Nathan Burwig
Math 87 HW 8
Due My Birthday :)

---

1 Spline Interpolation

Manually form a matrix for cubic spline interpolation of f(x) with natural and with clamped boundary conditions, if

$$f(0) = 0,\ f(1) = 2,\ f(2) = 1$$

You may use numpy to solve the resulting linear systems for coefficients.

1. In order to complete cubic spline interpolation there are several factors we will want to take into account. The first, are the conditions in place from our data points. Those are the points listed above. We can list the constraints they impart on our system as follows...

$$\begin{aligned} f_1(0) &= x^3a_1 + x^2b_1 + xc_1 + d_1 = d_1 &&= 0 && (1.1)\\ f_1(1) &= x^3a_1 + x^2b_1 + xc_1 + d_1 = a_1 + b_1 + c_1 + d_1 &&= 2 && (1.2)\\ f_2(1) &= x^3a_2 + x^2b_2 + xc_2 + d_2 = a_2 + b_2 + c_2 + d_2 &&= 2 && (1.3)\\ f_2(2) &= x^3a_2 + x^2b_2 + xc_2 + d_2 = 8a_2 + 4b_2 + 2c_2 + d_2 &&= 1 && (1.4) \end{aligned}$$

Along with these constraints, we also require that our system is continuous at both points, which means both the first and second derivatives must be equal to eachother at those points. The constraints for the first derivative are as follows...

$$\frac{df_1(x)}{dx} = \frac{df_2(x)}{dx}$$

Which implies..

$$0 = 3a_1 + 2b_1 + c_1 - 3a_2 - 2b_2 - c_2 \tag{1.5}$$

And the constraint on the second derivative we get...

$$\frac{d^2f_1(x)}{dx^2} = \frac{d^2f_2(x)}{dx^2}$$

Which further implies...

$$0 = 6a_1 + 2b_1 - 6a_2 - 2b_2 \tag{1.6}$$

We now turn to the final condition we express, which are the boundary constraints. These are either natural or clamped, and will have an impact in our final matrix. The *natural* constraint dictates that the second derivatives of the end points must be equal to zero. The *clamped* constriants dicate that the first derivatives must be equal to zero at the endpoints.

Natural:

$$\begin{aligned} f_1^{''}(x_0) &= 6a_1x + 2b_1 = 2b_1 &&= 0 && (1.7)\\ f_2^{''}(x_n) &= 6a_2x + 2b_2 = 2a_2 + 2b_2 &&= 0 && (1.8) \end{aligned}$$

Clamped:

$$f_1'(x_0) \;=\; 3a_1x^2 + 2b_1x + c_1 \;=\; c_1 \qquad\qquad = 0 \tag{1.9}$$

$$f_2'(x_n) \;=\; 3a_2x^2 + 2b_2x + c_2 \;=\; 12a_2 + 4b_2 + c_2 = 0 \tag{1.10}$$

Now that we have our boundary conditions setup, we can setup two different linear systems of these equations, one with the natural condition and one with the clamped. Then we can solve these systems for the coefficients of the cubic polynomials which satisfy all the constraints we've given.

Natural:

$$\left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 8 & 4 & 2 & 1 \\ \hline 3 & 2 & 1 & 0 & -3 & -2 & -1 & 0 \\ \hline 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 \end{array}\right] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

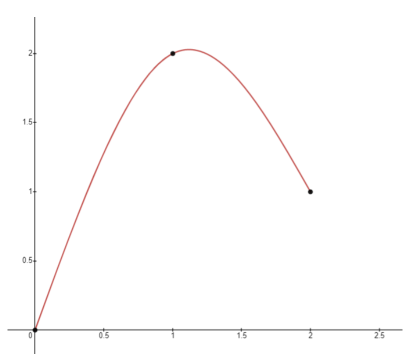
I have delineated the different constraints being used by the horizontal lines. The first four are associated with constraints 1.1-4, then 1.5, 1.6 and 1.7-8.

Clamped:

$$\left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 8 & 4 & 2 & 1 \\ \hline 3 & 2 & 1 & 0 & -3 & -2 & -1 & 0 \\ \hline 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 4 & 1 & 0 \end{array}\right] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we can solve these quickly using the `numpy.linalg.solve()` function in `python`. Doing so gives us two coefficient vectors. Below you will see the two plots given, along with the associated polynomials given for the two different boundary conditions.

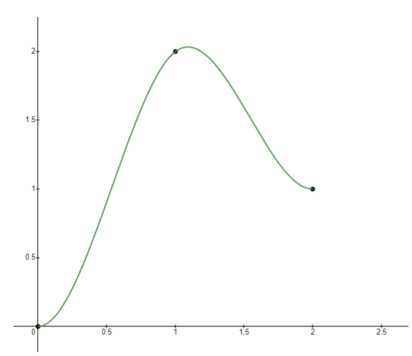
Natural:



