1 Preliminaries

1.1 Notation

The canonical basis vectors will be written $\mathbf{e}_k$ for valid values of k. Let $\mathbf{x} \in \mathbb{R}^n$. It will be customary to write:

$$x_k = \langle \mathbf{e}_k, \mathbf{x} \rangle$$

Given $A \in \mathbb{R}^{m \times d}$, Let A_i and A^j denote the rows and columns of A, respectively. Then A_i^j will denote the entry from A in the i-th row and j-th column. Matrix multiplication is then given by:

$$A\mathbf{x} = \sum_{j=1}^{d} A^j x_j = \begin{pmatrix} \langle A_1, \mathbf{x} \rangle \\ \vdots \\ \langle A_m, \mathbf{x} \rangle \end{pmatrix} = \sum_{i=1}^{m} \langle A_i, \mathbf{x} \rangle \, \mathbf{e}_i$$

Let $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^m$, and define the following notation for vectors in $\mathbb{R}^{d+m}$:

$$\left\langle \mathbf{e}_k, \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \right\rangle = \begin{cases} x_k & k \leq d \\ w_{k-d} & d+1 \leq k \leq d+m \end{cases}$$

1.2 Definitions

Definition 1 (H-Cone)**.** *Let* $A \in \mathbb{R}^{m \times d}$*, then define*

$$\mathcal{C}_{\mathcal{H}}(A) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$$

Definition 2 (V-Cone)**.** *Let* $V \in \mathbb{R}^{d \times n}$*, then define*

$$\mathcal{C}_{\mathcal{V}}(V) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid \exists \mathbf{t} \in \mathbb{R}^n \geq \mathbf{0},\, \mathbf{x} = V\mathbf{t} \right\}$$

A vector of the form $V\mathbf{t}$*, where* $\mathbf{t} \geq \mathbf{0}$ *is called a "conical combination of* V*".*

Definition 3 (Hyperplane)**.** *Let* $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$*. Then the set*

$$\{ \mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{a}, \mathbf{x} \rangle = b \}$$

Is known as a hyperplane. Let $H_k^n \subseteq \mathbb{R}^n$ *denote the hyperplane defined by:*

$$H_k^n = \{ \mathbf{x} \in \mathbb{R}^n \mid x_k = 0 \}$$

Definition 4 (Projection)**.** *Let* $\mathbf{x} \in \mathbb{R}^d$*. The vector* $\mathbf{x}' \in \mathbb{R}^{d-k}$ *formed by omitting* k *coordinates of* $\mathbf{x}$ *is called a projection of* $\mathbf{x}$*. In particular, let* $\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^m$*, and* $\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{w} \end{smallmatrix} \right) \in \mathbb{R}^{d+m}$*, and define* $\pi_{\mathbf{x}} : \mathbb{R}^{d+m} \to \mathbb{R}^d$ *as:*

$$\pi_{\mathbf{x}} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} = \mathbf{x}$$

Then $\pi_{\mathbf{x}}$ *is a projection onto the first* d*-coordinates.*

$$
\begin{tikzcd}
\boxed{A\mathbf{x} \leq \mathbf{0}} \arrow[d, "\text{lift}"'] & & \boxed{\mathcal{C}_{\mathcal{V}}(V)} \\
\boxed{\mathcal{C}_{\mathcal{V}}\{\pm\binom{\mathbf{e}_i}{A^i} \cup \binom{0}{\mathbf{e}_j}\}} \arrow[rr, "\bigcap_{d+1}^{d+m} H_k^{d+m}", "\text{drop}"'] & & \boxed{\{\mathbf{x} : \binom{\mathbf{x}}{\mathbf{0}} \in \mathcal{C}_{\mathcal{V}}(V'')\}} \arrow[u, "\pi_{\mathbf{x}}", "\text{project}"']
\end{tikzcd}
$$

Figure 1: Diagram of the proof ($\mathcal{C}_{\mathcal{H}} \to \mathcal{C}_{\mathcal{V}'} \to \mathcal{C}_{\mathcal{V}}$)

Remarks: In the following sections it will be proved that every **H-Cone** is a **V-Cone**, and every **V-Cone** is an **H-Cone**. This shows that these are two fundamentally different descriptions of the same object. Each has its own power and purpose, which will be discussed later.

