Midterm 1

Nathan Burwig
MATH 87 Math Modeling

October 5, 2022

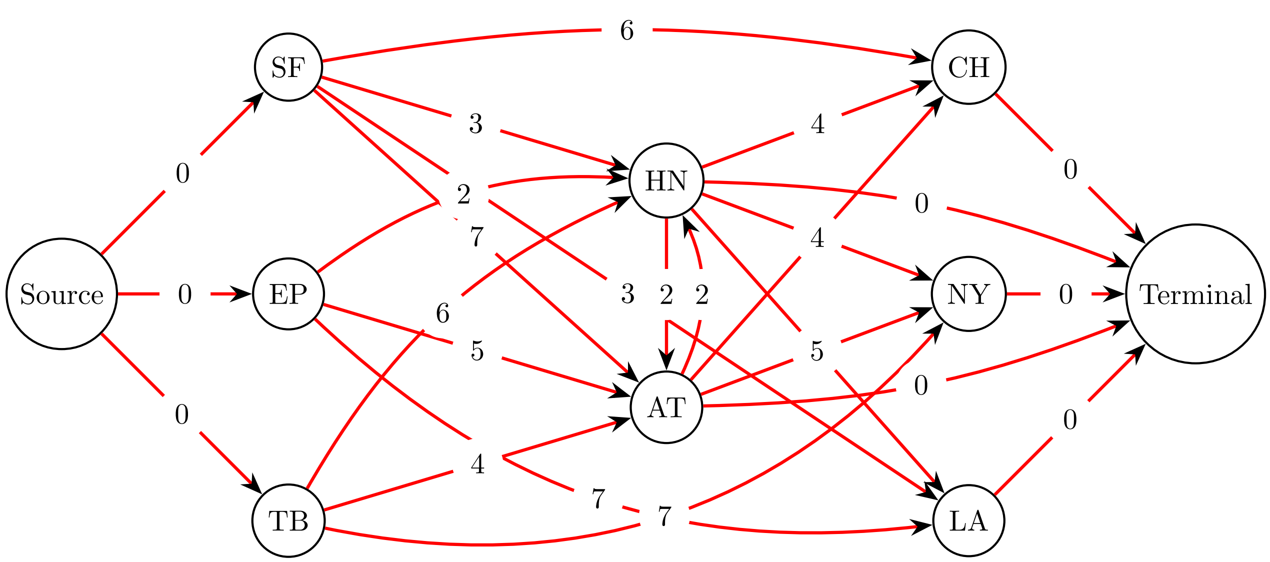
This is the midterm report of Nathan Burwig for the Fall Semester of 2022

1 The problem

We are given the problem of minimzing the costs of a certain (rubber) duck shipping and manufacturing company. What follows is my attempt to do so, along with a few other optimization problems related to this one

1.1 Section 1

We start this section, with a clearly labled network-flow model. Primarily, this section features the actual graph I will be using to model this network flow problem. Each node represents a major city where ducks are freighted between. The number associated with each edge is the price per duck in dollars that it costs to ship along that route. We have a source and terminal node to help process some of the data involved.



Note that the source and terminal nodes have no costs associated with them, as these notes are somewhat of a dummy variable. The Source node could represent something like a manufacturing facility, and the Terminal node would be the actual customers who recieve the ducks.

