1 Algebraic Formula

1.1 Expressions

Here is an incredibly useful algebraic derivation for dealing with polyhedra. Let $\alpha_i, \beta_j \in \mathbb{R}$, and $\mathbf{a}_i, \mathbf{b}^j$ be from some vector space. Then

$$\begin{aligned}
&\sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_j \beta_j \mathbf{b}^j \\
&= \left(1 + \frac{\sum_j \beta_j}{\sum_i \alpha_i}\right) \sum\nolimits_i \alpha_i \mathbf{a}^i - \frac{\sum_j \beta_j}{\sum_i \alpha_i} \sum\nolimits_i \alpha_i \mathbf{a}^i + \frac{\sum_i \alpha_i}{\sum_i \alpha_i} \sum\nolimits_j \beta_j \mathbf{b}^j \\
&= \left(1 + \frac{\sum_j \beta_j}{\sum_i \alpha_i}\right) \sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_{i,j} \frac{\alpha_i \beta_j}{\sum_i \alpha_i} (\mathbf{b}^j - \mathbf{a}^i)
\end{aligned}$$

If we let $\alpha = \sum_i \alpha_i$, and $\beta = \sum_j \beta_j$, then this can be written more legibly:

$$\sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_j \beta_j \mathbf{b}^j = \left(1 + \frac{\beta}{\alpha}\right) \sum\nolimits_i \alpha_i \mathbf{a}^i + \sum\nolimits_{i,j} \frac{\alpha_i \beta_j}{\alpha} (\mathbf{b}^j - \mathbf{a}^i)$$

A seemingly awkward but useful specialization of this identity is shown in the following proposition:

Proposition 1. *Suppose that* $(\forall \mathbf{u} \in N) f(\mathbf{u}) \neq 0$, $(\forall \mathbf{v} \in P) f(\mathbf{v}) \neq 0$. *Let* $\sigma_P = \sum_{\mathbf{v} \in P} t_v f(\mathbf{v})$, $\sigma_N = \sum_{\mathbf{u} \in N} t_u f(\mathbf{u})$. *Then*

$$\sum_{\mathbf{v} \in P} t_v \mathbf{v} + \sum_{\mathbf{u} \in N} t_u \mathbf{u} = \left(1 + \frac{\sigma_N}{\sigma_P}\right) \sum_{\mathbf{v} \in P} t_v \mathbf{v} + \sum_{\substack{\mathbf{u} \in N \\ \mathbf{v} \in P}} \frac{t_v t_u}{\sigma_P} (f(\mathbf{v})\mathbf{u} - f(\mathbf{v})\mathbf{v})$$

Proof. If in the above formula we let $\alpha_v = t_v f(\mathbf{v})$, $\mathbf{a}^v = \mathbf{v}/f(\mathbf{v})$, $\beta_u = t_u f(\mathbf{v})$, and $\mathbf{b}^u = \mathbf{u}/f(\mathbf{v})$, we get

$$\begin{aligned}
&\sum_{\mathbf{v} \in P} t_v \mathbf{v} + \sum_{\mathbf{u} \in N} t_u \mathbf{u} = \\
&= \sum_{\mathbf{v} \in P} (t_v f(\mathbf{v}))(\mathbf{v}/f(\mathbf{v})) + \sum_{\mathbf{u} \in N} (t_u f(\mathbf{v}))(\mathbf{u}/f(\mathbf{v})) \\
&= \sum_{\mathbf{v} \in P} \alpha_v \mathbf{a}^v + \sum_{\mathbf{u} \in N} \beta_v \mathbf{b}^u = \\
&= \left(1 + \frac{\beta}{\alpha}\right) \sum_{\mathbf{v} \in P} \mathbf{a}^v + \sum_{\substack{\mathbf{u} \in N \\ \mathbf{v} \in P}} \frac{\alpha_v \beta_u}{\alpha} (\mathbf{b}^u - \mathbf{a}^v) \\
&= \left(1 + \frac{\sigma_N}{\sigma_P}\right) \sum_{\mathbf{v} \in P} t_v \mathbf{v} + \sum_{\substack{\mathbf{u} \in N \\ \mathbf{v} \in P}} \frac{t_v t_u}{\sigma_P} (f(\mathbf{v}) f(\mathbf{u}))(\mathbf{u}/f(\mathbf{u}) - \mathbf{v}/f(\mathbf{v})) \\
&= \left(1 + \frac{\sigma_N}{\sigma_P}\right) \sum_{\mathbf{v} \in P} t_v \mathbf{v} + \sum_{\substack{\mathbf{u} \in N \\ \mathbf{v} \in P}} \frac{t_v t_u}{\sigma_P} (f(\mathbf{v})\mathbf{u} - f(\mathbf{v})\mathbf{v})
\end{aligned}$$

□

1.2 Functions

The following formulas will be useful for eliminating redundant information concerning sets of linear functions and their arguments. They are proved here in a somewhat more general form than will be required later.

Definition 1 (Functional)**.** *A real valued linear function f defined on a vector space is called a* functional.

In this paper, functionals can mostly be thought of as row vectors (dual vectors).

Definition 2 (Split-Sets)**.** *Let F be a set of functionals with $f \in F$, and V a set of vectors with $\mathbf{v} \in V$, define:*

$$\begin{aligned} P(F, \mathbf{v}) &= f \in F \mid (\forall v \in V) f(\mathbf{v}) > 0 \\ N(F, \mathbf{v}) &= f \in F \mid (\forall v \in V) f(\mathbf{v}) < 0 \\ Z(F, \mathbf{v}) &= f \in F \mid (\forall v \in V) f(\mathbf{v}) = 0 \end{aligned}$$

$$\begin{aligned} P(f, V) &= \mathbf{v} \in V \mid (\forall f \in F) f(\mathbf{v}) > 0 \\ N(f, V) &= \mathbf{v} \in V \mid (\forall f \in F) f(\mathbf{v}) < 0 \\ Z(f, V) &= \mathbf{v} \in V \mid (\forall f \in F) f(\mathbf{v}) = 0 \end{aligned}$$

The sets defined by P, N, and Z partition either F or V, depending on the object of focus. If f is a row vector, then $Z(f, V)$ is all the vectors from V orthogonal to f. Similarly, if F are all row vectors, then $Z(F, \mathbf{v})$ are all vectors from F orthogonal to $\mathbf{v}$.

