Midterm 2

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MATH 87 Math Modeling

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1 Introduction

This is the midterm for Math Modelling (MATH 087) where we explore the aspects of the page rank system in a couple of smaller models, as well as some large examples.

Google uses an algorithm called PageRank to rank web-pages by importance. The PageRank algorithm was originally formulated using a Markov Chain. Because of this, we can simulate it without a terrible amount of difficulty using our knoledge of Markov Chains.

We know Markov chains can be represented by directed graphs with weight over edges connecting nodes. Thus we can write an adjacency matrix that shows which nodes are connected to which. We then assume that there is an equal probability of moving from one node, to any of the next nodes. So the weight over the edges from one node, to three nodes, will be $\frac{1}{3}$ each.

We know this is meant to represent a web experience, thus we know that at any point the user can stop clicking links (switching between nodes) so we can have a sink node where other nodes go to. From the sink node, we can consider that the user can then go to any other node in the graph with equal probability. Thus, if our adjacency matrix contains a sink (ie a column of zeros) then we can replace it with a column of ones.

2 The Problems

1. **A brief bit of math**

   Given an adjacently matrix A, complete the following description of the i,j entry of the corresponding PageRank transition matrix C for the probability p.

   (a) If the j-th column of the matrix A is equal to $\mathbf{0}$, then for each i = 0,..., n−1: what is $C_{i,j}$?

   We know that $C = (1-p)T_1 + \frac{p}{n}\mathbf{J_n}$ for $\mathbf{J_n}$ being an $n \times n$ matrix of ones. If the j-th column of A is all zeros, then we can clearly see that T_1 will have a column of ones, thus we should anticipate our C matrix to have values $\frac{1}{n}$ in $C_{i,j}$.

   (b) Suppose now the sum of the entires of the j-th column of the matrix A is equal to $s > 0$, then...

   If $A_{i,j} = 0$, then $C_{i,j} = \frac{p}{n}$

If $A_{i,j} = 1$, then $C_{i,j} = \frac{n-(L-n)}{Ln}$

We get this just from expanding out the expression for C and generalizing.

The above will be useful to us when going to write the code for the next problem.

2. **Transition Matrices**

For this problem, we wish to develop a python function to create a transition matrix from an adjacenty matrix. For this, we need to keep in mind the things we solved in part one. Namely how the matrices react to having columns of zeros or zeros in the A matrix.

```python
## make_transition is a function that takes in an adjacency matrix and
## produces a transition matrix under a given sets of rules and using a
## probability damping factor in accordance with the pagerank system
def make_transition(A, p):
    (n, m) = A.shape
    big_c = np.zeros((n,n))

    # check if square
    if n == m:
        # get number of ones in a column
        for i in range(n):
            col = A[:, i]
            c = 0

            #count number of ones in a column
            for j in range(n):
                if col[j] == 1:
                    c = c + 1

            # a fix for if the column is of zeros, replaces with one
            if c == 0:
                col = np.ones(n)
                c = n

            # populate the big_C matrix with the correct values given
            # part 1
            for k in range(n):
                if col[k] == 1:
                    big_c[k][i] = (n + (c - n) * p)/(c * n)
                else:
                    big_c[k][i] = p/n

    # return the final array
    return big_c
```

Listing 1: make_transition

We can test this against any number of matrices, but testing against the test sample given in the assignment. Doing so gives the following output...

```python
print("Transition Matrix:")
print(P, "\n")

print("Adjacency Matrix:")
print(make_transition(P, .8))
```

Listing 2: testing make_transition

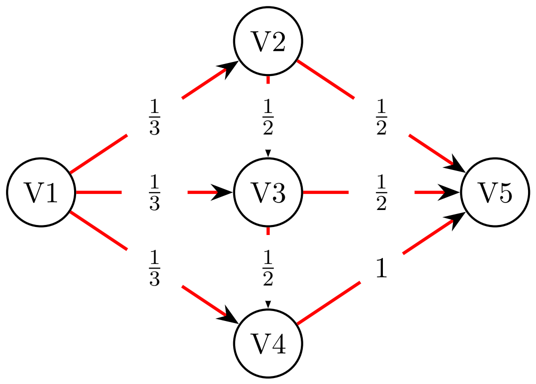
```
[output]
            Transition Matrix:
            [[0 0 0 0 1]
             [1 0 0 0 1]
             [1 1 0 0 1]
             [1 0 1 0 1]
             [0 1 0 1 1]]

            Adjacency Matrix:
            [[0.16000 0.16000 0.16000 0.16000 0.20000]
             [0.22667 0.16000 0.16000 0.16000 0.20000]
             [0.22667 0.26000 0.16000 0.16000 0.20000]
             [0.22667 0.16000 0.36000 0.16000 0.20000]
             [0.16000 0.26000 0.16000 0.36000 0.20000]]
```

