

## Homework 2: Dependent Randomized Rounding

Fall 2024

### 1 Problem 1: P=NP...?

1. In the traveling salesperson problem, we are given a complete graph  $G = (V, E)$  with edge costs  $c : E \rightarrow \mathbb{R}_{\geq 0}$  that form a metric, i.e.  $c_{\{u,w\}} \leq c_{\{u,v\}} + c_{\{v,w\}}$  for all  $u, v, w$ . Our goal is to find the minimum cost Hamiltonian cycle. Now where  $n = |V|$  and  $E(S)$  for  $S \subseteq V$  is the set of edges with both endpoints in  $S$ , let

$$P_{\text{sub}} = \begin{cases} \sum_{e \in E} x_e = n \\ \sum_{e \in E(S)} x_e \leq |S| - 1 & \forall S \subsetneq V \\ x_e \geq 0 & \forall e \in E \end{cases}$$

Prove that  $P_{\text{sub}} \cap \{0, 1\}^E$  is the set of all feasible solutions to the traveling salesperson problem, i.e. all Hamiltonian cycles in  $G$ .

2. Notice that  $P_{\text{sub}}$  is exactly the spanning tree polytope  $P_{\text{st}}$  except we have changed the  $n - 1$  to an  $n$ . So, it seems like we should be able to apply the proof from Lecture 4 to show that it has integral vertices. Either use this to prove P=NP or find a flaw in the argument from Lecture 4 when applied to  $P_{\text{sub}}$  instead of  $P_{\text{st}}$  (i.e. when we change the  $n - 1$  to an  $n$ ).
3. In Lecture 3, we mentioned that it is NP-Hard to obtain a 1.001 approximation for weighted  $k$ -ECSS for any  $k$ . Either explain where the proof fails and give an explicit counterexample showing that the  $1 + O(\sqrt{\frac{\log n}{k}})$  approximation does not work for weighted  $k$ -ECSS, or adapt the algorithm to prove that P=NP.

### 2 Problem 2: Scaling into Integral

Suppose  $P$  is a polytope in  $[0, 1]_{\geq 0}^n$  and  $\tilde{P}$  is the convex hull of  $P \cap \mathbb{Z}^n$ . Now, suppose that there is an  $\alpha \geq 1$  such that given any point  $x \in P$  there exists a randomized algorithm  $A$  that produces a random point  $\tilde{x} \in \tilde{P}$  such that  $\mathbb{E}[\tilde{x}_i] \leq \alpha x_i$  for every input, where the expectation is taken over the possible outputs  $\tilde{x}$  of  $A$  given  $x$ .

Given a polytope  $P$ ,  $P^\uparrow$  is called the *dominant* of  $P$  and consists of all points  $x$  such that there exists  $x' \in P$  for which  $x' - x \in \mathbb{R}_{\geq 0}^n$ .

1. Prove that  $\alpha \cdot P \subseteq \tilde{P}^\uparrow$  (where  $\alpha \cdot P$  consists of all points in  $x$  scaled entry-wise by  $\alpha$ ).
2. Prove that if  $P$  is defined by the constraints  $0 \leq x_i \leq 1$  and a collection of *covering constraints*, i.e. constraints of the form  $a^T x \geq b$  for  $a \in \mathbb{R}_{\geq 0}^n, b \in \mathbb{R}_{\geq 0}$ , then the integrality gap of the LP  $\min c^T x$  subject to  $x \in P$  is at most  $\alpha$  for any  $c \in \mathbb{R}_{\geq 0}^n$ .

### 3 Problem 3: Randomized Pipage Rounding for Matroids

A *matroid*  $M = (E, \mathcal{I})$  is defined by a collection of elements  $E$  and a collection of *independent sets*  $\mathcal{I} \subseteq 2^E$  with  $\emptyset \in \mathcal{I}$  and the properties:

- (i) **Downward Closed:** If  $I \in \mathcal{I}$ , then  $J \in \mathcal{I}$  for every  $J \subseteq I$ .
- (ii) **Augmentation Property:** If  $I, J \in \mathcal{I}$  and  $|I| < |J|$  then there exists some  $e \in J$  such that  $I \cup \{e\} \in \mathcal{I}$ .

