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Minkowski-Weyl Theorem

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Dedication.

Title: Minkowski-Weyl Theorem

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Abstract: The Minkowski-Weyl Theorem is proven for polyhedra by first showing the proof for cones, then the reductions from polyhedra to cones. The proof follows Ziegler [1], and uses Fourier-Motzkin elimination. A C++ implementation is given for the enumeration algorithm suggested by the proof, as well a means of testing the implementation against some special polyhedra. The Farkas Lemma is then proven and used to prove the validity of the testing methods.

Keywords: Minkowski-Weyl Theorem polyhedra Fourier-Motzkin C++

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Introduction

Polyhedra are fundamental mathematical objects. Two ways of describing polyhedra are:

1. A finite intersection of half-spaces
2. The *Minkowski-Sum* of the *convex-hull* of a finite set of rays and a finite set of points

The Minkowski-Weyl Theorem is a fundamental result in the theory of polyhedra. It states that both means of representation are equivalent. The proof given here is algorithmic in nature, using a technique known as *Fourier-Motzkin elimination*. The correctness of the algorithm also proves a result known as The Farkas Lemma.

This thesis is broken up into four chapters. Chapter 1 states the definitions necessary for Minkowski-Weyl Theorem, and states the theorem. Chapter 2 proves the theorem, by first considering the case that the polyhedron is a cone, then shows how to reduce the case of general polyhedra to that of cones. Chapter 3 shows a C++ implementation of the transformations described in Chapter 2. Chapter 4 presents a method of testing the program for special cases of polyhedra, known as either pointed or full-dimensional polyhedra. The Farkas Lemma is proven and extensively used to show the validity of the testing methods.

1. Minkowski-Weyl Theorem

1.1 Polyhedra

Definition 1.1.1 (Non-negative Linear Combination). Let $U \in \mathbb{R}^{d \times p}$, $\mathbf{t} \in \mathbb{R}^p$, $\mathbf{t} \geq \mathbf{0}$, then $\sum_{1 \leq j \leq p} t_j U^j = U\mathbf{t}$ is called a *non-negative linear combination* of U .

Definition 1.1.2 (V-Cone). Let $U \in \mathbb{R}^{d \times p}$. The set of all non-negative linear combinations of U is denoted $\text{cone}(U)$. Such a set is called a *V-Cone*.

Definition 1.1.3 (Convex Combination). Let $V \in \mathbb{R}^{d \times n}$, $\boldsymbol{\lambda} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \geq \mathbf{0}$, $\sum_{1 \leq j \leq n} \lambda_j = 1$, then $\sum_{1 \leq j \leq n} \lambda_j V^j$ is called a *convex combination* of V . The set of all convex combinations of V is denoted $\text{conv}(V)$.

Definition 1.1.4 (V-Polyhedron). Let $V \in \mathbb{R}^{d \times n}$, $U \in \mathbb{R}^{d \times p}$. Then the set

$$\{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \text{cone}(U), \mathbf{y} \in \text{conv}(V)\}$$

is called a *V-Polyhedron*.

Note: Given two sets P and Q , the set $P + Q = \{p + q \mid p \in P, q \in Q\}$ is called the *Minkowski Sum* of P and Q . Therefore, we will write a V-Polyhedron as $\text{cone}(U) + \text{conv}(V)$ for some U and V .

Definition 1.1.5 (H-Polyhedron). Let $A \in \mathbb{R}^{m \times d}$, $\mathbf{b} \in \mathbb{R}^m$. Then the set

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b} \right\}$$

is called an *H-Polyhedron*.

Definition 1.1.6 (H-Cone). Let $A \in \mathbb{R}^{m \times d}$. Then the set

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$$

is called an *H-Cone*.

A simple but useful property of cones is that they are closed under addition and positive scaling.

Proposition 1.1.1 (Closure Property of Cones). *Let C be either an H-Cone or a V-Cone, for each i $\mathbf{x}^i \in C$, and $c_i \geq 0$. Then:*

$$\sum_i c_i \mathbf{x}^i \in C$$

Proof. First we prove Proposition 1.1.1 for H-Cones, then for V-Cones. If, for each i , $A\mathbf{x}^i \leq \mathbf{0}$, then $A(c_i\mathbf{x}^i) = t_i A\mathbf{x}^i \leq \mathbf{0}$, and

$$A\left(\sum_i c_i \mathbf{x}^i\right) = \sum_i A(c_i \mathbf{x}^i) = \sum_i c_i A\mathbf{x}^i \leq \sum_i \mathbf{0} \leq \mathbf{0}$$

So, $\sum_i c_i \mathbf{x}^i \in C$ when C is an H-Cone. Next, suppose that $C = \text{cone}(U)$, and for each i , $\exists \mathbf{t}_i \geq \mathbf{0} : \mathbf{x}^i = U\mathbf{t}_i$. Then $c_i \mathbf{t}_i \geq \mathbf{0}$, and $\sum_i c_i \mathbf{t}_i \geq \mathbf{0}$. Therefore

$$\sum_i c_i \mathbf{x}^i = \sum_i c_i U\mathbf{t}_i = \sum_i U(c_i \mathbf{t}_i) = U\left(\sum_i c_i \mathbf{t}_i\right)$$

So, $\sum_i c_i \mathbf{x}^i \in C$ when C is a V-Cone. □

This proposition will be used in the following way: if we wish to show that $\sum_i c_i \mathbf{x}^i$ is in a member of some cone C , it suffices to show that, for each i , $c_i \geq 0$ and $\mathbf{x}^i \in C$.

1.2 Minkowski-Weyl Theorem

The following theorem is the basic result to be proved in this thesis, which states that V-Polyhedra and H-Polyhedra are two different representations of the same objects.

Theorem 1 (Minkowski-Weyl Theorem). *Every V-Polyhedron is an H-Polyhedron, and every H-Polyhedron is a V-Polyhedron.*

The proof proceeds by first showing that V-Cones are representable as H-Cones, and H-Cones are representable as V-Cones. Then it is shown that the case of polyhedra can be reduced to cones.

Theorem 2 (Minkowski-Weyl Theorem for Cones). *Every V-Cone is an H-Cone, and every H-Cone is a V-Cone.*

2. Proof of the Minkowski-Weyl Theorem

2.1 Every V-Cone is an H-Cone

Definition 2.1.1 (Coordinate Projection). Let I be the identity matrix. Then the matrix I' formed by deleting some rows from I is called a **coordinate-projection**.

The proof rests on the following two lemmas:

Lemma 2.1.1 (Lifting a V-Cone). *Every V-Cone is a coordinate-projection of an H-Cone.*

Lemma 2.1.2 (Projecting an H-Cone). *Every coordinate-projection of an H-Cone is an H-Cone.*

Proof. Given Lemma 2.1.1 and Lemma 2.1.2, the proof follows simply. Given a V-Cone, we use Lifting a V-Cone to get a description involving coordinate-projection of an H-Cone. Then we can apply Projecting an H-Cone in order to get an H-Cone. \square

Proof of Lifting a V-Cone. We prove that every V-Cone is a coordinate-projection of an H-Cone, by giving an explicit formula. Let $U \in \mathbb{R}^{d \times p}$, and observe that

$$\text{cone}(U) = \{U\mathbf{t} \mid \mathbf{t} \in \mathbb{R}^p, \mathbf{t} \geq \mathbf{0}\} = \left\{ \mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \in \mathbb{R}^p) \mathbf{x} = U\mathbf{t}, \mathbf{t} \geq \mathbf{0} \right\}$$

We will collect \mathbf{t} and \mathbf{x} on the left side of the inequality, treating \mathbf{t} as a variable and expressing its constraints as linear inequalities, then project away the coordinates corresponding to \mathbf{t} . The following expression takes one step:

$$\mathbf{t} \geq \mathbf{0} \Leftrightarrow -I\mathbf{t} \leq \mathbf{0} \tag{2.1}$$

Using the equality: $a = 0 \Leftrightarrow a \leq 0 \wedge -a \leq 0$, and block matrix notation, we take the second step.

$$\mathbf{x} = U\mathbf{t} \Leftrightarrow \mathbf{x} - U\mathbf{t} = \mathbf{0} \Leftrightarrow \begin{pmatrix} I & -U \\ -I & U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \tag{2.2}$$

Comparing (2.1) and (2.2), we define a matrix transform:

Transform 1 (V-Cone Lift).

$$T_V(U) = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix}$$

So we define $A' = T_V(U)$, then we can rewrite $\text{cone}(U)$:

$$\text{cone}(U) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid A' \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \right\}$$

Let $\Pi \in \{0, 1\}^{d \times (d+p)}$ be the identity matrix in $\mathbb{R}^{(d+p) \times (d+p)}$, but with the last p -rows deleted. Then Π is a coordinate projection, and the above expression can be written:

$$\text{cone}(U) = \Pi \left(\left\{ \mathbf{y} \in \mathbb{R}^{d+p} \mid A' \mathbf{y} \leq \mathbf{0} \right\} \right) \quad (2.3)$$

This is a coordinate projection of an H-Cone, and Lifting a V-Cone is shown. \square

To prove Projecting an H-Cone, we use two separate propositions.

Proposition 2.1.3 (Projecting Null Columns). *Let $B \in \mathbb{R}^{m' \times (d+p)}$, B' be B with the last p columns deleted, and Π the identity matrix with the last p rows deleted (i.e. $B' = \Pi B$). Furthermore, suppose that the last p columns of B are $\mathbf{0}$. Then*

$$\Pi \left(\left\{ \mathbf{y} \in \mathbb{R}^{d+p} \mid B \mathbf{y} \leq \mathbf{0} \right\} \right) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid B' \mathbf{x} \leq \mathbf{0} \right\}$$

Proof. Recall that $B \mathbf{y} \leq \mathbf{0}$ means that $(\forall i) \langle B_i, \mathbf{y} \rangle \leq 0$. Because the last p columns of B are $\mathbf{0}$, any row B_i of B can be written $(B'_i, \mathbf{0})$, with $\mathbf{0} \in \mathbb{R}^p$. We can also rewrite $\mathbf{y} \in \mathbb{R}^{d+p}$ as (\mathbf{x}, \mathbf{w}) with $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^p$, so that $\mathbf{x} = \Pi(\mathbf{y})$. Then

$$\langle B, \mathbf{y} \rangle = \langle (B'_i, \mathbf{0}), (\mathbf{x}, \mathbf{w}) \rangle = \langle B'_i, \mathbf{x} \rangle = \langle B'_i, \Pi(\mathbf{y}) \rangle$$

It follows that

$$\langle B_i, \mathbf{y} \rangle \leq 0 \Leftrightarrow \langle B'_i, \Pi(\mathbf{y}) \rangle \leq 0$$

Since B_i is an arbitrary row of B , the proposition is shown. \square

In order to use the above proposition, we need a matrix with columns which are $\mathbf{0}$. The next proposition shows us how to obtain such a matrix from another, while maintaining certain properties.

Proposition 2.1.4 (Fourier Motzkin Elimination for H-Cones). *Let $B \in \mathbb{R}^{m_1 \times (d+p)}$, $1 \leq k \leq (d+p)$, and $\mathbf{x} = \sum_{i \neq k} x_i \mathbf{e}_i$. Then there exists a matrix $B' \in \mathbb{R}^{m_2 \times (d+p)}$ with the following properties:*

1. Every row of B' is a positive linear combination of rows of B .
2. m_2 is finite.
3. The k -th column of B' is $\mathbf{0}$.
4. $(\exists t) B(\mathbf{x} + t \mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B' \mathbf{x} \leq \mathbf{0}$

Proof. Partition the rows of B as follows:

$$\begin{aligned} P &= i \mid B_i^k > 0 \\ N &= j \mid B_j^k < 0 \\ Z &= l \mid B_l^k = 0 \end{aligned}$$

Then let B' be a matrix with rows of the following forms:

$$\begin{aligned} C_l &= B_l & | \quad l \in Z \\ C_{ij} &= B_i^k B_j - B_j^k B_i & | \quad i \in P, j \in N \end{aligned}$$

1 and 2 are clear. 3 is satisfied for rows indexed by Z by definition. That it holds for the other rows, observe:

$$\langle C_{ij}, \mathbf{e}_k \rangle = \langle B_i^k B_j - B_j^k B_i, \mathbf{e}_k \rangle = B_i^k B_j^k - B_j^k B_i^k = 0$$

The right direction of 4 is shown in the following calculations. Because $B_l^k = 0$:

$$\langle B_l, \mathbf{x} + t\mathbf{e}_k \rangle = \langle B_l, \mathbf{x} \rangle + tB_l^k = \langle B_l, \mathbf{x} \rangle = \langle C_l, \mathbf{x} \rangle$$

So $\langle B_l, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0 \Rightarrow \langle C_l, \mathbf{x} \rangle \leq 0$. For rows indexed by P, N , because $B_i^k B_j^k - B_j^k B_i^k = 0$ we have:

$$\langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \rangle = \langle B_i^k B_j - B_j^k B_i, \mathbf{x} \rangle$$

Because B_i^k and $-B_j^k$ are non-negative:

$$\langle B_i, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0, \langle B_j, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0 \Rightarrow \langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0$$

Therefore $\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \rangle \leq 0$, and the right implication is shown.

Now suppose that $B'\mathbf{x} \leq \mathbf{0}$. The task is to find a t so that $B\mathbf{x} \leq \mathbf{0}$. Because rows indexed by Z have $B_l^k = 0$, $B'(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Rightarrow B\mathbf{x} \leq \mathbf{0}$. So the task is to find a t so that the inequality holds for rows indexed by P and N . Observe

$$\begin{aligned} \forall i \in P, \forall j \in N \quad & \langle B_i^k B_j - B_j^k B_i, \mathbf{x} \rangle \leq 0 & \Leftrightarrow \\ \forall i \in P, \forall j \in N \quad & \langle B_i^k B_j, \mathbf{x} \rangle \leq \langle B_j^k B_i, \mathbf{x} \rangle & \Leftrightarrow \\ \forall i \in P, \forall j \in N \quad & \langle B_i/B_i^k, \mathbf{x} \rangle \leq \langle B_j/B_j^k, \mathbf{x} \rangle & \Leftrightarrow \\ & \max_{i \in P} \langle B_i/B_i^k, \mathbf{x} \rangle \leq \min_{j \in N} \langle B_j/B_j^k, \mathbf{x} \rangle \end{aligned}$$

Note that the third inequality changes directions because $B_j^k < 0$. Now we choose t to lie in this last interval, and show that we can use it to satisfy all of the constraints given by B . So, we have a t such that

$$\max_{i \in P} \langle B_i/B_i^k, \mathbf{x} \rangle \leq t \leq \min_{j \in N} \langle B_j/B_j^k, \mathbf{x} \rangle$$

In particular,

$$(\forall j \in N) \quad \langle B_j/B_j^k, \mathbf{x} \rangle \geq t \Rightarrow \langle B_j, \mathbf{x} \rangle - B_j^k t \leq 0$$

Again, the inequality changes directions because $B_j^k < 0$. Now consider a row B_j from B :

$$\langle B_j, \mathbf{x} - t\mathbf{e}_k \rangle = B_j \mathbf{x} - B_j^k t \leq 0$$

Similarly,

$$(\forall i \in P) \quad t \geq B_i/B_i^k \mathbf{x} \Rightarrow 0 \geq B_i \mathbf{x} - B_i^k t$$

Now consider a row B_i from B :

$$\langle B_i, \mathbf{x} - t\mathbf{e}_k \rangle = B_i \mathbf{x} - B_i^k t \leq 0$$

So, we've demonstrated that $\mathbf{x} - t\mathbf{e}_k$ satisfies all the constraints from B , and the left implication is shown. So ∇ holds. \square

Remark 1 (Fourier Motzkin Matrix). Fourier Motzkin Elimination for H-Cones highlights the properties of the matrix B' . Upon close inspection, we can create a Matrix Y such that $B' = YB$, and every element of Y is non-negative. Create the following set of row vectors Y

$$\begin{aligned} & \mathbf{e}_l \quad | \quad l \in Z \\ & B_i^k \mathbf{e}_j - B_j^k \mathbf{e}_i \quad | \quad i \in P, j \in N \end{aligned}$$

Since the basis vectors simply select rows during matrix multiplication, it is clear that

$$B' = YB$$

Proof of Projecting an H-Cone. Here we prove the case that the coordinate projection is onto the first d of $d + p$ coordinates. Let $\{\mathbf{y} \in \mathbb{R}^{d+p} : A'\mathbf{y} \leq \mathbf{0}\}$ be the H-Cone we need to project, and Π the coordinate-projection we need to apply (the identity matrix with the last p columns deleted). For each $1 \leq k \leq p$ we can use Fourier Motzkin Elimination for H-Cones in an incremental manner, starting with A' .

```

let  $B_0 := A'$ 
for  $1 \leq k \leq p$ 
  let  $B_k :=$  result of proposition 2 applied to  $B_{k-1}, \mathbf{e}_{d+k}$ 
endfor
return  $B_p$ 

