1 Algebraic Formula

Here is an incredibly useful algebraic derivation for dealing with polyhedra. Let I, J be finite index sets, and agree that $i \in I$ and $j \in J$ where appropriate. Also let $\alpha_i, \beta_j \in \mathbb{R}$, and $\mathbf{a}_i, \mathbf{b}^j$ be from some vector space. Then

$$\begin{aligned}
&\sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_j \beta_j \mathbf{b}^j \\
&= \left(1 + \frac{\sum_j \beta_j}{\sum_i \alpha_i}\right) \sum\nolimits_i \alpha_i \mathbf{a}^i - \frac{\sum_j \beta_j}{\sum_i \alpha_i} \sum\nolimits_i \alpha_i \mathbf{a}^i + \frac{\sum_i \alpha_i}{\sum_i \alpha_i} \sum\nolimits_j \beta_j \mathbf{b}^j \\
&= \left(1 + \frac{\sum_j \beta_j}{\sum_i \alpha_i}\right) \sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_{i,j} \frac{\alpha_i \beta_j}{\sum_i \alpha_i} (\mathbf{b}^j - \mathbf{a}^i)
\end{aligned}$$

If we let $\alpha = \sum_i \alpha_i$, and $\beta = \sum_j \beta_j$, then this can be written more legibly:

$$\sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_j \beta_j \mathbf{b}^j = \left(1 + \frac{\beta}{\alpha}\right) \sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_{i,j} \frac{\alpha_i \beta_j}{\alpha} (\mathbf{b}^j - \mathbf{a}^i)$$

A seemingly awkward but useful specialization of this identity is shown in the following proposition:

Proposition 1. *Let i, j range over finite index sets I, J, and suppose that $\forall i, j,\ \mathbf{v}^i_k, \mathbf{v}^j_k \neq 0$. Let $\sigma_i = \sum_i t_i v^i_k$, $\sigma_j = \sum_j t_j v^j_k$. Then*

$$\sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_j t_j \mathbf{v}^j = \left(1 + \frac{\sigma_j}{\sigma_i}\right) \sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_{i,j} \frac{t_i t_j}{\sigma_i} (v^i_k \mathbf{v}^j - v^j_k \mathbf{v}^i)$$

Proof. If in the above formula we let $\alpha_i = t_i v^i_k$, $\mathbf{a}^i = \mathbf{v}^i / v^i_k$, $\beta_j = t_j v^j_k$, and $\mathbf{b}^j = \mathbf{v}^j / v^j_k$, we get

$$\begin{aligned}
&\sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_j t_j \mathbf{v}^j = \\
&= \sum\nolimits_i (t_i v^i_k)(\mathbf{v}^i / v^i_k) + \sum\nolimits_j (t_j v^j_k)(\mathbf{v}^j / v^j_k) \\
&= \sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_j \beta_j \mathbf{b}^j = \\
&= \left(1 + \frac{\beta}{\alpha}\right) \sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_{i,j} \frac{\alpha_i \beta_j}{\alpha} (\mathbf{b}^j - \mathbf{a}^i) \\
&= \left(1 + \frac{\sigma_j}{\sigma_i}\right) \sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_{i,j} \frac{t_i t_j}{\sigma_i} (v^i_k v^j_k)(\mathbf{v}^j / v^j_k - \mathbf{v}^i / v^i_k) \\
&= \left(1 + \frac{\sigma_j}{\sigma_i}\right) \sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_{i,j} \frac{t_i t_j}{\sigma_i} (v^i_k \mathbf{v}^j - v^j_k \mathbf{v}^i)
\end{aligned}$$

□

2 Preliminaries

2.1 Notation

The canonical basis vectors will be written $\mathbf{e}_k$ for valid values of k. Let $\mathbf{x} \in \mathbb{R}^n$. It will be customary to write:

$$x_k = \langle \mathbf{e}_k, \mathbf{x} \rangle$$

Given $A \in \mathbb{R}^{m \times d}$, Let A_i and A^j denote the rows and columns of A, respectively. Then A_i^j will denote the entry from A in the i-th row and j-th column. Matrix multiplication is then given by:

$$A\mathbf{x} = \sum_{j=1}^{d} A^j x_j = \begin{pmatrix} \langle A_1, \mathbf{x} \rangle \\ \vdots \\ \langle A_m, \mathbf{x} \rangle \end{pmatrix} = \sum_{i=1}^{m} \langle A_i, \mathbf{x} \rangle \, \mathbf{e}_i$$

Let $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^m$, and define the following notation for vectors in $\mathbb{R}^{d+m}$:

$$\left\langle \mathbf{e}_k, \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \right\rangle = \begin{cases} x_k & k \leq d \\ w_{k-d} & d+1 \leq k \leq d+m \end{cases}$$

2.2 Definitions

Definition 1 (H-Cone)**.** *Let $A \in \mathbb{R}^{m \times d}$, then define*

$$\mathcal{C}_{\mathcal{H}}(A) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$$

Definition 2 (V-Cone)**.** *Let $V \in \mathbb{R}^{d \times n}$, then define*

$$\mathcal{C}_{\mathcal{V}}(V) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} \in \mathbb{R}^n \geq \mathbf{0}, \, \mathbf{x} = V\mathbf{t} \right\}$$

A vector of the form $V\mathbf{t}$, where $\mathbf{t} \geq \mathbf{0}$ is called a "conical combination of V".

