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OBSERVATIONAL TESTS OF WORLD MODELS*

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Observational cosmologists owe a great debt to H. P. Robertson not only because his work clarified much of the confusion which surrounded the observational approach to the cosmological problem in the 1930's but also because his rigorous derivation of the two fundamental equations which connect theory and observation (i.e. the metrical and apparent luminosity relations) established beyond doubt their validity. During his career, Robertson returned again and again to the problem, each time with new insight, and each time to clarify the observational approach. He, together with Heckmann and McVittie, were foremost in casting the theoretical equations exclusively in terms of observables such as redshift, apparent luminosity, and angular diameter, which can be determined at the telescope.

The influence of Robertson on the observers was enormous. He took great interest in the new developments connected with the 200-inch project, and his council concerning theoretical interpretation of the observations was always given liberally.

It is my privilege in this memorial symposium to retrace some of the logical development of the theory, to show how observational relations follow from Robertson's methods in deriving the models, and finally to indicate where progress has been made in comparing the resulting predictions with the real world.

1. The metric. The cosmological problem concerns the large scale structure of the universe, its motion and geometry. Einstein's theory of gravitation provides a natural vehicle for construction of the various world models because the metric properties of the space-time continuum are directly related to the distribution of matter through the field equations of the theory. Given the energy content and the distribution of matter, the metric properties follow, or conversely, given the metric, the large scale material features are predicted. Observation shows that the actual universe is isotropic about ourselves. The redshift-apparent magnitude relation is the same in all directions and no departure from isotropy is evident in galaxy counts made to faint light levels. Furthermore, if there is nothing special about our position in the universe, this isotropy implies homoge-

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neity, i.e. observations from any point must give the same picture as from any other. Isotropy and homogeneity are usually taken as the initial mathematical condition of the problem. They are introduced into the theory by the adoption of line elements which possess these properties.

Robertson [1] in 1929 and A. G. Walker [2] in 1936 showed that the most general line element which satisfies the two conditions has the form

$$ds^2 = c^2\,dt^2 - R^2(t)\,du^2 \tag{1}$$

where du^2 is the metric of an auxiliary homogeneous and isotropic three-space of constant Riemannian curvature which is time independent, R is the space dilatation factor which can be a function of time giving rise to an expanding or contracting spatial metric, and c is the velocity of light. Furthermore, the most general form for the spatial part du^2 can be written as

$$du^2 = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{2}$$

where r, θ, and φ are dimensionless time independent variables relative to which the universe is at rest. That is, r, θ, and φ are tied to the fundamental particles of the continuum (the galaxies) and are therefore said to be co-moving as $R(t)$ unfolds. The number k is the signature of the space curvature and with proper normalization takes the values -1, 0, or $+1$ for hyperbolic, Euclidean, or elliptical space respectively, which then gives the Riemannian curvature of the manifold as k/R^2.

Equations (1) and (2) are fundamental to the problem. All properties of the models flow from them when the dynamics which determine $R(t)$ are specified by the field equations. But at this point we need not commit ourselves to any system of dynamics in order to derive some observational consequences of these general metrics.

The equation of a light track is the null geodesic $ds^2 = 0$ which then defines a metric distance u from (1) as

$$u = c\int_{t_1}^{t_0} \frac{dt}{R(t)} = \text{a function independent of time,} \tag{3}$$

where t_1 is the cosmic (proper) time of light emission from the galaxy in question and t_0 is the time of reception on the earth. Furthermore, the connection between the metric distance u and the coordinate r follows from (2) for radial tracks ($d\theta = d\varphi = 0$) as

$$u = \int_0^r \frac{dr}{(1 - kr^2)^{\frac{1}{2}}} = \begin{cases} \sin^{-1} r, & k = +1, \\ r, & k = 0, \\ \sinh^{-1} r, & k = -1, \end{cases} \tag{4}$$

where the observer is taken to be at $r = 0$. Thus r can be found using (3) and (4) once the function $R(t)$ is known.

