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Math 87 HW 5
Due 10/19/2022

Strolling Down an Infinite Street

Suppose you're standing on a street with buildings labelled by the integers (specifically, you're in front of the building labelled 0, and suppose that the indices are increasing to the right). Suppose that every minute you flip a coin. If the coin is heads you walk right and if the coin is tails you walk left.

1. **Explain why your position (i.e. the building you're in front of) as a function of time can be modeled as a Markov Chain**

   We know your position in this system is determined probabilistically. Ie, moving from one node to the node to the left has a 1/2 chance of happening, so our chance of being at a particular node at any time t is given by the probability of getting to the nodes we were previously on. We could make this into a stochastic matrix, but given that we are dealing with an integer line, that would make it an infinite matrix. In anycase, the values would be largely zero apart from adjacent nodes in which case the probabilities in the matrix would be 1/2.

   Since our system can be represented as a stochastic matrix, and we can draw out a finite state machine for this system, we know that it is in fact a Markov chain.

   I've also assumed that infinite markov chains are allowed, but I don't actually believe they are. At least, there doesn't seem to be any particular meaning to having an infinite Markov chain vs a very large chain. For all intents and purposes, I will assume out street ends at some point very far away from the start, such that I can avoid any infinity problems. I know the problem says infinite street I just also know that things get wonky at infinity, so in classic physicist fashion I will approximate infinity as large N :).

2. **Is the distance from where you started as a function of time a Markov Chain?**

   The distance from where we started, unlike the initial problem, is *not* an integer line, but instead a natural number line, as distance cannot be negative. Thus we have an FSM where we start at zero. After flipping a coin, we have a 100% chance of being at distance 1, and from then on, whatever node you are at, you have a 50% chance of moving either closer to or farther from zero. So you can still describe your next state, at time t + 1, as a probability from your current state, which means that this system is also a Markov chain.

3. **Now suppose that every minute you flip two coins. If both are heads, you move right, if both are tails you move left and otherwise you stay put. Is your distance from where you started a Markov chain in this scenario? How do you expect this to compare to the process described in part 2?**

   Your distance from zero would still be a Markov chain for the same reason as above. The only real difference is that you have self-cycles at each node where you can loop back onto the node you are on with a 1/2 probability. At node zero, you have a 50% chance of moving to distance 1, and from there, a 1/4 chance of moving either left or right, and 1/2 of staying put. Thus you can model this as a Markov chain.

4. For the single coin experiment, we know that at time zero we sit at node zero with 100% probability, thus there is a zero percent chance of being on an odd node. At t = 1, we are either on -1 or 1 with 100% probability, thus we are on odd with 100%. This pattern will continue, and if our time step is odd (ie $t \ \% \ 2 \neq 0$), then we are on odd with 100%, and otherwise we are on odd with 0%.

   The case of two coins is slightly more complex, but not terribly. We know that when we flip coins the first time, it is a 50 50 chance that we move at all. If we do move, we are on a an odd, which means at time step one, we are on an odd with 50%. At each time step following, we have a 50 50 chance of moving, which means that we will either remain on the odd we are on, or we will move onto an even, or vice versa. Thus at **every** timestep *apart from time* $t = 0$ we have a 50% chance of being on an odd.

   Just for fun, I went ahead and simulated 100 million people walking in this way, and was able to confirm that the probabilities above are in fact accurate.

Rain or Shine

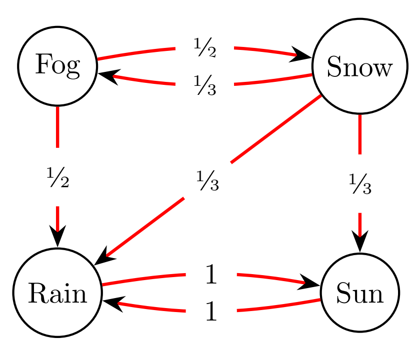
On Planet X, the weather is strangely predictable: The weather is always either sunny, rainy, foggy or snowy. If it rains today, its sunny tomorrow. If it is sunny today, its rainy tomorrow. If its foggy today, its not sunny tomorrow. Finally, the weather is never the same two days in a row. Apart from these rules, the weather is completely random, in that if e.g. its foggy today it is equally likely to be either rainy or snowy tomorrow. You live on Planet X and are trying to figure out what to wear this week, so you'd like to develop a model for the weather.

