No Hair Theorems in Scalar Tensor Theories of Gravity

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This paper aims to reiterate the work conducted over the last semester of my research rotation with Dr. Damien Easson. In general, this covers a broad spectrum of generally beginner calculations which were utliized to rampup my individual understanding. Later a discussion of scalar-tensor theories and their applications will occur, wherupon an analysis of a simple case of the no-hair theorem for a scalar tensor theory will be conducted.

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1 Introduction

Our discussion is going to start at a fairly basic point by first reiterating the results of generic classical field theory. This discussion will be brief but necessary as these results will be utilized in later sections. An example case for a simple scalar field in will be conducted wherupon we will turn to considering field theory as it applies to gravitation. The core ideas behind varying actions with respect to the metric will be explored, and Einstein's equations will be recovered via varying the Einstein Hilbert action. The ideas will then turn to discussing scalar field couplings to gravity where we may consider no-hair theorems and possible configurations which could produce scalar hair on a black hole.

This discussion will be fairly rudimentary at higher levels, however it serves to display large portions of what I have personally learned over the last semester and how this knowledge may be applied in research regarding scalar tensor theories of gravity which constitutes a not insignificant fraction of modern research.

It is also well worth noting that the following introductory sections will, to a certain extent, mimic that of Carroll's treatment of classical field theory in section 1.10 of his book [1]. This is simply due to the fact that this was the source from which I learned these concepts originally. For a more complete treatment of these concepts see section 1.10 of [1].

1.1 Classical Field Theory

In this section the basics of classical field theory are covered which will aid us later in our discussion of varying the Hilbert-Einstein action in section 2.1.

Classical field theory takes the ideas surrounding basic Lagrangian mechanics and allows us to generalize them to fields via Lagrange densities. The normal story for Lagrangians in elementary mechanics goes as follows.

Given the simple case of a particle in one dimension we can derive the equations of motion in terms of generalized coordinates $(q(t), q(\dot{t}))$ for that particle by making use of action principles. In general we can define an action S which takes a function as an argument and returns a scalar. For our purposes we consider the generic action S.

$$S = \int \mathcal{L}\left(q(t), \dot{q}(t)\right) dt$$

Hamilton's principle tells us that the path taken by a particle is one for which the action is stationary (ie minimized or possibly maximized). Generally, though, as opposed to varying the action for a given Lagrangian, one can instead simply take the Euler-Lagrange equations and determine the equivalent equations of motion.

As it turns out, these same principles can be generalized to fields which allows us a means of understanding the behavior of fields and how they change in time.

For instance, if we take a set of spacetime-dependent fields $\Phi^i(x^\mu)$, then our action becomes a functional of the fields as opposed to the generalized coordinates. This allows us to express our Lagrangian as an integral over space of some *Lagrange Density*. The Lagrange density itself is what contains the information regarding the fields Φ^i and their spacetime derivatives $\partial_\mu \Phi^i$.

$$L = \int d^3x \; \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

Which means that our action is then

$$S = \int dt \; L = \int d^4x \; \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

We then get the Euler-Lagrange equations in the same way as in mechanics, via considering that the action be unchanged under small variations of the fields. Consider

$$\Phi^i \to \Phi^i + \delta\Phi^i$$
$$\partial_\mu \Phi^i \to \partial_\mu \Phi^i + \delta\left(\partial_\mu \Phi^i\right)$$

and then also consider that δ commutes with the derivative so that $\delta\left(\partial_\mu \Phi^i\right) = \partial_\mu\left(\delta\Phi^i\right)$. Under small variations, we can expand the Lagrangian to first order which yields the following.

$$\begin{aligned}\mathcal{L}(\Phi^i, \partial_\mu \Phi^i) &\to \mathcal{L}(\Phi^i + \delta\Phi^i,\; \partial_\mu \Phi^i + \partial_\mu(\delta\Phi^i)) \\ &\to \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) + \frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu(\delta\Phi^i)\end{aligned}$$

