# TIME, ENERGY, RELATIVITY, AND COSMOLOGY

I. E. Segal

Mathematics Department
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

1. Introduction. One of Einstein's major messages is that nothing is necessarily *a priori*; even the most unlikely subjects can be discussed and analyzed, sometimes with revolutionary conclusions. Most notably, he showed that space, time, and of course gravitation, were not at all *a priori*. Mathematical progress -- at a sophisticated ideational level, rather than at an elementary or classical problem-solving level -- provided the tools for a cogent physical analysis having striking observational consequences, explaining existing anomalies and making predictions which have been precisely confirmed.

What I should like to do here is to discuss and analyze, in a similar spirit (albeit very briefly), the concepts of time-and-energy (which are even more closely connected than space-and-time), as well as some physical implications of the discussion in one area of observation -- cosmology. What makes the mathematical parameter $t$ which we use in so many ways the *time*; or certain expressions $E$, the *energy*? In short, apart from tradition, authority, and crude pragmaticism, what makes the time the time, or the energy the energy, as they are used in special and general relativity, in classical and quantum mechanics?

What are the intrinsic characteristics of time and energy; and to what extent are these characteristics uniquely definitive? Are there other forms of time and energy that are physically conceivable, besides the conventional ones?--just as, as been realized in the past century or two, there are other forms of space-time, besides the Newtonian, or even the special (or even general) relativistic? The answers will be both explicit and physically applicable.



B. Gruber et al. (eds.), *Symmetries in Science*

2. The Einstein-Minkowski cosmos. The Einstein-Minkowski time was a much more sophisticated concept than the Newtonian time, and disposed in a convincingly simple manner of the Michelson-Morley anomaly. But with the advent of quantum mechanics, the question arose of why time was not (or did not appear to be) an operator, like other observables. Further, what was the relation between time as a coordinate of space-time events, and time as a group parameter, as in the temporal evolution group $U(t) = e^{itH}$, $H$ being the quantum hamiltonian? What, in these terms, is the observed time, in laboratories?

To work towards answers to these questions I begin with the central discovery of Einstein-Minkowski, which in modern terms is stated simply as: space-time is a manifold; the physical time is defined only relative to a particular coordinate system, which is arbitrary within a stated group of transformations, which physically represents the possible relations between the totality of observers. Among the various coordinates, the time was distinguished by its connection with causality, as noted by Minkowski and emphasized in the work of Robb. The point of view here goes back at least as far as Maxwell, who regarded our perception of the serial order of events as the most primitive manifestation of time.

3. The Minkowski-Robb concept of causality. Let us try to formulate the causal structure of the world of events in general but succinct terms, following the philosophical ideas of Einstein, Minkowski, and Robb. Starting from the concept of the physical cosmos--i.e. 'space-time'--as a manifold, say $M$, the following axioms seem about as primitive and unexceptionable as, say, the axioms of Peano for the integers.

Axiom 1. The cosmos is a 4-dimensional manifold.

Axiom 2. At each point of the cosmos there exists an infinitesimal notion of causality: specifically, there is given in the tangent space $T_p$ at each point $p$, a non-trivial closed convex cone $C_p$ -- representing physically the totality of 'future' directions as perceived by an observer at $p$.

In Axiom 2, 'non-trivial' means that $C_p$ and $-C_p$ have only 0 in common, a formulation that excludes Newtonian causality and subtly insinuates Einstein's requirement that there is a limiting finite velocity to all phyical processes (that of light); but a slightly different formulation would admit the Newtonian concept, in which the cones $C_p$ are half-spaces. This matter will not be discussed here, nor will conventional points of mathematical grammar-- the degree of smoothness of the manifold, of the 'cone-field' $C_p$, etc. interesting and non-trivial as they are in certain connections, they are largely orthogonal to my present considerations. Let me remark also that space-time limitations make it impossible for me to pause to justify physically the convexity and closure of the cones $C_p$, and similar points in the future, but that such justification exists.

