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Minkowski-Weyl Theorem

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Dedication.

Title: Minkowski-Weyl Theorem

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Abstract: The Minkowski-Weyl Theorem is proven for polyhedra by first showing the proof for cones, then the reductions from polyhedra to cones. The proof follows Ziegler [1], and uses Fourier-Motzkin elimination. A C++ implementation is given for the enumeration algorithm suggested by the proof.

Keywords: Minkowski-Weyl Theorem polyhedra Fourier-Motzkin C++

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Introduction

Polyhedra are fundamental mathematical objects. Two ways of describing polyhedra are:

1. A finite intersection of half-spaces
2. The *Minkowski-Sum* of the *convex-hull* of a finite set of rays and a finite set of points

The Minkowski-Weyl Theorem is a fundamental result in the theory of polyhedra. It states that both means of representation are equivalent. The proof given here is algorithmic in nature, using a technique known as *Fourier-Motzkin elimination*. The algorithm implied by the proof is then implemented in C++.

1. Minkowski-Weyl Theorem

1.1 Polyhedra

Definition 1 (Non-negative Linear Combination). Let $U \in \mathbb{R}^{d \times p}$, $\mathbf{t} \in \mathbb{R}^p$, $\mathbf{t} \geq \mathbf{0}$, then $\sum_{1 \leq j \leq p} t_j U^j = U\mathbf{t}$ is called a *non-negative linear combination* of U .

Definition 2 (V-Cone). Let $U \in \mathbb{R}^{d \times p}$. The set of all non-negative linear combinations of U is denoted $\text{cone}(U)$. Such a set is called a *V-Cone*.

Definition 3 (Convex Combination). Let $V \in \mathbb{R}^{d \times n}$, $\boldsymbol{\lambda} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \geq \mathbf{0}$, $\sum_{1 \leq j \leq n} \lambda_j = 1$, then $\sum_{1 \leq j \leq n} \lambda_j V^j$ is called a *convex combination* of V . The set of all convex combinations of V is denoted $\text{conv}(V)$.

Definition 4 (V-Polyhedron). Let $V \in \mathbb{R}^{d \times n}$, $U \in \mathbb{R}^{d \times p}$. Then the set

$$\{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \text{cone}(U), \mathbf{y} \in \text{conv}(V)\}$$

is called a *V-Polyhedron*.

Definition 5 (H-Polyhedron). Let $A \in \mathbb{R}^{m \times d}$, $\mathbf{b} \in \mathbb{R}^m$. Then the set

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b} \right\}$$

is called an *H-Polyhedron*.

Definition 6 (H-Cone). Let $A \in \mathbb{R}^{m \times d}$. Then the set

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$$

is called an *H-Cone*.

A simple but useful property of cones is that they are closed under addition and positive scaling.

Proposition 1. Let C be either an H-Cone or a V-Cone, for each i $\mathbf{x}^i \in C$, and $c_i \geq 0$. Then:

$$\sum_i c_i \mathbf{x}^i \in C$$

Proof. First we prove Proposition 1 for H-Cones, then for V-Cones. If, for each i , $A\mathbf{x}^i \leq \mathbf{0}$, then $A(c_i \mathbf{x}^i) = c_i A\mathbf{x}^i \leq \mathbf{0}$, and

$$A\left(\sum_i c_i \mathbf{x}^i\right) = \sum_i A(c_i \mathbf{x}^i) = \sum_i c_i A\mathbf{x}^i \leq \sum_i \mathbf{0} \leq \mathbf{0}$$

So, $\sum_i c_i \mathbf{x}^i \in C$ when C is an H-Cone. Next, suppose that $C = \text{cone}(U)$, and for each i , $\exists \mathbf{t}_i \geq \mathbf{0} : \mathbf{x}^i = U\mathbf{t}_i$. Then $c_i \mathbf{t}_i \geq \mathbf{0}$, and $\sum_i c_i \mathbf{t}_i \geq \mathbf{0}$. Therefore

$$\sum_i c_i \mathbf{x}^i = \sum_i c_i U\mathbf{t}_i = \sum_i U(c_i \mathbf{t}_i) = U\left(\sum_i c_i \mathbf{t}_i\right)$$

So, $\sum_i c_i \mathbf{x}^i \in C$ when C is a V-Cone. □

This proposition will be used in the following way: if we wish to show that $\sum_i c_i \mathbf{x}^i$ is a member of some cone C , it suffices to show that, for each i , $c_i \geq 0$ and $\mathbf{x}^i \in C$.

1.2 Minkowski-Weyl Theorem

The following theorem is the basic result to be proved in this thesis, which states that V-Polyhedra and H-Polyhedra are two different representations of the same objects.

Theorem 1 (Minkowski-Weyl Theorem). *Every V-Polyhedron is an H-Polyhedron, and every H-Polyhedron is a V-Polyhedron.*

The proof proceeds by first showing that V-Cones are representable as H-Cones, and H-Cones are representable as V-Cones. Then it is shown that the case of polyhedra can be reduced to cones.

Theorem 2 (Minkowski-Weyl Theorem for Cones). *Every V-Cone is an H-Cone, and every H-Cone is a V-Cone.*

2. Proof of the Minkowski-Weyl Theorem

2.1 Every V-Cone is an H-Cone

Definition 7 (Coordinate Projection). Let I be the identity matrix. Then the matrix I' formed by deleting some rows from I is called a **coordinate-projection**.

The proof rests on the following two propositions:

(V1) Every V-Cone is a coordinate-projection of an H-Cone.

(V2) Every coordinate-projection of an H-Cone is an H-Cone.

Proof. Given (V1) and (V2), the proof follows simply. Given a V-Cone, we use (V1), to get a description involving coordinate-projection of an H-Cone. Then we can apply (V2) in order to get an H-Cone. \square

Proof of (V1). We prove that every V-Cone is a coordinate-projection of an H-Cone, by giving an explicit formula. Let $U \in \mathbb{R}^{d \times p}$, and observe that

$$\text{cone}(U) = \{U\mathbf{t} \mid \mathbf{t} \in \mathbb{R}^p, \mathbf{t} \geq \mathbf{0}\} = \left\{ \mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \in \mathbb{R}^p) \mathbf{x} = U\mathbf{t}, \mathbf{t} \geq \mathbf{0} \right\}$$

We will collect \mathbf{t} and \mathbf{x} on the left side of the inequality, treating \mathbf{t} as a variable and expressing its constraints as linear inequalities, then project away the coordinates corresponding to \mathbf{t} . The following expression takes one step:

$$\mathbf{t} \geq \mathbf{0} \Leftrightarrow -I\mathbf{t} \leq \mathbf{0} \tag{2.1}$$

And using the equality: $a = 0 \Leftrightarrow a \leq 0 \wedge -a \leq 0$, and block matrix notation, we take the second step.

$$\mathbf{x} = U\mathbf{t} \Leftrightarrow \mathbf{x} - U\mathbf{t} = \mathbf{0} \Leftrightarrow \begin{pmatrix} I & -U \\ -I & U \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \tag{2.2}$$

Comparing (2.1) and (2.2), we define a new matrix $A' \in \mathbb{R}^{(p+2d) \times (d+p)}$:

$$A' = \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix} \tag{2.3}$$

then we can rewrite $\text{cone}(U)$:

$$\text{cone}(U) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid A' \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \end{pmatrix} \leq \mathbf{0} \right\}$$

Let $\Pi \in \{0, 1\}^{d \times (d+p)}$ be the identity matrix in $\mathbb{R}^{(d+p) \times (d+p)}$, but with the last p -rows deleted. Then Π is a coordinate projection, and the above expression can be written:

$$\text{cone}(U) = \Pi \left(\left\{ \mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0} \right\} \right) \tag{2.4}$$

This is a coordinate projection of an H-Cone, and (V1) is shown. \square

To prove (V2), we use two separate propositions.

