Definition 1 (non-negative linear combination). *Let $U \in \mathbb{R}^{d \times p}$, $\mathbf{t} \in \mathbb{R}^p, \mathbf{t} \geq \mathbf{0}$, then $\sum_{1 \leq j \leq p} t_j U^j$ is called a **non-negative linear combination** of U.*

Definition 2 (V-Cone). *Let $U \in \mathbb{R}^{d \times p}$. The set of all non-negative linear combinations of U is denoted* cone(U)*. Such a set is called a **V-Cone**.*

Definition 3 (convex combination). *Let $V \in \mathbb{R}^{d \times n}$, $\boldsymbol{\lambda} \in \mathbb{R}^n, \boldsymbol{\lambda} \geq \mathbf{0}, \sum_{1 \leq j \leq n} \lambda_j = 1$, then $\sum_{1 \leq j \leq n} \lambda_j V^j$ is called a **convex combination** of V. The set of all convex combinations of V is denoted* conv(V).

Definition 4 (V-Polyhedron). *Let $V \in \mathbb{R}^{d \times n}$, $U \in \mathbb{R}^{d \times p}$. Then the set*

$$\{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \operatorname{cone}(U), \mathbf{y} \in \operatorname{conv}(V)\}$$

is called a V-Polyhedron.

Definition 5 (H-Polyhedron). *Let $A \in \mathbb{R}^{m \times d}$, $\mathbf{b} \in \mathbb{R}^m$. Then the set*

$$\left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; A\mathbf{x} \leq \mathbf{b}\right\}$$

is called an H-Polyhedron.

Definition 6 (H-Cone). *Let $A \in \mathbb{R}^{m \times d}$. Then the set*

$$\left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; A\mathbf{x} \leq \mathbf{0}\right\}$$

is called an H-Cone.

The following theorem is the basic result to be proved in this thesis, which states that V-Polyhedra and H-Polyhedra are two different representations of the same objects.

Theorem 1 (Minkowski-Weyl Theorem). *Every V-Polyhedron is an H-Polyhedron, and every H-Polyhedron is a V-Polyhedron.*

The proof proceeds by first showing that V-Cones are representable as H-Cones, and H-Cones are representable as V-Cones. Then it is shown that the case of polyhedra can be reduced to cones.

Theorem 2 (Minkowski-Weyl Theorem for Cones). *Every V-Cone is an H-Cone, and every H-Cone is a V-Cone.*

1 Every V-Cone is an H-Cone

Definition 7 (Coordinate Projection). *Let I be the identity matrix. Then the matrix I' formed by deleting some rows from I is called a **coordinate-projection**.*

The proof rests on the following two propostions:

(V1) Every V-Cone is a coordinate-projections of an H-Cone.

(V2) Every coordinate-projection of an H-Cone is an H-Cone.

Proof. Given (V1) and (V2), the proof follows simply. Given a V-Cone, we use (V1), to get a description involving coordinate-projection of an H-Cone. Then we can apply (V2) in order to get an H-Cone. □

Proof of (V1). We prove that every V-Cone is a coordinate-projection of an H-Cone, by giving an explicit formula. Let $U \in \mathbb{R}^{d\times p}$, and observe that

$$\operatorname{cone}(U) = \{U\mathbf{t} \mid \mathbf{t} \in \mathbb{R}^p,\, \mathbf{t} \geq \mathbf{0}\} = \left\{\mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \in \mathbb{R}^p)\, \mathbf{x} = U\mathbf{t},\, \mathbf{t} \geq \mathbf{0}\right\}$$

We will collect $\mathbf{t}$ and $\mathbf{x}$ on the left side of the inequality, treating $\mathbf{t}$ as a variable and expressing its contraints as linear inequalities, then project away the coordinates corresponding to $\mathbf{t}$. The following expression takes one step:

$$\mathbf{t} \geq \mathbf{0} \Leftrightarrow -I\mathbf{t} \leq \mathbf{0} \tag{1}$$

And using the equality: $a = 0 \Leftrightarrow a \leq 0 \wedge -a \leq 0$, and block matrix notation, we take the second step.

$$\mathbf{x} = U\mathbf{t} \Leftrightarrow \mathbf{x} - U\mathbf{t} = \mathbf{0} \Leftrightarrow \begin{pmatrix} I & -U \\ -I & U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \tag{2}$$

Comparing (1) and (2), we define a new matrix $A' \in \mathbb{R}^{(p+2d)\times(d+p)}$:

$$A' = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix}$$

then we can rewrite $\operatorname{cone}(U)$:

$$\operatorname{cone}(U) = \left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; A' \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0}\right\}$$

Let $\Pi \in \{0,1\}^{d\times(d+p)}$ be the identity matrix in $\mathbb{R}^{(d+p)\times(d+p)}$, but with the last p-rows deleted. Then Π is a coordinate projection, and the above expression can be written:

$$\operatorname{cone}(U) = \Pi\left(\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0}\right\}\right) \tag{3}$$

This is a coordinate projection of an H-Cone, and (V1) is shown. □

To prove (V2), we use two separate propositions.

