Nathan Burwig
Math 87 HW 3
Due 09/28/2022

---

1. **Hope for the best...** You have a barrel which can store seven gallons of food, and you decide to fill it with rice and dried beans. You estimate that each gallon of beans will provides enough nutrition for approximately 9 days of meals, whereas each gallon of rice only provides around 5 days. Each gallon of beans costs \$12 and each gallon of rice costs \$5. You have \$60 to spend, and would like to calculate how many gallons of rice and beans to buy in order to maximize the number of days your food stores will last

   (a) Write this problem as a dual linear program

   Before writting all of this out as a dual linear programming problem, I first want to make sure I understand how it is written out as primal linear program. We know we have certain constraints that go as follows:

$$B + R \leq 7$$
$$12B + 5R \leq 60$$

   For $R$ as Gallons of Rice and $B$ as Gallons of Beans. We also know that our "cost" function, or in this case, the function that describes how many days we survive based on the gallons of either rice or beans we buy, is given as follows:

$$D(x) = 9B + 5R$$

   With this, we can define our constraints in terms of matrices in the following way:

$$A = \begin{bmatrix} 1 & 1 \\ 12 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 60 \end{bmatrix}, \quad c = \begin{bmatrix} 9 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} B \\ R \end{bmatrix}$$

   Where we are trying to maximize our objective function (c) against our constraints. Ie maximize the following:

$$c \cdot \begin{bmatrix} B \\ R \end{bmatrix} \text{ Such that } A \cdot \begin{bmatrix} B \\ R \end{bmatrix} \leq b$$

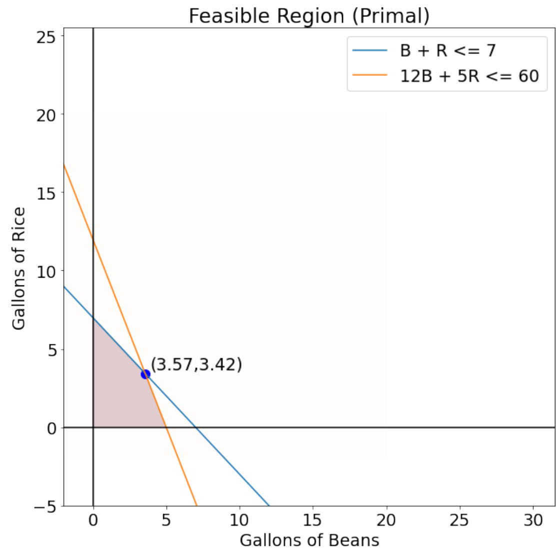
   Now that we can see how our matrices are formed for the primal linear program, we can simply take a couple of transposes to see how we want to setup the dual linear program. We know that, instead of maximizing a value for the primal linear program, in the dual linear program, we want to be minimzing our "costs" or the "prices" of our resources. So we will have a new variable of dual prices called $y$. The rest of our problem will then use the following variables.

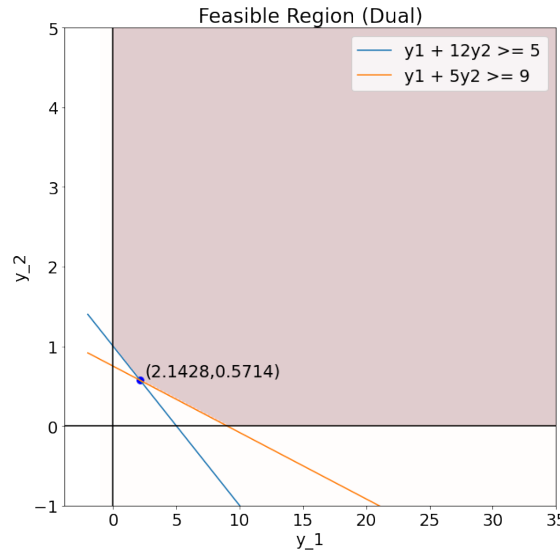
$$b^T = \begin{bmatrix} 7 & 60 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 12 \\ 1 & 5 \end{bmatrix}, \quad c^T = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

   Where now, the $c^T$ matrix represents our contraint values, $b^T \cdot y$ represents our objective function, and $A^T$ is stil our constraint matrix for given values of $c^T$. This means we are no longer maximizing the nutritional days, but instead miniming the costs associated with getting and saving food. Then the variable of our $y$ vector become $y_1$, the unit price of a gallon of food, and $y_2$ the unit price of nutritional days of food.

(b) Find solutions to primal and dual linear program by plotting their feasible spaces. Confirm strong duality and the complementary slackness theorems are satisfied. Write out the dual prices for each of our primal constraints.

We can start by plotting the primal linear program and dual linear program as follows.