Definition 3 (Elimination Set)**.** *Given a set F of functionals and a set V of arguments, let $f \in F$ and $\mathbf{v} \in V$.define the following operator $*$ on sets:*

$$\begin{aligned} P(f, V) * N(f, V) &= \{ f(\mathbf{v})\mathbf{u} - f(\mathbf{u})\mathbf{v} \mid \mathbf{v} \in P(f, V), \quad \mathbf{u} \in N(f, V) \} \\ P(F, \mathbf{v}) * N(F, \mathbf{v}) &= \{ f(\mathbf{v})g - g(\mathbf{v})f \mid f \in P(F, \mathbf{v}), \quad g \in N(F, \mathbf{v}) \} \end{aligned}$$

Note that $P(f, V) * N(f, V)$ maps a set of vectors to a set of vectors who lie in the kernel of f, while $P(F, \mathbf{v}) * P(F, \mathbf{v})$ maps a set of functionals to a set of functionals for which $\mathbf{v}$ is a member of their common kernels. If F is a set of row vectors, then the term "kernel" can be replaced with "orthogonal subspace."

This next proposition shows us that we may recover some information about the original sets P and N after we've "eliminated" $\mathbf{v}$ from them.

Proposition 2. *Let $P = P(F, \mathbf{v})$ and $N = N(F, \mathbf{v})$, then the following claim holds for any $\mathbf{x}$ of the domain of F:*

$$(\forall h \in P * N)\, h(\mathbf{x}) \leq 0 \Rightarrow (\exists t)(\forall f \in P \cup N)\, f(\mathbf{x} + t\mathbf{v}) \leq 0$$

Proof. The task is to find a t such that $(\forall f \in P \cup N) f(\mathbf{x} - t\mathbf{v}) \leq 0$. We have:

$$\begin{aligned} &(\forall f \in P, \forall g \in N) \quad f(\mathbf{v})g(\mathbf{x}) \quad - g(\mathbf{v})f(\mathbf{x}) \leq 0 \Rightarrow \\ &(\forall f \in P, \forall g \in N) \quad f(\mathbf{v})g(\mathbf{x}) \quad \leq g(\mathbf{v})f(\mathbf{x}) \\ &(\forall f \in P, \forall g \in N) \quad f(\mathbf{x})/f(\mathbf{v}) \leq g(\mathbf{x})/g(\mathbf{v}) \end{aligned}$$

Note that in the last line the inequality changes direction because $g(\mathbf{v}) < 0$. The last line also means that

$$\sup_{f\in P} f(\mathbf{x})/f(\mathbf{v}) \le \inf_{g\in Z} g(\mathbf{x})/g(\mathbf{v})$$

To see this, denote $F_f = \sup_{f\in P} f(\mathbf{x})/f(\mathbf{v})$ and $F_g = \inf_{g\in Z} g(\mathbf{x})/g(\mathbf{v})$, and suppose that $F_f > F_g$. Then there is an $\epsilon > 0$ such that $F_f - \epsilon > F_g + \epsilon$, but then

$$\begin{array}{ll} \exists f \in P & |f(\mathbf{x})/f(\mathbf{v}) - F_f| < \epsilon \Rightarrow F_f - \epsilon < f(\mathbf{x})/f(\mathbf{v}) \\ \exists g \in N & |g(\mathbf{x})/g(\mathbf{v}) - F_g| < \epsilon \Rightarrow F_g + \epsilon > g(\mathbf{x})/g(\mathbf{v}) \end{array}$$

This implies that $g(\mathbf{x})/g(\mathbf{v}) < f(\mathbf{x})/f(\mathbf{v})$, a contradiction.

Then, let $t \in [F_f, F_g]$. We have

$$\begin{array}{lrcll} (\forall f \in P) & f(\mathbf{x})/f(\mathbf{v}) & \le & t & \Rightarrow \\ (\forall f \in P) & f(\mathbf{x}) & \le & tf(\mathbf{v}) & \Rightarrow \\ (\forall f \in P) & f(\mathbf{x}) - tf(\mathbf{v}) & \le & 0 & \Rightarrow \\ (\forall f \in P) & f(\mathbf{x} - t\mathbf{v}) & \le & 0 & \end{array}$$

Similarly,

$$\begin{array}{lrcll} (\forall g \in P) & g(\mathbf{x})/g(\mathbf{v}) & \ge & t & \Rightarrow \\ (\forall g \in P) & g(\mathbf{x}) & \le & tg(\mathbf{v}) & \Rightarrow \\ (\forall g \in P) & g(\mathbf{x}) - tg(\mathbf{v}) & \le & 0 & \Rightarrow \\ (\forall g \in P) & g(\mathbf{x} - t\mathbf{v}) & \le & 0 & \end{array}$$

So this t accomplishes the task $(\forall f \in P \cup N) f(\mathbf{x} - t\mathbf{v}) \le 0$.

□

The following proposition tells us that the elimination procedure is "sound." We may reason about some set of functionals after eliminating $\mathbf{v}$ from them, because some useful property is maintained.