Let $A \subseteq \mathbb{R}^n$. Then $A \cap H_k^n$ is simply all of the points of A who have the k-th coordinate 0. Let π^k be the projection that omits the k-th coordinate. Then $\pi^k(A \cap H_k^n)$ are all the points of A whose k-th coordinate is 0, but without that 0 coordinate. If projections are thought of as "forgetting useless information," and intersections as "capturing only the useful information," then the sequence of projection and intersection is "capturing the useful information, and forgetting the useless information."

2 H-Cone $\to$ V-Cone

This section gives a proof that every H-Cone is also a V-Cone.

2.1 Introduction.

Discussing "what is hard" is helpful for understanding the proof and why it is formed the way it is. Proposition 1 (lift) is not terribly difficult, it mostly just takes a clever idea and attention to detail. It is reminiscent of techniques common to linear programming. Propositions 2, 3, and 5 (project) are very straightforward, they exist primarily to make the proofs of the other propositions less cluttered. Proposition 4 (drop) is really the "main" idea that needs proving. This is because intersecting a V-Cone with a set of the form H_k^{d+m} requires determining all vectors of the V-Cone $\mathcal{C}_{\mathcal{V}}(V')$ that have a 0 in the k-th coordinate. Only conical combinations of V' of a special form will have this property, determining this form is the heart of the proof. Using the language from the above remark, "forgetting the useless information is easy, capturing the important information is hard, and representing the information in a different way is tricky."

2.2 Lift $\mathcal{C}_\mathcal{H} \to \mathcal{C}_{\mathcal{V}'}$

Here we use some Linear Programming techniques to rewrite $\mathcal{C}_\mathcal{H}(A)$ as a subset of a V-Cone.

Proposition 1. *Let $A \in \mathbb{R}^{m\times d}$. Then there exists a $V' \in \mathbb{R}^{(d+m)\times(2d+m)}$ such that*

$$\mathcal{C}_\mathcal{H}(A) = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_\mathcal{V}(V') \right\}$$

Proof. Define V':

$$V' = \left\{ \pm \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \;\middle|\; 1 \le j \le d, 1 \le i \le m \right\}$$

Let $\mathbf{x} \in \mathcal{C}_\mathcal{H}(A)$. Then it is to be shown that:

$$A\mathbf{x} \le \mathbf{0} \Rightarrow \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_\mathcal{V}(V')$$

The task is to find a $t_j^+, t_j^-, w_i \ge 0$ such that:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix}$$

Consider the following assignments:

$$t_j^+ = \begin{cases} x_j & x_j \ge 0 \\ 0 & x_j < 0 \end{cases}$$

$$t_j^- = \begin{cases} 0 & x_j \ge 0 \\ -x_j & x_j < 0 \end{cases}$$

$$w_i = -\langle A_i, \mathbf{x} \rangle$$

By the way we've defined t_j^+ and t_j^-, $x_j = t_j^+ - t_j^-$. Then:

$$\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix}$$

Furthermore:

$$\sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} = -\sum_{i=1}^{m} \langle A_i, \mathbf{x} \rangle \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix}$$

It is clear that $t_j^+, t_j^- \ge 0$, and $w_i \ge 0$ follows from $A\mathbf{x} \le \mathbf{0}$. It follows that:

$$\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \in \mathcal{C}_\mathcal{V}(V')$$

Combining results, we have:

$$\sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -A\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

And we conclude that $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$.