We can now ask ourselves about complementary slackness and the strong duality theorems. We know that strong duality dictates the following...

$$c \cdot x^* = b^T \cdot y^*$$

Such that if the strong duality theorem holds for this optimization problem, this will be true. We can test this quickly by taking the following

$$c \cdot x^* = \begin{bmatrix} 9 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3.57 \\ 3.42 \end{bmatrix} = 49.286$$

And...

$$b^T \cdot y^* = \begin{bmatrix} 7 & 60 \end{bmatrix} \cdot \begin{bmatrix} 2.143 \\ 0.571 \end{bmatrix} = 49.286$$

So we can see that the strong duality theorem holds. We now care to test if complementary slackness upholds in this context. To test this, we can use complementary slackness theorem which states the following must be true.

$$(b - Ax)^T \cdot y = 0 = (y^T A - c) \cdot x$$

This can be tested, and by quickly plugging these into Numpy, we get the result that the complementary slackness theorem is upheld (up to computational error).

The dual price $y_1 = 2.143$ is associated with the primal constraint regarding the gallons of food (or $B + R \leq 7$) and the dual price $y_2 = 0.571$ is associated with the constraint regarding the unit price of nutritional days of food (or $12B + 5R \leq 60$).

(c) Suppose you increase the barrel size to accomodate $c$ gallons of food. Does the dual price for the modified constraint provide an accurate prediction of the increase in the primal objective function? Answer for $c = 1, 2, 4, 6$.

We know the dual price variable is given by $y^* = [\ 2.143,\ 0.571\ ]$, which, via the dual price lemma, means that we should expect that for every one gallon we increase our barrell size by, our primal objective function value should increase in approximate accordance with the value of the dual price variable. We can test this, by running the linear program against the values $7 + c$ for all given $c$. The results are given in the following table.

| Dual price predictions | | |
|---|---|---|
| 7 + c | Primal Obj. Func. Val | Difference |
| c = 0 | 49.29 | N/A |
| c = 1 | 51.42 | 2.13 |
| c = 2 | 53.57 | 2.15 |
| c = 4 | 57.85 | 4.28 |
| c = 6 | 60.00 | 2.15 |

We can see here that the dual variable is actually a rather accurate predictor of just how the primal objective function will increase. This is, of course, until the difference begins to become larger, in which case it seems to become less accurate. This is largely due to the fact that, up until the value of 13 for the gallons of food, the dual price remained the same, but after this value, it changes. I am not entirely certain why this is the case. So for now, it is simply an observed behavior.

2. You're at a yard sale and have spied four crates of goods. Crates A, B, C, and D. They are worth \$5000, \$600, \$3500, and \$6000 respectively, but you can buy them from the yard sale for \$24, \$76, \$43, \$754. You have \$800, and can carry 85 pounds. The crates weigh 75.5, 2.7, 3.3, and 6.7 pounds (A, B, C, and D). Given that there is only one of each crate, you wish to maximize your turn around profit.

   (a) Write the above as an integer linear program.
   To start, we should identify our objective function, and our constraints. We know that what we wish to maximize is our profit, which can be regarded as our revenue minus our costs (ie $P(A, B, C, D) = R(A, B, C, D) - q(A, B, C, D)$). This can be written as follows.

$$c = [\ 4976,\ 524,\ 3457,\ 5264\ ]$$

We also know that we are constrained in weight and buying power by the following relations:

$$24A + 76B + 43C + 754D \leq 800$$
$$75.5A + 2.7B + 3.3C + 6.7D \leq 85$$

We also know that for each crate, we can only by one of them. So we define a constraint matrix A as follows.