Definition 3 (Hyperplane)**.** *Let $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$. Then the set*

$$\{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{a}, \mathbf{x} \rangle = b\}$$

Is known as a hyperplane. Let $H_k^n \subseteq \mathbb{R}^n$ denote the hyperplane defined by:

$$H_k^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_k = 0\}$$

Definition 4 (Projection)**.** *Let $\mathbf{x} \in \mathbb{R}^d$. The vector $\mathbf{x}' \in \mathbb{R}^{d-k}$ formed by omitting k coordinates of $\mathbf{x}$ is called a projection of $\mathbf{x}$. In particular, let $\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^m$, and $\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{w} \end{smallmatrix}\right) \in \mathbb{R}^{d+m}$, and define $\pi_{\mathbf{x}} : \mathbb{R}^{d+m} \to \mathbb{R}^d$ as:*

$$\pi_{\mathbf{x}} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} = \mathbf{x}$$

Then $\pi_{\mathbf{x}}$ is a projection onto the first d-coordinates.

$$
\begin{tikzcd}
\boxed{A\mathbf{x} \leq \mathbf{0}} \arrow[d, "\text{lift}"'] & & \boxed{\mathcal{C}_{\mathcal{V}}(V)} \\
\boxed{\mathcal{C}_{\mathcal{V}}\{\pm\left(\begin{smallmatrix}\mathbf{e}_i \\ A^i\end{smallmatrix}\right) \cup \left(\begin{smallmatrix}0 \\ \mathbf{e}_j\end{smallmatrix}\right)\}} \arrow[rr, "\bigcap_{d+1}^{d+m} H_k^{d+m}", "\text{drop}"'] & & \boxed{\{\mathbf{x} : \left(\begin{smallmatrix}\mathbf{x} \\ \mathbf{0}\end{smallmatrix}\right) \in \mathcal{C}_{\mathcal{V}}(V'')\}} \arrow[u, "\pi_{\mathbf{x}}", "\text{project}"']
\end{tikzcd}
$$

Figure 1: Diagram of the proof ($\mathcal{C}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{V}'} \to \mathcal{C}_{\mathcal{V}}$)

Remarks: In the following sections it will be proved that every **H-Cone** is a **V-Cone**, and every **V-Cone** is an **H-Cone**. This shows that these are two fundamentally different descriptions of the same object. Each has its own power and purpose, which will be discussed later.

Let $A \subseteq \mathbb{R}^n$. Then $A \cap H_k^n$ is simply all of the points of A who have the k-th coordinate 0. Let π^k be the projection that omits the k-th coordinate. Then $\pi^k(A \cap H_k^n)$ are all the points of A whose k-th coordinate is 0, but without that 0 coordinate. If projections are thought of as "forgetting useless information," and intersections as "capturing only the useful information," then the sequence of projection and intersection is "capturing the useful information, and forgetting the useless information."

3 H-Cone $\to$ V-Cone

This section gives a proof that every H-Cone is also a V-Cone.

3.1 Introduction.

Discussing "what is hard" is helpful for understanding the proof and why it is formed the way it is. Proposition 2 (lift) is not terribly difficult, it mostly just takes a clever idea and attention to detail. It is reminiscent of techniques common to linear programming. Propositions 3, 4, and 6 (project) are very straightforward, they exist primarily to make the proofs of the other propositions less cluttered. Proposition 5 (drop) is really the "main" idea that needs proving. This is because intersecting a V-Cone with a set of the form H_k^{d+m} requires determining all vectors of the V-Cone $\mathcal{C}_{\mathcal{V}}(V')$ that have a 0 in the k-th coordinate. Only conical combinations of V' of a special form will have this property, determining this form is the heart of the proof. Using the language from the above remark, "forgetting the useless information is easy, capturing the important information is hard, and representing the information in a different way is tricky."

3.2 Lift $\mathcal{C}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{V}'}$

Here we use some Linear Programming techniques to rewrite $\mathcal{C}_{\mathcal{H}}(A)$ as a subset of a V-Cone.