Now let $\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V')$. The task is to show that $A\mathbf{x} \leq 0$. We have:

$$\begin{aligned} \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = \sum_{j=1}^{d} t_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} - \sum_{j=1}^{d} t_j^- \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = & \qquad \Big| \; t_j^+, t_j^-, w_i \geq 0 \\ \sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} & \qquad \Big| \; t_j^+, t_j^-, w_i \geq 0 \end{aligned}$$

Since the only contribution to the coordinate of x_j is $t_j^+ - t_j^-$, we may conclude that $x_j = t_j^+ - t_j^-$. Continuing the string of equalities:

$$\begin{aligned} \sum_{j=1}^{d} (t_j^+ - t_j^-) \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = & \qquad \Big| \; t_j^+, t_j^-, w_i \geq 0 \\ \sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{i=1}^{m} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_j \end{pmatrix} = & \qquad \Big| \; w_i \geq 0, x_j \in \mathbb{R} \\ \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} & \qquad \Big| \; \mathbf{w} \geq \mathbf{0}, \mathbf{x} \in \mathbb{R}^d \end{aligned}$$

This last line implies that $A\mathbf{x} + \mathbf{w} = \mathbf{0} \Rightarrow A\mathbf{x} = -\mathbf{w} \leq \mathbf{0}$. $\square$

2.3 Project $\pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{H}'})$

This proposition is not hard, it just writes the new form in a better way.

Proposition 2.

$$\left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \right\} = \pi_{\mathbf{x}} \left(\mathcal{C}_{\mathcal{V}}(V') \bigcap_{k=d+1}^{d+m} H_k^{d+m} \right)$$

Proof.

$$\begin{aligned}
\mathcal{C}_{\mathcal{V}}(V') \bigcap_{k=d+1}^{d+m} H_k^{d+m} &= \left\{ \mathbf{z} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; z_k = 0 : d+1 \le k \le d+m \right\} \\
&= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; \mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^m, w_k = 0 : 1 \le k \le m \right\} \\
&= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; \mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^m, \mathbf{w} = \mathbf{0} \right\} \\
&= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \;\middle|\; \mathbf{x} \in \mathbb{R}^d \right\}
\end{aligned}$$

Then

$$\pi_{\mathbf{x}} \left(\mathcal{C}_{\mathcal{V}}(V') \bigcap_{k=d+1}^{d+m} H_k^{d+m} \right) = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \in \mathcal{C}_{\mathcal{V}}(V') \right\}$$

□

2.4 Subcone

This proposition shows that a cone generated by a subset of an existing cone is a subset of the original cone. In other words, "conical combinations of conical combinations are again conical combinations."

Proposition 3. $U \subseteq \mathcal{C}_{\mathcal{V}}(V) \Rightarrow \mathcal{C}_{\mathcal{V}}(U) \subseteq \mathcal{C}_{\mathcal{V}}(V)$

Proof. For any $\mathbf{u}^i \in U$ we have $\mathbf{u}^i = \sum_j t_{ij} \mathbf{v}^j$. Then, for any conical combination of $\mathbf{u}^i$ we have:

$$\sum_i s_i \mathbf{u}^i = \sum_i s_i \left(\sum_j t_{ij} \mathbf{v}^j \right) = \sum_j \left(\sum_i s_i t_{ij} \right) \mathbf{v}^j$$

Since $s_i, t_{ij} \ge 0$, the final expression is a conical combination of V. □

2.5 Drop

Here is where the real work is done. This proposition shows how to remove one of the intersection terms from our new form. It uses a technique known as "Fourier Motzkin elimination." The definition of V_{out} and τ are the important ideas, the rest of the proof is just the ugly details showing that they are smart.

Note: V'_{in} and V'_{out} are intended to represent "input" and "output" sets. Notice that n_{in} and n_{out} may be different, i.e. the output may have (a lot) more vectors than the input.

Proposition 4. *Let $V'_{in} \in \mathbb{R}^{(d+m) \times n_{in}}$, then there exists a set $V'_{out} \in \mathbb{R}^{(d+m) \times n_{out}}$ such that*

$$\mathcal{C}_{\mathcal{V}}(V'_{in}) \cap H_k^{d+m} = \mathcal{C}_{\mathcal{V}}(V'_{out})$$

Proof. Suppose that the vectors of V_{in} are indexed by a set I. We partition the vectors of V_{in} based on their value at coordinate k.

$$\begin{aligned} P &= i \in I \mid v_k^i > 0 \\ N &= j \in I \mid v_k^j < 0 \\ Z &= l \in I \mid v_k^l = 0 \end{aligned}$$

