

CS 111 ASSIGNMENT 3

due February 5, 2023

Problem 1:

a) Consider the following linear homogeneous recurrence relation: $R_n = 4R_{n-1} - 3R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3 .

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.

d) Find the general form of the particular solution of the recurrence $B_n = 3B_{n-2} - 2B_{n-3} + 2$.

Solution 1:

(a) This linear recurrence has a characteristic equation of $x^2 - 4x + 3 = 0$. This can be factored into $(x - 3)(x - 1) = 0 \implies x = 1, 3$. From here, we can write the general solution:

$R_n = \alpha_1(1)^n + \alpha_2(3)^n = \alpha_1 + \alpha_2(3)^n$. We can now plug in R_0 and R_2 to solve for α .

$$R_0 = 1 = \alpha_1 + \alpha_2.$$

$$R_2 = 5 = \alpha_1 + \alpha_2(3)^2.$$

Solving this system of linear equations, we get $\alpha_{1,2} = \frac{1}{2}$. This means that the closed form for the recurrence is $R_n = \frac{1}{2}(3)^n + \frac{1}{2}$. We can plug in 3 and get $R_3 = \frac{1}{2}(3)^3 + \frac{1}{2} = 14$.

(b) A general solution of a recurrence with these roots would look like:

$$R_n = \alpha_1(1)^n + \alpha_2(-2)^n + \alpha_3n(-2)^n + \alpha_4(2)^n + \alpha_5(7)^n + \alpha_6n(7)^n.$$

(c) The characteristic equation of this recurrence $x^4 - 256 = 0 \implies (x - 4)(x + 4)(x^2 + 16) = 0 \implies (x - 4)(x + 4)(x + 4i)(x - 4i) = 0$. The roots are $-4, 4, -4i, 4i$. The general solution to this is $R_n = \alpha_1(-4)^n + \alpha_2(4)^n + \alpha_3(-4i)^n + \alpha_4(4i)^n$.

(d) TODO

Problem 2: Solve the following recurrence equations:

a)

$$\begin{aligned}f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\f_0 &= 0 \\f_1 &= 1 \\f_2 &= 4\end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

b)

$$\begin{aligned}t_n &= t_{n-1} + 2t_{n-2} + 2^n \\t_0 &= 0 \\t_1 &= 2\end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Solution 2:

(a) The characteristic equation of this recurrence is $x^3 - x^2 - 4x - 2 = 0$. Using the rational root theorem, we know that one root of this polynomial must be ± 1 , or ± 2 . Plugging in -1 yields a true statement, therefore we can factor out an $x + 1$ from the characteristic equation. Using synthetic division, we get: $(x + 1)(x^2 - 2x - 2) = 0$. Using the quadratic equation, we can simplify the second degree factor and get: $(x + 1)(x - (1 - \sqrt{3}))(x - (1 + \sqrt{3})) = 0$. Now that we have the roots, we can find the general solution to the recurrence: $f_n = \alpha_1(-1)^n + \alpha_2(1 - \sqrt{3})^n + \alpha_3(1 + \sqrt{3})^n$. We can now plug in the base cases to solve for the constants:

$$f_0 = 0 = \alpha_1 + \alpha_2 + \alpha_3$$

$$f_1 = 1 = -\alpha_1 + \alpha_2(1 - \sqrt{3}) + \alpha_3(1 + \sqrt{3})$$

$$f_2 = 4 = \alpha_1 + \alpha_2(1 - \sqrt{3})^2 + \alpha_3(1 + \sqrt{3})^2$$

Solving this system of linear equations gives $\alpha_{1,2,3} = 2, \frac{-6-5\sqrt{3}}{6}, \frac{-6+5\sqrt{3}}{6}$. Thus, the solution of the recurrence is $f_n = 2(-1)^n + \frac{-6-5\sqrt{3}}{6}(1 - \sqrt{3})^n + \frac{-6+5\sqrt{3}}{6}(1 + \sqrt{3})^n$.

Problem 3: We want to tile an $n \times 1$ strip with 1×1 tiles that are green (G), blue (B), and red (R), 2×1 purple (P) and 2×1 orange (O) tiles. Green, blue and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row (for ex., GGOBR is allowed, but GGGOBR, GROPP and PBOBR are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

Solution 3:

Just for reference, note that the following patterns are NOT allowed: GB, BG, GP, PG, BP, PB, PP, BBB, GGG. We now have to break up the tilings into its most basic components. Also, let's say that the number of tilings is T_n .

First, we can look at the easiest cases: inserting an R will give a tile with $n-1$ possible tilings left. Similarly, inserting an O will give $n-2$ possible tilings. Since R and O have no restrictions on what they can be next to, we don't have to do anything else with them.

Now let's look at the next simplest case: inserting a P. Since P cannot be next to a G, P, or B, it must be either next to a R or and O. This will leave you with $n-3$ and $n-4$ possible tilings respectively.

Green and Blue are more complicated, but they both behave the same in terms of the restrictions placed on them. Let's say you insert a G. Since R and O have no restrictions, we can freely place them next to G. Placing an R leaves $n-2$ possible tilings left, and placing an O will leave $n-3$ left. If you place a G next to the G, in order to prevent 3 G's in a row, we have to place either a R or an O. Placing them will leave $n-3$ and $n-4$ remaining tilings respectively.

Again, G and B behave exactly the same, so we can just multiply the tilings for the above paragraph by 2. Adding up all the tilings gives

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