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CS 111 ASSIGNMENT 4

due February 19, 2024

Problem 1: Give an asymptotic estimate, using the Θ-notation, of the number of letters printed by the algorithms given below. Give a complete justification for your answer, by providing an appropriate recurrence equation and its solution.

(a)
```
algorithm PrintAs(n)
    if n ≤ 1 then
        print("A")
    else
        for j ← 1 to n^3
            do print("A")
        for i ← 1 to 5 do
            PrintAs( ⌊n/2⌋ )
```

(b)
```
algorithm PrintBs(n)
    if n ≥ 4 then
        for j ← 1 to n^2
            do print("B")
        for i ← 1 to 6 do
            PrintBs( ⌊n/4⌋ )
        for i ← 1 to 10 do
            PrintBs( ⌈n/4⌉ )
```

(c)
```
algorithm PrintCs(n)
    if n ≤ 2 then
        print("C")
    else
        for j ← 1 to n
            do print("C")
        PrintCs( ⌊n/3⌋ )
        PrintCs( ⌊n/3⌋ )
        PrintCs( ⌊n/3⌋ )
        PrintCs( ⌊n/3⌋ )
```

(d)
```
algorithm PrintDs(n)
    if n ≥ 5 then
        print("D")
        print("D")
      if (x ≡ 0 (mod 2)) then
            PrintDs( ⌊n/5⌋ )
            PrintDs( ⌈n/5⌉ )
            x ← x + 3
        else
            PrintDs( ⌈n/5⌉ )
            PrintDs( ⌊n/5⌋ )
            x ← 5x + 3
```

In part (d), variable x is a global variable initialized to 1.

The solutions for all the problems will use the master theorem, so I will just state it here. Let $a \geq 1, b > 1, c > 0$ and $d \geq 0$. If $T(n)$ satisfies the recurrence $T(n) = aT(n/b) + cn^d$.

$$T(x) = \begin{cases} \Theta(n^{log_b(a)}) & a > b^d \\ \Theta(n^d log(n)) & a = b^d \\ \Theta(n^d) & a < b^d \end{cases}$$

Solution 1:
(a) This algorithm can be rewritten as the recurrence: $A(n) = n^3 + 5A(\lfloor n/2 \rfloor)$. Applying the master theorem, with $a = 5, b = 2, c = 1, d = 3$. Since $a < b^d$, $T(n) = \Theta(n^3)$.

(b) This algorithm can be rewritten as the recurrence: $B(n) = n^2 + 6B(\lfloor n/4 \rfloor) + 10B(\lceil n/4 \rceil) = 16B(n/4) + n^2 \implies a = 16, b = 4, c = 1, d = 2$. From the master theorem, since $a = b^d$, $B(n) = \Theta(n^2 \log(n))$.

(c) This algorithm can be rewritten as the recurrence: $C(n) = 4C(\lfloor n/3 \rfloor) + n$. Applying the master theorem, with $a = 4, b = 3, c = 1, d = 1 \implies a > b^d \implies C(n) = \Theta(n^{\log_3 4})$.

(d) This algorithm can be rewritten as the recurrence: $D(n) = 2 + 2D(n/5)$. The reason for the $2D(n/5)$ is because there even though there are four recursive calls in the code, only two are executed in any given iteration. Applying the master theorem, with $a = 2, b = 5, c = 2, d = 0 \implies a > b^d$, we have $B(n) = \Theta(n^{log_5(2)})$.

Problem 2: We have three sets A, B, C with the following properties:

(a) $|B| = 2|A|$, $|C| = 3|A|$,

(b) $|A \cap B| = 18$, $|A \cap C| = 20$, $|B \cap C| = 24$,

(c) $|A \cap B \cap C| = 11$,

(d) $|A \cup B \cup C| = 129$.

Use the inclusion-exclusion principle to determine the number of elements in A. Show your work.

Solution 2: According to the inclusion exclusion principle, we have:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.
We can substitute the values from (b), (c), (d) into the principle:
$129 = |A| + |B| + |C| - 18 - 20 - 24 + 11 \implies |A| + |B| + |C| = 180$. To solve this, we want to put everything in terms of a single variable. From (a), we know the relationships between the three variables. I will choose to express everything in terms of $|A|$. Plugging this in, we get
$|A| + 2|A| + 3|A| = 180 \implies |A| = 30$.

Problem 3: A company, Nice Inc., will award 45 fellowships to high-achieving UCR students from four different majors: computer science, biology, political science and history. They decided to give fellowship awards to at least 8 students majoring in computer science and at most 8 biology majors. The number of political science and history majors should be between 5 and 12 students each. How many possible lists of awardees are there? You need to give a complete derivation for the final answer, using the method developed in class. (Brute force listing of all lists will not be accepted.)

Solution 3: Let's say that the number of CS awards is C, biology is B, political science is P, history is H. We can express the conditions as:
$C \geq 8$
$B \leq 8$
$5 \leq P, H, \leq 12$
$C + B + P + H = 45$

We can substitute $B' = B - 5$ and $P' = P - 5$ $(0 \leq P', H' \leq 7)$ into the sum. Also, since C has no restrictions on the upper bound, we can kind of exclude it for now[1] (and subtract 8). We can now write:
$B + P' + H' = 27$.

We can also let the number of partitions that fit these conditions be the total number of partitions ($S()$) minus the number of invalid partitions:
$S(B \leq 8 \wedge P' \leq 7 \wedge H' \leq 7)$
$= S() - S(B \geq 9 \vee P' \geq 8 \vee H' \geq 8)$.

We know that the total number of partitions,
S() $= \binom{m+k-1}{k-1}, m = 27, k = 4 \implies S() = \binom{30}{3} = 4060$.
m is the total number of awards, and k is the number of groups.

We can also calculate the number of invalid partitions using the inclusion-exclusion principle. Also, the number of partitions with conditions is $\binom{m-A+k-1}{k-1}$, where A is the sum of the lower bounds.
$S(B \geq 9 \vee P' \geq 8 \vee H' \geq 8) =$
$S(B \geq 9) + S(P' \geq 8) + S(H' \geq 8)$
$- S(B \geq 9 \wedge P' \geq 8) - S(B \geq 9 \wedge P' \geq 8) - S(B \geq 8 \wedge P' \geq 8)$
$+ S(B \geq 9 \wedge P' \geq 8 \wedge H' \geq 8)$

$= \binom{27-9+4-1}{4-1} + \binom{27-8+4-1}{4-1} + \binom{27-8+4-1}{4-1}$
$- \binom{27-(9+8)+4-1}{4-1} - \binom{27-(9+8)+4-1}{4-1} - \binom{27-(8+8)+4-1}{4-1} - \binom{27-(9+8+8)+4-1}{4-1}$
$+ \binom{27-(8+8+9)+4-1}{4-1}$
$= 1330 + 1540 + 1540 - 286 - 286 - 364 + 10 = 3484$.
Subtracting the total with the number of invalid partitions, we get $4060 - 3484 = 576$.

[1] As you may notice, later on when we finding the number of integer partitions, we choose k=4, which includes C. So how could we account for C in k, yet exclude it in m? This is because we are essentially setting aside 8 slots for C (which is why we subtract 8 from 45), and then seeing how many integer partitions exist. Due to the way the partition formula works, there's no reason that $B + P' + H'$ cannot actually add up to less than 27. This is because we are assuming there is a fourth group (C) that will fill that void.

Academic integrity declaration. I did this homework by myself. I got help from office hours, though.