Proposition 3. *Let F be a set of linear functionals, and let $P = P(F, \mathbf{v})$, $N = N(F, \mathbf{v})$, and $Z = Z(F, \mathbf{v})$. Then we have:*

$$(\exists t)(\forall f \in F) f(\mathbf{x} - t\mathbf{v}) \le 0 \Leftrightarrow (\forall f \in Z \cup P * N) f(\mathbf{x}) \le 0$$

Proof. "$\Rightarrow$" If $g \in N$ and $f \in P$, then observe:

$$\begin{array}{l} f(\mathbf{x}) \le 0 \Rightarrow \quad f(\mathbf{v}) f(\mathbf{x}) \le 0 \\ f(\mathbf{x}) \le 0 \Rightarrow -g(\mathbf{v}) f(\mathbf{x}) \le 0 \end{array}$$

Combining these, we see that

$$f(\mathbf{x}) \le 0 \Rightarrow f(\mathbf{v}) f(\mathbf{x}) - g(\mathbf{v}) f(\mathbf{x}) \le 0$$

Suppose that $(\forall f \in F) f(\mathbf{x} - t\mathbf{v}) \leq 0$. Then if $f \in Z$ then:

$$f(\mathbf{x} - t\mathbf{v}) = f(\mathbf{x}) + tf(\mathbf{v}) = f(\mathbf{x}) \leq 0$$

If $h \in P * N$, then $h = f(\mathbf{v})g - g(\mathbf{v})f$ for some $f \in P, g \in N$. Then

$$\begin{aligned} h(\mathbf{x} - t\mathbf{v}) &= (f(\mathbf{v})g - g(\mathbf{v})f)(\mathbf{x} - t\mathbf{v}) \\ &= f(\mathbf{v})g(\mathbf{x} - t\mathbf{v}) - g(\mathbf{v})f(\mathbf{x} - t\mathbf{v}) \\ &= tf(\mathbf{v})g(\mathbf{v}) - tg(\mathbf{v})f(\mathbf{v}) + f(\mathbf{v})g(\mathbf{x}) - g(\mathbf{v})f(\mathbf{x}) \\ &= f(\mathbf{v})g(\mathbf{x}) - g(\mathbf{v})f(\mathbf{x}) \leq 0 \end{aligned}$$

"$\Leftarrow$" Next suppose that $(\forall f \in Z \cup P * N) f(\mathbf{x}) \leq 0$. If $f \in Z$ then:

$$f(\mathbf{x} - t\mathbf{v}) = f(\mathbf{x}) + tf(\mathbf{v}) = f(\mathbf{x}) \leq 0$$

Since functions from Z will be satisfied with whatever t we choose, we can apply proposition 2. $\square$

The next proposition is quite similar, but now instead of functions being the focus, vectors are.

Proposition 4. *Let V be a set of vectors, $P = P(F, \mathbf{v})$, $N = N(F, \mathbf{v})$, $Z = Z(F, \mathbf{v})$, $V' = P * N \cup Z$, $\bar{P} = \{\mathbf{v}/f(\mathbf{v}) \mid \mathbf{v} \in P\}$, and $r > 0$. Then*

$$(\exists t_v \geq 0) \left[\mathbf{x} = \sum_{\mathbf{v} \in V} t_v \mathbf{v},\ f(\mathbf{x}) = r \right] \Leftrightarrow$$

$$\left(\exists t_v \geq 0, \lambda_v \geq 0, \sum_{\mathbf{v} \in P} \lambda_v = r \right) \left[\mathbf{x} = \sum_{\mathbf{v} \in V'} t_v \mathbf{v} + \sum_{\mathbf{v} \in \bar{P}} \lambda_v \mathbf{v} \right]$$

Proof. "$\Rightarrow$" Let $\sigma_N = \sum_{\mathbf{u} \in N} t_u f(\mathbf{u})$, and $\sigma_P = \sum_{\mathbf{v} \in P} t_v f(\mathbf{v})$. Observe that $\mathbf{x}$ can be written:

$$\mathbf{x} = \sum_{\mathbf{u} \in N} t_u \mathbf{u} + \sum_{\mathbf{v} \in P} t_v \mathbf{v} + \sum_{\mathbf{w} \in Z} t_w \mathbf{w}$$

Then we get

$$f(\mathbf{x}) = r = \sum_{\mathbf{u} \in N} t_u f(\mathbf{u}) + \sum_{\mathbf{v} \in P} t_v f(\mathbf{v}) + \sum_{\mathbf{w} \in Z} t_w f(\mathbf{w}) = \sum_{\mathbf{u} \in N} t_u f(\mathbf{u}) + \sum_{\mathbf{v} \in P} t_v f(\mathbf{v})$$

This implies that $\sigma_P + \sigma_N = r$, and $1 + \sigma_N / \sigma_P = \frac{r}{\sigma_P}$. By proposition 1, $\mathbf{x}$ can be rewritten:

$$\begin{aligned} \mathbf{x} &= \sum_{\mathbf{v} \in P} \frac{t_v r}{\sigma_P} \mathbf{v} + \sum_{\substack{\mathbf{u} \in N \\ \mathbf{v} \in P}} \frac{t_u t_v}{\sigma_P} (f(\mathbf{v})\mathbf{u} - f(\mathbf{u})\mathbf{v}) + \sum_{\mathbf{w} \in Z} t_w \mathbf{w} \\ &= \sum_{\mathbf{v} \in P} \frac{t_v r f(\mathbf{v})}{\sigma_P} \frac{\mathbf{v}}{f(\mathbf{v})} + \sum_{\substack{\mathbf{u} \in N \\ \mathbf{v} \in P}} \frac{t_u t_v}{\sigma_P} (f(\mathbf{v})\mathbf{u} - f(\mathbf{u})\mathbf{v}) + \sum_{\mathbf{w} \in Z} t_w \mathbf{w} \end{aligned}$$

Note that $f(\mathbf{v})/\mathbf{v} \in \bar{P}$. Now observe that

$$\sum_{\mathbf{v}\in\bar{P}} \frac{t_v r f(\mathbf{v})}{\sigma_P} = r\frac{\sigma_P}{\sigma_P} = r$$

This shows that $\mathbf{x}$ is written as was to be shown.