Our action then goes to $S + \delta S$ where we can see

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu(\delta\Phi^i)\right]$$

Our desire is to express the functional derivative of the action ($\frac{\delta S}{\delta \Phi^i}$) as being equal to zero to derive the equations of motion for any general field. In order to do this we need to find a way to factor $\delta\Phi^i$ our of our expression for δS. Thus, we can take the second term of our integral and integrate by parts

$$-\int d^4x \; \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right)\delta\Phi^i + \int d^4x \; \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\delta\Phi^i\right)$$

The second term is a total derivative and thus can be considered as a boundary term so that it vanishes (ie we choose that the derivatives vanish at the boundary which we set to be infinity).

The above result leaves us with the following.

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \right) \right] \delta \Phi^i$$

Thus...

$$\frac{\delta S}{\delta \Phi^i} = \frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \right) = 0 \tag{1}$$

Which recovers the general result of classical field theory and opens up the ability to analyze a wide range of problems. This result is critical for the types of problems analyzed throughout the course of this paper and in general this process is one that will be repeated a handful of time as analyses of various actions are conducted.

1.2 Classical Scalar Field

Another result which we will find to be particularly useful throughout the rest of this paper is that of the equation of motion for a general scalar field. This calculation will also show another method for arriving at equations of motion, not by varying the action directly, but by using the result given in equation (1).

We consider a real scalar field $\phi(x^\mu)$, a function of the coordinates. The scalar field's energy density as a local function of the spacetime can be expressed in the Lorentz invariant form as follows.

$$-\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)$$

Where for now we restrict ourselves to manifestly flat spaces and thus utilize the Minkowski metric $\eta^{\mu\nu}$. Thus our Lagrangian in total is

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - V(\phi)$$

As stated previously, from here there are two options. One is to utilize principles of varying actions as done in 1.1. The method we will choose to utilize here, however, is perhaps the simpler one which is to utilize the result in equation (1).

For the ϕ term:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{dV}{d\phi}$$

For the $\partial_\mu \phi$ term:

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\mu \phi)}\left[-\frac{1}{2}\eta^{\rho\sigma}(\partial_\rho \phi)(\partial_\sigma \phi)\right] &= -\frac{1}{2}\eta^{\rho\sigma}\left[\delta^\mu_\rho(\partial_\sigma \phi) + (\partial_\rho \phi)\delta^\mu_\sigma\right] \\
&= -\frac{1}{2}\left[\eta^{\mu\sigma}(\partial_\sigma \phi) + \eta^{\rho\mu}(\partial_\rho \phi)\right] \\
&= -\frac{1}{2}\left[\partial^\mu \phi + \partial^\mu \phi\right] \\
&= -\eta^{\mu\nu}\partial_\nu \phi
\end{aligned}$$

Thus our final result is

$$\begin{aligned}
-\eta^{\mu\nu}(\partial_\mu)(\partial_\nu \phi) - \frac{dV}{d\phi} &= 0 \\
\Box\phi - \frac{dV}{d\phi} &= 0 \qquad (2)
\end{aligned}$$

2 Varying Actions

In the following passages we will turn towards considerations of gravitation. As it turns out, Gravity is also a theory of fields in a sense. General relativity relates the distribution of matter and energy to the curvature of spacetime or, more generally, how the metric tensor evolves on a manifold. Because of this, one can describe the curvature of the spacetime using field equations which is where our discussion will start. Then, considerations of alternate theories will begin wherupon scalar fields will be introduced into our action, allowing us to consider simple cases of modified gravity.

First, though, we need to arrive at the original field equations, which we will do via a similar method as before.

2.1 The Hilbert-Einstein Action

In general, it is not immediately clear where one should start when attempting to derive the field equations. If, however, you should like to consider an approach which allows you to utilize methods of varying actions then perhaps it is best to start with the Ricci scalar.