Proposition 2. *Let $B \in \mathbb{R}^{m' \times (d+p)}$, B' be B with the last p columns deleted, and Π the identity matrix with the last p rows deleted (i.e. $B' = \Pi B$). Furthermore, suppose that the last p columns of B are $\mathbf{0}$. Then*

$$\Pi \left(\left\{ \mathbf{y} \in \mathbb{R}^{d+p} \mid B\mathbf{y} \leq \mathbf{0} \right\} \right) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid B'\mathbf{x} \leq \mathbf{0} \right\}$$

Proof. Recall that $B\mathbf{y} \leq \mathbf{0}$ means that $(\forall i) \langle B_i, \mathbf{y} \rangle \leq 0$. By the way we've defined B , any row B_i of B can be written $(B'_i, \mathbf{0})$, with $\mathbf{0} \in \mathbb{R}^p$. Rewriting $\mathbf{y} \in \mathbb{R}^{d+p}$ as (\mathbf{x}, \mathbf{w}) with $\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^p$, so that $\mathbf{x} = \Pi(\mathbf{y})$. Then

$$\langle B, \mathbf{y} \rangle = \langle (B'_i, \mathbf{0}), (\mathbf{x}, \mathbf{w}) \rangle = \langle B'_i, \mathbf{x} \rangle = \langle B'_i, \Pi(\mathbf{y}) \rangle$$

It follows that

$$\langle B_i, \mathbf{y} \rangle \leq 0 \Leftrightarrow \langle B'_i, \Pi(\mathbf{y}) \rangle \leq 0$$

Since B_i is an arbitrary row of B , the proposition is shown. \square

In order to use the above proposition, we need a matrix with $\mathbf{0}$ columns. The next proposition shows us how to do so, one column at a time.

Proposition 3. *Let $B \in \mathbb{R}^{m_1 \times (d+p)}$, $1 \leq k \leq p$, and $\mathbf{x} = \sum_{i \neq k} x_i \mathbf{e}_i$. Then there exists a matrix $B' \in \mathbb{R}^{m_2 \times (d+p)}$ with the following properties:*

1. *Every row of B' is a positive linear combination of rows of B .*
2. *m_2 is finite.*
3. *The k -th column of B' is $\mathbf{0}$.*
4. *$(\exists t) B(\mathbf{x} + t\mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$*

Proof. Partition the rows of B as follows:

$$\begin{aligned} P &= i \mid B_i^k > 0 \\ N &= j \mid B_j^k < 0 \\ Z &= l \mid B_l^k = 0 \end{aligned}$$

Then let B' be a matrix with rows of the following forms:

$$\begin{aligned} C_l &= B_l & \mid l \in Z \\ C_{ij} &= B_i^k B_j - B_j^k B_i & \mid i \in P, j \in N \end{aligned}$$

1 and 2 are clear. 3 can be seen from:

$$\langle C_l, \mathbf{e}_k \rangle = 0$$

$$\langle C_{ij}, \mathbf{e}_k \rangle = \langle B_i^k B_j - B_j^k B_i, \mathbf{e}_k \rangle = B_i^k B_j^k - B_j^k B_i^k = 0 \quad (2.5)$$

The right direction of 4 is shown in the following calculations. Because $B_l^k = 0$:

$$\langle B_l, \mathbf{x} + t\mathbf{e}_k \rangle = \langle B_l, \mathbf{x} \rangle + tB_l^k = \langle B_l, \mathbf{x} \rangle = \langle C_l, \mathbf{x} \rangle$$

So:

$$\langle B_l, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0 \Rightarrow \langle C_l, \mathbf{x} \rangle \leq 0$$

For rows indexed by P, N , we observe (2.15), and have:

$$\langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \rangle = \langle B_i^k B_j - B_j^k B_i, \mathbf{x} \rangle$$

Now, we use property 1:

$$\langle B_i, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0, \langle B_j, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0 \Rightarrow \langle B_i^k B_j - B_j^k B_i, \mathbf{x} + t\mathbf{e}_k \rangle \leq 0$$

Therefore

$$\langle B_i^k B_j - B_j^k B_i, \mathbf{x} \rangle \leq 0$$

Now suppose that $B'\mathbf{x} \leq \mathbf{0}$. The task is to find a t so that $B\mathbf{x} \leq \mathbf{0}$. Looking at (2.15), any choice of t we make will be okay for rows indexed by Z . So the task is to find a t so that the inequality holds for rows indexed by P and N . Observe

$$\begin{aligned} \forall i \in P, \forall j \in N \quad & \langle B_i^k B_j - B_j^k B_i, \mathbf{x} \rangle \leq 0 \Leftrightarrow \\ \forall i \in P, \forall j \in N \quad & \langle B_i^k B_j, \mathbf{x} \rangle \leq \langle B_j^k B_i, \mathbf{x} \rangle \Leftrightarrow \\ \forall i \in P, \forall j \in N \quad & \langle B_j/B_j^k, \mathbf{x} \rangle \geq \langle B_i/B_i^k, \mathbf{x} \rangle \Leftrightarrow \\ & \min_{j \in N} \langle B_j/B_j^k, \mathbf{x} \rangle \geq \max_{i \in P} \langle B_i/B_i^k, \mathbf{x} \rangle \end{aligned}$$

Note that the third inequality changes directions because $B_j^k < 0$. Now we choose t to lie in this last interval, and show that we can use it to satisfy all of the constraints given by B . So, we have a t such that

$$\min_{j \in N} \langle B_j/B_j^k, \mathbf{x} \rangle \geq t \geq \max_{i \in P} \langle B_i/B_i^k, \mathbf{x} \rangle$$

In particular,

$$\begin{aligned} (\forall j \in N) \quad & \langle B_j/B_j^k, \mathbf{x} \rangle \geq t \Rightarrow \\ (\forall j \in N) \quad & \langle B_j, \mathbf{x} \rangle - B_j^k t \leq 0 \end{aligned}$$

Again, the inequality changes directions because $B_j^k < 0$. Now consider a row B_j from B :

$$\langle B_j, \mathbf{x} - t\mathbf{e}_k \rangle = B_j\mathbf{x} - B_j^k t \leq 0$$

Similarly,

$$\begin{aligned} (\forall i \in P) \quad & t \geq B_i/B_i^k \mathbf{x} \Rightarrow \\ (\forall i \in P) \quad & 0 \geq B_i\mathbf{x} - B_i^k t \end{aligned}$$

Now consider a row B_i from B :

$$\langle B_i, \mathbf{x} - t\mathbf{e}_k \rangle = B_i\mathbf{x} - B_i^k t \leq 0$$

So, we've demonstrated that $\mathbf{x} - t\mathbf{e}_k$ satisfies all the constraints from B , and the left implication is shown. So 4 holds. \square

Now to prove:

(V2) Every coordinate-projection of an H-Cone is an H-Cone.

proof of (V2). Here we prove the case that the coordinate projection is onto the first d of $d + p$ coordinates. Let $\{\mathbf{y} \in \mathbb{R}^{d+p} : A'\mathbf{y} \leq \mathbf{0}\}$ be the H-Cone we need to project, and Π the coordinate-projection we need to apply (the identity matrix with the last p columns deleted). For each $1 \leq k \leq p$ we can use proposition 3 in an incremental manner, starting with A' .

```

let  $B_0 := A'$ 
for  $1 \leq k \leq p$ 
  let  $B_k :=$  result of proposition 2 applied to  $B_{k-1}, \mathbf{e}_{d+k}$ 
endfor
return  $B_p$ 

```

Consider the resulting B . Property 2 holds throughout, so B is finite. After each iteration, property 3 holds for k , so the k -th column is $\mathbf{0}$. Since each iteration only results from non-negative combinations of the result of the previous iteration (property 1), once a column is $\mathbf{0}$ it remains so. Therefore, at the end of the process, the last p columns of B are all $\mathbf{0}$. Then, by proposition 2, we can apply Π to B by simply deleting the last p columns of B . Denote this resulting matrix A . We still need to check:

$$A'\mathbf{y} \leq \mathbf{0} \Leftrightarrow A(\Pi(\mathbf{y})) \leq \mathbf{0} \quad (2.6)$$

$$(\exists t_1) \dots (\exists t_p) A'(\mathbf{x} + t_1 \mathbf{e}_{d+1} + \dots + t_p \mathbf{e}_{d+p}) \leq \mathbf{0} \Leftrightarrow A\mathbf{x} \leq \mathbf{0} \quad (2.7)$$

Then, using (2.6) and (2.7), it is easy to see that:

$$\Pi \{ \mathbf{y} \in \mathbb{R}^{d+p} \mid A'\mathbf{y} \leq \mathbf{0} \} = \{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \} \quad (2.8)$$

