II

Mathematical development

1. Causal orientations

Unless otherwise specified, all manifolds will be taken as real, finite dimensional, and of class C^∞; and all groups as finite-dimensional Lie groups; there is, however, no essential difficulty in being considerably more general.

Definition 1 A *convex cone* in a real linear space $\mathbf{L}$ is a subset $\mathbf{C}$ with the property that if $x, y \in \mathbf{C}$, and if $a, b \geqq 0$, then $ax + by \in \mathbf{C}$. Such a cone is *trivial* if either $\mathbf{C} = \mathbf{L}$ or $\mathbf{C} = \{0\}$; otherwise it is *nontrivial*. It is *proper* if it contains no full straight lines and is not a direct product of a ray and a cone of codimension one. It is C^∞ (respectively analytic) if there exists a finite set Φ of C^∞ (respectively analytic) functions on $\mathbf{L}$ such that $x \in \mathbf{C}$ if and only if $f(x) \geqq 0$ for all $f \in \Phi$.

Definition 2 An *infinitesimal causal orientation* in a manifold M is an assignment $p \to C(p)$ to each point p of M of a nontrivial closed convex cone $C(p)$ in the tangent space at p, which is locally definable by a finite number of inequalities on continuous functions of p and the components of tangent vectors. Specifically, this means that each point p has a neighborhood N in which there exist local coordinates $x_1, \ldots, x_n$ and a finite set Φ of continuous functions on $N \times R^n$ such that if $q \in N$ and $\lambda \in T_q$ (the tangent space at q), then $l \in C(q)$ if and only if $l = \sum_k a_k(\partial/\partial x_k)|_q$, and $f(q, a_1, \ldots, a_n) \geqq 0$ for all

$f \in \Phi$. If M is C^∞ (respectively analytic or algebraic), the causal orientation is called C^∞ (respectively analytic or algebraic) if it is similarly defined by a finite number of C^∞ (respectively analytic or algebraic) functions.

Example 1 Let G be a Lie group, let $\mathbf{G}$ denote the Lie algebra of all right-invariant vector fields on G, and let $\mathbf{C}$ be a nontrivial closed convex cone in $\mathbf{G}$. Defining $C(p) = [X_p : X \in \mathbf{C}]$ is easily seen to give a causal orientation in G, which is C^∞ (respectively analytic) if the cone $\mathbf{C}$ is such.

Definition 3 An infinitesimal causal orientation in a group G is said to be *right-invariant* if of the form given in Example 1; *left-invariant* if the same except that left-invariant vector fields are employed; and simply *invariant* if both right and left invariant.

Example 2 Let $G = U(n)$, i.e., the group of all unitary $n \times n$ complex matrices. Then $\mathbf{G}$ can be identified with the linear space $\mathbf{H}(n)$ of all $n \times n$ complex Hermitian matrices, via the isomorphism: if $X \in \mathbf{G}$, then $X \to H$, where $H \in \mathbf{H}(n)$, if for all $C^\infty f$ on $U(n)$, $Xf = (d/dt) f(e^{itH}x)|_{t=0}$ (where x is a bound variable); and to avoid undue circumlocution, we shall use this identification. Let $\mathbf{C}$ denote the subset of $\mathbf{G}$ consisting of those H in $\mathbf{H}(n)$ for which $H \geqq 0$, in the usual sense that $\langle Hx, x\rangle \geqq 0$ for all vectors x, $\langle \cdot, \cdot \rangle$ denoting the usual complex Hermitian positive definite inner product in C^n. Then $\mathbf{C}$ is a closed convex algebraic cone in $\mathbf{G}$. It is evidently invariant under the action (induced) on $\mathbf{G}$ of inner automorphisms of G, and so defines an invariant infinitesimal causal orientation. It is not difficult to verify that if $n = 2$, this is the only quadratic such structure, apart from those defined by $-\mathbf{C}$, and by the one-dimensional cones $[lI : l \geqq 0]$ and $[lI : l \leqq 0]$.

Definition 4 *A finite globally causal orientation* in a Hausdorff topological space M is a transitive relation $x \prec y$ defined for suitable pairs $x, y \in M$, and called "precedence," having the properties that the set of all pairs (x, y) in $M \times M$ such that $x \prec y$ is closed, and that $x \prec x$ for all x. A *finite* (not necessarily globally) *causal orientation* in M is an assignment to each point $p \in M$ of a neighborhood N_p and a finite globally causal orientation in N_p such that if $q \in N_p$, then in a sufficiently small neighborhood of q, the given causal orientations in N_p and N_q coincide. If M is a C^∞ (respectively analytic) manifold, such a causal orientation is called C^∞ (respectively analytic) if it is locally definable by a finite set of C^∞ (respectively analytic) functions of x and y, in a fashion similar to that earlier indicated.

Example 3 Let $\mathbf{C}$ be a closed convex nontrivial cone in the real finite-dimensional linear space $\mathbf{L}$, and define $x \prec y$ to mean that $y \in x + \mathbf{C}$; $\mathbf{L}$ is thereby given a finite, globally causal orientation.

More generally, let G be an arbitrary topological (Hausdorff) group. A finite causal orientation in G is *right-invariant* in case $x \prec y$ if and only if $xa \prec ya$ for all $a \in G$; similarly for *left-invariant* and simply *invariant*. Let a *conoid* in G be defined as a closed subset K containing e such that $K^2 \subset K$. It is easily verified that for any conoid K, the relation $x \prec y$ if and only if $yx^{-1} \in K$ defines a finite globally causal right-invariant orientation in G; and that conversely every such orientation arises in this manner from the conoid $[x \in G : e \prec x]$; moreover such an orientation is fully invariant if and only if $aKa^{-1} \subset K$ for all $a \in G$.

More specifically, let $(\mathbf{L}, Q)$ be a *pseudo-Euclidean space*, defined as a pair consisting of a real linear space $\mathbf{L}$ of finite dimension together with a given nondegenerate symmetric bilinear form Q on $\mathbf{L}$; and suppose that Q is of type $(1, n)$, $n + 1$ being the dimension of $\mathbf{L}$, i.e., can be expressed in the form $x_0^2 - x_1^2 - \cdots - x_n^2$ in terms of suitable coordinates $x_0, x_1, \ldots, x_n$ on $\mathbf{L}$; such a pair $(\mathbf{L}, Q)$ will be called a *linear Lorentzian manifold*. The function $\varepsilon(X) = \operatorname{sgn} x_0$ defined on the closed subset $[X \in \mathbf{L}: Q(X, X) \geqq 0]$ is invariant under the component of the identity $O_0(\mathbf{L}, Q)$ of the automorphism group of $(\mathbf{L}, Q)$; and the subset in turn, $\mathbf{C} = [X \in \mathbf{L} : Q(X, X) \geqq 0, \varepsilon(X) \geqq 0]$, is a closed convex nontrivial algebraic cone in $\mathbf{L}$, which is invariant under $O_0(\mathbf{L}, Q)$. With the finite globally causal orientation determined by this cone, $(\mathbf{L}, Q)$ becomes the Minkowski space determined by $(\mathbf{L}, Q)$. It is also called the *$(n + 1)$-dimensional Minkowski space.*

The Minkowski spaces determined by two linear Lorentzian manifolds are isomorphic if and only if they have the same dimension. Here *isomorphism* means a one-to-one transformation preserving linearity, the form Q, and the precedence relation just defined; however, by the work of Alexandrov and Ovchinnikova, and Zeeman, cited earlier, the assumption of linearity is superfluous, if $n > 1$.

In case $n = 3$, the Minkowski space (of dimension 4) is isomorphic, in the same sense, to the space $\mathbf{H}(2)$ causally oriented by the cone $\mathbf{C}$ defined in Example 2. The isomorphism may be expressed in terms of the coordinates $t = x_0$, $x = x_1$, $y = x_2$, $z = x_3$, as follows:

$$X = (t, x, y, z) \to H = \begin{pmatrix} t - x & y + iz \\ y - iz & t + x \end{pmatrix}.$$

Since $X^2 = \det H$ and $2t = \operatorname{tr} H$, the causal cone in $\mathbf{L}$ (i.e. that defining in the indicated way the causal orientation, and definable as $[(t, x, y, z) : t^2 - x^2 - y^2 - z^2 \geqq 0,\ t \geqq 0]$) is mapped onto the set of all H such that $\det H \geqq 0$ and $\operatorname{tr} H \geqq 0$, i.e., the set $[H : H \geqq 0]$.

Definition 5 In a manifold M with an infinitesimal causal orientation, the set of all tangent vectors l at the point p such that $l \in C(p) \wedge -C(p)$ is called the *instantaneous present* at p, and denoted as $\mathbf{N}_p$. The causal orientation is called *Newtonian* if dim $\mathbf{N}_p$ (which is independent of p by continuity) is dim $M - 1$; *partially Newtonian* if dim $\mathbf{N}_p > 0$; *Einsteinian* if dim $\mathbf{N}_p = 0$; *Bergsonian* if dim $C(p) = 1$.

Similarly, in a manifold M with a finite globally causal orientation, the finite present P_x at a point x is defined as $[y \in M : x \prec y \text{ and } y \prec x]$. The orientation is called *Newtonian, partially Newtonian*, or *Einsteinian* near x_0 according as the equivalence classes relative to the relation $x \sim x'$ if $x \prec x'$ and $x' \prec x$ are totally ordered by the partial ordering canonically induced on them from that in M (for all x in some neighborhood of x_0) or $P_x \neq \{x\}$ (for some x in all sufficiently small neighborhoods) or $P_x = \{x\}$ (for all x in some neighborhood). It is *Bergsonian* if the union of the future and past of x is totally ordered. (We shall make little use of these definitions; they are included to suggest the conceptual basis and general scope of the theory.)

A (*causal*) isomorphism between causally oriented manifolds (of either the infinitesimal or finite type) is a manifold-isomorphism that carries the one causal orientation into the other. Similarly for the notion of (causal) automorphism of a causally oriented manifold; the group of all such, in the compact-open topology, will be called the (causal) *automorphism group* of the (causally structured) manifold. When the manifold is also a group, the term *causal morphism* may be used to avoid confusion with the notion of group automorphism.