```

Consider the resulting B . Property 2 holds throughout, so B is finite. After each iteration, property 3 holds for k , so the k -th column is $\mathbf{0}$. Since each iteration only results from non-negative combinations of the result of the previous iteration (property 1), once a column is $\mathbf{0}$ it remains so. Therefore, at the end of the process, the last p columns of B are all $\mathbf{0}$. Then, by Projecting Null Columns, we can apply Π to B by simply deleting the last p columns of B . Denote this resulting matrix A . We still need to check that

$$\Pi \{ \mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0} \} = \{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \} \quad (2.4)$$

This follows from the following:

$$A'\mathbf{y} \leq \mathbf{0} \Rightarrow A(\Pi(\mathbf{y})) \leq \mathbf{0} \quad (2.5)$$

$$A\mathbf{x} \leq \mathbf{0} \Rightarrow (\exists t_1) \dots (\exists t_p) A'(\mathbf{x} + t_1 \mathbf{e}_{d+1} + \dots + t_p \mathbf{e}_{d+p}) \leq \mathbf{0} \quad (2.6)$$

The key observation of this verification utilizes property 4 of Fourier Motzkin Elimination for H-Cones:

$$(\exists t)B(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$$

In what follows, let $\mathbf{x} = \sum_{1 \leq j \leq d} x_j \mathbf{e}_j$. The above property is applied sequentially to the sets B_k as follows:

$$\begin{array}{llll} (\exists t_p)(\exists t_{p-1}) \dots (\exists t_1) & B_0(\mathbf{x} + t_1 \mathbf{e}_p + t_2 \mathbf{e}_{p-1} + \dots + t_p \mathbf{e}_d) \leq \mathbf{0} & \Leftrightarrow & \\ (\exists t_p) \dots (\exists t_2) & B_1(\mathbf{x} + t_2 \mathbf{e}_{d+2} + \dots + t_p \mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow & \\ \vdots & \vdots & & \vdots \\ (\exists t_p) & B_{p-1}(\mathbf{x} + t_p \mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow & \\ & B_p \mathbf{x} \leq \mathbf{0} & & \end{array}$$

Because $A' = B_0$, and A is B_p with the last p columns deleted, (2.5) and (2.6) hold, therefore (2.4) holds, and the proof of Projecting an H-Cone is complete, and we've shown that a coordinate projection of an H-Cone is again an H-Cone. \square

With Lifting a V-Cone and Projecting an H-Cone proven, we are now certain that any V-Cone is also an H-Cone.

2.2 Every H-Cone is a V-Cone

Definition 2.2.1 (Coordinate Hyperplane). A set of the form

$$\{\mathbf{x} \in \mathbb{R}^{d+m} \mid \langle \mathbf{x}, \mathbf{e}_k \rangle = 0\} = \{\mathbf{x} \in \mathbb{R}^{d+m} \mid x_k = 0\}$$

is called a *coordinate-hyperplane*.

This is how coordinate hyperplanes will be used. We consider a V-Cone intersected with some coordinate hyperplanes, and write it in the following way:

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U' \mathbf{t} \right\} \quad (2.7)$$

If we suppose that $U' \subset \mathbb{R}^{d+m}$, and Π is the identity matrix with the last m rows deleted, then this is just a convenient way of writing:

$$\Pi \left(\text{cone}(U') \cap \{x_{d+1} = 0\} \cap \dots \cap \{x_{d+m} = 0\} \right) \quad (2.8)$$

The proof rests on the following three propositions:

Lemma 2.2.1 (Lifting an H-Cone). *Every H-Cone is a coordinate-projection of a V-Cone intersected with some coordinate hyperplanes.*

Lemma 2.2.2 (Intersecting a V-Cone). *Every V-Cone intersected with a coordinate-hyperplane is a V-Cone.*

Lemma 2.2.3 (Projecting a V-Cone). *Every coordinate-projection of a V-Cone is an V-Cone.*

Proof. Given lemma 2.2.1, lemma 2.2.2, and lemma 2.2.3, the proof follows simply. Given an H-Cone, we use Lifting an H-Cone to get a description involving the coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. We apply Intersecting a V-Cone as many times as necessary to eliminate the intersections, then we can apply Projecting a V-Cone in order to get a V-Cone. \square

Proof of Lifting an H-Cone. Let $A \in \mathbb{R}^{m \times d}$, we now show that the H-Cone

$$\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0}\}$$

can be written as the projection of a V-Cone intersected with some hyperplanes. We use the following transform.

Transform 2 (H-Cone Lift).

$$T_H(A) = \begin{pmatrix} \mathbf{0} & I & -I \\ I & A & -A \end{pmatrix}$$

Define

$$U' = T_H(A) = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}, \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}, 1 \leq j \leq d, 1 \leq i \leq m \right\}$$

We then claim:

$$\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0}\} = \left\{ \mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \geq \mathbf{0}) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t} \right\} \quad (2.9)$$

First, considering (2.7) and (2.8), observe that this is a coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. Next, we note that

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$$

We can write this as a sum with all positive coefficients if we split up the x_j as follows:

$$x_j^+ = \begin{cases} x_j & x_j \geq 0 \\ 0 & x_j < 0 \end{cases} \quad x_j^- = \begin{cases} 0 & x_j \geq 0 \\ -x_j & x_j < 0 \end{cases}$$

Then we have

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} \quad (2.10)$$

where $x_j^+, x_j^- \geq 0$. Also observe that

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$$

This can also be written

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \quad (2.11)$$

(2.10) and (2.11) together show

$$A\mathbf{x} \leq \mathbf{0} \Rightarrow (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

Conversely, suppose

$$(\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

We would like to show that $A\mathbf{x} \leq \mathbf{0}$. Let x_j^+, x_j^-, w_i take the values of \mathbf{t} that are coefficients of $\begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$, $\begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}$, and $\begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}$ respectively, and denote $x_j = x_j^+ - x_j^-$. Then we have

$$\begin{aligned} \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} &= \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\ &= \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} \end{aligned}$$

where $\mathbf{w} \geq \mathbf{0}$. By (2.11) we have $A\mathbf{x} \leq \mathbf{0}$. So (2.9) holds. \square

The proof of Intersecting a V-Cone relies upon the following proposition.

Proposition 2.2.4 (Fourier Motzkin Elimination for V-Cones). *Let $Y \in \mathbb{R}^{(d+m) \times n_1}$, $1 \leq k \leq m$, and \mathbf{x} satisfy $x_k = 0$. Then there exists a matrix $Y' \in \mathbb{R}^{(d+m) \times n_2}$ with the following properties:*

1. *Every column of Y' is a positive linear combination of columns of Y .*
2. *n_2 is finite.*
3. *The k -th row of Y' is $\mathbf{0}$.*
4. *$(\exists \mathbf{t} \geq \mathbf{0}) \mathbf{x} = Y\mathbf{t} \Leftrightarrow (\exists \mathbf{t}' \geq \mathbf{0}) \mathbf{x} = Y'\mathbf{t}'$*

Proof. We partition the columns of Y :

$$\begin{aligned} P &= i \mid Y_k^i > 0 \\ N &= j \mid Y_k^j < 0 \\ Z &= l \mid Y_k^l = 0 \end{aligned}$$

We then define Y' :

$$Y' = \{Y^l \mid l \in Z\} \cup \{Y_k^i Y^j - Y_k^j Y^i \mid i \in P, j \in N\}$$

1 and 2 are clear. 3 can be seen from:

$$\begin{aligned} \langle Y'^l, \mathbf{e}^k \rangle &= 0 \\ \langle Y'^{ij}, \mathbf{e}^k \rangle &= \langle Y_k^i Y^j - Y_k^j Y^i, \mathbf{e}^k \rangle = Y_k^i Y_k^j - Y_k^j Y_k^i = 0 \end{aligned} \tag{2.12}$$

Before moving on to the proof, we first note how we may write our vectors.

$$Y\mathbf{t} = \sum_{l \in Z} t_l Y^l + \sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j$$

$$Y'\mathbf{t} = \sum_{l \in Z} t_l Y^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (Y_k^i Y^j - Y_k^j Y^i)$$

Then, by Closure Property of Cones (cone closure), to show that the proposition is true, we need only show that, given any $t_i, t_j \geq 0$ ($t_{ij} \geq 0$), there exists $t_{ij} \geq 0$ ($t_i, t_j \geq 0$) such that

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} t_{ij} (Y_k^i Y^j - Y_k^j Y^i) \quad (2.13)$$

Proposition 2.2.5 (Sum Mixture). *Suppose that*

$$\sum_{i \in P} t_i Y_{d+1}^i + \sum_{j \in N} t_j Y_{d+1}^j = 0 \quad Y_k^j < 0 < Y_k^i$$

Then the following holds

$$(t_i, t_j \geq 0) \Rightarrow (\exists t_{ij} \geq 0) \quad \text{such that (2.13) holds}$$

$$(t_{ij} \geq 0) \Rightarrow (\exists t_i, t_j \geq 0) \quad \text{such that (2.13) holds}$$

Proof of proposition 2.2.5. First note that if all $t_i = 0, t_j = 0$, then choosing $t_{ij} = 0$ satisfies (2.13), likewise if all $t_{ij} = 0$, then $t_i = 0, t_j = 0$ satisfies (2.13). So suppose that some $t_i \neq 0, t_j \neq 0, t_{ij} \neq 0$.

The right hand side of (2.13) can be written

$$\sum_{j \in N} \left(\sum_{i \in P} t_{ij} Y_k^i \right) Y^j + \sum_{i \in P} \left(- \sum_{j \in N} t_{ij} Y_k^j \right) Y^i$$

This means, given $t_{ij} \geq 0$, we can choose $t_j = \sum_{i \in P} t_{ij} Y_k^i$, and $t_i = - \sum_{j \in N} t_{ij} Y_k^j$, both of which are greater than 0.

Now suppose we have been given $t_i \geq 0, t_j \geq 0$. First observe:

$$0 = \sum_{i \in P} t_i Y_k^i + \sum_{j \in N} t_j Y_k^j \Rightarrow \sum_{i \in P} t_i Y_k^i = - \sum_{j \in N} t_j Y_k^j$$

Denote the value in this equality as σ , and note that $\sigma > 0$. Then

$$\sum_{i \in P} t_i Y^i = \frac{- \sum_{j \in N} t_j Y_k^j}{\sigma} \sum_{i \in P} t_i Y^i = \sum_{\substack{i \in P \\ j \in N}} - \frac{t_i t_j}{\sigma} Y_k^j Y^i$$

$$\sum_{j \in N} t_j Y^j = \frac{\sum_{i \in P} t_i Y_k^i}{\sigma} \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma} Y_k^i Y^j$$

Combining these results, we have

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma} (Y_k^i Y^j - Y_k^j Y^i)$$

□

Finally, we can conclude that, given $\mathbf{t} \geq \mathbf{0}$, if $Y\mathbf{t}$ has a 0 in the final coordinate, then we can write it as $Y'\mathbf{t}'$ where $\mathbf{t}' \geq \mathbf{0}$, and any non-negative linear combination of vectors from Y' can be written as a non-negative linear combination of vectors from Y , and will necessarily have the k -th coordinate be 0 by property 3. So property 4 holds. \square

Proof of Intersecting a V-Cone. In Fourier Motzkin Elimination for V-Cones, the assumption that $x_k = 0$ in property 4 creates the set $\text{cone}(Y) \cap \{\mathbf{x} \mid x_k = 0\}$. This set, by property 4, is $\text{cone}(Y')$. \square

Proof of Projecting a V-Cone. We shall prove that the coordinate-projection of a V-Cone is again a V-Cone. Let Π be the relevant projection, then we have:

$$\Pi\{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U\mathbf{t}) \mid \mathbf{t} \geq \mathbf{0}\} = \{(\Pi U)\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\}$$

The last equality follows from associativity of matrix multiplication. Therefore,

$$\Pi(\text{cone}(U)) = \text{cone}(\Pi U)$$

\square

Having shown that H-Cones are V-Cones, the proof of the Minkowski-Weyl Theorem for cones is complete.

2.3 Reducing Polyhedra to Cones

Definition 2.3.1 (Hyperplane). Let $\mathbf{y} \in \mathbb{R}^d$, $c \in \mathbb{R}$. Then a set of the form

$$\{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{y}, \mathbf{x} \rangle = c\}$$

is called a *hyperplane*.

2.3.1 H-Polyhedra \leftrightarrow H-Cones

Proposition 2.3.1. *Every H-Polyhedron can be written as an H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$, and any H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$ is an H-Polyhedron.*

Proof. We begin by re-writing the expression:

$$A\mathbf{x} \leq \mathbf{b} \Leftrightarrow -\mathbf{b} + A\mathbf{x} \leq \mathbf{0} \Leftrightarrow \begin{bmatrix} -\mathbf{b} & A \end{bmatrix} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0}$$

Note that

$$\left\{ \mathbf{x} \mid \begin{bmatrix} -\mathbf{b} & A \end{bmatrix} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \mid \begin{bmatrix} -\mathbf{b} & A \end{bmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} \cap \{\mathbf{x} \mid x_0 = 1\}$$

It follows that

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \mid \begin{bmatrix} -\mathbf{b} & A \end{bmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} \cap \{\mathbf{x} \mid x_0 = 1\}$$

\square

2.3.2 V-Polyhedra \leftrightarrow V-Cone

Proposition 2.3.2 (V-Polyhedron \rightarrow V-Cone).

$$\text{cone}(U) + \text{conv}(V) = \text{cone} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ U & V \end{pmatrix} \cap \{\mathbf{x} \mid x_0 = 1\}$$

Proof. For the value 1 to appear in the first coordinate, a convex combination of the vectors from $(\mathbf{1}, V)$ must be taken. After that, any non-negative combination of $(\mathbf{0}, U)$ added to this vector won't affect the 1 in the first coordinate. \square

This shows that a V-Polyhedron may be written as an intersection of a V-Cone and the hyperplane $\{\mathbf{x} \mid x_0 = 1\}$. It is more difficult to show that, given a V-Cone, that you can intersect it with the hyperplane $\{\mathbf{x} \mid x_0 = 1\}$ and get a V-Polyhedron out of it.

Proposition 2.3.3 (V-Cone \rightarrow V-Polyhedron). *Let Π be the identity matrix with the first row deleted. Then, for any set $C_U = \text{cone}(U)$ there are sets W and V such that*

$$\text{cone}(V) \cap \{\mathbf{x} \mid x_0 = 1\} = \text{cone}(W) + \text{conv}(V)$$

Proof. We partition U into the sets:

$$\begin{aligned} P &= i \mid U_0^i > 0 \\ N &= j \mid U_0^j < 0 \\ Z &= l \mid U_0^l = 0 \end{aligned}$$

And define two new sets:

$$\begin{aligned} W &= \{U^l \mid l \in Z\} \cup \{U_0^i U^j - U_0^j U^i \mid i \in P, j \in N\} \\ V &= \{U^i / U_0^i \mid i \in P\} \end{aligned}$$

Then I claim that

$$C_U = \text{cone}(W) + \text{conv}(V)$$

Say $\mathbf{x} \in \text{cone}(W)$, and $\mathbf{y} \in \text{conv}(V)$. Then \mathbf{x} can be written

$$\begin{aligned} \mathbf{x} &= \sum_{l \in Z} t_l U^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (U_0^i U^j - U_0^j U^i) \\ &= \sum_{l \in Z} t_l U^l + \sum_{j \in N} \left(\sum_{i \in P} t_{ij} U_0^i \right) U^j + \sum_{i \in P} \left(\sum_{j \in N} -t_{ij} U_0^j \right) U^i \end{aligned}$$

So $\mathbf{x} \in C_U$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{x} \rangle = \sum_{l \in Z} t_l U_0^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (U_0^i U_0^j - U_0^j U_0^i) = 0$$

So $x_0 = 0$. Similarly, \mathbf{y} can be written:

$$\mathbf{y} = \sum_{i \in P} \lambda_i U^i / U_0^i, \quad \sum_{i \in P} \lambda_i = 1$$

So $\mathbf{y} \in C_U$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{y} \rangle = \sum_{i \in P} \lambda_i U_0^i / U_0^i = \sum_{i \in P} \lambda_i = 1$$

So $y_0 = 1$ and $x_0 + y_0 = 1$. It follows that $\mathbf{x} + \mathbf{y} \in C_U$.