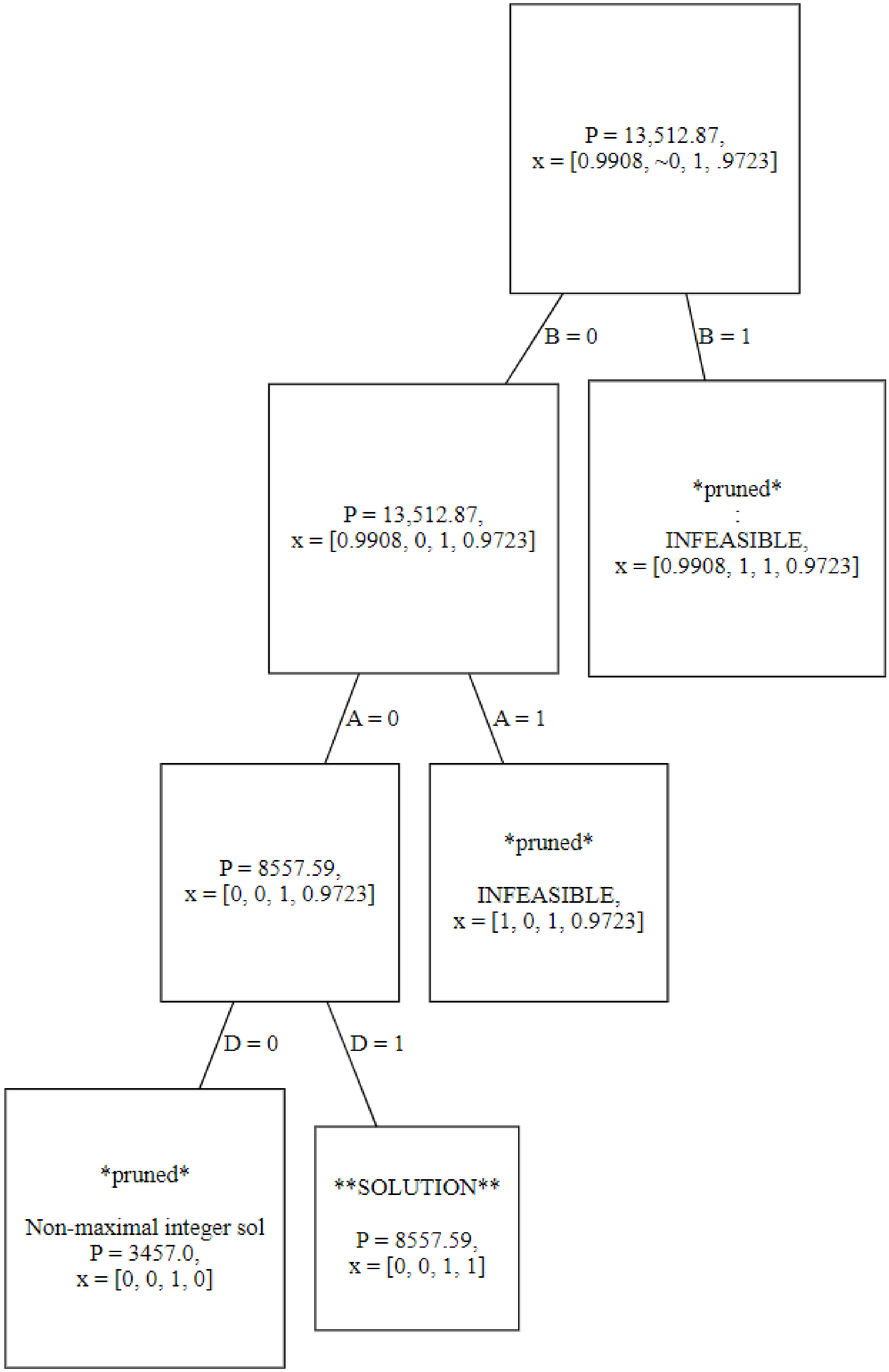
$$A = \begin{bmatrix} 24 & 76 & 43 & 754 \\ 75.5 & 2.7 & 3.3 & 6.7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 800 \\ 85 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Right off the bat, we can attempt to run this as a standard, primal linear program and we will see that we get $x^* = [\ .9908,\ 0,\ 1,\ .9723\ ]$. To make this an integer linear programming problem, we must take into account that A, B, C, and D can ONLY be 1 or 0 and no value inbetween. In this case, that means if the solution to the integer linear programming problem is $x^i$, then $x^i \geq [\ 1\ 0\ 1\ 1\ ] \longrightarrow x^i$ is infeasible. We can now use the branch and bound algorithm to find the optimal value.

(b) Execute Branch and Bound We can execute the entire branch and bound algorithm based on the constraints given and that we already have two integer values (in the actual example I work out, I treat B as a non-integer because I could not determine whether or not to account to computational error at this order of magnitude). We prune any node whose value change violated the constraints, and we prune any integer value that is not maximal. If we do the whole problem out and put the results into a graph, we get the following.

(c) Draw the branch and bound tree for the solution

I drew the tree on the following page so it could be blown up to a decent enough proportion. In the problem, I assumed that we only had one integer solution, as most of the others were only approximately 0 or 1, and thus I branched on every variable save for $D$. The final solution came out to be one of crate $C$ and one of crate $D$, which totalled \$8557.59.


P = 13,512.87,
x = [0.9908, ~0, 1, .9723]
B = 0
B = 1
P = 13,512.87,
x = [0.9908, 0, 1, 0.9723]
*pruned*
:
INFEASIBLE,
x = [0.9908, 1, 1, 0.9723]
A = 0
A = 1
P = 8557.59,
x = [0, 0, 1, 0.9723]
*pruned*
INFEASIBLE,
x = [1, 0, 1, 0.9723]
D = 0
D = 1
*pruned*
Non-maximal integer sol
P = 3457.0,
x = [0, 0, 1, 0]
**SOLUTION**
P = 8557.59,
x = [0, 0, 1, 1]