The key observation of this verification utilizes property 4 of proposition 3:

$$(\exists t) B(\mathbf{x} + t \mathbf{e}_k) \leq \mathbf{0} \Leftrightarrow B'\mathbf{x} \leq \mathbf{0}$$

In what follows, let $\mathbf{x} = \sum_{1 \leq j \leq d} x_j \mathbf{e}_j$. The above property is applied sequentially to the sets B_k as follows:

$$\begin{array}{lll}
(\exists t_p)(\exists t_{p-1}) \dots (\exists t_1) & B_0(\mathbf{x} + t_1 \mathbf{e}_p + t_2 \mathbf{e}_{p-1} + \dots + t_p \mathbf{e}_d) \leq \mathbf{0} & \Leftrightarrow \\
(\exists t_p) \dots (\exists t_2) & B_1(\mathbf{x} + t_2 \mathbf{e}_{d+2} + \dots + t_p \mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow \\
& \vdots & \vdots \\
(\exists t_p) & B_{p-1}(\mathbf{x} + t_p \mathbf{e}_{d+p}) \leq \mathbf{0} & \Leftrightarrow \\
& B_p \mathbf{x} \leq \mathbf{0} & \Leftrightarrow
\end{array}$$

Because $A' = B_0$, and A is B_p with the last p columns deleted, (2.6) and (2.7) hold, therefore (2.8) holds, and the proof of (V2) is complete, and we've shown that a coordinate projection of an H-Cone is again an H-Cone. \square

With (V1) and (V2) proven, we are now certain that any V-Cone is also an H-Cone.

2.2 Every H-Cone is a V-Cone

Definition 8 (Coordinate Hyperplane). A set of the form

$$\left\{ \mathbf{x} \in \mathbb{R}^{d+m} \mid \langle \mathbf{x}, \mathbf{e}_k \rangle = 0 \right\} = \left\{ \mathbf{x} \in \mathbb{R}^{d+m} \mid x_k = 0 \right\}$$

is called a *coordinate-hyperplane*.

We will use coordinate-hyperplanes in the following way. We consider a V-Cone intersected with some coordinate hyperplanes, and write it in the following way:

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U' \mathbf{t} \right\} \quad (2.9)$$

If we suppose that $U' \subset \mathbb{R}^{d+m}$, and Π is the identity matrix with the last m rows deleted, then this is just a convenient way of writing:

$$\Pi \left(\text{cone}(U') \cap \{x_{d+1} = 0\} \cap \cdots \cap \{x_{d+m} = 0\} \right) \quad (2.10)$$

The proof rests on the following three propositions:

H1 Every H-Cone is a coordinate-projection of a V-Cone intersected with some coordinate hyperplanes.

H2 Every V-Cone intersected with a coordinate-hyperplane is a V-Cone

H3 Every coordinate-projection of a V-Cone is an V-Cone.

Proof. Given *H1*, *H2*, and *H3*, the proof follows simply. Given an H-Cone, we use *H1* to get a description involving the coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. We apply *H2* as many times as necessary to eliminate the intersections, then we can apply *H3* in order to get a V-Cone. \square

Proof of H1. Let $A \in \mathbb{R}^{m \times d}$, we now show that the H-Cone

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\}$$

can be written as the projection of a V-Cone intersected with some hyperplanes. Define U' :

$$U' = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}, \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}, \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}, 1 \leq j \leq d, 1 \leq i \leq m \right\}$$

Then we claim:

$$\left\{ \mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{0} \right\} = \left\{ \mathbf{x} \in \mathbb{R}^d \mid (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U' \mathbf{t} \right\} \quad (2.11)$$

First, considering (2.9) and (2.10), observe that this is a coordinate-projection of a V-Cone intersected with some coordinate-hyperplanes. Next, we note that

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$$

We can write this as a sum with all positive coefficients if we split up the x_j as follows:

$$x_j^+ = \begin{cases} x_j & x_j \geq 0 \\ 0 & x_j < 0 \end{cases} \quad x_j^- = \begin{cases} 0 & x_j \geq 0 \\ -x_j & x_j < 0 \end{cases}$$

Then we have

$$\begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} = \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} \quad (2.12)$$

where $x_j^+, x_j^- \geq 0$. Also observe that

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid A\mathbf{x} + \mathbf{w} = \mathbf{0}$$

This can also be written

$$A\mathbf{x} \leq \mathbf{0} \Leftrightarrow (\exists \mathbf{w} \geq \mathbf{0}) \mid \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} \quad (2.13)$$

(2.12) and (2.13) together show

$$A\mathbf{x} \leq \mathbf{0} \Rightarrow (\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

Conversely, suppose

$$(\exists \mathbf{t} \geq 0) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} = U'\mathbf{t}$$

We would like to show that $A\mathbf{x} \leq \mathbf{0}$. Let x_j^+, x_j^-, w_i take the values of \mathbf{t} that are coefficients of $\begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix}$, $\begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix}$, and $\begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix}$ respectively, and denote $x_j = x_j^+ - x_j^-$. Then we have

$$\begin{aligned} \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix} &= \sum_{1 \leq j \leq d} x_j^+ \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq j \leq d} x_j^- \begin{pmatrix} -\mathbf{e}_j \\ -A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\ &= \sum_{1 \leq j \leq d} x_j \begin{pmatrix} \mathbf{e}_j \\ A^j \end{pmatrix} + \sum_{1 \leq i \leq n} w_i \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_i \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{x} \\ A\mathbf{x} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix} \end{aligned}$$

where $\mathbf{w} \geq \mathbf{0}$. By (2.13) we have $A\mathbf{x} \leq \mathbf{0}$. So (2.11) holds. \square

Note that U' can be written:

$$U' = \begin{pmatrix} \mathbf{0} & I & -I \\ I & A & -A \end{pmatrix} \quad (2.14)$$

The proof of $H2$ relies upon the following proposition.

Proposition 4. *Let $Y \in \mathbb{R}^{(d+m) \times n_1}$, $1 \leq k \leq m$, and \mathbf{x} satisfy $x_k = 0$. Then there exists a matrix $Y' \in \mathbb{R}^{(d+m) \times n_2}$ with the following properties:*

1. Every column of Y' is a positive linear combination of rows of B .
2. n_2 is finite.
3. The k -th row of Y' is $\mathbf{0}$.
4. $(\exists \mathbf{t} \geq \mathbf{0}) \mathbf{x} = Y\mathbf{t} \Leftrightarrow (\exists \mathbf{t}' \geq \mathbf{0}) \mathbf{x} = Y'\mathbf{t}'$

Recall that Y^i is the i -th column of Y , and Y_k^i is the element of Y in the i -th column and k -th row.

Proof. We partition the columns of Y :

$$\begin{aligned} P &= i \mid Y_k^i > 0 \\ N &= j \mid Y_k^j < 0 \\ Z &= l \mid Y_k^l = 0 \end{aligned}$$

We then define Y' :

$$Y' = \{Y^l \mid l \in Z\} \cup \{Y_k^i Y^j - Y_k^j Y^i \mid i \in P, j \in N\}$$

1 and 2 are clear. 3 can be seen from:

$$\langle Y^l, \mathbf{e}^k \rangle = 0$$

$$\langle Y^{t_{ij}}, \mathbf{e}^k \rangle = \langle Y_k^i Y^j - Y_k^j Y^i, \mathbf{e}^k \rangle = Y_k^i Y_k^j - Y_k^j Y_k^i = 0 \quad (2.15)$$

Before moving on to the proof, we first note how to write our vectors.

$$\begin{aligned} Y\mathbf{t} &= \sum_{l \in Z} t_l Y^l + \sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j \\ Y'\mathbf{t} &= \sum_{l \in Z} t_l Y^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (Y_k^i Y^j - Y_k^j Y^i) \end{aligned}$$

Then, by proposition 1, to show that the proposition is true, we need only show that, given any $t_i, t_j \geq 0$ ($t_{ij} \geq 0$), there exists $t_{ij} \geq 0$ ($t_i, t_j \geq 0$) such that

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} t_{ij} (Y_k^i Y^j - Y_k^j Y^i) \quad (2.16)$$

Proposition 5. *Suppose that*

$$\sum_{i \in P} t_i Y_{d+1}^i + \sum_{j \in N} t_j Y_{d+1}^j = 0 \quad Y_k^j < 0 < Y_k^i$$

Then the following holds

$$\begin{aligned} (t_i, t_j \geq 0) &\Rightarrow (\exists t_{ij} \geq 0) \text{ such that (2.16) holds} \\ (t_{ij} \geq 0) &\Rightarrow (\exists t_i, t_j \geq 0) \text{ such that (2.16) holds} \end{aligned}$$

Proof. First note that if all $t_i = 0, t_j = 0$, then choosing $t_{ij} = 0$ satisfies (2.16), likewise if all $t_{ij} = 0$, then $t_i = 0, t_j = 0$ satisfies (2.16). So suppose that some $t_i \neq 0, t_j \neq 0, t_{ij} \neq 0$.