In models where R varies with time, galaxies will exhibit wavelength shifts in their spectral lines which can be computed as follows. Consider a light wave emitted by a galaxy at time t_1 whose invariant metric distance is u. Let the time of emission of two successive crests of the wave be t_1 and $t_1 + \Delta t_1$. These same two wave crests will be received on earth at times t_0 and $t_0 + \Delta t_0$ which are connected to the emission times through (3) by

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{R(t)} = u = \text{constant.} \tag{5}$$

But

$$\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{R(t)} = \int_{t_1}^{t_0} \frac{dt}{R(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{R(t)} - \int_{t_1}^{t_1+\Delta t_1} \frac{dt}{R(t)}$$

which, when substituted into (5) gives

$$\frac{\Delta t_0}{\Delta t_1} = \frac{R(t_0)}{R(t_1)}\,. \tag{6}$$

The wavelengths of the emitted and arriving radiation are, by definition, λ (emitted wave) $= c\Delta t_1$ and λ (arriving wave) $= c\Delta t_0$, which require from (6) that

$$\frac{\lambda(\text{arriving})}{\lambda(\text{emitted})} = \frac{R(t_0)}{R(t_1)}\,.$$

But λ (arriving) $\equiv \lambda$ (emitted) $+ \Delta\lambda$, where $\Delta\lambda$ is the observed wavelength shift. Hence, if we assume that the atoms in the distant galaxy emit and absorb radiation at the same wavelength as in our laboratories on the earth, then

$$1 + \frac{\Delta\lambda}{\lambda_0} \equiv 1 + z = \frac{R(t_0)}{R(t_1)} \tag{7}$$

where λ_0 signifies the laboratory wavelength and is an intrinsic property of the atom and where z is defined as $\Delta\lambda/\lambda_0$. (Note the implicit assumption here that the atomic constants do not change over the light traveltime $t_0 - t_1$.) Equation (7), first derived by Lemaitre, shows that the wavelengths of spectral lines will be shifted either toward the red or blue in models where R varies with time.

At this point in the theory, (again without the introduction of dynamics), we can derive the apparent luminosity-redshift relation so familiar from the observational data on the expanding universe. The approach of greatest generality is to assume that $R(t)$ is a regular function which can be expanded in a Taylor series between the times t_0 and t_1. Equation (7) will then give z in terms of the function $R(t_0)$ (hereafter denoted by R_0), its first and higher time derivatives, and the light travel time $t_0 - t_1$. The light travel time can be eliminated by use of (3) where the integrand can also be expanded in terms of $(t_0 - t_1)$. Combining the two gives a redshift-metric distance relation of the form

$$z = \dot{R}_0(u/c) + \tfrac{1}{2}(\dot{R}_0^2 - R_0\ddot{R}_0)(u/c)^2 + O(u/c)^3. \tag{8}$$

The metric distance in (8) can be replaced by the apparent luminosity, which is observable, when the physical relation between u, luminosity, and redshift is known.

Robertson completely solved this latter problem [3] in 1938 and vindicated Tolman's earlier solution [4] which had been questioned by W. de Sitter and by H. Vogt. He showed that the apparent luminosity of a galaxy of absolute luminosity L with redshift z whose metric distance was u is given by

$$l = \frac{L}{4\pi R_0^2 r^2 (1 + z)^2} \tag{9}$$

where r and u are related by (4). Robertson's derivation was rigorous, employing the conservation laws of the radiation field in an expanding universe. An intuitive appreciation of (9) can be obtained by noting that $4\pi R_0^2 r^2$ is the area of the spherical wavefront of the emitted radiation at the time t_0 (when it is received on earth) and this factor gives the usual inverse square diminution of the light. The additional two factors of $1 + z$ can be understood as Hubble's "energy effect" and "number effect" [5]. Equation (9) can now be used to eliminate u from (8) giving a relation between l, z, and the time derivatives of R_0. The final result, common to the work of Robertson, Heckmann [6], and McVittie [7] is

$$m_{\text{bol}} = 5 \log z + 1.086(1 - q_0)\, z + \cdots + \text{const.}, \tag{10}$$

where m_{bol} is the bolometric luminosity of (9) expressed on the Pogson astronomical magnitude scale defined by $m = -2.5 \log l + \text{const.}$, and where q_0, known as the deceleration parameter, is defined by

$$q_0 \equiv -\frac{\ddot{R}_0 R_0}{\dot{R}_0^2}. \tag{11}$$

This function q_0 is fundamental to what follows because we shall show that knowledge of its value specifies the intrinsic geometry of the real world for a certain class of models.

The first term of (10) is completely verified by observation for $z < 0.1$ (see for example [8]). The second term is known to be small but detectable at the limit of the 200-inch telescope, and it is fair to say that its measurement was the chief observational problem for the telescope in cosmology in the minds of the builders.