The Lagrangian is taken to be a scalar function of the fields and their derivatives. Thus, when looking for an approach to deriving Einstein's field equations via utilization of action principles, we hope to find a function which may be taken to be the Lagrangian. This scalar function is indeed the Ricci scalar and, as this method was initially posed by Hilbert, it has since been known as the Hilbert, or Hilbert-Einstein action.

$$S_H = \int d^n x \sqrt{-g} R \tag{3}$$

From here, one might hope that we could immediately begin solving for the equations of motion via the Euler Lagrange equations, however, there is a subtlety here. It requires us to first consider the fact that we are not necessarily in a flat space, so we have to update equation (1) to a covariant formulation. As it turns out it generalizes quite easily. The first aspect is the formation of the action, which now looks as such.

$$S = \int \mathcal{L}(\Phi^i, \ \nabla_\mu \Phi^i) \ d^n x$$

Where $\mathcal{L}$ is defined as a Lagrange density which is given as $\mathcal{L} = \sqrt{-g}\check{\mathcal{L}}$ where $\check{\mathcal{L}}$ is a scalar. The covariant Euler Lagrange equations become:

$$\frac{\partial \check{\mathcal{L}}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \check{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) = 0 \tag{4}$$

With this in mind, we can see that if we take $g_{\mu\nu}$ to be out dynamical variable when we consider the Hilbert action, there will be a problem. Namely, it will be written out as covariant derivatives of the metric tensor which definitionally vanish. Thus we have to vary the Hilbert action directly.

The dependence of the Ricci scalar on the metric and other functions of the metric means that our action will split into three distinct pieces.

$$(a) \int d^n x \ R \ \delta\sqrt{-g} \qquad (b) \int d^n x \ \sqrt{-g} \ R_{\mu\nu} \delta g^{\mu\nu} \qquad (c) \int d^n x \ \sqrt{-g} \ g^{\mu\nu} \delta R_{\mu\nu}$$

We know that the functional derivative is $\frac{\delta S}{\delta g^{\mu\nu}}$ so we would like to isolate the $\delta g^{\mu\nu}$ term. (b) has already done this for us, but terms (a) and (c) will need a decent bit of persuasion. We start with the easier of the two, the variation of the determinant of the metric.

2.1.1 The Determinant

For this particular piece of the variation, we can start simply by considering a direct variation of $\sqrt{-g}$. In general we know that the variation of any object can be given by it's derivate with respect to the variational parameter, times the variation of that parameter. Thus we expect that $\delta\sqrt{-g}$ might yield

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta g$$

While this may at first seem helpful, we still have only g, the determinant of the metric tensor. We would like to find a more helpful way of expressing this result. We can utilize a helpful identity to see how this can connect to a more useful representation.

$$\ln(\det(M)) = tr(\ln(m))$$
$$\delta(\det(M)) = \det(M)\; tr(M^{-1}\delta(M))$$

Applying this to the variation of the determinant we see the following.

$$-\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}\left(g\left(g^{\mu\nu}\delta g_{\mu\nu}\right)\right)$$
$$= \frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\delta g_{\mu\nu} \tag{5}$$
$$= -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu} \tag{6}$$

Where we make the connection to the last equation via a double contraction (ie $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$).

2.1.2 Boundary Conditions

Here we determine the variation of the Ricci tensor, $\delta R_{\mu\nu}$. To start, we know that the Ricci tensor has dependence on the metric as it is a contraction of the Riemann tensor which itself is calculated from the Christoffel connection components. Thus, we anticipate having to iterate down to that level in order to make a determination about the variation of the Ricci tensor. We can start with the definition of the Ricci tensor.

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

However, in general

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{7}$$

So we may consider variations the Riemann tensor under the principle that $\Gamma \to \Gamma + \delta\Gamma$. If we attempt this on (7), then we will get several variational terms of the connection components, which themselves are given as follows.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right) \tag{8}$$

Where I will be using the common "," and ";" notation to represent gradients and covariant derivatives respectively.

