CS 599: Rounding Techniques in Approximation Algorithms

Homework 2: Dependent Randomized Rounding

Fall 2024

1 Problem 1: P=NP...?

1. In the traveling salesperson problem, we are given a complete graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}_{\geq 0}$ that form a metric, i.e. $c_{\{u,w\}} \leq c_{\{u,v\}} + c_{\{v,w\}}$ for all u, v, w. Our goal is to find the minimum cost Hamiltonian cycle. Now where $n = |V|$ and $E(S)$ for $S \subseteq V$ is the set of edges with both endpoints in S, let

$$P_{\text{sub}} = \begin{cases} \sum_{e \in E} x_e = n \\ \sum_{e \in E(S)} x_e \leq |S| - 1 & \forall S \subsetneq V \\ x_e \geq 0 & \forall e \in E \end{cases}$$

   Prove that $P_{\text{sub}} \cap \{0,1\}^E$ is the set of all feasible solutions to the traveling salesperson problem, i.e. all Hamiltonian cycles in G.

2. Notice that P_{sub} is exactly the spanning tree polytope P_{st} except we have changed the $n-1$ to an n. So, it seems like we should be able to apply the proof from Lecture 4 to show that it has integral vertices. Either use this to prove P=NP or find a flaw in the argument from Lecture 4 when applied to P_{sub} instead of P_{st} (i.e. when we change the $n-1$ to an n).

3. In Lecture 3, we mentioned that it is NP-Hard to obtain a 1.001 approximation for weighted k-ECSS for any k. Either explain where the proof of the $1 + O(\sqrt{\frac{\log n}{k}})$ approximation for the unweighted case fails and give an integrality gap example of $1 + \epsilon$ for some constant $\epsilon > 0$ and some $k \geq \log n$, or adapt the algorithm to prove that P=NP. **Hint:** The input may be a multigraph. Recall that the k-ECSS polytope is as follows:

$$P_{k-ECSS} = \begin{cases} x(\delta(S)) \geq k & \forall S \subset V \\ 0 \leq x_e \leq 1 & \forall e \in E \end{cases}$$

2 Problem 2: Scaling into Integral

Suppose P is a polytope in $[0,1]^n_{\geq 0}$ and $\tilde{P}$ is the convex hull of $P \cap \mathbb{Z}^n$. Now, suppose that there is an $\alpha \geq 1$ such that given any point $x \in P$ there exists a randomized algorithm A that produces a random point $\tilde{x} \in \tilde{P}$ such that $\mathbb{E}[\tilde{x}_i] \leq \alpha x_i$ for all i for every input, where the expectation is taken over the possible outputs $\tilde{x}$ of A given x.

Given a polytope P, $P^{\uparrow}$ is called the *dominant* of P and consists of all points x such that there exists $x' \in P$ for which $x - x' \in \mathbb{R}^n_{\geq 0}$.

1. Prove that $\alpha \cdot P \subseteq \tilde{P}^{\uparrow}$ (where $\alpha \cdot P$ consists of all points in x scaled entry-wise by α).

2. Prove that the integrality gap of the LP $\min c^T x$ subject to $x \in P$ is at most α for any $c \in \mathbb{R}^n_{\geq 0}$.

3 Problem 3: Randomized Pipage Rounding for Matroids

A *matroid* $M = (E, \mathcal{I})$ is defined by a collection of elements E and a collection of *independent sets* $\mathcal{I} \subseteq 2^E$ with $\emptyset \in \mathcal{I}$ and the properties:

(i) **Downward Closed**: If $I \in \mathcal{I}$, then $J \in \mathcal{I}$ for every $J \subseteq I$.

(ii) **Augmentation Property**: If $I, J \in \mathcal{I}$ and $|I| < |J|$ then there exists some $e \in J$ such that $I \cup \{e\} \in \mathcal{I}$.

A *basis* of a matroid is any maximal independent set. The *rank* of a collection of elements $F \subseteq E$ is the maximum possible size of $I \cap F$ for any $I \in \mathcal{I}$. Let $r : 2^E \to \mathbb{Z}_{\geq 0}$ be the rank function.

In this problem, we will consider the following polytope P_M for a matroid M:

$$P_M = \begin{cases} x(E) = r(E) & \\ x(S) \leq r(S) & \forall S \subseteq E \\ x_e \geq 0 & \forall e \in E \end{cases}$$

1. Argue that the set of forests of a graph forms a matroid M, and that in this case $P_M = P_{\text{st}}$.
2. Adapt the proof in class for the spanning tree polytope to show that for every matroid M, P_M as defined above is the convex hull of its (integral) bases, i.e. it is the base polytope of M. You may use that the rank function r is submodular, i.e. $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$ for $S, T \subseteq E$. Then briefly argue that randomized pipage rounding works for any matroid with the same guarantees as for spanning trees.
3. Use the method of conditional expectation to derandomize randomized pipage rounding so that given $x \in P_M$ and any cost function $c : E \to \mathbb{R}^E$, we can find a point of cost at most $c(x)$ in polynomial time.
4. When using given an independent distribution $\mu : 2^E \to \mathbb{R}^E_{\geq 0}$ with marginals x, we have used that $\mathbb{P}_{S \sim \mu}[|S \cap F| = 0] \leq e^{-x(F)}$ for any $F \subseteq E$. Show that this also holds for distributions with 0-negative correlation and thus for randomized pipage rounding.

   Show that the same bound does *not* hold for all distributions μ with only negative correlation, i.e. $\mathbb{E}\left[\prod_{i \in S} X_i\right] \leq \prod_{i \in S} \mathbb{E}[X_i]$ for all $S \subseteq E$, by exhibiting a negatively correlated distribution for which this does not hold.

4 Problem 4: Lottery

Use the above to show that we can design a multi-item lottery as follows. Suppose we have a collection of n goods $g_1, \ldots, g_n$, a collection of m people $w_1, \ldots, w_m$, and a specified $0 < \epsilon < 1$. Now, we will allow people to buy up to one lottery ticket. Before they purchase a ticket, they will specify the subset of the goods that they would be happy winning, and if we sell them a ticket we must promise them that their chance of getting one of their chosen goods is at least ϵ.

Using that P_M has a polynomial time separation oracle for every M, implement a polynomial time lottery system that (i) can determine when a ticket can be faithfully sold (this will depend on the subset of goods desired), (ii) will output a random assignment that respects the guarantee, and (iii) is *fair to all groups* in the sense that for any group of k people, the probability at least one of them wins something is at least $1 - e^{-k\epsilon}$.

5 Bonus Problems

1. Prove the parsimonious property from Lecture 6 using splitting off.

2. Show that the randomized rounding algorithm for the multi-commodity flow problem can be derandomized using the method of conditional expectation.

3. Give an example showing that the Chernoff bound upper tail does not hold for distributions with only *pairwise* negative correlation, i.e. $\mathbb{P}[e, f \in S] \leq \mathbb{P}[e \in S]\,\mathbb{P}[f \in S]$ (and not full negative correlation as we proved for pipage rounding).

4. (***) Prove that pipage rounding can be used to give a $1 + O(1/\sqrt{k})$ for k-ECSM. (Note: This is quite difficult. I think I vaguely see how to prove this but I'm not 100% sure. This could potentially even be false, although I don't think so. This could be an interesting final project.)