Proposition 2. *Let $A \in \mathbb{R}^{m \times d}$. Then there exists a $V' \in \mathbb{R}^{(d+m)\times(2d+m)}$ such that*

$$\mathcal{C}_{\mathcal{H}}(A) = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \right\}$$

Proof. Define V':

$$V' = \left\{ \pm \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \;\middle|\; 1 \le j \le d, 1 \le i \le m \right\}$$

Let $\mathbf{x} \in \mathcal{C}_{\mathcal{H}}(A)$. Then it is to be shown that:

$$A\mathbf{x} \le \mathbf{0} \Rightarrow \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$$

The task is to find a $t_j^+, t_j^-, w_i \ge 0$ such that:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix}$$

Consider the following assignments:

$$t_j^+ = \begin{cases} x_j & x_j \ge 0 \\ 0 & x_j < 0 \end{cases}$$

$$t_j^- = \begin{cases} 0 & x_j \ge 0 \\ -x_j & x_j < 0 \end{cases}$$

$$w_i = -\langle A_i, \mathbf{x} \rangle$$

By the way we've defined t_j^+ and t_j^-, $x_j = t_j^+ - t_j^-$. Then:

$$\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix}$$

Furthermore:

$$\sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} = -\sum_{i=1}^{m} \langle A_i, \mathbf{x} \rangle \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix}$$

It is clear that $t_j^+, t_j^- \ge 0$, and $w_i \ge 0$ follows from $A\mathbf{x} \le \mathbf{0}$. It follows that:

$$\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$$

Combining results, we have:

$$\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

And we conclude that $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$.

Now let $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$. The task is to show that $A\mathbf{x} \leq 0$. We have:

$$\begin{aligned} \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = & \quad \Big| \; t_j^+, t_j^-, w_i \geq 0 \\ \sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} & \quad \Big| \; t_j^+, t_j^-, w_i \geq 0 \end{aligned}$$

Since the only contribution to the coordinate of x_j is $t_j^+ - t_j^-$, we may conclude that $x_j = t_j^+ - t_j^-$. Continuing the string of equalities:

$$\begin{aligned} \sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = & \quad \Big| \; t_j^+, t_j^-, w_i \geq 0 \\ \sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = & \quad \Big| \; w_i \geq 0, x_j \in \mathbb{R} \\ \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} & \quad \Big| \; \mathbf{w} \geq \mathbf{0}, \mathbf{x} \in \mathbb{R}^d \end{aligned}$$

This last line implies that $A\mathbf{x} + \mathbf{w} = \mathbf{0} \Rightarrow A\mathbf{x} = -\mathbf{w} \leq \mathbf{0}$. □

3.3 Project $\pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{H}'})$

This proposition is not hard, it just writes the new form in a better way.

Proposition 3.

$$\left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \right\} = \pi_{\mathbf{x}} \left(\mathcal{C}_{\mathcal{V}}(V') \bigcap_{k=d+1}^{d+m} H_k^{d+m} \right)$$

Proof.

$$\mathcal{C}_{\mathcal{V}}(V') \bigcap_{k=d+1}^{d+m} H_k^{d+m} = \left\{ \mathbf{z} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; z_k = 0 : d+1 \le k \le d+m \right\}$$
$$= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; \mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^m, w_k = 0 : 1 \le k \le m \right\}$$
$$= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; \mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^m, \mathbf{w} = \mathbf{0} \right\}$$
$$= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; \mathbf{x} \in \mathbb{R}^d \right\}$$

Then

$$\pi_{\mathbf{x}} \left(\mathcal{C}_{\mathcal{V}}(V') \bigcap_{k=d+1}^{d+m} H_k^{d+m} \right) = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \right\}$$

□

3.4 Subcone

This proposition shows that a cone generated by a subset of an existing cone is a subset of the original cone. In other words, "conical combinations of conical combinations are again conical combinations."

Proposition 4. $U \subseteq \mathcal{C}_{\mathcal{V}}(V) \Rightarrow \mathcal{C}_{\mathcal{V}}(U) \subseteq \mathcal{C}_{\mathcal{V}}(V)$

Proof. For any $\mathbf{u}^i \in U$ we have $\mathbf{u}^i = \sum_j t_{ij} \mathbf{v}^j$. Then, for any conical combination of $\mathbf{u}^i$ we have:

$$\sum_i s_i \mathbf{u}^i = \sum_i s_i \left(\sum_j t_{ij} \mathbf{v}^j \right) = \sum_j \left(\sum_i s_i t_{ij} \right) \mathbf{v}^j$$

Since $s_i, t_{ij} \ge 0$, the final expression is a conical combination of V. □

3.5 Drop

Here is where the real work is done. This proposition shows how to remove one of the intersection terms from our new form. It uses a technique known as "Fourier Motzkin elimination." The definition of V_{out} and τ are the important ideas, the rest of the proof is just the ugly details showing that they are smart.

Note: V'_{in} and V'_{out} are intended to represent "input" and "output" sets. Notice that n_{in} and n_{out} may be different, i.e. the output may have (a lot) more vectors than the input.