4. Leray's causal treatment of partial differential equations. The concept of 'causal cosmos' defined by Axioms 1 and 2 is a very general 'soft' one, but it is cogent for at least one non-trivial scientific purpose -- the study of causality features of dynamical partial differential equations embodying finite propagation velocity. In fact, in terms of the work of Leray, 'finite propagation velocity' means essentially a dynamical development which is consistent with a certain causal structure, which defines domains of dependence and regions of influence comparable to those considered in the classical theory of hyperbolic partial differential equations. The very requirement that the dynamical development be defined by such equations appears in fact as not merely an academic or technological restriction, but to be implied by finiteness of the propagation velocity--as shown in a simple but representative case by S. Berman.

Leray begins with a given partial differential equation whose characteristics define a cone-field of the foregoing type, and treats its fundamental solutions. In order to have global advanced and retarded elementary solutions, and not merely local ones, some form of global causality is evidently required. (Evidently, 'if time winds back on itself,' there is no distinction between the advanced and retarded solutions.) Leray's concept of global hyperbolicity suffices; it could be adjoined as an additional axiom formulating global causality, but in the presence of symmetry conditions to be developed later, the following weaker assumption is adequate:

Axiom 3. The cosmos is globally causal in the sense that it admits no closed time-like loops.

The point of view here now reverses the roles of the basic partial differentiation equations and the causal structure of the cosmos, relative to the approach of Leray. The causal structure in the cosmos is here assumed given or determinable in some definite manner; the dynamics of objects in the cosmos is then constrained to be causal with respect to the given structure. On the other hand, the Einstein equations join the points of view by making the causal structure itself (apart from the scalar field defined by the scale) the object whose dynamics is sought. But empty, or reference, space-time, in terms of which gravitational effects are observed and operationally described, has a more a priori causal structure, related to its symmetries, to which I now turn.

5. Causal symmetries. In order to correlate theory with

physical observation in a well-defined manner, it seems necessary to treat causal symmetries, which must exist, at least the symmetries which define temporal evolution, if physics is to be based on measurements at all similar to those employed in laboratories today. For example, there is no clear theoretical notion of temporal duration with which the usual measurements of duration--one of the most precise and fundamental of all measurements--may be identified, in a generic causal manifold.

Intuitively, duration has to do with the flowing of time, much as Newton conceived it. In quantum mechanics, the paramter t which labels the temporal evolution operator $U(t) = e^{itH}$, H being the Hamiltonian, may be correlated with laboratory time, as measured e.g. by an atomic clock, quite readily; but the opposite is the case for the general time-like coordinate on a causal manifold. What then is the relation between these two conceptions of time?

To make a long and not necessarily conclusive discussion short, I shall simply describe a concept of time which I believe does the essential job of explicating the connection between time as a coordinate and time as a group parameter. In a very simple way it imposes a theoretically cogent limitation on time--physical time, that is, which has been presumed unique, within some definite class of equivalences, as opposed to an infinitude of subjective concepts of time, as treated by Bergson and his successors--which appears to correspond to the intrinsic usage in both micro- and macrophysics.

First, the general concept of a causal symmetry: this is defined as a transformation on a causal manifold which preseves causality. That is to say, a one-to-one transformation T on the causal manifold M is defined as causal if it carries a future tangent vector $\lambda$ at a point p (i.e. a vector in the future cone $C_p$) into a future tangent vector $dT(\lambda)$ at the point Tp. It would be the same to say that T preserves at least local temporal precedence--where, locally, x 'precedes' y in the manifold M (both being in a sufficiently small neighborhood of a given point) in case there exists a time-like arc from x to y--or in other words, that for any given point p, there is a neighborhood N of p such that if x and y are in N and $x \ll y$ ($\ll$ means 'precedes'), then $Tx \ll Ty$. On Minkowski space, any orthochronous Lorentz transformation has this property, and it is a theorem (Alexandrov-Ovchinnikova-Zeeman) that no others, apart from scale transformations and their composites with Lorentz transformations, do so globally. Presumably every truly physically possible transformation on the cosmos--as opposed to coordinate transformations, which merely relabel the points--is a causal symmetry. I note in passing, however, and this will be important later, that the local transformations (in a well-defined mathematical sense) which preserve causality form a properly larger local group, which is in fact the 15-dimensional conformal group, 4 dimensions greater than the globally causal group of Minkowski space.