Varying the connection component yields the following

$$\begin{aligned}
\delta\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}\delta g^{\lambda\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right) + \frac{1}{2}g^{\lambda\sigma}\left(\delta g_{\sigma\mu,\nu} + \delta g_{\sigma\nu,\mu} - \delta g_{\mu\nu,\sigma}\right) \\
&= -\frac{1}{2}g^{\alpha\mu}g^{\nu\beta}\delta g_{\mu\nu}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right) + \frac{1}{2}g^{\lambda\sigma}\left(\delta g_{\sigma\mu,\nu} + \delta g_{\sigma\nu,\mu} - \delta g_{\mu\nu,\sigma}\right) \\
&= g^{\nu\beta}\delta g_{\mu\nu}\Gamma^{\alpha}_{\mu\nu} + \frac{1}{2}g^{\lambda\sigma}\left(\delta g_{\sigma\mu,\nu} + \delta g_{\sigma\nu,\mu} - \delta g_{\mu\nu,\sigma}\right) \qquad (9)
\end{aligned}$$

Now we are left with the variations of the gradients of the metric. We note that, while typically the covariant derivative of the metric vanishes, the covariant derivative of the *variation* of the metric does not. Thus we can utilize the covariant derivative of the metric to determine a value for the gradients in terms of connection components and $\delta g_{\mu\nu}$.

$$\begin{aligned}
\nabla_\mu\left(\delta g_{\sigma\nu}\right) &= \partial_\mu(\delta g_{\sigma\nu}) - \Gamma^{\rho}_{\ \sigma\mu}\delta g_{\rho\nu} - \Gamma^{\rho}_{\ \nu\mu}\delta g_{\sigma\rho} \\
\nabla_\nu\left(\delta g_{\sigma\nu}\right) &= \partial_\nu(\delta g_{\sigma\mu}) - \Gamma^{\rho}_{\ \sigma\nu}\delta g_{\rho\mu} - \Gamma^{\rho}_{\ \mu\nu}\delta g_{\sigma\rho} \\
\nabla_\sigma\left(\delta g_{\mu\nu}\right) &= \partial_\sigma(\delta g_{\mu\nu}) - \Gamma^{\rho}_{\ \mu\sigma}\delta g_{\rho\nu} - \Gamma^{\rho}_{\ \nu\mu}\delta g_{\mu\rho}
\end{aligned}$$

Thus we arrive at a result which we can utilize in equation (9).

$$\partial_\mu\left(\delta g_{\sigma\nu}\right) + \partial_\nu\left(\delta g_{\sigma\mu}\right) - \partial_\sigma\left(\delta g_{\mu\nu}\right) = \nabla_\mu\left(\delta g_{\sigma\nu}\right) + \nabla_\nu\left(\delta g_{\sigma\mu}\right) - \nabla_\sigma\left(\delta g_{\mu\nu}\right) + 2\Gamma^{\rho}_{\ \mu\nu}\delta g_{\sigma\rho}$$

Which comes as a result of the symmetry relationship for swapping lower indices in the connection components. The last term in the equation above will cancel with the term in (9) which means we are left with the following as the variation of the connection components.

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_\nu\left(\delta g_{\sigma\mu}\right) + \nabla_\mu\left(\delta g_{\sigma\mu}\right) - \nabla_\sigma\left(\delta g_{\mu\nu}\right)\right) \qquad (10)$$