So we can clearly see that it is operating as expected given the output. This is good. We can now take any adjacency matrix and convert it into a transition matrix.

3. **Eigenvector Analysis**

We now want to apply some of these tools and ideas that we've built up here. We know we can make a transition matrix out of an adjacency matrix in a way that has an associated dampening factor such that we have an approximate PageRank setup.

We can do this setup for the following di-graph.



This has the same adjacency matrix as the example used above in the code. I have marked the probabilities associated with each edge here, and it is worth noting that this is not the graph representing the transition matrix with the column of ones included to account for the sink node. That would be represented by node five connecting to every node in the graph with a weight of $\frac{1}{5}$

We now wish to determine the 1-eigenvector of the system. This vector, normalized such that it is a probability vector, will give us the long-term probability that a given user will be on a given site. We can utilize the `numpy.linalg.eig()` function to determine the eigenvalues of our given matrix. From there, we can normalize the vector, and this will be a probability vector which we can then interpret.

```python
        ## b is associated with p=0.8 and c with p=0.4
        e_val_b, e_vec_b = np.linalg.eig(b)
        e_val_c, e_vec_c = np.linalg.eig(c)

        ## determine which eigenvalue is of value one
        print("eigen val for first entry in e_val_b:  ", e_val_b[0])
        print("eigen val for first entry in e_val_c:  ", e_val_c[0], "\n")

        ## name the vector of interest as the one eigenvector
        ovb = e_vec_b[:,0]
        print(ovb)
        ovc = e_vec_c[:,0]
        print(ovc)

```

Listing 3: Finding eigenvals and eigenvectors

We associate `ovb` with the eigenvector of the `p=.8` and `ovc` with `p=.4`. The output of this is then...

```
[output]
    eigen val for first entry in e_val_b:   (1+0j)
    eigen val for first entry in e_val_c:   (1.0000000000000004+0j)

    ovb:   [-0.3759169 +0.j -0.40097802+0.j -0.44107583+0.j
             -0.48919319+0.j -0.51385334+0.j]
    ovc:   [0.24682597+0.j 0.29619116+0.j 0.38504851+0.j
             0.52722026+0.j 0.65201547+0.j]
```

We can normalize these eigenvectors to get the following. I wrote a small piece of code that outputs information of interest relating to the normalized vectors, so i'll just print the output from that.

```
[output p=.8]
    NORMALISED VECTOR:
    [0.16925438-0.j
     0.180538   -0.j
     0.1985918  -0.j
     0.22025636-0.j
     0.23135945-0.j]

    NORMALISATION TEST:
    (1+0j)

    FORMATTED VECTOR:
    (0.1692543780465788-0j)
    (0.18053800324968394-0j)
    (0.19859180357465242-0j)
    (0.22025636396461457-0j)
    (0.23135945116447024-0j)
```

```
[output p=.4]
    NORMALISED VECTOR:
    [0.11712894+0.j
     0.14055472+0.j
     0.18272114+0.j
     0.25018741+0.j
     0.3094078  +0.j]

    NORMALISATION TEST:
    (1+0j)

    FORMATTED VECTOR:
    (0.11712893553223379+0j)
    (0.14055472263868063+0j)
    (0.1827211394302849+0j)
    (0.25018740629685166+0j)
    (0.309407796101949+0j)
```

We can do a power iteration of our adjacency matrix to see if the values line up, and we see, when running the power iteration 200 times...

```python
        aa = (np.linalg.matrix_power(a,200))
        cc = (np.linalg.matrix_power(c,200))
        print("**aa:\n", aa[:,0])
        print("**cc:\n", cc[:,0])
```

