The input is blocked from what I've done so far, I will attempt to do a rewrite of part of the proof to make sure I have the right "style"

0.1 Notation

The canonical basis vectors will be written $\mathbf{e}_k$ for valid values of k. Let $\mathbf{x} \in \mathbb{R}^n$. It will be customary to write $x_k = \langle \mathbf{e}_k, \mathbf{x} \rangle$. Given $A \in \mathbb{R}^{m \times d}$, Let A_i and A^j denote the rows and columns of A, respectively. Then A_i^j will denote the entry from A in the i-th row and j-th column. Matrix multiplication is then given by:

$$A\mathbf{x} = \sum_{j=1}^{d} A^j x_j = \begin{pmatrix} \langle A_1, \mathbf{x} \rangle \\ \vdots \\ \langle A_m, \mathbf{x} \rangle \end{pmatrix}$$

Let $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^m$, and define the following notation for vectors in $\mathbb{R}^{d+m}$:

$$\left\langle \mathbf{e}_k, \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \right\rangle = \begin{cases} x_k & k \leq d \\ w_{k-d} & d+1 \leq k \leq d+m \end{cases}$$

Let σ denote the sign function, i.e:

$$\sigma(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

0.2 Definitions

Definition 1. *Let $A \in \mathbb{R}^{m \times d}$, then define $\mathcal{C}_{\mathcal{H}}(A) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$.*

Definition 2. *Let $V \in \mathbb{R}^{d \times n}$, then define $\mathcal{C}_{\mathcal{V}}(V) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} \in \mathbb{R}^n \geq \mathbf{0},\, \mathbf{x} = V\mathbf{t} \right\}$.*

Definition 3. *Let $H_k^n \subseteq \mathbb{R}^n$ be defined as $\{\mathbf{x} \in \mathbb{R}^n \mid x_k = 0\}$.*

Definition 4. *Let $[n] = \{1, 2, \ldots, n\}$ $K = \{k_1, k_2, \ldots, k_l\} \subseteq [n]$. Define $\pi^K : \mathbb{R}^n \to \mathbb{R}^{n-l}$ as follows. Let $J = [n] \setminus K = \{j_1 < j_2 < \cdots < j_{n-l}\}$. Then*

$$\pi^K(x_1, x_2, \ldots, x_n) = (x_{j_1}, x_{j_2}, \ldots, x_{j_{n-l}})$$

Definition 5. *Let $A \subseteq \mathbb{R}^d$, $B \subseteq \mathbb{R}^{d+n}$. Then*

$$A \simeq B \Leftrightarrow \exists K \subseteq [d+n] : A = \pi^K(B)$$

0.3 H-Cone $\to$ V-Cone

Theorem 1. *Let $A \in \mathbb{R}^{m\times d}$, then for the set $\mathcal{C}_{\mathcal{H}}(A)$, there exists a $V \in \mathbb{R}^{(d+m)\times n}$ such that*

$$\mathcal{C}_{\mathcal{H}}(A) \simeq \mathcal{C}_{\mathcal{V}}(V)$$

The proof of this theorem follows from the following claims:

Claim 1. *Let $A \in \mathbb{R}^{m\times d}$. Then there exists a $V \in \mathbb{R}^{(d+m)\times(2d+m)}$ such that*

$$\mathcal{C}_{\mathcal{H}}(A) \simeq \mathcal{C}_{\mathcal{V}}(V) \bigcap_{k=d+1}^{d+m} H_k^{d+m}$$

Claim 2. *Let $V \in \mathbb{R}^{d\times n}$, then there exists a set $V' \in \mathbb{R}^{d\times n'}$ such that*

$$\mathcal{C}_{\mathcal{V}}(V) \cap H_k^d = \mathcal{C}_{\mathcal{V}}(V')$$

Proof of **Claim 1**. Define V:

$$V = \left\{ \pm \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \;\middle|\; 1 \le j \le d, 1 \le i \le m \right\}$$

Let $\mathbf{x} \in \mathcal{C}_{\mathcal{H}}(A)$. Then it is to be shown that:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V) \bigcap_{k=d+1}^{d+m} H_k^{d+m}$$

Consider the vector:

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \sum_{j=1}^{d} |x_j| \sigma(x_j) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V)$$

Membership follows from $|x_j| \ge 0$. We now need a vector from $\mathcal{C}_{\mathcal{V}}(V)$ to add to this vector to make the "bottom" $\mathbf{0}$. Note that $\forall i \, \langle A_i, \mathbf{x} \rangle \le 0$. Now consider the vector:

$$\begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix} = \sum_{i=1}^{m} -\langle A_i, \mathbf{x} \rangle \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V)$$

We now have:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix} = \sum_{j=1}^{d} |x_j| \sigma(x_j) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} -\langle A_i, \mathbf{x} \rangle \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V)$$

We have shown that

$$A\mathbf{x} \leq \mathbf{0} \Rightarrow \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V) \bigcap_{k=d+1}^{d+m} H_k^{d+m}$$

Now let $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V)$. The task is to show that $A\mathbf{x} \leq 0$. We have:

$$\begin{aligned}
\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} &= \sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{j=1}^{d} t_j^- - \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = && \Big| \, t_j^+, t_j^-, w_i \geq 0 \\
& \sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = && \Big| \, t_j^+, t_j^-, w_i \geq 0 \\
& \sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = && \Big| \, w_i \geq 0, x_j \in \mathbb{R} \\
& \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = && \Big| \, \mathbf{w} \geq \mathbf{0}, \mathbf{x} \in \mathbb{R}^d
\end{aligned}$$

