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a) Consider the following linear homogeneous recurrence relation: $R_n = 4R_{n-1} - 3R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3.

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.

d) Find the general form of the particular solution of the recurrence $B_n = 3B_{n-2} - 2B_{n-3} + 2$.

(a) This linear recurrence has a characteristic equation of $x^2 - 4x + 3 = 0$. This can be factored into $(x-3)(x-1) = 0 \implies x = 1, 3$. From here, we can write the general solution:
$R_n = \alpha_1(1)^n + \alpha_2(3)^n = \alpha_1 + \alpha_2(3)^n$. We can now plug in R_0 and R_2 to solve for α.
$R_0 = 1 = \alpha_1 + \alpha_2$.
$R_2 = 5 = \alpha_1 + \alpha_2(3)^2$.
We can subtract the two equations to get $4 = 8\alpha_2 \implies \alpha_2 = \frac{1}{2}$.
Plugging in α_2 into the first equation, we get $\alpha_{1,2} = \frac{1}{2}$. This means that the closed form for the recurrence is $R_n = \frac{1}{2}(3)^n + \frac{1}{2}$. We can plug in 3 and get $R_3 = \frac{1}{2}(3)^3 + \frac{1}{2} = 14$.

(b) A general solution of a recurrence with these roots would look like:

$$R_n = \alpha_1(1)^n + \alpha_2(-2)^n + \alpha_3 n(-2)^n + \alpha_4(2)^n + \alpha_5(7)^n + \alpha_6 n(7)^n.$$

(c) The characteristic equation of this recurrence $x^4 - 256 = 0 \implies (x-4)(x+4)(x^2+16) = 0$ $\implies (x-4)(x+4)(x+4i)(x-4i) = 0$. The roots are $-4, 4, -4i, 4i$. The general solution to this is $R_n = \alpha_1(-4)^n + \alpha_2(4)^n + \alpha_3(-4i)^n + \alpha_4(4i)^n$.

(d) At first glance, the general form of the particular solution is $B_n'' = \beta$. To be sure, though, we have to find the general form of the homogenous equation. The characteristic equation of the homogenous part is $x^3 - 3x + 2 = 0$. Using the rational root theorem and synthetic divison, we can factor this into $(x-1)^2(x+2) = 0$. This means that $x = -2, 1$ (multiplicity 2). This means that the general form of the homogeneous solution is $B_n' = \alpha_1(-2)^n + \alpha_3 + \alpha_4 n$. Going back to the particular solution, we can see that $B_n'' = \beta$ will not work because it is a solution to the homogenous form. By the same logic, we can see that $B_n'' = \beta n$ will not work either. This means that the general form of the particular solution is $B_n'' = \beta n^2$.

Solve the following recurrence equations:

a)

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 4 \end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

b)

$$\begin{aligned} t_n &= t_{n-1} + 2t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

(a) The characteristic equation of this recurrence is $x^3 - x^2 - 4x - 2 = 0$. Using the rational root theorem, we know that one root of this polynomial must be ± 1, or ± 2. Pluggin in -1 yields a true statement, therefore we can factor out an $x + 1$ from the characteristic equation. Using synthetic division, we get: $(x+1)(x^2 - 2x - 2) = 0$. Using the quadratic equation, we can simplify the second degree factor and get: $(x + 1)(x - (1 - \sqrt{3}))(x - (1 + \sqrt{3})) = 0$. Now that we have the roots, we can find the general solution to the recurrence: $f_n = \alpha_1(-1)^n + \alpha_2(1 - \sqrt{3})^n + \alpha_3(1 + \sqrt{3})^n$. We can now plug in the base cases to solve for the constants:
$f_0 = 0 = \alpha_1 + \alpha_2 + \alpha_3$
$f_1 = 1 = -\alpha_1 + \alpha_2(1 - \sqrt{3}) + \alpha_3(1 + \sqrt{3})$
$f_2 = 4 = \alpha_1 + \alpha_2(1 - \sqrt{3})^2 + \alpha_3(1 + \sqrt{3})^2$
To solve this system, we can use Gauss-Jordan elimination:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & (1-\sqrt{3}) & (1+\sqrt{3}) & 1 \\ 1 & (1-\sqrt{3})^2 & (1+\sqrt{3})^2 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{-6-5\sqrt{3}}{6} \\ 0 & 0 & 1 & \frac{-6+5\sqrt{3}}{6} \end{pmatrix}$$

This means that the solutions are: $\alpha_{1,2,3} = 2, \frac{-6-5\sqrt{3}}{6}, \frac{-6+5\sqrt{3}}{6}$. Thus, the solution of the recurrence is $f_n = 2(-1)^n + \frac{-6-5\sqrt{3}}{6}(1 - \sqrt{3})^n + \frac{-6+5\sqrt{3}}{6}(1 + \sqrt{3})^n$.

