Advanced Algorithms

Lecture 23: The Price of Anarchy

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1 The Price of Anarchy

Last lecture, we looked at pure and mixed Nash equilibria. We saw some examples where the Nash equilibrium is not the best overall solution for the players: in particular, this was the case for the Prisoner's Dilemma and the similar hockey helmet problem.

	Confess	Don't Confess
Confess	(2, 2)	(5, 0)
Don't Confess	(0, 5)	(3, 3)

Table 1: Payoff matrix for the prisoner's dilemma.

In particular, there was a pure Nash here in which both players confess. However it would be in both of their interests to try to achieve the solution (2,2). In general, we may be interested in studying the *social welfare* of Nash equilibria, and comparing them to solutions which maximize the social welfare.

Social welfare in this lecture will simply be the sum of the expected utilities for a Nash equilibrium. Given a Nash Q or any assignment of strategies to players Q, let $w(Q)$ be the social welfare of that assignment. We assume social welfare is always positive.

Definition 1.1 (Price of Anarchy)**.** *The price of anarchy in a game G with a social welfare function w and set of Nash equilibria E is the value of:*

$$\textsc{Price-of-Anarchy}(G) = \min_{Q \in E} \frac{w(Q)}{w(OPT)}$$

where OPT is the strategy assignment that maximizes the social welfare.

This is called the price of anarchy (PoA) because it measures the largest deviation of social welfare when comparing top-down solutions and "anarchy," the states that the game can reach without outside influence. For the prisoner's dilemma payoff matrix above, the PoA is $\frac{2}{3}$, because the only Nash has welfare 4 and OPT has welfare 6. If a scenario has a PoA of significantly below 1, it indicates that it is beneficial for some kind of top-down solution to be imposed. In the hockey example, the PoA was significant enough that a rule was imposed, causing the equilibrium to shift to everyone wearing helmets.

A similar concept is the price of stability, which measures the price of the best Nash.

Definition 1.2 (Price of Stability)**.** *The price of stability in a game G with a social welfare function w and set of Nash equilibria E is the value of:*

$$\textsc{Price-of-Stability}(G) = \max_{Q \in E} \frac{w(Q)}{w(OPT)}$$

where OPT is the strategy assignment that maximizes the social welfare.

In the prisoner's dilemma, these two notions are the same as there is a unique Nash. However in some settings there are many equilibria. Here, you can imagine an outside force may be able to encourage the *best* equilibrium, and then the system will stay there on its own. For example, consider the following variation of the prisoner's dilemma where if neither player confesses, they get zero years in jail (so a high utility of 5) because the police can't prove anything, and they get a light sentence if they confess and the other player does not:

	Confess	Don't Confess
Confess	(2, 2)	(4, 0)
Don't Confess	(0, 4)	(5, 5)

Table 2: Payoff matrix for the modified prisoner's dilemma.

Here there are *two* Nash equilibria: in the first, both players confess, and in the second, both players don't confess. The optimal social welfare is for both players not to confess, which is one of the equilibria. The price of anarchy is $\frac{2}{5}$, since the worst Nash has welfare 4 and OPT is 10. The price of stability, on the other hand, is 1, because the best equilibrium is OPT itself.

And indeed, in this situation the price of stability looks like a good proxy for how rational players would act. If both players know this is the payoff matrix, there is a high likelihood they will individually both decide not to confess (versus the previous matrix, which would probably lead to confession).

1.1 n Player Games and Flows

We will study n player games in this lecture. As in the two player case, each player has a set of possible actions and may pick a distribution over those actions. The result is a distribution over assignments of strategies Q. The game gives us a function (a generalization of a payoff matrix) which gives us the utility of each player at an assignment Q which we can use to compute $w(Q)$, the sum of expected utilities or the social welfare.

In particular, we will study *congestion games*, and in particular the special case of flows from s to t. Here we have a directed graph with a marked starting point s and a destination t. As this will model the flow of traffic between two points, we will assume the number of players n is very large. In this way, we can assume every player has a single strategy and not a distribution over strategies, as if a player chooses a distribution over paths we can instead think of splitting that player into multiple players, each with a deterministic choice.