Example 4 (a) Any left translation on a group with a left-invariant causal orientation (infinitesimal or finite) is a causal automorphism; but is not a group automorphism, except in the case of translation by the unit element. If the causal orientation is fully invariant, inner automorphisms are causal as well as group automorphisms.

(b) If in the first paragraph of Example 3 the cone $\mathbf{C}$ is proper, then according to a theorem of Alexandrov (1967), every causal automorphism is an affine transformation, whose homogeneous part leaves $\mathbf{C}$ invariant. In an arbitrary topological group G with invariant conoid K, any right or left translation is a causal-morphism, as is any group-automorphism of G which carries K into K; but in general, for a Lie group, the converse (i.e., the analogue of Alexandrov's theorem) is not valid.

(c) Note that the set of all "forward" vector fields (i.e., vector fields X such that $X_p \in C(p)$ for all p) is a convex cone in the space of all vector fields, M being taken here to be C^∞. Moreover, every tangent vector $l \in C(p)$ at some point p is the value at p of some forward vector field on M. Thus a causal orientation on a C^∞ manifold may equally well be defined by specification of the forward vector fields; and our axioms can be changed to the assumption that there is given a convex cone **C** in the space of all vector fields, which is closed in the topology of convergence on compact sets.

(d) Let S be an arbitrary C^∞ manifold, and set $M = R^1 \times S$. Define the vector field $a(\partial/\partial t) \times I_S + I_{R^1} \times X$, where X is any vector field on S, to be forward if and only if $a \geqq 0$. This defines a causal orientation on M which is evidently Newtonian. Note that the group of all causal automorphisms is infinite dimensional, for it includes all the transformations $I_{R^1} \times T$, for T an arbitrary diffeomorphism on S. (Here I_S denotes the identity operator on the space S.)

Scholium 2.1 A C^∞ manifold admits an infinitesimal causal orientation if and only if it admits a nonvanishing vector field.

If the C^∞ manifold M admits the nonvanishing vector field X, then defining $C(p)$ as $[aX_p : a \geqq 0]$ defines a causal orientation. To prove the "only if" part of the scholium, take a Riemannian metric on the manifold, thus obtaining in each tangent space a corresponding euclidean structure. Now note

Lemma 2.1.1 Let E denote a finite-dimensional Euclidean space, and K the set of all nontrivial closed convex cones in E. Then there exists a continuous Euclidean-invariant function defined on K, which maps each cone in K into a nonvanishing vector in the cone.

The topology on the space of cones is here the usual one, definable as that obtained from the Hausdorff metric applied to the intersections of the cones in question with the unit ball, the cones all being taken as having vertex at the origin. I am indebted to Professor W. Fenchel for the observation that this lemma is deducible from a result of Shepard (1966). To complete the proof of the scholium, assign to each point p the vector in $C(p)$ which is given by a map having the properties given in the conclusion of the lemma, the tangent space being taken as Euclidean in the indicated way. This shows the existence of a continuous nonvanishing vector field on M, whence a smooth such field also exists.

Remark 1 The variant of this result in which "causal orientation" is replaced by "Lorentzian structure" (i.e., hyperbolic pseudo-Riemannian metric) is well known; cf. Lichnerowicz (1971) for further developments in this direction. It is also well known that a compact manifold admits a nonvanishing vector field if and only if its Euler characteristic vanishes (Hopf–Samelson).

Example 5 Any covering manifold of a causally oriented manifold is naturally causally oriented itself, by virtue of the pullback from the local homeomorphisms into the covered manifold. It is easily seen that any covering manifold of a globally causal manifold is itself globally causal; but the covering manifold may be globally causal when the covered manifold is not, as e.g. in the case of $U(n)$, oriented as in Example 2. $U(1)$ is not globally causal, for the timelike arc $t \to e^{2\pi i t}$, $t \in [0, 1]$, is closed; but evidently the universal covering group $\tilde{U}(1) \cong R^1$ is globally causal, and the same is true for $\tilde{U}(n)$ (cf. below).

Definition 6 A *timelike arc* in a manifold with an infinitesimal causal orientation is a continuous, piecewise C^1, oriented arc whose forward tangent at each point p of the arc lies in $C(p)$; if in an extreme direction of the boundary of $C(p)$ for all p, the arc is called *lightlike*. Thus, if the arc is parametrized by the mapping $s \to p(s)$, $s \in [0, 1]$, then the tangent vector at $p(s_0)$,

$$f \to \lim_{\varepsilon \to 0} \varepsilon^{-1}[f(p(s_0 + \varepsilon)) - f(p(s_0))],$$

f being an arbitrary C^1 function near $p(s_0)$, is in $C(p)$, in the case of a timelike arc. The arc is *strictly timelike* if at each point the forward tangent lies in the interior of $C(p)$.

If p and q are points in the manifold M with infinitesimal causal orientation, we say "p precedes q" and write $p \prec q$ if there exists a timelike arc whose initial point is p and terminal point is q. It is evident that this relation is transitive: if $p \prec q$ and $q \prec r$, then $p \prec r$. If it has the property that $p \prec q$ and $q \prec p$ implies that $p = q$—alternatively, if every closed timelike curve is trivial, i.e. a single point—the causal orientation is said to be *semiglobal*. In such a manifold, the *future* F_x of any point x (respectively the *past* P_x or finite *present* N_x) is defined as the union of all points preceded by x (respectively which precede x, or both precede and are preceded by x).

Two points p and q in a manifold with infinitesimal causal orientation are called (relatively) *spacelike* if neither $p \prec q$ nor $q \prec p$. A *spacelike submanifold* is one any two points of which are relatively spacelike.

Example 6 (a) A compact manifold may admit a semiglobal causal orientation, an example being the n-dimensional torus, $n > 1$, oriented by taking $\mathbf{C}$ in its Lie algebra as displacement in the positive direction along an irrational one-parameter subgroup.

(b) On the other hand, *A compact C^∞ manifold with an infinitesimal causal orientation cannot be semiglobal if the causal cone at each point has nonvanishing interior.* For by the lemma cited earlier, there then exists a vector field X on the manifold M such that for each p, X_p is an interior point of $C(p)$. (Compare the proof of Scholium 2.1.) Let p_0 be a nonwandering point relative to the flow on M defined by X; by compactness, such a point exists. According to a version of the "closing lemma" due to Pugh,† there is a vector field X' on M which is arbitrarily close to X, (in the C^1 topology) and has a closed orbit. Such a vector field is however again timelike, in the sense that $X'(p)$ is interior to $C(p)$ for all p, i.e., M is not semiglobal.

(c) In Example 4c, the (finite) present at any point (t, q) consists of all points (t, q') with $q' \in S$.

(d) Any Einsteinian infinitesimal causal orientation whose cones $C(p)$ have nonvanishing interiors determines a finite causal orientation, by virtue of the relation $x \prec y$ earlier defined. This is a consequence basically of work of Zaremba and Marchaud as developed by Leray (1952); cf. also Choquet-Bruhat (1971). The concept of "global hyperbolicity" due to Leray and further developed by Choquet-Bruhat is a strengthening of the condition of global causality leading to global existence theorems of associated linear hyperbolic equations.

2. Causality in groups

A *causal group* is defined as a Lie group with an invariant causal orientation. Although in a vector group there are continuum many invariant causal orientations, in general Lie groups do not admit invariant causal orientations. We shall be particularly interested in cases in which they do admit such, but shall first discuss the general existence question.

Scholium 2.2 An open simple Lie group G admits an invariant causal orientation if and only if there exists an element $X \in \mathbf{G}$ such that if $a_1, \ldots, a_n$ are arbitrary in G and $c_1, \ldots, c_n$ are arbitrary nonnegative numbers, and if $\sum_j c_j \operatorname{ad}(a_j)(X) = 0$, then all $c_j = 0$.

Proof Note first that the instantaneous present of a simple group with invariant causal orientation necessarily vanishes. For it determines a linear subspace of the Lie algebra which is invariant under all inner automorphisms, and hence an ideal.

Now to show the "if" part of the scholium, define $\mathbf{C}$ as the closure of the set of all $\sum_j c_j \operatorname{ad}(a_j)(X)$, X being the fixed element of $\mathbf{G}$ which is given, and the a_j and c_j being as described in the scholium, and otherwise arbitrary. Then $\mathbf{C}$ is a closed convex invariant subset of the Lie algebra $\mathbf{G}$. It is nontrivial because if a convex set is dense in a finite-dimensional linear space, it must be all of the space. Thus if $\mathbf{C}$ is all of $\mathbf{G}$, every vector in $\mathbf{G}$ has the form $\sum_j c_j \operatorname{ad}(a_j)(X)$ for some nonnegative $c_1, \ldots, c_n$ and suitable $a_1, \ldots, a_n$ in G. But if $-X$ has this form, a contradiction to the hypothesis regarding X follows.

† Pugh (1967). It is possible to avoid the use of this result by a direct elementary argument due to L. Hörmander.

To prove the "only if" part of the scholium, let **C** be a nontrivial cone in **G** defining an invariant causal orientation, and let X be an arbitrary nonzero element of **C**. If $\sum_j c_j \operatorname{ad}(a_j)(X) = 0$ and not all $c_j = 0$, it follows that $-X = \sum_k c'_k \operatorname{ad}(a'_k)(X)$ for suitable nonnegative c'_k and elements a'_k of G, showing that $-X$ is in the instantaneous present. By the initial observation, this is in contradiction with the assumed simplicity of G.

Corollary 2.2.1 The group $O(n, 1)$ admits no invariant causal orientation if $n \geqq 3$.

Lemma Every element of the Lie algebra of $O(n, 1)$, $n \geqq 3$, is contained in the Lie algebra of some $O(3, 1)$ subgroup.

This follows by infinitesimalization of results of Wigner (1939) (cf. also Philips and Wigner, 1968).

