

CS 111 ASSIGNMENT 2

due February 5

Problem 1: Prove the following statement:

If $p > 5$ and $\gcd(p, 20) = 1$, then $(p^2 - 21)(p^2 + 16) \equiv 0 \pmod{20}$.

Hint: The product of any k consecutive integers is divisible by k .

Solution 1:

We are trying to prove that $(p^2 - 21)(p^2 + 16)$ is divisible by 20. First, we should remember one of the properties of modular arithmetic: $ab \pmod{c} \equiv (a \pmod{c})(b \pmod{c})$. Notice that within each factor, we can add or subtract c as much as we like, and the value would not change. We will use this to simplify the original problem:

$$(p^2 - 21)(p^2 + 16) \equiv (p^2 - 21 + 20)(p^2 + 16 - 20) \equiv (p^2 - 1)(p^2 - 4) \pmod{20}.$$

We can simplify this further: $(p^2 - 1)(p^2 - 4) = (p + 1)(p - 1)(p + 2)(p - 2) = (p - 2)(p - 1)(p + 1)(p + 2)$.

For now, let's look at a different, but similar product: $(p - 2)(p - 1)(p)(p + 1)(p + 2)$. This is a product of 5 consecutive integers. We know that a product of 5 consecutive integers is divisible by 5 (by the hint). We also know that p is not divisible by 5 and 4, since p is coprime with 20. This means that one of the factors other than p must be divisible by 5. following similar logic, we also know that one of the non- p factors must be divisible by 4, since there is also a product of 4 consecutive integers. Going back to $(p - 2)(p - 1)(p + 1)(p + 2)$, Since there is a factor that is divisible by 4, and another by 5, the product must be divisible by 20.

Therefore, the statement $(p^2 - 21)(p^2 + 16) \equiv 0 \pmod{20}$ holds.

Problem 2:

Alice's RSA public key is $P = (e, n) = (7, 4453)$. Bob sends Alice the message by encoding it as follows. First he assigns numbers to characters: A is 8, B is 9, ..., Z is 33, a blank is 34, quotation marks: 35, a coma: 36, a period: 37, an apostrophe: 38. Then he uses RSA to encode each number separately.

Bob's encoded message is:

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1400 2218 99 2088 4191 84 843 99 4191 3780 764 4191 2979 2269 99 764
2218 2269 2088 843 3015 99 2970 1443 1655 99 3237 2979 99 447 1443 3237
1032 2382 871 843 1655 99 871 1443 99 4242 843 99 4191 2269 99 843
4191 2269 2979 99 871 1443 99 2382 2269 843 99 4191 2269 99 3237 2979
99 871 843 3780 843 1032 2088 1443 2962 843 2916 99 3237 2979 99 764
2218 2269 2088 99 2088 4191 2269 99 447 1443 3237 843 99 871 1655 2382
843 99 4242 843 447 4191 2382 2269 843 99 2218 99 447 4191 2962 99
2962 1443 99 3780 1443 2962 1294 843 1655 99 2970 2218 1294 2382 1655 843
99 1443 2382 871 99 2088 1443 764 99 871 1443 99 2382 2269 843 99
3237 2979 99 871 843 3780 843 1032 2088 1443 2962 843 2916 1400
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Decode Bob's message. Notice that you only know Alice's public key, but don't know the private key. So you need to "break" RSA to decrypt Bob's message. For the solution, you need to provide the following:

- Describe step by step how you arrived at the solution: show how to find p and q , $\phi(n)$ and d .
- Show your work for one integer in the message ($M = 2218$): the expression, the decrypted integer, the character that it is mapped to.
- To decode the remaining numbers, you need to write a program in C++ (see below), test it in Gradescope, and append the code to HW 2, Problem 2 solutions.
- Give the decoded message (in integers).
- Give Bob's message in plaintext. What does it mean and who said it?

For part (c). Your program should :

- Take three integers, e , n (the public key for RSA), and m (the number of characters in the message) as input to your program. Next, input the ciphertext.
- Test whether the public key is valid. If not, output a single line "Public key is not valid!" and quit the program.
- If the public key is valid, decode the message.
- Output p and q , $\phi(n)$ and d .
- On a new line, output the decoded message in integers.

- (vii) On a new line, output the decoded message in English. The characters should be all uppercase. You can assume that the numbers will be assigned to characters according to the mapping above.

More information and specifications will be provided separately.

Upload your code to Gradescope to test. There will be 15-16 (open and hidden) test cases. Your score for the RSA code will be based on the score that you received in Gradescope. If you have any questions, post them on Slack.

