Nathan Burwig
Math 87 HW 1
Due 2/02/22

---

1. (a) How much of a discount will maximize the gyms profits on this special? Model the question as a single-variable optimization problem

Note: for all calculations, I assume n = 100
We know we want to maximize the profits of the gym with this discount. Thus, we need to find out what the optimal discount would be to maximize profts. The first step to doing this is to find the profit function, in terms of the variable we wish to optimize ($k =$ \$ amount of discount, $n =$ number of people purchasing membership)
This relationship can be modelled as follows:

$$P(k) = n\,(1800 - k)\left(1 + \frac{0.15}{100}k\right) \tag{1.1}$$

From here, we want to maximize profit against our variable k, so we can differentiate and solve to see...

$$\begin{aligned} \frac{dP(k)}{dk} &= n(1.7 - 0.003k) \\ 0 &= n(1.7 - 0.003k) \\ k &= 566.667 \end{aligned}$$

Thus our optimal discount is \$566.67 leading to a total subscription price of \$1,233.00 and a net profit of \$226,166.65.

(b) Compute the sensitivity of the optimal discount and the corresponding profit to the 15% assumption.

We want to know the sensitivity of the profit relative to change in the percentage increase of subscriptions as a result of the discount. In order to find this, we will need to take our first profit formula and parameterize the 15% assumption as a variable. Doing so yields...

$$P(k, m) = n\,(1800 - k)\left(1 + \frac{m}{100}k\right)$$

We can differentiate this formula and solve for $k(m)$:

$$k(m) = \frac{50(18m - 1)}{m} \tag{1.2}$$

Now we can find the sensitivity of the profit to the discount using the following formula:

$$S(k, m) = \frac{dk(m)}{dm}\frac{m}{k(m)}$$

Doing so yields a formula which we can solve for the 15% assumption to find a sensitivity of...

$$S(k, m) = \frac{1}{18m - 1} \Rightarrow S(k, m) = 0.588$$

(c) Suppose that the special only generates a 10% increase in sales per \$100. What is the effect?

We can calculate the exact effect that would be had on the profit and the optimal discount by utilizing equation (1.2) and plugging in for $m = .10$ then calculating the profit from the optimal discount. Doing so yields...

$$k(0.10) = 400.00$$

Thus the optimal discount is \$400 meaning our total profit is \$196,000.00. So changing our assumption from .15 to .10, decreased our profit, but in total was a net profit gain from the \$180000.00 that would have been earned if not for the discount.

(d) Under what circumstances would an offer of a special cause a reduction in profit (your answer should be quantitative)?

If we want to know when the discount would result in a decrease in profit, we can just calculate where the critical point of the profit function is with respect to our parameter m given the optimal discount.

$$\begin{aligned}
\frac{dP}{dm} &= \frac{d}{dm}\left[n(1800-k)(1+\frac{m}{100}k)\right] \\
0 &= 25\left(324 - \frac{1}{m^2}\right)n \\
m &= \frac{1}{18} \approx .055
\end{aligned}$$

So when the increase in customers per \$100 discount is at approximately 5.5%, having a discount would no longer be profitable.

2. (a) Find the optimal temperature and pH level in the allowed range.

Given that we already know what function we desire to optimize, we can immediately jump into optimization. We can optimize by differentiating with respect to each variable and setting each equation to zero. This will give us a system of equations which we can solve to find our optimal value. I will hold off on considering the bounds of this function, until after an initial optimal value has been found. A boundary analysis can be completed if the optimal value is not within bounds.
We start be differentiating the following function...

$$F(H,T) = -0.038T^2 - 0.223TH - 10.982H^2 + 7.112T + 60.912H - 328.896 \tag{2.1}$$

Differentiating with respect to both variables yields...

$$\begin{aligned}
\frac{\partial F}{\partial T} &= -0.076T - 0.223H + 7.112 \\
\frac{\partial F}{\partial H} &= -0.223T - 21.964H + 60.912
\end{aligned}$$

This system can be arranged into a matrix equation that looks like the following:

$$\begin{bmatrix} 0.076 & 0.223 \\ 0.223 & 21.964 \end{bmatrix} \begin{bmatrix} T \\ H \end{bmatrix} = \begin{bmatrix} 7.112 \\ 60.912 \end{bmatrix}$$

This system can be solved (chose to do so using numpy arrays) and doing to yeilds an optimal T and H value of $T = 88.065$, $H = 1.879$.

Notably, these values are completely within the bounds given by the problem, so further analysis of optimal values is not required.

(b) Use matplotlib to produce a graph and a contour plot of the percentage of the powder function F (H, T).