Proposition 1. *Let $B \in \mathbb{R}^{m'\times(d+p)}$, B' be B with the last p columns deleted, and Π the identity matrix with the last p rows deleted (i.e. $B' = \Pi B$). Furthermore, suppose that the last p columns of B are $\mathbf{0}$. Then*

$$\Pi\left(\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid B\mathbf{y} \leq \mathbf{0}\right\}\right) = \left\{\mathbf{x} \in \mathbb{R}^d \mid B'\mathbf{x} \leq \mathbf{0}\right\}$$

Proof. Recall that $B\mathbf{y} \leq \mathbf{0}$ means that $(\forall i)\,\langle B_i, \mathbf{y}\rangle \leq 0$. By the way we've defined B, any row B_i of B can be written $(B'_i, \mathbf{0})$, with $\mathbf{0} \in \mathbb{R}^p$. Rewriting $\mathbf{y} \in \mathbb{R}^{d+p}$ as $(\mathbf{x}, \mathbf{w})$ with $\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^p$, so that $\mathbf{x} = \Pi(\mathbf{y})$. Then

$$\langle B, \mathbf{y}\rangle = \langle (B'_i, \mathbf{0}), (\mathbf{x}, \mathbf{w})\rangle = \langle B'_i, \mathbf{x}\rangle = \langle B'_i, \Pi(\mathbf{y})\rangle$$

It follows that

$$\langle B_i, \mathbf{y}\rangle \leq 0 \Leftrightarrow \langle B'_i, \Pi(\mathbf{y})\rangle \leq 0$$

Since B_i is an arbitrary row of B, the proposition is shown. □

In order to use the above proposition, we need a matrix with $\mathbf{0}$ columns. The next proposition shows us how to do so, one column at a time.

Proposition 2. *Let $B \in \mathbb{R}^{m_1 \times (d+p)}$, $1 \leq k \leq p$, and $\mathbf{x} = \sum_{i \neq k} x_i \mathbf{e}_i$. Then there exists a matrix $B' \in \mathbb{R}^{m_2 \times (d+p)}$ with the following properties:*

1. *Every row of B' is a postive linear combination of rows of B.*
2. *m_2 is finite.*
3. *The k-th column of B' is $\mathbf{0}$.*
4. *$(\exists t) B(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$*

Proof. Partition the rows of B as follows:

$$\begin{aligned} P &= i \mid B_i^k > 0 \\ N &= j \mid B_j^k < 0 \\ Z &= l \mid B_l^k = 0 \end{aligned}$$

Then let B' be a matrix with rows of the following forms:

$$\begin{aligned} C_l &= B_l && \mid l \in Z \\ C_{ij} &= B_i^k B_j - B_j^k B_i && \mid i \in P, j \in N \end{aligned}$$

1 and *2* are clear. *3* can be seen from:

$$\langle C_l, \mathbf{e}_k\rangle = 0$$

$$\langle C_{ij}, \mathbf{e}_k\rangle = \left\langle B_i^k B_j - B_j^k B_i, \mathbf{e}_k \right\rangle = B_i^k B_j^k - B_j^k B_i^k = 0 \tag{4}$$

The right direction of *4* is shown in the following calculations. Because $B_l^k = 0$:

$$\langle B_l, \mathbf{x} + t\mathbf{e}_k\rangle = \langle B_l, \mathbf{x}\rangle + tB_l^k = \langle B_l, \mathbf{x}\rangle = \langle C_l, \mathbf{x}\rangle$$

So:

$$\langle B_l, \mathbf{x} + t\mathbf{e}_k\rangle \leq 0 \Rightarrow \langle C_l, \mathbf{x}\rangle \leq 0$$

For rows indexed by P, N, we observe (13), and have:

$$\left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \right\rangle = \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \right\rangle$$

Now, we use property *1*:

$$\langle B_i, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0,\ \langle B_j, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0 \Rightarrow \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \right\rangle \leq 0$$