Before proceeding to the exact solution by using the dynamics implicit in the field equations, it is well to remark that the second term of (10) arises solely because the speed of light is finite. We cannot observe all parts of the universe at the same cosmic time but rather observe to successively earlier historical epochs when we observe to successively greater distances. Therefore, we sample the expansion factor $R(t)$ at different times. It is easy to show that if observations could be made of all parts of the universe simultaneously, then a rigorously linear velocity-distance relation would obtain, i.e. at *any given* cosmic time the distance Ru and the redshift z are connected by the exact equation

$$cz = \dot{R}u = HRu \tag{12}$$

where $H \equiv \dot{R}/R$ is the Hubble expansion parameter.[1] But our inability to observe all Ru simultaneously shows that (12) cannot rigorously represent the data. A deviation from linearity in the theoretical redshift-distance relation must exist which is due entirely to the finite travel-time of light. This makes clear why observations to detect the non-linear term must reach to very great distances where $t_0 - t_1$ is the order of R/c.

2. The intrinsic geometry from q_0. Equation (10) is the series expansion of an exact expression which can be derived once $R(t)$ is found using the dynamics of the problem which are here assumed to be given by the field equations of general relativity. It is well known that the two ordinary differential equations for $R(t)$ which result from introducing the components of the metric tensor from (1) and (2) into the Einstein equations are

$$\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \frac{8\pi G p}{c^2} = -\frac{kc^2}{R^2} + \Lambda c^2 \tag{13}$$

and

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G\rho}{3} = -\frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \tag{14}$$

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[1] We note in passing that the logarithmic term of the right member of (10) gives just such a relation when $m_{\rm bol}$ is converted to distance by the usual astronomical equation

$$m = M - 5 + 5 \log (Ru)$$

where M is the absolute magnitude of the source, Ru is the distance in parsecs, and where the constant of (10) contains c and H. Note also that (12) is just (8) when the second order term in u is neglected.

where p is the isotropic hydrodynamic pressure of matter and radiation, ρ is the density of matter and energy, Λ is the cosmological constant, k/R^2 is the Riemannian spatial curvature, and k is the signature of the space. Equations (13) and (14) hold for all time and, in particular, hold for the present epoch of observation, denoted by subscript zero. They can, without integration, be made to yield important results if we introduce the present value of the Hubble expansion parameter $H_0 = \dot{R}_0/R_0$ and the present value of the deceleration parameter $q_0 = -\ddot{R}_0/R_0H_0^2$ both of which are observable.

Subtraction of (14) from (13) and, following Einstein, treating those models with $\Lambda = 0$, gives

$$\rho_0 + \frac{3p_0}{c^2} = \frac{3H_0^2q_0}{4\pi G}. \tag{15}$$

Substitution of the present value of the Hubble constant of 75 km./sec. per million parsecs (see Sandage [9]) into (15) gives

$$\rho_0 + \frac{3p_0}{c^2} = 2.06 \times 10^{-29} q_0 \text{ gm./cm.}^3.$$

A second result is found by putting H_0 into (14) and simplifying with (15) giving

$$\frac{kc^2}{R_0^2} = \frac{4\pi G}{3q_0}\left[\rho_0(2q_0 - 1) - \frac{3p_0}{c^2}\right] \tag{16}$$

which expresses one of the major themes of relativity, namely that the intrinsic geometry, given here by the curvature k/R_0^2, is determined by the total density and pressure.

The pressure term p_0 can be shown to be negligible at the present epoch compared with ρ_0 (see [10]), and therefore (16), with the help of (15), reduces to

$$\frac{kc^2}{R_0^2} = H_0^2(2q_0 - 1) \tag{17}$$

which shows that if $q_0 > \frac{1}{2}$, $k = +1$ and space is elliptical; if $q_0 = \frac{1}{2}$, $k = 0$ and space is Euclidean; if $q_0 < \frac{1}{2}$, $k = -1$ and space is hyperbolic.