Proposition 5. *Let* $V'_{in} \in \mathbb{R}^{(d+m) \times n_{in}}$*, then there exists a set* $V'_{out} \in \mathbb{R}^{(d+m) \times n_{out}}$ *such that*

$$\mathcal{C}_{\mathcal{V}}(V'_{in}) \cap H_k^{d+m} = \mathcal{C}_{\mathcal{V}}(V'_{out})$$

Proof. Suppose that the vectors of V_{in} are indexed by a set I. We partition the vectors of V_{in} based on their value at coordinate k.

$$\begin{aligned} P &= i \in I \mid v_k^i > 0 \\ N &= j \in I \mid v_k^j < 0 \\ Z &= l \in I \mid v_k^l = 0 \end{aligned}$$

Here, we've used different indices i, j, l. This is purely for convenience, and in what follows we'll follow the convention that $i \in P$, $j \in N$, and $l \in Z$. Next, let

$$V_{out} = \{\mathbf{v}^l \mid l \in Z\} \cup \{v_k^i \mathbf{v}^j - v_k^j \mathbf{v}^i \mid i \in P, j \in N\}$$

There are two critical properties of the set V_{out}. First, every $\mathbf{v} \in V_{out}$ is formed by taking conical combinations of vectors from V_{in}. Also, for any $\mathbf{v} \in V_{out}$, we have that $v_k = 0$. These two properties, along with proposition 4 gives

$$\mathcal{C}_{\mathcal{V}}(V_{out}) \subseteq \mathcal{C}_{\mathcal{V}}(V_{in}) \cap H_k^{d+m}$$

Next, say $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V_{in}) \cap H_k^{d+m}$, then:

$$\mathbf{x} = \sum_{i \in P} t_i \mathbf{v}^i + \sum_{j \in N} t_j \mathbf{v}^j + \sum_{l \in Z} t_l \mathbf{v}^l$$

Let $\sigma_i = \sum_i t_i v_k^i$, and $\sigma_j = \sum_j t_j v_k^j$. Because $x_k = 0$, and $v_k^l = 0$ for each $l \in Z$, we have

$$\begin{aligned} 0 &= \sum_{i \in P} t_i v_k^i + \sum_{j \in N} t_j v_k^j + \sum_{l \in Z} t_l v_k^l \\ &= \sum_{i \in P} t_i v_k^i + \sum_{j \in N} t_j v_k^j \\ &= \sigma_i + \sigma_j \end{aligned}$$

If $\sigma_i = 0$, then $\sigma_j = 0$, and all t_i, t_j must also be 0. This would mean that $\mathbf{x}$ is formed soley with vectors from $\mathbf{v}^l : l \in Z$, and $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V_{out})$.

Suppose that $\sigma_i \neq 0$, then note that

$$0 = 1 + \frac{\sigma_j}{\sigma_i}$$

By Proposition 1, $\mathbf{x}$ can be rewritten as:

$$\mathbf{x} = \sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_{i,j} \frac{t_i t_j}{\sigma_i} (v_k^i \mathbf{v}^j - v_k^j \mathbf{v}^i)$$

This shows $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V_{out})$, so we conclude

$$\mathcal{C}_{\mathcal{V}}(V_{in}) \cap H_k^{d+m} \subseteq \mathcal{C}_{\mathcal{V}}(V_{out})$$

$\square$

3.6 Projecting a V-Cone

Another simple proposition, this just shows that instead of taking all conical combinations of a set of vectors and then projecting them, we can just project the generators beforhand. I.e., it doesn't matter if we forget about some coordinates before or after we take conical combinations of them.

Proposition 6. *Let* $V' \in \mathbb{R}^{(d+m)\times n}$*, then*

$$\pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V') = \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$$

Proof. This follows from the fact that projections are linear transformations. Take the projection:

$$\pi_{\mathbf{x}} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} = \mathbf{x}$$

Then

$$\pi_{\mathbf{x}} \left(\alpha \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} \right) = \pi_{\mathbf{x}} \begin{pmatrix} \alpha\mathbf{x} + \mathbf{y} \\ \alpha\mathbf{w} + \mathbf{z} \end{pmatrix} = \alpha\mathbf{x} + \mathbf{y} = \alpha\pi_{\mathbf{x}} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + \pi_{\mathbf{x}} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix}$$

Let J index the vectors of V'. Say $\mathbf{v} \in \mathcal{C}_{\mathcal{V}}(V')$, then

$$\pi_{\mathbf{x}}(\mathbf{v}) = \pi_{\mathbf{x}} \left(\sum_{j\in J} t_j \mathbf{v}^j \right) = \sum_{j\in J} t_j \pi_{\mathbf{x}}(\mathbf{v}^j)$$

This is apparently a conical combination of vectors from $\pi_{\mathbf{x}}(V')$, so $\mathbf{v} \in \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$.