Proof of corollary In view of the lemma, it suffices to show that every nonzero element of the Lie algebra of $O(3, 1)$ violates the condition of the scholium. Indeed, for every such element X there exists an element a of $O_0(3, 1)$ (where here and henceforth the subscript 0 to a group indicates the connected component containing e) such that $\operatorname{ad}(a)(X) = -X$. For as a Lie group, $O(3, 1)$ is locally isomorphic to $SL(2, C)$, and its Lie algebra correspondingly to that constituted by the 2×2 matrices of zero trace. Any such matrix is similar either to one of the form

$$\begin{pmatrix} l & 0 \\ 0 & -l \end{pmatrix} \quad \text{or to} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that any such matrix is similar to its negative. But a similarity transformation on such a matrix corresponds precisely to the action of $\operatorname{ad}(a)$ on the Lie algebra, for suitable a.

After this chapter was written, a general criterion for the case of semisimple groups was obtained by B. Kostant. With his permission, a slight modification of his treatment is given here.

Theorem (*Kostant*) Let G be a semisimple Lie group, and let R be a representation of G on the real finite-dimensional linear vector space **V**. Let K be a Lie subgroup of G such that $R(K)$ is maximal compact in $R(G)$. Then there exists an $R(G)$-invariant closed convex cone **C** in **V** such that $\mathbf{C} \cap -\mathbf{C} = \{0\}$ if and only if there exists a nonvanishing $R(K)$-invariant vector in **V**.

Proof It is evidently no essential loss of generality to assume that R is faithful, and to take R as the representation $A \to A$. Now suppose that **C** is a given cone with the indicated properties. Then there exists a linear functional λ on V such that $\lambda(x) \geqq 0$ for all $x \in \mathbf{C}$ and $\lambda(x_0) > 0$ for some $x_0 \in \mathbf{C}$.

Let $w = \int_{R(K)} Ax\, dA$; then w is K-invariant, and $\lambda(w) > 0$, showing that $w \neq 0$.

Conversely, suppose there exists a nonvanishing K-invariant vector w in $\mathbf{V}$. For the proof, a positive definite bilinear from β on $\mathbf{V}$ with the property that $\beta(Aw, w) \geqq 0$ for all $A \in G$ will be constructed. Toward this end, let $\mathbf{G} = \mathbf{K} + \mathbf{P}$ be the Cartan decomposition of $\mathbf{G}$, the Lie algebra of G, so that $G = PK$ is the polar decomposition of G, where $P = \exp \mathbf{P}$. Then $\mathbf{K} + i\mathbf{P} \equiv \mathbf{G}_u$ is a compact form of the complexification $\mathbf{G} + i\mathbf{G}$, and hence the complexification $\mathbf{V}_c = \mathbf{V} + i\mathbf{V}$ of $\mathbf{V}$ can be given a complex Hilbert space structure in such a way that the elements of $\mathbf{G}_u$ are skew-Hermitian. Let B denote the restriction to $\mathbf{V} \times \mathbf{V}$ of the Hilbert space inner product. Then all $X \in \mathbf{P}$ are Hermitian, which implies that A is positive definite for $A \in P$, i.e., $\langle Aw, w\rangle \geqq 0$ for all $A \in P$ and $w \in \mathbf{V}$. But any $B \in G$ is of the form $B = AU$, where $A \in P$ and $U \in K$. Since $Bw = Aw$, $\beta(Bw, w) \geqq 0$ for all $B \in G$.

Now let $\mathbf{C}_0$ denote the set of all finite linear combinations with positive coefficients of the Aw with $A \in G$. Then $\mathbf{C}_0$ is a G-invariant convex cone, and $\beta(v, w) \geqq 0$ for all $v \in \mathbf{C}_0$; it follows that $\beta(v, v') \geqq 0$ for all v and v' in $\mathbf{C}_0$. The closure $\mathbf{C}$ of $\mathbf{C}_0$ has the same properties. Finally, if z is both in $\mathbf{C}$ and $-\mathbf{C}$, then $\beta(z, z)$ and $\beta(z, -z)$ are both nonnegative, implying that $\beta(z, z) = 0$, and hence that $z = 0$.

Corollary If $\mathbf{G}$ is simple, then there exists a closed convex cone $\mathbf{C}$ in $\mathbf{G}$ such that $\mathbf{C} \cap -\mathbf{C} = 0$, and which is invariant under the adjoint representation, if and only if G/K is Hermitian symmetric, or equivalently, if the center of K has dimension 1.

Proof It is known that when $\mathbf{V}$ is irreducible, there exists at most one $R(K)$-invariant vector (within a scalar factor). The theorem then implies that the indicated cone $\mathbf{C}$ exists if and only if the dimension of the centralizer of $\mathbf{K}$ in $\mathbf{G}$ is 1. But according to a result of E. Cartan, as a real space $\mathbf{P}$ is irreducibly invariant under ad $\mathbf{K}$, and so contains no nonzero elements which commute with $\mathbf{K}$. It follows that $\mathbf{C}$ exists if and only if the center of $\mathbf{K}$ is one-dimensional, which, by another result of Cartan, is equivalent to G/K being Hermitian symmetric.

Discussion The maximal compact subgroup of $SO(p, q)$ is $SO(p) \times SO(q)$ implying that the Lie algebra of $SO(p, q)$ contains a cone $\mathbf{C}$ of the indicated type if and only if either p or q is two. On the other hand, it follows that $SU(p, q)$ (whose maximal compact subgroup is $SU(p) \times SU(q) \times U(1)$ for $pq \neq 0$), and $Sp(2n, R)$ (whose maximal compact subgroup is $U(n)$), and exceptional cases corresponding to E_6 and E_7 always admit such a cone.

In the case of a simple group, if there exists any nontrivial invariant convex cone **C**, it is necessarily of the indicated type, since $\mathbf{C} \cap -\mathbf{C}$ is an ideal. In all probability, if **C** exists at all, it is unique (the integration argument given earlier shows it to be minimal), contains interior points, and coincides with the positive-energy cone for suitable unitary representations, i.e., the subset $\mathbf{C} = [X : iU(X) \geqq 0]$ for the representation U. In general, the conoid C in G generated by **C**, i.e., the closure of $\bigcup_{n=1}(\exp \mathbf{C})^n$ in G lacks the important property that $C \cap C^{-1} = \{e\}$, i.e., G lacks a nontrivial *finite* sense of future displacement corresponding to the infinitesimal one defined by **C**; but this is conjecturally the case if G is simply connected.

Scholium 2.3 In a causal group, the exponential map is locally causality-preserving (the Lie algebra being linearly causally oriented by its given cone): if $X \prec Y$, then $e^X \prec e^Y$ and if $e^{tX} \prec e^{tY}$ for all small $t > 0$, then $X \prec Y$.

Proof Since this is a local question, it is no essential loss of generality to take the group G to be a group of matrices (by Ado's theorem); the Lie algebra may then be identified with a Lie algebra of matrices. It is evidently sufficient (since dT is a linear isomorphism at each point in a sufficiently small neighborhood) to show that if T denotes the map $X \to e^X$, then dT carries any vector in the cone at X, $X + \mathbf{C}$ (identified by virtue of linearity with a subset of the tangent space at X) into a vector in the forward cone $C(e^X)$.

Consider the ray $[X + \varepsilon Z : \varepsilon \geqq 0]$, where Z is a fixed element of **G**. This ray maps into the arc $e^{X+\varepsilon Z}$, $\varepsilon \geqq 0$, in G. To say that this arc is in a forward direction at $\varepsilon = 0$ is to say that $(\partial/\partial\varepsilon)e^{X+\varepsilon Z}e^{-X}|_{\varepsilon=0}$ lies in **C**. In fact, by Duhamel's principle,

$$e^{X+\varepsilon Z} = e^X + \int_0^1 e^{(1-s)X}(\varepsilon Z)e^{s(X+\varepsilon Z)}\,ds.$$

From this it follows that $(\partial/\partial\varepsilon)e^{X+\varepsilon Z}e^{-X} = \int_0^1 e^{sX}Ze^{-sX}\,ds$. Since **C** is invariant and closed, the last integral has its value in **C**.

To show that $e^{tX} \prec e^{tY}$ for all small t implies that $X \prec Y$; it suffices, noting that $e^{-tX}e^{tY} = e^{t(Y-X)} + O(t^2)$, to treat the case $X = 0$, which follows from the Leray (1952) theory.

Corollary 2.3.1 The unicover $\tilde{U}(n)$ of $U(n)$ (in the causal orientation earlier indicated) is globally causal.

Proof The unicover $\tilde{U}(n)$ is isomorphic to $R^1 \times SU(n)$, the covering transformation being $(t, u) \to e^{2\pi i t}u$. To show that $\tilde{U}(n)$ is globally causal it suffices to show that if (t_j, u_j), $j = 1, 2$, are any two points such that $(t_1, u_1) \prec (t_2, u_2)$, then $t_1 < t_2$, for there can then exist no nontrivial closed timelike arc. By compactness, it suffices to show this when the two points are arbitrarily close; they may then be taken to be in $U(n)$ rather than in the unicover, and by invariance it is no essential loss of generality to take one of the points as the group unit I. The result then reduces to showing that if w is sufficiently near e in $U(n)$ and if $e \prec w$, then $0 < t(w)$, where $t(w) = (2\pi ni)^{-1} \log \det w$; this is again a consequence of the Leray theory.

To show that the future and past of any point is closed, it suffices to use a criterion of Choquet-Bruhat (1971) for global hyperbolicity, according to which this is implied by the existence of a complete Riemannian metric on the manifold such that the timelike arcs from one point to another have bounded length. Using the direct product metric on $R^1 \times SU(n)$ (the usual one on R^1, any on $SU(n)$), this follows from the compactness of $SU(n)$ and what has been shown above.

***Definition** 7* A *forward displacement* in a causally oriented manifold M is a causal automorphism T such that $x \prec Tx$ for all $x \in M$.