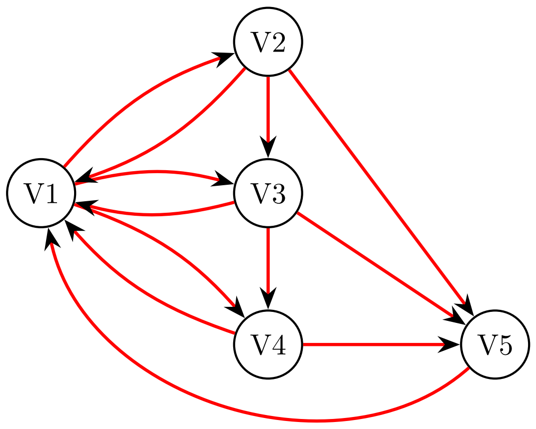
Listing 4: Power iteration

```
[output]
    **aa:
    [0.16925 0.18054 0.19859 0.22026 0.23136]
    **cc:
    [0.11713 0.14055 0.18272 0.25019 0.30941]
```

So we can see that in the end we recover the same eigenvectors regardless of power iteration or more typical eigenvector analysis. This is good! It means our matrix is stochastic and following all the rules we exepect.

4. **A New Graph**

We now look at the following diagram...



We can create an adjacency matrix for this diagram that looks as follows (note, I've indexed the nodes such that index zero is node one and so on for each row and column, in the same way that the initial matrix was created.)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We can run the same set analyses on this as we ran on the previous parts. I'll save us the hassle of writing out all the same code again and skip to the conclusions...

```
[output p=.8]
    NORMALISED VECTOR:
    [0.24840514+0.j
     0.17656034+0.j
     0.18833103+0.j
     0.19539345+0.j
     0.19131003+0.j]

    NORMALISATION TEST:
    (0.9999999999999999+0j)

    FORMATTED VECTOR:
    (0.248405144227276+0j)
    (0.17656034294848497+0j)
    (0.18833103247838412+0j)
    (0.1953934461963234+0j)
    (0.19131003414953138+0j)
```

```
[output p=.4]
    NORMALISED VECTOR:
    [0.31989529-0.j
     0.14397906-0.j
     0.17277487-0.j
     0.19581152-0.j
     0.16753927-0.j]

    NORMALISATION TEST:
    (1+0j)

    FORMATTED VECTOR:
    (0.31989528795811517-0j)
    (0.14397905759162308-0j)
    (0.17277486910994766-0j)
    (0.195811518324607 3-0j)
    (0.16753926701570687-0j)
```

We can note a distinct difference bewtween the analysis in question 2, vs this question. The most notable feature is that the higher values of the eigenvectors are not associated with the lower numbered nodes (nodes 1, 2, and 3) whereas in the previous example we see the higher probabilities are associated with the higher numbered nodes (nodes 3, 4, and 5).

Just by looking at the graph, this makes sense. We can see the graph is significantly more connected in this case around nodes 1-4, with the ability to go back and forth between 1 and 2, 3, and 4 as many times as you wish. Before, we could not do that, so it makes sense that on average you wouldn't expect to be on the lower numbered nodes as frequently. Now, however, there are more links to these nodes, so we would expect that in the long term, we have a higher probability of being on these nodes.

5. **The Perron-Frobenius Theorem**

For any directed graph, explain whey the corresponding `PageRank` transition matrix of `p > 0` is a stochastic matrix corresponding to a strongly connected aperiodic transition diagram. In particular, explain why the conclusion of the *Perron-Frobenius Theorem* holds for this matrix.

We start with an explanation of why the `PageRank` Transition matrix is *stochastic*.

We know the transition matrix must be stochastic, as the damping factor ensures that the sum of each of the columns of the transition matrix is exactly 1 by dividing by the total number of nodes. This is the definition of a stochastic matrix, and thus we know the transition matrix is stochastic.

We know the transition matrix is *strongly connected* as the damping factor is the probability that you stop and move to a random website in the group. Thus, at any point in the graph there is some chance that you will move to any other node in the graph, and because $p > 0$, we know that every node is connected with some probability p, thus the chain *must* be strongly connected.

Further, we know that there is some chance of moving between any nodes, including (possibly) the node you are currently on, there must be self cycles of length 1 at every node. Thus the graph is *aperiodic.*

Given that the graph is aperiodic and strongly connected, we know the Perron-Frobenius Theorem must hold, and that there is a single largest eigenvalue no greater than 1, and that all other eigenvalues have an absolute value between 1 and 0. This is exactly what we see in our `PageRank` simulations here, so this all makes sense.