Next, suppose that $\mathbf{z} \in C_U$, then \mathbf{z} can be written

$$\mathbf{z} = \sum_{l \in Z} t_l U^l + \sum_{i \in P} t_i U^i + \sum_{j \in N} t_j U^j$$

It will be convenient to use shorter notation for these sums. Define the following:

$$\begin{aligned} \boldsymbol{\sigma}_Z &= \sum_{l \in Z} t_l U^l, & \sigma_l &= \sum_{l \in Z} t_l U_0^l = 0 \\ \boldsymbol{\sigma}_P &= \sum_{i \in P} t_i U^i, & \sigma_i &= \sum_{i \in P} t_i U_0^i \\ \boldsymbol{\sigma}_N &= \sum_{j \in N} t_j U^j, & \sigma_j &= \sum_{j \in N} t_j U_0^j \end{aligned}$$

Then it holds that

$$\begin{aligned} \langle \mathbf{e}_0, \mathbf{z} \rangle &= \sigma_l + \sigma_i + \sigma_j = \sigma_i + \sigma_j = 1 \quad \Rightarrow \quad -\sigma_j / \sigma_i = 1 - 1 / \sigma_i \\ \boldsymbol{\sigma}_P &= \boldsymbol{\sigma}_P / \sigma_i + (1 - 1 / \sigma_i) \boldsymbol{\sigma}_P = \boldsymbol{\sigma}_P / \sigma_i - (\sigma_j / \sigma_i) \boldsymbol{\sigma}_P \end{aligned}$$

Using the new notation, we can rewrite \mathbf{z} :

$$\mathbf{z} = \boldsymbol{\sigma}_Z + \boldsymbol{\sigma}_P + \boldsymbol{\sigma}_N = \boldsymbol{\sigma}_Z + \frac{\boldsymbol{\sigma}_P}{\sigma_i} - \frac{\sigma_j}{\sigma_i} \boldsymbol{\sigma}_P + \frac{\sigma_i}{\sigma_i} \boldsymbol{\sigma}_N = \boldsymbol{\sigma}_Z + \frac{\boldsymbol{\sigma}_P}{\sigma_i} + \frac{\sigma_i \boldsymbol{\sigma}_N - \sigma_j \boldsymbol{\sigma}_P}{\sigma_i}$$

Using ‘Closure Property of Cones’ on page 3, we need only show that

1. $\boldsymbol{\sigma}_Z \in \text{cone}(W)$
2. $(\sigma_i \boldsymbol{\sigma}_N - \sigma_j \boldsymbol{\sigma}_P) \in \text{cone}(W)$
3. $\boldsymbol{\sigma}_P / \sigma_i \in \text{conv}(V)$

Since each $U^l : l \in Z$ is in C_V , (1) holds. We also have:

$$\sigma_i \boldsymbol{\sigma}_N - \sigma_j \boldsymbol{\sigma}_P = \sum_{i \in P} t_i \sum_{j \in N} t_j U_0^i U^j - \sum_{j \in N} t_j \sum_{i \in P} t_i U_0^j U^i = \sum_{\substack{i \in P \\ j \in N}} t_i t_j (U_0^i U^j - U_0^j U^i)$$

So (2) holds. Finally,

$$\boldsymbol{\sigma}_P / \sigma_i = \sum_{i \in P} t_i U^i / \sigma_i = \sum_{i \in P} (t_i U_0^i / \sigma_i) (U^i / U_0^i)$$

Since $\sum_{i \in P} (t_i U_0^i / \sigma_i) = \sigma_i / \sigma_i = 1$, it follows that $\boldsymbol{\sigma}_P / \sigma_i \in \text{conv}(V)$. □

2.4 Picture of the Proof

Here we show a diagram that represent the proof of the Minkowski-Weyl Theorem.

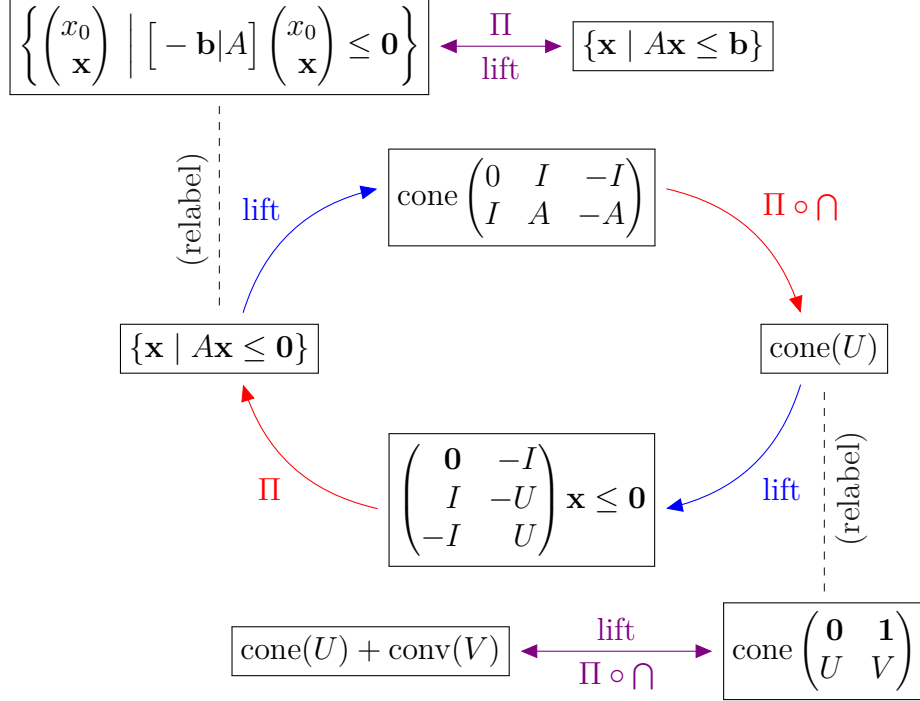


Figure 2.1: Diagram of the proof $P_H \leftrightarrow P_V$

Figure 2.1 shows the flow from an H-Polyhedron to a V-Polyhedron and back. There are **arrows** for transformations back and forth from polyhedra to cones, **arrows** to show the transformation between cones and intermediate representation, and **arrows** to show where Fourier Motzkin elimination is applied to reduce these intermediate representations to standard cones. V-Cones are **lifted** to H-Cones which need to be **projected** (Π), and H-Cones are **lifted** to V-Cones which need to be **intersected and projected** ($\Pi \circ \cap$).

3. C++ Implementation

The above transformations have been implemented in C++. Program `main.cpp` takes one argument specifying the type of input object. It reads the description of the object from standard input, and writes the result of the implied transformation to standard output. If no arguments are supplied, then a `usage` message is given. The `usage` message, which also contains the input format for the objects, is:

```
usage: ./main input_type
```

The input object is read on `stdin`, and the result of the transform is sent to `stdout`. `input_type` determines the type of input and output:

```
-vc # transforms a vcone into an hcone
-vp # transforms a vpolyhedron into an hpolyhedron
-hc # transforms an hcone into a vcone
-hp # transforms an hpolyhedron into a vpolyhedron
```

input format is as follows:

```
hcone := dimension ws (vector ws)*
vcone := dimension ws (vector ws)*
hpoly := dimension+1 ws (vector ws constraint ws)*
vpoly := dimension ws ('U' | 'V') ws vpoly_vecs*

ws      := whitespace, as would be read by "cin >> ws;"
dimension := a positive integer. For hpoly, add one to
            the dimension of the space (this extra
            dimension is for the constraint)
vector    := (dimension) doubles separated by whitespace
constraint := a double (the value b_i in <A_i,x> <= b_i)
'V' | 'U'  := the literal character 'U' or 'V'
vpoly_vecs := ([ 'U' ] ws vector) | ([ 'V' ] ws vector)
```

VPOLY ONLY:

```
vpoly contains two matrices:
  U - contains the rays of the vpolyhedron
  V - contains the points of the vpolyhedron
```

On input, enter 'U' or 'V' to indicate which matrix should receive the vectors that follow. You can switch back and forth as you like, but either 'U' or 'V' must be entered before starting to input vectors.

EXAMPLES:

```
$ ./main -vc <<< "2 1 0"
```

OUTPUT:

```
2
-0 -1
0 1
0 0
-1 0
```

```
$ ./main -hc <<< "2 1 0 0 1"
```

OUTPUT:

```
2
-1 0
0 0
0 0
0 -1
```

```
$ ./main -vp <<< "2 U 1 0 V 0 0 1 1"
```

OUTPUT:

```
3
0 0 -0
0 0 -0
0 1 1
0 0 1
-1 1 -0
-1 0 -0
0 0 -0
0 -1 -0
$ ./main -hp <<< "3 0 -1 0 0 1 1 -1 1 0"
```

OUTPUT:

```
2
U
1 0
0 0
0 0
0 0
V
0 0
1 1
```

The files pertaining to the implementation will be discussed in the following sections, but here is a table showing the include dependencies followed by a short

summary of the files.

file	includes
linear_algebra.h	
fourier_motzkin.h	linear_algebra.h
polyhedra.h	fourier_motzkin.h
main.cpp	polyhedra.h
test_functions.h	linear_algebra.h
test.cpp	test_functions.h, polyhedra.h

Here is a very brief summary of the files mentioned in the above table, more details are given in sequent sections.

- `linear_algebra.h`
Types `Vector` and `Matrix`, and some basic functionality for them
- `fourier_motzkin.h`
Fourier Motzkin elimination, Minkowski-Weyl Theorem for cones
- `polyhedra.{cpp,h}`
Transforms between polytopes and polyhedra, Minkowski-Weyl Theorem
- `test_functions.h`
Types and functions for testing the algorithms
- `test.cpp`
Test cases for the algorithms and the functions from `test_functions.h`

3.1 Code

The relevant code will be displayed with commentary below. Some of the code relating to C++ specific technicalities and I/O is omitted.

3.2 `linear_algebra.h`

The types `Vector` and `Vectors` are used in the representation of polyhedra. The `std::valarray` template is used because it has built-in vector-space operations (sum and scaling). `std::vector`, is used as a container of `Vectors`, however other containers could be used.

```
10 using Vector = std::valarray<double>;
11 using Vectors = std::vector<Vector>;
```

The `class` `Matrix` implements a subset of what a *C++ Container* should. It is the primary type for representing polyhedra, and directly represents Cones, as well as H-Polyhedra. The interface is designed to enforce the following invariant:

$$(\forall v \in \text{vectors}) v.\text{size}() == d$$

The factory function `read_Matrix` is provided to read a `Matrix` from an `istream`. It is necessary because the value of `d` can't be known before reading some of the stream.

```

13 class Matrix {
14 // invariant: d >= 0
15 // invariant: (forall valid i) vectors[i].size() == d
16 public:
17     const size_t d; // size of all Vectors
18 private:
19     Vectors vectors;
20 public:
21     // needed for back_insert_iterator
22     using value_type = Vector;
23
24     Matrix(size_t d);
25     Matrix(std::initializer_list<Vector>&&);
26     bool check() const; // checks each Vector has size d
27
28     //defaults don't work because of const member
29     Matrix(const Matrix&);
30     Matrix(Matrix&&);
31     Matrix &operator=(const Matrix&);
32     Matrix &operator=(Matrix&&);
33     Matrix &operator=(std::initializer_list<Vector>&&);
34
35     static Matrix read_Matrix(std::istream&);
36
37     Vectors::iterator      begin();
38     Vectors::iterator      end();
39     Vectors::const_iterator begin() const;
40     Vectors::const_iterator end()   const;
41
42     bool    empty() const;
43     size_t  size()   const;
44     Vector& back();
45
46     Vector& add_Vector();
47     void push_back(const Vector &v);
48     void push_back(Vector &&v);
49 };

```

The `struct VPoly` gather two `Matrix`s needed to represent a V-Polyhedron. The `Matrix U` corresponds to the rays that generate the cone, and the `Matrix V` corresponds to the points, i.e.

$$\text{vpoly} = \text{cone}(\text{vpoly}.U) + \text{conv}(\text{vpoly}.V)$$

```

51 struct VPoly {
52     const size_t d;
53     Matrix U; // rays
54     Matrix V; // points
55
56     VPoly(size_t d) : d{d}, U{d}, V{d} {}
57     VPoly(std::initializer_list<Vector>&&,
58           std::initializer_list<Vector>&&);

```



```

59     bool check() const;
60
61     static VPoly read_VPoly(std::istream&);
62 };

```

The `class` `input_error` is thrown to indicate an invalid input to the program, and provide some clue as to why it failed. Here are two command line examples:

```

$ ./main -vc <<< "0"
terminate called after throwing an instance of 'input_error'
  what():  bad d: 0
Aborted (core dumped)
$ ./main -vc <<< "2 1"
error reading matrix, vector 1
terminate called after throwing an instance of 'input_error'
  what():  failed to read vector: istream failed
Aborted (core dumped)

```

```

64 class input_error : public std::runtime_error {
65 public:
66     input_error(const char*s);
67     input_error(const std::string &s);
68 };

```

`operator>>` and `operator<<` implement the input format described in `usage.txt`.

```

70 std::istream& operator>>(std::istream&, Vector&);
71 std::istream& operator>>(std::istream&, Matrix&);
72 std::istream& operator>>(std::istream&, VPoly&);

74 std::ostream& operator<<(std::ostream& o, const Vector&);
75 std::ostream& operator<<(std::ostream& o, const Matrix&);
76 std::ostream& operator<<(std::ostream& o, const VPoly&);

```

`usage()` outputs the usage message shown above.

```

78 int usage();

```

3.3 linear_algebra.cpp

`e_k` creates the canonical basis Vector $e_k \in \mathbb{R}^d$.

```

232 Vector e_k(size_t d, size_t k) {
233     Vector result(d);
234     result[k] = 1;
235     return result;
236 }

```

`concatentate` takes the Vectors $l \in \mathbb{R}^{l.size()}$ and $r \in \mathbb{R}^{r.size()}$ and `returns` the Vector $(l,r) \in \mathbb{R}^{l.size() + r.size()}$

```

239 Vector concatenate(const Vector &l, const Vector &r) {
240     Vector result(l.size() + r.size());
241     copy(begin(l), end(l), begin(result));
242     copy(begin(r), end(r), next(begin(result), l.size()));
243     return result;
244 }

```

`get_column` returns the k -th column of the Matrix M . Note that while a Matrix may logically represent either a collection of row or column Vectors, `get_column` is only used in the function `transpose`, where this distinction is unimportant.

```

249 Vector get_column(const Matrix &M, size_t k) {
250     if (!(0 <= k && k < M.d)) {
251         throw std::out_of_range("k < 0 || M.d <= k");
252     }
253     Vector result(M.size());
254     size_t result_row{0};
255     for (auto &&row : M) {
256         result[result_row++] = row[k];
257     }
258     return result;
259 }

```

`transpose` returns the transpose of Matrix M .

```

262 Matrix transpose(const Matrix &M) {
263     if (M.empty()) {
264         return M;
265     }
266     Matrix result{M.size()};
267     // for every column of M,
268     for (size_t k = 0; k < M.d; ++k) {
269         result.push_back(get_column(M,k));
270     }
271     return result;
272 }

```

A slice object can be used to conveniently obtain a subset of a valarray. `slice_matrix` returns the Matrix obtained by applying the slice s to each Vector of the Matrix.

```

275 Matrix slice_matrix(const Matrix &M, const std::slice &s) {
276     Matrix result{s.size()};
277     transform(M.begin(), M.end(), back_inserter(result),
278         [s](const Vector &v) { return v[s]; });
279     return result;
280 }

```

3.4 fourier_motzkin.cpp

A slice object is determined by three fields: `start`, `size`, and `stride`, and implicitly represents all indices of the form:

$$\sum_{0 \leq k < \text{size}} \text{start} + k \cdot \text{stride}$$

Therefore:

$$i \in \text{slice} \Leftrightarrow i - \text{start} \equiv 0 \pmod{\text{stride}}, \quad \text{start} \leq i \leq \text{start} + \text{stride} \cdot \text{size}$$

```

11 bool index_in_slice(size_t index, const slice &s) {
12     return ((index - s.start()) % s.stride() == 0) &&
13           s.start() <= index &&
14           index <= s.start() + s.stride()*(s.size()-1);
15 }

