Rounding Techniques in Approximation Algorithms

Lecture 7: An $O(\log n/\log\log n)$ Approximation for ATSP

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1 The Asymmetric Traveling Salesperson Problem (ATSP)

In ATSP, we are given as input a set of vertices V and a metric $c : V \times V \to \mathbb{R}_{\geq 0}$ which may be asymmetric, so that $c_{(u,v)}$ is not necessarily equal to $c_{(v,u)}$. Our goal is to return a minimum cost directed Hamiltonian cycle. Note that by the metric property, it is sufficient to find an Eulerian connected graph.

While TSP (the version of this problem in which c is symmetric) has had a constant factor approximation since the 1970s [Chr76], an $O(1)$ approximation for ATSP was not found until very recently [STV20]. This lecture will focus on a different line of work initiated by [Asa+10] on using *thin trees* to approximate ATSP. [Asa+10] used this to give a $O(\log n/\log\log n)$ approximation, improving upon $O(\log n)$. It is still open whether this approach can also lead to an $O(1)$ approximation for ATSP, and indeed its connection to the thin tree conjecture (defined in Section 1.2) makes it an appealing open problem.

1.1 LP Relaxation

As in the spirit of this course, we begin with a linear programming relaxation. Our polytope will be as follows, where $G = (V, A)$ is the complete directed graph:

$$P_{ATSP} = \begin{cases} x(\delta^+(v)) = x(\delta^-(v)) & \forall v \in V \\ x(\delta^+(S)) \geq 1 & \forall S \subset V \\ x_a \geq 0 & \forall a \in A \end{cases}$$

It is not difficult to see that $P_{ATSP} \cap \{0,1\}^A$ is the set of all directed Hamiltonian cycles. This polytope is exponential size, but has a polynomial time approximation oracle for the constraints over all $S \subset V$: find the minimum directed cut of the graph.

So, now we find $x \in P_{ATSP}$ such that $\sum_{a\in A} x_a c_a$ is minimized. Our goal is now to round this to an integral solution. As mentioned, we already have developed most of the tools we need to do this! But it is not at all clear how to start. In comes thin trees.

1.2 Thin Trees

Definition 1.1 (α-Thin Tree)**.** *Given a graph $G = (V, E)$, a spanning tree T of G is α-thin (for $\alpha > 0$) if*

$$|\delta_T(S)| \leq \alpha|\delta(S)| \qquad \forall S \subset V$$

A famous conjecture of Goddyn is as follows:

Conjecture 1.2 (Thin Tree Conjecture [God04]). *There exists a constant k such that every k-edge-connected graph has an 0.99-thin tree.*[1]

At a first glance this looks relatively straightforward. Given a 1000-edge-connected graph, by Nash-Williams we know there are 500 edge disjoint spanning trees. All we want to find is a *single* tree that doesn't have almost all of the edges across any cut. And yet, this is been open for 20 years despite significant efforts to prove it, e.g. [AO15]. A strengthening of the conjecture is as follows:

Conjecture 1.3 (Strong Thin Tree Conjecture). *Every k-edge-connected graph has an $O(1/k)$-thin tree.*

This is the best possible result asymptotically since in a cut of size k, a tree must take at least one edge. Amazingly, it turns out that the strong thin tree conjecture implies an $O(1)$ approximation for ATSP. We will show this holds if one can find a strongly thin trees of small cost, but by using the dual of the ATSP relaxation, one can show it actually holds for any strongly thin tree. This boils down to showing that actually every thin tree is also cheap.

1.3 Thin Trees for ATSP

The main goal of this section is as follows:

Theorem 1.4. *Given $x \in P_{ATSP}$, let T be a tree of G such that*

1. $|T \cap \delta(S)| \leq \alpha \cdot x(\delta(S))$, *where* $\delta(S) = \delta^-(S) \cup \delta^+(S)$, *and*
2. $c(T) \leq \beta \cdot c(x)$

Then, there is an ATSP solution of cost at most $(2\alpha + \beta)c(x)$.

We will use the following fact that we leave as an exercise. It can be proved using Hoffman's circulation theorem.

Fact 1.5. *Let $\ell \in \mathbb{Z}^A_{\geq 0}$. Then, the following polytope has integral vertices.*

$$P_{circ} = \begin{cases} y(\delta^+(v)) = y(\delta^-(v)) & \forall v \in V \\ \ell_a \leq y_a & \forall a \in A \end{cases}$$

Furthermore,

$$P^{\uparrow}_{circ} = \begin{cases} y(\delta^+(S)) \geq \ell(\delta^-(S)) & \forall S \subset V \\ \ell_a \leq x_a & \forall a \in A \end{cases}$$

Where recall that $P^{\uparrow}$ (the ***dominant*** *of P) is the set of points $y \in \mathbb{R}^n$ such that there exists $y' \in P$ with $y - y' \in \mathbb{R}^n_{\geq 0}$.*

[1]This conjecture also implies that for any constant $\alpha > 0$ there exists a k such that all k-edge-connected graphs have an α-thin tree. This is the original statement of the conjecture, but I prefer the one written above since it sounds like it must be true!

