The Minkowski-Weyl Theorem

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Outline

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V/H Polyhedra/Cones

Let $U \in \mathbb{R}^{d \times l}$, $V \in \mathbb{R}^{d \times m}$, $A \in \mathbb{R}^{m \times d}$

- V-Cone:
  $\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} \;\equiv\; \text{cone}(U)$
- V-Polytope:
  $\{V\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^T\boldsymbol{\lambda} = 1\} \;\equiv\; \text{conv}(V)$
- V-Polyhedron:
  $\{U\mathbf{t} + V\boldsymbol{\lambda} \mid \mathbf{t}, \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^T\boldsymbol{\lambda} = 1\} \;\equiv\; \text{cone}(U) + \text{conv}(V)$
- H-Cone:
  $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$
- H-Polyhedron:
  $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$

Minkowski-Weyl Theorem

- General Statement:
  V-Polyhedra and H-Polyhedra are the same things
- For Cones:
  V-Cones and H-Cones are the same things

First, the proof is done for cones, then polyhedra are reduced to cones.

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V-Cones are H-Cones

Basic idea:

- rewrite V-Cone as a projection of an H-Cone
- show that a projection of an H-Cone is an H-Cone

Say we are given $\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} \ldots$

- First, the non-negativity constraint

Say we are given $\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} \ldots$

- $\mathbf{t} \geq \mathbf{0} \qquad \Leftrightarrow \begin{pmatrix} \mathbf{0} & -I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0}$

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- Next, capture the points

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- $\mathbf{x} = U\mathbf{t} \quad \Leftrightarrow \quad \begin{pmatrix} I & -U \\ -I & U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0}$

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- Finally, project the combined constraints

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- $\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} \quad = \left\{ \mathbf{x} \;\middle|\; \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \right\}$

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The final expression is a projection of an H-Cone

We project an H-Cone one coordinate at a time with Fourier-Motzkin Elimination

- Observation:
$$\begin{pmatrix} A & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \Leftrightarrow A\mathbf{x} \leq \mathbf{0}$$
- Therefore:
$$\left\{ \mathbf{x} \,\middle|\, \begin{pmatrix} A & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \right\} = \{ \mathbf{x} \mid A\mathbf{x} \leq \mathbf{0} \}$$
- So, projecting **0** columns is easy.

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- So, projecting **0** columns is easy.

Suppose we are projecting column k from $A\mathbf{x} \leq \mathbf{0}$.

- First, partition the rows as follows:
  Z: rows with k-th column 0
  P: rows with k-th column positive
  N: rows with k-th column negative
- Create a new matrix A' with rows as follows:
  $$\begin{cases} A_z & A_z \in Z \\ A_p^k A_n - A_n^k A_p & A_n \in N, A_p \in P \end{cases}$$
- You can show:
  $(\exists t)\ A(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow A'\mathbf{x} \leq \mathbf{0}$

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H-Cones are V-Cones

Basic idea:

- rewrite V-Cone as a projection of intersection of an H-Cone with hyperplanes
- show that a intersection of a V-Cone with a hyperplane is a V-Cone
- projecting V-Cones is easy

Say we are given $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}\ldots$

- First, take up the slack

Say we are given $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}\ldots$

- $A\mathbf{x} \leq \mathbf{0} \quad \Leftrightarrow \quad \exists \mathbf{w} \geq \mathbf{0} \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$

Say we are given $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \ldots$

- $A\mathbf{x} \leq \mathbf{0} \qquad \Leftrightarrow \quad \exists \mathbf{w} \geq \mathbf{0} \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$
- Next, capture the points

Say we are given $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}\ldots$

- $A\mathbf{x} \leq \mathbf{0} \quad \Leftrightarrow \quad \exists \mathbf{w} \geq \mathbf{0} \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$
- $A\mathbf{x} + \mathbf{w} = \mathbf{0} \quad \Leftrightarrow \quad \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$

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- Finally, split the positive/negative contributions:

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- $A\mathbf{x} \leq \mathbf{0} \quad \Leftrightarrow \quad \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \operatorname{cone} \begin{pmatrix} I & -I & \mathbf{0} \\ A & -A & I \end{pmatrix}$

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The final expression is a intersection of a V-Cone with some hyperplanes

We intersect a V-Cone one coordinate at a time with **Dual-Fourier-Motzkin Elimination**

- We need to characterize elements of $\mathrm{cone}(U)$ with the k-th element 0.

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- We use the following algebraic trick...