```

`fourier_motzkin` takes a Matrix `M` and a coordinate `k` and creates the set which either corresponds to a projection of an H-Cone (without actually doing the projection), or the intersection of a V-Cone with a coordinate-hyperplane.

```

20 Matrix fourier_motzkin(Matrix M, size_t k) {
21     Matrix result{M.d};
22     // Partition into Z,P,N
23     const auto z_end = partition(M.begin(), M.end(),
24                                 [k](const Vector &v) { return v[k] == 0; });
25     const auto p_end = partition(z_end, M.end(),
26                                 [k](const Vector &v) { return v[k] > 0; });
27     // Move Z to result
28     move(M.begin(), z_end, back_inserter(result));
29     // convolute vectors from P,N
30     for (auto p_it = z_end; p_it != p_end; ++p_it) {
31         for (auto n_it = p_end; n_it != M.end(); ++n_it) {
32             result.push_back(
33                 (*p_it)[k]*(*n_it) - (*n_it)[k]*(*p_it));
34         }
35     }
36     return result;
37 }

```

The lines:

```

23 const auto z_end = partition(M.begin(), M.end(),
24                             [k](const Vector &v) { return v[k] == 0; });
25 const auto p_end = partition(z_end, M.end(),
26                             [k](const Vector &v) { return v[k] > 0; });

```

Partition `M` into logical sets Z, P, N that satisfy the following:

set	range	property
Z	$[M.begin(), z_end)$	$it \in Z \Leftrightarrow (*it)[k] = 0$
P	$[z_end, p_end)$	$it \in P \Leftrightarrow (*it)[k] > 0$
N	$[p_end, M.end())$	$it \in N \Leftrightarrow (*it)[k] < 0$

The line:

```

28 move(M.begin(), z_end, back_inserter(result));

```

Moves Z into the result. The lines:

```

30 for (auto p_it = z_end; p_it != p_end; ++p_it) {
31     for (auto n_it = p_end; n_it != M.end(); ++n_it) {
32         result.push_back(

```

```

33         (*p_it)[k]*(*n_it) - (*n_it)[k]*(*p_it));
34     }
35 }

```

convolute the vectors in the way described in ‘Fourier Motzkin Elimination for H-Cones’ on page 6 and ‘Fourier Motzkin Elimination for V-Cones’ on page 11 (concerning projecting an H-Cone and intersecting a V-Cone with a coordinate-hyperplane), and push them into the result `Matrix`. In particular, it creates the sets which correspond to

$$\{B_i^k B_j - B_j^k B_i \mid i \in P, j \in N\}, \quad \{Y_k^i Y^j - Y_k^j Y^i \mid i \in P, j \in N\}$$

`sliced_fourier_motzkin` applies `fourier_motzkin` to `Matrix M` for each $k \notin \mathbf{s}$, then slices the resulting `Matrix` using `slice_matrix` and `s`. This is the realization of the algorithms indicated by the proofs of either direction of the Minkowski-Weyl Theorem for cones.

```

40 Matrix sliced_fourier_motzkin(Matrix M, const slice &s) {
41     for (size_t k = 0; k < M.d; ++k) {
42         if (!index_in_slice(k,s)) {
43             M = fourier_motzkin(M, k);
44         }
45     }
46     return slice_matrix(M, s);
47 }

```

When transforming an H-Cone to a V-Cone, it first must be written as a V-Cone of a new matrix, then it is intersected with coordinate-hyperplanes and projected. Similarly, when a V-Cone is transformed into an H-Cone, it must be written as an H-Cone of a new matrix then projected with coordinate-projections. The transformations are described in V-Cone Lift and H-Cone Lift, and summarized here:

$$T_H(A) = \begin{pmatrix} \mathbf{0} & I & -I \\ I & A & -A \end{pmatrix} \quad T_V(U) = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix}$$

Note that the transformation of U can be written:

$$T_H(A) = \begin{pmatrix} \mathbf{0} & I & -I \\ -I & -U & U \end{pmatrix}^T$$

Remembering that a `Matrix` is either a collection of row *or* column `Vectors`, it is not surprising that these two transformations can be written as one function of a `Matrix` and some coefficients. In `generalized_lift`, the coefficients are given as an `array<double, 5> C`, so the overall transformation can be illustrated as:

$$\text{Matrix } M \rightarrow \begin{pmatrix} \mathbf{0} & C[0]I \\ C[1]I & C[2]M \\ C[3]I & C[4]M \end{pmatrix}$$

where `Matrix M` is a collection of row `Vectors`, or

$$\text{Matrix } M \rightarrow \begin{pmatrix} \mathbf{0} & C[1]I & C[3]I \\ C[0]I & C[2]M & C[4]M \end{pmatrix}$$

where `Matrix M` is a collection of column `Vectors`.

```

64 Matrix generalized_lift(const Matrix &cone,
65                        const array<double,5> &C) {
66     const size_t d = cone.d;
67     const size_t n = cone.size();
68     Matrix result{d+n};
69     Matrix cone_t = transpose(cone);
70     // |0  C[0]*I|  |0      |
71     //      |C[0]*I|
72     for (size_t i = 0; i < n; ++i) {
73         result.add_Vector()[d+i] = C[0];
74     }
75     size_t k = 0;
76     // |C[1]*I C[2]*U|  |C[1]*I|
77     //      |C[2]*A|
78     for (auto &&row_t : cone_t) {
79         result.push_back(
80             concatenate(C[1]*e_k(d,k++), C[2]*row_t));
81     }
82     k = 0;
83     // |C[3]*I C[4]*U|  |C[3]*I|
84     //      |C[4]*A|
85     for (auto &&row_t : cone_t) {
86         result.push_back(
87             concatenate(C[3]*e_k(d,k++), C[4]*row_t));
88     }
89     return result;
90 }

```

lift_vccone and lift_hccone implement the appropriate transformation using generalized_lift and providing the appropriate coefficients in array<double, 5> C.

```

98 Matrix lift_vccone(const Matrix &vccone) {
99     return generalized_lift(vccone, {-1,1,-1,-1,1});
100 }

```

```

107 Matrix lift_hccone(const Matrix &hccone) {
108     return generalized_lift(hccone, {1,1,1,-1,-1});
109 }

```

cone_transform consolidates the logic of the V-Cone \rightarrow H-Cone and H-Cone \rightarrow V-Cone transformations by accepting a Matrix cone and a Lift.

```

112 Matrix cone_transform(const Matrix &cone,
113                      LiftSelector lift) {
114     if (cone.empty()) {
115         throw logic_error{"empty cone for transform"};
116     }
117     switch (lift) {
118     case LiftSelector::lift_vccone: {
119         return sliced_fourier_motzkin(
120             lift_vccone(cone), slice(0, cone.d, 1));
121     } break;
122     case LiftSelector::lift_hccone: {

```

```

123     return sliced_fourier_motzkin(
124         lift_hcone(cone), slice(0, cone.d, 1));
125     } break;
126     default: {
127         throw std::logic_error{"invalid LiftSelector"};
128     }
129 }
130 }

```

`vccone_to_hcone` and `hcone_to_vccone` specialize `cone_transform` by providing the appropriate `Lift`.

```

132 Matrix vccone_to_hcone(Matrix vccone) {
133     return cone_transform(vccone, LiftSelector::lift_vccone);
134 }

136 Matrix hcone_to_vccone(Matrix hcone) {
137     return cone_transform(hcone, LiftSelector::lift_hcone);
138 }

```

3.5 polyhedra.cpp

`hpoly_to_hcone` and `hcone_to_hpoly` implement the `Matrix` transforms:

$$\text{hpoly_to_hcone} : (A|b) \rightarrow (-b|A), \quad \text{hcone_to_hpoly} : (-b|A) \rightarrow (A|b)$$

These very simple transforms are done with the `cshift` function, which “circularly shifts” the elements of a `Vector` (provided as part of the interface to `valarray`).

```

13 Matrix hpoly_to_hcone(Matrix hpoly) {
14     transform(hpoly.begin(), hpoly.end(), hpoly.begin(),
15         [](Vector v) {
16             v[v.size()-1] *= -1;
17             return v.cshift(-1);
18         });
19     return hpoly;
20 }

24 Matrix hcone_to_hpoly(Matrix hcone) {
25     transform(hcone.begin(), hcone.end(), hcone.begin(),
26         [](Vector v) {
27             v[0] *= -1;
28             return v.cshift(1);
29         });
30     return hcone;
31 }

```

`vpoly_to_vccone` implements the `VPoly` transform:

$$\text{vpoly} \rightarrow \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \text{vpoly.U} & \text{vpoly.V} \end{pmatrix}$$

```

36 Matrix vpoly_to_vccone(VPoly vpoly) {
37     //requires increase in dimension
38     Matrix result{vpoly.d+1};
39     for (auto &&u : vpoly.U) {
40         result.push_back(concatenate({0},u));
41     }
42     for (auto &&v : vpoly.V) {
43         result.push_back(concatenate({1},v));
44     }
45     return result;
46 }

```

normalized_P takes the members of U that have $x_0 > 0$, scaled by $1/x_0$. Let Π be the identity matrix with the 0-th row deleted, and $P = \{\mathbf{u} \in U : u_0 > 0\}$. then this is the result of:

$$\Pi(\{\mathbf{x}/x_0 : \mathbf{x} \in P\} \cap \{x_0 = 1\})$$

```

50 Matrix normalized_P(const Matrix &U) {
51     if (U.d <= 1) {
52         throw std::logic_error{"can't normalize U!"};
53     }
54     Matrix result{U.d-1};
55     std::slice s{1,result.d,1};
56     for (auto &&v : U) {
57         // select the vectors with positive 0-th coordinate
58         if (v[0] <= 0) { continue; }
59         // normalize the selected vectors,
60         result.push_back(v[0] == 1 ? v[s] : (v / v[0])[s]);
61     }
62     return result;
63 }

```

vccone_to_vpoly implements V-Cone \rightarrow V-Polyhedron.

```

67 VPoly vccone_to_vpoly(Matrix vccone) {
68     VPoly result{vccone.d-1};
69     result.U = sliced_fourier_motzkin(
70         vccone, slice(1,vccone.d-1,1));
71     result.V = normalized_P(vccone);
72     return result;
73 }

```

hpoly_to_vpoly and vpoly_to_hpoly implement the complete transformations promised by the file.

```

77 VPoly hpoly_to_vpoly(Matrix hpoly) {
78     return vccone_to_vpoly(
79         hccone_to_vccone(
80             hpoly_to_hccone(move(hpoly))));
81 }

```

```

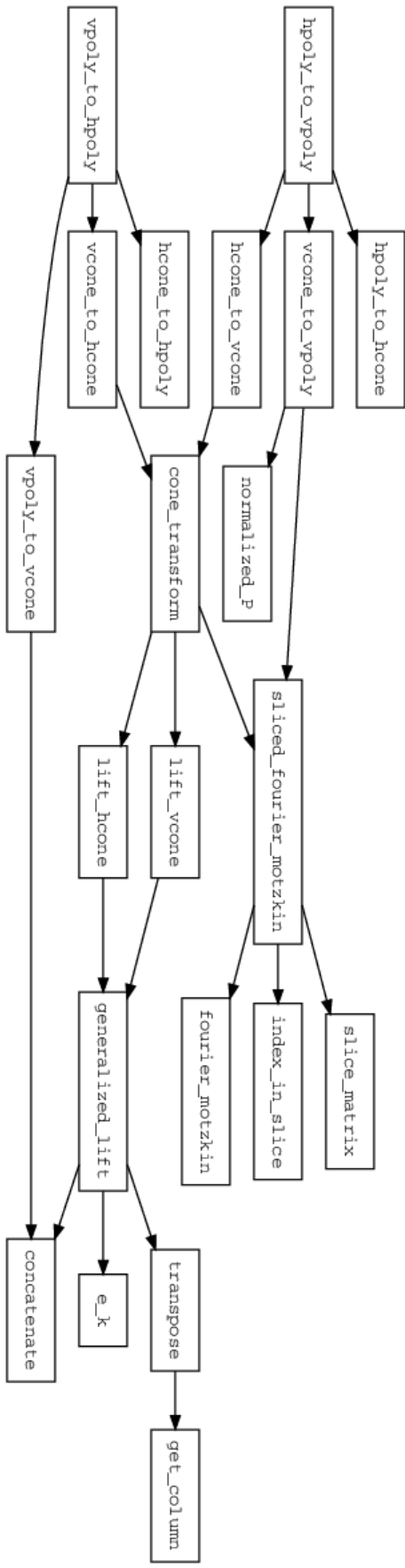
83 Matrix vpoly_to_hpoly(VPoly vpoly) {
84     return hccone_to_hpoly(

```

```
85         vcone_to_hcone(  
86             vpoly_to_vcone(move(vpoly))));  
87     }
```

3.6 Picture of the Program

In the following diagram, the nodes represent functions, and the edges can be read as “calls.” Such a diagram is known as a “callgraph,” and is only intended to give an overview of the program.



4. Testing

In the next sections, the methods used for testing the program described above will be discussed. It will be convenient to assume that sets representing row vectors and cone-generators do not contain $\mathbf{0}$. This results in no loss of generality, only the annoyance of constantly assuming some triviality does not occur.

Notation: Let $AU \leq \mathbf{b}$ be shorthand for $(\forall \mathbf{u} \in U) A\mathbf{u} \leq \mathbf{b}$.

4.1 Testing H-Cone \rightarrow V-Cone

Suppose we have an H-Cone $C_A = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, and would like to test if a V-Cone $C_{V'} = \text{cone}(V')$ represents the same set. It's easy to check that

$$AV' \leq \mathbf{0} \Rightarrow C_{V'} \subseteq C_A$$

It's not clear what to do to check if $C_A \subseteq C_{V'}$. Suppose we had a set V , and we knew that $C_A = \text{cone}(V)$, and that $C_A = C_{V'} \Rightarrow V \subseteq V'$. Then we'd have the following situation:

$$\begin{aligned} AV' \leq \mathbf{0} &\Rightarrow C_{V'} \subseteq C_A \\ V \subseteq V' &\Rightarrow C_A \subseteq C_{V'} \\ C_{V'} = C_A &\Rightarrow V \subseteq V' \\ C_{V'} = C_A &\Rightarrow AV' \leq \mathbf{0} \end{aligned}$$

The problem is now to come up with such a set V , and to determine when such a set may or may not exist for a given cone. We will need to relax the requirements on V a little bit, but not in a way that reduces its utility. First, we consider a *minimal* set generating a cone.

Definition 4.1.1 (Minimal Set). A set V is called *minimal* for $\text{cone}(V)$ if

$$(\forall \mathbf{v} \in V) \text{ cone}(V \setminus \{\mathbf{v}\}) \subset \text{cone}(V)$$

Proposition 4.1.1. If a set V is not minimal for $\text{cone}(V)$ then

$$\exists \mathbf{v} \in V, \mathbf{t} \geq \mathbf{0}, \mathbf{v} = V\mathbf{e}_i, \mathbf{t} \neq \mathbf{e}_i : \quad \mathbf{v} = V\mathbf{t}$$

That is, there is a member of V which is a non-trivial non-negative linear combination of elements of V .

Proof. Say $\text{cone}(V \setminus \{\mathbf{v}\}) = \text{cone}(V)$ where $\mathbf{v} = V\mathbf{e}_i$. Then $\exists \mathbf{t} \geq \mathbf{0}$ such that $\mathbf{v} = (V \setminus \{\mathbf{v}\})\mathbf{t}$. Let \mathbf{t}' be \mathbf{t} with a 0 in the position corresponding to \mathbf{v} in V . Then $\mathbf{v} = V\mathbf{t}$. \square

Is the converse true? That is, is it true that, if V is minimal, then

$$\mathbf{t} \geq \mathbf{0}, \mathbf{v} = V\mathbf{e}_i, [\mathbf{v} = V\mathbf{t} \Rightarrow \mathbf{t} = \mathbf{e}_i] \tag{4.1}$$

Not quite. There is one catch, if there is some

$$\mathbf{t} \geq \mathbf{0}, \mathbf{t} \neq \mathbf{0}, V\mathbf{t} = \mathbf{0} \quad (4.2)$$

then (4.1) fails. So, for what cones does (4.2) fail? It turns out that there is a useful class of cones called *pointed* having this property.

Definition 4.1.2 (Vertex). Let P be a polyhedron. A point $\mathbf{v} \in P$ is called a *vertex* if, for any $\mathbf{u} \neq \mathbf{0}$, at least one of the following is true:

$$\begin{aligned} \mathbf{v} + \mathbf{u} &\notin P \\ \mathbf{v} - \mathbf{u} &\notin P \end{aligned}$$

Definition 4.1.3 (Pointed Cones). A cone is called *pointed* if it has a vertex.