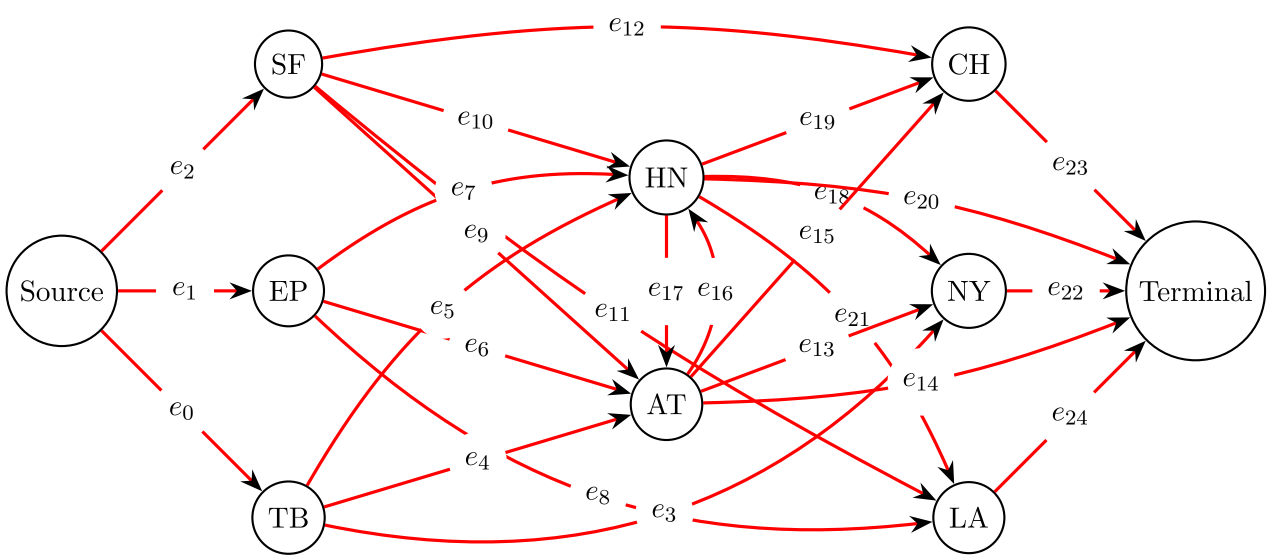
There are also a handful of constraints that we have to consider given this problem. We know that no single route (edge) can have more than 200 ducks go over it. We also know that there are specific demands for each node. These demans are not shown in the above diagram, simply because the diagram would become too chaotic. If we know we don't want to send any more than a given cities demand, though, (as we don't want to have ducks shipped that don't sell) then we know the demand can be express in the following inequality constraints.

$$
\begin{aligned}
\text{Source} &\leq 0 \\
\text{Santa Fe (SF)} &\leq -700 \\
\text{El Paso (EP)} &\leq -200 \\
\text{Tampa Bay (TB)} &\leq -200 \\
\text{Chicago (CH)} &\leq 200 \\
\text{LA (LA)} &\leq 200 \\
\text{NY (NY)} &\leq 250 \\
\text{Houston (HN)} &\leq 300 \\
\text{Atlanta (AT)} &\leq 150 \\
\text{Terminal} &\leq 0
\end{aligned}
$$

We assign a negative value to the constraints of the warehouse cities as they need to get rid of ducks that are stored, and we have 0's on the source and terminal as, in this context, the ducks going through them is not of significance to the total system.

1.2 Formulating the Linear Program

We know that we want to minimize the costs associated with shipping ducks, or to find the best way to ship the ducks with the minimum cost associated. To do this, we want to find some set of constraints that gives us the conservation quantities for our system of nodes. In this case, we know that (because we have a terminal and source node) anything that goes into a node, must come out of the node, unless it is terminal or source. We can use this to write a set of constraint equations that will form our constraint matrix A. In order to do this, though, we will want there to be a mapping between some useful variables (ie e_i) and the edge between two cities. The mapping will go as follows...



Wow that's a mess. In any case, there is in fact a unique labelling for all 25 variables. We know the each node has a conservation law that can be written as something like the following:

$$\text{Conservation for NY} \rightarrow e_3 + e_{13} + e_{18} - e_{22} = 0$$

One will find that they can actually write a great deal of these conservation laws. To spare the space and time it would take to write them all out, I will use this time to display some of the code utilized in making this all possible, which includes the matrix of interest (the matrix A). The rows of the array represent each node, and the columns are edges. A negative value indicates an outgoing edge, while positive is incoming.

```
import numpy as np
from scipy.optimize import linprog
A = np.array([ [-1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
               [ 1,  0, 0,-1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
               [0, 1, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
               [0, 0, 1, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
               [0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, -1, -1, -1, -1, 1, 0, 0, 0, 0, 0, 0, 0],
               [0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, -1, -1, -1, -1, -1, 0, 0, 0],
               [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0],
               [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, -1, 0],
               [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1],
               [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1]])

b = np.array([0, -200, -200, -700, 150, 300, 250, 200, 200, 0])
c = np.array([0,0,0,7,4,6,5,2,7,7,3,3,6,5,0,4,2,2,6,4,0,5,0,0,0])

Aub = np.identity(25)
bub = np.array([1100, 1100, 1100, 200, 200, 200, 200, 200, 200, 200, 200, 200, 200, 200, 1100,
                200, 200, 200, 200, 200, 1100, 200, 1100, 1100, 1100])

res = linprog(c, A_eq = A, b_eq = b, A_ub = Aub, b_ub = bub)
print(res)
```