Scholium 2.4 Let M be a manifold with infinitesimal causal orientation, whose corresponding finite relation $p \prec q$ defines a finite globally causal orientation (respectively, manifold with finite causal orientation). Let G be any C^1 Lie transformation group on M, which is represented by causal automorphisms of M. Let $\mathbf{C}$ denote the set of all elements X in $\mathbf{G}$ such that $\exp(tX)$ is a forward displacement for all $t > 0$ (respectively, near each point p is a forward displacement for all sufficiently small $t \geqq 0$). If $\mathbf{C} \neq 0$, then G is invariantly causally oriented by the designation of $\mathbf{C}$ as causal cone; and is globally causal if M is such.

Proof If $X, Y \in \mathbf{C}$, then $\exp[t(X + Y)] = \lim_n(\exp(tX/n)\exp(tY/n))^n$, which represents $\exp[t(X + Y)]$ as a limit of products of forward displacements; by the results just cited, any such product is again a forward displacement, as is any limit of such. Since $\mathbf{C}$ is invariant under multiplication by positive scalars by its definition, it follows that $\mathbf{C}$ is convex. Another application of the fact that the future of a point is a closed set shows that $\mathbf{C}$ is closed.

The elements of G act as causal automorphisms, and so transform by conjugation any forward displacement into another forward displacement. It follows that $\mathbf{C}$ is invariant under $\mathrm{ad}(G)$. Now if g is a function from $[0, 1]$ to G defining a timelike arc, and if x is arbitrary in M, then gx defines a timelike arc in M. If M is globally causal and if g is closed, it follows that gx is constant on $[0, 1]$, i.e., g is constant on $[0, 1]$, which means that G is globally causal.

***Example** 7* The causal automorphism group of Minkowski space M is the 11-parameter group consisting of the inhomogeneous Lorentz transformations augmented by scale transformations. This is a Lie group, which acts analytically, and so is globally causally oriented by the designation of $\mathbf{C}$ as those Lie algebra elements that generate forward displacements. This orientation is invariant, and consists of the sums of infinitesimal scale transformations with forward vector displacements.

Scholium 2.5 Let M be a C^∞ causal manifold admitting a connected Lie group G of causal automorphisms, of which M is a homogeneous space; and suppose that the subgroup H of G leaving fixed one point of M has finitely many components.

Then the universal covering space $\tilde{M}$ of M is $\tilde{G}/H'$, where H' is the connected subgroup of $\tilde{G}$ whose Lie algebra is (locally) the same as that of H, and $\tilde{G}$ acts causally on $\tilde{M}$.

Lemma 2.5.1 If G is a connected and simply connected Lie group and H is a closed connected subgroup, then $M = G/H$ is simply connected.

Proof Let $t \to m(t)$, $t \in [0, 1]$, be a continuous arc in M, with $m(0) = m(1) = e'$, where $e' = \phi(e)$, ϕ being the canonical map of G onto G/H; it must be shown that $m(\cdot)$ is homotopic to a trivial map. From the known local form of G/H, for any $s \in [0, 1]$ and for t sufficiently near to s, there exists a smooth arc $t \to g(t)$ in G such that $m(t) = \phi(g(t))$. Combining this with the simple connectivity of G, it follows that there exists a continuous arc $t \to g(t)$ defined for all $t \in [0, 1]$, such that $m(t) = \phi(g(t))$, $t \in [0, 1]$. Since the exponential map from the Lie algebra **G** to G has dense range, it is no essential loss of generality, for the purpose of showing that the arc $m(\cdot)$ is homotopically trivial, to assume that $g(1)$ lies on a one-parameter subgroup of H; otherwise, $m(1)$ and $g(1)$ may be displaced by arbitrarily little to achieve this situation, without affecting the homotopic character of the arcs in question. Now let $g'(t) = \exp(tX)$, where X is an element of the Lie algebra of **H** of H such that $\exp(X) = g(1)$. Then $g(\cdot)$ and $g'(\cdot)$ are homotopic in G; it follows that $\phi \circ g(\cdot)$ and $\phi \circ g'(\cdot)$ are homotopic in M; but the latter path is trivial.

Proof of scholium Let D be the discrete central subgroup of $\tilde{G}$ such that $\tilde{G}/D$ is isomorphic to G, and let θ denote the canonical homomorphism of $\tilde{G}$ onto G. Then $\theta(H') \doteq H_0$, for $\theta(H')$ is a connected subgroup of G with the same Lie algebra as H_0, where H_0 is the component of the identity of H. It follows that the map $gH' \to \theta(g)H$ is well defined from $\tilde{G}/H'$ onto G/H_0. By general Lie theory and the fact that θ is a local isomorphism near the group unit, the indicated map is also a local homeomorphism, and hence is a covering transformation of $\tilde{G}/H'$ onto G/H_0, which in turn covers M finitely. Since $\tilde{G}/H'$ is simply connected by the lemma, it is the unicover of M.

In order to establish the causality of the action of $\tilde{G}$ on $\tilde{M}$, it suffices to show that every one-parameter subgroup of $\tilde{G}$ acts causally on $\tilde{M}$, for $\tilde{G}$ is generated by these, and any product of causality-preserving transformations is again such. Consider first the case in which $H = H_0$. Let ϕ denote the indicated covering transformation of $\tilde{M}$ onto M: $\phi(\tilde{g}H') = \theta(\tilde{g})H$. Then it follows that $\phi(\tilde{g}_1 \tilde{g}_2 H') = \theta(\tilde{g}_1)\phi(\tilde{g}_2 H')$. Now observe the

Lemma 2.5.2 Define a diffeomorphism T at a point p of a causal manifold M as being *causal* at p in case dT_p carries C_p into C_{Tp}. Now let T_t be a one-parameter C^∞ group of diffeomorphisms of M, and suppose that for each point $p \in M$, there exists an $\varepsilon(p) > 0$ such that T_t is causal at p if $|t| < \varepsilon(p)$. Then T_t is causality-preserving, for every t.

Proof Let $\bar{t}$ denote the supremum of the values $t > 0$ such that T_s is causal at p for all $s \in [0, t]$; if $\bar{t} = \infty$, the conclusion of the lemma is valid, so suppose $\bar{t} < \infty$. Now T_s is causal at $T_{\bar{t}}p$ if $|s| < \varepsilon'$ for some $\varepsilon' > 0$. But this means that $T_{\bar{t}+s}$ is causal at p, contradicting the assumption that $\bar{t} < \infty$, and completing the proof.

To conclude the proof for the case $H = H_0$ it now suffices to show

Lemma 2.5.3 Every transformation on $\tilde{M}$ corresponding to an element of a one-parameter subgroup of $\tilde{G}$ is causality-preserving.

Proof Let $\tilde{p} = \tilde{g}_0 H'$ be an arbitrary point of $\tilde{M}$, and let $\tilde{X}$ be arbitrary in $\tilde{\mathfrak{G}}$. Then ϕ is locally a diffeomorphism near $\tilde{p}$, say ϕ is a diffeomorphism of the open set $\tilde{R}$ having $\tilde{p}$ in its interior onto the open subset R in M having $p = \phi(\tilde{p})$ in its interior. Specializing the relation indicated above, for any real t, $\phi(\exp(t\tilde{X})\tilde{g}_0 H') = \exp(tX)\phi(\tilde{g}_0 H')$, where $X = d\theta(\tilde{X})$. Let $\varepsilon > 0$ be so small that $\exp(t\tilde{x})\tilde{p} \in \tilde{R}$ and $\exp(tX)p \in R$ for $|t| < \varepsilon$. The two local one-parameter groups involved here then have equivalent action near $\tilde{p}$ and p, as implemented by ϕ; since $\exp(tX)$ is causality-preserving on M, this shows that $\exp(t\tilde{X})$ is causal at $\tilde{p}$, for sufficiently small t. It follows from the immediately preceding lemma that T_t is causality-preserving on $\tilde{M}$ for all t.

The general case reduces to the case in which $H = H_0$ once it is shown that the action of G on G/H_0 is causal; but the local action of G on G/H_0 is identical with its local action on M.

3. Causal morphisms of groups

We now consider groups of causality-preserving transformations on specific classes of causal groups. These transformations are not necessarily group automorphisms (e.g., vector translations on Minkowski space preserve causality but are not automorphisms of the vector group by which the space may be represented as a causal group); to avoid confusion, we shall use the term *causal morphism* rather than causal automorphisms to refer briefly to a causality-preserving analytic homeomorphism on a causal group.

The notation $\mathbf{H}(n)$ will refer to the real linear space of all $n \times n$ complex Hermitian matrices, as a causal linear manifold, the cone $\mathbf{C}(n)$ being taken as the matrices H which are $\geqq 0$. When indicated by the context, $\mathbf{H}(n)$ will be identified in the usual way with the Lie algebra of the $n \times n$ unitary group $U(n)$, whose causal orientation will be taken as that defined by the indicated cone.

Scholium 2.6 If $n \geqq 2$, every one-to-one transformation T of $\mathbf{H}(n)$ onto itself leaving 0 fixed and such that $T(H) \leqq T(H')$ if and only if $H \leqq H'$ is of the form $T(H) = G^*F(H)G$, where G is an arbitrary nonsingular matrix and F is either the map $F(H) = H$ or $F(H) = \bar{H}$.

Proof According to a theorem of Alexandrov (1967), any such transformation T is necessarily affine, in the real linear space $\mathbf{H}(n)$. The proof is concluded by reference to the result that any linear transformation on $\mathbf{H}(n)$ which is an isomorphism for the order relation has the indicated form.

Scholium 2.7 For any transformation $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $SU(n, n)$, let $\rho(T)$ denote the transformation $U \to (AU + B)(CU + D)^{-1}$ on $U(n)$. The map $T \to \rho(T)$ is a homomorphism of $SU(n, n)$ into the group of all causal morphisms of $U(n)$.