1. **Explain why the weather can be modeled as a Markov chain. Write out the transition matrix, and draw the corresponding finite state machine**

   We can model the weather as a markov chain because we know the probabilities of going from a given state to another state. Our next state is probabilistically determined from our current state, and all the sates are possible to arrive at (depending on where you start I suppose). It has the following transition matrix.

$$S = \begin{bmatrix} 0 & 1 & 1/2 & 1/3 \\ 1 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

   Which we arrive at based on the following finite state machine...



2. **Check whether the conditions for the Perron-Frobenius theorem is satisfied for this problem (aperiodic and strongly connected). Explain your reasoning.**

   We know that the Perron-Frobenius theorem is satisfied if the transition diagram for a Markov Chain is strongly connected and aperiodic. We can note, that if we are at a rainy day, or a sunny day, there is no possible way we can arrive at a snowy or foggy day, thus the graph is not strongly connected as there is not a sequence of edges that can take us from Rainy to Foggy. Thus, the Perron-Frobenius theorem is not satisfied for this Markov Chain.

3. **Do you expect power iteration to be effective for computing the greatest eigenvector of your transition matrix?**

   I do not expect power iteration to be an effective method for computing the greatest eigenvector of the transition matrix. If we use power iteration, at some point the terms that are less than 1 will go to zero, and we will be left with two terms that are just 1, which will not help us determine the greatest eigenvector as there will be two with the same value.

4. **Find the eigenvalue decomposition for the transition matrix, and the associated eigenvectors. Explain why these values confirm your answer to part 2.**

   We can find the eigenvalue decomposition of our stochastic matrix by using wolfram! Because I find it easier than doing it using python. We find the following values for our eigenvalues...

   $$\lambda_1 = -1 \quad \lambda_2 = 1 \quad \lambda_3 = -\frac{1}{\sqrt{6}} \quad \lambda_4 = \frac{1}{\sqrt{6}}$$

   With corresponding eigenvectors...

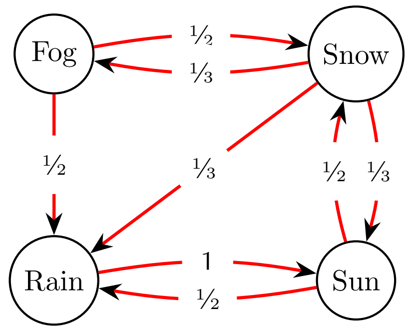
   $$v_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}, \;\; v_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

   $$v_3 = \begin{bmatrix} -.4367 & .2532 & -.8165 & 1 \end{bmatrix}, \;\; v_4 = \begin{bmatrix} -.7640 & -1.053 & 0.8165 & 1 \end{bmatrix}$$

   We can immediately see that the eigenvalues of this transition matrix show our response to part 2 was, in fact, correct. We know that Perron-Frobenius dictates that a Markov chain satisfying the theorem's conditions will only have one eigenvalue with value $|\lambda| = 1$ and all other eigenvalues whose absolute values are less than one. This clearly isn't true for the case of λ_1 and λ_2.

5. **Suppose that the "weather rules" change so that if its sunny today, it is equally likely to be snowy or rainy tomorrow. Write out the new transition matrix, associated finite state machine, and determine whether the conditions for the Perron-Frobenius are satisfied. Compute the eigenvalue decomposition and compare to the previous set of eigenvalues.**

   We can remodel our state machine to reflect the change in the rules. It will look as follows...



And our transition matrix will also reflect these changes in the following ways...

$$S' = \begin{bmatrix} 0 & 1/2 & 1/2 & 1/3 \\ 1 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

Now we ask if the Perron-Frobenius theorem is satisfied. We can see know that the graph is strongly connected, as you can devise a path from one node to any other node.

We do have to determine whether or not this graph is aperiodic. We can see that it must be, as no integer greater than 1 can divide the longest cycle on the graph, so certainly no integer greater than one can divide any smaller length paths. The longest path on this graph does not exceed four, and is only ever a fraction, and never a whole integer greater than 1. Thus no integer greater than one could divide it, and it is aperiodic and strongly connected, satisfying Perron-Frobenius.

We now expect to see only one eigenvalue with value 1 and the rest all being less than one. Let us see.

$$\lambda_1' = 1 \quad \lambda_2' \approx -.789 \quad \lambda_3' \approx -.211 \quad \lambda_4' = 0$$

So we can see that the eigenvalues for this new problem, are a bit different from the original and, as expected, the largest value of the eigenvalues is λ_1' which is exactly 1, all the rest having abosolute values less than 1. So we can see here how Perron-Frobenius plays out.