Next the concept of a temporal (forward/backward) displacement: this is defined as a causal symmetry T such that for all points p, either $p \ll Tp$ (forward displacement) or $Tp \ll p$ (backward displacement). For example, on Minkowski space every vector translation by a vector in the interior of the future light cone is a forward temporal displacement, and conversely, these are the only such. In the case of a general causal manifold, there is a convex cone not in the manifold itself, but in the infinitesimal group of causal symmetries which corresponds to the forward displacements. More specifically, if G denotes the group of all causal symmetries of the cosmos M, the totality F of forward displacements of M forms an 'invariant semigroup,' having the properties:

$$F^2 \subset F, \quad aFa^{-1} \subset F \quad (a \in G), \quad F \cap F^{-1} = \{e\}$$

There is then a closed convex cone $\underline{F}$ in the infinitesimal group $\underline{G}$ which generates F.

6. <u>Temporal duration and invariant clocks</u>. With this sketch of mathematical background, I define the notion of a <u>temporally invariant clock</u> on the given causal manifold M--physically, one for which there is an invariant notion of duration, independent of the instants of measurement, as the following couple:

1) a function $\tau$, say, which assigns to each point p of M a real number $\tau(p)$ (its 'time' coordinate, physically speaking);

2) a one-parameter temporal evolution group $T_t$: i.e. each $T_t$ is a forward displacement for $t > 0$, and $T_{t+t'} = T_t T_{t'}$ for all real values of t and t'; which are connected by the following basic constraint:

3) for all points p and all real values t,

$$\tau(T_t p) = \tau(p) + t.$$

The last condition in effect states that the clock gives an invariant notion of duration, and severely limits the possible transformations of the time coordinate; for example, an arbitrary smooth monotone transformation $t \to f(t)$ is no longer permitted.

In the case of Minkowski space, for example, the following definitions

$$\tau(x_0,x_1,x_2,x_3) = x_0, \quad T_t(x_0,x_1,x_2.x_3) = (x_0+t,x_1,x_2,x_3)$$

satisfy the foregoing conditions. Conversely, every invariant clock on Minkowski space takes this form relative to a suitable

Lorentz frame and scale in the space of the parameter t. Together, these two mathematical facts provide some validation for the philosophically reasonable but physically tentative notion of clock just proposed.

A general class of examples of invariant clocks is obtainable in a similar fashion by the replacement of the flat euclidean space by an arbitrary 3-dimensional Riemannian manifold, say S, with the causal structure on the cosmos $R^1 \times S$ defined from the analogous Lorentzian structure $dx_0^2 - ds^2$, where ds denotes the element of distance on S. The special case in which $S = S^3$, the 3-sphere (surface of the unit sphere in 4-dimensional euclidean space) will be seen to have some remarkable properties, but more clearly to the point at the moment is the fact that direct observation of the cosmos is necessarily local; that the notion of invariant clock extends directly to a correspondingly local notion, mathematically entirely well-defined; and that there exist invariant local clocks on Minkowski space that are essentially different from any global clock, i.e. the presently standard one. "Local" here is on a scale far exceeding even the solar system, so the question must arise of whether the standard clock is entirely correct.

In fact, an alternative local clock exists which cannot be distinguished from the standard clock by direct laboratory experiments of the present order of precision, and yet would have natural large-scale implications that are quite different from those of the standard clock, which implications nevertheless are not in disagreement with observation. This means that it is a material physical issue, whether the standard clock or an alternative clock, is correct on a large scale. To describe explicitly this alternative clock, consider the case of $R^1 \times SU(2)$ with the unique invariant metric on SU(2) (under both left and right group translations), and the causal structure indicated in the preceding paragraph. Locally, say in the vicinity of the point (0,I), where I is the unit matrix in SU(2), this cosmos is identical, as a causal manifold with Minkowski space; specifically, the mapping which carries the point (t,U) of $R^1 \times SU(2)$ into the inverse Cayley transform of $e^{it}U$ yields a $2 \times 2$ hermitian matrix H, which corresponds to a unique point $(x_0,x_1,x_2,x_3)$ of Minkowski space via the representation $H = \begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix}$; and this mapping is causal.