Then if we take it back to equation (7) and simply hit it with the variation operator

$$\delta R^{\rho}_{\ \sigma\mu\nu} = \partial_\mu\delta\Gamma^{\rho}_{\ \nu\sigma} - \partial_\nu\Gamma^{\rho}_{\ \mu\sigma} + \delta\Gamma^{\rho}_{\ \mu\lambda}\Gamma^{\lambda}_{\ \nu\sigma} + \Gamma^{\rho}_{\ \mu\lambda}\delta\Gamma^{\lambda}_{\ \nu\sigma} - \delta\Gamma^{\rho}_{\ \nu\lambda}\Gamma^{\lambda}_{\ \mu\sigma} - \Gamma^{\rho}_{\ \nu\lambda}\delta\Gamma^{\lambda}_{\ \mu\sigma} \qquad (11)$$

From here it might seem difficult to pick a direction, however, we can determine an expression for $\partial_\mu\Gamma$ in terms of it's covariant derivative which may offer a simplification.

$$\begin{aligned}
\nabla_\mu\Gamma^{\rho}_{\ \nu\sigma} &= \partial_\mu\delta\Gamma^{\rho}_{\ \nu\sigma} + \Gamma^{\rho}_{\ \mu\lambda}\delta\Gamma^{\lambda}_{\ \nu\sigma} - \Gamma^{\lambda}_{\ \mu\nu}\delta\Gamma^{\rho}_{\ \lambda\sigma} - \Gamma^{\lambda}_{\ \mu\sigma}\delta\Gamma^{\rho}_{\ \nu\lambda} \\
\nabla_\nu\Gamma^{\rho}_{\ \mu\sigma} &= \partial_\nu\delta\Gamma^{\rho}_{\ \mu\sigma} + \Gamma^{\rho}_{\ \nu\lambda}\delta\Gamma^{\lambda}_{\ \mu\sigma} - \Gamma^{\lambda}_{\ \nu\mu}\delta\Gamma^{\rho}_{\ \lambda\sigma} - \Gamma^{\lambda}_{\ \nu\sigma}\delta\Gamma^{\rho}_{\ \mu\lambda}
\end{aligned}$$

The first approach may simply be to solve for $\partial_\mu\Gamma$ and substitute into equation (11), however one can note a similarity between the terms in the two expression immediately. In order to get

$\partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma}$, we would need to subtract the two covariant derivatives of the connection components, thus we simply attempt that substitution and it is immediately found that

$$\delta R^\rho{}_{\sigma\mu\nu} = \nabla_\mu \Gamma^\rho{}_{\nu\sigma} - \nabla_\nu \Gamma^\rho{}_{\mu\sigma} \tag{12}$$

Thus our term for the variation becomes

$$\int d^n x \, \sqrt{-g} \, g^{\mu\nu} \left[\nabla_\mu \Gamma^\rho{}_{\nu\sigma} - \nabla_\nu \Gamma^\rho{}_{\mu\sigma} \right]$$

Where generically we can factor out the covariant derivative so long as we are careful to mind the metric tensor and the dummy indices. Doing so leaves the integral in the form of a total derivative which *should* allow us to consider this term as vanishing as it is a boundary term. However, there is some nuance here. The boundary term may include not just the metric variation, but also derivatives of the metric variation which are not necessarily trivially zero [1]. For a more in depth discussion, see Carroll chapter 4 section 3 where he will surely direct you to Wald (1984).

2.2 The Einstein Equations

Now that we have the non-vanishing terms of the variation of the Hilbert action, we can piece these things together to see the resulting equation of motion.

$$\begin{aligned} \delta S_H &= \int d^n x \, \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] \delta g^{\mu\nu} \\ \frac{1}{\sqrt{-g}} \frac{\delta S_H}{\delta g^{\mu\nu}} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\mu} \end{aligned}$$

Thus we arrive at the Einstein equations for a vacuum.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{13}$$

3 Scalar-Tensor Theories of Gravity

A demonstration of the derivation of the Einstein equation for the vacuum has now been conducted, which opens possibilities of considering alternative theories of gravity. There are many alternative theories which have been explored in the literature, but some which have prevailed in maintaining interest over the last several years are scalar-tensor theories of gravity. These theories are so-called as they contain a scalar field coupling in the metric, and thus can change the resulting equations of motion.

