Rounding Techniques in Approximation Algorithms

Lecture 5: Randomized Pipage Rounding

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1 Pipage Rounding

In this lecture, we will discuss a different randomized rounding method called the pipage rounding method. In the coming lectures will use this as a tool to obtain a better approximation algorithm for the k-ECSM problem as well as an approximation algorithm for ATSP, the Asymmetric Traveling Salesperson Problem.

To lead our discussion towards the algorithms for k-ECSM and ATSP, we will discuss the method for the case of spanning trees, but it also works for other settings. In particular, it can be used for any matroid polytope, as you will see in the second homework.

1.1 Spanning Tree Polytope

Firstly, we define the spanning tree polytope and analyze some of its properties. Given a graph $G = (V, E)$, the spanning tree polytope is the convex hull of the indicator vectors of all spanning trees of G. We have a variable x_e for each edge $e \in E$. For any subset of edges $S \subseteq E$, let $x(S)$ denote $\sum_{e \in S} x_e$. For any subset of vertices $A \subseteq V$, denote $E(A)$ as the set of edges in the subgraph induced by vertices of A. Then, the spanning tree polytope P_{st} is defined by the following constraints:

$$P_{\text{st}} = \begin{cases} x(E) = |V| - 1 & \\ x(E(S)) \leq |S| - 1 & \forall S \subseteq V \\ x_e \geq 0 & \forall e \in E. \end{cases}$$

It is not hard to check that if $1_T \in \mathbb{R}^E$ is the indicator variable of some spanning tree T, then 1_T is in P_{st}. Indeed, the first constraint is just saying that there are $|V| - 1$ edges in the spanning tree. The second constraint is true because for any subset of vertices $S \subseteq V$, the number of edges in T in the subgraph induced by S is at most $|S| - 1$. The third constraint clearly holds. It turns out that the vertices of P_{st} are precisely the set of all possible 1_Ts over all spanning trees T (which we will prove soon).

We now define the notion of tight sets. A set of vertices $S \subseteq V$ is tight if $x(E(S)) = |S| - 1$, i.e. the constraint (2) is tight for S. For example, V and all one-element sets are tight. Call a tight set nontrivial if it has at least 2 elements. We show the following lemma:

Lemma 1.1. *If A and B are tight sets for a given x and $A \cap B$ is nonempty, then $A \cap B$ is a tight set.*

Proof. We have

$$\begin{aligned} |A| - 1 + |B| - 1 &= x(E(A)) + x(E(B)) \\ &\leq x(E(A \cup B)) + x(E(A \cap B)) \\ &\leq |A \cup B| - 1 + |A \cap B| - 1 \\ &= |A| - 1 + |B| - 1, \end{aligned}$$

where the first inequality holds because if an edge is in both $E(A)$ and $E(B)$, then it must be in both $E(A \cup B)$ and $E(A \cap B)$; the second inequality holds from the feasibility of x. Hence, all inequalities are equalities, which implies $x(E(A \cup B)) = |A \cup B| - 1$ and $x(E(A \cap B)) = |A \cap B| - 1$, as desired. $\square$

1.2 Pipage Rounding Algorithm

We can now begin discussing the randomized pipage rounding algorithm (see [AS04] for the initial deterministic version and [CVZ10] for the randomized variant we analyze here). Our goal is to round a fractional spanning tree $x \in P_{\text{st}}$ to an integral spanning tree. The idea of the algorithm is simple: while x contains some non-integral coordinates, we find a direction d such that $x + \epsilon d, x - \delta d \in P_{\text{st}}$ for sufficiently small $\epsilon, \delta > 0$. Then, we move randomly in the direction d until we hit a new tight constraint. After repeating this process finitely many (in fact, $O(|V||E|)$) times, we will obtain an integral solution.

To show that the algorithm works, we first need to prove the following lemma:

Lemma 1.2. *Suppose that $x \in P_{\text{st}}$ has some non-integral coordinates. Then, there exists some direction d such that $x + \epsilon d, x - \epsilon d \in P_{\text{st}}$ for sufficiently small $\epsilon > 0$.*

Proof. Contract all edges with $x_e = 1$ and delete all edges with $x_e = 0$ so that $0 < x_e < 1$ for all edges. Since x has some non-integral coordinates, the remaining graph is nonempty. Let S be any *minimal* nontrivial tight set.