Proposition 4.1.2. *The following statements are equivalent.*

1. $\text{cone}(V)$ is pointed.
2. $\mathbf{t} \geq \mathbf{0}, \mathbf{t} \neq \mathbf{0}, [V\mathbf{t} = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}]$

Proof. First, observe that, due to Closure Property of Cones, if a cone has a vertex, then it is the origin. Suppose that the origin is a vertex, but that (2) fails. Since $\mathbf{0} \notin V$, \mathbf{t} has at least two non-zero elements, let one be t_i . Then $\mathbf{0} = V(t_i \mathbf{e}_i) + V(\mathbf{t} - t_i \mathbf{e}_i)$. Let $\mathbf{u} = V(t_i \mathbf{e}_i)$. Clearly $\mathbf{u} \neq \mathbf{0}$, and also $\mathbf{u}, -\mathbf{u} \in C$. Then the origin is not a vertex, a contradiction.

Next, suppose that $\mathbf{0}$ is not a vertex, then $\exists \mathbf{t}_1, \mathbf{t}_2 \geq \mathbf{0}, \mathbf{t}_{1,2} \neq \mathbf{0}, \mathbf{u} = V\mathbf{t}_1, -\mathbf{u} = V\mathbf{t}_2$. Then $\mathbf{t}_1 + \mathbf{t}_2 \geq \mathbf{0}, \mathbf{t}_1 + \mathbf{t}_2 \neq \mathbf{0}$, and $V(\mathbf{t}_1 + \mathbf{t}_2) = \mathbf{0}$. \square

Now we can consider the converse of Proposition 4.1.1.

Proposition 4.1.3 (Minimal V-Cone Generators). *Suppose that $\text{cone}(V)$ is pointed. Then the following two statements are equivalent:*

1. V is minimal
2. $\mathbf{t} \geq \mathbf{0}, \mathbf{v} = V\mathbf{e}_i, [\mathbf{v} = V\mathbf{t} \Rightarrow \mathbf{t} = \mathbf{e}_i]$

Proof. $(\neg 1 \Rightarrow \neg 2)$ is Proposition 4.1.1. So suppose that $\mathbf{t} \geq \mathbf{0}, \mathbf{v} = V\mathbf{e}_i$, and $\mathbf{v} = V\mathbf{t}$. If $0 \leq t_i < 1$, then $\mathbf{v} = V(\mathbf{t} - t_i \mathbf{e}_i)/(1 - t_i)$, and $\mathbf{v} \in \text{cone}(V \setminus \{\mathbf{v}\})$, which would mean that V is not minimal. Suppose that $t_i \geq 1$. Then $\mathbf{t} - \mathbf{e}_i \geq \mathbf{0}$, and $\mathbf{0} = V(\mathbf{t} - \mathbf{e}_i)$. Because V is pointed, by Proposition 4.1.2 $\mathbf{0} = \mathbf{t} - \mathbf{e}_i$, so $\mathbf{t} = \mathbf{e}_i$. \square

Proposition 4.1.3 gives us a way to characterize the minimal sets generating V-Cones. Clearly, there is not a unique minimal set generating any V-Cone. However, we can relax the requirement of unicity to equivalence, in the following way.

Definition 4.1.4 (vector equivalence). Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, non-zero, and suppose that $\mathbf{u}/\|\mathbf{u}\| = \mathbf{v}/\|\mathbf{v}\|$. Then say that \mathbf{u}, \mathbf{v} are *equivalent*, and write $\mathbf{u} \simeq \mathbf{v}$. If for every $\mathbf{u} \in U$ there is a $\mathbf{v} \in V$ such that $\mathbf{u} \simeq \mathbf{v}$, write $U \sqsubseteq V$. Write $U \simeq V$ if $U \sqsubseteq V$ and $V \sqsubseteq U$.

Proposition 4.1.4. *Then the following two statements are equivalent:*

1. $\mathbf{v} \simeq \mathbf{u}$
2. $(\exists t > 0) \mathbf{v} = t\mathbf{u}$

Proof. $(1 \Rightarrow 2)$. Let $t = \|\mathbf{v}\| / \|\mathbf{u}\|$. Then $t > 0$, and $\mathbf{v} = t\mathbf{u}$.

$(2 \Rightarrow 1)$. $\mathbf{v} / \|\mathbf{v}\| = t\mathbf{u} / \|t\mathbf{u}\| = \mathbf{u} / \|\mathbf{u}\|$ □

We now show that the minimal sets generating pointed V-Cones are essentially unique.

Proposition 4.1.5 (Minimal Generators of a Pointed Cone). *Suppose that V is minimal, and $\text{cone}(V) = \text{cone}(V')$ is pointed. Then $V \sqsubseteq V'$. It follows that if V' is also minimal, then $V \simeq V'$.*

We'll use this short lemma in the proof of the above proposition.

Lemma 4.1.6. *Suppose A is a non-negative matrix, $\mathbf{b} \geq \mathbf{0}$, and $A\mathbf{b} = \mathbf{e}_i$. Then there exists an l , $t > 0$ such that $A(t\mathbf{e}_l) = \mathbf{e}_i$*

Proof. Since A and \mathbf{b} are non-negative, the following holds:

$$(\forall j, k \neq i) b_j > 0 \Rightarrow A_k^j = 0 \quad (4.3)$$

Since $A\mathbf{b} = \mathbf{e}_i$, there is some $b_l > 0$, and $A_k^l > 0$. (4.3) shows that the entire column is zero except for the entry in row i , so $A(\mathbf{e}_l / A_k^l) = \mathbf{e}_i$. □

Proof of Proposition 4.1.5. Let $\mathbf{v} \in V$, $\mathbf{v} = V\mathbf{e}_i$. If we can show that there is some $\mathbf{v}' \in V'$ such that $\mathbf{v} \simeq \mathbf{v}'$, then we're done. Since $\text{cone}(V) = \text{cone}(V')$, there is a non-negative matrix A such that $V' = VA$. Furthermore, there is a non-negative vector \mathbf{b} such that $\mathbf{v} = V'\mathbf{b}$. Then $\mathbf{v} = V'\mathbf{b} = (VA)\mathbf{b} = V(A\mathbf{b})$. By Proposition 4.1.3, $A\mathbf{b} = \mathbf{e}_i$. By Lemma 4.1.6, there is a $t > 0, l$ such that $A\mathbf{b} = A(t\mathbf{e}_l)$. Then $\mathbf{v} = VA(t\mathbf{e}_l) = tV'\mathbf{e}_l = t\mathbf{v}'$ where $\mathbf{v}' \in V'$. By Proposition 4.1.4, $\mathbf{v} \simeq \mathbf{v}'$. □

So now we know that pointed cones have essentially unique generating sets. We now turn to the question of using this knowledge to create a test for the program. We suppose that we have a minimal generating set V for some pointed V-Cone C , and have created a matrix A so that $C = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(V)$. We run the program and get a set V' , and let $C' = \text{cone}(V')$. We must check that $C' = C$.

$$\begin{aligned} AV' \leq \mathbf{0} &\Rightarrow C' \subseteq C \\ V \sqsubseteq V' &\Rightarrow C \subseteq C' \\ C' = C &\Rightarrow V \sqsubseteq V' \\ C' = C &\Rightarrow AV' \leq \mathbf{0} \end{aligned}$$

Equivalence Criteria 1 (H-Cone \rightarrow V-Cone). *Say V is a minimal generating set for the pointed V-Cone C , and suppose $C = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(V)$. Then*

$$C = \text{cone}(V') \Leftrightarrow AV' \leq \mathbf{0}, V \sqsubseteq V'$$

Test 1 (H-Cone \rightarrow V-Cone). We now have a method for testing the program. First, we hand-craft an H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ based on minimal set V for some pointed V-Cone. We then run our program to get a set V' , with the alleged property that $\text{cone}(V') = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. If we confirm Equivalence Criteria 1, then our program has succeeded.

Remark 2. Can we test the program for non-pointed cones? Yes, but it is slightly more complicated. Instead of prior knowledge of a minimal generating set for the cone, we also need to know what the largest linear subspace L contained in the cone. If we project away this linear subspace, then we will have a pointed cone. Given another set V' , we may project away this subspace from V' using a projection matrix, and use Test 1. Then we need to see if $\text{cone}(V')$ spans L . This can be done with a modified fourier-motzkin elimination, but unfortunately we are trying to test the implementation of fourier-motzkin elimination.

It may still be worthwhile to do such tests, but it should be noted that a test isn't designed to prove a program correct, only prove it incorrect. If we analyze the program well and test the fourier-motzkin elimination extensively, then the added complexity of the more general testing may not be worth it. As of now this is left as a possible future extension of the program.

Remark 3. While not important for testing the program, one may ask if pointed V-Cones are the only cones with essentially unique generating sets. The answer is no, for any line has an essentially unique generating set, but is not pointed. However, this is the only exception. It isn't hard to see that, given a non-pointed cone, if it occupies more than one-dimension, then it must at least occupy a half-plane, and a halfplane has uncountably many non-equivalent generators. So, technically, the Test 1 would work for one-dimensional non-pointed cones (lines).

4.2 Testing V-Cone \rightarrow H-Cone

In this section we create a method in the vein of H-Cone \rightarrow V-Cone, but for testing the program transforming V-Cones to H-Cones. This section is almost identical to the previous, with the exception of requiring the Farkas Lemma.

Definition 4.2.1 (Minimal Set of Constraints). A set A is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ if

$$(\forall A_i \in A) \{\mathbf{x} \mid A \setminus \{A_i\} \mathbf{x} \leq \mathbf{0}\} \supset \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$$

Proposition 4.2.1. If a set A is not minimal for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ then

$$\exists A_i \in A, \mathbf{t} \geq \mathbf{0}, A_i = \mathbf{e}_i^T A, \mathbf{t} \neq \mathbf{e}_i : A_i = \mathbf{t}^T V$$

That is, there is a member of A which is a non-trivial non-negative linear combination of elements of V .

In order to prove Proposition 4.2.1, we require the Farkas Lemma.

4.2.1 Farkas Lemma

Proposition 4.2.2 (The Farkas Lemma). *Let $U \in \mathbb{R}^{d \times n}$. Precisely one of the following is true:*

$$\begin{aligned} (\exists \mathbf{t} \geq \mathbf{0}) : \mathbf{x} &= U\mathbf{t} \\ (\exists \mathbf{y}) : U^T \mathbf{y} &\leq \mathbf{0}, \langle \mathbf{x}, \mathbf{y} \rangle > 0 \end{aligned}$$

Proof. That both can't be true can be seen by:

$$\mathbf{x} = U\mathbf{t} \Rightarrow \mathbf{y}^T \mathbf{x} = \mathbf{y}^T U\mathbf{t} \Rightarrow 0 \neq 0$$

To see that at least one is true we must reconsider the process of converting a V-Cone to an H-Cone. First, from $\text{cone}(U)$ we create the following matrix:

$$A = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix}$$

By the way A is constructed,

$$(\exists \mathbf{t}) : A \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{t} \geq \mathbf{0}) \mathbf{x} = U\mathbf{t} \quad (4.4)$$

In the proof of the transformation, we use 'Fourier Motzkin Elimination for H-Cones' on page 6 to transform that matrix A . The 'Fourier Motzkin Matrix' on page 8 promises a sequence of matrices Y_{d+1}, \dots, Y_{d+n} with certain properties. Let $Y = (Y_{d+n})(Y_{d+(n-1)}) \dots (Y_{d+1})$, then it can be said of Y :

1. Every element of Y is non-negative.
2. Y is finite.
3. The last n columns of YA are all $\mathbf{0}$.
4. $(\exists t_{d+1}, \dots, t_{d+n}) A(\mathbf{x} + \sum_{i=d+1}^{d+n} t_i \mathbf{e}_i) \leq \mathbf{0} \Leftrightarrow (YA)\mathbf{x} \leq \mathbf{0}$

Note that here $\mathbf{x} \in \mathbb{R}^{d+n}$. A has three blocks of rows, which can be labeled with Z, P, N in a fairly obvious way. Then, Y can be broken up into three blocks of columns, so that

$$Y = (Y_Z \ Y_P \ Y_N)$$

Where each of $Y_Z, Y_P, Y_N \geq \mathbf{0}$. Consolidating what is known about A and Y ,

$$YA = (Y_Z \ Y_P \ Y_N) \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix} = (Y' \ \mathbf{0})$$

Here, we have let $Y' = Y_P - Y_N$. Then it follows that

$$\mathbf{0} = -Y_Z - Y_P(U) + Y_N(U) = -Y_Z - Y'(U) \Rightarrow Y_Z = -Y'U \Rightarrow Y'U \leq \mathbf{0}$$

Then it holds that, for any row $\mathbf{y}' \in Y'$:

$$\mathbf{y}'U \leq \mathbf{0} \quad (4.5)$$

It is also true that

$$(YA) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} = (Y' \mathbf{0}) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} = Y'\mathbf{x}$$

We also have

$$(\exists \mathbf{t}) : A \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \Leftrightarrow (YA) \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \Leftrightarrow Y'\mathbf{x} \leq \mathbf{0} \quad (4.6)$$

Note that here $\mathbf{x} \in \mathbb{R}^d$. So, if given some \mathbf{x} , the left side of (4.6) is not satisfied, then neither is the right, and there must be some row $\mathbf{y}' \in Y'$ such that the following holds:

$$\langle \mathbf{y}', \mathbf{x} \rangle > 0 \quad (4.7)$$

Then we conclude that, if the right side of (4.4) fails, then there is a vector $\mathbf{y}' \in Y'$ satisfying (4.5) and (4.7). \square

Remark 4. The Farkas Lemma above can be equivalently stated:

$$(\exists \mathbf{t} \geq \mathbf{0}) : \mathbf{t}^T A = \mathbf{y} \quad \Leftrightarrow \quad \neg(\exists \mathbf{x}) : A\mathbf{x} \leq \mathbf{0}, \langle \mathbf{y}, \mathbf{x} \rangle > 0$$

This way of writing it makes it clear that, if $\mathbf{y}^T \mathbf{x} \leq 0$ holds for some H-Cone $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, then \mathbf{y} is a non-negative linear combination of the rows of A .

Proof of Proposition 4.2.1. Say $\{\mathbf{x} \mid (A \setminus \{A_i\})\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. Then, by Remark 4, $A_i^T \mathbf{x} \leq 0$ holds for $\{\mathbf{x} \mid (A \setminus \{A_i\})\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, so A_i is a non-negative linear combination of some rows of $A \setminus \{A_i\}$. \square

As before, the converse will fail if we can combine rows of A in a non-trivial way to get $\mathbf{0}$. For what cones does this occur? Well, it would be necessary that the following holds for some \mathbf{y} :

$$\mathbf{y}^T \mathbf{x} \leq 0, \quad -\mathbf{y}^T \mathbf{x} \leq 0$$

But this means that $\mathbf{y}^T \mathbf{x} = 0$ holds for every member of the cone. We can prevent this from occurring by forcing the cone to contain a basis.

Definition 4.2.2 (Full-Dimensional Cones). A cone is called *full-dimensional* if it contains a basis (i.e. d linearly-independent vectors).

The most important property of a basis B that we shall use is:

$$\mathbf{y}^T B = \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0} \quad (4.8)$$

Proposition 4.2.3. *The following statements are equivalent.*

1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is full-dimensional.
2. $\mathbf{t} \geq \mathbf{0}, \mathbf{t} \neq \mathbf{0}, [\mathbf{t}^T A = \mathbf{0} \Rightarrow \mathbf{t} = \mathbf{0}]$

Proof. $(\neg 1 \Rightarrow \neg 2)$. If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is not full-dimensional, then there is some \mathbf{y} so that for every \mathbf{x} the cone $\mathbf{y}^T \mathbf{x} = 0$. Then, by Remark 4, we'd have some non-negative $\mathbf{t}_1, \mathbf{t}_2$ such that $\mathbf{t}_1^T A = \mathbf{y}$ and $\mathbf{t}_2^T A = -\mathbf{y}$, in which case $\mathbf{t}_1 + \mathbf{t}_2$ is a counter example to (2).

