

Homework 2: Dependent Randomized Rounding

Fall 2024

1 Problem 1: P=NP...?

1. In the traveling salesperson problem, we are given a complete graph $G = (V, E)$ with edge costs $c : E \rightarrow \mathbb{R}_{\geq 0}$ that form a metric, i.e. $c_{\{u,w\}} \leq c_{\{u,v\}} + c_{\{v,w\}}$ for all u, v, w . Our goal is to find the minimum cost Hamiltonian cycle. Now where $n = |V|$ and $E(S)$ for $S \subseteq V$ is the set of edges with both endpoints in S , let

$$P_{\text{sub}} = \begin{cases} \sum_{e \in E} x_e = n \\ \sum_{e \in E(S)} x_e \leq |S| - 1 & \forall S \subsetneq V \\ x_e \geq 0 & \forall e \in E \end{cases}$$

Prove that $P_{\text{sub}} \cap \{0, 1\}^E$ is the set of all feasible solutions to the traveling salesperson problem, i.e. all Hamiltonian cycles in G .

2. Notice that P_{sub} is exactly the spanning tree polytope P_{st} except we have changed the $n - 1$ to an n . So, it seems like we should be able to apply the proof from Lecture 4 to show that it has integral vertices. Either use this to prove P=NP or find a flaw in the argument from Lecture 4 when applied to P_{sub} instead of P_{st} (i.e. when we change the $n - 1$ to an n).
3. In Lecture 3, we mentioned that it is NP-Hard to obtain a 1.001 approximation for weighted k -ECSS for any k . Either explain where the proof of the $1 + O(\sqrt{\frac{\log n}{k}})$ approximation for the unweighted case fails and give an integrality gap example of $1 + \epsilon$ for some constant $\epsilon > 0$ and some $k \geq \log n$, or adapt the algorithm to prove that P=NP.

2 Problem 2: Scaling into Integral

Suppose P is a polytope in $[0, 1]_{\geq 0}^n$ and \tilde{P} is the convex hull of $P \cap \mathbb{Z}^n$. Now, suppose that there is an $\alpha \geq 1$ such that given any point $x \in P$ there exists a randomized algorithm A that produces a random point $\tilde{x} \in \tilde{P}$ such that $\mathbb{E}[\tilde{x}_i] \leq \alpha x_i$ for all i for every input, where the expectation is taken over the possible outputs \tilde{x} of A given x .

Given a polytope P , P^\uparrow is called the *dominant* of P and consists of all points x such that there exists $x' \in P$ for which $x' - x \in \mathbb{R}_{\geq 0}^n$.

1. Prove that $\alpha \cdot P \subseteq \tilde{P}^\uparrow$ (where $\alpha \cdot P$ consists of all points in x scaled entry-wise by α).
2. Prove that the integrality gap of the LP $\min c^T x$ subject to $x \in P$ is at most α for any $c \in \mathbb{R}_{\geq 0}^n$.

3 Problem 3: Randomized Pipage Rounding for Matroids

A *matroid* $M = (E, \mathcal{I})$ is defined by a collection of elements E and a collection of *independent sets* $\mathcal{I} \subseteq 2^E$ with $\emptyset \in \mathcal{I}$ and the properties:

- (i) **Downward Closed:** If $I \in \mathcal{I}$, then $J \in \mathcal{I}$ for every $J \subseteq I$.
- (ii) **Augmentation Property:** If $I, J \in \mathcal{I}$ and $|I| < |J|$ then there exists some $e \in J$ such that $I \cup \{e\} \in \mathcal{I}$.

A *basis* of a matroid is any maximal independent set. The *rank* of a collection of elements $F \subseteq E$ is the maximum possible size of $I \cap F$ for any $I \in \mathcal{I}$. Let $r : 2^E \rightarrow \mathbb{Z}_{\geq 0}$ be the rank function.

In this problem, we will consider the following polytope P_M for a matroid M :

$$P_M = \begin{cases} x(E) = r(E) \\ x(S) \leq r(S) \quad \forall S \subseteq E \\ x_e \geq 0 \quad \forall e \in E \end{cases}$$

1. Argue that the set of forests of a graph forms a matroid M , and that in this case $P_M = P_{\text{st}}$.
2. Adapt the proof in class for the spanning tree polytope to show that for every matroid M , P_M as defined above is the convex hull of its (integral) bases, i.e. it is the base polytope of M . You may use that the rank function r is submodular, i.e. $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$ for $S, T \subseteq E$. Then briefly argue that randomized pipage rounding works for any matroid with the same guarantees as for spanning trees.
3. Use the [method of conditional expectation](#) to derandomize randomized pipage rounding so that given $x \in P_M$ and any cost function $c : E \rightarrow \mathbb{R}^E$, we can find a point of cost at most $c(x)$ in polynomial time.
4. When using given an independent distribution $\mu : 2^E \rightarrow \mathbb{R}_{\geq 0}^E$ with marginals x , we have used that $\mathbb{P}_{S \sim \mu} [|S \cap F| = 0] \leq e^{-x(F)}$ for any $F \subseteq E$. Show that this also holds for distributions with 0-negative correlation and thus for randomized pipage rounding.

Show that the same bound does *not* hold for all distributions μ with only negative correlation, i.e. $\mathbb{E} [\prod_{i \in S} X_i] \leq \prod_{i \in S} \mathbb{E} [X_i]$ for all $S \subseteq E$, by exhibiting a negatively correlated distribution for which this does not hold.

4 Problem 4: Lottery

Use the above to show that we can design a multi-item lottery as follows. Suppose we have a collection of n goods g_1, \dots, g_n , a collection of m people w_1, \dots, w_m , and a specified $0 < \epsilon < 1$. Now, we will allow people to buy up to one lottery ticket. Before they purchase a ticket, they will specify the subset of the goods that they would be happy winning, and if we sell them a ticket we must promise them that their chance of getting one of their chosen goods is at least ϵ .

Using that P_M has a polynomial time separation oracle for every M , implement a polynomial time lottery system that (i) can determine when a ticket can be faithfully sold (this will depend on the subset of goods desired), (ii) will output a random assignment that respects the guarantee, and (iii) is *fair to all groups* in the sense that for any group of k people, the probability at least one of them wins something is at least $1 - e^{-k\epsilon}$.

5 Bonus Problems

1. Prove the parsimonious property from Lecture 6 using splitting off.
2. Show that the randomized rounding algorithm for the multi-commodity flow problem can be derandomized using the method of conditional expectation.
3. Give an example showing that the Chernoff bound upper tail does not hold for distributions with only *pairwise* negative correlation, i.e. $\mathbb{P}[e, f \in S] \leq \mathbb{P}[e \in S] \mathbb{P}[f \in S]$ (and not full negative correlation as we proved for pipage rounding).
4. (***) Prove that pipage rounding can be used to give a $1 + O(1/\sqrt{k})$ for k -ECSM. (Note: This is quite difficult. I think I vaguely see how to prove this but I'm not 100% sure. This could potentially even be false, although I don't think so. This could be an interesting final project.)