"$\Leftarrow$" Let $\mathbf{x}$ be written:

$$\mathbf{x} = \sum_{\mathbf{v}\in\bar{P}} \lambda_v \mathbf{v} + \sum_{\substack{\mathbf{u}\in N\\ \mathbf{v}\in P}} t_{uv}(f(\mathbf{v})\mathbf{u} - f(\mathbf{u})\mathbf{v}) + \sum_{\mathbf{w}\in Z} t_w \mathbf{w}$$

First, we rewrite $\mathbf{v} \in \bar{P}$:

$$\sum_{\mathbf{v}\in\bar{P}} \lambda_v \mathbf{v} = \sum_{\mathbf{v}\in P} \frac{\lambda_v}{f(\mathbf{v})}\mathbf{v}$$

Also, rewrite

$$\begin{aligned}
&\sum_{\substack{\mathbf{u}\in N\\ \mathbf{v}\in P}} t_{uv}(f(\mathbf{v})\mathbf{u} - f(\mathbf{u})\mathbf{v}) \\
&= \sum_{\substack{\mathbf{u}\in N\\ \mathbf{v}\in P}} t_{uv} f(\mathbf{v})\mathbf{u} - \sum_{\substack{\mathbf{u}\in N\\ \mathbf{v}\in P}} t_{uv} f(\mathbf{u})\mathbf{v} \\
&= \sum_{\mathbf{u}\in N} \left(\sum_{v\in P} t_{uv} f(\mathbf{v})\right)\mathbf{u} + \sum_{\mathbf{v}\in P}\left(\sum_{u\in N} -t_{uv} f(\mathbf{u})\right)\mathbf{v}
\end{aligned}$$

Denote $\sum_{v\in P} t_{uv} f(\mathbf{v})$ as α_v, and $\sum_{u\in N} -t_{uv} f(\mathbf{u})$ as α_u. Note that $\alpha_u, \alpha_v \geq 0$. Then $\mathbf{x}$ can be rewritten

$$\begin{aligned}
x &= \sum_{\mathbf{v}\in P} \frac{\lambda_v}{f(\mathbf{v})}\mathbf{v} + \sum_{\mathbf{v}\in P} \alpha_v \mathbf{v} + \sum_{\mathbf{u}\in U} \alpha_u \mathbf{u} + \sum_{\mathbf{w}\in Z} t_w \mathbf{w} \\
&= \sum_{\mathbf{v}\in P} (\frac{\lambda_v}{f(\mathbf{v})} + \alpha_v)\mathbf{v} + \sum_{\mathbf{u}\in U} \alpha_u \mathbf{u} + \sum_{\mathbf{w}\in Z} t_w \mathbf{w}
\end{aligned}$$

This shows $\mathbf{x}$ as a positive linear combination of vectors from V, completing the proof. □

Finally, we note a simple specialization of proposition 5.

Proposition 5. *Let V be a set of vectors, $P = P(F, \mathbf{v})$, $N = N(F, \mathbf{v})$, $Z = Z(F, \mathbf{v})$, and $V' = P * N \cup Z$. Then*

$$(\exists t_v \geq 0)\left[\mathbf{x} = \sum_{\mathbf{v}\in V} t_v \mathbf{v},\ f(\mathbf{x}) = 0\right] \Leftrightarrow (\exists t_v \geq 0)\left[\mathbf{x} = \sum_{\mathbf{v}\in V'} t_v \mathbf{v}\right]$$

Proof. This follows immediately from proposition 4. □

2 Preliminaries

2.1 Notation

The canonical basis vectors will be written $\mathbf{e}_k$ for valid values of k. Let $\mathbf{x} \in \mathbb{R}^n$. It will be customary to write:

$$x_k = \langle \mathbf{e}_k, \mathbf{x} \rangle$$

Given $A \in \mathbb{R}^{m \times d}$, Let A_i and A^j denote the rows and columns of A, respectively. Then A_i^j will denote the entry from A in the i-th row and j-th column. Matrix multiplication is then given by:

$$A\mathbf{x} = \sum_{j=1}^{d} A^j x_j = \begin{pmatrix} \langle A_1, \mathbf{x} \rangle \\ \vdots \\ \langle A_m, \mathbf{x} \rangle \end{pmatrix} = \sum_{i=1}^{m} \langle A_i, \mathbf{x} \rangle \, \mathbf{e}_i$$

This shows a matrix can either be considered as a collection of vectors, with multiplication combining them, or a collection of functionals, with multiplication applying them.

Let $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^m$, and define the following notation for vectors in $\mathbb{R}^{d+m}$:

$$\left\langle \mathbf{e}_k, \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \right\rangle = \begin{cases} x_k & k \leq d \\ w_{k-d} & d+1 \leq k \leq d+m \end{cases}$$

This is standard "concatenation" procedure.

2.2 Definitions

Definition 4 (H-Cone)**.** *Let $A \in \mathbb{R}^{m \times d}$, then define*

$$\mathcal{C}_{\mathcal{H}}(A) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$$

Definition 5 (V-Cone)**.** *Let $V \in \mathbb{R}^{d \times n}$, then define*

$$\mathcal{C}_{\mathcal{V}}(V) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} \in \mathbb{R}^n \geq \mathbf{0}, \, \mathbf{x} = V\mathbf{t} \right\}$$

A vector of the form $V\mathbf{t}$, where $\mathbf{t} \geq \mathbf{0}$ is called a "conical combination of V".