6. **A somewhat larger example**

We now turn our heads to a different example. We have been given code to process a large `.json` file which contains information simulating the searchs for some animals as webpages, and we can use the `PageRanks` setup we have developed thus far, to run an analysis on this, and determine which nodes are most likely to be visited over time.

Using the code given in the assignment, we can initialize the adjacancy matrix from the `.json` file. After that, we can feed the array into our `make_transition` function, to get the transition matrix, and then do the usual eigen vector method to get the long term probabilities for each website.

The code to do all of this is given in the following.

```python
## The code to initialize the matrix A
(ll, A) = adj_from_json("data.json")

## make the transition matrix and get it's eigen values
A_a = make_transition(A, 0.8)
e_vlA, e_vcA = np.linalg.eig(A_a)

## get the 1-eigenvector and normalize it, then print info
## using a previously created function
eA = e_vcA[:,0]
e_nA = normalize(eA)
norm_inf(e_nA)
```

Listing 5: The analysis

After doing this, it becomes clear that the eigenvector has been normalized. I will be putting some of the output below, but it takes up a rather large amount of space so I will be cutting out some of the middling values in order to save space.

```
[output p=.8]
    NORMALISED VECTOR:
    [0.01353617-0.j
     0.01061748-0.j
     0.01035092-0.j
        . . .
     0.01049602-0.j
     0.01059754-0.j
     0.01032602-0.j]

    NORMALISATION TEST:
    (1+0j)

        . . .
```

So now we have our normalized eigenvector. We now need a way to figure out the top ten values in it, and which values are associated in the `ll` matrix. In order to do this, we can write a quick funciton that finds the top ten max values, by creating a copy of the array, then finding the max value within that copy, and storing it in a new list. Then, we can set that value to 0, such that we never find the same max twice. We can do this 10 times in a for loop, and then we will have our list of top ten websites ranked by `PageRank`.

The function we outlined above, looks as follows.

```python
    def top_ten(e_nA):
        ## copy array to temp array so we don't change values in eigenvector
        temp = []
        for i in range(len(e_nA)):
            temp.append(e_nA[i])

        ## array of max values
        maxes = []

        ## loop through 10 times finding max val in array, and copying
        for i in range(10):

            ## find max val and index of max val
            max = np.amax(temp)
            ind = np.where(temp == max)[0]

            ## find the associated animal in ll
            animal = ll[int(ind)]

            ## set index to zero so it doesn't find the same max twice
            temp[int(ind)] = 0

            ## put the animal in the aray maxes
            maxes.append(animal)

        print(maxes)
```

Listing 6: Finding the top ten

Calling this on our A array yields the following...

```
['Blue Whale', 'Ant', 'Donkey', 'Squirrel', 'Rook', 'Grouse',
 'Fowl', 'Gerbil', 'Carp', 'Albatross']
```

Thus we have our top ten list.

When we reduce the value of p we find that we get the *mostly* the same animals in the top ten, just in a different ordering. The reason, I suspect, for this has to do with the actual number of edges. If there are many more edges leading to the animals we found in our top ten, then we would expect that, regardless of our damping factor, you will still have a higher chance of being on those websites. The damping factor would just mean that you have a higher chance of randomly moving to one of any sites more frequently, so the lower you go, the less probably it is for that to happen.

7. **Citation Station**

Does this system work the same way for something like paper citations where the edges are citations and the nodes papers containing those citations?

I think this would work almost exactly the same. You could still have your damping factor enfore the Perron-Frobenious Theorem, and thus you still have a workable stochastic matrix. The setup is almost identical as you have your citations working as links. The p factor would just be the chance that at any given point the user stops sifting through papers and decided to move to some random paper in the stack.

One thing worth noting, though, is that, while I think this proposal would work in theory, I don't know if it is exactly the best setup. If you go looking for papers on google, I feel like this would be appropriate, but the 1-eigenvector associated with this setup would be ranking the long term average probability of a paper being read, and not necessarily sort via merit or peer review. Just because a paper has plenty of citations, doesn't necessarily make it accurate, and thus I don't know if this system would be the best for sorting things by academic merit. However, it could work just fine in the sense that one could certainly setup a system that looks like this.