Similarly, say $\mathbf{u} \in \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$, then

$$\mathbf{u} = \sum_{j\in J} t_j \pi_{\mathbf{x}}(\mathbf{v}^j) = \pi_{\mathbf{x}} \left(\sum_{j\in J} t_j \mathbf{v}^j \right)$$

So $\mathbf{u} \in \pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V'))$. □

3.7 Main Theorem

Theorem 1. *Let* $A \in \mathbb{R}^{m\times d}$*. Then there exists a* $V \in \mathbb{R}^{d\times n}$ *such that*

$$\mathcal{C}_{\mathcal{H}}(A) = \mathcal{C}_{\mathcal{V}}(V)$$

Proof. Theorem 1 follows from the propositions as follows. First, apply proposition 2 to A to get a set V' of vectors in $\mathbb{R}^{d+m}$. Proposition 3 shows us that the set $\mathcal{C}_{\mathcal{H}}(A)$ can be formed by first taking a finite number of intersections of $\mathcal{C}_{\mathcal{V}}(V')$ with hyperplanes of the form H_k^{d+m}, then projecting this set onto $\mathbf{x}$. Proposition 5 gives us a method to eliminate all of the intersections in the form given by proposition 3, and end up with a set V'' with the following useful property:

$$\mathcal{C}_{\mathcal{V}}(V'') \bigcap_{k=d+1}^{d+m} H_k^{d+m} = \mathcal{C}_{\mathcal{V}}(V'')$$

Proposition 6, along with the set V'', allows the following calculation:

$$\mathcal{C}_{\mathcal{H}}(A) = \pi_{\mathbf{x}}\left(\mathcal{C}_{\mathcal{V}}(V'') \bigcap_{k=d+1}^{d+m} H_k^{d+m}\right) = \pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V'')) = \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V''))$$

Letting $V = \pi_{\mathbf{x}}(V'')$, we then have

$$\mathcal{C}_{\mathcal{H}}(A) = \mathcal{C}_{\mathcal{V}}(V)$$

$\square$

4 V-Cone $\rightarrow$ H-Cone

This section gives a proof that every V-Cone is also a H-Cone.

4.1 Introduction.

The proof going the other direction has a likeness to the former, due to the two representations being "dual" to one another. First, recall that the expression $A\mathbf{x} \leq \mathbf{0}$ can be written:

$$(\forall A_i \in A)\, \langle A_i, \mathbf{x}\rangle \leq 0\}$$

Here, A goes from being a matrix to being a set of row vectors.

4.2 Lift $\mathcal{C}_{\mathcal{V}} \rightarrow \mathcal{C}_{\mathcal{H}'}$

The first order of business is to get a new representation of our V-Cone.

Proposition 7. *For every set $V \in \mathbb{R}^{d\times n}$ there exists an $A \in \mathbb{R}^{(n+2d)\times(n+d)}$ such that*

$$\mathcal{C}_{\mathcal{V}}(V) = \{\mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} : (\forall A_i \in A)\, \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \leq 0\}$$

Proof. Observe that a V-Cone can be written:

$$\mathcal{C}_{\mathcal{V}}(V) = \{\mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} \geq \mathbf{0} : \mathbf{x} = V\mathbf{t}\}$$

Our operation here is to let $\mathbf{t}$ be a variable, and represent its contraints and relationship with $\mathbf{x}$ in terms of linear inequalities. Consider the following systems of inequalities:

$$\begin{aligned} \mathbf{t} &\geq \mathbf{0} & \Leftrightarrow && -I\mathbf{t} &\leq \mathbf{0} \\ \mathbf{x} &= V\mathbf{t} & \Leftrightarrow && I\mathbf{x} - V\mathbf{t} &\leq \mathbf{0} \\ &&&& -I\mathbf{x} + V\mathbf{t} &\leq \mathbf{0} \end{aligned}$$

We can use this as a template for a new matrix $A \in \mathbb{R}^{(n+2d)\times(n+d)}$:

$$A = \begin{pmatrix} \mathbf{0} & -I \\ -V & I \\ V & -I \end{pmatrix}$$

Then we have that

$$\mathcal{C}_{\mathcal{V}}(V) = \{\mathbf{x} \in \mathbb{R}^{d+n} \mid \exists \mathbf{t} : (\forall A_i \in A)\, \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \leq \mathbf{0}\}$$

□

4.3 Drop

Our new representation of the V-Cone is almost an H-Cone, the only problem is the $\exists$ qualifier. Note that the expression $\exists \mathbf{t}$ is equivalent to $\exists t_1, \exists t_2, \ldots, \exists t_n$. The method of dealing with these is to show that they can be elimnated one at at time, then conclude that we can get rid of all of them.

Proposition 8. *For every set $A_{in} \in \mathbb{R}^{m_{in}\times(d+n)}$ there is a set $A_{out} \in \mathbb{R}^{m_{out}\times(d+n)}$ such that*

$$\begin{aligned} &\exists t\, (\forall A_i \in A_{in})\ \ \langle A_i, \mathbf{x} - t\mathbf{e}_k\rangle \leq 0 \Leftrightarrow \\ &\forall t\, (\forall A_i \in A_{out})\, \langle A_i, \mathbf{x}\rangle \leq 0 \end{aligned}$$

Proof. Define the following subsets of A_{in}:

$$\begin{aligned} P &= A_i \mid A_i^k > 0 \\ N &= A_j \mid A_j^k < 0 \\ Z &= A_l \mid A_l^k = 0 \end{aligned}$$