The right hand side of (2.16) can be written

$$\sum_{j \in N} \left(\sum_{i \in P} t_{ij} Y_k^i \right) Y^j + \sum_{i \in P} \left(- \sum_{j \in N} t_{ij} Y_k^j \right) Y^i$$

This means, given $t_{ij} \geq 0$, we can choose $t_j = \sum_{i \in P} t_{ij} Y_k^i$, and $t_i = - \sum_{j \in N} t_{ij} Y_k^j$, both of which are greater than 0.

Now suppose we have been given $t_i \geq 0, t_j \geq 0$. First observe:

$$0 = \sum_{i \in P} t_i Y_k^i + \sum_{j \in N} t_j Y_k^j \Rightarrow \sum_{i \in P} t_i Y_k^i = - \sum_{j \in N} t_j Y_k^j$$

Denote the value in this equality as σ , and note that $\sigma > 0$. Then

$$\begin{aligned} \sum_{i \in P} t_i Y^i &= \frac{- \sum_{j \in N} t_j Y_k^j}{\sigma} \sum_{i \in P} t_i Y^i = \sum_{\substack{i \in P \\ j \in N}} - \frac{t_i t_j}{\sigma} Y_k^j Y^i \\ \sum_{j \in N} t_j Y^j &= \frac{\sum_{i \in P} t_i Y_k^i}{\sigma} \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma} Y_k^i Y^j \end{aligned}$$

Combining these results, we have

$$\sum_{i \in P} t_i Y^i + \sum_{j \in N} t_j Y^j = \sum_{\substack{i \in P \\ j \in N}} \frac{t_i t_j}{\sigma} (Y_k^i Y^j - Y_k^j Y^i)$$

□

Finally, we can conclude that, given $\mathbf{t} \geq \mathbf{0}$, if $Y\mathbf{t}$ has a 0 in the final coordinate, then we can write it as $Y'\mathbf{t}'$ where $\mathbf{t}' \geq \mathbf{0}$, and any non-negative linear combination of vectors from Y' can be written as a non-negative linear combination of vectors from Y , and will necessarily have the k -th coordinate be 0 by property 3. So property 4 holds. □

Proof of H2. In proposition 4, the assumption that $x_k = 0$ in property 4 creates the set $\text{cone}(Y) \cap \{\mathbf{x} \mid x_k = 0\}$. This set, by property 4, is $\text{cone}(Y')$. □

Proof of H3. We shall prove that the coordinate-projection of a V-Cone is again a V-Cone. Let Π be the relevant projection, then we have:

$$\Pi \{U\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U\mathbf{t}) \mid \mathbf{t} \geq \mathbf{0}\} = \{\Pi(U)\mathbf{t} \mid \mathbf{t} \geq \mathbf{0}\}$$

The last equality follows from associativity of matrix multiplication. Therefore,

$$\Pi(\text{cone}(U)) = \text{cone}(\Pi(U))$$

□

2.3 Reducing Polyhedra to Cones

Definition 9 (Hyperplane). Let $\mathbf{y} \in \mathbb{R}^d$, $c \in \mathbb{R}$. Then a set of the form

$$\{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{y}, \mathbf{x} \rangle = c\}$$

is called a *hyperplane*.

2.3.1 H-Polyhedra \leftrightarrow H-Cones

We show that an H-Polyhedron can be represented as the projection of an H-Cone intersected with a hyperplane. We begin by re-writing the expression:

$$A\mathbf{x} \leq \mathbf{b} \Leftrightarrow -\mathbf{b} + A\mathbf{x} \leq \mathbf{0} \Leftrightarrow \begin{bmatrix} -\mathbf{b} & A \end{bmatrix} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \leq \mathbf{0} \quad (2.17)$$

Proposition 6. *Every H-Polyhedron can be written as an H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$, and any H-Cone intersected with the set $\{\mathbf{x} \mid x_0 = 1\}$ is an H-Polyhedron.*

Proof. We extend (2.17):

$$\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\} \Leftrightarrow \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \in \{\mathbf{y} \in \mathbb{R}^{d+1} \mid \begin{bmatrix} -\mathbf{b} & A \end{bmatrix} \mathbf{y} \leq \mathbf{0}\}$$

We conclude, given an H-Polyhedron, we can add an extra coordinate and prepend the vector \mathbf{b} to the left of A , and later we can just move this column back to the right side of the inequality and drop the extra coordinate. \square

2.3.2 V-Polyhedra \leftrightarrow V-Cone

We show that a V-Polyhedra can be represented as a projection of a V-Cone intersected with the hyperplane $\{\mathbf{y} \in \mathbb{R}^{d+1} \mid y_0 = 1\}$. Given two sets $V \in \mathbb{R}^{d \times n}$ and $U \in \mathbb{R}^{d \times p}$, the V-Polyhedron is given by:

$$P_V = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \text{cone}(U), \mathbf{y} \in \text{conv}(V)\}$$

It isn't hard to see that

$$\mathbf{x} \in P_V \Leftrightarrow \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \in \text{cone} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ U & V \end{pmatrix}$$

For the value 1 to appear in the first coordinate, a convex combination of the vectors from $(\mathbf{1}, V)$ must be taken. After that, any non-negative combination of $(\mathbf{0}, U)$ added to this vector won't affect the 1 in the first coordinate.

It is more difficult to show that, given a V-Cone, that you can intersect it with the hyperplane $\{\mathbf{y} \in \mathbb{R}^{d+1} \mid y_0 = 1\}$ and get a V-Polytope out of it. So let

$$C_V = \text{cone}(U) \cap \{\mathbf{y} \in \mathbb{R}^{d+1} \mid y_0 = 1\}$$

We partition U into the sets:

$$\begin{aligned} P &= i \mid U_0^i > 0 \\ N &= j \mid U_0^j < 0 \\ Z &= l \mid U_0^l = 0 \end{aligned}$$

And define two new sets:

$$\begin{aligned} U' &= \{U^l \mid l \in Z\} \cup \{U_0^i U^j - U_0^j U^i \mid i \in P, j \in N\} \\ V &= \{U^i / U_0^i \mid i \in P\} \end{aligned}$$

Then I claim that

$$C_V = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \text{cone}(U'), \mathbf{y} \in \text{conv}(V)\} \quad (2.18)$$

Say $\mathbf{x} \in \text{cone}(U')$, \mathbf{x} can be written

$$\begin{aligned} \mathbf{x} &= \sum_{l \in Z} t_l U^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (U_0^i U^j - U_0^j U^i) \\ &= \sum_{l \in Z} t_l U^l + \sum_{j \in N} \left(\sum_{i \in P} t_{ij} U_0^i \right) U^j + \sum_{i \in P} \left(\sum_{j \in N} -t_{ij} U_0^j \right) U^i \end{aligned}$$

So $\mathbf{x} \in \text{cone}(U)$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{x} \rangle = \sum_{l \in Z} t_l U_0^l + \sum_{\substack{i \in P \\ j \in N}} t_{ij} (U_0^i U_0^j - U_0^j U_0^i) = 0$$

So $x_0 = 0$. Similarly, for \mathbf{y} ,

$$\mathbf{y} = \sum_{i \in P} \lambda_i U^i / U_0^i, \quad \sum_{i \in P} \lambda_i = 1$$

So $\mathbf{y} \in \text{cone}(U)$, and then $\mathbf{x} + \mathbf{y} \in \text{cone}(U)$. Furthermore,

$$\langle \mathbf{e}_0, \mathbf{y} \rangle = \sum_{i \in P} \lambda_i U_0^i / U_0^i = 1$$

So $y_0 = 1$ and $x_0 + y_0 = 1$. Then, by proposition 1, $\mathbf{x} + \mathbf{y} \in C_V$.

