{b}\big\} \Leftrightarrow \begin{pmatrix}1\\ \mathbf{x}\end{pmatrix} \in \big\{\mathbf{y} \in \mathbb{R}^{d+1} \mid \big[-\mathbf{b}|A\big]\mathbf{y} \le \mathbf{0}\big\}$$

We conclude, given an H-Polyhedron, we can add an extra coordinate and prepend the vector $\mathbf{b}$ to the left of A, and later we can just move this column back to the right side of the inequality and drop the extra coordinate. □

4.2 V-Polyhedra $\leftrightarrow$ V-Cone

We show that a V-Polyhedra can be reprented as a projection of a V-Cone intersected with the hyperplane $\big\{\mathbf{y} \in \mathbb{R}^{d+1} \mid y_0 = 1\big\}$. Given two sets $V \in \mathbb{R}^{d\times n}$ and $U \in \mathbb{R}^{d\times p}$, the V-Polyhedron is given by:

$$P_V = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \text{cone}(U),\, \mathbf{y} \in \text{conv}(V)\}$$

It isn't hard to see that

$$\mathbf{x} \in P_V \Leftrightarrow \begin{pmatrix}1\\ \mathbf{x}\end{pmatrix} \in \text{cone}\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ U & V\end{pmatrix}$$

For the value 1 to appear in the first coordinate, a convex combination of the vectors from $(\mathbf{1}, V)$ must be taken. After that, any non-negative combination of $(\mathbf{0}, U)$ added to this vector won't affect the 1 in the first coordinate.

It is more difficult to show that, given a V-Cone, that you can intersect it with the hyperplane $\big\{\mathbf{y} \in \mathbb{R}^{d+1} \mid y_0 = 1\big\}$ and get a V-Polytope out of it. So let

$$C_V = \text{cone}(U) \cap \big\{\mathbf{y} \in \mathbb{R}^{d+1} \mid y_0 = 1\big\}$$

We partition U into the sets:

$$\begin{aligned} P &= i \mid U_0^i > 0\\ N &= j \mid U_0^j < 0\\ Z &= l \mid U_0^l = 0 \end{aligned}$$

And define two new sets:

$$U' = \left\{U^l \mid l \in Z\right\} \cup \left\{U_0^i U^j - U_0^j U^i \mid i \in P,\, j \in N\right\}$$
$$V = \left\{U^i / U_0^i \mid i \in P\right\}$$

Then I claim that

$$C_V = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathrm{cone}(U'),\, \mathbf{y} \in \mathrm{conv}(V)\} \tag{16}$$

Say $\mathbf{x} \in \mathrm{cone}(U')$, $\mathbf{x}$ can be written

$$\begin{aligned}\mathbf{x} &= \sum_{l \in Z} t_l U^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij}(U_0^i U^j - U_0^j U^i) \\ &= \sum_{l \in Z} t_l U^l + \sum_{j \in N} \left(\sum_{i \in P} t_{ij} U_0^i\right) U^j + \sum_{i \in P} \left(\sum_{j \in N} -t_{ij} U_0^j\right) U^i\end{aligned}$$

So $\mathbf{x} \in \mathrm{cone}(U)$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{x} \rangle = \sum_{l \in Z} t_l U_0^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij}(U_0^i U_0^j - U_0^j U_0^i) = 0$$

So $x_0 = 0$. Similarly, for $\mathbf{y}$,

$$\mathbf{y} = \sum_{i \in P} \lambda_i U^i / U_0^i, \quad \sum_{i \in P} \lambda_i = 1$$

So $\mathbf{y} \in \mathrm{cone}(U)$, and then $\mathbf{x} + \mathbf{y} \in \mathrm{cone}(U)$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{y} \rangle = \sum_{i \in P} \lambda_i U_0^i / U_0^i = 1$$

So $y_0 = 1$ and $x_0 + y_0 = 1$. Then, by proposition 1, $\mathbf{x} + \mathbf{y} \in C_V$.

Next, suppose that $\mathbf{z} \in C_V$, then $\mathbf{z}$ can be written

$$\mathbf{z} = \sum_{l \in Z} t_l U^l + \sum_{i \in P} t_i U^i + \sum_{j \in N} t_j U^j$$

It will be convenient to use shorter notation for these sums. Define the following:

$$\begin{aligned}\sigma_Z &= \sum_{l \in Z} t_l U^l, & \sigma_l &= \sum_{l \in Z} t_l U_0^l = 0 \\ \sigma_P &= \sum_{i \in P} t_i U^i, & \sigma_i &= \sum_{i \in P} t_i U_0^i \\ \sigma_N &= \sum_{j \in N} t_j U^j, & \sigma_j &= \sum_{j \in N} t_j U_0^j\end{aligned}$$

Then it holds that

$$\langle \mathbf{e}_0, \mathbf{z} \rangle = \sigma_l + \sigma_i + \sigma_j = \sigma_i + \sigma_j = 1 \quad \Rightarrow \quad -\sigma_j/\sigma_i = 1 - 1/\sigma_i$$

$$\sigma_P = \sigma_P/\sigma_i + (1 - 1/\sigma_i)\sigma_P = \sigma_P/\sigma_i - (\sigma_j/\sigma_i)\sigma_P$$

Using the new notation, we can rewrite $\mathbf{z}$:

$$\mathbf{z} = \sigma_Z + \sigma_P + \sigma_N = \sigma_Z + \frac{\sigma_P}{\sigma_i} - \frac{\sigma_j}{\sigma_i}\sigma_P + \frac{\sigma_i}{\sigma_i}\sigma_N = \sigma_Z + \frac{\sigma_P}{\sigma_i} + \frac{\sigma_i\sigma_N - \sigma_j\sigma_P}{\sigma_i}$$

Using proposition 1, we need only show that

1. $\sigma_Z \in \text{cone}(U')$
2. $(\sigma_i\sigma_N - \sigma_j\sigma_P)/\sigma_i \in \text{cone}(U')$
3. $\sigma_P/\sigma_i \in \text{conv}(V)$

Since each $U^l : l \in Z$ is in C_V, (1) holds. We also have:

$$\sigma_i\sigma_N - \sigma_j\sigma_P = \sum_{i \in P} t_i \sum_{j \in N} t_j U_0^i U^j - \sum_{j \in N} t_j \sum_{i \in P} t_i U_0^j U^i = \sum_{\substack{i \in P \\ j \in N}} t_i t_j (U_0^i U^j - U_0^j U^i)$$

So (2) holds. Finally,

$$\sigma_P/\sigma_i = \sum_{i \in P} t_i U^i/\sigma_i = \sum_{i \in P} (t_i U_0^i/\sigma_i)(U^i/U_0^i)$$

Since $\sum_{i \in P}(t_i U_0^i/\sigma_i) = \sigma_i/\sigma_i = 1$, it follows that $\sigma_P/\sigma_i \in \text{conv}(V)$.