As before, we adopt the convention that $A_i \in P$, $A_j \in N$, and $A_l \in Z$. Now define

$$A_{out} = Z \cup \{A_i^k A_j - A_j^k A_i \mid A_i \in P, A_j \in N\}$$

Suppose that $\exists t : A_{in}(\mathbf{x} - t\mathbf{e}_k) \leq \mathbf{0}$. For convenience, let $\mathbf{x}' = \mathbf{x} - t\mathbf{e}_k$.

$$\begin{aligned} \langle A_i, \mathbf{x}'\rangle \leq 0 &\Rightarrow -A_j^k \langle A_i, \mathbf{x}'\rangle \leq 0 \\ \langle A_j, \mathbf{x}'\rangle \leq 0 &\Rightarrow \quad A_i^k \langle A_j, \mathbf{x}'\rangle \leq 0 \end{aligned}$$

Summing the last two inequalities:

$$\left\langle A_i^k A_j - A_j^k A_i, \mathbf{x}' \right\rangle \leq 0$$

Furthermore, since

$$\left\langle A_i^k A_j - A_j^k A_i, \mathbf{e}_k \right\rangle = A_i^k A_j^k - A_j^k A_i^k = 0$$

We also have

$$\begin{aligned}
&\left\langle A_i^k A_j - A_j^k A_i, \mathbf{x}' \right\rangle \\
&= \left\langle A_i^k A_j - A_j^k A_i, \mathbf{x} - t\mathbf{e}_k \right\rangle \\
&= \left\langle A_i^k A_j - A_j^k A_i, \mathbf{x} \right\rangle \leq 0
\end{aligned}$$

So we conclude that $\exists t : A_{in}(\mathbf{x} - t\mathbf{e}_k) \leq \mathbf{0} \Rightarrow A_{out}\mathbf{x} \leq \mathbf{0}$.

Next suppose that $(\forall t) A_{out}(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0}$. We must find a t such that $A_{in}(\mathbf{x} - t\mathbf{e}_k) \leq \mathbf{0}$. Then we have

$$\begin{aligned}
&(\forall i, j) \left\langle A_i^k A_j - A_j^k A_i, \mathbf{x} \right\rangle \leq 0 \;\Rightarrow \\
&(\forall i, j) \left\langle A_i^k A_j, \mathbf{x} \right\rangle \;\; \leq \left\langle A_j^k A_i, \mathbf{x} \right\rangle \;\Rightarrow \\
&(\forall i, j) \left\langle A_j / A_j^k, \mathbf{x} \right\rangle \geq \left\langle A_i / A_i^k, \mathbf{x} \right\rangle
\end{aligned}$$

In particular, we have that

$$\max_i \left\langle A_i / A_i^k, \mathbf{x} \right\rangle \leq \min_j \left\langle A_j / A_j^k, \mathbf{x} \right\rangle$$

So, let $t \in [\max_i \left\langle A_i / A_i^k, \mathbf{x} \right\rangle, \min_j \left\langle A_j / A_j^k, \mathbf{x} \right\rangle]$. Then:

$$\begin{aligned}
&\forall i : tA_i^k \geq \langle A_i, \mathbf{x} \rangle \\
&\forall j : tA_j^k \geq \langle A_j, \mathbf{x} \rangle
\end{aligned}$$

Note that the second inequality holds because $A_j^k < 0$. Now consider $\mathbf{x} - t\mathbf{e}_k$:

$$\begin{aligned}
&\langle A_i, \mathbf{x} - t\mathbf{e}_k \rangle = \langle A_i, \mathbf{x} \rangle - tA_i^k \leq 0 \\
&\langle A_j, \mathbf{x} - t\mathbf{e}_k \rangle = \langle A_j, \mathbf{x} \rangle - tA_j^k \leq 0
\end{aligned}$$

Therefore, $\exists t : A_{in}(\mathbf{x} - t\mathbf{e}_k) \leq \mathbf{0}$ □

Proposition 9. *For every set $A \in \mathbb{R}^{(d+n)\times n}$ there is a set $A' \in \mathbb{R}^{(d+n)\times n'}$ such that*

$$\begin{aligned}
&\{\mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} (\forall A_i \in A) \;\; \langle A_i, (\mathbf{x}, \mathbf{t}) \rangle \leq 0\} = \\
&\{\mathbf{x} \in \mathbb{R}^d \mid \forall \mathbf{t} (\forall A_i \in A') \, \langle A_i, (\mathbf{x}, \mathbf{t}) \rangle \leq 0\}
\end{aligned}$$

Proof. The set A' is formed by repeatedly applying proposition 8. □

Proposition 10. *Let $A_i \in \mathbb{R}^{d+n}, \mathbf{x} \in \mathbb{R}^d, \mathbf{t} \in \mathbb{R}^n$. Then*

$$\forall \mathbf{t} \, \langle A_i, (\mathbf{x}, \mathbf{t}) \rangle \leq 0 \Rightarrow (A_i^k = 0 \mid d + 1 \leq k \leq d + n)$$