Next, suppose that $\mathbf{z} \in C_V$, then \mathbf{z} can be written

$$\mathbf{z} = \sum_{l \in Z} t_l U^l + \sum_{i \in P} t_i U^i + \sum_{j \in N} t_j U^j$$

It will be convenient to use shorter notation for these sums. Define the following:

$$\begin{aligned} \sigma_Z &= \sum_{l \in Z} t_l U^l, & \sigma_l &= \sum_{l \in Z} t_l U_0^l = 0 \\ \sigma_P &= \sum_{i \in P} t_i U^i, & \sigma_i &= \sum_{i \in P} t_i U_0^i \\ \sigma_N &= \sum_{j \in N} t_j U^j, & \sigma_j &= \sum_{j \in N} t_j U_0^j \end{aligned}$$

Then it holds that

$$\langle \mathbf{e}_0, \mathbf{z} \rangle = \sigma_l + \sigma_i + \sigma_j = \sigma_i + \sigma_j = 1 \quad \Rightarrow \quad -\sigma_j/\sigma_i = 1 - 1/\sigma_i$$

$$\sigma_P = \sigma_P/\sigma_i + (1 - 1/\sigma_i)\sigma_P = \sigma_P/\sigma_i - (\sigma_j/\sigma_i)\sigma_P$$

Using the new notation, we can rewrite \mathbf{z} :

$$\mathbf{z} = \sigma_Z + \sigma_P + \sigma_N = \sigma_Z + \frac{\sigma_P}{\sigma_i} - \frac{\sigma_j}{\sigma_i}\sigma_P + \frac{\sigma_i}{\sigma_i}\sigma_N = \sigma_Z + \frac{\sigma_P}{\sigma_i} + \frac{\sigma_i\sigma_N - \sigma_j\sigma_P}{\sigma_i}$$

Using proposition 1, we need only show that

1. $\sigma_Z \in \text{cone}(U')$
2. $(\sigma_i\sigma_N - \sigma_j\sigma_P)/\sigma_i \in \text{cone}(U')$
3. $\sigma_P/\sigma_i \in \text{conv}(V)$

Since each $U^l : l \in Z$ is in C_V , (1) holds. We also have:

$$\sigma_i\sigma_N - \sigma_j\sigma_P = \sum_{i \in P} t_i \sum_{j \in N} t_j U_0^i U^j - \sum_{j \in N} t_j \sum_{i \in P} t_i U_0^j U^i = \sum_{\substack{i \in P \\ j \in N}} t_i t_j (U_0^i U^j - U_0^j U^i)$$

So (2) holds. Finally,

$$\sigma_P/\sigma_i = \sum_{i \in P} t_i U^i / \sigma_i = \sum_{i \in P} (t_i U_0^i / \sigma_i) (U^i / U_0^i)$$

Since $\sum_{i \in P} (t_i U_0^i / \sigma_i) = \sigma_i / \sigma_i = 1$, it follows that $\sigma_P/\sigma_i \in \text{conv}(V)$.

2.4 Picture of the Proof

Here we show a diagram that represent the proof of the Minkowski-Weyl Theorem.

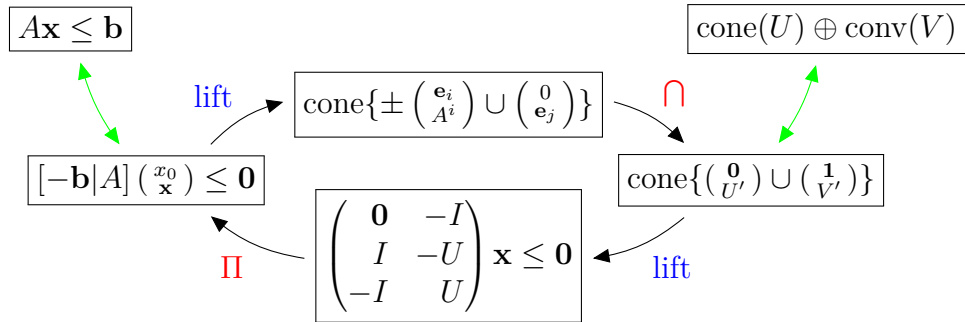


Figure 2.1: Diagram of the proof $P_H \leftrightarrow P_V$

Figure 2.1 shows the flow from an H-Polyhedron to a V-Polyhedron and back. The **colored arrows** are the transformations back and forth from polyhedra to cones. The black arrows show the transformation between cones. V-Cones are **lifted** to H-Cones which need to be **projected** (Π), and H-Cones are **lifted** to V-Cones which need to be **intersected** (\cap) with some coordinate-hyperplanes then projected.

3. C++ Implementation

The above transformation have been implemented in C++. Program `main.cpp` takes on argument specifying the type of input object. It reads the description of the object from standard input, and writes the result of the implied transformation to standard output. If no arguments are supplied, then a `usage` message is given. The `usage` message, which also contains the input format for the objects, is:

```
usage: ./main input_type
```

The input object is read on `stdin`, and the result of the transform to sent to `stdout`. `input_type` determines the type of input and output:

```
-vc # transforms a vcone into an hcone
-vp # transforms a vpolyhedron into an hpolyhedron
-hc # transforms an hcone into a vcone
-hp # transforms an hpolyhedron into a vpolyhedron
```

input format is as follows:

```
hcne := dimension (vector ws)*
vcne := dimension (vector ws)*
hpoly := dimension+1 (vector ws constraint ws)*
vpoly := dimension ('U' | 'V') ws vpoly_vecs*
```

```
ws      := whitespace, as would be read by "cin >> ws;"
dimension := a positive integer. For hpoly, add one to the dimension of
           the space (this extra dimension is for the constraint)
vector    := (dimension) doubles separated by whitespace
constraint := a double (the value b_i in <A_i,x> <= b_i)
'V' | 'U' := the literal character 'U' or 'V'
vpoly_vecs := (['U'] ws vector) | (['V'] ws vector)
```

VPOLY ONLY:

```
vpoly contains two matrices:
U - contains the rays of the vpolyhedron
V - contains the points of the vpolyhedron
```

On input, enter 'U' or 'V' to indicate which matrix should receive the vectors that follow. You can switch back and forth as you like, but either 'U' or 'V' must be entered before starting to input vectors.

EXAMPLES:

```
$ ./main -vc <<< "2 1 0"
```

OUTPUT:

```
2
-0 -1
0 1
0 0
-1 0
```

```
$ ./main -hc <<< "2 1 0 0 1"
```

OUTPUT:

```
2
-1 0
0 0
0 0
0 -1
```

```
$ ./main -vp <<< "2 U 1 0 V 0 0 1 1"
```

OUTPUT:

```
3
0 0 -0
0 0 -0
0 1 1
0 0 1
-1 1 -0
-1 0 -0
0 0 -0
0 -1 -0
```

```
$ ./main -hp <<< "3 0 -1 0 0 1 1 -1 1 0"
```

OUTPUT:

```
2
U
1 0
0 0
0 0
0 0
V
0 0
1 1
```

The files pertaining to the implementation will be discussed in the following sections, but here is a table showing the include dependencies followed by a short summary of the files.

file	includes
linear_algebra.h	
fourier_motzkin.h	linear_algebra.h
polyhedra.h	fourier_motzkin.h
main.cpp	polyhedra.h
test_functions.h	linear_algebra.h
test.cpp	test_functions.h, polyhedra.h

Here is a very brief summary of the files mentioned in the above table, more details are given in sequent sections.

- `linear_algebra.h`
Types `Vector` and `Matrix`, and some basic functionality for them
- `fourier_motzkin.h`
Fourier Motzkin elimination, Minkowski-Weyl Theorem for cones
- `polyhedra.{cpp,h}`
Transforms between polytopes and polyhedra, Minkowski-Weyl Theorem
- `test_functions.h`
Types and functions for testing the algorithms, e.g. if an h-polyhedra contains some points
- `test.cpp`
Test cases for functions from `test_functions.h` and algorithms for Minkowski-Weyl Theorem

3.1 Code

The relevant code will be displayed with commentary below. Some of the code relating to C++ specific technicalities and I/O is omitted.