It is well known that the indicated transformation $\rho(T)$ does indeed act on $U(n)$, and that ρ is a homomorphism. To show that $\rho(T)$ is a causal morphism it suffices to show that for arbitrary $U \in U(n)$ and arbitrary Hermitian $H \geqq 0$, then

$$-i(\partial/\partial\varepsilon)T(e^{i\varepsilon H}U)T(U)^{-1}|_{\varepsilon=0} \geqq 0,$$

where $T(U)$ denotes $(AU + B)(CU + D)^{-1}$. By straightforward differentiation, the indicated derivative is

$$\begin{aligned} AHU(AU + B)^{-1} &- (AU + B)(CU + D)^{-1}CHU(AU + B)^{-1} \\ &= [A - (AU + B)(CU + D)^{-1}C]HU(AU + B)^{-1}. \end{aligned}$$

To show that the last expression is nonnegative, it suffices to show that

$$A - (AU + B)(CU + D)^{-1} - C = [U(AU + B)^{-1}]^*.$$

This putative equality transforms by simple reversible operations into the putative equalities

$$\begin{aligned} A - (AU + B)(CU + D)^{-1}C &= (U^*A^* + B^*)^{-1}U^* \\ AU - (AU + B)(CU + D)^{-1}CU &= (U^*A^* + B^*)^{-1}; \\ (U^*A^* + B^*)^{-1}[(U^*A^* + B^*)AU - (U^*C^* + D^*)CU] &= (U^*A^* + B^*)^{-1}; \\ (U^*A^* + B^*)AU - (U^*C^* + D^*)CU &= I, \end{aligned}$$

and this last equation follows from the relations $A^*A - C^*C = I$, $B^*A - D^*C = 0$, which are implied by the assumption that $T \in SU(n, n)$.

Scholium 2.8 The Lie algebra of $U(n)$, as identified with $\mathbf{H}(n)$, is causally isomorphic to the open dense subset

$$U_*(n) = [U \in U(n) : \det(I - U) \neq 0],$$

via the Cayley transform $H \to (H - iI)(H + iI)^{-1}$.

Proof It suffices to show that if H and F are any fixed Hermitian matrices, and if $H(\varepsilon) = H + \varepsilon F$, then setting

$$U(H) = (H - iI)(H + iI)^{-1} \lim_{\varepsilon \to 0} (i\varepsilon)^{-1}[U(H(\varepsilon))U(H)^{-1} - I] \geqq 0$$

if and only if $F \geqq 0$, for the indicated limit relation means that the arc $\varepsilon \to U(H(\varepsilon))$ ($\varepsilon \geqq 0$) has a timelike forward direction at $\varepsilon = 0$. Now

$$\begin{aligned}(i\varepsilon)^{-1}[U(H(\varepsilon))U(H)^{-1} - I] &= -2(i\varepsilon)^{-1}[i(H + iI)^{-1} \\ &\quad - i(H(\varepsilon) + iI)^{-1}]U(H)^{-1} \\ &= 2\varepsilon^{-1}(H(\varepsilon) + iI)^{-1}[(H + iI) \\ &\quad - (H(\varepsilon) + iI)](H + iI)^{-1}U(H)^{-1} \\ &\quad - 2(H + iI)^{-1}F(H - iI)^{-1}.\end{aligned}$$

The last expression is of the form G^*FG where G is nonsingular, and so is nonnegative if and only if F is such.

Remark 2 It is very likely that all causal morphisms of $U(n)$ are of the form treated in Scholium 2.7. It is also likely that, within conjugacy, the Cayley transform is the most general open causal transformation from a linear causal manifold (i.e., one admitting a linear structure in such a way that the future of any point x has the form $x + C$ for some closed convex cone C) into $U(n)$, provided $n > 1$. (It is easily seen that this is not the case when $n = 1$.) This would follow directly if the Alexandrov theorem were true as a local theorem, but it is not: local conformal transformations in Minkowski space are local causal automorphisms, without necessarily being locally affine.

In a different but related vein, it is probable that any one-to-one transformation which preserves the causal structure in a smooth causal manifold is necessarily smooth, provided the defining causal cones are proper. (This means they should contain no full lines and should not be the direct product of a ray and a cone of lower dimension.) A partial result in this direction has been given by Choquet-Bruhat (1971); for manifolds which are globally hyperbolic in the sense of Leray, automorphisms in the indicated sense are necessarily continuous. It also seems probable that the group of all such automorphisms is finite-dimensional, again in the proper case. That some such restriction is necessary is shown by the case of two-dimensional Minkowski space; see Zeeman (1964).

Corollary 2.8.1 For any transformation $T \in \tilde{S}\tilde{U}(n, n)$, let $\tilde{\rho}(T)$ denote the action on the unicover $\tilde{U}(n)$ earlier indicated. Then $\tilde{\rho}$ is a homomorphism of $\tilde{S}\tilde{U}(n, n)$ into the group of all causal automorphisms of the globally causal space $\tilde{U}(n)$; and $\tilde{S}\tilde{U}(n, n)$ is itself globally causal with respect to the causal orientation naturally induced on it.

Proof This follows directly from earlier scholia, together with the results concerning $U(n)$ just obtained.

4. Causality and conformality

As already seen, Minkowski space is closely related to the space $U(2)$ in the series $U(n)$ of causal groups. It may also be represented in terms of projective quadrics in a way that brings out the relations between Minkowski spaces of different dimensions. Instead of the series $SU(n, n)$ of groups, the series $O(n, 2)$ intervenes. Considerations of conformality play a general role for this series.

Definition 8 A *conformal linear space* is a pair $(\mathbf{L}, \tilde{Q})$, where $(\mathbf{L}, Q)$ is a pseudo-Euclidean space (i.e., $\mathbf{L}$ is a real finite-dimensional linear vector space; Q is a nondegenerate symmetric bilinear form on $\mathbf{L}$), and $\tilde{Q}$ denotes the equivalence class of symmetric forms containing the given form Q, equivalence being defined as proportionality via a nonzero constant. For any given pseudo-Euclidean space $(\mathbf{L}, Q)$, the conformal linear space $(\mathbf{L}, \tilde{Q})$ is called the *induced* (or associated, or corresponding) conformal linear space; and $\tilde{Q}$ is called a linear conformal structure on $\mathbf{L}$.

A pseudo-Riemannian space is a pair (S, g) where S is a real C^∞ manifold and g assigns to each point x of S a nondegenerate symmetric bilinear form g_x on the tangent space S_x to S at x, in a C^∞ manner (i.e. in terms of local coordinates, the coefficients of g are C^∞). A conformal transformation from one pseudo-Riemannian space (S, g) into another (S', g') is a C^∞ homeomorphism T from S into S' such that for every point $x \in S$, the differential dT_x of T at x is a linear conformal isomorphism of $(S_x, \tilde{g}_x)$ into $(S'_{Tx}, \tilde{g}'_{Tx})$. When such a transformation T exists, the pseudo-Riemannian spaces (S, g) and (S', g') are said to be *conformally equivalent.*

A conformal space is a pair (S, q) consisting of a C^∞ manifold S together with a mapping q from each point x of S to a linear conformal structure on S_x, which near each point x_0 has the form $q_x = \tilde{g}_x$ for some pseudo-Riemannian structure g near x_0. Conformal transformations between conformal spaces, conformal equivalence of conformal spaces, etc. are defined correspondingly.

(The foregoing definitions are basically very well known; but because of slight variations in the literature, it has seemed desirable to make them explicit here; this serves also to indicate some notations.)

Now let $(\mathbf{L}, Q)$ be an arbitrary pseudo-Euclidean space. Let θ denote the mapping $x \to \tilde{x}$ from $\mathbf{L}$ onto the corresponding projective space $\tilde{\mathbf{L}}$ of all lines of $\mathbf{L}$. The manifold $\mathbf{Q} = [\tilde{x} \varepsilon \tilde{\mathbf{L}} : Q(x, x) = 0]$ is called the *projective quadric determined by Q.*

The orthogonal group $\mathbf{O}(Q)$ is defined to consist of all linear transformations T on $\mathbf{L}$ leaving invariant the form Q. If T is any nonsingular linear transformation on $\mathbf{L}$, the transformation $\tilde{T} : \tilde{x} \to \tilde{T}\tilde{x}$ is the projectivity induced by T. If $T \in \mathbf{O}(Q)$, then $\tilde{T}$ leaves M invariant, and the map $T \to \tilde{T} \mid M$ is a homomorphism of $\mathbf{O}(Q)$ into the group of all projectivities of $\tilde{\mathbf{L}}$ which leave M invariant. The image of $\mathbf{O}(Q)$ will be called the projective group on M, and denoted $\mathbf{P}(Q)$; locally, it is isomorphic with $\mathbf{O}(Q)$ via the indicated mapping.

Scholium 2.9 For any given real nondegenerate quadratic form on a finite-dimensional linear space $\mathbf{L}$, there exists a unique C^∞ conformal structure on the associated projective quadric which is invariant under the projective group of the quadric.

Specifically, this structure is given as follows in the notation just indicated:

Every tangent vector λ to $\mathbf{Q}$ at a point $\tilde{x}$ has the form $d\theta(\lambda')$ for some tangent vector λ' to $\mathbf{L}$ at x such that $Q(\lambda', x) = 0$ (making the canonical identification of the tangent space at x with $\mathbf{L}$), and conversely every vector of the form $d\theta(\lambda')$ is tangent to $\mathbf{Q}$ at $\tilde{x}$; and the linear conformal structure in the tangent space $T_{\tilde{x}}$ is determined by the quadratic form $g(\lambda_1, \lambda_2) = Q(\lambda'_1, \lambda'_2)$, where $\lambda_j = d\theta(\lambda'_j)$ $(j = 1, 2)$.

Proof That a unique conformal structure on $\mathbf{Q}$ is obtained in the indicated fashion is a matter of elementary calculus on manifolds, with the use of Euler's theorem on homogeneous functions. That this structure is invariant under the projective group follows from its invariant form. To show it is the unique such structure, it suffices to show that at each point $\tilde{x}$ of $\mathbf{Q}$, there exists a unique linear conformal structure in $T_{\tilde{x}}$ which is invariant under the induced action of the group of projectivities of $\mathbf{Q}$ which leave $\tilde{x}$ fixed. Taking the point at infinity (as is no essential restriction, the projective group being transitive on $\mathbf{Q}$), this is a matter of showing that the pseudo-Euclidean group on R^k, extended by magnifications, relative to a nondegenerate quadratic form Q' on R^k, leaves invariant no linear conformal structure other than that determined by Q'. This is elementary.