Proof. Suppose that $A_i^k > 0$, and let $\alpha = \langle A_i, (\mathbf{x}, \mathbf{0} \rangle$. Then $\langle A_i, (\mathbf{x}, t\mathbf{e}_k) \rangle = \alpha + tA_i^k$. By setting $t > -\alpha / A_i^k$, we arrive at a contradiction. □

4.4 Projection

Proposition 11. *For every set* $A' \in \mathbb{R}^{(d+n)\times n'}$ *there exists a set* $A'' \in \mathbb{R}^{d\times n'}$ *such that*

$$\{\mathbf{x} \in \mathbb{R}^d \mid (\forall A_i \in A')\, \forall \mathbf{t}\, \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \le 0\} =$$
$$\{\mathbf{x} \in \mathbb{R}^d \mid (\forall A_i \in A'') \quad \langle A_i, \mathbf{x}\rangle \le 0\}$$

Proof. Let $A'' = \pi_{\mathbf{x}}(A')$, that is, the set formed by projecting each from from A' onto the first d coordinates. From proposition 10, we have

$$\langle A_i, (\mathbf{x}, \mathbf{t})\rangle = \langle \pi_{\mathbf{x}}(A_i), \mathbf{x}\rangle$$

Therefore, $A'(\mathbf{x}, \mathbf{t}) \le \mathbf{0} \Leftrightarrow A''\mathbf{x} \le \mathbf{0}$. □

4.5 Main Theorem

Theorem 2. *Let* $V \in \mathbb{R}^{d\times n}$. *Then there exists an* $A \in \mathbb{R}^{m\times d}$ *such that:*

$$\mathcal{C}_{\mathcal{V}}(V) = \mathcal{C}_{\mathcal{H}}(A)$$

Proof. First, apply proposition 7 to V to get a set A'. Then apply proposition 9 to A' to get a set A''. Finally, let A be the result of applying 11 to A''. □

5 Main Theorem

Theorem 3 (Minkowski-Weyl Theorem)**.** *Every V-Cone is an H-Cone, and every H-Cone is a V-Cone.*

6 Polyhedra

Generalizing to Polyhedra In this section, we will define polyhedra, and expand theorem 3 to include them.

Definition 5 (Convex Combination)**.** *Let* $\mathbf{x}_i \in \mathbb{R}^n, \lambda_i \ge 0, \sum_i \lambda_i = 1$. *Then an expression of the form:*

$$\sum_i \lambda_i \mathbf{x}_i$$

is called a convex combination of the vectors $\mathbf{x}_i$.

Definition 6 (Convex Hull)**.** *Let* $A \subseteq \mathbb{R}^n$. *Then let*

$$\mathrm{conv}(A)$$

denote the set of all convex combinations of members of A.

Definition 7 (Minkowski Sum)**.** *Let $A, B \subseteq \mathbb{R}^n$. Then denote*

$$A \oplus B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$$

The set $A \oplus B$ is called the Minkowski Sum of A and B.

Definition 8 (V-Polyhedron $\mathcal{P}_{\mathcal{V}}$)**.** *Let U, V be finite subsets of $\mathbb{R}^d$. Then the set*

$$\operatorname{cone}(U) \oplus \operatorname{cone}(V)$$

is called a V-Polyhedron.

Definition 9 (H-Polyhedron $\mathcal{P}_{\mathcal{H}}$)**.** *Let $A \in \mathbb{R}^{m \times d}, \mathbf{b} \in \mathbb{R}^m$. Then the set*

$$\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}$$

is called an H-Polyhedron.

Definition 10 (Set Similarity)**.** *Let $A \subseteq \mathbb{R}^n, x \in \mathbb{R}$. Write*

$$\{x\} \times A \simeq A$$

The purpose of this notation is to avoid constantly writing and checking that sets differ only by a product with a point in space. This is because a polyhedron doesn't "gain anything" from taking its product with a point.

Remark In order to generalize the Minkowski-Weyl Theoremto polyhedra, we need only provide a procedure to turn a polyhedron into a cone and back again, for both V and H polyhedra.

6.1 $\mathcal{P}_{\mathcal{H}} \leftrightarrow \mathcal{C}_{\mathcal{H}}$

Proposition 12 ($\mathcal{P}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{H}}$)**.** *Every H-Polyhedron can be represented by the intersection of an H-Cone and a hyperplane.*

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \simeq \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; (-\mathbf{b}\, A) \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} \cap \{x_0 = 1\}$$

Proof. First note that $(-\mathbf{b}\, A) \left(\begin{smallmatrix} x_0 \\ \mathbf{x} \end{smallmatrix} \right) = -x_0\mathbf{b} + A\mathbf{x}$. Then

$$A\mathbf{x} \leq \mathbf{b} \Leftrightarrow -1\mathbf{b} + A\mathbf{x} \leq \mathbf{0}$$

□

Proposition 13 ($\mathcal{C}_{\mathcal{H}} \to \mathcal{P}_{\mathcal{H}}$)**.** *Every intersection of an H-Cone with a hyperplane $\{x_0 = 1\}$ is a polyhedron.*