Definition 6 (Hyperplane)**.** *Let $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$. Then the set*

$$\{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{a}, \mathbf{x} \rangle = b\}$$

Is known as a hyperplane. Let $H_k^n \subseteq \mathbb{R}^n$ denote the hyperplane defined by:

$$H_k^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_k = 0\}$$

Definition 7 (Projection)**.** *Let $\mathbf{x} \in \mathbb{R}^d$. The vector $\mathbf{x}' \in \mathbb{R}^{d-k}$ formed by omitting k coordinates of $\mathbf{x}$ is called a projection of $\mathbf{x}$. In particular, let $\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^m$, and $\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{w} \end{smallmatrix}\right) \in \mathbb{R}^{d+m}$, and define $\pi_{\mathbf{x}} : \mathbb{R}^{d+m} \to \mathbb{R}^d$ as:*

$$\pi_{\mathbf{x}} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} = \mathbf{x}$$

Then $\pi_{\mathbf{x}}$ is a projection onto the first d-coordinates.

$$
\begin{tikzcd}
\boxed{A\mathbf{x} \leq \mathbf{0}} \arrow[d, "\text{lift}"'] & & \boxed{\mathcal{C}_{\mathcal{V}}(V)} \\
\boxed{\mathcal{C}_{\mathcal{V}}\{\pm\left(\begin{smallmatrix}\mathbf{e}_i \\ A^i\end{smallmatrix}\right) \cup \left(\begin{smallmatrix}0 \\ \mathbf{e}_j\end{smallmatrix}\right)\}} \arrow[rr, "\bigcap_{d+1}^{d+m} H_k^{d+m}", "\text{drop}"'] & & \boxed{\{\mathbf{x} : \left(\begin{smallmatrix}\mathbf{x} \\ \mathbf{0}\end{smallmatrix}\right) \in \mathcal{C}_{\mathcal{V}}(V'')\}} \arrow[u, "\pi_{\mathbf{x}}", "\text{project}"']
\end{tikzcd}
$$

Figure 1: Diagram of the proof ($\mathcal{C}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{V}'} \to \mathcal{C}_{\mathcal{V}}$)

Remarks: In the following sections it will be proved that every **H-Cone** is a **V-Cone**, and every **V-Cone** is an **H-Cone**. This shows that these are two fundamentally different descriptions of the same object. Each has its own power and purpose, which will be discussed later.

Let $A \subseteq \mathbb{R}^n$. Then $A \cap H_k^n$ is simply all of the points of A who have the k-th coordinate 0. Let π^k be the projection that omits the k-th coordinate. Then $\pi^k(A \cap H_k^n)$ are all the points of A whose k-th coordinate is 0, but without that 0 coordinate. If projections are thought of as "forgetting useless information," and intersections as "capturing only the useful information," then the sequence of projection and intersection is "capturing the useful information, and forgetting the useless information."

3 H-Cone $\to$ V-Cone

This section gives a proof that every H-Cone is also a V-Cone.

3.1 Introduction.

Discussing "what is hard" is helpful for understanding the proof and why it is formed the way it is. Proposition 6 (lift) is not terribly difficult, it mostly just takes a clever idea and attention to detail. It is reminiscent of techniques common to linear programming. Propositions **??**, **??**, and 10 (project) are very straightforward, they exist primarily to make the proofs of the other propositions less cluttered. Proposition **??** (drop) is really the "main" idea that needs proving. This is because intersecting a V-Cone with a set of the form H_k^{d+m} requires determining all vectors of the V-Cone $\mathcal{C}_{\mathcal{V}}(V')$ that have a 0 in the k-th coordinate. Only conical combinations of V' of a special form will have this property, determining this form is the heart of the proof. Using the language from the above remark, "forgetting the useless information is easy, capturing the important information is hard, and representing the information in a different way is tricky."

3.2 Lift $\mathcal{C}_\mathcal{H} \to \mathcal{C}_{\mathcal{V}'}$

Here we use some Linear Programming techniques to rewrite $\mathcal{C}_\mathcal{H}(A)$ as a subset of a V-Cone.

Proposition 6. *Let $A \in \mathbb{R}^{m \times d}$. Then there exists a $V' \in \mathbb{R}^{(d+m)\times(2d+m)}$ such that*

$$\mathcal{C}_\mathcal{H}(A) = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_\mathcal{V}(V') \right\}$$

Proof. Define V':

$$V' = \left\{ \pm \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \;\middle|\; 1 \le j \le d, 1 \le i \le m \right\}$$

Let $\mathbf{x} \in \mathcal{C}_\mathcal{H}(A)$. Then it is to be shown that:

$$A\mathbf{x} \le \mathbf{0} \Rightarrow \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_\mathcal{V}(V')$$

The task is to find a $t_j^+, t_j^-, w_i \ge 0$ such that:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \sum_{j=1}^d t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^d t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^m w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix}$$

Consider the following assignments:

$$t_j^+ = \begin{cases} x_j & x_j \ge 0 \\ 0 & x_j < 0 \end{cases}$$

$$t_j^- = \begin{cases} 0 & x_j \ge 0 \\ -x_j & x_j < 0 \end{cases}$$

$$w_i = -\langle A_i, \mathbf{x} \rangle$$

By the way we've defined t_j^+ and t_j^-, $x_j = t_j^+ - t_j^-$. Then:

$$\sum_{j=1}^d t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^d t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \sum_{j=1}^d (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \sum_{j=1}^d x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix}$$

Furthermore:

$$\sum_{i=1}^m w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} = -\sum_{i=1}^m \langle A_i, \mathbf{x} \rangle \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix}$$

It is clear that $t_j^+, t_j^- \ge 0$, and $w_i \ge 0$ follows from $A\mathbf{x} \le \mathbf{0}$. It follows that:

$$\sum_{j=1}^d t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^d t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^m w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \in \mathcal{C}_\mathcal{V}(V')$$

Combining results, we have:

$$\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

And we conclude that $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$.