Definition 9 A real nondegenerate quadratic form is said to be of type (a, b) if it may be expressed in terms of suitable linear coordinates as $x_{-1}^2 + \cdots + x_{-a}^2 - x_1^2 - \cdots - x_b^2$. Types of pseudo-Riemannian and conformal structures are similarly defined.

Scholium 2.10 With the same notation as in the preceding scholium, if Q is of type (a, b), with $0 < a \leqq b$, then $\mathbf{Q}$ is analytically conformal with the direct product of two spheres $S^{a-1} \times S^{b-1}$ modulo the direct product of the corresponding antipodal maps.

Proof Taking $\mathbf{Q}$ to have the form given in the preceding definition, every point of $\mathbf{Q}$ is of the form $\tilde{x}$ with x such that

$$x_{-1}^2 + \cdots + x_{-a}^2 = x_1^2 + \cdots + x_b^2 = 1.$$

With $S^{a-1} = [(x_{-1}, \ldots, x_{-a}) : x_{-1}^2 + \cdots + x_{-a}^2 = 1]$, the mapping from $S^{a-1} \times S^{b-1}$ into $\mathbf{Q}$,

$$\Pi: x = [(x_{-1}, \ldots, x_{-a}), (x_1, \ldots, x_b)] \to \tilde{x},$$

is therefore onto M. Evidently, $\tilde{x} = \tilde{y}$ with x and y in $S^{a-1} \times S^{b-1}$ if and only if $x = \pm y$, so that Π is a twofold covering of $\mathbf{Q}$ by $S^{a-1} \times S^{b-1}$. The antipodal map $A: x \to -x$ is thus such that $S^{a-1} \times S^{b-1}$ modulo the two-element group $\{1, A\}$ is analytically isomorphic to $\mathbf{Q}$ via the indicated mapping.

Using the fact that a tangent vector to S^{a-1} at $(x_1, \ldots, x_a)$ has the form $\sum_{i=1}^a u_i(\partial/\partial x_i)$ with coefficients u_i such that $\sum_{i=1}^a u_i x_i = 0$, the u_i being unique, and that the length of this tangent vector according to the standard Riemannian structure on S^{a-1} is $\sum_{i=1}^a u_i^2$, it is straightforward to compute the conformal structure given by the preceding scholium in terms of the x_i as coordinates, and verify that it agrees locally with that on the direct product $S^{a-1} \times S^{b-1}$ (and so is the same as that at the quotient of this product modulo A).

Definition 10 A conformal space (S, q) is said to be *conformally* [globally] *causal* in case q is of type $(1, c)$ $(c \geqq 1)$, and if S admits a [global] causal orientation whose cone at any point x consists of one of the two cones in the tangent space at x on which $g(\lambda, \lambda) \geqq 0$, where g is any pseudo-Riemannian structure defined near x which induces the conformal structure q.

Remark 3 Any covering space of a conformal manifold is again a conformal manifold, in a unique way so that the defining covering local homeomorphism is locally conformal; and is conformally [globally] causal if the original manifold is such. Compare the earlier remark on the lifting of causal structures to covering manifolds.

Scholium 2.11 Let $(\mathbf{L}, Q)$ be a given pseudo-Euclidean space such that Q is of type $(2, n + 1)$, $n > 1$. Then the projective quadric $\mathbf{Q}$ defined above is conformally causal; and its unicover $\tilde{\mathbf{Q}}$ is globally so.

Proof Taking Q to be of the form $x_{-1}^2 + x_0^2 - \sum_{i=1}^{n+1} x_i^2$ relative to suitable coordinates x_i on **L**, and setting $\mathbf{Q}' = S^1 \times S^n$, then $\mathbf{Q}'$ has a natural conformal structure, i.e., the direct product of the structures on its factors, and as shown earlier, covers **Q** twice via a local homeomorphism which is also conformal. This conformal structure on $\mathbf{Q}'$ is also causal, as may be seen in the following way. The tangent space T_x at any point $x \in \mathbf{Q}'$ is a linear subspace of the set of all tangent vectors to R^{n+3}, where $\mathbf{Q}'$ is imbedded in R^{n+3} via the mapping

$$[(x_{-1}, x_0), (x_1, \ldots, x_{n+1})] \to (x_{-1}, x_0, \ldots, x_{n+1}).$$

This subspace consists of all tangent vectors to R^{n+3} of the form $\lambda = \sum_{i=-1}^{n+1} u_i(\partial/\partial x_i)$ such that $u_{-1}x_{-1} + u_0 x_0 = \sum_{i=1}^{n+1} u_i x_i = 0$. The conformal structure on $\mathbf{Q}'$ at x may be correspondingly determined by the pseudo-Riemannian structure g given by the equation

$$g(\lambda, \lambda) = u_{-1}^2 + u_0^2 - \sum_{i=1}^{n+1} u_i^2.$$

Now let $C'(x)$ denote the set of all vectors λ tangent to $\mathbf{Q}'$ at x, such that $g(\lambda, \lambda) \geqq 0$ and $u_{-1}x_0 - u_0 x_{-1} \geqq 0$. It is easily seen that $C'(x)$ is a closed convex cone, and that it is C^∞ as a function of x in the sense earlier indicated. Thus $\mathbf{Q}'$ is conformally causal. The conformal structure on $\mathbf{Q}'$ is invariant under A, and the same is true of its causal structure. For the tangent vector $\lambda = \sum_i u_i(\partial/\partial x_i)$ at any point $x \in \mathbf{Q}'$ is carried by (the induced action of) A into the tangent vector $\mu = -\sum_i u_i(\partial/\partial x_i)$ at $-x$. Evidently

$$g_{-x}(\mu, \mu) = g_x(\lambda, \lambda)$$

and

$$u_{-1}x_0 - u_0 x_{-1} = (-u_{-1})(-x_0) - (-u_0)(-x_{-1}),$$

showing that $\mu \in C'(-x)$ if and only if $\lambda \in C'(x)$.

The quotient manifold $\mathbf{Q} = \mathbf{Q}'/\{1, A\}$ therefore acquires both the conformal and causal structure of M, by taking $C(y)$ for $y \in M$ as $d\eta(C'(x))$, where η denotes the canonical map from $\mathbf{Q}'$ onto $\mathbf{Q}$. (Note that a conformal–causal structure is determined by its causal structure alone.)

The universal covering manifold $\tilde{M}$ is evidently $R^1 \times S^{b-1}$, the projection map from $\tilde{M}$ to Q' being

$$(t, (x_1, \ldots, x_b)) \to (\cos t, \sin t, x_1, \ldots, x_b).$$

To show that no timelike arc in $\tilde{\mathbf{Q}}$ is closed, let $s \to y(s)$, $s \in [0, 1]$ be an arbitrary such arc, and let $u(s)$ and $v(s)$ denote the components of $y(s)$ in R^1

and S^{b-1} respectively. It suffices to show that either $u(1) > u(0)$, or $y(s) = y(0)$ for all s, for it then follows that the timelike arc is closed only if it is trivial. To this end it will suffice in turn to show that $u'(s) \geqq 0$, and $u'(s) = 0$ for all $s \in [0, 1]$ only if $y(s) = y(0)$ for all such s.

Observe in this connection that the transformations on $R^1 \times S^n$ of the form $T_1 \times T_2$, where T_1 is a translation in R^1 and T_2 is a rotation on S^n, are causal morphisms. In the case of T_1, translation through s is for sufficiently small s an action which locally is a causal morphism on **Q**, i.e., its differential maps the defining cones $C(p)$ appropriately; it follows by continuity that this is true globally on $\tilde{M}$, for all s. In the case of T_2, the argument is similar. Thus in order to show that $u'(s) \geqq 0$ and that $u'(s) = 0$ only if $y'(s) = 0$, it suffices to consider the case in which $y(s) = 0 \times (1, 0, \ldots, 0)$. Further, since the question is a local one, it may equally be determined in $\mathbf{Q}'$ rather than $\tilde{\mathbf{Q}}$, with $y(s)$ taken as the point of $\mathbf{Q}'$ covered by $0 \times (1, 0, \ldots, 0)$, i.e., $(1, 0) \times (1, 0, \ldots, 0)$. (Note that the covering of **Q** by $\tilde{\mathbf{Q}}$ may be factored into the covering of $\mathbf{Q}'$ by $\tilde{\mathbf{Q}}$, followed by η.) Now writing

$$y(t) = (x_{-1}(t), x_0(t), \ldots,)$$

near $t = s$, where $y(s) = (1, 0, 1, 0, \ldots, 0)$, then $x_0(t) = \sin u(t)$, showing that $u'(t) = (\cos u(t))^{-1}x_0'(t)$ near this point. Observing that $\cos u(t) = x_{-1}(t) > 0$ near $s = t$, and that for a timelike arc from $y(s)$, $x_0'(t) \geqq 0$ by the requirement that $u_0 x_{-1} - u_{-1} x_0 \geqq 0$ for a tangent vector in the cone $C(y(t))$, it results that $u'(t) \geqq 0$. Further, $u(s) = 0$ only if $x_0'(s) = 0$; but then $x_{-1}'(s) = -\sin u(s)u'(s)$ showing that $x_{-1}'(s) = 0$; and the requirement that $u_{-1}^2 + u_0^2 \geqq \sum_{i=1}^n u_i^2$ for a tangent vector $\sum u_j(\partial/\partial x_j)$ in $C(y(s))$ then implies that $x_j'(s) = 0$ for all j, i.e., $y'(s) = 0$.

Corollary 2.11.1 $\tilde{\mathbf{Q}}$ is globally hyperbolic in the sense of Leray.