This allows us to think of an assignment Q as simply a set of directed $s-t$ paths, where Q_e is the percentage of players who travel on arc e in their path. Therefore, any distribution over strategies Q can be mapped to a feasible flow f from s to t, where $f_e = Q_e$. This means we have a flow $f \in \mathbb{R}^E_{\geq 0}$ which obeys flow constraints. Namely, for each vertex $v \in V \setminus \{s,t\}$ we have $f(\delta^+(v)) = f(\delta^-(v))$ (flow conservation) and for s, we have $f(\delta^+(s)) = 1$.

Furthermore, given a flow f we can determine a corresponding strategy by decomposing it into paths and assigning a corresponding percentage of players to each path. So, these are equivalent.

1.2 Congestion Games

It remains to define the utility function (and therefore welfare) of an assignment Q, or equivalently, a flow $f \in \mathbb{R}^E_{\geq 0}$. For every arc e, we can define the *congestion* on that arc, which will be a function of the number of players using it. In particular:

$$c_e(x) = \alpha_e x + \ell_e$$

where $\alpha, \ell \in \mathbb{R}^E_{\geq 0}$ correspond to arc-specific quantities: ℓ_e is the length of arc e, or the time to travel on it when there are no other cars, and α_e measures the slowdown when usage on the arc is high. For example, α_e may roughly encode the inverse of the width of the road: roads with many lanes are not as impacted by high usage.

Our goal is to minimize the overall travel time. This is given by $\sum_{e \in E} f_e \cdot c_e(f_e)$. However, here it doesn't quite make sense to use a welfare function, since we want to minimize this quantity instead of maximize it. Instead we want to use a "social cost" function, which will be exactly

$$\textsc{cost}(f) = \sum_{e \in E} f_e \cdot c_e(f_e)$$

There is then an analogous price of anarchy for a graph G with respect to social cost given a set of equilibria E, here represented as flows:

$$\textsc{Price-of-Anarchy}(G) = \max_{f \in E} \frac{\textsc{cost}(f)}{\textsc{cost}(OPT)}$$

where OPT is the flow minimizing the overall travel time. In this setting, the lower the PoA, the better the Nash equilibria are. We can similarly define the price of stability (PoS).

Lemma 1.3. *A flow is a Nash equilibrium if and only if for all paths P from s to t supported by f, $\sum_{e \in P} c_e(f_e)$ is minimal among all s-t paths.*

This is immediate, as by definition it means no player wants to deviate from their path. It crucially uses that we think of each player as having no impact themselves on the network's congestion, which is justified by our assumption that the number of players is large ($n \to \infty$).

1.3 Braess' Paradox

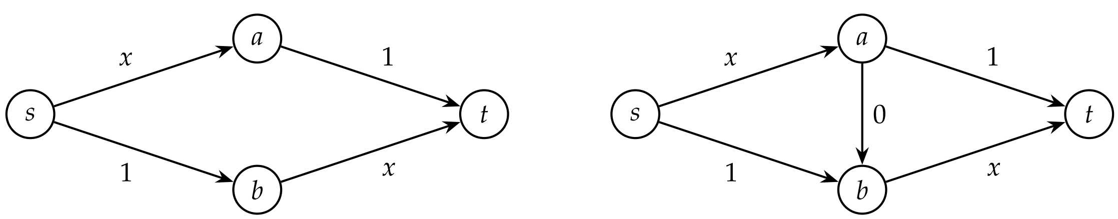

Figure 1: An illustration of Braess' paradox. Here the congestion function on the edges with label 1 is constant: $c_e(x) = 1$, and similarly for the 0 edge $c_e(x) = 0$. On the other edges, $c_e(x) = x$.

To familiarize ourselves with this setup, we will investigate a surprising phenomenon known as Braess' paradox seen in Fig. 1. Let's compute the Nash equilibria for each graph. In the left graph, the flow will clearly just split: we will send half the flow on the top half and half on the bottom. Here no one has an incentive to deviate. In this way all paths have length 1.5, and the social cost is 1.5.

What about the right graph? The optimum solution is the same as the left graph and has social cost 1.5 again. But this is not the Nash! The Nash will be to put all the flow on the edges with congestion function x for a total social cost of 2. This is because the edges with function x will always be no more congested than the edges with constant congestion of 1, so any greedy player will pick the edges with label x. So this is a Nash, and clearly it is the only Nash, leading to a PoA and PoS of $\frac{4}{3}$ in the right graph.