A *basis* of a matroid is any maximal independent set. The *rank* of a collection of elements  $F \subseteq E$  is the maximum possible size of  $I \cap F$  for any  $I \in \mathcal{I}$ . Let  $r : 2^E \rightarrow \mathbb{Z}_{\geq 0}$  be the rank function.

In this problem, we will consider the following polytope  $P_M$  for a matroid  $M$ :

$$P_M = \begin{cases} x(E) = r(E) \\ x(S) \leq r(S) \quad \forall S \subseteq E \\ x_e \geq 0 \quad \forall e \in E \end{cases}$$

1. Argue that the set of forests of a graph forms a matroid  $M$ , and that in this case  $P_M = P_{\text{st}}$ .
2. Adapt the proof in class for the spanning tree polytope to show that for every matroid  $M$ ,  $P_M$  as defined above is the convex hull of its (integral) bases, i.e. it is the base polytope of  $M$ . You may use that the rank function  $r$  is submodular, i.e.  $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$  for  $S, T \subseteq E$ . Then briefly argue that randomized pipage rounding works for any matroid with the same guarantees as for spanning trees.
3. Use the [method of conditional expectation](#) to derandomize randomized pipage rounding so that given  $x \in P_M$  and any cost function  $c : E \rightarrow \mathbb{R}^E$ , we can find a point of cost at most  $c(x)$  in polynomial time.
4. When using given an independent distribution  $\mu : 2^E \rightarrow \mathbb{R}_{\geq 0}^E$  with marginals  $x$ , we have used that  $\mathbb{P}_{S \sim \mu} [|S \cap F| = 0] \leq e^{-x(F)}$  for any  $F \subseteq E$ . Show that this also holds for distributions with 0-negative correlation and thus for randomized pipage rounding.

Show that the same bound does *not* hold for all distributions  $\mu$  with only negative correlation, i.e.  $\mathbb{E} [\prod_{i \in S} X_i] \leq \prod_{i \in S} \mathbb{E} [X_i]$  for all  $S \subseteq E$ , by exhibiting a negatively correlated distribution for which this does not hold.

### 4 Problem 4: Lottery

Use the above to show that we can design a multi-item lottery as follows. Suppose we have a collection of  $n$  goods  $g_1, \dots, g_n$ , a collection of  $m$  people  $w_1, \dots, w_m$ , and a specified  $0 < \epsilon < 1$ . Now, we will allow people to buy up to one lottery ticket. Before they purchase a ticket, they will specify the subset of the goods that they would be happy winning, and if we sell them a ticket we must promise them that their chance of getting one of their chosen goods is at least  $\epsilon$ .

Using that  $P_M$  has a polynomial time separation oracle for every  $M$ , implement a polynomial time lottery system that (i) can determine when a ticket can be faithfully sold (this will depend on the subset of goods desired), (ii) will output a random assignment that respects the guarantee, and (iii) is *fair to all groups* in the sense that for any group of  $k$  people, the probability at least one of them wins something is at least  $1 - e^{-k\epsilon}$ .

## 5 Bonus Problems

1. Prove the parsimonious property from Lecture 6 using splitting off.
2. Show that the randomized rounding algorithm for the multi-commodity flow problem can be derandomized using the method of conditional expectation.
3. Give an example showing that the Chernoff bound upper tail does not hold for distributions with only *pairwise* negative correlation, i.e.  $\mathbb{P}[e, f \in S] \leq \mathbb{P}[e \in S] \mathbb{P}[f \in S]$  (and not full negative correlation a.k.a the NCP as we proved for pipage rounding).
4. (\*\*\*) Prove that swap rounding gives a  $1 + O(1/\sqrt{k})$  for  $k$ -ECSM. (Note: This is quite difficult. I think I vaguely see how to prove this but I'm not 100% sure. This could potentially even be false, although I don't think so. This could be an interesting final project.)