By a theorem of Choquet-Bruhat, it suffices to show that for every fixed pair of points, the set of all timelike arcs from one to the other is bounded relative to a complete Riemannian metric on $\tilde{M}$. The direct product of the usual (translation-invariant) metrics on R^1 and S^n is such. The argument just given shows that a timelike arc from (t_1, p_1) to (t_2, p_2) is such that the component is monotone increasing, while the S^n component describes an arc whose length over any interval is bounded by the length of the R^1 component. Thus the total length is bounded by $2|t_2 - t_1|$.

5. Relation to Minkowski space

We next show that the quadric **Q** just considered has imbedded in it (algebraically, conformally, and chronogeometrically) Minkowski space as an open dense subset. The imbedding transformation is an analogue to the Cayley transform† in its chronogeometric properties.

The concept of "lightlike point" has chronogeometric significance but we give here the following purely algebraic

Definition 11 Let $x, y \in \mathbf{L}$ be nonzero and such that $Q(x, x) = Q(y, y) = 0$. Then "$\tilde{y}$ is lightlike relative to $\tilde{x}$" means that $Q(x, y) = 0$.

In order to describe the cited imbedding quite explicitly it is helpful to recall some aspects of spherical geometry. A *sphere* in the pseudo-Euclidean space (M, F) is defined as a subset of the form $[X \in M : F(X - X_0, X - X_0) = k$, where X_0 and k are fixed in M and R^1, respectively; a *null-sphere* is one for which $k = 0$. Note that the equation defining a sphere can be put in the form

$$aF(X, X) - 2F(X, X_0) + c = 0,$$

where $a \neq 0$, and that conversely all such equations define spheres; null-spheres are characterized by the condition that $ac - F(X_0, X_0) = 0$.

A *conformal sphere* in M is a subset of the form $[X \in M : aF(X, X) - 2F(X, X_0) + c = 0]$, where a and c are fixed in R^1, X_0 is fixed in M, and not all of a, c, and X_0 vanish (in other words, a conformal sphere is either a sphere in the usual sense, relative to the Minkowski metric, or a hyperplane); a conformal null-sphere is one for which $ac - F(X_0, X_0) = 0$. Denoting as $\mathbf{L}$ the vector space of dimension $\dim M + 2$ whose components are a, the vector X_0, and c; and as Q the form

$$Q(a, c, X_0; a', c', X_0') = F(X_0, X_0') - (ac' + a'c)/2,$$

it follows that the set of all conformal null-spheres in M is in one-to-one correspondence with the projective quadric $\mathbf{Q}$. The canonical mapping from $\mathbf{L}$ onto the projective space of all its rays will be denoted as θ. X^2 will signify $F(X, X)$.

Scholium 2.12 The mapping j: $X \to \theta((1, X, X^2))$ from M into $\mathbf{Q}$ is conformal, and has range equal to the set of all points of $\mathbf{Q}$ that are not lightlike relative to the point $\theta((0, 0, 1))$.

Proof To say that $\theta((a, X, c))$ is lightlike with respect to $P_0 = \theta((0, 0, 1))$ is to say that $a = 0$. Thus $\theta((1, X, X^2))$ is never lightlike with respect to P_0.

† From an abstract standpoint the present treatment may in part be regarded as a chronogeometrical interpretation of the generalized Cayley transform known for symmetric spaces, applicable to certain Siegel domains.

Conversely, if $\theta((a, X, c))$ is not lightlike with respect to P, then it is no essential loss of generality to take $a = 1$, and then $c = X^2$.

To show conformality of the mapping j, note first that for any fixed vector A in M, the mapping $T: X \to X + A$, is conformal. Observe next that there exists a transformation T' on **Q**, in the conformal group treated earlier, such that $jT = T'j$. Indeed, T' is the transformation

$$(a, X, c) \to (a', X', c')$$

where $a' = a$, $X' = X + aA$, $c' = c + 2X \cdot A + aA^2$; this transformation is easily seen to be a projectivity which is in the group defined earlier, leaving **Q** invariant, for any value of A. Now since the totality of transformations of the form $X \to X + A$ is transitive on M, it follows that it is sufficient to show conformality at one point, say the point $X = 0$.

At this point, the differential of the mapping j: $X \to \theta(1, X, X^2))$ is the mapping dj: $Y \to d\theta((0, Y, 0))$, by a simple computation, with the usual identification of tangent vectors in M with vectors in M; and reference to an earlier scholium shows this to be conformal.

Scholium 2.13 A conformal transformation of a conformally causal manifold into a conformal manifold that admits a causal orientation is causal if the latter manifold is suitably oriented.

Proof Observe first that a conformal transformation from one conformal-causal manifold into another is either causal or anticausal (the latter meaning that the precedence relation is reversed). For if T denotes the transformation and $C^{\pm}(x)$ the infinitesimal future and past cones at x, then dT_x carries $C^+(x)$ into either $C^+(Tx)$ or $C^-(Tx)$. The set of all points x such that the former eventuality holds is open and closed by continuity, and the same is true of the latter eventuality; and "manifold" is always connected in the present usage.

Corollary 2.13.1 If F is of type $(1, n)$, then Q is of type $(2, n + 1)$, and j is causal if **Q** is suitably oriented causally.

Proof Choosing coordinates $x_0, x_1, \ldots, x_n$ such that $F(X, X) = x_0^2 - x_1^2 - \cdots - x_n^2$, and introducing variables x_{-1} and x_{n+1} by the equations $a = x_{-1} + x_{n+1}$, $c = x_{n+1} - x_{-1}$, then Q takes the form

$$Q(a, B, c; a, B, c) = x_{-1}^2 + x_0^2 - x_1^2 - \cdots - x_{n+1},$$

where $B = (x_0, \ldots, x_n)$. Thus **Q** admits a causal orientation, and j is causal by the preceding scholium, for a suitable choice of one of the two possible orientations.

Definition 12 When endowed with the conformal-causal orientation such that j is causal, **Q** will be called *n-dimensional conformal space–time*, or the *conformal compactification of n-dimensional Minkowski space–time.*

6. Observers and clocks

We now analyze mathematically the concept of observer, at increasing levels of specificity. In this connection one is naturally led to treat such concomitants of observers as "clocks" and "rods." The concepts developed coincide with mathematical forms of the usual physical notions in the case of Minkowski space; further examples are given in the cases of the two series of causal manifolds earlier considered.

Definition 13 Let M be a given globally causal manifold. A *spatio-temporal factorization* of M (for brevity, simply *factorization*) is an equivalence class of prefactorizations, where a prefactorization is a pair (S, ϕ) consisting of a C^∞ manifold S and a diffeomorphism ϕ of $T \times S$ onto M, where T is a real interval having the properties that:

(i) For any fixed $x \in S$, the map $t \to \phi(t, x)$ is a timelike arc in M;
(ii) For any fixed $t \in T$, the map $x \to \phi(t, x)$ defines a spacelike submanifold of M.

Two such prefactorizations (S, ϕ) and (S', ϕ') are *equivalent* if there exist diffeomorphisms f and g of R^1 onto R^1 and S onto S' such that f is orientation-preserving, and

$$\phi(f \times g)^{-1} = \phi'.$$

(Thus, corresponding to any factorization there are trivial fiberings of M by timelike arcs on the one hand, and by spacelike submanifolds (automatically maximal, as such), on the other. Conversely, two factorizations are the same if and only if the corresponding fiberings of M are the same.)

If a is any causal morphism of M, the *transform* of any prefactorization (S, ϕ) (respectively factorization represented by this prefactorization) is defined as the prefactorization (S, ϕ') (respectively factorization represented by this prefactorization), where $\phi'(t, x) = a(\phi(t, x))$; and the prefactorizations (respectively corresponding factorizations) are said to be *conjugate.*

Example 8 If M is $(n + 1)$-dimensional Minkowski space, and if $x_0, x_1, \ldots, x_n$ are linear coordinates such that the fundamental quadratic form $F(X, X) = x_0^2 - x_1^2 - \cdots - x_n^2$, a prefactorization may be defined by taking $S = R^n$ and defining $\phi(t, \mathbf{x})$ for arbitrary $t \in R^1$ and $\mathbf{x} \in R^n$ as the point of M having coordinates $(t, x_1, \ldots, x_n)$, where $\mathbf{x} = (x_1, \ldots, x_n)$. A scale transformation on M (i.e., similarity transformation), or a Euclidean transformation of the space component $(x_1, \ldots, x_n)$ (as a causal transformation on M) carries this prefactorization into a different one, which is, however, equivalent. Further, if $t \to t'$ and $\mathbf{x} \to \mathbf{x}'$ are diffeomorphisms of R^1 onto R^1 and R^n onto R^n, the former being orientation-preserving, then defining $\phi'(t, \mathbf{x}) = \phi(t', \mathbf{x}')$, the pair (R^n, ϕ') is equivalent to the pair (R^n, ϕ).

Definition 14 If S is a C^∞ manifold, and if R^1 and S have given Finsler structures (in the sense of norms in each tangent space, the norms being only positive-homogeneous), the *causal product* of R^1 and S (with the given structures) is the manifold $R^1 \times S$ with the causal structure which assigns to the point (t, x) the cone consisting of all tangent vectors $a(\partial/\partial t) + X$ (X being a tangent vector to S at x) for which $a \geqq 0$ and $\|X\|_x \leqq a\|\partial/\partial t\|_t$ (the subscripts indicating evaluation of the norms in the corresponding tangent spaces). A *metric prefactorization* of a causal manifold consists of a prefactorization (S, ϕ) together with given Finsler structures on R^1 and S such that ϕ is a causal morphism of $R^1 \times S$ onto M. A *metric observer* is an equivalence class of such, where (S', ϕ') with given Finsler structures on R^1 and S' is equivalent to the preceding one if and only if there exist maps f and g as earlier, with the additional property that f and g are Finslerian isometries.