Listing 1: The Program

I know that this looks like a lot, but each row represents the conservation rules associated for a given node. This makes it our conservation matrix, or at least on part of it. There is another large matrix here, the `Aub` matrix which is an identity matrix. This brings us into another part of our constraints, namely that each edge (apart from edges associated with the source or terminal node) can only ship 200 items per month, as the problem says. We can then make our inequality constraint vector `bub` to enforce that constraint. You'll notice, that for terminal and source connections, I put a maximum of 1100, simply because there are only 1100 ducks in the system. You could actually change these and it would not impact the problem too greatly, so long as you stay above the number of ducks that actually go from source to SF, EP, and TB respectively and similarly for the terminal nodes.

The reason we keep these values separate from the first set of arrays and matrices, is because we will pass them in as separate arguments to the linprog function. This is a neat piece of functionality from Scipy that we will use throughout this project.

We can also notice, in the above, that we have a `b` vector, which represents the demands of the given nodes, which has already been discussed.

Finally we have the objective function `c`. This vector is representative of the costs associated with travelling over each edge. This is because we wish to minimize this value over our whole system. What this amounts to is a vector of 25 edges, all with the associated traversal cost. With all of these systems in place, we can actually just run it!

1.3 Running the Code

When we run the code, we get an encouraging response! It actually ran!

I wrote a small function that actually takes cares of mapping the edges back to the actual routes, and shows how many ducks we pass along a given route. The output of that, for this example, is shown below.

```
total costs: 5300
e_0     S/TB    ------> 0
e_1     S/EP    ------> 0
e_2     S/SF    ------> 0
e_3     TB/NY   ------> 200
e_4     TB/AT   ------> 0
e_5     TB/HN   ------> 0
e_6     EP/AT   ------> 0
e_7     EP/HN   ------> 200
e_8     EP/LA   ------> 0
e_9     SF/AT   ------> 100
e_10    SF/HN   ------> 200
e_11    SF/LA   ------> 200
e_12    SF/CH   ------> 200
e_13    AT/NY   ------> 0
e_14    AT/T    ------> 0
e_15    AT/CH   ------> 0
e_16    AT/HN   ------> 0
e_17    HN/AT   ------> 50
e_18    HN/NY   ------> 50
e_19    HN/CH   ------> 0
e_20    HN/T    ------> 0
e_21    HN/LA   ------> 0
e_22    NY/T    ------> 0
e_23    CH/T    ------> 0
e_24    LA/T    ------> 0
```

So we can see that our total costs equate to $5300. I've also taken the time to print out the actual mapping that was used in a slightlymore readable fashion now, and also to illustrate exactly which routes get how many ducks, as this is the actual information of interest for someone working in a company like this.

The code that was utilized to run this linear program is exactly what is seen above, and no different. You can see the use of the `Aub` and `bub` matrices which make working this problem much much easier.

We can actually note that the total number of ducks apparently shipped is greater than the total number of ducks in our system, which is accounted for in the Houston and Atlanta routes, each of which shipped fifty ducks individually. Thus we should (and do in fact) have a total number of ducks shipped being 1200.

1.4 Uh oh, LA is upset

We now turn to consider the scenarios in which the workers in LA are quite unhappy. The first scenario we consider is that of the workers in LA cause the shipping prices to be increased by a factor of 2. This is, of course, only for routes **to** LA. We can change this by simply multiplying the associated nodes in the objective function by 2. We then have the following as our linear program...

```
c_la = np.array([0, 0, 0, 7, 4, 6, 5, 2, 7*2, 7, 3, 3*2, 6, 5, 0, 4, 2, 2, 6, 4, 0, 5*2, 0, 0, 0])

res_la = linprog(c_la, A_eq = A, b_eq = b, A_ub = Aub, b_ub = bub)
```

Listing 2: LA routes double in price

Running this program, we get that the total costs associated with this cost doubling comes out to be $5900.