$(\neg 2 \Rightarrow \neg 1)$. Suppose $\mathbf{t} \geq \mathbf{0}$, $\mathbf{t}^T A = \mathbf{0}$, and $\mathbf{t} \neq \mathbf{0}$. Since $\mathbf{0} \notin A$, at least two elements of \mathbf{y} are non-zero, say one is y_i . Then $\mathbf{0} = y_i A_i + (\mathbf{y} - y_i \mathbf{e}_i)^T A$, which then means both $A_i \mathbf{x} \leq 0$ and $-A_i \mathbf{x} \leq 0$ holds for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$, in which case it is not full dimensional. \square

Proposition 4.2.4. *Suppose that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is full-dimensional. Then the following two statements are equivalent:*

1. A is minimal
2. $\mathbf{t} \geq \mathbf{0}$, $[A_i = \mathbf{t}^T A \Rightarrow \mathbf{t} = \mathbf{e}_i]$

Proof. $(\neg 1 \Rightarrow \neg 2)$ is Proposition 4.2.1. So suppose that $\mathbf{t} \geq \mathbf{0}$, and $A_i = \mathbf{t}^T A$. If $0 \leq t_i < 1$, then $A_i = (\mathbf{t} - t_i \mathbf{e}_i)^T A / (1 - t_i)$, and $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid (A \setminus \{A_i\})\mathbf{x} \leq \mathbf{0}\}$, which would mean that A is not minimal. Suppose that $t_i \geq 1$. Then $\mathbf{t} - \mathbf{e}_i \geq \mathbf{0}$, and $\mathbf{0} = (\mathbf{t} - \mathbf{e}_i)^T A$. Because A is full-dimensional, by Proposition 4.2.3, $\mathbf{0} = \mathbf{t} - \mathbf{e}_i$, so $\mathbf{t} = \mathbf{e}_i$. \square

Proposition 4.2.5. *The following two statements are equivalent:*

1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ is full dimensional and A is minimal
2. $\text{cone}(A^T)$ is pointed and A is minimal

Proof. This follows from the nearly identical form of (2) in Proposition 4.2.4 and Proposition 4.1.3. \square

In order to create an equivalence criterion like $\text{H-Cone} \rightarrow \text{V-Cone}$, we use the following result.

Theorem 3 (Dual Cone).

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow \text{cone}(A^T) = \text{cone}(A'^T)$$

Proof. First suppose that $\text{cone}(A^T) = \text{cone}(A'^T)$. Then there exists a non-negative matrix B such that $A'^T = A^T B$. Then $A\mathbf{x} \leq \mathbf{0} \Rightarrow B^T A\mathbf{x} \leq \mathbf{0} \Rightarrow A'\mathbf{x} \leq \mathbf{0}$. Precisely the same reasoning shows that $A'\mathbf{x} \leq \mathbf{0} \Rightarrow A\mathbf{x} \leq \mathbf{0}$, and we conclude that $\text{cone}(A^T) = \text{cone}(A'^T) \Rightarrow \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$.

Next suppose that $\text{cone}(A^T) \neq \text{cone}(A'^T)$, that is, let $\mathbf{z} \in \text{cone}(A^T), \mathbf{z} \notin \text{cone}(A'^T)$. We must show that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$. By the Farkas Lemma, we have a \mathbf{y} such that $\langle \mathbf{y}, \mathbf{z} \rangle > 0$, $A'\mathbf{y} \leq \mathbf{0}$. Clearly this means that $\mathbf{y} \in \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$. Since $\mathbf{z} \in \text{cone}(A)$, there is some $(\mathbf{t} \geq \mathbf{0}) : \mathbf{z}^T = \mathbf{t}^T A$. Then if $A\mathbf{y} \leq \mathbf{0}$, we would have $\langle \mathbf{y}, \mathbf{z} \rangle = \mathbf{t}^T A\mathbf{y} \leq 0 < \langle \mathbf{y}, \mathbf{z} \rangle$, a contradiction. So we conclude that $\mathbf{y} \notin \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. \square

Proposition 4.2.6. *Suppose that A is minimal, and $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$ is full-dimensional. Then $A \subseteq A'$. It follows that if A' is also minimal, then $A \simeq A'$.*

Proof. By Proposition 4.2.5 and Theorem 3, Proposition 4.2.6 is true if it is true for cones, which is shown in Minimal Generators of a Pointed Cone. \square

Say we know that $C = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(V)$ is full-dimensional, with A minimal. We have another set A' and let $C' = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$. Then we can test if $C' = C$. The following summarizes the situation:

$$\begin{aligned} A'V \leq \mathbf{0} &\Rightarrow C \subseteq C' \\ V \sqsubseteq V' &\Rightarrow C' \subseteq C \\ C' = C &\Rightarrow A \sqsubseteq A' \\ C' = C &\Rightarrow A'V \leq \mathbf{0} \end{aligned}$$

Equivalence Criteria 2 (V-Cone \rightarrow H-Cone). *Say H is a minimal generating set of constraints for the full-dimensional H-Cone C , and suppose $C = \text{cone}(V) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$. Then*

$$C = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \Leftrightarrow A'V \leq \mathbf{0}, A \sqsubseteq A'$$

Test 2 (H-Cone \rightarrow V-Cone). We now have a method for testing the program. First, we hand-craft a V-Cone $\text{cone}(V)$ based on minimal set A for some pointed H-Cone. We then run our program to get a set A' , with the alleged property that $\text{cone}(V) = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$. If we confirm Equivalence Criteria 2, then our program has succeeded.

Remark 5. Can we test the program for non-full-dimensional cones? (TODO finish this remark).

As of now this is left as a possible future extension of the program.

Remark 6. While not important for testing the program, one may ask if full-dimensional H-Cones are the only cones with essentially unique generating set of constraints. The answer is no, for any set of the form $\mathbf{y}^T \mathbf{x} = c$ has an essentially unique generating set of constraints. However, this is the only exception. It isn't hard to see that, given independent constraints of the form $A\mathbf{x} = \mathbf{0}$, if A has more than two rows, then, for any non-singular B , $BA\mathbf{x} = \mathbf{0}$ is an equivalent constraint. So, technically, the Test 1 would work for hyperplanes.

Generalizing to Polyhedra In the following sections we generalize Test 1 and Test 2 to polyhedra.

4.3 Testing H-Polyhedron \rightarrow V-Polyhedron

Say we have an H-Polyhedron $P_{A,\mathbf{b}} = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$, and wish to check that our program correctly calculates a V' and U' such that $P_{A,\mathbf{b}} = \text{cone}(U') + \text{conv}(V')$. Again, we shall use the notion of minimality and show that under certain circumstances we can use minimal sets to demonstrate the validity of our algorithm. First, we consider the case of a V-Polyhedron with no cone.

4.3.1 Polytopes

First we consider the special case of a V-Polyhedron given by $P = \text{conv}(V)$.

Definition 4.3.1 (Minimal Set for Polytopes). A set V is called *minimal* for the polytope $\text{conv}(V)$ if:

$$(\forall \mathbf{v} \in V) \text{ conv}(V \setminus \{\mathbf{v}\}) \subset \text{conv}(V)$$

Proposition 4.3.1. V is minimal for $\text{conv}(V)$ if and only if V is the set of vertices of $\text{conv}(V)$.

We will need:

Proposition 4.3.2. A convex combination of convex combinations is another convex combination

Proof. Let Λ represent a collection of convex combinations, that is, $\mathbf{1}^T \Lambda = \mathbf{1}^T$, and let $\boldsymbol{\lambda} \geq \mathbf{0}$, $\mathbf{1}^T \boldsymbol{\lambda} = 1$ be a convex combinator. Then $\Lambda \boldsymbol{\lambda} = \boldsymbol{\lambda}'$ where $\boldsymbol{\lambda}' \geq \mathbf{0}$, $\mathbf{1}^T \boldsymbol{\lambda}' = 1$. That $\boldsymbol{\lambda}' \geq \mathbf{0}$ is clear, then just note that $\mathbf{1}^T \boldsymbol{\lambda}' = \mathbf{1}^T \Lambda \boldsymbol{\lambda} = \mathbf{1}^T \boldsymbol{\lambda} = 1$. \square

Proof of Proposition 4.3.1. First, suppose that V is not minimal. Then there is a $\mathbf{v} \in V$ that satisfies $\text{conv}(V \setminus \{\mathbf{v}\}) = \text{conv}(V)$. Denote $V' = V \setminus \{\mathbf{v}\}$. Then $\mathbf{v} = V' \boldsymbol{\lambda}'$, where there is some λ_i with $0 < \lambda_i < 1$. Let $\mathbf{u} = V'(\mathbf{e}_i - \boldsymbol{\lambda}')$. Then

$$\mathbf{v} - \lambda_i \mathbf{u} = V' \boldsymbol{\lambda}' - \lambda_i V'(\mathbf{e}_i - \boldsymbol{\lambda}') = (1 - \lambda_i) V' \boldsymbol{\lambda}' + \lambda_i V' \mathbf{e}_i$$

By Proposition 4.3.2, the right hand side of this equation is a convex combination of members of V' , so $\mathbf{v} - \lambda_i \mathbf{u} \in \text{conv}(V')$. Similarly,

$$\mathbf{v} + \lambda_i \mathbf{u} = V' \boldsymbol{\lambda}' - \lambda_i V'(\mathbf{e}_i - \boldsymbol{\lambda}') = V'(\boldsymbol{\lambda}' - \lambda_i \mathbf{e}_i) + V'(\lambda_i \boldsymbol{\lambda}')$$

Consider $(\boldsymbol{\lambda}' - \lambda_i \mathbf{e}_i)$. Note that this expression is non-negative, and sums to $1 - \lambda_i$. Next note that $\lambda_i \boldsymbol{\lambda}'$ is non-negative, and sums to λ_i . This means that the right hand side of the equation is a convex combination of V' , so $\mathbf{v} + \lambda_i \mathbf{u} \in \text{conv}(V')$, and \mathbf{v} is not a vertex.

Next, suppose that $\mathbf{v} \in V$ is not a vertex, and let $V' = V \setminus \{\mathbf{v}\}$. Then there is some non-zero \mathbf{u} such that $\mathbf{v} + \mathbf{u} \in \text{conv}(V)$, $\mathbf{v} - \mathbf{u} \in \text{conv}(V)$. First, let $\alpha, \beta > 0$, and consider $\mathbf{v} + \alpha \mathbf{u}$ and $\mathbf{v} - \beta \mathbf{u}$. Observe that

$$\frac{\beta(\mathbf{v} + \alpha \mathbf{u})}{\alpha + \beta} + \frac{\alpha(\mathbf{v} - \beta \mathbf{u})}{\alpha + \beta} = \frac{\alpha \mathbf{v} + \beta \mathbf{v}}{\alpha + \beta} = \mathbf{v}$$

This shows that we can positively scale α and β , and still get \mathbf{v} as a convex combination of the result. So we search for positive α and β that give a point of $\text{conv}(V')$, which by Proposition 4.3.2 shows that $\mathbf{v} \in \text{conv}(V')$ so V is not minimal. First observe that $\mathbf{v} + \mathbf{u} = V \boldsymbol{\lambda}$ for some $\boldsymbol{\lambda}$. Then let $\boldsymbol{\lambda}' = \boldsymbol{\lambda} - \lambda_i \mathbf{e}_i$, so $\mathbf{u} + \mathbf{v} = V' \boldsymbol{\lambda}' + \lambda_i \mathbf{v}$, and $\mathbf{u} = V'(\boldsymbol{\lambda}') + (\lambda_i - 1) \mathbf{v}$. Then

$$\mathbf{v} + \alpha \mathbf{u} = \mathbf{v} + \alpha(\lambda_i - 1) \mathbf{v} + \alpha V' \boldsymbol{\lambda}'$$

So we let $\alpha = 1/(1 - \lambda_i)$, and the term in \mathbf{v} disappears, while $\boldsymbol{\lambda}'/(1 - \lambda_i)$ is a convex combination. Similarly, we have $\mathbf{v} - \mathbf{u} = V\boldsymbol{\mu}$, and $\mathbf{u} = \mathbf{v}(1 - \mu_i) - V'\boldsymbol{\mu}'$. Then

$$\mathbf{v} - \beta\mathbf{u} = \mathbf{v}(1 - \beta(1 - \mu_i)) + \beta V'\boldsymbol{\mu}'$$

So let $\beta = 1/(1 - \mu_i)$, so the right hand side is a convex combination of members of V' . \square

4.3.2 Characterstic Cone

Now we consider the set $\text{cone}(U)$ in $\text{cone}(U) + \text{conv}(V)$. The next proposition shows that it is essntially unique for any given Polyhedron.

Proposition 4.3.3 (Characterstic Cone). *Suppose that $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \text{cone}(U) + \text{conv}(V)$. Then the following three statements are equivalent:*

1. $A\mathbf{r} \leq \mathbf{0}$
2. $(\forall \mathbf{x} \in P)(\forall \alpha > 0) \mathbf{x} + \alpha\mathbf{r} \in P$
3. $\mathbf{r} \in \text{cone}(U)$

Proof. $(1 \Rightarrow 2)$. $\mathbf{x} \in P$ means that $A\mathbf{x} \leq \mathbf{b}$, and $A\mathbf{r} \leq \mathbf{0}$ means that $A(\mathbf{x} + \alpha\mathbf{r}) \leq A\mathbf{x} \leq \mathbf{b}$.

$(\neg 1 \Rightarrow \neg 2)$. Suppose $\langle A_i, \mathbf{r} \rangle > 0$, then let $\alpha > (b_i - \langle A_i, \mathbf{x} \rangle) / \langle A_i, \mathbf{r} \rangle$. We have:

$$\langle A_i, \mathbf{x} + \alpha\mathbf{r} \rangle > \langle A_i, \mathbf{x} \rangle + \frac{b_i \langle A_i, \mathbf{r} \rangle - \langle A_i, \mathbf{x} \rangle \langle A_i, \mathbf{r} \rangle}{\langle A_i, \mathbf{r} \rangle} = b_i$$

$(3 \Rightarrow 2)$. This is essentially the definition of $\text{cone}(U) + \text{conv}(V)$.

$(2 \Rightarrow 3)$. Now for the real work. Suppose that (2) holds, but $\mathbf{r} \notin \text{cone}(U)$. Then by the Farkas Lemma, we have a \mathbf{y} that satisfies $(\forall \mathbf{r} \in U) \langle \mathbf{r}, \mathbf{y} \rangle \leq 0$, $\langle \mathbf{y}, \mathbf{r} \rangle > 0$. From (2) we construct a sequence: $(\mathbf{x}_n) = \mathbf{v} + n \cdot \mathbf{r}$. Then it is clear that the sequence $\langle \mathbf{y}, \mathbf{x}_n \rangle \rightarrow \infty$. It is also clear that $(\forall n) \mathbf{x}_n \in P$. We now need the following:

Proposition 4.3.4. *A linear, real-valued function on the set $\text{conv}(V)$ achieves its maximal value at some $\bar{\mathbf{v}} \in V$.*

Proof. To see this is true, suppose that the linear function is given by $\langle \mathbf{y}, \cdot \rangle$, and that $\bar{\mathbf{v}}$ is an element of V such that $(\forall \mathbf{v} \in V) \langle \mathbf{y}, \bar{\mathbf{v}} \rangle \geq \langle \mathbf{y}, \mathbf{v} \rangle$. Then, for any $\mathbf{r} \in \text{conv}(V)$, $\mathbf{r} = \sum_{\mathbf{v} \in V} \lambda_v \mathbf{v}$ where $\sum \lambda_v = 1 \Rightarrow \lambda_v \leq 1$, and it follows

$$\langle \mathbf{y}, \mathbf{r} \rangle = \left\langle \mathbf{y}, \sum_{\mathbf{v} \in V} \lambda_v \mathbf{v} \right\rangle = \sum_{\mathbf{v} \in V} \lambda_v \langle \mathbf{y}, \mathbf{v} \rangle \leq \sum_{\mathbf{v} \in V} \lambda_v \langle \mathbf{y}, \bar{\mathbf{v}} \rangle = \langle \mathbf{y}, \bar{\mathbf{v}} \rangle$$

\square

Now consider the maximum value of the function $\langle \mathbf{y}, \cdot \rangle$ on P . Since any element of P can be written $\mathbf{r} + \mathbf{v} \mid \mathbf{r} \in \text{cone}(U)$, $\mathbf{v} \in \text{conv}(V)$, and $(\forall \mathbf{r} \in U) \langle \mathbf{y}, \mathbf{r} \rangle \leq 0$, we can find the maximum value on $\text{conv}(V)$. However, $\langle \mathbf{y}, \cdot \rangle$ achieves its maximal value on $\text{conv}(V)$ at some $\bar{\mathbf{v}} \in V$, which is a contradiction with the fact that $\langle \mathbf{y}, \mathbf{x}_n \rangle \rightarrow \infty$, so we conclude that $\mathbf{r} \in \text{cone}(U)$. \square

Remark 7 (Characteristic Cone). Note that (2) in the proof above is independent of A and U . This means that the cone of a polyhedron is independent of its representation, i.e. if $\text{cone}(U) + \text{conv}(V) = \text{cone}(U') + \text{conv}(V')$, then $\text{cone}(U) = \text{cone}(U')$, while it is not necessarily true that $\text{conv}(V) = \text{conv}(V')$. Similarly, if $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$, then it holds that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$.