This last line implies that $A\mathbf{x} + \mathbf{w} = \mathbf{0} \Rightarrow A\mathbf{x} = -\mathbf{w} \leq \mathbf{0}$. Thus,

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V) \bigcap_{k=d+1}^{d+m} H_k^{d+m} \Rightarrow A\mathbf{x} \leq \mathbf{0}$$

To complete the proof, take $K = \{d+1, d+2, \ldots, d+m\}$, and observe that

$$\pi^K \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \mathbf{x}$$

It follows that

$$\pi^K \left(\mathcal{C}_{\mathcal{V}}(V) \bigcap_{k=d+1}^{d+m} H_k^{d+m} \right) = \mathcal{C}_{\mathcal{H}}(A)$$

$\square$

Proof of **Claim 2**. Define:

$$\begin{aligned}
P &= \mathbf{u} \in V \mid u_k > 0 \\
N &= \mathbf{v} \in V \mid v_k < 0 \\
Z &= \mathbf{w} \in V \mid w_k = 0
\end{aligned}$$

Next, let

$$V' = Z \cup \{u_k \mathbf{v} - v_k \mathbf{u} \mid \mathbf{u} \in P, \, \mathbf{v} \in N\}$$

That $\mathcal{C}_{\mathcal{V}}(V') \subseteq \mathcal{C}_{\mathcal{V}}(V)$ follows from the fact that every vector in $\mathcal{C}_{\mathcal{V}}(V')$ is a positive linear combination of vectors from V. In detail, say $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V')$, then

$$
\begin{aligned}
\mathbf{x} &= \sum_{\substack{\mathbf{u}\in U \\ \mathbf{v}\in V}} t_{uv}(u_k\mathbf{v} - v_k\mathbf{u}) + \sum_{\mathbf{w}\in Z} t_w\mathbf{w} \\
&= \sum_{\substack{\mathbf{u}\in U \\ \mathbf{v}\in V}} t_{uv}u_k\mathbf{v} + \sum_{\substack{\mathbf{u}\in U \\ \mathbf{v}\in V}} -t_{uv}v_k\mathbf{u} + \sum_{\mathbf{w}\in Z} t_w\mathbf{w} \\
&= \sum_{\mathbf{v}\in N}\left(\sum_{\mathbf{u}\in P} t_{uv}u_k\right)\mathbf{v} + \sum_{\mathbf{u}\in P}\left(\sum_{\mathbf{v}\in N} -t_{uv}v_k\right)\mathbf{u} + \sum_{\mathbf{w}\in Z} t_w\mathbf{w}
\end{aligned}
$$

This last line witness $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V)$.

Next, say $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V) \cap H_k^d$. Then $x_k = 0$, and

$$
\begin{aligned}
\mathbf{x} &= \sum_{\mathbf{v}\in P} t_v\mathbf{v} + \sum_{\mathbf{u}\in N} t_u\mathbf{u} + \sum_{\mathbf{w}\in Z} t_w\mathbf{w} \Rightarrow \\
0 &= \sum_{\mathbf{v}\in P} t_v v_k + \sum_{\mathbf{u}\in N} t_u u_k + \sum_{\mathbf{w}\in Z} t_w w_k \\
&= \sum_{\mathbf{v}\in P} t_v v_k + \sum_{\mathbf{u}\in N} t_u u_k
\end{aligned}
$$

This final line implies that the sums have opposite values. Denote this value by τ, that is

$$
\tau = \sum_{\mathbf{v}\in P} t_v v_k = -\sum_{\mathbf{u}\in N} t_u u_k
$$

Because of the way N and P are defined, we have that $0 \leq \tau$. We can now rewrite $\mathbf{x}$ as

$$
\begin{aligned}
\mathbf{x} &= \sum_{\mathbf{v}\in P} t_v\mathbf{v} + \sum_{\mathbf{u}\in N} t_u\mathbf{u} + \sum_{\mathbf{w}\in Z} t_w\mathbf{w} = \\
&= \frac{-1}{\tau}\sum_{\mathbf{u}\in N} t_u u_k \sum_{\mathbf{v}\in P} t_v\mathbf{v} + \frac{1}{\tau}\sum_{\mathbf{v}\in P} t_v v_k \sum_{\mathbf{u}\in N} t_u\mathbf{u} + \sum_{\mathbf{w}\in Z} t_w\mathbf{w} \\
&= \frac{-1}{\tau}\sum_{\substack{\mathbf{u}\in U \\ \mathbf{v}\in V}} t_u t_v u_k\mathbf{v} + \frac{1}{\tau}\sum_{\substack{\mathbf{u}\in U \\ \mathbf{v}\in V}} t_u t_v v_k\mathbf{u} + \sum_{\mathbf{w}\in Z} t_w\mathbf{w} \\
&= \sum_{\substack{\mathbf{u}\in U \\ \mathbf{v}\in V}} \frac{t_u t_v}{\tau}(v_k\mathbf{u} - u_k\mathbf{v}) + \sum_{\mathbf{w}\in Z} t_w\mathbf{w}
\end{aligned}
$$

This last line witnesses $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V')$. □

Proof of **Theorem 1**. Theorem 1 follows by first applying claim 1 to A, then successively applying claim 2 to each H_k^{d+m} for $d+1 \leq k \leq d+m$ to eliminate the intersection terms. □