For instance, one could consider minimally coupling a scalar field to the Ricci scalar in the Hilbert-Einstein action.

$$S = \int d^n x \, \sqrt{-g} \left(R - \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \tag{14}$$

Here we can see we have simply added the gradient/kinetic energy and the potential energy of a generic scalar field to the action. If we choose to vary this action to determine the equations of motion, we will find a familiar result with a few caveats.

Since the variation of this action will break into three separate pieces, we can immediately substitute our result from 2.1.2. However, now considerations of the variation of the scalar field terms must come into play. Unlike in sections 1.1 and 1.2 though, the variational parameter is the metric tensor and not the scalar field.

The first step is a simple reduction of the covariant derivatives of the scalar field to their gradients, as that is definitionally what the covariant derivative of a scalar field is.

$$\begin{aligned} S_\phi &= \int d^n x \, \left(-\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) - V(\phi) \right) \sqrt{-g} \\ &\to \int d^n x \, \frac{1}{2} \, \delta g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) - \int d^n x \, \frac{1}{2} \, g^{\mu\nu} \delta \sqrt{-g} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) - \int d^n x \, V(\phi) \delta \sqrt{-g} \end{aligned}$$

The first term is fine on it's own, the second and third terms only pickup a variation of the action through the root of its negative determinant which is readily known by equation (5) thus

$$\delta S_\phi = -\frac{1}{2} \int \nabla_\mu \phi \nabla_\nu \phi \, d^n x + \frac{1}{4} g_{\mu\nu} \delta g^{\mu\nu} \int g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - V(\phi) d^n x$$

Or equivalently.

$$T^{(\phi)}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi) \tag{15}$$

Which, after also varying the action with respect to ϕ in order to get the equation of motion for ϕ, leads us to a new set of equations of motion.

$$G_{\mu\nu} = T^{(\phi)}_{\mu\nu} \tag{16}$$

$$\Box \phi - \frac{dV}{d\phi} = 0 \tag{2}$$

Which in and of itself is not the most impressive outcome but it certainly is fascinating. We get that scalar field addition, even if only coupled minimally, will have effects on the field equations. This implies that certain alternate theories of gravity which contain scalar fields could have detectable deviations from measured astrophysical values, and thus can be effectively ruled out based on nothing but *how much* the scalar field changes the effective field equations [3].

One particular example of the usefulness of this method can be seen by considering black hole spacetimes to determine how deviations from the standard picture can arise by adding different scalar configurations. In particular, the rest of this paper will focus on how adding scalar field configurations could produce *scalar hair* in black hole spacetimes.

4 The No-hair Theorem

The no-hair theorem is perhaps one of the most poorly named theorems in all of physics. Not only is the naming convention obscure, but it also is not a theorem in the typical sense of the word. It is perhaps slightly better put as "The No-hair Theorems." This is as, there is no one size fits all determination, but simply a set of proofs for various blackhole spacetimes that all point to the same conclusion; black holes can only be characterized by three quantities (or "charges").

This is to say that while most matter in the universe is distinguishable based on any number of factors, black holes can only be characterized by their mass, spin, and charge. However, the existence of a theorem in physics guarantees only one outcome; that physicists will try quite hard to prove it wrong. This is of course not without reason, as determining if black holes could have a nontrivial scalar field configuration would imply a black hole solution that deviates from the standard picture. This deviation could then agree with predictions from any number of models and thus black holes could become optimal testing grounds for various scalar-tensor theories or otherwise [4].

For example, in [5] it is shown that scalar tensor theories which contain the Guass-Bonnet invariant nonminimally coupled to a scalar field do in fact produce scalar hair under certain energy and symmetry conditions.

Thus, the remainder paper is an exploration of a simple case of the no-hair theorem utilizing the tools built up in the previous sections. We will find that in some cases black holes can make for excellent testing grounds for alternative theories, and also that even for the simple cases, showing the no hair theorem is not necessarily a trivial proof when approached in certain ways.