Therefore

$$\left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \right\rangle \leq 0$$

Now suppose that $B'\mathbf{x} \leq \mathbf{0}$. The task is to find a t so that $B\mathbf{x} \leq \mathbf{0}$. Looking at (13), any choice of t we make will be okay for rows indexed by Z. So the task is to find a t so that the inequality holds for rows indexed by P and N. Observe

$$\begin{aligned}
&\forall i \in P, \forall j \in N \quad \left\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \right\rangle \leq 0 \Leftrightarrow \\
&\forall i \in P, \forall j \in N \quad \left\langle B_i^k B_j, \mathbf{x} \right\rangle \ \leq \ \left\langle B_j^k B_i, \mathbf{x} \right\rangle \ \Leftrightarrow \\
&\forall i \in P, \forall j \in N \quad \left\langle B_j / B_j^k, \mathbf{x} \right\rangle \geq \left\langle B_i / B_i^k, \mathbf{x} \right\rangle \Leftrightarrow \\
&\quad \min_{j \in N} \left\langle B_j / B_j^k, \mathbf{x} \right\rangle \ \geq \ \max_{i \in P} \left\langle B_i / B_i^k, \mathbf{x} \right\rangle
\end{aligned}$$

Note that the third inequality changes directions because $B_j^k < 0$. Now we choose t to lie in this last interval, and show that we can use it to satisfy all of the constraints given by B. So, we have a t such that

$$\min_{j \in N} \left\langle B_j / B_j^k, \mathbf{x} \right\rangle \geq t \geq \max_{i \in P} \left\langle B_i / B_i^k, \mathbf{x} \right\rangle$$

In particular,

$$\begin{aligned}
&(\forall j \in N) \quad \left\langle B_j / B_j^k, \mathbf{x} \right\rangle \quad \geq t \Rightarrow \\
&(\forall j \in N) \quad \langle B_j, \mathbf{x} \rangle - B_j^k t \leq 0
\end{aligned}$$

Again, the inequality changes directions because $B_j^k < 0$. Now consider a row B_j from B:

$$\langle B_j, \mathbf{x} - t\mathbf{e}_k \rangle = B_j \mathbf{x} - B_j^k t \leq 0$$

Similarly,

$$\begin{aligned}
&(\forall i \in P) \quad t \geq B_i / B_i^k \mathbf{x} \quad \Rightarrow \\
&(\forall i \in P) \quad 0 \geq B_i \mathbf{x} - B_i^k t
\end{aligned}$$

Now consider a row B_i from B:

$$\langle B_i, \mathbf{x} - t\mathbf{e}_k \rangle = B_i \mathbf{x} - B_i^k t \leq 0$$

So, we've demonstrated that $\mathbf{x} - t\mathbf{e}_k$ satisfies all the constraints from B, and the left implication is shown. So *4* holds. $\square$

Now to prove:

(V2) Every coordinate-projection of an H-Cone is an H-Cone.

proof of (V2). Here we prove the case that the coordinate projection is onto the first d of $d+p$ coordinates. Let $\left\{\mathbf{y} \in \mathbb{R}^{d+p} : A'\mathbf{y} \leq \mathbf{0}\right\}$ be the H-Cone we need to project, and Π the coordinate-projection we need to apply (the identity matrix with the last p columns deleted). For each $1 \leq k \leq p$ we can use proposition 2 in an incremental manner, starting with A'.

let $B_0 := A'$

for $1 \leq k \leq p$

  let $B_k :=$ result of proposition 2 applied to B_{k-1}, $\mathbf{e}_{d+k}$

endfor

return B_p

Consider the resulting B. Property *2* holds throughout, so B is finite. After each iteration, property *3* holds for k, so the k-th column is $\mathbf{0}$. Since each iteration only results from non-negative combinations of the result of the previous iteration (property *1*), once a column is $\mathbf{0}$ it remains so. Therefore, at the end of the process, the last p columns of B are all $\mathbf{0}$. Then, by proposition 1, we can apply Π to B by simply deleting the last p columns of B. Denote this resulting matrix A. We still need to check:

$$A'\mathbf{y} \leq \mathbf{0} \Leftrightarrow A(\Pi(\mathbf{y})) \leq \mathbf{0} \tag{5}$$

$$(\exists t_1)\ldots(\exists t_p)A'(\mathbf{x} + t_1\mathbf{e}_{d+1} + \cdots + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} \Leftrightarrow A\mathbf{x} \leq \mathbf{0} \tag{6}$$