(b) To solve this recurrence, we need to first solve the homogeneous part and the nonhomogenous parts of the recurrence. The characteristic equation of the homogeneous part is $x^2 - x - 2 = 0 \implies x_{1,2} = -1, 2$. This gives us $t'_n = \alpha_1(-1)^n + \alpha_2(2)^n$.

For the nonhomogenous, we should recognize that the general form of the particular solution is $t''_n = \beta 2^n$. The homogenous equation already has a 2^n multiplied by a constant, so we need to multiply the particular solution by n: $t''_n = \beta n 2^n$. To solve for β, we should plug in the particular solution into the recurrence: $\beta n 2^n = \beta(n-1)2^{n-1} + 2\beta(n-2)2^{n-2} + 2^n$. Dividing this equation by 2^{n-2} yields: $4\beta n = 2\beta(n-1) + 2\beta(n-2) + 4$. Solving for β gives $\beta = \frac{2}{3}$. This means that $t''_n = \frac{2}{3} n 2^n$.

Combining the two, we find that the general form of the solution is $t_n = \alpha_1(-1)^n + \alpha_2(2^n) + \frac{2}{3} n 2^n$.
Plugging in the base cases gives:
$t_0 = 0 = \alpha_1 + \alpha_2$
$t_1 = 2 = -\alpha_1 + 2\alpha_2 + \frac{4}{3}$.
Add the two equations to get $2 = 3\alpha_2 + \frac{4}{3} \implies \alpha_2 = \frac{2}{9}$.
Plugging α_2 into the first equation gives us: $\alpha_1 = -\frac{2}{9}$. This means that the final solution is: $t_n = -\frac{2}{9}(-1)^n + \frac{2}{9}(2^n) + \frac{2}{3} n 2^n$.

We want to tile an $n \times 1$ strip with 1×1 tiles that are green (G), blue (B), and red (R), 2×1 purple (P) and 2×1 orange (O) tiles. Green, blue and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row (for ex., GGOBR is allowed, but GGGOBR, GROPP and PBOBR are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

Just for reference, note that the following patterns are NOT allowed: GB, BG, GP, PG, BP, PB, PP, BBB, GGG. We now have to break up the tilings into its most basic components. Also, let's say that the number of tilings is T_n.

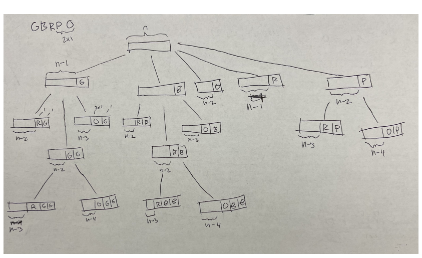
First, we can look at the easiest cases: inserting an R will give a tile with T_{n-1} possible tilings left. Similarly, inserting an O will give T_{n-2} possible tilings. Since R and O have no restrictions on what they can be next to, we don't have to do anything else with them.

Now let's look at the next simplest case: inserting a P. Since P cannot be next to a G,P, or B, it must be either next to a R or and O. This will leave you with T_{n-3} and T_{n-4} possible tilings respectively.

Green and Blue are more complicated, but they both behave the same in terms of the restrictions placed on them. Let's say you insert a G. Since R and O have no restrictions, we can freely place them next to G. Placing an R leaves T_{n-2} possible tilings left, and placing an O will leave T_{n-3} left. If you place a G next to the G, in order to prevent 3 G's in a row, we have to place either a R or an O. Placing them will leave T_{n-3} and T_{n-4} remaining tilings respectively.

Again, G and B behave exactly the same, so we can just multiply the tilings for the above paragraph by 2. Adding up all the possible tilings gives:
$T_n = 2(T_{n-2} + T_{n-3} + T_{n-3} + T_{n-4}) + T_{n-2} + T_{n-1} + T_{n-3} + T_{n-4}$
$= 3T_{n-4} + 5T_{n-3} + 3T_{n-2} + T_{n-1}$.

Since this recurrence is 4th degree, we need 4 base cases. We can say $T_0 = 1$ because there is only one way to tile 0 tiles. $T_1 = 3$ because there are three tiles that will fit in a 1x1 slot. $T_2 = 9$ because there are two 2x1 tiles plus seven ways to arrange 1x1 tiles (RG, RB, GG, BB, RR, GR, GB). Following similar logic, we can see that $T_3 = 23$. [1] Below is a tree representation of this recursion.



[1] The possible combinations for this case are OG, GO, OB, BO, OR, RO, PR, RP, GGR, GRG, GRR, RGG, RGR, RRG, RRR, BBR, BRB, BRR, RBB, RBR, RRB, BRG, GRB.

Academic integrity declaration. I did this homework on my own. I got a lot of help during office hours, though.