Here, we've used different indices i, j, l. This is purely for convenience, and in what follows we'll follow the convention that $i \in P$, $j \in N$, and $l \in Z$. Next, let

$$V_{out} = \{\mathbf{v}^l \mid l \in Z\} \cup \{v_k^i \mathbf{v}^j - v_k^j \mathbf{v}^i \mid i \in P, j \in N\}$$

There are two critical properties of the set V_{out}. First, every $\mathbf{v} \in V_{out}$ is formed by taking conical combinations of vectors from V_{in}. Also, for any $\mathbf{v} \in V_{out}$, we have that $v_k = 0$. These two properties, along with proposition 3 gives

$$\mathcal{C}_{\mathcal{V}}(V_{out}) \subseteq \mathcal{C}_{\mathcal{V}}(V_{in}) \cap H_k^{d+m}$$

Next, say $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V_{in}) \cap H_k^{d+m}$, then:

$$\mathbf{x} = \sum_{i \in P} t_i \mathbf{v}^i + \sum_{j \in N} t_j \mathbf{v}^j + \sum_{l \in Z} t_l \mathbf{v}^l$$

Because $x_k = 0$, and $v_k^l = 0$ for each $l \in Z$, we have

$$\begin{aligned} 0 &= \sum_{i \in P} t_i v_k^i + \sum_{j \in N} t_j v_k^j + \sum_{l \in Z} t_l v_k^l \\ &= \sum_{i \in P} t_i v_k^i + \sum_{j \in N} t_j v_k^j \end{aligned}$$

This final line implies that the sums have opposite values. Denote this value by τ, that is

$$\tau = \sum_{i \in P} t_i v_k^i = - \sum_{j \in N} t_j v_k^j$$

If $\tau = 0$, then each $t_i, t_j = 0$ and $\mathbf{x}$ is a conical combination of vectors from $\mathbf{v}^l : l \in Z$, and therefore $\mathbf{x} \in \mathcal{C}_{\mathcal{V}}(V_{out})$. Suppose that $\tau > 0$. Then we have

$$\sum_{i \in P} t_i \mathbf{v}^i = \frac{-1}{\tau} \sum_{j \in N} t_j v_k^j \sum_{i \in P} t_i \mathbf{v}^i = - \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\tau} v_k^j \mathbf{v}^i$$

$$\sum_{j \in N} t_j \mathbf{v}^j = \frac{1}{\tau} \sum_{i \in N} t_i v_k^i \sum_{j \in P} t_j \mathbf{v}^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\tau} v_k^i \mathbf{v}^j$$

Combining these results, we have:

$$\begin{aligned}\sum_{i\in P} t_i\mathbf{v}^i + \sum_{j\in N} t_j\mathbf{v}^j &= \sum_{\substack{i\in P\\ j\in N}} \frac{t_it_j}{\tau} v_k^i\mathbf{v}^j - \sum_{\substack{i\in P\\ j\in N}} \frac{t_it_j}{\tau} v_k^j\mathbf{v}^i \\ &= \sum_{\substack{i\in P\\ j\in N}} \frac{t_it_j}{\tau}(v_k^i\mathbf{v}^j - v_k^j\mathbf{v}^i)\end{aligned}$$

Because $\tau > 0, t_i, t_j \geq 0$, it follows that $\frac{t_it_j}{\tau} \geq 0$. This shows that the sum above can be written as a conical combination of vectors from V_{out}. We can now rewrite $\mathbf{x}$:

$$\begin{aligned}\mathbf{x} &= \sum_{i\in P} t_i\mathbf{v}^i + \sum_{j\in N} t_j\mathbf{v}^j + \sum_{l\in Z} t_l\mathbf{v}^l \\ &= \sum_{\substack{i\in P\\ j\in N}} \frac{t_it_j}{\tau}(v_k^i\mathbf{v}^j - v_k^j\mathbf{v}^i) + \sum_{l\in Z} t_l\mathbf{v}^l\end{aligned}$$

This shows $\mathbf{x}$ is a conical combination of vectors from V_{out}, so

$$\mathcal{C}_{\mathcal{V}}(V_{in}) \cap H_k^{d+m} \subseteq \mathcal{C}_{\mathcal{V}}(V_{out})$$

□

2.6 Projecting a V-Cone

Another simple proposition, this just shows that instead of taking all conical combinations of a set of vectors and then projecting them, we can just project the generators beforhand. I.e., it doesn't matter if we forget about some coordinates before or after we take conical combinations of them.

