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Math 87 HW 4
Due 10/7/2022

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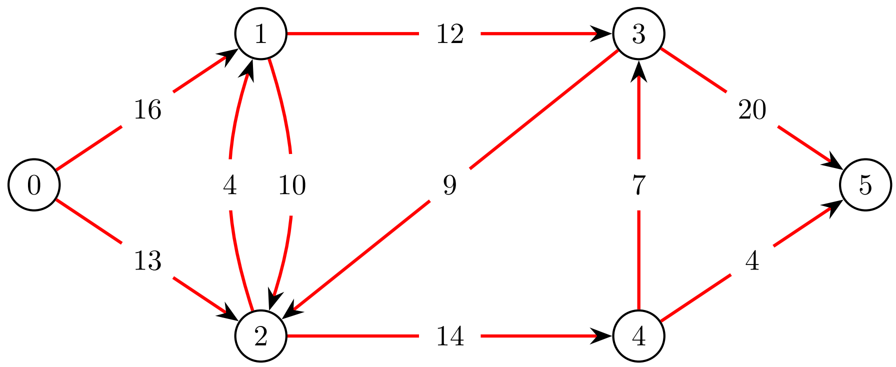
1. Compute the max possible flow from source (node 0) to sink (node 5) for the following graph. Also, identify the minimum cut.



We would like to know the max flow of this diagram from source to terminal node. We can start this by naming our edges and equally listing out our inequality constrainings. These are the values on the edges themselves, and will be listed out as the values over the edges being maximum values for the edge.

$$
\begin{aligned}
e_0 &\leq 16\\
e_1 &\leq 13\\
e_2 &\leq 10\\
e_3 &\leq 4\\
e_4 &\leq 12\\
e_5 &\leq 9\\
e_6 &\leq 14\\
e_7 &\leq 7\\
e_8 &\leq 4\\
e_9 &\leq 20
\end{aligned}
$$

Inequality constraint

We can see the inequality constraints off to the left here, and these will be neatly put into a vector called `b`. We will see shortly, that we need to add a few zeroes to that vector when we go to take the dual linear program. For now these equality constraints are fine.

We also have certain equality constraints that are based on the conservation rules of the graph above. We know each node will have certain edges going in and certain edges going out. We can write conservation equations for them. Notably, we will not be writing conservation equations for source or terminal, as they either destroy or create values that go over the edges, thus they don't conserve values. These conservation equations go as follows:

$$
\begin{aligned}
1)&\quad 0 = e_0 + e_2 - e_3 - e_4\\
2)&\quad 0 = e_1 - e_2 + e_3 + e_5 - e_6\\
3)&\quad 0 = e_5 + e_6 + e_8 - e_9\\
4)&\quad 0 = e_6 - e_7 - e_8
\end{aligned}
$$

From here, we can begin setting up the A matrix, using our equality constraints, as well as our inequality constraints. The equality represented by the node conservation rules, and the inequality by the inequalities shown in upper left.

We know our constraints now, and can setup a matrix that looks like the following...

$$A = \begin{bmatrix} I_{10} \\ \text{Conservation for node 1} \\ \text{Conservation for node 2} \\ \text{Conservation for node 3} \\ \text{Conservation for node 4} \end{bmatrix}$$

Lastly, we need an objective function, which we maximize over. We know that the edges connected to the source node, are the only edges that will be ”producing” whatever items will be traversing these edges. Because of this, they are the only edges that will have nonzero values in our objective function, as we are optimizing over the paths that come from the source node. Thus...

$$c = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now have all the values required to produce a linear program of the primal variety. But, we are going to take this and turn it into a dual linear program, as we also want to find the minimal cut. The dual linear program will give us both the value of the maximal flow, as well as the minimal cut. We know that a dual linear program looks something like the following:

$$b^T \cdot y \to max$$
$$A^T y \geq c^T$$

So, we can now see the general setup and can begin putting this all together.

Taking the transpose of the large A matrix results in another large matrix that I will not be printing the entirety of as it would simply take far too long and too much space. I will, however, be showing the code that sets up the A matrix, and it's general form becomes the following.

$$\begin{bmatrix} I_{10} & [\text{Conservation for 1, 2, 3, 4}]^T \end{bmatrix}$$

And thus our matrix goes from being a 14x10 to at 10x14. We then have that times our vector y which has the general form $y = \begin{bmatrix} \mathbf{d} \\ \mathbf{p} \end{bmatrix}$ where $d \in \mathbb{R}^{10}$ and $p \in \mathbb{R}^4$ all constrained against c^T.

Right, now we can keep talking setup, or we can go ahead and show some results, and a little code to further solidify what is going on here.

```python
import numpy as np
from scipy.optimize import linprog

def sbv(index,size):
    return np.array([1.0 if i == index-1 else 0.0 for i in range(size)])

# constraints | inequality | equality                |
A = np.block([[ sbv(1,10)  , np.array([ 1, 0, 0, 0]) ],
              [ sbv(2,10)  , np.array([ 0, 1, 0, 0]) ],
              [ sbv(3,10)  , np.array([1, -1, 0, 0]) ],
              [ sbv(4,10)  , np.array([ -1, 1, 0, 0]) ],
              [ sbv(5,10)  , np.array([ -1, 0, 1, 0]) ],
              [ sbv(6,10)  , np.array([ 0, 1, -1, 0]) ],
              [ sbv(7,10)  , np.array([ 0, -1, 0, 1]) ],
              [ sbv(8,10)  , np.array([ 0, 0, 1, -1]) ],
              [ sbv(9,10)  , np.array([ 0, 0, 0, -1]) ],
              [ sbv(10,10) , np.array([ 0, 0, -1, 0]) ]])

c = np.array([1, 1, 0, 0, 0, 0, 0, 0, 0, 0])

b = np.array([16, 13, 10, 4, 12, 9, 14, 7, 4, 20, 0, 0, 0, 0])
bounds = np.array(10*[(0,None)] +  4*[(None,None)])


res = linprog(b, A_ub = (-1)*A, b_ub = (-1)*c, bounds=bounds)
print(res)
```

Listing 1: The dual linear program

A couple notes about the setup here, is the use of the `bounds` parameter which, frankly is something I didn't know existed until looking at the example code. After looking it up, it appears to be the upper and lower bounds for each decision variable.

We can also note that b^T has a couple extra zeros on the end of it, which are as they account for the inequality constraints on the nodes, which are part of the $\mathbf{y}$ vector, represented in the $\mathbf{p}$ half of the vector.

Solving this results in the following output.

```
fun:  23.0
   p_0     0
   p_1     0
   p_2     0
   p_3     0
   p_4     1
   p_5     0
   p_6     0
   p_7     1
   p_8     1
   p_9     0
   d_10    1
   d_11    1
   d_12    0
   d_13    1
```

The output of linprog

The variables past p_9 are associated with the nodes, so we will simply look at the first ten. These values will lign up exactly with the edges in our graph. We know, via the min cut theorem, the values which are equal to one, are associated with the edges that are a part of the min cut. Thus, edge e_4, e_7 and e_8 are part of the min cut. Thus, the minimal sum of the two groups is the optimal value. That optimal value is 23.0 which was found when running the linear program.

I will show the graph after making the min cut, down below. We can see that this cut does in fact separate the grpah as it creates an impassable disconnect between the two sets of nodes.