$f_1(x) = -3.25x^3 + 5.25x^2 \qquad [0 \le x \le 1]$

$f_2(x) = 2.75x^3 - 12.75x^2 + 18x - 6 \qquad [1 \le x \le 2]$

Clamped

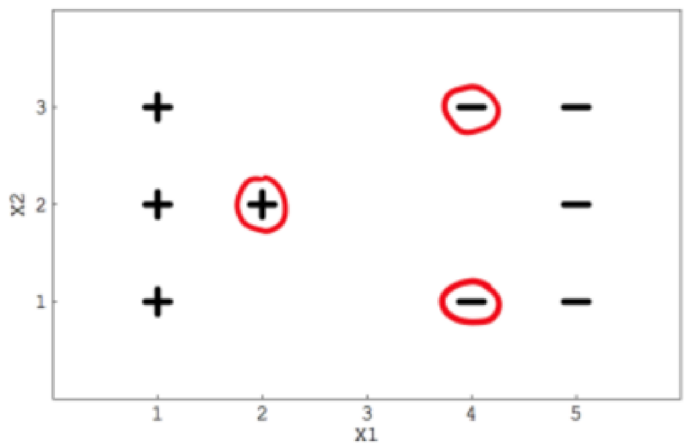


$f_1(x) = -3.25x^3 + 5.25x^2 \qquad [0 \le x \le 1]$

$f_2(x) = 2.75x^3 - 12.75x^2 + 18x - 6 \qquad [1 \le x \le 2]$

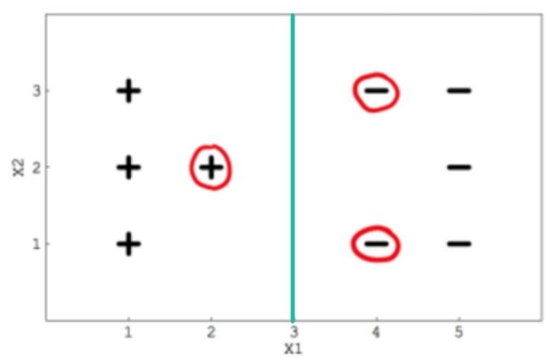
2 Comprehending SVM

Suppose we have data drawn from two different populations, shown in the figure below as (+) and (−). Support vector machines could be simply used to linearly separate such classes of data.



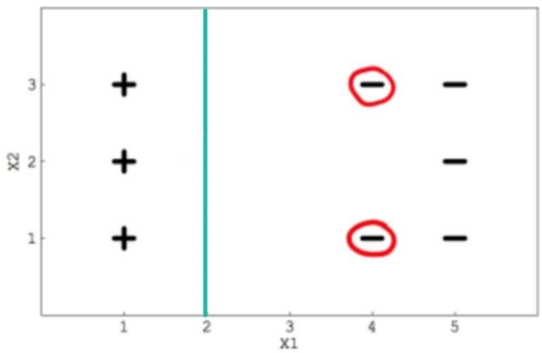
1. **Draw your approximation of the separating line SVM would generate to separate these two classes of data**

   We know that SVM cares about the perpendicular distance between nearby points on the plot. The average then would form a line that looks as follows.



2. **Suppose that the (+) in the red circle was deleted from the data set. Would the supporting line change? If so, draw it.**

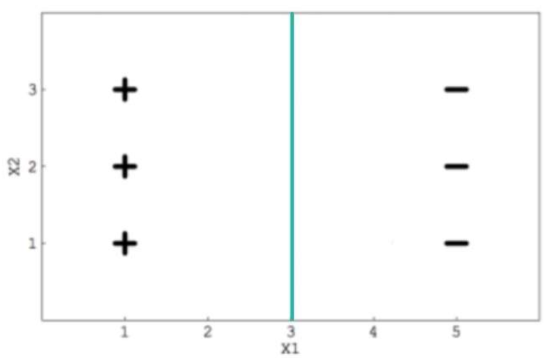
   If the plus disappears, it moves our line as follows...



My line being at 2 is perhaps a bit misleading, looking at it now it should probably be closer to 2.5 or so.

3. **Suppose that all red circled data points were deleted. Would the supporting line change? If so, draw it**

   Removing all red points would move the line as follows...



3 Computing Kernels

Let $\hat{x} = \langle x_1, x_2 \rangle$ and $\hat{y} = \langle y_1, y_2 \rangle$ both $\in \mathbb{R}^2$. Suppose we use the kernel function $K(\hat{x}, \hat{y}) = (\hat{x} \cdot \hat{y} + c)^2$. Compute the higher dimensional embedding (i.e. the feature map) of $\hat{x}$ corresponding to this kernel.

1. We would like to determine the higher dimensional embedding imposed by the kernel given above. We know that the dimensionality will be $\begin{pmatrix} n+d \\ d \end{pmatrix}$ Where n is the original dimension of the problem, and d is the power to which we are raising our kernel. So, we expect this projection to go into $\mathbb{R}^6$

   We can start by just expanding out the inner product we have in our kernel function.

$$\begin{aligned} K(\hat{x}, \hat{y}) &= (\hat{x} \cdot \hat{y} + c)^2 \\ &= (x_1 y_1 + x_2 y_2 + c)^2 \\ &= ((x_1 y_1)^2 + (x_2 y_2)^2 + 2x_1 x_2 y_1 y_2 + 2c x_1 y_1 + 2c x_2 y_2 + c^2 \end{aligned}$$

   It is here we can note that this whole expression can be written out in terms of a rather long winded dot product. They can be arranged in a couple of different orders, but in general I will separate the x's from the y's.

$$\begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2} x_1 x_2 \\ \sqrt{2c} x_2 \\ \sqrt{2c} x_1 \\ c \end{bmatrix}^T \begin{bmatrix} y_1^2 \\ y_2^2 \\ \sqrt{2} y_1 y_2 \\ \sqrt{2c} y_2 \\ \sqrt{2c} y_1 \\ c \end{bmatrix}$$

   We can manually multiply this out and we see...

$$(x_1 y_1)^2 + (x_2 y_2)^2 + 2x_1 x_2 y_1 y_2 + 2c x_2 y_2 + 2c x_1 y_1 + c^2$$

   Which is exactly what we started with. Thus we can define the higher dimensional embedding from our kernel as the following feature map.

$$\phi(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2} x_1 x_2 \\ \sqrt{2c} x_2 \\ \sqrt{2c} x_1 \\ c \end{bmatrix}$$