We can run a similar analysis on what would happen if, instead of the price of the routes doubling, the number of ducks that can be shipped over given routes changes anything compared to what we just solved. Running that analysis requires us to change not the objective funciton, but the variable `bub` as this is the upper bound innequality constraint for the number of ducks that can be shipped over a single route. That would look like the following...

```
bub_la = np.array([1100, 1100, 1100, 200, 200, 200, 200, 200, 100, 200, 200, 100, 200,
                   200, 1100, 200, 200, 200, 200, 200, 1100, 100, 1100, 1100, 1100])

res_la2 = linprog(c, A_eq = A, b_eq = b, A_ub = Aub, b_ub = bub_la)
```

Listing 3: LA routes cut volume in half

We get that the total costs of shipping given this new constraint, comes out to be \$6050 which is certainly more than the previous case.

For the sake of completeness, I will also display the number of ducks that went over each individual route.

For the values associated with the cost of shipping doubling we have...

```
fun:
  5899.999997911287
e_0     S/TB    ---> 0
e_1     S/EP    ---> 0
e_2     S/SF    ---> 0
e_3     TB/NY   ---> 200
e_4     TB/AT   ---> 0
e_5     TB/HN   ---> 0
e_6     EP/AT   ---> 0
e_7     EP/HN   ---> 200
e_8     EP/LA   ---> 0
e_9     SF/AT   ---> 100
e_10    SF/HN   ---> 200
e_11    SF/LA   ---> 200
e_12    SF/CH   ---> 200
e_13    AT/NY   ---> 0
e_14    AT/T    ---> 0
e_15    AT/CH   ---> 0
e_16    AT/HN   ---> 0
e_17    HN/AT   ---> 50
e_18    HN/NY   ---> 50
e_19    HN/CH   ---> 0
e_20    HN/T    ---> 0
e_21    HN/LA   ---> 0
e_22    NY/T    ---> 0
e_23    CH/T    ---> 0
e_24    LA/T    ---> 0
```

For the values associated with the cost of shipping doubling we have...

```
fun:
  6049.9999911648265
e_0     S/TB    ---> 0
e_1     S/EP    ---> 0
e_2     S/SF    ---> 0
e_3     TB/NY   ---> 200
e_4     TB/AT   ---> 0
e_5     TB/HN   ---> 0
e_6     EP/AT   ---> 0
e_7     EP/HN   ---> 145
e_8     EP/LA   ---> 55
e_9     SF/AT   ---> 200
e_10    SF/HN   ---> 200
e_11    SF/LA   ---> 100
e_12    SF/CH   ---> 200
e_13    AT/NY   ---> 50
e_14    AT/T    ---> 0
e_15    AT/CH   ---> 0
e_16    AT/HN   ---> 0
e_17    HN/AT   ---> 0
e_18    HN/NY   ---> 0
e_19    HN/CH   ---> 0
e_20    HN/T    ---> 0
e_21    HN/LA   ---> 45
e_22    NY/T    ---> 0
e_23    CH/T    ---> 0
e_24    LA/T    ---> 0
```

We can see that in the first case, not much actually changes when we alter the prices, but if we alter the maximum amount allowed to flow to LA, we get some alternative routes being taken.

1.5 Houston, we have a problem

We wish to run a very similar analysis, but instead of running it on LA, we run it on Houston. First we take the doubling of the prices of routes going to Houston. When we do this, we run the program as follows:

```
c_hn = np.array([0, 0, 0, 7, 4, 6*2, 5, 2*2, 7, 7, 3*2, 3, 6, 5, 0, 4, 2*2, 2 ,6 ,4 ,0 ,5 ,0 ,0, 0])

res_hn = linprog(c_hn, A_eq = A, b_eq = b, A_ub = Aub, b_ub = bub)
```

Then we can run the same anaylysis and limit the number of ducks that can go to Houston. When we print the result, we get that the costs associated with doubling the price of routes to houston become $6250.

We run the same analysis as in the LA case yet again and cut the shipping volume to Houston in half to determine what the change is. We see that, we get the exact same costs! Which is fascinating. We of course get this by running the following linear program:

```
bub_hn = np.array([1100, 1100, 1100, 200, 200, 200*(.5), 200, 200*(.5), 200, 200,
                   200*(.5), 200, 200, 200, 1100, 200, 200*(.5), 200, 200, 200,
                   1100, 200, 1100, 1100, 1100])

res_hn2 = linprog(c, A_eq = A, b_eq = b, A_ub = Aub, b_ub = bub_hn)
```

While these linear programs have the same output, they don't necessarily take the same routs, as we can see in the following.