We say that two sets A and B cross if $A \cap B, A - B, B - A$ are all nonempty. Observe that if $T \neq S$ is a tight set that crosses S, then $|T \cap S| = 1$. Indeed, if $|T \cap S| \geq 2$, then by Lemma 1.1, we know that $T \cap S$ is a smaller nontrivial tight set, which is a contradiction. Hence, $|T \cap S| = 1$.

Now, fix any direction $d \in \mathbb{R}^E$ such that $d_e = 0$ if $e \notin E(S)$ and $\sum_{e \in E(S)} d_e = 0$. We show that for small enough $\epsilon > 0$, we have $x + \epsilon d, x - \epsilon d \in P_{\text{st}}$. Note that by choosing ϵ small enough, we can always ensure that all currently non-tight constraints are still satisfied. Hence, it remains to check that moving in the direction d does not change the value of all tight constraints. For any tight set T, either $|S \cap T| \leq 1$ (in which case moving in the direction d doesn't change $x(e(T))$), or $|S \cap T| > 1$, which by our previous observation implies that $S \subseteq T$ (since S and T do not cross and $|S|$ is minimal). Since $\sum_{e \in E(S)} d_e = 0$, the constraint T is still satisfied after moving. This proves the claim. $\square$

Note that this immediately implies that the vertices of P_{st} are precisely the integral spanning trees of G.

Corollary 1.3. *Every vertex of P_{st} is integral.*

Now, given $x \in P_{\text{st}}$ which has fractional coordinates, we can perform the following update: find a direction d *with exactly two nonzero coordinates* so that $x + \epsilon d \in P_{\text{st}}$ and has a new tight constraint (this could include an edge becoming integral) and that $x - \delta d \in P_{\text{st}}$ and has a new tight constraint for $\epsilon, \delta > 0$. Now, move to $x + \epsilon d$ with probability p and $x - \delta d$ with probability $(1 - p)$ so that $x = p(x + \epsilon d) + (1 - p)(x - \delta d)$ (to preserve $\mathbb{E}[x]$), i.e. choosing $p = \frac{\delta}{\epsilon + \delta}$. We repeat this process until x has no more fractional coordinates. We show that this algorithm takes $O(|V||E|)$ steps.

Lemma 1.4. *After $O(|V||E|)$ steps, the solution x becomes integral.*

Proof. We show that every $O(|V|)$ steps, at least one new edge of x becomes integral, which will imply the claim.

After performing a rounding step, a new constraint of x becomes tight. If a new edge of x becomes integral, we are done. Otherwise, we have some new tight set T. Let S be the minimal nontrivial tight set before the rounding step. We know that S and T intersect in at least one edge and $S \subsetneq T$ (or else $x(e(T))$ would not have changed). Hence, $S \cap T$ is a nontrivial tight set for the new x. This implies that if no new edge becomes integral, the size of the minimal tight set decreases. Thus, after at most $|V|$ steps, some new edge must become integral, as desired. □

In future lectures we will use this framework to prove the following:

Theorem 1.5. *Let $x \in P_{\text{st}}$. Then, if T is the random tree produced by randomized pipage rounding, the following holds:*

1. $\mathbb{P}[e \in T] = x_e$ *for all* $e \in E$.
2. $\mathbb{P}[S \subseteq T] \leq \prod_{e \in S} x_e$ *for all* $S \subseteq E$, *i.e. the distribution over trees produced by swap rounding is negatively correlated. More precisely, this is often called the* ***negative cylinder property.***

Furthermore, randomized swap rounding runs in deterministic polynomial time.

We have already proved that the algorithm runs in polynomial time. Therefore, in the next lecture we will focus on proving (1) and (2) as well as show an application of swap rounding.

References

[AS04] A. A. Ageev and M. I. Sviridenko. "Pipage Rounding: A New Method of Constructing Algorithms with Proven Performance Guarantee". In: *Journal of Combinatorial Optimization* 8.3 (2004), pp. 307–328. ISSN: 1573-2886. DOI: 10.1023/B:JOCO.0000038913.96607.c2 (cit. on p. 2).

[CVZ10] Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. "Dependent Randomized Rounding via Exchange Properties of Combinatorial Structures". In: *FOCS*. 2010, pp. 575–584 (cit. on p. 2).