Varying Actions in String Frame and Einstein Frame to Compare EOM and Phase Space Diagrams

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In this section of the paper we will aim to recover the string frame equations of motion from Scott Watson's papers on string cosmology.

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1 Varying The String Frame Action

In this section we will attempt to derive the string frame equations of motion from a general action and in doing so we will derive the Hamiltonian constraint (ie one of the Friendmann Equations). We will use these equations of motion to formulate a phase space for the dilaton and string frame Hubble parameter, wherupon we will consider the same action in the Einstein frame. A phase space in the Einstein frame for the actual dilaton and the actual hubble parameter will be produced, and interpretations on early universe (ie Hagerdorn and possibly post hagerdorn) phase dynamics will be offered.

We will start with the string frame action in it's full form before considering reductions and simplifications of the action.

1.1 The Full Action

The typical action considered in string cosmologies (as is done in [**?**] and [**?**]) is a general N-dimensional action which allows us to include any general number of extensive dimensions (typically here we will consider d=3).

$$S = \frac{1}{16\pi G_n} \int d^n x \sqrt{-g} e^{-2\phi} \left(R + 4g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) + \int d^n x \mathcal{L}_m \tag{1}$$

As is typical of the literature, the three form Einstein-Maxwell tensor H will not be important for this discussion and thus will be ignored from the action. Considerations of the matter Lagrangian will in fact be important, however alternatively it may be excluded from explicit calculation in favor of simply asserting a form of the energy momentum tensor which will also be explored to some extent.

If we ignore the three form tensor H, then we are left with the much simpler action

$$S = \frac{1}{2\kappa_N} \int d^{N+1}x \, \sqrt{-g} \, e^{-\phi_s} \left(R + (\partial\phi_s)^2 - \mathcal{L}_m \right) \tag{2}$$

Which we can vary with respect to the string frame metric g in a typical fashion. The nontrivial coupling with the scalar field ϕ_s, however it is not disimilar to a Brans-Dicke action and thus it's variation is well known to some extent.

We consider the action in three parts, the variation of the Ricci scalar which we know results in the typical field equations, then the variation of the scalar field derivatives and matter Lagrangian with the determinant of the metric. Considerations of the shape of the metric will be important, as well as the specific form of the Ricci scalar as we can simply take the timelike component of the action to determine the equations of motion. The pure timelike component will result in the equivalent of the Friedmann equation in the string frame (ie the Hamiltonian constraint) while the equations of motion for ϕ_s and λ_s (where $\lambda_s = \ln(a)$) will not generically depend on the variation of the action with respect to the metric and thus it doesn't matter if we consider the purely timelike term or the whole action.

For what seems to be historical reasons it is typical to start with the anisotropic FLRW metric

$$ds^2 = -n(t)^2 dt^2 + \sum_{i=1}^{N} e^{2\lambda_s^{(i)}(t)} dx_i^2 \tag{3}$$

And then consider specializations to isometric metrics (ie all $\lambda_s^(i)$ in the metric are identical) and with $a_s^{(i)} = e^{\lambda_s^{(i)}(t)}$. Then we can consider a new form of the metric by determining the form of the Ricci scalar.