Now let $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$. The task is to show that $A\mathbf{x} \leq 0$. We have:

$$\begin{aligned} \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = &\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = && \Big| \; t_j^+, t_j^-, w_i \geq 0 \\ &\sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} && \Big| \; t_j^+, t_j^-, w_i \geq 0 \end{aligned}$$

Since the only contribution to the coordinate of x_j is $t_j^+ - t_j^-$, we may conclude that $x_j = t_j^+ - t_j^-$. Continuing the string of equalities:

$$\begin{aligned} &\sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = && \Big| \; t_j^+, t_j^-, w_i \geq 0 \\ &\sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = && \Big| \; w_i \geq 0, x_j \in \mathbb{R} \\ &\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} && \Big| \; \mathbf{w} \geq \mathbf{0}, \mathbf{x} \in \mathbb{R}^d \end{aligned}$$

This last line implies that $A\mathbf{x} + \mathbf{w} = \mathbf{0} \Rightarrow A\mathbf{x} = -\mathbf{w} \leq \mathbf{0}$. □

3.3 Drop $\mathcal{C}_{\mathcal{H}'} \to \mathcal{C}_{\mathcal{V}}$

Observe that the following set can be written:

$$\begin{aligned} &\left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \right\} = \\ &\left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \exists \mathbf{t} \geq \mathbf{0} : \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} = \sum_{\mathbf{v} \in V'} t_v \mathbf{v},\ \left\langle \mathbf{e}_k, \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \right\rangle = 0,\ d+1 \leq k \leq d+m \right\} \end{aligned}$$

This is almost a cone, except for the requirements given by $\langle \mathbf{e}_k, \left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{t} \end{smallmatrix}\right) \rangle = 0$. What we need is, for each k, a way to get a new set V'' such that

$$(\exists \mathbf{t} \geq \mathbf{0}) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} = \sum_{\mathbf{v} \in V'} t_v \mathbf{v},\ \left\langle \mathbf{e}_k, \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \right\rangle = 0 \Leftrightarrow (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} = \sum_{\mathbf{v} \in V''} t_v \mathbf{v}$$

Then, for each $d+1 \leq k \leq d+m$, we can apply this method to eliminate all of those constraints.

Proposition 7. *Let $V'_{in} \in \mathbb{R}^{(d+m)\times n_{in}}$, then there exists a set $V'_{out} \in \mathbb{R}^{(d+m)\times n_{out}}$ such that*

$$\left\{ \mathbf{x} \;\middle|\; \exists \mathbf{t} \geq \mathbf{0} : \mathbf{x} = \sum_{\mathbf{v}\in V'_{in}} t_v \mathbf{v},\ \langle \mathbf{e}_k, \mathbf{x} \rangle = 0 \right\} = \left\{ \mathbf{x} \;\middle|\; \exists \mathbf{t} \geq \mathbf{0} : \mathbf{x} = \sum_{\mathbf{v}\in V'_{out}} t_v \mathbf{v} \right\}$$

Proof. This is a direct result of proposition 5. □

Proposition 8. *For each $V' \in \mathbb{R}^{(d+m)\times n'}$, there exists a $V'' \in \mathbb{R}^{(d+m)\times n''}$ such that:*

$$\left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \right\} = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V'') \right\}$$

Proof. Repeatedly apply proposition 7. □

3.4 Projecting a V-Cone

Another simple proposition, this just shows that instead of taking all conical combinations of a set of vectors and then projecting them, we can just project the generators beforhand. I.e., it doesn't matter if we forget about some coordinates before or after we take conical combinations of them.

Proposition 9 (Projections are linear)**.** *Projections are linear.*

Proof. Take the identity matrix, and remove all rows corresponding to which coordinates you want to remove. Clearly, this is a linear map. □

Proposition 10. *Let $V' \in \mathbb{R}^{(d+m)\times n}$, then*

$$\pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V') = \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$$

Proof. This follows from the fact that projections are linear transformations. Let J index the vectors of V'. Say $\mathbf{v} \in \mathcal{C}_{\mathcal{V}}(V')$, then

$$\pi_{\mathbf{x}}(\mathbf{v}) = \pi_{\mathbf{x}} \left(\sum_{j\in J} t_j \mathbf{v}^j \right) = \sum_{j \in J} t_j \pi_{\mathbf{x}}(\mathbf{v}^j)$$

This is apparently a conical combination of vectors from $\pi_{\mathbf{x}}(V')$, so $\mathbf{v} \in \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$.

Similarly, say $\mathbf{u} \in \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$, then

$$\mathbf{u} = \sum_{j\in J} t_j \pi_{\mathbf{x}}(\mathbf{v}^j) = \pi_{\mathbf{x}} \left(\sum_{j\in J} t_j \mathbf{v}^j \right)$$

So $\mathbf{u} \in \pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V'))$. □

3.5 Main Theorem

Theorem 1. *Let $A \in \mathbb{R}^{m \times d}$. Then there exists a $V \in \mathbb{R}^{d \times n}$ such that*

$$\mathcal{C}_{\mathcal{H}}(A) = \mathcal{C}_{\mathcal{V}}(V)$$

Proof. Theorem 1 follows from the propositions as follows. First, apply proposition 6 to A to get a set V' of vectors in $\mathbb{R}^{d+m}$. Proposition 8 gives us a method to eliminate all of constraints of the form $\langle \mathbf{e}_k, \left(\begin{smallmatrix}\mathbf{x}\\ \mathbf{t}\end{smallmatrix}\right)\rangle = 0$, and the result is a new set V''. Proposition 10 allows us to forget about the useless coordinates from V'' using an operator of the form $\pi_{\mathbf{x}}$. Letting $V = \pi_{\mathbf{x}}(V'')$, we then have

$$\mathcal{C}_{\mathcal{H}}(A) = \mathcal{C}_{\mathcal{V}}(V)$$

□

4 V-Cone → H-Cone

This section gives a proof that every V-Cone is also a H-Cone.

4.1 Introduction.

The proof going the other direction has a likeness to the former, due to the two representations being "dual" to one another. First, recall that the expression $A\mathbf{x} \leq \mathbf{0}$ can be written:

$$(\forall A_i \in A)\, \langle A_i, \mathbf{x} \rangle \leq 0\}$$

Here, A goes from being a matrix to being a set of row vectors.

4.2 Lift $\mathcal{C}_{\mathcal{V}} \to \mathcal{C}_{\mathcal{H}'}$

The first order of business is to get a new representation of our V-Cone.