This feels paradoxical because by *adding* a road to the network, we actually make the congestion *worse*. This goes against our intuition, since in any network, adding edges only reduces the cost of the optimal solution. Braess' paradox demonstrates that the same is not true for Nash equilibria.

A corollary in the real world might be that in some situations, you might solve traffic problems by removing roads or lanes instead of adding them. Counter-intuitive, but true!

1.4 Bounding the Price of Anarchy

It turns out that this is the worst case PoA over all graphs: no graph has PoA greater than $\frac{4}{3}$.[1] The key fact is as follows:

Fact 1.4. *Define a new congestion function $c'_e(x) = 2\alpha_e x + \ell_e$, where recall $c_e(x) = \alpha_e x + \ell_e$. Then any flow f is optimal for c if and only if it is a Nash for c'.*

Proof. First we will show that if f is optimal for c, it is a Nash for c'. Suppose it is not a Nash for c'. Then for some path P supported by f and some P', we have:

$$\sum_{e \in P'} 2\alpha_e f_e + \ell_e < \sum_{e \in P} 2\alpha_e f_e + \ell_e$$

But this shows a direction in which f can move to reduce its cost in c. To see this, let's examine the derivatives of the social cost function, which recall is:

$$c(f) = \sum_{e \in E} f_e(\alpha_e f_e + \ell_e) = \sum_{e \in E} \alpha_e f_e^2 + \ell_e f_e$$

So the derivative in each direction f_e is

$$\frac{\partial c}{\partial f_e} = \frac{\partial \alpha_e f_e^2 + \ell_e f_e}{\partial f_e} = 2\alpha_e f_e + \ell_e$$

Now, consider the directional derivative in the direction d which is $+1$ in the direction of the path P' and -1 in the direction of the path P. This is a valid direction as it respects flow conservation and P is supported by f. Then:

$$\nabla_d c(f) = \sum_{e \in P'} (2\alpha_e f_e + \ell_e) - \sum_{e \in P} (2\alpha_e f_e + \ell_e) < 0$$

using the earlier inequality, which is a contradiction with the optimality of f for c. The other direction is similar and we leave it as an exercise to verify. □

[1] We follow the proof outline given in these lecture notes by Mądry.

Theorem 1.5. *The price of anarchy in any graph is at most $\frac{4}{3}$.*

Proof. Let f be a Nash on G. Create a new graph G' which is identical to G except it has two copies of every edge. We will extend the congestion function to G' so that the first copy of each edge e will have cost function c_e, and the second copy e' will have $c_{e'}(x) = c_e(f_e)$ so that $\alpha_{e'} = 0$ and $\ell_{e'} = c_e(f_e)$.

Note that f is a Nash in G' with extended congestion function c, as there is no reason for any flow to use a copy of an edge. But now, consider a new flow g in G' so that $g_e = \frac{1}{2} f_e$ and $g_{e'} = \frac{1}{2} f_e$. Now, it turns out that g is *optimal* in G' by Fact 1.4.

To argue this, we need to demonstrate that it is a Nash with respect to the cost function $c'_e(x) = 2\alpha_e x + \ell_e$ and then apply the fact on G' with c and c'. This follows from the fact that the congestion $c'_e(g)$ at any edge is equal to $c_e(f)$, since for any original edge e we have:

$$c'_e(g) = 2\alpha_e(f_e/2) + \ell_e = c_e(f_e)$$

and for any copied edge e' we have:

$$c'_{e'}(g) = c_e(f_e)$$

So g is an optimal solution in G'. However, clearly the OPT in G' is at most the OPT in G since we have only added links. So:

$$\begin{aligned} OPT(G) \geq OPT(G') = c(g) &= \sum_{e \in E} (\alpha_e \left(\frac{f_e}{2}\right)^2 + \frac{f_e}{2}\ell_e + \frac{f_e}{2} c_e(f_e)) \\ &= \sum_{e \in E} \alpha_e \frac{3}{4} f_e^2 + f_e \ell_e \geq \frac{3}{4} c(f) \end{aligned}$$

Or in other words, the Nash divided by OPT is at most $\frac{4}{3}$, as desired. $\square$