Notice that the criteria $y(\delta^+(S)) \geq \ell(\delta^-(S))$ is necessary, as in any circulation x, we will have $y(\delta^+(S)) = y(\delta^-(S)) \geq \ell(\delta^-(S))$ for all $S \subset V$.

Now let's use this to prove Theorem 1.4. We will pick $\ell = \mathbb{I}\{T\}$ and define $y \in P^{\uparrow}_{circ}$. As long as $c(y) \leq (2\alpha + \beta)c(x)$, we are done, as we get an integral circulation of cost at most $c(y)$. Furthermore, this is an ATSP solution! Why? The graph is Eulerian, as we have a circulation, and it must be connected since T had at least one edge going across every cut (in some direction).

So it remains to define a cheap y. But we can use $y = \mathbb{I}\{T\} + 2\alpha \cdot x$. We clearly have $y_a \geq \ell_a$ for all $a \in A$. So we only need to verify $y(\delta^+(S)) \geq \ell(\delta^-(S))$. So let's calculate this.

$$\begin{aligned} y(\delta^+(S)) &\geq 2\alpha \cdot x(\delta^+(S)) && \text{By definition} \\ &= \alpha \cdot x(\delta(S)) && \text{By definition of } P_{ATSP} \\ &\geq \alpha \cdot \frac{1}{\alpha}|T \cap \delta^-(S)| && T \text{ is } \alpha\text{-thin} \\ &= |T \cap \delta^-(S)| = \ell(\delta^-(S)) \end{aligned}$$

So, now it suffices to show that we can find a thin tree.

2 Finding a Thin Tree

It turns out that swap rounding together with Chernoff and Karger's cut counting bound gives us an $O(\log n / \log\log n)$ thin tree of expected cost $c(x)$ with high probability.

The only detail we do not quite have yet is how to sample a spanning tree with pipage rounding. Let's do that now.

Lemma 2.1. *Let $x \in P_{ATSP}$. Let $x'_{\{u,v\}} = x_{(u,v)} + x_{(v,u)}$. Then $(1 - \frac{1}{n})x' \in P_{\mathrm{st}}$.*

Proof. Since $x \in P_{ATS}$, we have $x^+(\delta(S)) + x^-(\delta(S)) \geq 2$. So, $x' \in P_{2-EC}$ and is also parsimonious, i.e. $x'(\delta(v)) = 2$ for all $v \in V$. But we proved in Lecture 6 that $(1 - \frac{1}{n})x' \in P_{\mathrm{st}}$ given a parsimonious $x' \in P_{2-EC}$, so we are done. □

This leads us to the algorithm:

Algorithm for ATSP

1. Find $x \in P_{ATSP}$ minimizing $\sum_{a \in A} x_a c_a$.
2. Let $x' \in P_{\mathrm{st}}$ be defined on the same vertex set with $x'_{\{u,v\}} = (1 - \frac{1}{n})(x_{(u,v)} + x_{(v,u)})$.
3. Sample a tree T using randomized pipage rounding starting from x'.
4. Find a minimum cost circulation that contains the edges of T in the original directed graph (and shortcut it to a Hamiltonian cycle).

Now the analysis follows analogously to Karger's algorithm for k-edge-connectivity, except we are using the upper tail Chernoff instead of the lower tail so we get $O(\log n / \log\log n)$ instead of $O(\log n)$.

Slightly more formally:

1. Fixing a cut of size α, the probability it has more than $O(\log n/\log\log n)\cdot x(\delta(S))$ edges across it is at most $n^{-\Omega(\alpha)}$.

2. Union bounding over all at most $n^{O(\alpha)}$ cuts of size between α and $\alpha+1$ (using Karger's cut counting bound), our probability of failure is still $n^{-\Omega(\alpha)}$.

3. So, union bounding over all cuts, our probability of failure is at most $\sum_{\alpha=1}^{\infty} n^{-\Omega(\alpha)} \leq \frac{1}{n}$ for the right choice of constants.

In other words, we get an $O(\log n/\log\log n)$ thin tree with high probability. We leave the details of this argument as an exercise.

References

[AO15] Nima Anari and Shayan Oveis Gharan. "Effective-Resistance-Reducing Flows, Spectrally Thin Trees, and Asymmetric TSP". In: *Proceedings of the 56th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. 2015, pp. 20–39. DOI: 10.1109/FOCS.2015.11 (cit. on p. 2).

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[Chr76] Nicos Christofides. *Worst Case Analysis of a New Heuristic for the Traveling Salesman Problem*. Report 388. Pittsburgh, PA: Graduate School of Industrial Administration, Carnegie-Mellon University, 1976 (cit. on p. 1).

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[STV20] Ola Svensson, Jakub Tarnawski, and László A. Végh. "A Constant-Factor Approximation Algorithm for the Asymmetric Traveling Salesman Problem". In: *Journal of the ACM* 67.6 (2020). DOI: 10.1145/3424306 (cit. on p. 1).