Homework 2 Problem 2

In this problem we are asked to find values corresponding to constant ξ, γ, or ϕ along with the value of d and either prolate of spheroidal coordinates that correspond to the following.

Part a

In[∘]:=
```
$Assumptions = ξ > 0 && γ > 0 && ϕ > 0 && a > 0 && b > 0 && d > 0;
```

Here we find constant coordinate values for a prolate spheroid defined by the equation

$$\frac{x^2+y^2}{b^2}+\frac{z^2}{a^2}=1$$

where a > b.

We know that for the prolate spheroid, we have the following coordinates

In[1]:=
```
x = d ξ Sin[γ] Cos[ϕ];
y = d ξ Sin[γ] Sin[ϕ];
z = d Sqrt[1 + ξ^2] Cos[γ];
```

So we should be able to now just solve for each coordinate of constant value.

In[∘]:=
$$\text{eqn} = \frac{(x^2+y^2)}{b^2}+\frac{z^2}{a^2} == 1;$$

Okay so now we can try and solve for ξ

In[∘]:=
```
eqn
```

Out[∘]=
$$\frac{d^2\,(1+\xi^2)\,\text{Cos}[\gamma]^2}{a^2}+\frac{d^2\,\xi^2\,\text{Cos}[\phi]^2\,\text{Sin}[\gamma]^2+d^2\,\xi^2\,\text{Sin}[\gamma]^2\,\text{Sin}[\phi]^2}{b^2} == 1$$

In[∘]:=
```
FullSimplify[PowerExpand[Solve[eqn, ξ]]]
```

Out[∘]=
$$\left\{\left\{\xi \to \frac{b\,\sqrt{-2\,a^2+d^2+d^2\,\text{Cos}[2\,\gamma]}}{d\,\sqrt{-a^2-b^2+(a-b)\,(a+b)\,\text{Cos}[2\,\gamma]}}\ \text{if condition}\right\}\right\}$$

And we can also find that the variable d is given as

In[∘]:=
```
FullSimplify[PowerExpand[Solve[eqn, d]]]
```

Out[∘]=
$$\left\{\left\{d \to \frac{\sqrt{2}\,a\,b}{\sqrt{b^2+(a^2+b^2)\,\xi^2+\left(b^2+(-a+b)\,(a+b)\,\xi^2\right)\text{Cos}[2\,\gamma]}}\right\}\right\}$$

Great, now moving onto part b

Part b

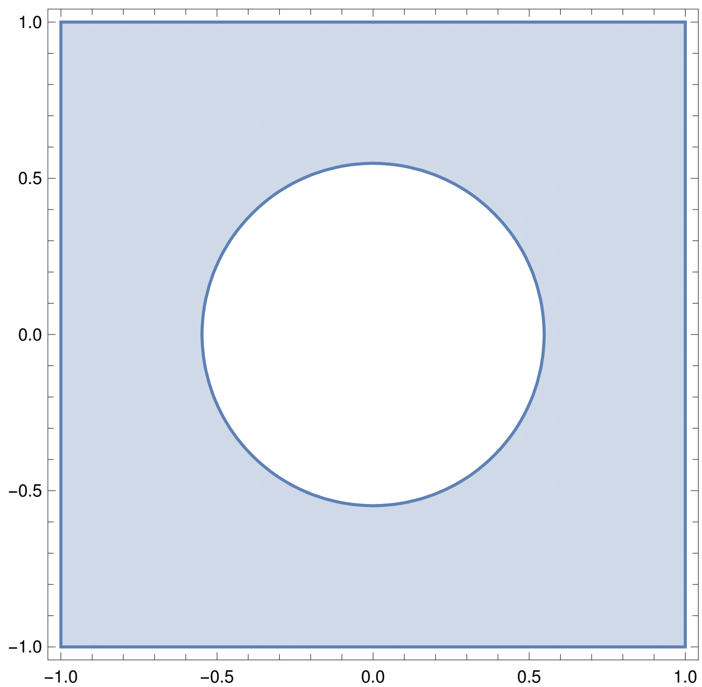
Here we want to find lines of constant coordinates relating to a plane with a circular hole of radius a. We can define a general equation or inequality as follows.

$$x^2 + y^2 > 0$$

Below is a plot of the region for the case of a=0.3, just for intuitions sake.

In[∘]:=
```
RegionPlot[p^2 + m^2 > 0.3`, {p, -1, 1}, {m, -1, 1}]
```

Out[∘]=



So now we can do just as before, setup and equation and solve.