Proof.

$$\{\mathbf{x} \mid A\mathbf{x} \leq 0\} \cap \{x_0 = 1\} = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; \begin{pmatrix} 1 & \mathbf{0} \\ -1 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \begin{pmatrix} 1 \\ -1 \\ \mathbf{b} \end{pmatrix} \right\}$$

□

6.2 $\mathcal{P}_\mathcal{V} \leftrightarrow \mathcal{C}_\mathcal{V}$

Proposition 14 ($\mathcal{P}_\mathcal{V} \to \mathcal{C}_\mathcal{V}$)**.** *Every V-Polyhedron can be represented as the intersection of a V-Cone and a hyperplane.*

$$\mathrm{conv}(V) \oplus \mathrm{cone}(U) \simeq \mathrm{cone}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ V & U \end{pmatrix} \cap \{x_0 = 1\}$$

Proof. Say $\mathbf{x} \in \mathrm{cone}(U) + \mathrm{conv}(V)$, $\mathbf{u}^j \in U$, $\mathbf{v}^i \in V$, $\lambda_i \geq 0, \sum_i \lambda_i = 1, t_j \geq 0$. If $\mathbf{x} = \sum_i \lambda_i \mathbf{v}^i + \sum_j t_j \mathbf{u}^j$, then

$$\sum\nolimits_i \lambda_i \begin{pmatrix} 1 \\ \mathbf{v}^i \end{pmatrix} + t_i \sum\nolimits_j \begin{pmatrix} 0 \\ \mathbf{u}^j \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \in \mathrm{cone}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ V & U \end{pmatrix} \cap \{x_0 = 1\}$$

Similarly, if

$$\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \in \mathrm{cone}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ V & U \end{pmatrix} \cap \{x_0 = 1\}$$

Then the sum of all λ_i corresponding to V must sum to one, and is the sum of a convex combination of members of $\left(\begin{smallmatrix} 1 \\ V \end{smallmatrix}\right)$, and a member of cone $\left(\begin{smallmatrix} 0 \\ U \end{smallmatrix}\right)$, then $\mathbf{x}$ is a sum of vectors from these two sets. □

Proposition 15 ($\mathcal{C}_\mathcal{V} \to \mathcal{P}_\mathcal{V}$)**.** *A V-Cone $\mathcal{C}_\mathcal{V}(V')$ intersected with a hyperplane of the from $\{x_0 = 1\}$ is a V-Polyhedron.*

Proof. Let I index V', and define the sets

$$\begin{aligned} P &= i \in I \mid v_0^i > 0 \\ N &= j \in I \mid v_0^j < 0 \\ Z &= l \in I \mid v_0^l = 0 \end{aligned}$$

Then define U and V:

$$\begin{aligned} U &= V'_Z \cup \{v_0^i \mathbf{v}^j - v_0^j \mathbf{v}^i \mid i \in P, j \in N\} \\ V &= \{\mathbf{v}^i / v_0^i \mid i \in P\} \end{aligned}$$

Let $\mathbf{x} \in \mathcal{C}_\mathcal{V}$, and denote $\sigma_i = \sum_i t_i v_0^i$, $\sigma_j = \sum_j t_j v_0^j$. Then

$$\begin{aligned} &\mathbf{x} = \sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_j t_j \mathbf{v}^j + \sum\nolimits_l t_l \mathbf{v}^l \\ &\sigma_i + \sigma_j = 1 \Rightarrow 1 + \frac{\sigma_j}{\sigma_i} = \frac{1}{\sigma_i} \end{aligned}$$

By proposition 1, we can write $\mathbf{x}$ as:

$$\begin{aligned} \mathbf{x} &= \left(1 + \frac{\sigma_j}{\sigma_i}\right) \sum\nolimits_i t_i \mathbf{v}^i + \sum\nolimits_{i,j} \frac{t_i t_j}{\sigma_i} (v_k^i \mathbf{v}^j - v_k^j \mathbf{v}^i) + \sum\nolimits_l t_l \mathbf{v}^l \\ &= \sum\nolimits_i \frac{t_i v_0^i}{\sigma_i} \frac{\mathbf{v}^i}{v_0^i} + \sum\nolimits_{i,j} \frac{t_i t_j}{\sigma_i} (v_k^i \mathbf{v}^j - v_k^j \mathbf{v}^i) + \sum\nolimits_l t_l \mathbf{v}^l \end{aligned}$$

This final expression shows that $\mathbf{x} \in \mathrm{cone}(U) \oplus \mathrm{conv}(V)$. □

6.3 Main Theorem for Polyhedra

Theorem 4 (Minkowski-Weyl Theoremfor polyhedra)**.** *Every V-Polyhedron is an H-Polyhedron, and every H-Polyhedron is a V-Polyhedron.*

Proof. Using the propositions of this section we can transfer a polyhedron to a cone, then use theorem 3 to change its representation, then again use the propositions of this section to recover the original cone. □