We intersect a V-Cone one coordinate at a time with Dual-Fourier-Motzkin Elimination

- We need to characterize elements of $\text{cone}(U)$ with the k-th element 0.
- Suppose that $\mathbf{x}_i, \mathbf{x}_j \in \text{cone}(U)$, $x_j^k < 0 < x_i^k$, $\sum_i x_i^k + \sum_j x_j^k = 0$, and let $1/\sigma := \sum_i x_i^k = -\sum_j x_j^k$.
- We can rewrite $\sum_j \mathbf{x}_j + \sum_i \mathbf{x}_i = \sigma \sum_{i,j} x_i^k \mathbf{x}_j - x_j^k \mathbf{x}_i$
- Note that: $x_i^k \mathbf{x}_j - x_j^k \mathbf{x}_i \in \text{cone}(U) \cap \{\mathbf{x} \mid x^k = 0\}$

We intersect a V-Cone one coordinate at a time with Dual-Fourier-Motzkin Elimination

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Suppose we are intersecting $\{x_k = 0\}$ with $\mathrm{cone}(U)$.

- First, partition the columns as follows:
  Z: columns with k-th row 0
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Reducing Polyhedra to Cones

- For H-Polyhedra, move constraints to column 0 and set that coordinate to 1
- $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; [-\mathbf{b}|A] \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} \cap \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; x_0 = 1 \right\}$
- For V-Polyhedra, use an intersection to enforce convex combinations
- $\text{cone}(U) + \text{conv}(V) = \Pi \left(\text{cone} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ U & V \end{pmatrix} \cap \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; x_0 = 1 \right\} \right)$

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Reducing Polyhedra to Cones

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- This intersection is a slight complication, but handled in essentially the same way as before. The basic idea:
- Denote $\sigma_i = \sum_i x_i^0$, $\sigma_j = \sum_j x_j^0$
- If $\sigma_i + \sigma_j = 1$, then $-\frac{\sigma_j}{\sigma_i} = 1 - \frac{1}{\sigma_i}$
- Then $\sum_i \mathbf{x}_i = \frac{\sum_i \mathbf{x}_i}{\sigma_i} + \left(1 - \frac{1}{\sigma_i}\right)\sum_i \mathbf{x}_i = \frac{1}{\sigma_i}\sum_i \mathbf{x}_i - \frac{\sigma_j}{\sigma_i}\sum_i \mathbf{x}_i$
- We can rewrite $\sum_j \mathbf{x}_j + \sum_i \mathbf{x}_i = \frac{1}{\sigma_i}\sum_i \mathbf{x}_i + \frac{1}{\sigma_i}\sum_{i,j} x_i^0 \mathbf{x}_j - x_j^0 \mathbf{x}_i$
- The left term is in $\text{conv}\begin{pmatrix} \mathbf{1} \\ V \end{pmatrix}$, while the right term is in the cone but has 0 in 0-th coordinate.

Reducing Polyhedra to Cones

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- Then $\sum_i \mathbf{x}_i = \frac{\sum_i \mathbf{x}_i}{\sigma_i} + \left(1 - \frac{1}{\sigma_i}\right) \sum_i \mathbf{x}_i = \frac{1}{\sigma_i} \sum_i \mathbf{x}_i - \frac{\sigma_j}{\sigma_i} \sum_i \mathbf{x}_i$
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Reducing Polyhedra to Cones

$$\mathrm{cone}(U) + \mathrm{conv}(V) = \Pi\left(\mathrm{cone}\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ U & V \end{pmatrix} \cap \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; x_0 = 1 \right\}\right)$$

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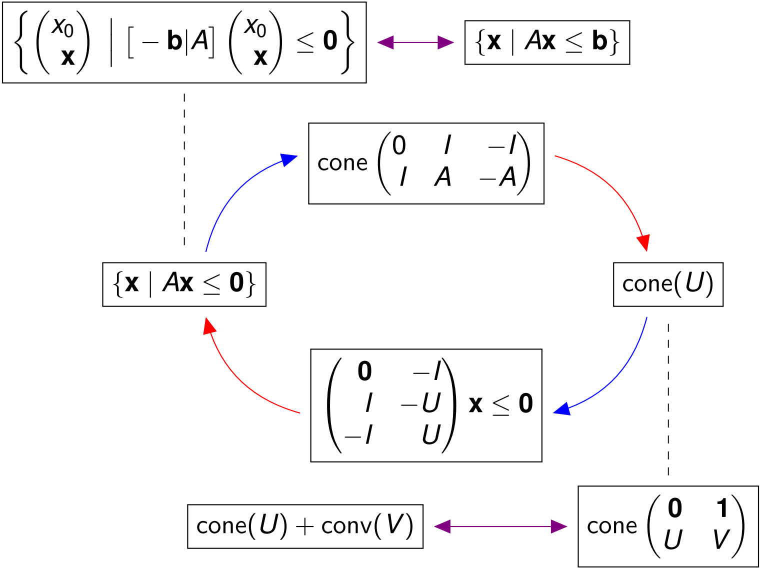