Then, using (5) and (6), it is easy to see that:

$$\Pi\left\{\mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0}\right\} = \left\{\mathbf{x} \in \mathbb{R}^{d} \mid A\mathbf{x} \leq \mathbf{0}\right\} \tag{7}$$

The key observation of this verification utilizes property *4* of proposition 2:

$$(\exists t)B(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$$

In what follows, let $\mathbf{x} = \sum_{1 \leq j \leq d} x_j\mathbf{e}_j$. The above property is applied sequentially to the sets B_k as follows:

$$\begin{array}{rll}
(\exists t_p)(\exists t_{p-1})\ldots(\exists t_1) & B_0(\mathbf{x} + t_1\mathbf{e}_p + t_2\mathbf{e}_{p-1} + \cdots + t_p\mathbf{e}_d) \leq \mathbf{0} & \Leftrightarrow \\
(\exists t_p)\ldots(\exists t_2) & B_1(\mathbf{x} + t_2\mathbf{e}_{d+2} + \cdots + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow \\
\vdots & \vdots & \vdots \\
(\exists t_p) & B_{p-1}(\mathbf{x} + t_p\mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow \\
 & B_p\mathbf{x} \leq \mathbf{0} & \Leftrightarrow
\end{array}$$

Because $A' = B_0$, and A is B_p with the last p columns deleted, (5) and (6) hold, therefore (7) holds, and the proof of (V2) is complete, and we've shown that a coordinate projection of an H-Cone is again an H-Cone. □

With (V1) and (V2) proven, we are now certain that any V-Cone is also an H-Cone.

2 Every H-Cone is a V-Cone

Definition 8 (Coordinate Hyperplane). *A set of the form*

$$\{\mathbf{x} \in \mathbb{R}^{d+m} \mid \langle \mathbf{x}, \mathbf{e}_k \rangle = 0\} = \{\mathbf{x} \in \mathbb{R}^{d+m} \mid x_k = 0\}$$

is called a coordinate-hyperplane.

We will use coordinate-hyperplanes in the following way. We consider a V-Cone intersected with some coordinate hyperplanes, and write it in the following way:

$$\left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t} \right\} \tag{8}$$

If we suppose that $U' \subset \mathbb{R}^{d+m}$, and Π is the identity matrix with the last m rows deleted, then this is just a convenient way of writing:

$$\Pi\big(\operatorname{cone}(U') \cap \{x_{d+1} = 0\} \cap \cdots \cap \{x_{d+m} = 0\}\big) \tag{9}$$

The proof rests on the following three propostions:

H1 Every H-Cone is a coordinate-projection of a V-Cone intersected with some coordinate hyperplanes.

H2 Every V-Cone intersected with a coordinate-hyperplane is a V-Cone

H3 Every coordinate-projection of a V-Cone is an V-Cone.

Proof. Given *H1*, *H2*, and *H3*, the proof follows simply. Given an H-Cone, we use *H1* to get a description involving the coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. We apply *H2* as many times as necessary to elimintate the intersections, then we can apply *H3* in order to get a V-Cone. □

Proof of H1. Let $A \in \mathbb{R}^{m \times d}$, we now show that the H-Cone

$$\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0}\}$$

can be written as the projection of a V-Cone intersected with some hyperplanes. Define U':

$$U' = \left\{ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}, 1 \leq j \leq d,\, 1 \leq i \leq m \right\}$$

Then we claim:

$$\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0}\} = \left\{\mathbf{x} \in \mathbb{R}^d \;\middle|\; (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t} \right\} \tag{10}$$

First, considering (8) and (9), observe that this is a coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. Next, we note that

$$\begin{pmatrix}\mathbf{x}\\A\mathbf{x}\end{pmatrix} = \sum_{1\le j\le d} x_j \begin{pmatrix}\mathbf{e}_j\\A^j\end{pmatrix}$$

We can write this as a sum with all positive coefficients if we split up the x_j as follows:

$$x_j^+ = \begin{cases} x_j & x_j \ge 0 \\ 0 & x_j < 0 \end{cases} \qquad x_j^- = \begin{cases} 0 & x_j \ge 0 \\ -x_j & x_j < 0 \end{cases}$$

Then we have

$$\begin{pmatrix}\mathbf{x}\\A\mathbf{x}\end{pmatrix} = \sum_{1\le j\le d} x_j^+ \begin{pmatrix}\mathbf{e}_j\\A^j\end{pmatrix} + \sum_{1\le j\le d} x_j^- \begin{pmatrix}-\mathbf{e}_j\\-A^j\end{pmatrix} \tag{11}$$

where $x_j^+, x_j^- \ge 0$. Also observe that

$$A\mathbf{x} \le \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \ge \mathbf{0}) \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$$