4.3.3 Minimal V-Polyhedra Pairs

Definition 4.3.2. A pair (U, V) is said to be *minimal* for $\text{cone}(U) + \text{conv}(V)$ if

$$(\forall \mathbf{u} \in U) \text{ cone}(U \setminus \{\mathbf{u}\}) \subset \text{cone}(U) + \text{conv}(V) \quad (4.9)$$

$$(\forall \mathbf{v} \in V) \text{ cone}(V \setminus \{\mathbf{v}\}) \subset \text{cone}(U) + \text{conv}(V) \quad (4.10)$$

As before, the pair may not be essentially unique. This can happen if U is not pointed. So we will consider only pointed cones for $\text{cone}(U)$.

Proposition 4.3.5. *If (U, V) is minimal, then U is minimal for $\text{cone}(U)$.*

Proof. By Remark 7, $\text{cone}(U) + \text{conv}(V) = \text{cone}(U') + \text{conv}(V)$ if and only if $\text{cone}(U) = \text{cone}(U')$. So this means that the minimality of U is only dependent on U . \square

Now we consider the vertices of $\text{cone}(U) + \text{conv}(V)$.

Proposition 4.3.6. *If \mathbf{v} is a vertex of $\text{cone}(U) + \text{conv}(V)$, then $[\mathbf{v} = U\mathbf{t} + V\boldsymbol{\lambda}] \Rightarrow \mathbf{t} = \mathbf{0}$.*

Proof. If \mathbf{v} can be written with some non-zero contribution from $\text{cone}(U)$, then you may decrease this contribution by some amount while staying in $\text{cone}(U) + \text{conv}(V)$, and you may increase the contribution by the same amount, so \mathbf{v} is not a vertex. \square

It will be useful to refer to the property of a set V such that no member of V may be written with a non-zero contribution from $\text{cone}(U)$. We will call it U -free.

Proposition 4.3.7. *If \mathbf{v} is a vertex of $\text{cone}(U) + \text{conv}(V)$, then \mathbf{v} is a vertex of $\text{conv}(V)$.*

Proof. By Proposition 4.3.6, $\mathbf{v} \in \text{conv}(V)$. If \mathbf{v} is not a vertex of $\text{conv}(V)$, then because $\text{conv}(V) \subseteq \text{cone}(U) + \text{conv}(V)$ it can't be a vertex of P . \square

Now we can show the following.

Proposition 4.3.8. *Suppose that (U, V) is a minimal pair for $\text{cone}(U) + \text{conv}(V)$. Then V is the set of vertices of $\text{cone}(U) + \text{conv}(V)$.*

Proof. By Proposition 4.3.6, if \mathbf{v} is a vertex of P , then it must be a vertex of V . Clearly, V is minimal for V , and is precisely the vertices of $\text{conv}(V)$. The only question is if the vertices of V are the vertices of P . Suppose that \mathbf{v} is a vertex of $\text{conv}(V)$. Then we must show that for any $\mathbf{u} \neq \mathbf{0}$, if $\mathbf{v} + \mathbf{u} \in P$ then $\mathbf{v} - \mathbf{u} \notin P$. Suppose that $\mathbf{u} \in \text{cone}(U)$. Then, since V is U -free, $\mathbf{v} - \mathbf{u} \notin P$, otherwise $\mathbf{v} = (\mathbf{v} - \mathbf{u}) + \mathbf{u}$ and V is not U -free. If $\mathbf{u} \notin \text{cone}(U)$, then if $\mathbf{v} + \mathbf{u} \in P$, $\mathbf{v} + \mathbf{u} \in \text{conv}(V)$. Then because \mathbf{v} is a vertex of $\text{conv}(V)$, $\mathbf{v} - \mathbf{u} \notin \text{conv}(V)$. \square

Proposition 4.3.9. *Let $P = \text{cone}(U) + \text{cone}(V)$. Then the following are equivalent*

1. (U, V) is minimal for P
2. U is minimal for $\text{cone}(U)$, V is the vertex set of P
3. U is minimal for $\text{cone}(U)$, V is the vertex set of $\text{conv}(V)$, and V is U -free

Proof. (1 \Rightarrow 2). This combines the results of Proposition 4.3.5 and Proposition 4.3.8

(2 \Rightarrow 3). That V is U -free follows from Proposition 4.3.6. By Proposition 4.3.7, the vertex set of P is a subset of the vertices of $\text{conv}(V)$. Let \mathbf{v} be a vertex of $\text{conv}(V)$, we must show that it is a vertex of P . Because it is a vertex of $\text{conv}(V)$, if $\mathbf{v} + \mathbf{u} \in \text{conv}(V)$ then $\mathbf{v} - \mathbf{u} \notin \text{conv}(V)$. Say $\mathbf{v} + \mathbf{u} \in \text{conv}(V) + \text{cone}(U)$. Then \mathbf{u} must have some non-zero contribution of $\text{cone}(U)$. If $\mathbf{v} - \mathbf{u} \in P$, then \mathbf{v} could be written as $(\mathbf{v} + \mathbf{u})/2 + (\mathbf{v} - \mathbf{u})/2$, which has an overall positive contribution from $\text{cone}(U)$, meaning that V is not U -free.

(3 \Rightarrow 1). Since V is the vertex set of $\text{conv}(V)$, if $\mathbf{v} \in V$ is also in $\text{cone}(U) + \text{conv}(V \setminus \{\mathbf{v}\})$, then \mathbf{v} can be written with a non-negative contribution from $\text{cone}(U)$, so V is not U -free. Next let $\mathbf{u} \in U$, and $U' = U \setminus \{\mathbf{u}\}$. We must find a point in $\text{cone}(U) + \text{conv}(V)$ that is not in $\text{cone}(U') + \text{conv}(V)$. Because $\mathbf{u} \notin \text{cone}(U')$, there is an \mathbf{x} that satisfies: $\mathbf{x}^T U' \leq \mathbf{0}$, and $\mathbf{x}^T \mathbf{u} > 0$. By Proposition 4.3.4, there is some maximum value c such that $(\forall \mathbf{x} \in \text{conv}(V)) \mathbf{x}^T \mathbf{u} \leq c$. This means that $\{\mathbf{x}^T \mathbf{y} : \mathbf{y} \in \text{cone}(U') + \text{conv}(V)\}$ is upper-bounded by c . But the set $\{\mathbf{x}^T \mathbf{y} : \mathbf{y} \in \text{cone}(U) + \text{conv}(V)\}$ is unbounded, since $\mathbf{x}^T \mathbf{u} > 0$. So we can conclude that $\text{cone}(U') + \text{conv}(V) \subset \text{cone}(U) + \text{conv}(V)$. We conclude that (U, V) are minimal. \square

Now we see that the minimal pairs for V-Polyhedra are essentially unique.

Proposition 4.3.10. *Let (U, V) be minimal for $P = \text{cone}(U) + \text{conv}(V) = \text{cone}(U') + \text{conv}(V')$. Then $U \sqsubseteq U'$, and $V \subseteq V'$.*

Proof. Since $\text{cone}(U) = \text{cone}(U')$, and U is minimal for $\text{cone}(U)$, by Equivalence Criteria 1 $U \sqsubseteq U'$. By Proposition 4.3.7 every vertex of P must be a vertex of V' , and because V contains precisely the vertices of P , $V \subseteq V'$. \square

Say we know that $P = \text{cone}(U) + \text{conv}(V) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$, with U, V minimal, and U pointed. We have another pair (U', V') , and let $P' = \text{cone}(U') + \text{conv}(V')$. We want to test if $P = P'$. We have the following:

$$\begin{aligned}
AU' \leq \mathbf{0} &\Rightarrow \text{cone}(U') \subseteq \text{cone}(U) \\
AV' \leq \mathbf{b} &\Rightarrow \text{conv}(V') \subseteq P \\
U \sqsubseteq U' &\Rightarrow \text{cone}(U) \subseteq \text{cone}(U') \\
V \subseteq V' &\Rightarrow \text{conv}(V) \subseteq \text{conv}(V') \\
P' = P &\Rightarrow AU \leq \mathbf{0} \\
P' = P &\Rightarrow AV \leq \mathbf{b} \\
P' = P &\Rightarrow U \sqsubseteq U' \\
P' = P &\Rightarrow V \subseteq V'
\end{aligned}$$

The first two lines imply that $P' \subseteq P$, while the next two imply that $P \subseteq P'$. We now have the ability to create an equivalence criteria.

Equivalence Criteria 3 (V-Cone \rightarrow H-Cone). *Say (U, V) is a minimal pair for $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ (with pointed $\text{cone}(U)$), and suppose $P' = \text{cone}(U') + \text{conv}(V')$. Then*

$$P = P' \Leftrightarrow AU \leq \mathbf{0}, AV \leq \mathbf{b}, V \subseteq V', U \subseteq U'$$

Test 3 (H-Polyhedron \rightarrow V-Polyhedron). We now have a method for testing the program. First, we hand-craft an H-Polyhedron $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ based on a minimal pair (U, V) for some pointed V-Polyhedron. We then run our program to get a pair (U', V') , with the alleged property that $\text{cone}(U') + \text{conv}(V') = \text{cone}(U) + \text{conv}(V)$. If we confirm Equivalence Criteria 3, then our program has succeeded.

4.4 Testing V-Polyhedron \rightarrow H-Polyhedron

Now we suppose we have a V-Polyhedron $P_{U,V} = \text{cone}(U) + \text{conv}(V)$, and would like to test the program which returns a matrix-vector pair A', \mathbf{b}' where supposedly $P_{U,V} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. Again, we will start off with a pair A, \mathbf{b} where we know that $P_{U,V} = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$, where A, \mathbf{b} satisfy some nice properties, and use those properties to test if $P_{U,V} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$.

Definition 4.4.1 (Minimal H-Pair). The pair (A, \mathbf{b}) is called *minimal* for $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$, if, for any row (A_i, b_i) , letting (A', \mathbf{b}') be (A, \mathbf{b}) with the i -th row deleted, $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \supset \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$.

To characterize Minimal H-Pairs, we will need a new form of the Farkas Lemma.

Theorem 4 (Farkas Lemma 2).

$$(\exists \mathbf{t} \geq \mathbf{0}) \mathbf{t}^T A = \mathbf{y}, \mathbf{t}^T \mathbf{b} \leq c \Leftrightarrow \begin{cases} (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{y}^T \mathbf{x} \leq 0 \text{ and} \\ (\forall \mathbf{x}) A\mathbf{x} \leq \mathbf{b} \Rightarrow \mathbf{y}^T \mathbf{x} \leq c \end{cases}$$

Proof. First we note that

$$\begin{aligned} \exists \mathbf{t} \geq \mathbf{0}, \mathbf{t}^T A = \mathbf{y}, \mathbf{t}^T \mathbf{b} \leq c &\Leftrightarrow \\ \exists \mathbf{t} \geq \mathbf{0}, \mathbf{t}^T \begin{pmatrix} A \\ 0 \end{pmatrix} = \mathbf{y}, \mathbf{t}^T \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} = c &\Leftrightarrow \\ \exists \mathbf{t} \geq \mathbf{0}, \mathbf{t}^T \begin{pmatrix} -\mathbf{b} & A \\ -1 & 0 \end{pmatrix} = (-c, \mathbf{y}) \end{aligned}$$

If we negate the right hand side of The Farkas Lemma, then we get that

$$\begin{aligned} \exists \mathbf{t} \geq \mathbf{0}, \mathbf{t}^T \begin{pmatrix} -\mathbf{b} & A \\ -1 & 0 \end{pmatrix} = (-c, \mathbf{y}) &\Leftrightarrow \\ \forall \mathbf{x}, x_0 \begin{pmatrix} -\mathbf{b} & A \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \Rightarrow \langle (-c, \mathbf{y}), (x_0, \mathbf{x}) \rangle \leq \mathbf{0} &\Leftrightarrow \\ \forall x_0 \geq 0, A\mathbf{x} \leq x_0 \mathbf{b} \Rightarrow \mathbf{y}^T \mathbf{x} \leq x_0 \cdot c \end{aligned}$$

Taking the case that $x_0 = 0$ and $x_0 > 0$ separately, you get the proposition. \square

Proposition 4.4.1. Suppose that (A, \mathbf{b}) is not minimal for some $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$. Then there is a row (A_i, b_i) for which the following holds. Let (A', \mathbf{b}') be (A, \mathbf{b}) with the i -th row deleted. Then there is a $\mathbf{t}' \geq \mathbf{0}$ such that $\mathbf{t}'^T A' = A_i$, $\mathbf{t}'^T \mathbf{b}' \leq b_i$.

Proof. Since (A, \mathbf{b}) is not minimal, there is such an (A_i, b_i) and (A', \mathbf{b}') such that $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. In the right hand side of Farkas Lemma 2, the conditions are satisfied with $A := A'$, $\mathbf{b} := \mathbf{b}'$, $\mathbf{y} := A_i$, $c := b_i$. \square

Is the converse true? Does it hold that given a (A, \mathbf{b}) which is minimal, the implication of Proposition 4.4.1 fails? In general, no. For example, the hyperplane $\langle \mathbf{y}, \mathbf{x} \rangle = c$ has a minimal representation $\{\langle \mathbf{y}, \mathbf{x} \rangle \leq c, \langle -\mathbf{y}, \mathbf{x} \rangle \leq -c\}$, but the sum of the rows is 0, and so $\mathbf{t}' := (2, 1)$ satisfies the claim. In general, we need all $d+1$ rows to be *affinely independent*.

Definition 4.4.2 (Affine Dependence). A set V of vectors is called *affinely independent* if, given any $\mathbf{y} \neq \mathbf{0}$, $c \in \mathbb{R}$, there is some $\mathbf{v} \in V$ such that $\mathbf{y}^T \mathbf{v} \neq c$.

Definition 4.4.3 (Full-Dimensional). A polyhedron is called *full-dimensional* if it contains an affinely independent set of vectors.

Proposition 4.4.2. If $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is full dimensional, and $\mathbf{y}^T A = \mathbf{0}$ with $\mathbf{y} \geq \mathbf{0}$, then either $\mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T \mathbf{b} > 0$.

Proof. Say $\mathbf{y}^T A = \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, and $\mathbf{y}^T \mathbf{b} = 0$. Then suppose $y_i \neq 0$. Then

$$y_i A_i + \sum_{j \neq i} y_j A_j = \mathbf{0}, \quad y_i b_i + \sum_{j \neq i} y_j b_j = 0$$

So, there are non-negative $\mathbf{t}_1, \mathbf{t}_2$ such that

$$\mathbf{t}_1^T A = -\mathbf{t}_2^T A, \quad \mathbf{t}_1^T \mathbf{b} = -\mathbf{t}_2^T \mathbf{b}$$

It follows that, for any \mathbf{x} satisfying $A\mathbf{x} \leq \mathbf{b}$:

$$\mathbf{t}_1^T A\mathbf{x} \leq \mathbf{t}_1^T \mathbf{b}, \quad -\mathbf{t}_2^T A\mathbf{x} \geq -\mathbf{t}_2^T \mathbf{b}$$

But then for any $\mathbf{x} \in P$ it must hold that $\mathbf{t}_1^T A\mathbf{x} = \mathbf{t}_1^T \mathbf{b}$. Since P is full dimensional, it must be that $\mathbf{t}_1 = \mathbf{0}$, then $\mathbf{y} = \mathbf{0}$. \square

Proposition 4.4.3. Suppose that (A, \mathbf{b}) is minimal for some full-dimensional $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$. Then let (A_i, b_i) be any row. If, for some $\mathbf{t} \geq \mathbf{0}$ $\mathbf{t}^T A = A_i$, then either $\mathbf{t} = \mathbf{e}_i$ or $\mathbf{t}^T \mathbf{b} > b_i$.

Proof. Suppose that $A_i = t_i A_i + \mathbf{t}'^T A'$, and $b_i = t_i b_i + \mathbf{t}'^T \mathbf{b}'$. If $0 \leq t_i < 1$, then $A_i = \mathbf{t}'^T A' / (1 - t_i)$, and $b_i = \mathbf{t}'^T \mathbf{b}' / (1 - t_i)$. But then (A, \mathbf{b}) is not minimal (the i -th row may be deleted without changing the polyhedron). Say $1 \leq t_i$. Then there is a non-negative \mathbf{t}'' with $\mathbf{t}''^T A = 0$, $\mathbf{t}''^T \mathbf{b} = 0$. By Proposition 4.4.2, $\mathbf{t}'' = \mathbf{0}$, and $\mathbf{t} = \mathbf{e}_i$. \square

Now we intend to use these properties of full-dimensionality and minimality to let us reduce the problem to one of cones.