3.1.1 `linear_algebra.h`

The types `Vector` and `Matrix` are used in the representation of polyhedra. The `std::valarray` template is used because it has built-in vector-space operations (sum and scaling). `std::vector`, is used, however other sequence containers could be used.

```
10 using Vector = std::valarray<double>;
11 using Vectors = std::vector<Vector>;
```

The `class Matrix` implements a subset of what a *C++ Container* should. It is the primary type for representing polyhedra, and directly represents Cones, as well as H-Polyhedra. The interface is designed to enforce the following invariant:

invariant: $(\forall v \in \text{vectors}) v.\text{size}() == d$

The *factory* function `read_Matrix` is provided to read a `Matrix` from an `istream`. It is necessary because the value of `d` can't be known before reading some of the stream.

```

13 class Matrix {
14 // invariant: d >= 0
15 // invariant: (forall valid i) vectors[i].size() == d
16 public:
17     const size_t d; // size of all Vectors
18 private:
19     Vectors vectors;
20 public:
21     // needed for back_insert_iterator
22     using value_type = Vector;
23
24     Matrix(size_t d);
25     Matrix(std::initializer_list<Vector>&&);
26     bool check() const; // checks each Vector has size d
27
28     //defaults don't work because of const member
29     Matrix(const Matrix&);
30     Matrix(Matrix&&);
31     Matrix &operator=(const Matrix&);
32     Matrix &operator=(Matrix&&);
33     Matrix &operator=(std::initializer_list<Vector>&&);
34
35     static Matrix read_Matrix(std::istream&);
36
37     Vectors::iterator      begin();
38     Vectors::iterator      end();
39     Vectors::const_iterator begin() const;
40     Vectors::const_iterator end()   const;
41
42     bool    empty() const;
43     size_t  size()   const;
44     Vector& back();
45
46     Vector& add_Vector();
47     void push_back(const Vector &v);
48     void push_back(Vector &&v);
49 };

```

The `struct` `VPoly` gather two `Matrix`s needed to represent a V-Polyhedron. The `Matrix` `U` corresponds to the rays that generate the cone, and the `Matrix` `V` corresponds to the points, i.e.

$$\text{Vpoly vpoly} = \{p + q \mid p \in \text{cone}(\text{vpoly}.U), q \in \text{conv}(\text{vpoly}.V)\}$$

```

51 struct VPoly {
52     const size_t d;
53     Matrix U; // rays
54     Matrix V; // points
55
56     VPoly(size_t d) : d{d}, U{d}, V{d} {}
57     VPoly(std::initializer_list<Vector>&&,
58           std::initializer_list<Vector>&&);

```

```

59     bool check() const;
60
61     static VPoly read_VPoly(std::istream&);
62 };

```

The `class` `input_error` is thrown to indicate an invalid input to the program, and provide some clue as to why it failed. Here are two command line examples:

```

$ ./main -vc <<< "0"
terminate called after throwing an instance of 'input_error'
  what():  bad d: 0
Aborted (core dumped)
$ ./main -vc <<< "2 1"
error reading matrix, vector 1
terminate called after throwing an instance of 'input_error'
  what():  failed to read vector: istream failed
Aborted (core dumped)

```

```

64 class input_error : public std::runtime_error {
65 public:
66     input_error(const char*s);
67     input_error(const std::string &s);
68 };

```

`operator>>` and `operator<<` implement the input format described in `usage.txt`.

```

70 std::istream& operator>>(std::istream&, Vector&);
71 std::istream& operator>>(std::istream&, Matrix&);
72 std::istream& operator>>(std::istream&, VPoly&);

74 std::ostream& operator<<(std::ostream& o, const Vector&);
75 std::ostream& operator<<(std::ostream& o, const Matrix&);
76 std::ostream& operator<<(std::ostream& o, const VPoly&);

```

`usage()` outputs the usage message shown above.

```

78 int usage();

```

3.1.2 linear_algebra.cpp

`e_k` creates the canonical basis Vector $\mathbf{e}_k \in \mathbb{R}^d$.

```

232 Vector e_k(size_t d, size_t k) {
233     Vector result(d);
234     result[k] = 1;
235     return result;
236 }

```

`concatenate` takes the Vectors $\mathbf{l} \in \mathbb{R}^{\mathbf{l}.size()}$ and $\mathbf{r} \in \mathbb{R}^{\mathbf{r}.size()}$ and **returns** the Vector $(\mathbf{l}, \mathbf{r}) \in \mathbb{R}^{\mathbf{l}.size() + \mathbf{r}.size()}$

```

239 Vector concatenate(const Vector &l, const Vector &r) {
240     Vector result(l.size() + r.size());
241     copy(begin(l), end(l), begin(result));

```



```

242     copy(begin(r), end(r), next(begin(result), l.size()));
243     return result;
244 }

```

`get_column` returns the k -th column of the Matrix `M`. Note that while a Matrix may logically represent either a collection of row or column Vectors, `get_column` is only used in the function `transpose`, where this distinction is unimportant.

```

249 Vector get_column(const Matrix &M, size_t k) {
250     if (!(0 <= k && k < M.d)) {
251         throw std::out_of_range("k<0 || M.d<=k");
252     }
253     Vector result(M.size());
254     size_t result_row{0};
255     for (auto &&row : M) {
256         result[result_row++] = row[k];
257     }
258     return result;
259 }

```

`transpose` returns the transpose of Matrix `M`.

```

262 Matrix transpose(const Matrix &M) {
263     if (M.empty()) {
264         return M;
265     }
266     Matrix result{M.size()};
267     // for every column of M,
268     for (size_t k = 0; k < M.d; ++k) {
269         result.push_back(get_column(M,k));
270     }
271     return result;
272 }

```

An slice object can be used to conveniently obtain a subset of an `valarray`. `slice_matrix` returns the Matrix obtained by applying the slice `s` to each Vector of the Matrix.

```

275 Matrix slice_matrix(const Matrix &M, const std::slice &s) {
276     Matrix result{s.size()};
277     transform(M.begin(), M.end(), back_inserter(result),
278         [s](const Vector &v) { return v[s]; });
279     return result;
280 }

```

3.1.3 `fourier_motzkin.h`

`Lift` is a function pointer `typedef` that is used in a generic cone-transformation function.

```

43 typedef Matrix(*Lift)(const Matrix&);

```

3.1.4 fourier_motzkin.cpp

A slice object is determined by three fields: `start`, `size`, and `stride`, and implicitly represents all indices of the form:

$$\sum_{0 \leq k < \text{size}} \text{start} + k \cdot \text{stride}$$

Therefore:

$$i \in \text{slice} \Leftrightarrow i - \text{start} \equiv 0 \pmod{\text{stride}}, \quad \text{start} \leq i \leq \text{start} + \text{stride} \cdot \text{size}$$

```

11 bool index_in_slice(size_t index, const slice &s) {
12     return ((index - s.start()) % s.stride() == 0) &&
13           s.start() <= index &&
14           index <= s.start() + s.stride()*(s.size()-1);
15 }

```

`fourier_motzkin` takes a Matrix `M` and a coordinate `k` and creates the set which either corresponds to a projection of an H-Cone (without actually doing the projection), or the intersection of a V-Cone with a coordinate-hyperplane.

```

20 Matrix fourier_motzkin(Matrix M, size_t k) {
21     Matrix result{M.d};
22     // Partition into Z,P,N
23     const auto z_end = partition(M.begin(), M.end(),
24                                 [k](const Vector &v) { return v[k] == 0; });
25     const auto p_end = partition(z_end, M.end(),
26                                 [k](const Vector &v) { return v[k] > 0; });
27     // Move Z to result
28     move(M.begin(), z_end, back_inserter(result));
29     // convolute vectors from P,N
30     for (auto p_it = z_end; p_it != p_end; ++p_it) {
31         for (auto z_it = p_end; z_it != M.end(); ++z_it) {
32             result.push_back(
33                 (*p_it)[k]*(*z_it) - (*z_it)[k]*(*p_it));
34         }
35     }
36     return result;
37 }