3. The solution for $R(t)$ and the exact redshift-apparent magnitude relation. The general solutions of (13) and (14) with $p = 0$, $\Lambda = 0$ are most easily given in parametric form as

$$R = a(1 - \cos\theta), \qquad t = \frac{a}{c}(\theta - \sin\theta) \tag{18}$$

for $k = +1$,

$$R = a(\cosh\theta - 1), \qquad t = \frac{a}{c}(\sinh\theta - \theta) \tag{19}$$

for $k = -1$, and

$$R = (6\pi G\rho R^3)^{\frac{1}{3}} t^{\frac{2}{3}} \tag{20}$$

for $k = 0$. Here $a = \dfrac{4\pi G\rho R^3}{3c^2}$ where we note that ρR^3 is a constant and c is the velocity of light. Equation (18) is the parametric representation of a cycloid where a is the radius of the generating circle, and θ is the angle through which the circle has rolled. For the $k = +1$ model, R returns again and again to zero each time $\theta = 2\pi n$ (where n is an integer) and therefore the closed elliptical model represents an oscillating universe which alternately expands and contracts. The many interesting properties of this model are discussed elsewhere [11], [12].

We now have the necessary information to express the luminosity l of (9) in terms of the other two observable quantities z and q_0. We shall consider only the $k = +1$ case in detail since the $k = -1$ and $k = 0$ models can be derived in a similar fashion. Consider first the parameter r which occurs in (9). From (3), (4), and (18) it follows that

$$r = \sin u = \sin(\theta_0 - \theta_1)$$

where θ_1 is the development angle when light left the galaxy in question and θ_0 is the angle at light reception. Substitution of R_0 from (18) into (9) and converting to the Pogson magnitude scale gives

$$\begin{aligned} m = 5 \log\,[a(1 - \cos\theta_0)\sin(\theta_0 - \theta_1)(1 + z)] \\ + 2.5 \log 4\pi - 2.5 \log L. \end{aligned} \tag{21}$$

The angles θ_0 and θ_1 can be eliminated from (21) and replaced by H_0, q_0, and z by the substitutions

$$H_0 \equiv \frac{\dot{R}_0}{R_0} = \frac{c}{a}\frac{\sin\theta_0}{(1 - \cos\theta_0)^2}, \tag{22}$$

$$q_0 \equiv -\frac{\ddot{R}_0}{H_0^2 R_0} = \frac{1 - \cos\theta_0}{\sin^2\theta_0}, \tag{23}$$

and, from (7),

$$1 + z = \frac{R_0}{R_1} = \frac{1 - \cos\theta_0}{1 - \cos\theta_1},$$

all of which follow from (18). Considerable reduction then gives

$$m = 5 \log q_0^{-2}\{q_0 z + (q_0 - 1)[(2q_0 z + 1)^{\frac{1}{2}} - 1]\} + C \tag{24}$$

for the answer, where the constant $C = 2.5 \log 4\pi - 2.5 \log L + 5 \log c/H_0$. And it can be shown that (24) also holds for the $k = 0$ and $k = -1$ space curvatures. Equation (24), first given by Mattig using a different derivation [13], has been tabulated by Sandage [10] and forms the basis for the observational determination of q_0 to be discussed in section 5. It is of interest that series expansion of (24) gives (10) as is required.

In the past decade an entirely new world model has been introduced by Hoyle and by Bondi and Gold. Their well-known steady state universe is not only homogeneous in space but in time as well. The Hubble parameter does not vary as the universe expands, contrary to the models we have just discussed. Furthermore, the mean density of matter does not change as the expansion proceeds, a condition which requires that *new* matter is injected into space at just the rate necessary to compensate for the dilution effect of the expansion. The only $R(t)$ function which will accomplish these goals is $R(t) \alpha \exp Ht$ which then requires from (11) that $q_0 = -1$. This difference in the value of q_0 between the steady state and the evolving models of relativity provides an observational test. The redshift-apparent magnitude relation for a steady state world can be derived using the required form for $R(t)$ in (3) and then using (3) and (4) in (9) to give

$$m = 5 \log z\,(1 + z) + C. \tag{25}$$

Equations (24) and (25) are compared in section 5. In closing this section we note that series expansion of both (24) and (25) gives (10) as expected.

4. Galaxy counts. Hubble's classical method [14] of finding the intrinsic geometry of space was to determine the volume, V, enclosed by any distance u and thereby to see if V increased more or less rapidly than $\frac{4}{3}\pi R_0^3 u^3$. His mensuration method was to assume that galaxies are distributed *uniformly* in space and to count the number of such galaxies to successively fainter luminosity limits. The relation between luminosity and distance u then gives the volume encompassed in surveys made to any magnitude m. The method, so beautiful in principle, is almost impossible to apply in practice because (1) galaxies in the general field show a large range in intrinsic luminosity [i.e. L has a wide distribution and cannot be considered constant in (9)] and (2), the distance surveyed by the 200-inch is not large enough by using field galaxies as the sampling objects because their average luminosity is relatively small compared to clusters of galaxies used in the redshift test of section 3. Nevertheless, it is of interest to derive the equations of the theory if only to show that deviations from the Euclidean approximation occur only when observations are pushed to enormous distances Ru such that $u = 1$, which means by (12) and (17) that the redshifts must attain a value close to one.