Double the shipping cost...

```
fun:
 6249.999918839532
e_0     S/TB   ——> 0
e_1     S/EP   ——> 0
e_2     S/SF   ——> 0
e_3     TB/NY  ——> 200
e_4     TB/AT  ——> 0
e_5     TB/HN  ——> 0
e_6     EP/AT  ——> 74
e_7     EP/HN  ——> 126
e_8     EP/LA  ——> 0
e_9     SF/AT  ——> 105
e_10    SF/HN  ——> 195
e_11    SF/LA  ——> 200
e_12    SF/CH  ——> 200
e_13    AT/NY  ——> 29
e_14    AT/T   ——> 0
e_15    AT/CH  ——> 0
e_16    AT/HN  ——> 0
e_17    HN/AT  ——> 0
e_18    HN/NY  ——> 21
e_19    HN/CH  ——> 0
e_20    HN/T   ——> 0
e_21    HN/LA  ——> 0
e_22    NY/T   ——> 0
e_23    CH/T   ——> 0
e_24    LA/T   ——> 0
```

Cutting max over route in half

```
fun:
 6249.999809297354
e_0     S/TB   ——> 0
e_1     S/EP   ——> 0
e_2     S/SF   ——> 0
e_3     TB/NY  ——> 100
e_4     TB/AT  ——> 0
e_5     TB/HN  ——> 100
e_6     EP/AT  ——> 100
e_7     EP/HN  ——> 100
e_8     EP/LA  ——> 0
e_9     SF/AT  ——> 200
e_10    SF/HN  ——> 100
e_11    SF/LA  ——> 200
e_12    SF/CH  ——> 200
e_13    AT/NY  ——> 150
e_14    AT/T   ——> 0
e_15    AT/CH  ——> 0
e_16    AT/HN  ——> 0
e_17    HN/AT  ——> 0
e_18    HN/NY  ——> 0
e_19    HN/CH  ——> 0
e_20    HN/T   ——> 0
e_21    HN/LA  ——> 0
e_22    NY/T   ——> 0
e_23    CH/T   ——> 0
e_24    LA/T   ——> 0
```

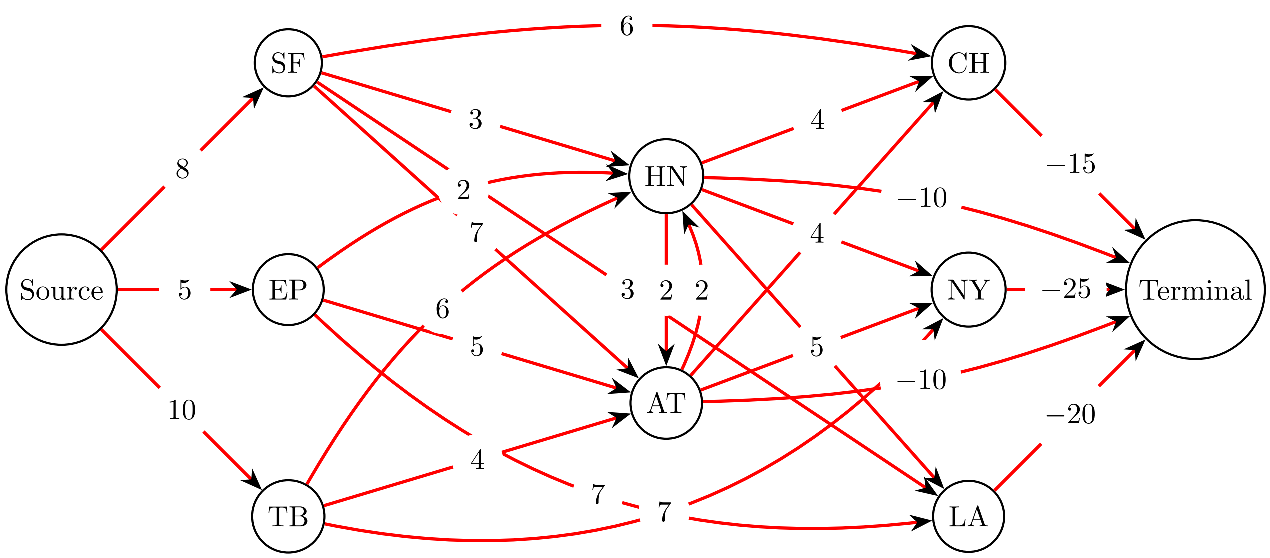
1.6 Woah, these ducks sell for money??