```

The lines:

```

23 const auto z_end = partition(M.begin(), M.end(),
24                             [k](const Vector &v) { return v[k] == 0; });
25 const auto p_end = partition(z_end, M.end(),
26                             [k](const Vector &v) { return v[k] > 0; });

```

Partition `M` into logical sets `Z`, `P`, `N` that satisfy the following:

set	range	property
Z	$[M.begin(), z_end)$	$it \in Z \Leftrightarrow (*it)[k] = 0$
P	$[z_end, p_end)$	$it \in P \Leftrightarrow (*it)[k] > 0$
N	$[p_end, M.end())$	$it \in N \Leftrightarrow (*it)[k] < 0$

The line:

```
28 move(M.begin(), z_end, back_inserter(result));
```

Moves Z into the result. The lines:

```
30 for (auto p_it = z_end; p_it != p_end; ++p_it) {
31     for (auto z_it = p_end; z_it != M.end(); ++z_it) {
32         result.push_back(
33             (*p_it)[k]*(*z_it) - (*z_it)[k]*(*p_it));
34     }
35 }
```

Convolutes the vectors in the way described in Propositions 3 and 4 (concerning projecting an H-Cone and intersecting a V-Cone with a coordinate-hyperplane), and push them into the result `Matrix`. In particular, it creates the sets which correspond to

$$B_i^k B_j - B_j^k B_i \mid i \in P, j \in N$$

`sliced_fourier_motzkin` applies `fourier_motzkin` to `Matrix M` for each $k \notin \text{slice } s$, then slices the resulting `Matrix` using `slice_matrix` and s . This is the realization of the algorithms indicated by the proofs of either direction of the Minkowski-Weyl Theorem for cones.

```
40 Matrix sliced_fourier_motzkin(Matrix M, const slice &s) {
41     for (size_t k = 0; k < M.d; ++k) {
42         if (!index_in_slice(k,s)) {
43             M = fourier_motzkin(M, k);
44         }
45     }
46     return slice_matrix(M, s);
47 }
```

When transforming an H-Cone to a V-Cone, it first must be written as a V-Cone of a new matrix, then it is intersected with coordinate-hyperplanes and projected. Similarly, when a V-Cone is transformed into an H-Cone, it must be written as and H-Cone of a new matrix then projected with coordinate-projections. The transformations are described in (2.3) and (2.14), and summarized here:

$$A \rightarrow \begin{pmatrix} \mathbf{0} & -I \\ I & -U \\ -I & U \end{pmatrix} \quad U \rightarrow \begin{pmatrix} \mathbf{0} & I & -I \\ I & A & -A \end{pmatrix}$$

Note that the tranformation of U can be written:

$$U \rightarrow \begin{pmatrix} \mathbf{0} & I \\ I & A \\ -I & -A \end{pmatrix}^T$$

Remembering that a `Matrix` is either a collection of row *or* column `Vectors`, it is not surprising that these two transformations can be written as one function of a `Matrix` and some coefficients. In `generalizedlift`, the coefficients are given as an `array<double, 5> C`, so the overall transformation can be illustrated as:

$$\text{Matrix } M \rightarrow \begin{pmatrix} \mathbf{0} & C[0]I \\ C[1]I & C[2]M \\ C[3]I & C[4]M \end{pmatrix}$$

Where Matrix M is a collection of row Vectors, or

$$\text{Matrix } M \rightarrow \begin{pmatrix} \mathbf{0} & C[1]I & C[3]I \\ C[0]I & C[2]M & C[4]M \end{pmatrix}$$

Where Matrix M is a collection of column Vectors.

```

64 Matrix generalized_lift(const Matrix &cone,
65                          const array<double,5> &C) {
66     const size_t d = cone.d;
67     const size_t n = cone.size();
68     Matrix result{d+n};
69     Matrix cone_t = transpose(cone);
70     // |0   C[0]*I|   |0       |
71     //           |C[0]*I|
72     for (size_t i = 0; i < n; ++i) {
73         result.add_Vector()[d+i] = C[0];
74     }
75     size_t k = 0;
76     // |C[1]*I C[2]*U|   |C[1]*I|
77     //           |C[2]*A|
78     for (auto &&row_t : cone_t) {
79         result.push_back(
80             concatenate(C[1]*e_k(d,k++), C[2]*row_t));
81     }
82     k = 0;
83     // |C[3]*I C[4]*U|   |C[3]*I|
84     //           |C[4]*A|
85     for (auto &&row_t : cone_t) {
86         result.push_back(
87             concatenate(C[3]*e_k(d,k++), C[4]*row_t));
88     }
89     return result;
90 }

98 Matrix lift_vccone(const Matrix &vccone) {
99     return generalized_lift(vccone, {-1,1,-1,-1,1});
100 }

107 Matrix lift_hccone(const Matrix &hccone) {
108     return generalized_lift(hccone, {1,1,1,-1,-1});
109 }

112 Matrix cone_transform(const Matrix &cone, Lift lift) {
113     if (cone.empty()) {
114         throw logic_error{"empty_cone_for_transform"};
115     }
116     // the idea of the entire mwt is this one line
117     return sliced_fourier_motzkin(
118         lift(cone), slice(0, cone.d, 1));
119 }

```

```

121 Matrix vccone_to_hcone(Matrix vccone) {
122     return cone_transform(vccone, lift_vccone);
123 }

```

```

125 Matrix hcone_to_vccone(Matrix hcone) {
126     return cone_transform(hcone, lift_hcone);
127 }

```

3.1.5 polyhedra.cpp

```

13 Matrix hpoly_to_hcone(Matrix hpoly) {
14     transform(hpoly.begin(), hpoly.end(), hpoly.begin(),
15         [](Vector v) {
16             v[v.size()-1] *= -1;
17             return v.cshift(-1);
18         });
19     return hpoly;
20 }

```

```

24 Matrix hcone_to_hpoly(Matrix hcone) {
25     transform(hcone.begin(), hcone.end(), hcone.begin(),
26         [](Vector v) {
27             v[0] *= -1;
28             return v.cshift(1);
29         });
30     return hcone;
31 }

```

```

36 Matrix vpoly_to_vccone(VPoly vpoly) {
37     //requires increase in dimension
38     Matrix result{vpoly.d+1};
39     for (auto &&u : vpoly.U) {
40         result.push_back(concatenate({0}, u));
41     }
42     for (auto &&v : vpoly.V) {
43         result.push_back(concatenate({1}, v));
44     }
45     return result;
46 }

```

```

50 Matrix normalized_P(const Matrix &U) {
51     if (U.d <= 1) {
52         throw std::logic_error{"can't normalize U!"};
53     }
54     Matrix result{U.d-1};
55     std::slice s{1, result.d, 1};
56     for (auto &&v : U) {
57         // select the vectors with positive 0-th coordinate
58         if (v[0] <= 0) { continue; }
59         // normalize the selected vectors,
60         result.push_back(v[0] == 1 ? v[s] : (v / v[0])[s]);

```

```

61     }
62     return result;
63 }

67 VPoly vcone_to_vpoly(Matrix vcone) {
68     VPoly result{vcone.d-1};
69     result.U = sliced_fourier_motzkin(
70         vcone, slice(1,vcone.d-1,1));
71     result.V = normalized_P(vcone);
72     return result;
73 }

77 VPoly hpoly_to_vpoly(Matrix hpoly) {
78     return vcone_to_vpoly(
79         hccone_to_vcone(
80             hpoly_to_hccone(move(hpoly))));
81 }

83 Matrix vpoly_to_hpoly(VPoly vpoly) {
84     return hccone_to_hpoly(
85         vccone_to_hccone(
86             vpoly_to_vccone(move(vpoly))));
87 }

```

3.1.6 test_functions.h

```

7 struct hccone_test_case {
8     std::string name;
9     Matrix hccone; // vectors for H or V cone
10    Matrix key; // minimal generating set of dual vectors
11
12    bool run_test() const;
13 };

15 struct vccone_test_case {
16     std::string name;
17     Matrix vccone; // vectors for H or V cone
18     Matrix key; // minimal generating set of dual vectors
19
20     bool run_test() const;
21 };

23 struct hpoly_test_case {
24     std::string name;
25     Matrix hpoly; // vectors for H-Polyhedron
26     VPoly key; // minimal generating set of dual vectors
27
28     bool run_test() const;
29 };

31 struct vpoly_test_case {
32     std::string name;
33     VPoly vpoly; // vectors for V-Polyhedron