The volume enclosed within distance Rr at the time t_0 of observation when $R = R_0$ is given by (2) as

$$V(r) = 2R_0^3 \int_0^r \frac{r^2\,dr}{(1 - kr^2)^{\frac{1}{2}}} \int_0^{\pi/2} \sin\theta\,d\theta \int_0^{2\pi} d\varphi$$

(26)
$$\begin{aligned} &= 2\pi R_0^3[\sin^{-1} r - r(1 - r^2)^{\frac{1}{2}}] \qquad &\text{for } k = +1 \\ &= \tfrac{4}{3}\pi R_0^3 r^3 \qquad &\text{for } k = 0 \\ &= 2\pi R_0^3[r(1 + r^2)^{\frac{1}{2}} - \sinh^{-1} r] \qquad &\text{for } k = -1. \end{aligned}$$

We shall carry the $k = +1$ case through in detail in the following and later only state the result for the other geometries.

By using (4) the $k = +1$ relation becomes

$$V(u) = 2\pi R_0^3[u - \sin u \cos u]$$

which, for interest, can be expanded for small u to

$$V(u) = \frac{4\pi R_0^3 u^3}{3}\left[1 - \frac{u^2}{5} + \cdots\right]$$

showing, as previously stated, that u must be close to one before deviations from Euclidean geometry can be detected.

Equations (26) can be expressed in terms of apparent magnitude and q_0 alone by using the precepts of section 3. Eliminating a from (21) by using (22), then exponentiating the result, and defining a new *observable* quantity A by

(27)
$$A \equiv 10^{0.2(m-C)},$$

where C is the constant in (24), and finally eliminating the trigonometric functions using (23), gives

(28)
$$r = \sin u = \frac{A(2q_0 - 1)^{\frac{1}{2}}}{1 + z}.$$

Because we wish (26) to be expressed in terms of apparent magnitudes alone, the $1 + z$ can be eliminated by substitution of $1 + z$ derived from (24) as

(29)
$$1 + z = q_0(1 + A) - (q_0 - 1)(1 + 2A)^{\frac{1}{2}}$$

to give finally

(30)
$$r = \frac{A(2q_0 - 1)^{\frac{1}{2}}}{q_0(1 + A) - (q_0 - 1)(1 + 2A)^{\frac{1}{2}}}$$

as a formula for the photometric distance. To put (26) completely in terms of observables, R_0^3 is expressed in terms of H_0 and q_0 by (17) which

