# Homework 2 problem 1

A note for you Phillip: Apologies, I saw the canvas announcement about not wanting the code as our work as well as your feedback on my previous homework where I submitted notebooks. However, given the timing of the announcement I didn't have time to transcribe all of the work I've done for this assignment onto paper, so I've gone ahead and just tried to add as much latex as I can to hopefully make it easier to read. Next assignment I will do the problem setup and solution on paper, and have the code attached as a sort of additional piece to show that I did put more work in than purely what I write on paper.

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## Problem 1 a

This problem has us looking at the metric for an oblate and prolate spheroid . We will start with the oblate case .
In order to get the metric tensor, we want to determine the basis vectors of our system (not necessary but could prove helpful and provides an easy way to get the metric tensor) .

We are given the following coordinate transformation from cartesian to the oblate spheroid .

In[∘]:=
```
$Assumptions = ξ > 0 && γ > 0 && ϕ > 0;
x = d Sqrt[1 + ξ^2] Sin[γ] Cos[ϕ];
y = d Sqrt[1 + ξ^2] Sin[γ] Sin[ϕ];
z = d ξ Cos[γ];
```

We can easily get the basis vectors by knowing that the basis vectors transform as derivatives.

In[∘]:=
```
eξ = D[{x, y, z}, ξ]
eγ = D[{x, y, z}, γ]
eϕ = D[{x, y, z}, ϕ]
```

Out[∘]= $\left\{\frac{d\,\xi\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\gamma]}{\sqrt{1+\xi^2}},\ \frac{d\,\xi\,\mathrm{Sin}[\gamma]\,\mathrm{Sin}[\phi]}{\sqrt{1+\xi^2}},\ d\,\mathrm{Cos}[\gamma]\right\}$

Out[∘]= $\left\{d\sqrt{1+\xi^2}\,\mathrm{Cos}[\gamma]\,\mathrm{Cos}[\phi],\ d\sqrt{1+\xi^2}\,\mathrm{Cos}[\gamma]\,\mathrm{Sin}[\phi],\ -d\,\xi\,\mathrm{Sin}[\gamma]\right\}$

Out[∘]= $\left\{-d\sqrt{1+\xi^2}\,\mathrm{Sin}[\gamma]\,\mathrm{Sin}[\phi],\ d\sqrt{1+\xi^2}\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\gamma],\ 0\right\}$

I'm gonna try writing them out in latex but the mathematica latex compiler seems to randomly insert characters like the \hat{a}-.. characters on my gamma basis vector, sorry about that.

$$\hat{e}_\xi = \left\{\frac{d\,\xi \sin(\gamma)\cos(\phi)}{\sqrt{\xi^2+1}}, \frac{d\,\xi \sin(\gamma)\sin(\phi)}{\sqrt{\xi^2+1}}, d\cos(\gamma)\right\}$$

$$\hat{e}_\gamma = \text{â-..}\left\{d\sqrt{\xi^2+1}\cos(\gamma)\cos(\phi), d\sqrt{\xi^2+1}\cos(\gamma)\sin(\phi), -d\,\xi\sin(\gamma)\,\text{â-..}\right\}$$

$$\hat{e}_\phi = \left\{-d\sqrt{\xi^2+1}\sin(\gamma)\sin(\phi), d\sqrt{\xi^2+1}\sin(\gamma)\cos(\phi), 0\right\}$$

Great, now that we have these new basis vectors, we can solve for the components of the metric tensor simply by finding their inner products . We can speed this up in mathematica by looping .

To be clear, all that is happening here is I am taking the inner product of each of the above basis vectors and automatically inserting them into the array . Then I simplify the contents and display them . I' ve included all this info into the following function G which just calculates the metric tensor .

In[∘]:=
```
G[basisVecs_List] := Module[{dimension, metricTensor},
    dimension = Length[basisVecs];
    metricTensor = ConstantArray[0, {dimension, dimension}];

    For[i = 1, i ≤ dimension, i++,
    For[j = 1, j ≤ dimension, j++,
        metricTensor[[i]][[j]] = Dot[basisVecs[[i]], basisVecs[[j]]]
        ];
    ];

    Return[metricTensor];
]
```

So now we can use this for both of the geometries ..

In[∘]:=
```
oblateBasis = {eξ, eγ, eϕ};
oblateMetric = FullSimplify[G[oblateBasis]];
MatrixForm[oblateMetric]
```

## The oblate metric tensor

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{d^2\left(1+2\,\xi^2+\text{Cos}[2\,\gamma]\right)}{2\,(1+\xi^2)} & 0 & 0 \\ 0 & \frac{1}{2}\,d^2\left(1+2\,\xi^2+\text{Cos}[2\,\gamma]\right) & 0 \\ 0 & 0 & d^2\,(1+\xi^2)\,\text{Sin}[\gamma]^2 \end{pmatrix}$$

Great, now that we have that, we can go ahead and find it for the prolate case.