It follows that the clock on $R^1 \times SU(2)$ earlier indicated--according to which $\tau(t,U) = t$ and $T_s(t,U) = (t+s,U)$--is carried locally into a local clock on Minkowski space. This local clock is not only distinct from any standard clock, it is not even transformable into a standard clock by any local causal transformation. It is simply physically essentially different; but this does not necessarily mean that it is physically incorrect. An analysis and computations which can only be summarized here indicate that in order to distinguish the non-standard from the standard clock by a local measurement of photon frequencies, a photon would have to be kept alive for 30 years (or travel the corresponding distance), assuming the rather optimistic precision of measurement of 1 part in $10^{15}$. The connection with photon frequencies is however the road to an elucidation of the large-scale physical differences between the standard clocks, and this question in turn raises the issue of what is the energy? to which I now turn.

7. Canonical energies for a given cosmos. In non-relativistic physics, quantum or classical, the energy may be specified rather arbitrarily, as e.g. a self-adjoint operator, or a function on phase space. This is no longer the case for relativistic physics, or for a dynamics on a general cosmos, since the energy determines the temporal evolution, which must be compatible with the causal structure of the cosmos.

It appears that the only 'natural' definition of the energy, i.e. one in which it is not simply given by external considerations imposed on the given system, essentially as in non-relativistic theory, is--as is also generally valid in non-relativistic theory, but not always (as e.g. in Hoyle's steady-state model of the universe)--as the generator of temporal evolution (also described as the 'dual' of the time). This again defines both the classical and the quantum energy.

What this means concretely may be illustrated by the treatment of the energy of the highly simplified model for photons defined by the two-dimensional wave equation $\phi_{x_0 x_0} - \phi_{x_1 x_1} = 0$. The classical energy is the functional

$$E(\phi) = \int [\partial\phi/\partial x_0)^2 + (\partial\phi/\partial x_1)^2]dx_1 \;;$$

the quantum energy is simply the operator H: $\phi \to \frac{\hbar}{i}(\partial\phi/\partial x_0)$, which is self-adjoint in the Hilbert space of all normalizable solutions of the wave equation. Now the wave equation is invariant under all causal transformations, local or global, and in particular under the non-standard temporal evolution group described earlier. Accordingly, it will have a non-standard energy associated with the alternative invariant clock. One finds for example the following which will be relevant later:

(1) The non-standard energy (say H' or E') always exceeds the standard energy (H or E);

(2) While the standard quantum energy H depends only on the immediate vicinity of the wave function $\phi$--is 'intensive,' so to

speak--and is in fact simply the frequency for a wave function which is locally a plane wave (within the attainable accuracy of physical measurement)--the excess non-standard energy, H' - H, is independent of the wave function in the immediate vicinity of any point--is 'extensive' so to speak--and varies inversely with the frequency for a cutoff plane wave of a given number of oscillations; the excess is thus 'delocalized' energy, intuitively speaking, and in fact is too small to be conceivably measurable in the case of a localized wave function.

Yet all of the general principles of theoretical physics, including quantum field theory, apply equally to both the standard and non-standard energies, as to the respective associated invariant clocks. The same is equally true in 4 space-time dimensions, and moreover applies also to Maxwell's equation, or any other relativistic equation left invariant by both standard and non-standard temporal evolution.

Could the non-standard clock and energy conceivably be the physically correct one? What sort of observational implications would it have which might be checked against experiment? One sees that the circumstance that the excess non-standard energy is delocalized and vanishes within the limits of observation for a localized wave function, and as if this were not enough, varies inversely with frequency (for a given number of oscillations), would make it extremely difficult to observe directly the excess energy. But one sees also, that since the total energy is conserved in the course of temporal evolution--trivially so, by virture of the definition of the energy as the generator of temporal evolution--the localized standard energy would decrease in the course of time, and be transformed partially into the not directly observable delocalized energy. This means that laboratory (i.e., local) observations on photons propagated over large distances would show a redshift. Since the redshift of light emitted from apparently distant galaxies is one of the striking facts of nature, an investigation of the quantitative observable implications regarding the redshift is thereby suggested.

