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CS 111 ASSIGNMENT 3
due February 5, 2023

Problem 1:
a) Consider the following linear homogeneous recurrence relation: $R_n = 4R_{n-1} - 3R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3.

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.

d) Find the general form of the particular solution of the recurrence $B_n = 3B_{n-2} - 2B_{n-3} + 2$.

Solution 1:
(a) This linear recurrence has a characteristic equation of $x^2 - 4x + 3 = 0$. This can be factored into $(x-3)(x-1) = 0 \implies x = 1, 3$. From here, we can write the general solution:
$R_n = \alpha_1(1)^n + \alpha_2(3)^n = \alpha_1 + \alpha_2(3)^n$. We can now plug in R_0 and R_2 to solve for α.
$R_0 = 1 = \alpha_1 + \alpha_2$.
$R_2 = 5 = \alpha_1 + \alpha_2(3)^2$.
Solving this system of linear equations, we get $\alpha_{1,2} = \frac{1}{2}$. This means that the closed form for the recurrence is $R_n = \frac{1}{2}(3)^n + \frac{1}{2}$. We can plug in 3 and get $R_3 = \frac{1}{2}(3)^3 + \frac{1}{2} = 14$.

(b) A general solution of a recurrence with these roots would look like:
$$R_n = \alpha_1(1)^n + \alpha_2(-2)^n + \alpha_3 n(-2)^n + \alpha_4(2)^n + \alpha_5(7)^n + \alpha_6 n(7)^n.$$

(c) The characteristic equation of this recurrence $x^4 - 256 = 0 \implies (x-4)(x+4)(x^2+16) = 0 \implies (x-4)(x+4)(x+4i)(x-4i) = 0$. The roots are $-4, 4, -4i, 4i$. The general solution to this is $R_n = \alpha_1(-4)^n + \alpha_2(4)^n + \alpha_3(-4i)^n + \alpha_4(4i)^n$.

(d) TODO

Problem 2: Solve the following recurrence equations:

a)

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 4 \end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

b)

$$\begin{aligned} t_n &= t_{n-1} + 2t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Solution 2:
(a) The characteristic equation of this recurrence is $x^3 - x^2 - 4x - 2 = 0$. Using the rational root theorem, we know that one root of this polynomial must be ± 1, or ± 2. Pluggin in -1 yields a true statement, therefore we can factor out an $x + 1$ from the characteristic equation. Using synthetic division, we get: $(x+1)(x^2 - 2x - 2) = 0$. Using the quadratic equation, we can simplify the second degree factor and get: $(x + 1)(x - (1 - \sqrt{3}))(x - (1 + \sqrt{3})) = 0$. Now that we have the roots, we can find the general solution to the recurrence: $f_n = \alpha_1(-1)^n + \alpha_2(1 - \sqrt{3})^n + \alpha_3(1 + \sqrt{3})^n$. We can now plug in the base cases to solve for the constants:
$f_0 = 0 = \alpha_1 + \alpha_2 + \alpha_3$
$f_1 = 1 = -\alpha_1 + \alpha_2(1 - \sqrt{3}) + \alpha_3(1 + \sqrt{3})$
$f_2 = 4 = \alpha_1 + \alpha_2(1 - \sqrt{3})^2 + \alpha_3(1 + \sqrt{3})^2$
Solving this system of linear equations gives $\alpha_{1,2,3} = 2, \frac{-6-5\sqrt{3}}{6}, \frac{-6+5\sqrt{3}}{6}$. Thus, the solution of the recurrence is $f_n = 2(-1)^n + \frac{-6-5\sqrt{3}}{6}(1 - \sqrt{3})^n + \frac{-6+5\sqrt{3}}{6}(1 + \sqrt{3})^n$.

(b) To solve this recurrence, we need to first solve the homogeneous part and the nonhomogenous parts of the recurrence. The characteristic equation of the homogeneous part is $x^2 - x - 2 = 0 \implies x_{1,2} = -1, 2$. This gives us $t'_n = \alpha_1(-1)^n + \alpha_2(2)^n$.

