Homework 2/3

Nathan Burwig

April 16, 2022

Problem 1

Take a vector $x = x^1 e_1 + x^2 e_2 + ... + x^n e_n$ with components x^i in the basis e_i. Let e'_i be a new basis and a transformation matrix A between e_i and e'_i given by a set of linear equations:

$$
\begin{aligned}
&e'_1 = a_{11} e_1 + a_{12} e_2 + ... + a_{1n} e_n \\
&. \\
&. \\
&e'_n = a_{n1} e_1 + a_{n2} e_2 + ... + a_{nn} e_n
\end{aligned}
$$

In the new basis, vector x can be written as $x' = x'^1 e'_1 + x'^2 e'_2 + ... + x'^n e'_n$, it will have different components, but it is the same vector $x = x'$.

Find how the components of vector transform under base change given by A - substitute for e'_i and find the transformation between x_1, $x_2 ... x_n$ and $x'_1, x'_2 ... x'_n$

Problem 2

The notation is x^k for components of a contravariant vector (tensor of rank 1 - tensor with 1 contravariant index and 0 covariant indices) and x_k for a covariant vector.

If base vectors transform like:

$$e_{k'} = A^i_{k'} e_i, \text{ or } e' = Me, \text{ or } e = M^{-1} e'$$

Then a contravariant vector (like a position vector) transforms like:

$$x_{k'} = A^i_{k'} x_i, \text{ or } x' = Mx$$

in the same way as coefficents of a linear 1-form $f(x) \to \mathbb{R}$:

$$f_{k'} = A^i_{k'} f_i, \text{ or } f' = Mf$$

Show explicitly, that the gradient of a scalar function taken with respect to a contravariant vector transforms like a covariant vector, and a derivative of a scalar function with respect to a covariant vector transforms transforms like a contravariant vector.

Problem 3

Take an anti-symmetric bilinear form $f_{i,j} = -f_{i,j}$ in 3-dimensions - see the handout where it is shown how coefficients of a 1-form transform, and do the same for an anti-symmetric 2-form $f(a, b) = -f(b, a) \to \mathbb{R}$, where a, b two vectors (a linear two-form is a function which takes two contravariant vectors as arguments and gives a number in a field over which the vector field is defined - in our case it is $\mathbb{R}$)

Make the identification $g^1 = f_{23}, g^2 = f_{31}, g^3 = f_{12}$

Show that components g^1 transform under a base transformation $e_k \to e_{k'}$ given by $e_{k'} = A^i_{k'} e_i$, or $e' = Me$

Like $g' = det(M)M^{-1}g$

Notice that a covariant vector transforms like $x' = M^{-1}x$.

What we call a vector product in 3D of two vectors $\vec{a}$ and $\vec{b}$, $\vec{c} = \vec{a}$ is really an object which is a result of identifying the coefficients of an anti-symmetric two form $f_{i,j}(a, b).a$ with $f^k = \epsilon_{ijk} f_{ij} = c$, where ϵ_{ijk} is the completely antisymmetric Levi-Civirta symbol (or tensor).