On the other hand, conceivably there are many other non-standard clocks; a priori in fact it is quite conceivable that there exist such a multi-parameter family of non-standard clocks that by adjustment of parameters one could fit an enormous variety of observable relations involving the redshift. But this is not the case; highly elementary and well-documented symmetry restrictions--'relativity,' as it is called, although 'invariance' would have been preferred by Einstein, it seems, and is more apt--cut down the (already quite limited) possibilities to the Minkowski clock and the non-standard one already cited, leading to observable redshift relations devoid of any free parameter, and thus highly vulnerable to confrontation with observation.

8. Generalized relativity. Minkowski space as a causal manifold leads to the observed addition laws for large velocities, and otherwise agrees with local observations, confirming it as a model for the cosmos, but not necessarily uniquely affirming it. In principle, there has always been a possibility that some other model for the (empty-reference) cosmos could do as well or even better in certain respects. But is there any reasonably simple and unique way to restrict conceivable models that is both physically well-grounded, and mathematically cogent?

Physically, the most conservative restrictions involve observation at one point of the cosmos. Among such restrictions, supported by a great variety of quantitative evidence in the small as in the large, are spatial and temporal isotropy. Spatial isotropy means that there are no preferred space-like directions; that given any two space-like directions, at a given point of observation, there is a physically admissible transformation which carries one direction into the other; 'physically admissible' certainly involves causality, so that a highly conservative formulation of spatial isotropy is the requirement that for any two space-like directions, there exists a causal transformation on the cosmos mapping one into the other. Observationally, spatial isotropy is strongly indicated in the large by observations on the cosmic background radiation, and microscopically by conservation of angular momentum.

'Temporal isotropy' is the equivalence between observers in relative motion at the same point. In Minkowski space, it takes the form of Lorentz invariance. On a general cosmos, it becomes the equivalence of any two time-like directions. Just as in the case of spatial isotropy, this may be formulated conservatively as the requirement that given any two future directions at a point of the cosmos, there exist a causal transformation carrying one point into the other.

Adding to these two observable restrictions the philosophical one of anti-anthropocentrism, which originated the Copernican revolution and appears vital today, one obtains the requirement of spatial and temporal isotropy at every point of the cosmos. In spite of the conservatism, simplicity, and naturalness of these restrictions, they are mathematically extremely cogent, as shown by the important results of J. L. Tits and E. Vinberg. The work of Vinberg classifying homogeneous cones limits the causal cones $C_p$ to cones definable in each tangent space by a quadratic equation, and thus implies that the causal structure must be induced from a Lorentzian metric. The work of Tits classifying Lorentzian manifolds enjoying various isotropy features shows that the cosmos must be locally Minkowskian, and in fact one of an explicitly enumerated set of possibilities. The absence of closed time-like paths earlier postulated then limits the possible cosmos to just 3: Minkowski space, whose causal group is the Poincare group augmented by scale transformations; the space $R^1 \times S^3$ earlier cited, whose causal group is locally the conformal group on Minkowski space (i.e. the group O(4,2)); and an hyperbolic space, whose causal group is locally O(3,2). All of these cosmos are imbeddable into $R^1 \times S^3$, in such a way that their groups become subgroups of that of $R^1 \times S^3$, and in fact they may be defined as orbits in $R^1 \times S^3$ under these subgroups.

All three of these possible global cosmos are identical locally, and so have the same local invariant clocks. If we demand further of a physical clock that locally, at least, the cosmos may be split into time and space components, in such a way that temporal evolution affects only the first factor--in accordance with standard physical thought, according to which a specific observation takes place at a particular time on objects distributed in space--then there are only two possible local physical clocks, namely those already cited. Corresponding to each of these there is moreover a global separation into time and space of one of the three models, in which temporal evolution changes only the time component, and spatial transformations (i.e. causal transformations leaving invariant the space component) act isotropically and traisitively on space. (The hyperbolic space does not separate in the large into time and space factors.)

In summary, then, with the one additional

Axiom 4. The cosmos admits an invariant clock relative to which it can be separated locally into time and space components (in the sense indicated above),

the cosmos is mathematically limited to 3 possibilities, all locally Minkowskian; and the corresponding invariant clock is limited locally to just two possibilities. Only one of these possibilities gives rise to an intrinsic redshift arising from the partial delocalization of the photon energy after propagation through a large distance.