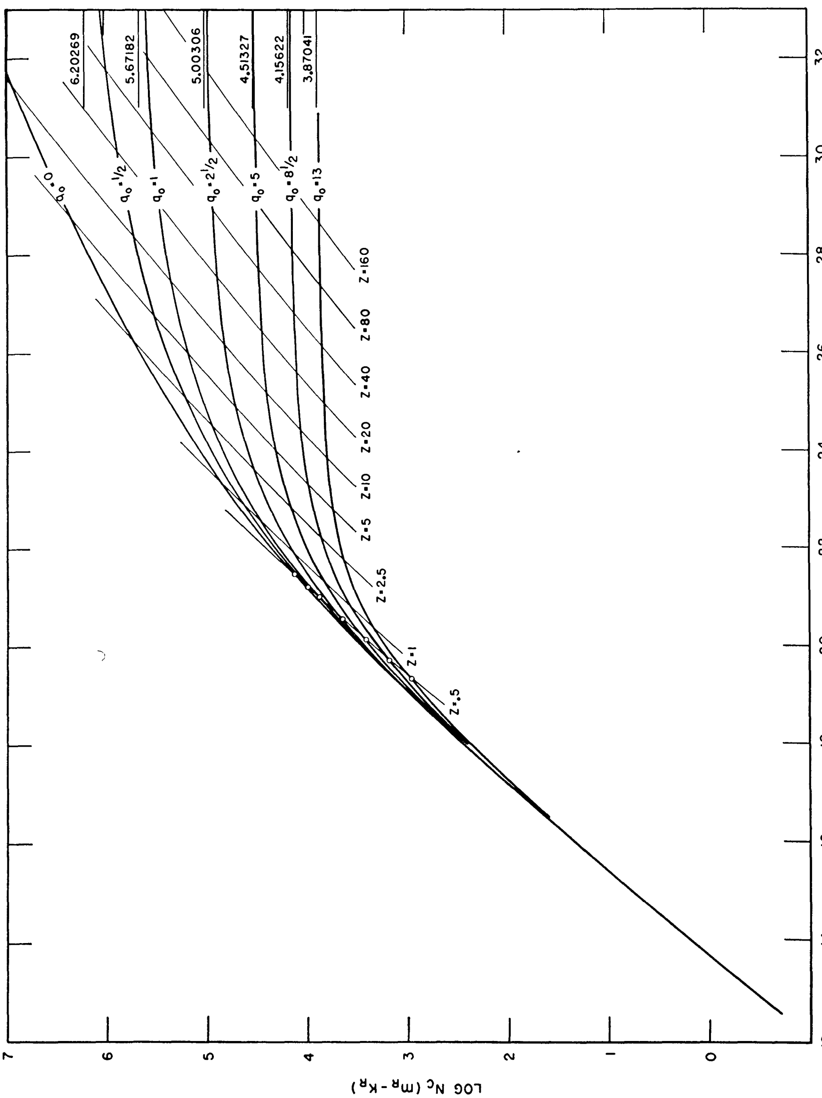

FIGURE 1. *The galaxy count-apparent magnitude relation for models with various* q_0 *values.*[2]

then reduces (26) to

$$N(m) = \frac{2\pi c^3 n}{H_0^3(2q_0 - 1)^{3/2}} [\sin^{-1} r - r(1 - r^2)^{\frac{1}{2}}], \qquad k = +1, \tag{31}$$

for the answer, where $N(m)$ is the number of galaxies projected on the plane of the sky brighter than apparent magnitude m, n is the number of galaxies per unit volume, and r is given parametrically by (27) and (30) with C determined by fitting (24) to the observations. In a similar fashion

$$N(m) = \frac{2\pi c^3 n}{H_0^3(1 - 2q_0)^{3/2}} [r(1 + r^2)^{\frac{1}{2}} - \sinh^{-1} r], \qquad k = -1, \tag{32}$$

and

$$N(m) = \frac{4\pi c^3 A^3 n}{3H_0^3} \{\tfrac{1}{2}[1 + A + (1 + 2A)^{\frac{1}{2}}]\}^{-3} \tag{33}$$

for the Euclidean case $k = 0$. Equations (31), (32), and (33), first derived by Mattig in a different way [15], have been tabulated by Sandage [10] with m as the argument and q_0 as the parameter. Figure 1 shows the result, where the heterochromatic magnitudes observed on a red system m_R and corrected for the wave length dependent effects of the redshift by k_R (see Appendix B in [8]) are plotted against log $N(m)$ per square degree. Lines of constant redshift z are also shown as determined from (29). The numbers on the right ordinate are the asymptotic values of log $N(m)$ per square degree for models with various q_0 values. This figure clearly shows that even at the limit of the 200-inch telescope in this ideal case where galaxies are assumed to have identical absolute magnitudes, the differences in log $N(m)$ between the models is extremely small, thus making the test impractical. From these considerations it can be shown that at $m_R = 22$, which is surely the observational limit for galaxy counts, an error of only 0.28 magnitudes will shift the conclusion from a closed oscillating model with $q_0 = 1$ to the open, empty model with $q_0 = 0$. Such a small magnitude difference would certainly be buried in the probable errors of any observational data, which forces the conclusion that galaxy counts, so straight forward in principle, cannot be used in practice to decide between world models.

5. Data for the redshift-magnitude relation. The situation appears to be different using the apparent magnitude-redshift test of (24) and

[2] The magnitudes $m_R - k_R$ are on a relative bolometric scale. The ordinate is the log of the number of galaxies per square degree which are brighter than magnitude $m_R - k_R$. Lines of constant redshift are shown. Numbers on the right are the asymptotic values of log $N(m)$ at the observable horizon.