Proposition 4.4.4. *The following statements are equivalent:*

1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$
2. $\left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \mid \begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b} & A \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\} = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \mid \begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b}' & A' \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \right\}$

Proof. $(2 \Rightarrow 1)$. Just set $x_0 = 1$, and move \mathbf{b}, \mathbf{b}' to the right side of the inequalities. $(\neg 2 \Rightarrow \neg 1)$. Suppose that:

$$\begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b} & A \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0}, \quad \begin{pmatrix} -1 & \mathbf{0} \\ -\mathbf{b}' & A' \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \not\leq \mathbf{0}$$

Observe that, by the way these sets are constructed, $x_0 \geq 0$. If $x_0 = 0$, then we have $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$, which, by Proposition 4.3.3 means that that A and A' don't create the same characteristic cone, so $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. If $x_0 > 0$, then we have:

$$A\mathbf{x} \leq x_0\mathbf{b}, \quad A'\mathbf{x} \not\leq x_0\mathbf{b}' \Rightarrow A(\mathbf{x}/x_0) \leq \mathbf{b}, \quad A'(\mathbf{x}/x_0) \not\leq \mathbf{b}'$$

So $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \neq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. □

Now, combining the results of 'Characteristic Cone' on page 39 and proposition 4.4.4, we have the following result:

Proposition 4.4.5. *The following two statement are equivalent:*

1. $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$
2. $\text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} = \text{cone} \begin{pmatrix} -\mathbf{b}'^T & -1 \\ A'^T & \mathbf{0} \end{pmatrix}$

Proposition 4.4.6. *If $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ is minimal and full-dimensional, then either*

1. $\text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ *is minimal and pointed, or*
2. $\text{cone} \begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix}$ *is minimal and pointed, and* $\text{cone} \begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix} = \text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$

Proof. To see that they are pointed, let

$$\text{cone} \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix} (\mathbf{t}, t_0) = \mathbf{0}$$

If (\mathbf{t}, t_0) is non-zero, then $\mathbf{t}^T A = \mathbf{0}$, which by Proposition 4.4.2 means that $-\mathbf{t}^T \mathbf{b} < 0$. This means that $-\mathbf{t}^T \mathbf{b} - t_0 < 0$. So $(\mathbf{t}, t_0) = \mathbf{0}$.

To show that it they are minimal, by 'Minimal V-Cone Generators' on page 31, we only need to show that

$$\mathbf{t} \geq \mathbf{0}, \quad [(A_i = \mathbf{t}^T A, \mathbf{t}^T \mathbf{b} + t_0 = b_i) \Rightarrow \mathbf{t} = \mathbf{e}_i]$$

By Proposition 4.4.3, if $\mathbf{t} \neq \mathbf{e}_i$, and $\mathbf{t}^T A = A_i$, then $\mathbf{t}^T \mathbf{b} > b_i$, so $\mathbf{t}^T \mathbf{b} + t_0 > b_i$. So that means that if $\mathbf{t}^T A = A_i$, $\mathbf{t}^T \mathbf{b} = b_i$, then \mathbf{t} must be \mathbf{e}_i . If there is a $\mathbf{t} \neq \mathbf{0}$ such that $\mathbf{t}^T A = \mathbf{0}$, then it follows that $-\mathbf{t}^T \mathbf{b} < 0$, and that $\text{cone} \begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix} = \begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$. Then $\begin{pmatrix} -\mathbf{b}^T \\ A^T \end{pmatrix}$ is minimal. If there is no such \mathbf{t} , that is, if $\mathbf{t}^T A = \mathbf{0}$ then $\mathbf{t} = \mathbf{0}$, then $\begin{pmatrix} -\mathbf{b}^T & -1 \\ A^T & \mathbf{0} \end{pmatrix}$ is minimal. \square

Proposition 4.4.7. *Say (A, \mathbf{b}) is minimal for full-dimensional $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$. Then if $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$, then $(A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')$.*

Proof. Proposition 4.4.5 shows that the equality of the H-Cones is predicated on the equality of the V-Cones $\text{cone} \begin{pmatrix} -\mathbf{b} \\ A \end{pmatrix}$ and $\text{cone} \begin{pmatrix} -\mathbf{b}' \\ A' \end{pmatrix}$. Proposition 4.4.6 shows that we end up with the situation that either $\text{cone} \begin{pmatrix} -\mathbf{b} \\ A \end{pmatrix} = \text{cone} \begin{pmatrix} -\mathbf{b}' \\ A' \end{pmatrix}$ with $\text{cone} \begin{pmatrix} -\mathbf{b} \\ A \end{pmatrix}$ minimal, in which case $\begin{pmatrix} -\mathbf{b} \\ A \end{pmatrix} \sqsubseteq \begin{pmatrix} -\mathbf{b}' \\ A' \end{pmatrix}$, or $\text{cone} \begin{pmatrix} -\mathbf{b} & -1 \\ A & \mathbf{0} \end{pmatrix}$ is minimal. In this case, since for no $A_i \in A$ is $A_i \simeq \mathbf{0}$, we conclude that $\begin{pmatrix} -\mathbf{b} \\ A \end{pmatrix} \sqsubseteq \begin{pmatrix} -\mathbf{b}' \\ A' \end{pmatrix}$ \square

Say we know that $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \text{cone}(U) + \text{conv}(V)$, with (A, \mathbf{b}) minimal, and P full-dimensional. We have another pair (A', \mathbf{b}') , and let $P' = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. We want to test if $P = P'$. We have the following:

$$\begin{aligned} A'U \leq \mathbf{0} & \Rightarrow \text{cone}(U') \subseteq \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\} \\ A'V \leq \mathbf{b} & \Rightarrow \text{conv}(V') \subseteq P' \\ (A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}') & \Rightarrow P' \subseteq P \\ P' = P & \Rightarrow A'U \leq \mathbf{0} \\ P' = P & \Rightarrow A'V \leq \mathbf{b} \\ P' = P & \Rightarrow (A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}') \end{aligned}$$

The first two lines imply that $P \subseteq P'$, so the first three mean that $P = P'$. We then have the following equivalence criteria:

Equivalence Criteria 4. *Say (A, \mathbf{b}) is a minimal pair for $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \text{cone}(U) + \text{conv}(V)$, and suppose $P' = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. Then*

$$P = P' \Leftrightarrow A'U \leq \mathbf{0}, A'V \leq \mathbf{b}, (A, \mathbf{b}) \sqsubseteq (A', \mathbf{b}')$$

Test 4 (V-Polyhedron \rightarrow H-Polyhedron). We now have a method for testing the program. First, we hand-craft a V-Polyhedron $\text{cone}(U) + \text{conv}(V)$ based on some minimal pair (A, \mathbf{b}) , then run our program to get the pair (A', \mathbf{b}') , with the alleged property that $\text{cone}(U) + \text{conv}(V) = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$. If we confirm Equivalence Criteria 4, then our program has succeeded.

4.5 test_functions.h

The following types are defined for running tests of the different algorithms. They are expected to be given a descriptive name, the object on which the test will be run, and a **key** with which the result of the test will be compared. The **key** object is one of the minimal objects described above.

```
7 struct hcône_test_case {
8     std::string name;
9     Matrix hcône; // vectors for H or V cone
10    Matrix key;    // minimal generating set
11
12    bool run_test() const;
13 };
14
15 struct vcône_test_case {
16     std::string name;
17     Matrix vcône; // vectors for H or V cone
18     Matrix key;    // minimal generating set
19
20    bool run_test() const;
21 };
22
23 struct hpoly_test_case {
24     std::string name;
25     Matrix hpoly; // vectors for H-Polyhedron
26     VPoly key;    // minimal generating set
27
28    bool run_test() const;
29 };
30
31 struct vpoly_test_case {
32     std::string name;
33     VPoly vpoly; // vectors for V-Polyhedron
34     Matrix key;    // minimal generating set
35
36    bool run_test() const;
37 };
```

4.6 test_functions.cpp

The dot-product and norm (in terms of dot product).

```
28 double operator*(const Vector &l, const Vector &r) {
29     if (l.size() > r.size()) {
30         throw runtime_error{"inner product: l > r"};
31     }
32     return inner_product(begin(l), end(l), begin(r), 0.);
33 }
34
35 double norm(const Vector &v) {
36     return sqrt(v*v);
37 }
```

`approximately_zero` is used during tests to avoid issues involving floating point rounding errors. For example, $1/6.0 * 2.5 - 5/12.0 == 0$ will give `false`, while `approximately_zero(1/6.0 * 2.5 - 5/12.0)` will return `true`. Test cases are used where intermediate calculations don't depend on such high accuracy, and these discrepancies can be ignored.

`approximately_zero(c) == true` is denoted $c \approx 0$.

```

39 bool approximately_zero(double d) {
40     const double error = .000001;
41     bool result = abs(d) < error;
42     if (d != 0 && result) {
43         ostringstream oss;
44         oss << scientific << d;
45         log("approximately_zero " + oss.str(), 1);
46     }
47     return result;
48 }

```

Tests $c < 0 \vee c \approx 0$.

```

50 bool approximately_lt_zero(double d) {
51     return d < 0 || approximately_zero(d);
52 }

```

Tests $\|v\| \approx 0$. This is denoted $v \approx 0$.

```

55 bool approximately_zero(const Vector &v) {
56     return approximately_zero(norm(v));
57 }

```

Tests $u/\|u\| - v/\|v\| \approx 0$. This is denoted $u \simeq v$.

```

59 bool is_equivalent(const Vector &l, const Vector &r) {
60     if (l.size() != r.size()) return false;
61     if (norm(l) == 0 || norm(r) == 0) {
62         return norm(l) == 0 && norm(r) == 0;
63     }
64     return approximately_zero(l / norm(l) - r / norm(r));
65 }

```

Tests $u - v \approx 0$. This is denoted $u \approx v$.

```

67 bool is_equal(const Vector &l, const Vector &r) {
68     if (l.size() != r.size()) return false;
69     return approximately_zero(l - r);
70 }

```

Tests $(\exists u \in U) \mid v \simeq u$.

```

72 bool has_equivalent_member(const Matrix &M,
73                             const Vector &v) {
74     if (!any_of(M.begin(), M.end(),
75                 [&](const Vector &u) {
76                     return is_equivalent(u, v);
77                 })) {
78         ostringstream oss;
79         oss << dashes
80             << " no equivalent member found for:\n"

```

```

80         << v << endl;
81         log(oss.str(),1);
82         return false;
83     }
84     return true;
85 }

```

Tests $(\exists \mathbf{u} \in U) \mid \mathbf{v} \approx \mathbf{u}$.

```

87 bool has_equal_member(const Matrix &M,
88                      const Vector &v) {
89     if (!any_of(M.begin(), M.end(),
90                [&](const Vector &u) { return
91                    is_equal(u,v); }))) {
92         ostringstream oss;
93         oss << dashes
94             << " no equal member found for:\n"
95             << v << endl;
96         log(oss.str(),1);
97         return false;
98     }
99     return true;
100 }

```

Tests $(\forall v \in V)(\exists \mathbf{u} \in U) \mid \mathbf{v} \simeq \mathbf{u}$. This is be denoted $V \sqsubseteq U$.

```

103 bool subset_mod_eq(const Matrix &generators,
104                  const Matrix &vcone) {
105     return all_of(generators.begin(), generators.end(),
106                  [&](const Vector &g) {
107                     return has_equivalent_member(vcone, g); }));
108 }

```

Tests $(\forall v \in V)(\exists \mathbf{u} \in U) \mid \mathbf{v} \approx \mathbf{u}$. This is be denoted $V \subseteq U$.

```

111 bool subset(const Matrix &generators,
112            const Matrix &vcone) {
113     return all_of(generators.begin(), generators.end(),
114                  [&](const Vector &g) {
115                     return has_equal_member(vcone, g); }));
116 }

```

Given a Vector constraint and Vector ray, tests if `approximately_lt_zero(ray * constraint)`. Note that if the constraint is of the form $\langle A_i, \mathbf{v} \rangle \leq b$ for some value b , then this tests $\langle A_i, \mathbf{ray} \rangle \leq 0$.

```

120 bool ray_satisfied(const Vector &constraint,
121                  const Vector &ray) {
122     if (constraint.size() != ray.size() &&
123         constraint.size()-1 != ray.size()) {
124         throw runtime_error{"bad ray vs constraint"};
125     }
126     double ip = ray * constraint;
127     if (!(approximately_lt_zero(ip))) {
128         ostringstream oss;
129         oss << dashes << " ray not satisfied!\n"

```

```

130         << "ray: " << ray
131         << "\nconstraint: " << constraint
132         << "\n ray * constraint = " << ip << endl;
133         log(oss.str(), 1);
134         return false;
135     }
136     return true;
137 }

```

Test $A\mathbf{v} \leq 0$

```

139 bool ray_satisfied(const Matrix &constraints,
140                   const Vector &ray) {
141     return all_of(constraints.begin(), constraints.end(),
142                  [&](const Vector &cv) {
143                     return ray_satisfied(cv, ray); });
144 }

```

Test $AV \leq 0$

```

146 bool rays_satisfied(const Matrix &constraints,
147                    const Matrix &rays) {
148     return all_of(rays.begin(), rays.end(),
149                  [&](const Vector &ray) {
150                     return ray_satisfied(constraints, ray); });
151 }

```

Test $\langle A_i, \mathbf{v} \rangle \leq b_i$

```

154 bool vec_satisfied(const Vector &constraint,
155                   const Vector &vec) {
156     size_t cback_i = constraint.size()-1;
157     if (cback_i != vec.size()) {
158         throw runtime_error{"bad vec vs constraint"};
159     }
160     double ip = vec * constraint;
161     double c_val = constraint[cback_i];
162     if (!(approximately_lt_zero(ip - c_val))) {
163         ostringstream oss;
164         oss << dashes << " vec not satisfied!\n"
165             << "vec: " << vec
166             << "\nconstraint: " << constraint
167             << "\n vec * constraint = " << ip << endl;
168         log(oss.str(), 1);
169         return false;
170     }
171     return true;
172 }

```

Test $A\mathbf{v} \leq \mathbf{b}$

```

174 bool vec_satisfied(const Matrix &constraints,
175                   const Vector &vec) {
176     return all_of(constraints.begin(), constraints.end(),
177                  [&](const Vector &cv) {
178                     return vec_satisfied(cv, vec); });

```

```
179 }
```

Test $AV \leq \mathbf{b}$

```
181 bool vecs_satisfied(const Matrix &constraints,
182                   const Matrix &vecs) {
183     return all_of(vecs.begin(), vecs.end(),
184                  [&](const Vector &vec) {
185                     return vec_satisfied(constraints, vec); });
186 }
```

Given an H-Cone $C = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\} = \text{cone}(U)$ where U is minimal, and a Matrix U' , determines if $C = \text{cone}(U')$. Similarly, given a V-Cone $C = \text{cone}(U) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{0}\}$ where A is minimal, and a Matrix A' , determines if $C = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{0}\}$.

```
190 bool equivalent_cone_rep(const Matrix &cone,
191                        const Matrix &key,
192                        const Matrix &alt_rep) {
193     return rays_satisfied (cone, alt_rep) &&
194            subset_mod_eq (key, alt_rep);
195 }
```

Given an H-Polytope $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} = \text{cone}(U) + \text{conv}(V)$ where U and V are minimal, and a pair (U', V') , determines if $P = \text{cone}(U') + \text{conv}(V')$.

```
197 bool equivalent_hpoly_rep(const Matrix &hpoly,
198                        const VPoly &key,
199                        const VPoly &vpoly) {
200     return rays_satisfied (hpoly, vpoly.U) &&
201            vecs_satisfied (hpoly, vpoly.V) &&
202            subset_mod_eq (key.U, vpoly.U) &&
203            subset (key.V, vpoly.V);
204 }
```

Given a V-Polytope $P = \text{cone}(U) + \text{conv}(V) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$ where A is minimal, and a Matrix (A', \mathbf{b}') , determines if $P = \{\mathbf{x} \mid A'\mathbf{x} \leq \mathbf{b}'\}$.

```
206 bool equivalent_vpoly_rep(const VPoly &vpoly,
207                        const Matrix &key,
208                        const Matrix &hpoly) {
209     return rays_satisfied (hpoly, vpoly.U) &&
210            vecs_satisfied (hpoly, vpoly.V) &&
211            subset_mod_eq (key, hpoly);
212 }
```

Bibliography

- [1] ZIEGLER, Gunter. *Lectures on Polytopes*. Springer-Verlag, New York, 1995.
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