Proposition 11. *For every set $V \in \mathbb{R}^{d \times n}$ there exists an $A \in \mathbb{R}^{(n+2d)\times(n+d)}$ such that*

$$\mathcal{C}_{\mathcal{V}}(V) = \{\mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} : (\forall A_i \in A)\, \langle A_i, (\mathbf{x}, \mathbf{t}) \rangle \leq 0\}$$

Proof. Observe that a V-Cone can be written:

$$\mathcal{C}_{\mathcal{V}}(V) = \{\mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} \geq \mathbf{0} : \mathbf{x} = V\mathbf{t}\}$$

Our operation here is to let $\mathbf{t}$ be a variable, and represent its contraints and relationship with $\mathbf{x}$ in terms of linear inequalities. Consider the following systems of inequalities:

$$\begin{aligned}
\mathbf{t} &\geq \mathbf{0} & \Leftrightarrow && -I\mathbf{t} &\leq \mathbf{0} \\
\mathbf{x} &= V\mathbf{t} & \Leftrightarrow && I\mathbf{x} - V\mathbf{t} &\leq \mathbf{0} \\
&&&& -I\mathbf{x} + V\mathbf{t} &\leq \mathbf{0}
\end{aligned}$$

We can use this as a template for a new matrix $A \in \mathbb{R}^{(n+2d)\times(n+d)}$:

$$A = \begin{pmatrix} \mathbf{0} & -I \\ -V & I \\ V & -I \end{pmatrix}$$

Then we have that

$$\mathcal{C}_{\mathcal{V}}(V) = \{\mathbf{x} \in \mathbb{R}^{d+n} \mid \exists \mathbf{t} : (\forall A_i \in A)\, \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \leq \mathbf{0}\}$$

□

4.3 Drop

Our new representation of the V-Cone is almost an H-Cone, the only problem is the $\exists$ qualifier. Note that the expression $\exists \mathbf{t}$ is equivalent to $\exists t_1, \exists t_2, \ldots, \exists t_n$. The method of dealing with these is to show that they can be elimnated one at at time, then conclude that we can get rid of all of them.

Proposition 12. *For every set $A_{in} \in \mathbb{R}^{m_{in}\times(d+n)}$ there is a set $A_{out} \in \mathbb{R}^{m_{out}\times(d+n)}$ such that*

$$\exists t\, (\forall A_i \in A_{in}) \;\; \langle A_i, \mathbf{x} - t\mathbf{e}_k\rangle \leq 0 \Leftrightarrow$$
$$\forall t\, (\forall A_i \in A_{out})\, \langle A_i, \mathbf{x}\rangle \leq 0$$

Proof. Remembering that each A_i is a linear functional, we simply apply proposition 3. □

Proposition 13. *For every set $A \in \mathbb{R}^{(d+n)\times n}$ there is a set $A' \in \mathbb{R}^{(d+n)\times n'}$ such that*

$$\{\mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t}(\forall A_i \in A) \;\; \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \leq 0\} =$$
$$\{\mathbf{x} \in \mathbb{R}^d \mid \forall \mathbf{t}(\forall A_i \in A')\, \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \leq 0\}$$

Proof. The set A' is formed by repeatedly applying proposition 12. □

4.4 Projection

Proposition 14. *Let $A_i \in \mathbb{R}^{d+n}, \mathbf{x} \in \mathbb{R}^d, \mathbf{t} \in \mathbb{R}^n$. Then*

$$\forall \mathbf{t}\, \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \leq 0 \Rightarrow (A_i^k = 0 \mid d+1 \leq k \leq d+n)$$

Proof. Suppose that $A_i^k > 0$, and let $\alpha = \langle A_i, (\mathbf{x}, \mathbf{0}\rangle$. Then $\langle A_i, (\mathbf{x}, t\mathbf{e}_k)\rangle = \alpha + tA_i^k$. By setting $t > -\alpha/A_i^k$, we arrive at a contradiction. □

Proposition 15. *For every set $A' \in \mathbb{R}^{(d+n)\times n'}$ there exists a set $A'' \in \mathbb{R}^{d\times n'}$ such that*

$$\{\mathbf{x} \in \mathbb{R}^d \mid (\forall A_i \in A')\, \forall \mathbf{t}\, \langle A_i, (\mathbf{x}, \mathbf{t})\rangle \leq 0\} =$$
$$\{\mathbf{x} \in \mathbb{R}^d \mid (\forall A_i \in A'') \quad \langle A_i, \mathbf{x}\rangle \leq 0\}$$

Proof. Let $A'' = \pi_{\mathbf{x}}(A')$, that is, the set formed by projecting each from from A' onto the first d coordinates. From proposition 14, we have

$$\langle A_i, (\mathbf{x}, \mathbf{t}) \rangle = \langle \pi_{\mathbf{x}}(A_i), \mathbf{x} \rangle$$

Therefore, $A'(\mathbf{x}, \mathbf{t}) \leq \mathbf{0} \Leftrightarrow A''\mathbf{x} \leq \mathbf{0}$. □

4.5 Main Theorem

Theorem 2. *Let $V \in \mathbb{R}^{d \times n}$. Then there exists an $A \in \mathbb{R}^{m \times d}$ such that:*

$$\mathcal{C}_{\mathcal{V}}(V) = \mathcal{C}_{\mathcal{H}}(A)$$

Proof. First, apply proposition 11 to V to get a set A'. Then apply proposition 13 to A' to get a set A''. Finally, let A be the result of applying 15 to A''. □

5 Main Theorem

Theorem 3 (Minkowski-Weyl Theorem)**.** *Every V-Cone is an H-Cone, and every H-Cone is a V-Cone.*

6 Polyhedra

Generalizing to Polyhedra In this section, we will define polyhedra, and expand theorem 3 to include them.