Proposition 5. *Let $V' \in \mathbb{R}^{(d+m)\times n}$, then*

$$\pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V') = \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$$

Proof. This follows from the fact that projections are linear transformations. Take the projection:

$$\pi_{\mathbf{x}}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} = \mathbf{x}$$

Then

$$\pi_{\mathbf{x}}\left(\alpha\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} + \begin{pmatrix}\mathbf{y}\\ \mathbf{z}\end{pmatrix}\right) = \pi_{\mathbf{x}}\begin{pmatrix}\alpha\mathbf{x}+\mathbf{y}\\ \alpha\mathbf{w}+\mathbf{z}\end{pmatrix} = \alpha\mathbf{x}+\mathbf{y} = \alpha\pi_{\mathbf{x}}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} + \pi_{\mathbf{x}}\begin{pmatrix}\mathbf{y}\\ \mathbf{z}\end{pmatrix}$$

Let J index the vectors of V'. Say $\mathbf{v} \in \mathcal{C}_{\mathcal{V}}(V')$, then

$$\pi_{\mathbf{x}}(\mathbf{v}) = \pi_{\mathbf{x}}\left(\sum_{j\in J} t_j\mathbf{v}^j\right) = \sum_{j\in J} t_j\pi_{\mathbf{x}}(\mathbf{v}^j)$$

This is apparently a conical combination of vectors from $\pi_{\mathbf{x}}(V')$, so $\mathbf{v} \in \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$.

Similarly, say $\mathbf{u} \in \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V'))$, then

$$\mathbf{u} = \sum_{j \in J} t_j \pi_{\mathbf{x}}(\mathbf{v}^j) = \pi_{\mathbf{x}} \left(\sum_{j \in J} t_j \mathbf{v}^j \right)$$

So $\mathbf{u} \in \pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V'))$. □

2.7 Main Theorem

Theorem 1. *Let $A \in \mathbb{R}^{m \times d}$, then for the set $\mathcal{C}_{\mathcal{H}}(A)$, there exists a $V \in \mathbb{R}^{d \times n}$ such that*

$$\mathcal{C}_{\mathcal{H}}(A) = \mathcal{C}_{\mathcal{V}}(V)$$

Proof. Theorem 1 follows from the propositions as follows. First, apply proposition 1 to A to get a set V' of vectors in $\mathbb{R}^{d+m}$. Proposition 2 shows us that the set $\mathcal{C}_{\mathcal{H}}(A)$ can be formed by first taking a finite number of intersections of $\mathcal{C}_{\mathcal{V}}(V')$ with hyperplanes of the form H_k^{d+m}, then projecting this set onto $\mathbf{x}$. Proposition 4 gives us a method to eliminate all of the intersections in the form given by proposition 2, and end up with a set V'' with the following useful property:

$$\mathcal{C}_{\mathcal{V}}(V'') \bigcap_{k=d+1}^{d+m} H_k^{d+m} = \mathcal{C}_{\mathcal{V}}(V'')$$

Proposition 5, along with the set V'', allows the following calculation:

$$\mathcal{C}_{\mathcal{H}}(A) = \pi_{\mathbf{x}} \left(\mathcal{C}_{\mathcal{V}}(V'') \bigcap_{k=d+1}^{d+m} H_k^{d+m} \right) = \pi_{\mathbf{x}}(\mathcal{C}_{\mathcal{V}}(V'')) = \mathcal{C}_{\mathcal{V}}(\pi_{\mathbf{x}}(V''))$$

Letting $V = \pi_{\mathbf{x}}(V'')$, we then have

$$\mathcal{C}_{\mathcal{H}}(A) = \mathcal{C}_{\mathcal{V}}(V)$$

□