Example 9 With the usual metrics on R^1 and R^n, the preceding factorization of Minkowski space is metric. As another example, consider the universal covering group $\tilde{U}(n)$ of $U(n)$, with the causal structure earlier indicated. The representation $\tilde{U}(n) \cong R^1 \times SU(n)$ determines a metric factorization, which is relative to the usual metric on R^1, and the following Finslerian metric on $SU(n)$ (which is Riemannian only for $n = 2$): with the identification of the Lie algebra of $SU(n)$ with $H_0(n)$ earlier indicated, $\|H\| = \inf[t : tI + H \geqq 0]$. It is easily verified that this defines a norm (positively, although not fully homogeneous, in general). Now if $H \geqq 0$ and $H = tI + H_0$ where H_0 is of zero trace, then necessarily $t \geqq 0$ and $\|H_0\| \leqq t$. Conversely, if $t \geqq 0$ and $\|H_0\| \leqq t$, then $tI + H_0 \geqq 0$ by the definition of $\|H_0\|$. Note that 0 is an interior point of the closed convex set in $H_0(n)$ consisting of elements of norm $\leqq 1$, although this set is not symmetric about the origin.

Definition 15 For any metric observer (S, ϕ) on a causal manifold M, the mapping from M into R^1 endowed with the given Finsler structure, defined by the equation $X \to t$ if $X = \phi(t, x)$ for some x, is called the *clock* of the observer and a clock on M is a mapping of the indicated type which is the clock of some observer. The mapping from M into S endowed with the given Finsler structure, defined by the equation $X \to x$ if $X = \phi(t, x)$ for some t, is called the *chart* of the observer; and a chart on M is a mapping from M to a C^∞ manifold endowed with a given Finsler structure, which is the chart of some observer. A chart and a clock are said to *match* if they are the chart and clock of one observer.

A causal morphism A is *forward* if $x \prec Ax$ for $x \in M$; *backward* if $Ax \prec x$ for all $x \in M$; *temporal* if either forward or backward; and *spatial* if x and Ax are relatively spacelike for all x. A group of causal morphisms of M is called temporal if it consists entirely of temporal transformations, and spatial if it consists entirely of spatial transformations.

A prefactorization (respectively factorization) is said to be *temporally homogeneous* if the map $\phi(t, x) \to \phi(t + t', x)$ is a temporal morphism, for all $t' \in R^1$; *spatially homogeneous* (respectively, and *isotropic*) if there exists a group G_s of causal morphisms of M (necessarily spatial) and an isomorphism ψ of G_s into the group of all diffeomorphisms of S, such that if $g \in G_s$, g: $\phi(t, x) \to \phi(t, \psi(g)x)$, and if the group $\psi(G_s)$ is transitive (respectively transitive and isotropic, or transitive on directions at any fixed point) on S; *homogeneous* if both temporally homogeneous and spatially homogeneous and isotropic; (respectively if a representative prefactorization is such). A *homogeneous observer* is an equivalence class of homogeneous prefactorizations, equivalence being defined as earlier. A metric and homogeneous observer whose prefactorization is homogeneous, and whose temporal and spatial groups (i.e., respectively the group $T_{t'}$: $\phi(t, x) \to \phi(t + t', x)$, or the group of all causal morphisms g such that for some diffeomorphism ψ of S, g: $\phi(t, x) \to \phi(t, \psi x)$, for all t and x) leave invariant the respective Finsler structures, is called *physical*.

A causal manifold is *covariant* if it admits a homogeneous factorization and in addition, the subgroup leaving one point p fixed is transitive on the strictly timelike directions (i.e., those in the interior of $C(p)$) at p. A *covariant observer* is a physical observer whose factorization is of this type.

If G_t is a given continuous one-parameter group of temporal transformations on M, with normalized parameter, a *G_t-clock* on M (where G_t is short for the transformation group (G_t, M)) is a function F from M to R^1 such that $F(T_t y) = F(y) + t$ for all $t \in R^1$ and $y \in M$. Similarly, if G_s is a given continuous group of spatial transformations on M, a *G_s-chart* consists of a C^∞ transformation group (G_s, S), the action of G_s on the C^∞ manifold S, being faithful, together with a map F from M to S such that $F(gx) = g(F(x))$.

Example 10 (a) *Minkowski space* A physical observer on this space M is defined by the earlier factorization, together with the unique invariant Riemannian metric on R^n invariant under the Euclidean group, which acts on M as a group of spatial transformations through its action on the component **x**, and the unique translation-invariant metric on R^1, scaled so that causal cones $C(p)$ have the requisite form. Any two such observers, defined by coordinate systems of the indicated types, are conjugate within the causal morphism group. No other physical observers are known. As is well known, M is covariant, as are the indicated physical observers.

(b) *The spaces* $\tilde{U}(n)$ Taking $S = SU(n)$ and defining $\phi(t, u) = (t, u)$ for $t \in R^1$ and $u \in R^1$ and $u \in SU(n)$, then with the usual metric on R^1 and the Finsler metric on $SU(n)$ earlier indicated, and with the temporal group $T_{t'}$: $(t, u) \to (t + t', u)$; and spatial group $(t, u) \to (t, vuw)$ $(v, w \in SU(n))$, we have a physical observer. In addition to the Finsler metric on $SU(n)$ there is a unique invariant Riemannian metric on $SU(n)$, but this cannot in general be used to describe the causal structure on $\tilde{U}(n)$ in the way familiar in the case $n = 2$. $\tilde{U}(n)$ is covariant, for the causal morphism group is evidently transitive, so that it suffices to show that this group is appropriately transitive on the directions at a fixed point, say at the identity. This causal morphism group includes the action of $\tilde{S}\tilde{U}(n, n)$ earlier indicated, which is locally identical to the action of $SU(n, n)$ on $U(n)$. Locally the Cayley transform is causal, so that this action can be transferred to the Lie algebra $\mathbf{H}(n)$, and then includes the transformations $H \to K^*HK + L$, where K is an arbitrary nonsingular matrix, and L is arbitrary in $H(n)$. Those transformations for which $L = 0$ leave 0 invariant, and the strictly timelike directions are those of a nonsingular $H \in H(n)$ such that $H > 0$. It is evident that if H' is another such direction, then there exists a nonsingular K such that $H' = K^*HK$. The spatial isotropy follows similarly and more readily.

(c) *The spaces* $\tilde{Q}_n$ This also is covariant and admits a physical observer; the Finsler metric on the space component is in this case Riemannian. With $S = S^n$ and $\phi(t, u) = (t, u)$ in our earlier notation, we have a prefactorization; with the usual metric on R^1 and the unique orthogonally invariant one on S, we have an observer. The temporal group $t \to t + t'$ acts appropriately to establish temporal homogeneity. The spatial group includes the action of the orthogonal group on S, lifted up to $\mathbf{Q}_n$, and this is evidently transitive and isotropic. Covariance follows from the facts that: (a) locally $\tilde{Q}_n$ is causally identical to Minkowski space; (b) the global causal morphisms of Minkowski space may all be lifted up to $\mathbf{Q}_n$; (c) the Lorentz group acts transitively on the strictly timelike directions at any point of Minkowski space.

(d) It should be noted that if F is any finite central subgroup of $SU(n)$, then $R^1 \times SU(n)/F$ is locally isomorphic to $\tilde{U}(n)$, and thereby defines a chronogeometry having *locally* all of its key symmetry properties; but that these properties may fail to be valid *globally*. For example, as pointed out by J. W. Milnor, if $n = 2$ and $F = \{\pm 1\}$, the corresponding factor space admits only a 7-dimensional causal morphism group, in contrast to the 15-dimensional group admitted by $U(2)$ itself. In particular, the factor space lacks global temporal isotropy. The proof reduces by general considerations to the determination of the causal vector fields on $U(2)$, which commute with F and thereby to a simple matrix computation.

Remark 4 (a) No other known (nonconjugate) covariant observers on the foregoing causal manifolds exist.

(b) It is likely that similar results hold for the Shilov boundaries of arbitrary Hermitian symmetric spaces, with suitable causal orientations; more precisely, for the unicovers of such manifolds. In all likelihood, the component of the identity of the causal morphism group is the induced action on the boundary of the group of the Hermitian space; and the chronogeometric features of the Cayley transform, which has already been extended to the general setting of such spaces, carry over.

7. Local observers

It is essentially straightforward to extend the foregoing considerations to local, rather than global, observers, using the usual concepts of the theory of local transformation groups. One may arrive in this way at a mathematical counterpart to the familiar physical concept of "local Lorentz frame."

If (S, ϕ) is a prefactorization on M, and if T and U are connected open subsets of R^1 and S, then with $\phi_1 = \phi \mid T \times U$, (U, ϕ_1) is an observer on $T \times U$, except that T is only diffeomorphic to R^1. We call this prefactorization on $\phi(T \times U)$ the restriction to this region of the given prefactorization on M. A local prefactorization at a point p in a manifold M is defined as prefactorization on some neighborhood N_p of p; two such are (locally) equivalent if their restrictions to some common neighborhood are (globally) equivalent; a local factorization at p is finally an equivalence class of such local prefactorizations.

Local observers may also be metric, homogeneous, physical, or covariant, the definitions being straightforward adaptations of the corresponding global ones. Conjugacy of local observers is defined as conjugacy via a local rather than global causal morphism, the causal morphism in question being one which preserves the relevant structure (metric, physical, or covariant).

Example 11 $\tilde{U}(n)$ is locally causally isomorphic to $\mathbf{H}(n)$, and Minkowski space M_n is locally causally isomorphic to $\tilde{Q}_n$. Hence, restricting to a suitable neighborhood the global observers on M_n and $H(n)$ previously given, and then transferring these observers via the aforementioned local causal isomorphisms to $\tilde{U}(n)$ and $\tilde{Q}_n$, one obtains certain local observers on these latter manifolds. In no case are these locally conjugate to the earlier given global observers, even as observers.

Remark 5 It is interesting to note that Haantjes (1937) has shown that any smooth local conformal transformation on a pseudo-Euclidean space can be extended to a global conformal transformation on its conformal compactification. In particular, any local smooth one-parameter group of local causality-preserving transformations on $\tilde{M}$ can be extended to a global such group acting globally on $\tilde{M}$. Conjecturally, this is valid for other cosmos associated with simple Lie groups, such as the series $SU(n, n)$, but this appears not to be known.