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CS 111 ASSIGNMENT 3

due February 5, 2023

Problem 1:
a) Consider the following linear homogeneous recurrence relation: $R_n = 4R_{n-1} - 3R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3.

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.

d) Find the general form of the particular solution of the recurrence $B_n = 3B_{n-2} - 2B_{n-3} + 2$.

Solution 1:
(a) This linear recurrence has a characteristic equation of $x^2 - 4x + 3 = 0$. This can be factored into $(x-3)(x-1) = 0 \implies x = 1, 3$. From here, we can write the general solution:
$R_n = \alpha_1(1)^n + \alpha_2(3)^n = \alpha_1 + \alpha_2(3)^n$. We can now plug in R_0 and R_2 to solve for α.
$R_0 = 1 = \alpha_1 + \alpha_2$.
$R_2 = 5 = \alpha_1 + \alpha_2(3)^2$.
Solving this system of linear equations, we get $\alpha_{1,2} = \frac{1}{2}$. This means that the closed form for the recurrence is $R_n = \frac{1}{2}(3)^n + \frac{1}{2}$. We can plug in 3 and get $R_3 = \frac{1}{2}(3)^3 + \frac{1}{2} = 14$.

(b) A general solution of a recurrence with these roots would look like:

$$R_n = \alpha_1(1)^n + \alpha_2(-2)^n + \alpha_3 n(-2)^n + \alpha_4(2)^n + \alpha_5(7)^n + \alpha_6 n(7)^n.$$

(c) The characteristic equation of this recurrence $x^4 - 256 = 0 \implies (x-4)(x+4)(x^2+16) = 0$ $\implies (x-4)(x+4)(x+4i)(x-4i) = 0$. The roots are $-4, 4, -4i, 4i$. The general solution to this is $R_n = \alpha_1(-4)^n + \alpha_2(4)^n + \alpha_3(-4i)^n + \alpha_4(4i)^n$.

(d) TODO

Problem 2: Solve the following recurrence equations:

a)

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 4 \end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

b)

$$\begin{aligned} t_n &= t_{n-1} + 2t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Solution 2:

(*a*) The characteristic equation of this recurrence is $x^3 - x^2 - 4x - 2 = 0$. Using the rational root theorem, we know that one root of this polynomial must be ± 1, or ± 2. Pluggin in -1 yields a true statement, therefore we can factor out an $x + 1$ from the characteristic equation. Using synthetic division, we get: $(x+1)(x^2-2x-2) = 0$. Using the quadratic equation, we can simplify the second degree factor and get: $(x+1)(x-(1-\sqrt{3}))(x-(1+\sqrt{3})) = 0$. Now that we have the roots, we can find the general solution to the recurrence: $f_n = \alpha_1(-1)^n + \alpha_2(1-\sqrt{3})^n + \alpha_3(1+\sqrt{3})^n$. We can now plug in the base cases to solve for the constants:

$f_0 = 0 = \alpha_1 + \alpha_2 + \alpha_3$

$f_1 = 1 = -\alpha_1 + \alpha_2(1-\sqrt{3}) + \alpha_3(1+\sqrt{3})$

$f_2 = 4 = \alpha_1 + \alpha_2(1-\sqrt{3})^2 + \alpha_3(1+\sqrt{3})^2$

Solving this system of linear equations gives $\alpha_{1,2,3} = 2, \frac{-6-5\sqrt{3}}{6}, \frac{-6+5\sqrt{3}}{6}$. Thus, the solution of the recurrence is $f_n = 2(-1)^n + \frac{-6-5\sqrt{3}}{6}(1-\sqrt{3})^n + \frac{-6+5\sqrt{3}}{6}(1+\sqrt{3})^n$.

Problem 3: PROBLEM 3 GOES HERE

Solution 3: SOLUTION 3 GOES HERE