I'm gonna wipe the current variables and then we should be able to make the process pretty quick ...

In[∘]:=
```
Clear["Global`*"]
```

```
(*define the coordinate system*)
x = d ξ Sin[γ] Cos[ϕ];
y = d ξ Sin[γ] Sin[ϕ];
z = d Sqrt[1 + ξ^2] Cos[γ];
(*now we calculate the basis vectors*)
eξ = D[{x, y, z}, ξ];
eγ = D[{x, y, z}, γ];
eϕ = D[{x, y, z}, ϕ];
prolateBasis = {eξ, eγ, eϕ};
(*now we can just call the function from before*)
prolateMetric = FullSimplify[G[prolateBasis]];
```

In[∘]:=
```
MatrixForm[prolateMetric]
```

The prolate metric tensor

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{d^2\left(1+2\,\xi^2-\text{Cos}[2\,\gamma]\right)}{2\,(1+\xi^2)} & 0 & 0 \\ 0 & -\frac{1}{2}\,d^2\left(-1-2\,\xi^2+\text{Cos}[2\,\gamma]\right) & 0 \\ 0 & 0 & d^2\,\xi^2\,\text{Sin}[\gamma]^2 \end{pmatrix}$$

We can see that the diagonal structure of a metric indicates an orthogonal coordinate system as the components of the metric tensor are themselves the inner products of the basis vectors of the coordinate system. Thus it is definitionally orthogonal if the offdiagonal components are zero as those are the Subscript[g, ij] components where i != j . Since clearly the offdiagonals are zero, our coordinate basis is orthogonal.

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# Problem 1b

In this problem we would like to calculate the Laplacian from the following formula

$$\nabla^2 = \frac{1}{h_1\,h_2\,h_3}\sum_{\text{cyclic}}\frac{\partial}{\partial u_i}\left(\frac{h_j\,h_k}{h_i}\right)\frac{\partial}{\partial u_i}$$

Where it is conveniently defined that the h_ijk are the root elements of the metric for whichever laplacian we hope to calculate and the u_i are the coordinates of the metric.

We can start with the prolate case, and start by getting the inverse h_i coefficients in front of the cyclic summation. The h_i are the diagonal components of the metric square rooted, so we can just grab those for the prolate.

## Calculating coefficients from generalized Laplacian cyclic sum for prolate

In[∘]:=
```
inverseH = TrigReduce[FullSimplify[TrigReduce[
    (Sqrt[prolateMetric[[1]][[1]]] Sqrt[prolateMetric[[2]][[2]]] Sqrt[prolateMetric[[3]][[3]]])^-1]]]
```

Out[∘]= $\frac{2\sqrt{d^2(1+\xi^2)\,\text{Csc}[\gamma]^2}}{d^4\,\xi\,(1+2\,\xi^2-\text{Cos}[2\,\gamma])}$

Now we need to do the more tedious terms. We need to execute a cyclic sum over these rooted diagonal elements. We should be able to do this fairly easily with a little loop or a quick function, but I think maybe it would just be easier to do each term explicitly, if not a little lazy.

In[∘]:=
```
h123 =
 FullSimplify[Sqrt[prolateMetric[[1]][[1]] × prolateMetric[[2]][[2]] * (prolateMetric[[3]][[3]])^-1]]
```

Out[∘]= $\sqrt{\frac{d^2\,(\xi^2\,\text{Csc}[\gamma]+\text{Sin}[\gamma])^2}{\xi^2+\xi^4}}$

In[∘]:=
```
h312 =
 FullSimplify[Sqrt[prolateMetric[[3]][[3]] × prolateMetric[[1]][[1]] * (prolateMetric[[2]][[2]])^-1]]
```

Out[∘]= $\xi\sqrt{\frac{d^2}{1+\xi^2}}\,\text{Abs}[\text{Sin}[\gamma]]$

In[∘]:=
```
h231 =
 FullSimplify[Sqrt[prolateMetric[[2]][[2]] × prolateMetric[[3]][[3]] * (prolateMetric[[1]][[1]])^-1]]
```

Out[∘]= $\xi\sqrt{d^2\,(1+\xi^2)}\,\text{Abs}[\text{Sin}[\gamma]]$

Great, now we should be able to put this all together. I will type this bit up in latex (fingers crossed this actually works).