```

```

34     Matrix key;    // minimal generating set of dual vectors
35
36     bool run_test() const;
37 };

```

3.1.7 test_functions.cpp

```

28 double operator*(const Vector &l, const Vector &r) {
29     if (l.size() > r.size()) {
30         throw runtime_error{"inner_product: l > r"};
31     }
32     return inner_product(begin(l), end(l), begin(r), 0.);
33 }

```

```

35 double norm(const Vector &v) {
36     return sqrt(v*v);
37 }

```

```

39 bool approximately_zero(double d) {
40     const double error = .000001;
41     bool result = d < error;
42     if (d != 0 && result) {
43         char d_buf[64]{};
44         std::sprintf(d_buf, "%.9e", d);
45         log("approximately_zero s + d_buf, 1);
46     }
47     return result;
48 }

```

```

50 bool approximately_lt_zero(double d) {
51     return d < 0 || approximately_zero(d);
52 }

```

```

55 bool approximately_zero(const Vector &v) {
56     return approximately_zero(norm(v));
57 }

```

```

59 bool is_equivalent(const Vector &l, const Vector &r) {
60     if (l.size() != r.size()) return false;
61     if (norm(l) == 0 || norm(r) == 0) {
62         return norm(l) == 0 && norm(r) == 0;
63     }
64     return approximately_zero(l / norm(l) - r / norm(r));
65 }

```

```

67 bool is_equal(const Vector &l, const Vector &r) {
68     if (l.size() != r.size()) return false;
69     return approximately_zero(l - r);
70 }

```

```

72 bool has_equivalent_member(const Matrix &M,
73                             const Vector &v) {
74     if (!any_of(M.begin(), M.end(),

```

```

75     [&](const Vector &u) { return is_equivalent(u,v); }) {
76     ostringstream oss;
77     oss << dashes << "no equivalent member found for:\n"
78         << v << endl;
79     log(oss.str(),1);
80     return false;
81 }
82 return true;
83 }

85 bool has_equal_member(const Matrix &M, const Vector &v) {
86     if (!any_of(M.begin(), M.end(),
87         [&](const Vector &u) { return is_equal(u,v); }))) {
88         ostringstream oss;
89         oss << dashes << "no equal member found for:\n"
90             << v << endl;
91         log(oss.str(),1);
92         return false;
93     }
94     return true;
95 }

98 bool subset_mod_eq(const Matrix &generators,
99                   const Matrix &vcone) {
100     return all_of(generators.begin(), generators.end(),
101         [&](const Vector &g) {
102             return has_equivalent_member(vcone, g); }));
103 }

106 bool subset(const Matrix &generators,
107             const Matrix &vcone) {
108     return all_of(generators.begin(), generators.end(),
109         [&](const Vector &g) {
110             return has_equal_member(vcone, g); }));
111 }

115 bool ray_satisfied(const Vector &constraint,
116                  const Vector &ray) {
117     if (constraint.size() != ray.size() &&
118         constraint.size()-1 != ray.size()) {
119         throw runtime_error{"bad ray vs constraint"};
120     }
121     double ip = ray * constraint;
122     if (!(approximately_lt_zero(ip))) {
123         ostringstream oss;
124         oss << dashes << "ray not satisfied!\n"
125             << "ray:" << ray
126             << "\nconstraint:" << constraint
127             << "\nray*constraint=" << ip << endl;
128         log(oss.str(), 1);
129         return false;
130     }

```



```

131     return true;
132 }

134 bool ray_satisfied(const Matrix &constraints,
135                   const Vector &ray) {
136     return all_of(constraints.begin(), constraints.end(),
137                 [&](const Vector &cv) {
138         return ray_satisfied(cv, ray); });
139 }

141 bool rays_satisfied(const Matrix &constraints,
142                    const Matrix &rays) {
143     return all_of(rays.begin(), rays.end(),
144                 [&](const Vector &ray) {
145         return ray_satisfied(constraints, ray); });
146 }

149 bool vec_satisfied(const Vector &constraint,
150                   const Vector &vec) {
151     size_t cback_i = constraint.size()-1;
152     if (cback_i != vec.size()) {
153         throw runtime_error{"bad_vec_vs_constraint"};
154     }
155     double ip = vec * constraint;
156     if (!(approximately_lt_zero(ip - constraint[cback_i]))) {
157         ostringstream oss;
158         oss << dashes << "vec not satisfied!\n"
159             << "vec: " << vec
160             << "\nconstraint: " << constraint
161             << "\nvec * constraint = " << ip << endl;
162         log(oss.str(), 1);
163         return false;
164     }
165     return true;
166 }

168 bool vec_satisfied(const Matrix &constraints,
169                   const Vector &vec) {
170     return all_of(constraints.begin(), constraints.end(),
171                 [&](const Vector &cv) {
172         return vec_satisfied(cv, vec); });
173 }

175 bool vecs_satisfied(const Matrix &constraints,
176                    const Matrix &vecs) {
177     return all_of(vecs.begin(), vecs.end(),
178                 [&](const Vector &vec) {
179         return vec_satisfied(constraints, vec); });
180 }

184 bool equivalent_cone_rep(const Matrix &cone,
185                          const Matrix &key,
186                          const Matrix &alt_rep) {

```

```

187     return rays_satisfied (cone, alt_rep) &&
188         subset_mod_eq (key, alt_rep);
189 }

191 bool equivalent_hpoly_rep(const Matrix &hpoly,
192                          const VPoly &key,
193                          const VPoly &vpoly) {
194     return rays_satisfied (hpoly, vpoly.U) &&
195         vecs_satisfied (hpoly, vpoly.V) &&
196         subset_mod_eq (key.U, vpoly.U) &&
197         subset (key.V, vpoly.V);
198 }

200 bool equivalent_vpoly_rep(const VPoly &vpoly,
201                          const Matrix &key,
202                          const Matrix &hpoly) {
203     return rays_satisfied (hpoly, vpoly.U) &&
204         vecs_satisfied (hpoly, vpoly.V) &&
205         subset_mod_eq (key, hpoly);
206 }

208 bool hccone_test_case::run_test() const {
209     auto alt_rep = hccone_to_vccone(hccone);
210     if (!equivalent_cone_rep(hccone, key, alt_rep)) {
211         log(dashes + "hccone_TEST_FAILED:" + name, 0);
212         ostringstream oss;
213         oss << "hccone:" << hccone << "key:" << key
214             << "alt_rep:" << alt_rep;
215         log (oss.str(), 2);
216         return false;
217     }
218     return true;
219 }

221 bool vccone_test_case::run_test() const {
222     auto alt_rep = vccone_to_hccone(vccone);
223     if (!equivalent_cone_rep(vccone, key, alt_rep)) {
224         log(dashes + "vccone_TEST_FAILED:" + name, 0);
225         ostringstream oss;
226         oss << "vccone:" << vccone << "key:" << key
227             << "alt_rep:" << alt_rep;
228         log (oss.str(), 2);
229         return false;
230     }
231     return true;
232 }

234 bool hpoly_test_case::run_test() const {
235     auto vpoly = hpoly_to_vpoly(hpoly);
236     if (!equivalent_hpoly_rep(hpoly, key, vpoly)) {
237         log(dashes + "hpoly_TEST_FAILED:" + name, 0);
238         ostringstream oss;

```

```

239     oss << "hpoly:_" << hpoly << "key:_" << key
240         << "vpoly:_" << vpoly;
241     log (oss.str(), 2);
242     return false;
243 }
244 return true;
245 }

247 bool vpoly_test_case::run_test() const {
248     auto hpoly = vpoly_to_hpoly(vpoly);
249     if (!equivalent_vpoly_rep(vpoly, key, hpoly)) {
250         log(dashes + "vpoly_TEST_FAILED:_" + name, 0);
251         ostringstream oss;
252         oss << "vpoly:_" << vpoly << "key:_" << key
253             << "hpoly:_" << hpoly;
254         log (oss.str(), 2);
255         return false;
256     }
257     return true;
258 }

```

Bibliography

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