Definition 8 (Convex Combination)**.** *Let $\mathbf{x}_i \in \mathbb{R}^n, \lambda_i \geq 0, \sum_i \lambda_i = 1$. Then an expression of the form:*

$$\sum_i \lambda_i \mathbf{x}_i$$

is called a convex combination of the vectors $\mathbf{x}_i$.

Definition 9 (Convex Hull)**.** *Let $A \subseteq \mathbb{R}^n$. Then let*

$$\operatorname{conv}(A)$$

denote the set of all convex combinations of members of A.

Definition 10 (Minkowski Sum)**.** *Let $A, B \subseteq \mathbb{R}^n$. Then denote*

$$A \oplus B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$$

The set $A \oplus B$ is called the Minkowski Sum of A and B.

Definition 11 (V-Polyhedron $\mathcal{P}_{\mathcal{V}}$)**.** *Let U, V be finite subsets of $\mathbb{R}^d$. Then the set*

$$\operatorname{cone}(U) \oplus \operatorname{cone}(V)$$

is called a V-Polyhedron.

Definition 12 (H-Polyhedron $\mathcal{P}_{\mathcal{H}}$). *Let* $A \in \mathbb{R}^{m\times d}, \mathbf{b} \in \mathbb{R}^m$. *Then the set*

$$\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}$$

is called an H-Polyhedron.

Definition 13 (Set Similarity). *Let* $A \subseteq \mathbb{R}^n, x \in \mathbb{R}$. *Write*

$$\{x\} \times A \simeq A$$

The purpose of this notation is to avoid constantly writing and checking that sets differ only by a product with a point in space. This is because a polyhedron doesn't "gain anything" from taking its product with a point.

Remark In order to generalize the Minkowski-Weyl Theoremto polyhedra, we need only provide a procedure to turn a polyhedron into a cone and back again, for both V and H polyhedra.

6.1 $\mathcal{P}_{\mathcal{H}} \leftrightarrow \mathcal{C}_{\mathcal{H}}$

Proposition 16 ($\mathcal{P}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{H}}$). *Every H-Polyhedron can be represented by the intersection of an H-Cone and a hyperplane.*

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \simeq \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; (-\mathbf{b}\, A) \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} \cap \{x_0 = 1\}$$

Proof. First note that $(-\mathbf{b}\, A) \left(\begin{smallmatrix} x_0 \\ \mathbf{x} \end{smallmatrix} \right) = -x_0\mathbf{b} + A\mathbf{x}$. Then

$$A\mathbf{x} \leq \mathbf{b} \Leftrightarrow -1\mathbf{b} + A\mathbf{x} \leq \mathbf{0}$$

□

Proposition 17 ($\mathcal{C}_{\mathcal{H}} \to \mathcal{P}_{\mathcal{H}}$). *Every intersection of an H-Cone with a hyperplane* $\{x_0 = 1\}$ *is a polyhedron.*

Proof.

$$\{\mathbf{x} \mid A\mathbf{x} \leq 0\} \cap \{x_0 = 1\} = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \;\middle|\; \begin{pmatrix} 1 & \mathbf{0} \\ -1 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \begin{pmatrix} 1 \\ -1 \\ \mathbf{b} \end{pmatrix} \right\}$$

□

6.2 $\mathcal{P}_{\mathcal{V}} \leftrightarrow \mathcal{C}_{\mathcal{V}}$

Proposition 18 ($\mathcal{P}_{\mathcal{V}} \to \mathcal{C}_{\mathcal{V}}$). *Every V-Polyhedron can be represented as the intersection of a V-Cone and a hyperplane.*

$$\mathrm{conv}(V) \oplus \mathrm{cone}(U) \simeq \mathrm{cone} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ V & U \end{pmatrix} \cap \{x_0 = 1\}$$

Proof. Say $\mathbf{x} \in \text{cone}(U) + \text{conv}(V)$, $\mathbf{u}^j \in U$, $\mathbf{v}^i \in V$, $\lambda_i \geq 0, \sum_i \lambda_i = 1, t_j \geq 0$. If $\mathbf{x} = \sum_i \lambda_i \mathbf{v}^i + \sum_j t_j \mathbf{u}^j$, then

$$\sum\nolimits_i \lambda_i \begin{pmatrix} 1 \\ \mathbf{v}^i \end{pmatrix} + t_i \sum\nolimits_j \begin{pmatrix} 0 \\ \mathbf{u}^j \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \in \text{cone} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ V & U \end{pmatrix} \cap \{x_0 = 1\}$$

Similarly, if

$$\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \in \text{cone} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ V & U \end{pmatrix} \cap \{x_0 = 1\}$$

Then the sum of all λ_i corresponding to V must sum to one, and is the sum of a convex combination of members of $\left(\begin{smallmatrix} 1 \\ V \end{smallmatrix} \right)$, and a member of cone $\left(\begin{smallmatrix} 0 \\ U \end{smallmatrix} \right)$, then $\mathbf{x}$ is a sum of vectors from these two sets. $\square$

Proposition 19 ($\mathcal{C}_\mathcal{V} \to \mathcal{P}_\mathcal{V}$)**.** *A V-Cone $\mathcal{C}_\mathcal{V}(V')$ intersected with a hyperplane of the from $\{x_0 = 1\}$ is a V-Polyhedron.*

Proof. The Cone intersected with the hyperplane can be written:

$$\{\mathbf{x} \mid \exists \mathbf{t} : V'\mathbf{t} = \mathbf{x},\ \langle \mathbf{e}_0, \mathbf{x} \rangle = 1\}$$

We apply proposition 4 with $r = 1$. $\square$

6.3 Main Theorem for Polyhedra

Theorem 4 (Minkowski-Weyl Theoremfor polyhedra)**.** *Every V-Polyhedron is an H-Polyhedron, and every H-Polyhedron is a V-Polyhedron.*

Proof. Using the propositions of this section we can transfer a polyhedron to a cone, then use theorem 3 to change its representation, then again use the propositions of this section to recover the original cone. $\square$