Okay so it doesn’t work so I’m going to just paste a screenshot into mathematica because apparently the Latex compiler isn’t just bad, it’s completely useless.

$$\begin{aligned}\nabla^2 &= \frac{1}{h_1h_2h_3}\sum_{\text{cyclic}}\frac{\partial}{\partial u_i}\left(\frac{h_jh_k}{h_i}\right)\frac{\partial}{\partial u_i}\\ &= \frac{2\sqrt{d^2(1+\xi^2)\csc(\gamma)^2}}{d^4\xi(1+2\xi^2-\cos(2\gamma)}\left(\partial_\xi\left(\xi\sqrt{d^2(1+\xi^2)}\sin(\gamma)\partial_\xi\right)+\partial_\gamma\left(\xi\sqrt{\frac{d^2}{1+\xi^2}}\sin(\gamma)\partial_\gamma\right)+\partial_\phi\left(\sqrt{\frac{d^2(\xi^2\csc(\gamma)+\sin(\gamma))^2}{xi^2+xi^4}}\right)\partial\phi\right)\end{aligned}$$

After a truly considerable amount of pain and suffering, this can be found to be exactly equivalent to the following.

$$\nabla^2 = \frac{\sqrt{1+\xi^2}}{d^2(\xi^2+\sin^2(\gamma))\xi}\left(\partial_\xi\left(\xi\sqrt{1+\xi^2}\partial_\xi\right)+\frac{1}{d^2(\xi^2+\sin^2(\gamma))\sin(\gamma)}\partial_\gamma\left(\sin(\gamma)\partial_\gamma\right)+\frac{1}{d^2\xi^2\sin^2(\gamma)}\partial_\phi^2\right)$$

Which is of course, exactly as we had hoped

We can of course execute this same process for the oblate case.

## Calculating coefficients from generalized Laplacian cyclic sum for oblate

In[•]:=
```
Clear[inverseH, h123, h312, h231]
inverseH = TrigReduce[FullSimplify[TrigReduce[
    (Sqrt[oblateMetric[[1]][[1]]] Sqrt[oblateMetric[[2]][[2]]] Sqrt[oblateMetric[[3]][[3]]])^-1]]]
```

Out[•]=
$$\frac{4\,\mathrm{Sign}[\mathrm{Sin}[\gamma]]}{(d^2)^{3/2}\left(\mathrm{Sin}[\gamma]+4\,\xi^2\,\mathrm{Sin}[\gamma]+\mathrm{Sin}[3\,\gamma]\right)}$$

In[•]:=
```
h123 = FullSimplify[Sqrt[oblateMetric[[1]][[1]] × oblateMetric[[2]][[2]] * (oblateMetric[[3]][[3]])^-1]]
```

Out[•]=
$$\frac{\left(1+2\,\xi^2+\mathrm{Cos}[2\,\gamma]\right)\sqrt{d^2\,\mathrm{Csc}[\gamma]^2}}{2\,(1+\xi^2)}$$

In[•]:=
```
h312 = FullSimplify[Sqrt[oblateMetric[[3]][[3]] × oblateMetric[[1]][[1]] * (oblateMetric[[2]][[2]])^-1]]
```

Out[•]=
$$\sqrt{d^2}\ \mathrm{Abs}[\mathrm{Sin}[\gamma]]$$

In[•]:=
```
h231 = FullSimplify[Sqrt[oblateMetric[[2]][[2]] × oblateMetric[[3]][[3]] * (oblateMetric[[1]][[1]])^-1]]
```

Out[•]=
$$\sqrt{d^2}\ (1+\xi^2)\,\mathrm{Abs}[\mathrm{Sin}[\gamma]]$$

So then putting this all together we get the following

$$\frac{4}{d^3(\sin(\gamma)+4\xi^2\sin(\gamma)+\sin(3\gamma))}\left(\partial_\xi\left(d(1+\xi^2)\sin(\gamma)\right)\partial_\xi+\partial_\gamma\left(d\sin(\gamma)\partial_\gamma\right)+\partial_\phi\left(\frac{(1+2\xi^2+\cos(2\gamma))\sqrt{d^2\csc^2(\gamma)}}{2(1+\xi^2)}\partial_\phi\right)\right)$$

Which, yet again after a long time and a lot of algebra gives a the familiar and expected form from the problem statement.

$$\nabla^2=\frac{1}{d^2(\xi^2+\cos^2(\gamma))}\left(\partial_\xi\left((\xi^2+1)\partial_\xi\right)+\frac{1}{\sin(\gamma)}\partial_\gamma\left(\sin(\gamma)\partial_\gamma\right)\right)+\frac{1}{d^2(\xi^2+1)\sin^2(\gamma)}\partial_\phi^2$$