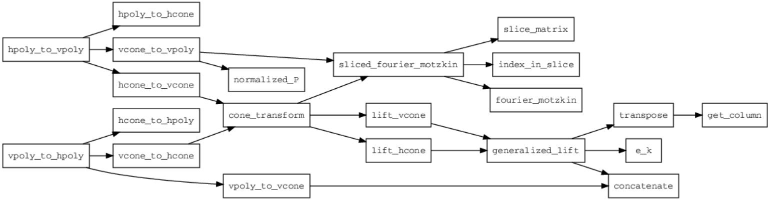
Figure: Diagram of the proof $P_H \leftrightarrow P_V$

Outline

Files and Includes

file	includes
linear_algebra.h	<C++ standard library>
fourier_motzkin.h	linear_algebra.h
polyhedra.h	fourier_motzkin.h
main.cpp	polyhedra.h
test_functions.h	linear_alebra.h
test.cpp	test_functions.h, polyhedra.h

Callgraph


hpoly_to_hcone
hpoly_to_vpoly
vcone_to_vpoly
hcone_to_vcone
normalized_P
sliced_fourier_motzkin
slice_matrix
index_in_slice
fourier_motzkin
cone_transform
lift_vcone
lift_hcone
hcone_to_hpoly
vpoly_to_hpoly
vcone_to_hcone
generalized_lift
transpose
get_column
e_k
vpoly_to_vcone
concatenate

Matrix fourier_motzkin(Matrix,k)

```
const auto z_end = partition(M.begin(), M.end(),
    [k](const Vector &v) { return v[k] == 0; });
const auto p_end = partition(z_end, M.end(),
    [k](const Vector &v) { return v[k] > 0; });
```

Partition `M` into logical sets Z, P, N that satisfy the following:

set	range	property
Z	[`M.begin()`, `z_end`)	`it` $\in Z \Leftrightarrow$ `(*it)[k]` $= 0$
P	[`z_end`, `p_end`)	`it` $\in P \Leftrightarrow$ `(*it)[k]` > 0
N	[`p_end`, `M.end()`)	`it` $\in N \Leftrightarrow$ `(*it)[k]` < 0

Matrix fourier_motzkin(Matrix,k)

```
for (auto p_it = z_end; p_it != p_end; ++p_it) {
  for (auto n_it = p_end; n_it != M.end(); ++n_it) {
    result.push_back(
      (*p_it)[k]*(*n_it) - (*n_it)[k]*(*p_it));
  }
}
```

This function creates the sets which correspond to:

$$\left\{ B_i^k B_j - B_j^k B_i \mid i \in P, j \in N \right\}, \quad \left\{ Y_k^i Y^j - Y_k^j Y^i \mid i \in P, j \in N \right\}$$

Outline

Pointed/ Full-Dimensional Polyhedra

- These are basically non-degeneracy constraints.
  - Pointed Polyhedra have at least one vertex
  - Full dimensional polyhedra contain an affine independent set
- Pointed V-Polyhedra and Full-Dimensional H-Polyhedra have “essentially unique” sets of generators
- These “essentially unique” sets make it easy to test for equivalence
- The characterizations are similar to “linear independence”

Pointed/ Full-Dimensional Polyhedra

- These "essentially unique" sets are "minimal"
- A set V is called *minimal* for $\text{cone}(V)$ if

$$(\forall \mathbf{v} \in V) \;\; \text{cone}(V \setminus \{\mathbf{v}\}) \subset \text{cone}(V)$$

- A set A is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ if

$$(\forall A_i \in A) \;\; \{\mathbf{x} \mid A \setminus \{A_i\}\,\mathbf{x} \leq \mathbf{0}\} \supset \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$

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- Suppose we have an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, know that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \mathrm{cone}(V)$, and would like to test if $\mathrm{cone}(V) = \mathrm{cone}(V')$

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\begin{aligned}
AV' \leq \mathbf{0} &\Rightarrow \mathrm{cone}(V') \subseteq \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \\
V \subseteq V' &\Rightarrow \mathrm{cone}(V) \subseteq \mathrm{cone}(V') \\
\mathrm{cone}(V') = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} &\Rightarrow AV' \leq \mathbf{0} \\
\mathrm{cone}(V') = \mathrm{cone}(V) &\overset{?}{\Rightarrow} V \subseteq V'
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- The last item would create an equivalence
  - Must relax "$\subseteq$" (vectors vs rays)
  - Requires notion of "essentially unique" representation

Testing Methods

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Outline

Pointed V-Cones

- The following statements are equivalent.
  1. $\mathrm{cone}(V)$ is pointed.
  2. $\mathbf{t} \geq \mathbf{0},\ [V\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}]$
- A set V is called *minimal* for $\mathrm{cone}(V)$ if: $(\forall \mathbf{v} \in V)\ \mathrm{cone}(V \setminus \{\mathbf{v}\}) \subset \mathrm{cone}(V)$
- Suppose that $\mathrm{cone}(V)$ is pointed. Then the following two statements are equivalent:
  1. V is minimal
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Full-Dimensional H-Cones