We are now going to take some interest in the actual profits of such a rubber duckie company. Up until this point we have only considered the costs of shipping, but now we actually care to know exactly how much money one could make by selling these as we ship them.

What this problem inevitably equates to, is a change in our graph edge values. Previously, we had all nodes connected to the terminal node equal to zero, but now those nodes will have a *negative* value associated with them. The negative is simply because we have associated costs with positive values, so in order to not have to sign switch our entire setup, we can simply make profit negative. This will mean we want to minimize this value, in order to maiximize profits.

We also have costs associated with with the production of ducks, which can be represented as costs associated with the edges connecting to source nodes.

With this in mind, we can now see that *mostly* all that has to actually change, are the values of our objective function. I'll redraw the graph below to show exactly what I mean when talking about the new values of source and terminal edges.



We can see now that our program looks a little bit different, with our objective function no longer having any zeros as it did before. Another value we will want to change is actually withing our `b` vector, where we want to change the value associated with the demand of each node. Previously we had the deman of EP, SF, and TB as being negative values, but now we will make them zero. This is because, in order for a price to be associated with the production of the ducks, we need to have them traverse the edges connected to the source node. So now, we give the source node a demand of -1100. The Terminal node value we can keep the same, as it will still only get the number of ducks entered into the system.

It is worth noting, that technically we could set the -1100 value to anything, but, considering the demand of our cities is exactly 1100 ducks, sending any more than 1100 ducks would reusult in a higher cost as those ducks wouldn't actuall by sold.

With all of these factors in mind, we get a linear program that looks like the following...

```
c_prof = np.array([10, 5, 8, 7, 4, 6, 5, 2, 7, 7, 3, 3, 6, 5, -10,
                   4, 2, 2, 6, 4, -10, 5, -25, -15, -20])
b_prof = np.array([-1100, 0, 0, 0, 150, 300, 250, 200, 200, 0])

res_prof = linprog((-1) * c_prof, A_eq = A, b_eq = b_prof, A_ub = Aub, b_ub = bub)
```

Note, we are running this program with the same values for the A matrix, as the overal conservation of the nodes should remain the same. We are also running it with the same `bub` as that should not change for this problem either. This gives us the following results:

```
fun:
  -18249.999931370083
e_0     S/TB    ----> 600
e_1     S/EP    ----> 100
e_2     S/SF    ----> 400
e_3     TB/NY   ----> 200
e_4     TB/AT   ----> 200
e_5     TB/HN   ----> 200
e_6     EP/AT   ----> 100
e_7     EP/HN   ----> 0
e_8     EP/LA   ----> 0
e_9     SF/AT   ----> 200
e_10    SF/HN   ----> 200
e_11    SF/LA   ----> 0
e_12    SF/CH   ----> 0
e_13    AT/NY   ----> 50
e_14    AT/T    ----> 0
e_15    AT/CH   ----> 200
e_16    AT/HN   ----> 200
e_17    HN/AT   ----> 100
e_18    HN/NY   ----> 0
e_19    HN/CH   ----> 0
e_20    HN/T    ----> 0
e_21    HN/LA   ----> 200
e_22    NY/T    ----> 0
e_23    CH/T    ----> 0
e_24    LA/T    ----> 0
```

So, there are many things to note about this output, the most important of which, is that it is almost certainly completely incorrect. We can notice that the edges leading to the terminal nodes have *no ducks traversing them.* This is an immediate red flag, as our program relies on ducks traversing this in order to actually produce a profit. I tried many different things to get this to work, including changing values for the matrices involved, but I couldn't get the program to work as hoped. We can see, however, that ducks do in fact traverse the source node edges, which is as expected.

We can also note the negative function value. This, I though, was good at first, before I realized that I was actually maximizing this problem as seen by the negative one. So a negative value means something else is going wrong here, or at least, certainly a negative value of this magnitude is indicative of something going wrong here.

If we look at the production of various cities changes as compared to the previous examples. Namely TB produces less, as with EP, but SF produces more than previous examples.

So, unfortunately this part of the problem was a bit of a failure, but hopefully I can bug fix it in the future, because I really do want to know what this problem would result in!

As a side note, thank you so much for letting me take some extra time. I was able to sleep a little easier (not terribly well however) knowing that I had a little extra time to finish the report.

fin