In[6]:=
```
eqn1 = x^2 + y^2 ≥ a
Reduce[Solve[eqn1, ξ]]
```

Out[6]= $d^2\, \xi^2 \operatorname{Cos}[\phi]^2 \operatorname{Sin}[\gamma]^2 + d^2\, \xi^2 \operatorname{Sin}[\gamma]^2 \operatorname{Sin}[\phi]^2 \geq a$

Mathematic won’t solve this one for whatever reason but it’s pretty easy to see that in this case

$$\xi > \frac{\sqrt{a}}{d \sin(\gamma)} \text{ and also } d > \frac{\sqrt{a}}{\xi \sin(\gamma)}$$

Part c

Now we tackle the oblate cases so we need to clear our coordinates

In[13]:=
```
Clear[x, y, z]
x = d Sqrt[1 + ξ^2] Sin[γ] Cos[ϕ];
y = d Sqrt[1 + ξ^2] Sin[γ] Sin[ϕ];
z = d ξ Cos[γ];
```

The equation we get constant values of ξ against is $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$

In[18]:=
$$\text{FullSimplify}\left[\text{PowerExpand}\left[\text{Solve}\left[\frac{(x^2+y^2)}{a^2}+\frac{z^2}{b^2}==1,\ \xi\right]\right]\right]$$

Out[18]=
$$\left\{\left\{\xi\to-\frac{\sqrt{2\,a^2-d^2+d^2\,\text{Cos}[2\,\gamma]}}{\sqrt{\frac{d^2\left(a^2+b^2+(a-b)\,(a+b)\,\text{Cos}[2\,\gamma]\right)}{b^2}}}\right\},\left\{\xi\to\frac{\sqrt{2\,a^2-d^2+d^2\,\text{Cos}[2\,\gamma]}}{\sqrt{\frac{d^2\left(a^2+b^2+(a-b)\,(a+b)\,\text{Cos}[2\,\gamma]\right)}{b^2}}}\right\}\right\}$$

Where we take either result to be true but obvious preference for the positive valued ξ. The variable d then becomes

In[19]:=
$$\text{FullSimplify}\left[\text{PowerExpand}\left[\text{Solve}\left[\frac{(x^2+y^2)}{a^2}+\frac{z^2}{b^2}==1,\ d\right]\right]\right]$$

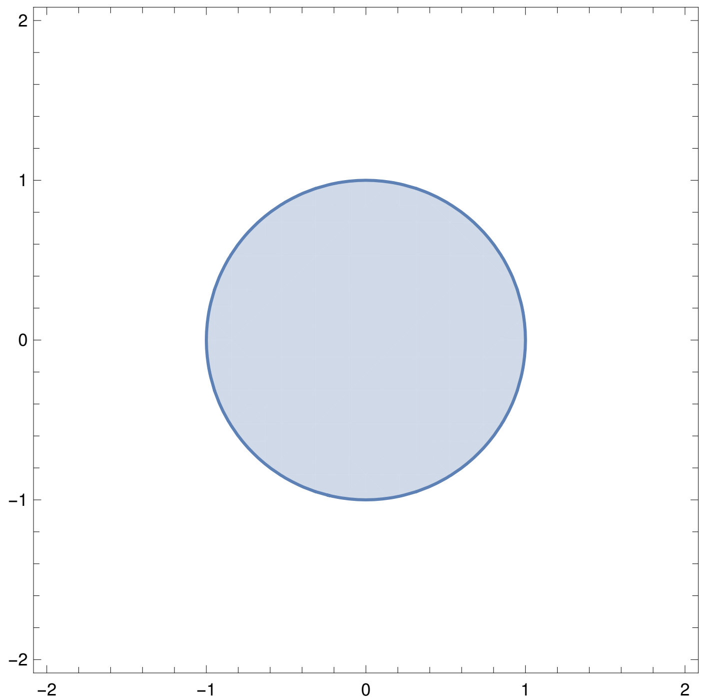
Out[19]=
$$\left\{\left\{d\to-\frac{\sqrt{2}}{\sqrt{\frac{b^2+(a^2+b^2)\,\xi^2-\left(b^2+(-a+b)\,(a+b)\,\xi^2\right)\text{Cos}[2\,\gamma]}{a^2\,b^2}}}\right\},\left\{d\to\frac{\sqrt{2}}{\sqrt{\frac{b^2+(a^2+b^2)\,\xi^2-\left(b^2+(-a+b)\,(a+b)\,\xi^2\right)\text{Cos}[2\,\gamma]}{a^2\,b^2}}}\right\}\right\}$$

Part d

Now we consider a disk of radius a using the same coordinates

In[21]:=
```
RegionPlot[m^2 + p^2 < 1, {m, -2, 2}, {p, -2, 2}]
```

Out[21]=



So we set this up against the equation $x^2 + y^2 < a$

In[23]:=
```
x^2 + y^2 < a
```

Out[23]= $d^2 (1 + \xi^2) \text{Cos}[\phi]^2 \text{Sin}[\gamma]^2 + d^2 (1 + \xi^2) \text{Sin}[\gamma]^2 \text{Sin}[\phi]^2 < a$

Okay great so mathematica still doesn't like this but it is pretty clear the inequality will obey the same exact general relationship, just with a slight change due to the coordinates being slightly different. We find

$$\xi = \frac{\sqrt{a}}{d \sin(\gamma)} - 1 \text{ and for d we find } d = \frac{\sqrt{a}}{(1 + \xi^2) \sin(\gamma)}$$