- All definitions are nearly identical, but we use the Farkas' lemma to consider the "dual cone"
- Farkas Lemma: Let $U \in \mathbb{R}^{d \times n}$. Precisely one of the following is true:

$$(\exists \mathbf{t} \geq \mathbf{0}) : \mathbf{x} = U\mathbf{t}$$
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- i.e. a point is contained in a cone or can be separated from it with a hyperplane
- The Farkas Lemma can be used to prove the following:

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \text{cone}(A^T) = \text{cone}(A'^T)$$

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Outline

- General Polyhedra are decomposed into a "characteristic-cone" and polytope
- Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \leq b\} = \text{cone}(U) + \text{conv}(V)$. Then the following three statements are equivalent:
  1. $A\mathbf{r} \leq \mathbf{0}$
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- Note that (2) in the proof above is independent of A and U.

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- In $P = \text{cone}(U) + \text{conv}(V)$ we need U to be minimal as for V-Cones
- Minimality for the set $\text{conv}(V)$ – a polytope – is given by the vertex set
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- If $\mathbf{v}$ is a vertex of $\mathrm{cone}(U) + \mathrm{conv}(V)$, then $[\mathbf{v} = U\mathbf{t} + V\lambda] \Rightarrow \mathbf{t} = \mathbf{0}$ (I call $\mathbf{v}$ here *U-free*)
- If $\mathbf{v}$ is a vertex of $\mathrm{cone}(U) + \mathrm{conv}(V)$, then $\mathbf{v}$ is a vertex of $\mathrm{conv}(V)$
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  1. (U, V) is minimal for P
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Full-Dimensional H-Polyhedra

- We need another form of the Farkas Lemma:

$$(\exists \mathbf{t} \geq \mathbf{0})\mathbf{t}^T A = \mathbf{y},\ \mathbf{t}^T\mathbf{b} \leq c \Leftrightarrow \begin{cases} (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{y}^T\mathbf{x} \leq 0 \textbf{ and} \\ (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{b} \Rightarrow \mathbf{y}^T\mathbf{x} \leq c \end{cases}$$

- Basically, a constraint is valid for a polyhedron if and only if it is a non-negative combination of rows of constraints (plus some change)
- If $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full dimensional, and $\mathbf{y}^T A = \mathbf{0}$ with $\mathbf{y} \geq \mathbf{0}$, then either $\mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T\mathbf{b} > \mathbf{0}$.
- $\mathbf{y}^T A = \mathbf{0},\ \mathbf{y}^T\mathbf{b} > \mathbf{0}$ occurs when two of the bounding hyperplanes are parallel

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Dual Homogenization Cone

- For H-Polyhedra switching to the dual setting is slightly more complicated
- The following two statement are equivalent:
  1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$
  2. $\text{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} = \text{cone}\begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix}$
- If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is minimal and full-dimensional, then either
  1. $\text{cone}\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ is minimal and pointed, or
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Pointed H-Polyhedra and Full-Dimensional V-Polyhedra

- $\mathbf{t}$ is a non-negative vector, $V \neq \emptyset$, and abbreviate linear-independent as LI. $\bar{V}$ denotes $\{\mathbf{v} - \mathbf{v}' \mid \mathbf{v}, \mathbf{v}' \in V\}$.

	Pointed	Full-Dimensional
$\mathrm{cone}(U)$	$U\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$	d LI vectors in U
$\mathrm{cone}(U) + \mathrm{conv}(V)$	$U\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$	d LI vectors in $U \cup \bar{V}$
$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$	d LI row vectors in A	$\mathbf{t}^T A = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}$
$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$	d LI row vectors in A	$\mathbf{t}^T A = \mathbf{0} \Rightarrow \mathbf{t}^T \mathbf{b} > 0$

Not much to see here

- The proofs are more tedious than enlightening
- The implementation is straightforward

Summary

- A V-Polyhedron could easily be represented as an H-Polyhedron with an infinite number of constraints
- The Minkowski-Weyl Theorem tells us that a finite number are enough (an “obvious” fact)
- The Farkas Lemma is a nice combinatorial result that encapsulates this fact
- Pointed and Full-Dimensional polyhedra are “non-degenerate” in some sense that have even better properties for determining their representations

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Further Considerations

- Better algorithms (Dual-Description)
- Other interpretations...
  - ...Systems of logical deduction
  - ...Systems where “lift and drop” creates a dual representation
- Are there any useful implications for “polyhedra complexes” or “chains”?