This can also be written

$$A\mathbf{x} \le \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \ge \mathbf{0}) \mid \begin{pmatrix}\mathbf{x}\\A\mathbf{x}\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\\mathbf{w}\end{pmatrix} = \begin{pmatrix}\mathbf{x}\\\mathbf{0}\end{pmatrix} \tag{12}$$

(11) and (12) together show

$$A\mathbf{x} \le \mathbf{0} \Rightarrow (\exists \mathbf{t} \ge 0) \begin{pmatrix}\mathbf{x}\\\mathbf{0}\end{pmatrix} = U'\mathbf{t}$$

Conversely, suppose

$$(\exists \mathbf{t} \ge 0) \begin{pmatrix}\mathbf{x}\\\mathbf{0}\end{pmatrix} = U'\mathbf{t}$$

We would like to show that $A\mathbf{x} \le \mathbf{0}$. Let x_j^+, x_j^-, w_i take the values of $\mathbf{t}$ that are coefficients of $\begin{pmatrix}\mathbf{e}_j\\A^j\end{pmatrix}$, $\begin{pmatrix}-\mathbf{e}_j\\-A^j\end{pmatrix}$, and $\begin{pmatrix}\mathbf{0}\\\mathbf{e}_i\end{pmatrix}$ respectively, and denote $x_j = x_j^+ - x_j^-$. Then we have

$$\begin{aligned}\begin{pmatrix}\mathbf{x}\\\mathbf{0}\end{pmatrix} &= \sum_{1\le j\le d} x_j^+ \begin{pmatrix}\mathbf{e}_j\\A^j\end{pmatrix} + \sum_{1\le j\le d} x_j^- \begin{pmatrix}-\mathbf{e}_j\\-A^j\end{pmatrix} + \sum_{1\le i\le n} w_i \begin{pmatrix}\mathbf{0}\\\mathbf{e}_i\end{pmatrix}\\ &= \sum_{1\le j\le d} x_j \begin{pmatrix}\mathbf{e}_j\\A^j\end{pmatrix} + \sum_{1\le i\le n} w_i \begin{pmatrix}\mathbf{0}\\\mathbf{e}_i\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{x}\\A\mathbf{x}\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\\mathbf{w}\end{pmatrix}\end{aligned}$$

where $\mathbf{w} \ge \mathbf{0}$. By (12) we have $A\mathbf{x} \le \mathbf{0}$. So (10) holds. $\square$

The proof of *H2* relies upon the following proposition.

Proposition 3. *Let $Y \in \mathbb{R}^{(d+m)\times n_1}$, $1 \le k \le m$, and $\mathbf{x}$ satisfy $x_k = 0$. Then there exists a matrix $Y' \in \mathbb{R}(d+m) \times n_2$ with the following properties:*

1. *Every column of Y' is a postive linear combination of rows of B.*
2. *n_2 is finite.*
3. *The k-th row of Y' is $\mathbf{0}$.*
4. $(\exists \mathbf{t} \ge \mathbf{0})\mathbf{x} = Y\mathbf{t} \Leftrightarrow (\exists \mathbf{t}' \ge \mathbf{0})\mathbf{x} = Y'\mathbf{t}'$

Recall that Y^i is the i-th column of Y, and Y^i_k is the element of Y in the i-th column and k-th row.

Proof. We partition the columns of Y:

$$\begin{aligned} P &= i \;\mid Y^i_k > 0 \\ N &= j \;\mid Y^j_k < 0 \\ Z &= l \;\mid Y^l_k = 0 \end{aligned}$$

We then define Y':

$$Y' = \left\{Y^l \mid l \in Z\right\} \cup \left\{Y^i_k Y^j - Y^j_k Y^i \mid i \in P,\, j \in N\right\}$$

1 and *2* are clear. *3* can be seen from:

$$\left\langle Y'^l, \mathbf{e}^k \right\rangle = 0$$

$$\left\langle Y'^{ij}, \mathbf{e}^k \right\rangle = \left\langle Y^i_k Y^j - Y^j_k Y^i, \mathbf{e}^k \right\rangle = Y^i_k Y^j_k - Y^j_k Y^i_k = 0 \tag{13}$$

Before moving on to the proof, we first note how to write our vectors.