For the nonhomogenous, we should recognize that the general form of the particular solution is $t''_n = \beta 2^n$. The homogenous equation already has a 2^n multiplied by a constant, so we need to multiply the particular solution by n: $t''_n = \beta n 2^n$. To solve for β, we should plug in the particular solution into the recurrence: $\beta n 2^n = \beta(n-1)2^{n-1} + 2\beta(n-2)2^{n-2} + 2^n$. Dividing this equation by 2^{n-2} yields: $4\beta n = 2\beta(n-1) + 2\beta(n-2) + 4$. Solving for β gives $\beta = \frac{2}{3}$. This means that $t''_n = \frac{2}{3}n2^n$.

Combining the two, we find that the general form of the solution is $t_n = \alpha_1(-1)^n + \alpha_2(2^n) + \frac{2}{3}n2^n$. Plugging in the base cases gives:
$t_0 = 0 = \alpha_1 + \alpha_2$
$t_1 = 2 = -\alpha_1 + 2\alpha_2 + \frac{2}{3}$.
Solving this system of linear equations gives $\alpha_1 = -\frac{2}{9}$, $\alpha_2 = \frac{2}{9}$. This means that the final solution is: $t_n = -\frac{2}{9}(-1)^n + \frac{2}{9}(2^n) + \frac{2}{3}n2^n$.

Problem 3: We want to tile an $n \times 1$ strip with 1×1 tiles that are green (G), blue (B), and red (R), 2×1 purple (P) and 2×1 orange (O) tiles. Green, blue and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row (for ex., GGOBR is allowed, but GGGOBR, GROPP and PBOBR are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

Solution 3:
Just for reference, note that the following patterns are NOT allowed: GB, BG, GP, PG, BP, PB, PP, BBB, GGG. We now have to break up the tilings into its most basic components. Also, let's say that the number of tilings is T_n.

First, we can look at the easiest cases: inserting an R will give a tile with T_{n-1} possible tilings left. Similarly, inserting an O will give T_{n-2} possible tilings. Since R and O have no restrictions on what they can be next to, we don't have to do anything else with them.

Now let's look at the next simplest case: inserting a P. Since P cannot be next to a G,P, or B, it must be either next to a R or and O. This will leave you with T_{n-3} and T_{n-4} possible tilings respectively.

Green and Blue are more complicated, but they both behave the same in terms of the restrictions placed on them. Let's say you insert a G. Since R and O have no restrictions, we can freely place them next to G. Placing an R leaves T_{n-2} possible tilings left, and placing an O will leave T_{n-3} left. If you place a G next to the G, in order to prevent 3 G's in a row, we have to place either a R or an O. Placing them will leave T_{n-3} and T_{n-4} remaining tilings respectively.

Again, G and B behave exactly the same, so we can just multiply the tilings for the above paragraph by 2. Adding up all the possible tilings gives:
$T_n = 2(T_{n-2} + T_{n-3} + T_{n-3} + T_{n-4}) + T_{n-2} + T_{n-1} + T_{n-3} + T_{n-4}$
$= 3T_{n-4} + 5T_{n-3} + 3T_{n-2} + T_{n-1}$.

Since this recurrence is 4th degree, we need 4 base cases. We can say $T_0 = 1$ because there is only one way to tile 0 tiles. $T_1 = 3$ because there are three tiles that will fit in a 1x1 slot. $T_2 = 6$ because there are two 2x1 tiles plus four ways to arrange 1x1 tiles (RG, RB, GG, BB). Following similar logic, we can see that $T_3 = 10$ (GRG, OG, RGG, BRB OB, RBB, GO, BO, RO, RRR, BBR, GGR,).

Academic integrity declaration. The homework papers must include at the end an academic integrity declaration. This should be a brief paragraph where you state *in your own words* (1) whether you did the homework individually or in collaboration with a partner student (if so, provide the name), and (2) whether you used any external help or resources.