4.1 The No-hair Theorem for Minimal Scalar Field Coupling

The first setting will be the most familiar, that given by our discussion section 3, the minimal scalar field coupling. The particular method chosen to conduct this proof is one that utilizes tools from sections 1 and 2.

The general methodology utilized in these sections will be to consider a spherically symmetric ansatz

$$ds^2 = -n(r)dt^2 + p(r)dr^2 + r^2 d\Omega^2 \tag{17}$$

where $d\Omega^2$ is the usual spherically symmetric component. We can substitute this into our equations of motion and consider the behavior as $r \to \infty$. This will give us some idea of the behavior of the fields at distance from the blackhole background and, so long as the scalar field picks up no nontrivial terms in the limit (ie $\phi(r) \to \phi_0$), then it can be said that there is no *scalar hair*.

For the first approach we will consider the ansatz as substituted into the equations of motion given in equations (2) and (16), sometimes called the covariant method. In the second case we will attempt to determine the equations of motion via the covariant Euler-Lagrange equations then substitute the ansatz again. In both cases we will get expressions for the equations of motion with the metric ansatz substituted into them before conducting an asymptotic analysis on the solutions to the equations of motion as $r \to \infty$.

4.1.1 Covariant method

The first step is to determine the components of tensors given in the equation of motion. In order to do this, an object oriented programming package for general relativity calculations in mathematica called OGRe was used [2]. The components of the Einstein Tensor are found to be the following:

$$\begin{aligned}
tt &: \ \frac{n(r)((p(r)-1)p(r)+rp'(r))}{r^2p(r)^2} \\
rr &: \ \frac{n(r)-n(r)p(r)+rn'(r)}{r^2n(r)} \\
\theta\theta &: \ \frac{r(-rp(r)n'(r)^2-2n(r)^2p'(r)+n(r)(-rn'(r)p'(r)+2p(r)(n'(r)+rn''(r))))}{4n(r)^2p(r)^2} \\
\phi\phi &: \ \theta\theta
\end{aligned}$$

The equations for the energy-momentum tensor can equally be determined.

$$\begin{aligned} tt &: \ \frac{n(r)\phi'(r)^2}{2p(r)} \\ rr &: \ \frac{1}{2}\phi'(r)^2 \\ \theta\theta &: \ -\frac{r^2\phi'(r)^2}{2p(r)} \\ \phi\phi &: \ -\frac{r^2\sin^2(\theta)\phi'(r)^2}{2p(r)} \end{aligned}$$

This allows a component-wise determination of the equations of motion which is in terms of the ansatz function $n(r)$ and $p(r)$.

Finally, the component for the scalar field can be determined. To start, considerations of a constant potential energy will be utilized, which causes the $V'(\phi)$ term to vanish.

$$\frac{p(r)(2\phi'' + \phi'(\frac{4}{r} + \frac{n'(r)}{n(r)})) - \phi'(r)p'(r)}{2p(r)^2} = 0$$

This offers us a wide range of equations to solve for 3 variables of interest. Initial inspection would indicate the t, r and ϕ (ϕ as in the equation of motion for the field, not the coordinate) terms would likely be the simplest equations to solve.

However, upon attempting a variety of methods to solve this set of equations, it is cautiously concluded that there there is no analytic solution available for this particular approach. Approximate solutions are certainly achievable (see [5] for a more in depth discussion of a substantially more complex problem) but go beyond the scope of our discussion. For now it will suffice to consider an alternative approach, ie considerations of the Euler-Lagrange equations for this action.

4.1.2 A second attempt

In this approach, we will attack the problem from only a slightly different position. The action given in equation (14) does allow us to consider a more simplistic approach when utilizing the ansatz equation (17). Ie, we can simply hit the Lagrangian with the Euler-Lagrange equations to determine the equations of motion.