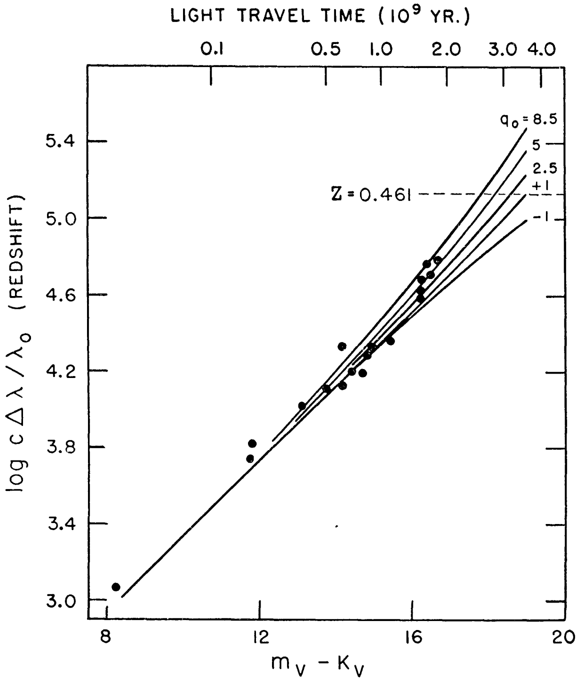

FIGURE 2. *The theoretical redshift-apparent magnitude relation from* (24) *and* (25) *for different* q_0 *values.*[3]

(25) because the theoretical lines, given in Figure 2, show that large differences occur in $m_v - k_v$ at redshifts which are within the range of the 200-inch. (Here m_v is the apparent magnitude on a visual wavelength system and k_v is the corresponding correction for selective redshift effects). Observational data for 18 clusters of galaxies [8] are plotted for comparison, and a formal solution for q_0, using (10) as the equation of condition, gives $q_0 = +2.45 \pm 0.8$ (see [8] and [10]). However, the scatter of the observed points in Figure 2 is so large that little confidence can be placed in this result. A more recent preliminary determination by Baum [16], [17] using his new observations made by a refined photoelectric technique, gives $q_0 = +1 \pm 0.5$ from data on about 7 clusters which include the crucial object 3C295 which Minkowski [18] found to have a redshift of $z = 0.461$. Baum's photometry has not yet been reported in detail but his

[3] Data for 18 clusters are plotted. The redshift for Minkowski's cluster is shown as a broken line for which no apparent magnitude data are presently available. The travel-time for light signals to reach the earth from clusters at given redshifts are shown along the top for an assumed Hubble time of $H_0^{-1} = 13 \times 10^9$ years, and for an assumed model $q_0 = +1$.

preliminary announcement [17] indicates that $m_v - k_v = 19.0$ as required by $q_0 = 1 \pm \frac{1}{2}$. The redshift of 3C295 is shown in Figure 2 as a broken line. It might seem that these two determinations of q_0 formally reject the steady state model because both are far removed from $q_0 = -1$. But such is not necessarily the case because all apparent magnitudes in Figure 2 must be corrected for an effect neglected so far. The time for light to travel from these distant clusters to the earth is enormous, and over this interval the stellar content of the galaxies must have changed due to stellar evolution. Light travel-times can be formally computed [11] from the models with the results for $q_0 = +1$, assuming $H_0^{-1} = 13 \times 10^9$ years, shown on the upper scale of Figure 2. Evolutionary corrections will be the largest for the most distant clusters and will decrease to almost zero for the nearby galaxies. But we yet know so little about the evolution of the total stellar content of galaxies that any final correction is out of the question at this time. However, an estimate of the effect has been made [11], [19] which suggests that Baum's observed q_0 value could be reduced to $q_0 = +0.2 \pm 0.5$.

Clearly the fit of observations to the formal theory has only begun. But, if only to illustrate the methods of this report, we shall give the answers based on $q_0 = +0.2$ and $H_0^{-1} = 13 \times 10^9$ years. These values, substituted in (15) and (17), require that (a) space is open and infinite with $k = -1$, (b) the expanding universe is decelerating ($\ddot{R} < 0$), (c) the radius of curvature is 1.6×10^{28} cm. (17×10^9 light years), (d) the mean density of matter plus radiation is 4×10^{-30} gm./cm.^{3}, and (e) the expansion will continue forever.

These numbers must not be taken seriously. Too many observational problems and unknown corrections yet remain. But the theory, thanks in large measure to the work of H. P. Robertson and to his unpublicized influence on others, is in such a form that the observers now know what to measure.

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