

CS 111 ASSIGNMENT 3

due February 5, 2023

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**Problem 1:**

- a) Consider the following linear homogeneous recurrence relation:  $R_n = 4R_{n-1} - 3R_{n-2}$ . It is known that:  $R_0 = 1$ ,  $R_2 = 5$ . Find  $R_3$ .
- b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.
- c) Determine the general solution of the recurrence equation  $A_n = 256A_{n-4}$ .
- d) Find the general form of the particular solution of the recurrence  $B_n = 3B_{n-2} - 2B_{n-3} + 2$ .

**Solution 1:**

(a) This linear recurrence has a characteristic equation of  $x^2 - 4x + 3 = 0$ . This can be factored into  $(x - 3)(x - 1) = 0 \implies x = 1, 3$ . From here, we can write the general solution:

$R_n = \alpha_1(1)^n + \alpha_2(3)^n = \alpha_1 + \alpha_2(3)^n$ . We can now plug in  $R_0$  and  $R_2$  to solve for  $\alpha$ .

$$R_0 = 1 = \alpha_1 + \alpha_2.$$

$$R_2 = 5 = \alpha_1 + \alpha_2(3)^2.$$

Solving this system of linear equations, we get  $\alpha_{1,2} = \frac{1}{2}$ . This means that the closed form for the recurrence is  $R_n = \frac{1}{2}(3)^n + \frac{1}{2}$ . We can plug in 3 and get  $R_3 = \frac{1}{2}(3)^3 + \frac{1}{2} = 14$ .

(b) A general solution of a recurrence with these roots would look like:

$$R_n = \alpha_1(1)^n + \alpha_2(-2)^n + \alpha_3n(-2)^n + \alpha_4(2)^n + \alpha_5(7)^n + \alpha_6n(7)^n.$$

(c) The characteristic equation of this recurrence  $x^4 - 256 = 0 \implies (x - 4)(x + 4)(x^2 + 16) = 0 \implies (x - 4)(x + 4)(x + 4i)(x - 4i) = 0$ . The roots are  $-4, 4, -4i, 4i$ . The general solution to this is  $R_n = \alpha_1(-4)^n + \alpha_2(4)^n + \alpha_3(-4i)^n + \alpha_4(4i)^n$ .

(d) TODO

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**Problem 2:** Solve the following recurrence equations:

a)

$$\begin{aligned}f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\f_0 &= 0 \\f_1 &= 1 \\f_2 &= 4\end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

b)

$$\begin{aligned}t_n &= t_{n-1} + 2t_{n-2} + 2^n \\t_0 &= 0 \\t_1 &= 2\end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

**Solution 2:**

(a) The characteristic equation of this recurrence is  $x^3 - x^2 - 4x - 2 = 0$ . Using the rational root theorem, we know that one root of this polynomial must be  $\pm 1$ , or  $\pm 2$ . Plugging in -1 yields a true statement, therefore we can factor out an  $x + 1$  from the characteristic equation. Using synthetic division, we get:  $(x + 1)(x^2 - 2x - 2) = 0$ . Using the quadratic equation, we can simplify the second degree factor and get:  $(x + 1)(x - (1 - \sqrt{3}))(x - (1 + \sqrt{3})) = 0$ . Now that we have the roots, we can find the general solution to the recurrence:  $f_n = \alpha_1(-1)^n + \alpha_2(1 - \sqrt{3})^n + \alpha_3(1 + \sqrt{3})^n$ . We can now plug in the base cases to solve for the constants:

$$f_0 = 0 = \alpha_1 + \alpha_2 + \alpha_3$$

$$f_1 = 1 = -\alpha_1 + \alpha_2(1 - \sqrt{3}) + \alpha_3(1 + \sqrt{3})$$

$$f_2 = 4 = \alpha_1 + \alpha_2(1 - \sqrt{3})^2 + \alpha_3(1 + \sqrt{3})^2$$

Solving this system of linear equations gives  $\alpha_{1,2,3} = 2, \frac{-6-5\sqrt{3}}{6}, \frac{-6+5\sqrt{3}}{6}$ . Thus, the solution of the recurrence is  $f_n = 2(-1)^n + \frac{-6-5\sqrt{3}}{6}(1 - \sqrt{3})^n + \frac{-6+5\sqrt{3}}{6}(1 + \sqrt{3})^n$ .

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**Problem 3:** We want to tile an  $n \times 1$  strip with  $1 \times 1$  tiles that are green (G), blue (B), and red (R),  $2 \times 1$  purple (P) and  $2 \times 1$  orange (O) tiles. Green, blue and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row (for ex., GGOBR is allowed, but GGGOBR, GROPP and PBOBR are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

**Solution 3:**

Just for reference, note that the following patterns are NOT allowed: GB, BG, GP, PG, BP, PB, PP, BBB, GGG. We now have to break up the tilings into its most basic components. Also, let's say that the number of tilings is  $T_n$ .

First, we can look at the easiest cases: inserting an R will give a tile with  $T_{n-1}$  possible tilings left. Similarly, inserting an O will give  $T_{n-2}$  possible tilings. Since R and O have no restrictions on what they can be next to, we don't have to do anything else with them.

Now let's look at the next simplest case: inserting a P. Since P cannot be next to a G,P, or B, it must be either next to a R or and O. This will leave you with  $T_{n-3}$  and  $T_{n-4}$  possible tilings respectively.

Green and Blue are more complicated, but they both behave the same in terms of the restrictions placed on them. Let's say you insert a G. Since R and O have no restrictions, we can freely place them next to G. Placing an R leaves  $T_{n-2}$  possible tilings left, and placing an O will leave  $T_{n-3}$  left. If you place a G next to the G, in order to prevent 3 G's in a row, we have to place either a R or an O. Placing them will leave  $T_{n-3}$  and  $T_{n-4}$  remaining tilings respectively.

Again, G and B behave exactly the same, so we can just multiply the tilings for the above paragraph by 2. Adding up all the possible tilings gives:

$$T_n = 2(T_{n-2} + T_{n-3} + T_{n-3} + T_{n-4}) + T_{n-2} + T_{n-1} + T_{n-3} + T_{n-4} \\ = 3T_{n-4} + 5T_{n-3} + 3T_{n-2} + T_{n-1}.$$

Since this recurrence is 4th degree, we need 4 base cases. We can say  $T_0 = 1$  because there is only one way to tile 0 tiles.  $T_1 = 3$  because there are three tiles that will fit in a  $1 \times 1$  slot.  $T_2 = 6$  because there are two  $2 \times 1$  tiles plus four ways to arrange  $1 \times 1$  tiles (RG, RB, GG, BB). Following similar logic, we can see that  $T_3 = 10$  (GRG, OG, RGG, BRB OB, RBB, GO, BO, RO, RRR, BBR, GGR,).

**Academic integrity declaration.** The homework papers must include at the end an academic integrity declaration. This should be a brief paragraph where you state *in your own words* (1) whether you did the homework individually or in collaboration with a partner student (if so, provide the name), and (2) whether you used any external help or resources.