This is technically not entirely distinct from the analysis conducted in 4.1.1 as the equations of motion should encode the same information, but it is possible that the form of the expressions will lend itself to be more easily calculated via a variety of computational methods.

Thus, we consider our starting point to be the following Lagrangian.

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) \tag{18}$$

Where we are again considering the simplest case of $V(\phi) = 0$ and utilizing the spherically symmetric ansatz of equation (17). Since we consider $\phi = \phi(r)$, we know that our Lagrangian simplifies down to the following.

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{2p(r)}\phi'(r)^2\right)$$

Using OGRe again to make quick work of the Ricci scalar, we can take the Euler-Lagrange equations for the above. The Ricci scalar is second order of functions of the ansatz, thus, after utilizing the second order Euler-Lagrange equations we land on the three following equations.

$$\begin{aligned}
&n: \ 4rp'(r) + p(r)(r^2\phi'(r)^2 + 2p(r) - 2) = 0 \\
&p: \ -2rn'(r) + n(r)(r^2\phi'(r)^2 + 2p(r) - 2) = 0 \\
&\phi: \ (rn(r)p'(r) - p(r)(4n(r) + rn'(r)))\phi'(r) - 2rn(r)p(r)\phi''(r) = 0
\end{aligned}$$

From here, simply attempting to solve this system of equations for the variables of interest should return solutions for n, p, and ϕ which we can analyze in the limit as $r \to \infty$. However, yet again we find that analytic solutions do not come easily for this particular system of equations. It seems that, if an analysis of this variety is to be conducted for the minimal scalar field, a slightly stronger assertion may be in order.

At this point it may be worth noting that, while in the confines of this paper there seems to be some issue with determining the no-hair theorem for a fairly trivial scalar field configuration, the solution is generally well known. In [3] it is shown that the minimal scalar field coupling produces no scalar hair, or that ϕ is constant at infinity. However, the method utilized is motivated not by the equations of motion (as is the case in [5]) but rather from more nuanced conditions. Thus, it is simply a matter of showing the no-hair theorem in this *particular* way which seems to require a stronger condition on the system.

4.1.3 A Stronger Condition

In this section, we attempt to determine if any statement is *easily* made about the no-hair theorem for a minimal scalar field coupling by introducing a new factor. Instead of working directly from a metric ansatz, we will start by first assuming that our solution is generally Schwarzschild, then determine the nature of the scalar field at infinity.

The general shape of the Schwarzschild solution is given by setting the function n and p in equation (17) to

$$n(r) = \frac{r+c}{r}$$
$$p(r) = \frac{r}{r+c}$$

where the constant c is typically given as the Schwarzschild radius r_s which is determined when taking the limit to Newtonian mechanics. For now, however, simply keeping it set to some arbitrary constant is fine.

When we insert this shape of solution into the equation of motion for ϕ determined in 4.1.2, we find that a solution does present itself.

$$\phi(r) = c_1 + c_2 \left(\frac{\ln(r)}{c} - \frac{\ln(c+r)}{c} \right)$$

Immediately we can see that in the limit as $r \to \infty$, the coefficient on the c_2 term will vanish leaving us with whatever constant value c_1 takes on which verifies the trivial solution that the scalar field has no nontrivial configuration at infinity (ie $c_2 \to \phi_0$).

5 Concluding remarks

Throughout the course of this paper a discussion of a wide variety of topics has been conducted, all starting from what are in hindsight fairly basic concepts. However, the immediate results of working from the most basic concepts up to more complex and interesting problems aptly displays my learning process over the course of the Fall 2023 semester.

While, eventually, a conclusion was reached regarding a somewhat stronger condition for the no-hair theorem in a minimally coupled scalar field background, a more nuanced discussion of the conditions for no-hair theorems likely would have benefitted the analysis. For instance, concepts like the energy condition and symmetry assumptions were not explored in depth.

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