We can use matplotlib to plot our graph and get a couple angles like so:

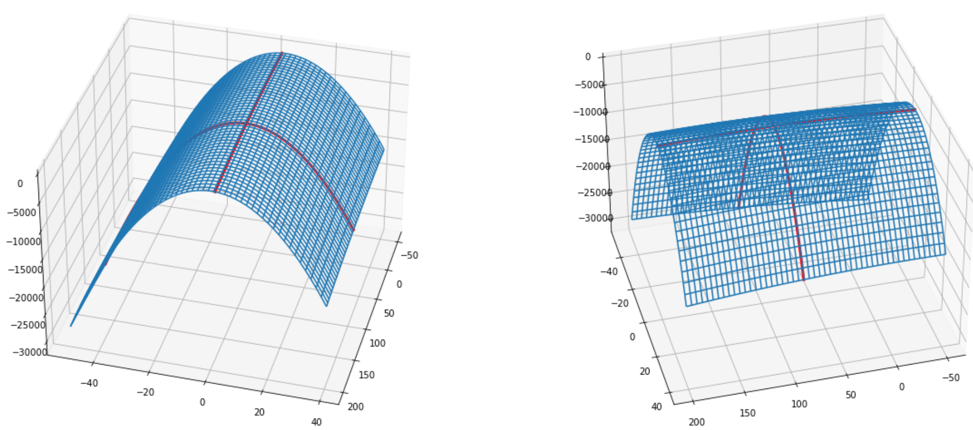

Figure 1: A couple of plots of $F(H, T)$ to give an idea of the shape and maximum point

Now we can include a contour plot, which is helpful, in this case, for verifying our optimal value is in fact in the correct area.

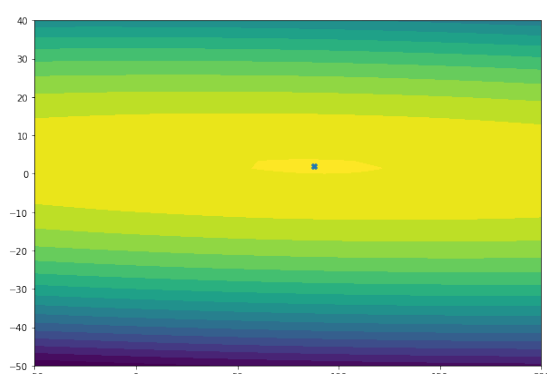

Figure 2: A contour plot with the corresponding maximal value marked by the blue x

3. (a) The Hardy-Weinberg principle states that p, q, and r are fixed from generation to generation, as are the frequencies of the different genotypes. Under this assumption, what is the probability that an individual has genotype AA? BB? OO? What is the probability of an individual having two different genes?i

Given the Hardy-Weinberg principle, we know that the total poplution can be represented in as $(p+q+r)^2$, which means that the total probability of an individual having two different genes can be represented in any of the following ways:

$$\frac{2pq}{(p+q+r)^2} = \frac{2pr}{(p+q+r)^2} = \frac{2qr}{(p+q+r)^2}$$

And the probability of having either AA, BB, or OO can be given as...

$$\frac{p^2}{(p+q+r)^2} = \frac{q^2}{(p+q+r)^2} = \frac{r^2}{(p+q+r)^2}$$

(b) Find the maximum percentage of the population that can have two different genes under the Hardy-Weinberg principle in two different ways, by directly maximizing a function of only two variables and by using the method of Lagrange multipliers.

We will first show how to solve this using the normal mehtod of optimization...
We know:

$$F(p,q,r) = 2pq + 2pr + 2qr$$

But also that, due to the Hardy-Weinberg principle, $r = 1 - p - q$, thus...

$$F(p,q) = 2pq + 2p(1-p-q) + 2q(1-p-q)$$

We can then differentiate in terms of p and q to get a system of equations which we can then solve.

$$\frac{\partial F}{\partial q} = -4q - 2p + 2$$
$$\frac{\partial F}{\partial p} = -2q - 4p + 2$$

Setting the above system equal to zero results in a solvable system of equations. The solutions to which are $p = 0.333$ and $q = 0.333$, which, when plugged back into $p+r+q = 1$ means that $r = 0.333$ also.
We can now solve by method of Lagrange multipliers. The first step will be to set up our expression in terms of Lagrange multipliers.

$$\begin{aligned} F(p,q,r,\lambda) &= f(p,q,r) - \lambda(c - g(p,q,r)) \\ &= 2(pq + pr + qr) - \lambda(1 - p - q - r) \end{aligned}$$

We can then differentiate this with respect to p, q, r, and λ...

$$\frac{\partial F}{\partial p} = 2q + 2r + \lambda, \quad \frac{\partial F}{\partial q} = 2p + 2r + \lambda$$
$$\frac{\partial F}{\partial r} = 2p + 2q + \lambda, \quad \frac{\partial F}{\partial \lambda} = p + q + r - 1$$

From here, we set all of these equations to 0 and solve the system as a matrix equation (again, done through numpy). The result is the following vector $\hat{p}$.

$$\begin{bmatrix} 0.333 \\ 0.333 \\ 0.333 \\ -1.333 \end{bmatrix}$$

Which, from top to bottom, are the values of p, q, r, and λ.

We can see immediately, that this result is in agreement with our previous result using an alternative method.

(c) Can you say what the Lagrange multiplier represents in the above example?

To be completely honest, I'm not entirely sure I understand what the Lagrange multiplier represents exactly in this context. I know that the lagrange multiplier, is equal to the "shadow price" which means that formally λ in this case is rate of change of the probability relative to our constraing (the Hardy-Weinberg Principle).

What that means in this context, though, I'm not exactly sure. I suppose it could indicate the expected change in the optimal percentages if there were a fourth blood type, but given the infinitesimal nature of the value of $\frac{\partial F}{\partial c}$ I don't entirely understand how that would work.