$$Y\mathbf{t} = \sum_{k\in Z} t_k Y^k + \sum_{i \in P} t_i Y^i + \sum_{j\in N} t_j Y^j$$

$$Y'\mathbf{t} = \sum_{k\in Z} t_k Y^k + \sum_{\substack{i\in P\\ j\in N}} t_{ij}(Y^i_k Y^j - Y^j_k Y^i)$$

Then, to show that the proposition is true, we need only show that, given any $t_i, t_j \ge 0$ ($t_{ij} \ge 0$), there exists $t_{ij} \ge 0$ ($t_i, t_j \ge 0$) such that

$$\sum_{i\in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i\in P\\ j\in N}} t_{ij}(Y^i_k Y^j - Y^j_k Y^i) \tag{14}$$

Proposition 4. *Suppose that*

$$\sum_{i\in P} t_i Y^i_{d+1} + \sum_{j\in N} t_j Y^j_{d+1} = 0 \qquad Y^j_k < 0 < Y^i_k$$

Then the following holds

$$\begin{aligned} (t_i, t_j \geq 0) &\Rightarrow (\exists t_{ij} \geq 0) \quad \text{such that (14) holds} \\ (t_{ij} \geq 0) &\Rightarrow (\exists t_i, t_j \geq 0) \text{ such that (14) holds} \end{aligned}$$

Proof. First note that if all $t_i = 0, t_j = 0$, then choosing $t_{ij} = 0$ satisfies (14), likewise if all $t_{ij} = 0$, then $t_i = 0, t_j = 0$ satisfies (14). So suppose that some $t_i \neq 0, t_j \neq 0, t_{ij} \neq 0$.

The right hand side of (14) can be written

$$\sum_{j\in N} \left(\sum_{i\in P} t_{ij} Y^i_k \right) Y^j + \sum_{i\in P} \left(-\sum_{j\in N} t_{ij} Y^j_k \right) Y^i$$

This means, given $t_{ij} \geq 0$, we can choose $t_j = \sum_{i\in P} t_{ij} Y^i_k$, and $t_i = -\sum_{j\in N} t_{ij} Y^j_k$, both of which are greater than 0.

Now suppose we have been given $t_i \geq 0, t_j \geq 0$. First observe:

$$0 = \sum_{i\in P} t_i Y^i_k + \sum_{j\in N} t_j Y^j_k \Rightarrow \sum_{i\in P} t_i Y^i_k = -\sum_{j\in N} t_j Y^j_k$$

Denote the value in this equality as σ, and note that $\sigma > 0$. Then

$$\begin{aligned} \sum_{i\in P} t_i Y^i &= \frac{-\sum_{j\in N} t_j Y^j_k}{\sigma} \sum_{i\in P} t_i Y^i = \sum_{\substack{i\in P\\ j\in N}} -\frac{t_i t_j}{\sigma} Y^j_k Y^i \\ \sum_{j\in N} t_j Y^j &= \frac{\sum_{i\in P} t_i Y^i_k}{\sigma} \sum_{j\in N} t_j Y^j = \sum_{\substack{i\in P\\ j\in N}} \frac{t_i t_j}{\sigma} Y^i_k Y^j \end{aligned}$$

Combining these results, we have

$$\sum_{i\in P} t_i Y^i + \sum_{j\in N} t_j Y^j = \sum_{\substack{i\in P\\ j\in N}} \frac{t_i t_j}{\sigma} (Y^i_k Y^j - Y^j_k Y^i)$$

□

Finally, we can conclude that, given $\mathbf{t} \geq \mathbf{0}$, if $Y\mathbf{t}$ has a 0 in the final coordinate, then we can write it as $Y'\mathbf{t}'$ where $\mathbf{t}' \geq \mathbf{0}$, and any non-negative linear combination of vectors from Y' can be written as a non-negative linear combination of vetors from Y, and will necessarily have the k-th coordinate be 0 by property *3*. So property *4* holds. □

Proof of H2. In proposition 3, the assumption that $x_k = 0$ in property *4* creates the set $\text{cone}(Y) \cap \{\mathbf{x} \mid x_k = 0\}$. This set, by property *4*, is $\text{cone}(Y')$. $\square$

Proof of H3. We shall prove that the coordinate-projection of a V-Cone is again a V-Cone. Let Π be the relevant projection, then we have:

$$\Pi \{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U\mathbf{t}) \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U)\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\}$$

The last equality follows from associativity of matrix multiplication. Therefore,

$$\Pi\big(\text{cone}(U)\big) = \text{cone}\left(\Pi(U)\right)$$

$\square$