Solution 2: SOLUTION 2 GOES HERE

Problem 3:

- (a) Compute $5^{1627} \pmod{12}$. Show your work.
- (b) Compute $8^{-1} \pmod{17}$ by listing the multiples. Show your work.
- (c) Compute $8^{-1} \pmod{17}$ using Fermat's Little Theorem. Show your work.
- (d) Compute $8^{-11} \pmod{17}$ using Fermat's Little Theorem. Show your work.
- (e) Find an integer x , $0 \leq x \leq 40$, that satisfies the following congruence: $31x + 54 \equiv 16 \pmod{41}$. Show your work. You should not use brute force approach.

Solution 3:

$$\begin{aligned}
 & \text{(a) } 5^{1627} \pmod{12} \\
 & \equiv 5 \cdot (5^2)^{813} \pmod{12} \\
 & \equiv 5 \cdot (25)^{813} \pmod{12} \\
 & \equiv 5 \cdot (1)^{813} \pmod{12} \\
 & \equiv 5 \pmod{12}.
 \end{aligned}$$

(b) $8^{-1} \pmod{17} \implies 8a \equiv 1 \pmod{17} \implies 8a = 17b + 1$. We need to find an a to make this equation true.

Multiples of 8: 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, 112, 120

Multiples of 17 (and then +1): 18, 35, 52, 69, 86, 103, 120

We can see that the equation is true when $a = 15$ and $b = 7$. This means that $8^{-1} \pmod{17} \equiv 15$.

(c) According to Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod{p}$, where p is prime. Since $8^{-1} \pmod{17}$, and 17 is prime, then $8x \equiv 1 \pmod{17}$, for some a . By Fermat's Little Theorem, we can say: $8^{16} \equiv 8^{17-1} \equiv 1 \pmod{17} \implies 8 \cdot 8^{15} \equiv 1 \pmod{17}$

$$\begin{aligned}
 & \implies 8^{-1} \equiv 8^{15} \pmod{17} \\
 & \equiv 8(8^2)^7 \pmod{17} \\
 & \equiv 8(64)^7 \pmod{17} \\
 & \equiv 8(13)^7 \pmod{17} \\
 & \equiv (8)(13)(13^2)^3 \pmod{17} \\
 & \equiv 104(169)^3 \pmod{17} \\
 & \equiv 2(16)^3 \pmod{17} \\
 & \equiv 2(16)(16)^2 \pmod{17} \\
 & \equiv 32(256) \pmod{17} \\
 & \equiv 15(256) \pmod{17} \\
 & \equiv 15(1) \pmod{17} \\
 & \equiv 15 \pmod{17}.
 \end{aligned}$$

Therefore, $8^{-1} \pmod{17} = 15$.

(d) $8^{-11} \pmod{17}$ is the same as $(8^{-1})^{11} \pmod{17}$. we know that, by the result found in parts b and c, that $8^{-1} \pmod{17} = 15$. This means that $(8^{-1})^{11} \equiv 15^{11} \pmod{17}$. From here, since we want to use Fermat's Little Theorem, we should multiply by $(8^5)(8^{-1})^5 \equiv (8^5)(15)^5 \equiv 1 \pmod{17}$. We know this to be true because of the properties of inverses. We now have $8^5(15^5)(15^{11}) \equiv 8^5(15)^{16} \pmod{17}$. By Fermat's Little Theorem (as stated in part c), we can substitute 15^{16} with 1: $8^5(15)^{16} \pmod{17} \equiv 8^5 \pmod{17}$. Now, we can simplify by squaring:

$$\begin{aligned} 8^5 &\equiv 8 \cdot 64^2 \pmod{17} \\ &\equiv 8 \cdot 13^2 \pmod{17} \\ &\equiv 8 \cdot 169 \pmod{17} \\ &\equiv 8 \cdot 16 \pmod{17} \\ &\equiv 128 \pmod{17} \\ &\equiv 9 \pmod{17}. \end{aligned}$$

(e) $31x + 54 \equiv 16 \pmod{41}$. First, we can subtract both sides by 54: $31x \equiv -38 \pmod{41} \equiv 3 \pmod{41}$. Now, we have to find the inverse $31^{-1} \pmod{41}$. We can list multiples to find the solution of $31a = 41k + 1$:

multiples of 31: 31, 62, 93, 124

multiples of 41(+1): 42, 83, 124

As we can see, the equation is true when $a = 4$ and $k = 3$. This means that the inverse of 31 is 4.

Going back to the original equation, we can multiply both sides by 31^{-1} :

$$31 \cdot 31^{-1}x \equiv 31^{-1} \cdot 3 \pmod{41} \implies x \equiv 4 \cdot 3 \pmod{41} \equiv 12 \pmod{41}. \text{ Therefore, } x = 12.$$