9. Cosmology and general relativity. The question of the quantitative implications regarding the redshift of photons propagated over large distances, on the assumption that the correct physical time is that derived from the local factorization as $R^1 \times S^3$, is facilitated by the conformal invariance of Maxwell's equations, and in fact the unitarity of the action of both standard and non-standard time evolution groups. The predicted redshift z after time t is then defined by the equation $(1+z)^{-1} = \langle e^{itH'} He^{-itH'}\rangle/\langle H\rangle$ , which remarkably is independent of the form or frequency of the localized photon state with respect to which the expectations are formed, leading to the redshift-distance relation,

$$z = \tan^2(r/2).$$

The distance is not an observable quantity, but the purely geometric relations between apparent luminosity and distance (as well as other observed quantities, such an angular diameter) permit the distance to be eliminated, and purely observable stochastic relations derived. These relations may then be tested on the large samples of galaxies, quasars, and radio sources which are now available.

Comprehensive systematic tests of this nature have been conducted in accordance with contemporary statistical procedures, and remarkably good agreement between prediction and observation, especially by historical standards of observational cosmology, have been found. With rare exceptions, which may in fact arise from non-randomness of the relatively small samples involved in these cases, the redshift theory indicated (called the 'chronometric' theory, in view of its derivation from a general analysis of the nature of time) leads to a much better fit with observation than do Friedman-Lemaitre models with their two free parameter, i.e. $q_0$ and $\Lambda$. Counts of the numbers of galaxies, quasars, etc. below given redshifts or brighter than given luminosities are also found to be in very good agreement with the natural postulate of spatial homogeneity for their distribution, if the physical cosmos is assumed to take the form $R^1 \times S^3$.

In addition, a number of isolated anomalies within the Friedman-model cosmology are simply eliminated by the chronometric theory; among these are, for example, the apparent supperrelativistic lateral velocities of a number of sources, and the extraordinary luminosity and apparent evolution of quasars. The cosmic background radiation is predicted as the temporally homogeneous equilibrium photon gas established by the diffusion and scattering of electromagnetic radiation around $S^3$ in accordance with energy conservation. The deviation from a pure Planck law detected in recent observations by Woody and Richards then becomes explicable as a consequence of a non-trivial level of isotropic angular momentum in the background radiation. The rough apparent coincidence of very large time scales is understandable from the relation between the Minkowski and chronometric times, $x_0 = 2 \tan (t/2)$ at a given point of space from which a uniform distribution in t implies a Cauchy distribution for $x_0$ whose median is of the order of the radius of the universe, $S^3$ (in units of c).

How is general relativity and its relation to cosmology affected? The postulated infinitesimal structure of space-time in general relativity, i.e. of reference or empty space-time, is changed from a Minkowski space, formed from the tangent space at the point of observation, to a chronometric space, $R^1 \times S^3$, invariantly attached to the point as the universal covering space of the conformal compactification of the tangent space with respect to the metric given in it. As far as is now known, the radius of the $S^3$ is too large (in conventional units; in natural units, the $S^3$ is of unit radius) to produce any presently observable effects in the small, and local observable aspects of general relativity are therefore unaffected.

In the large, because of the compactness of $S^3$ it is necessary, as Einstein proposed, to add the cosmological term to his equation. Overall, the resulting universe departs widely from the Friedman-Lemaitre model--any expansion, if present at all, must be slight--but in its gross features is consistent with Einstein's original static conception. One is reminded of Einstein's original misgivings about Friedman's work; these now appear as another example of his prescience.

## REFERENCES


1. A. Einstein (1917). Kosmologische Betrachtungen zur Allgemeinen Relativitatstheorie. Sitzungsberichte, Preuss. Akad. d. Wissenschaften.
2. J. Leray (1952). Hyperbolic partial differential equations. Ins. Adv. Study, Princeton, N.J.
3. I. E. Segal (1976). Mathematical cosmology and estragalactic astronomy. Academic Press, New York.
4. H. P. Jakobsen, M. Kon, and I. E. Segal (1979). Angular momentum of the cosmic background radiation. Phys. Rev. Lett. 42, pp. 1788-91.
5. H. P. Jackobsen, B. Orsted, I. E. Segal, B. Speh, and M. Vergne (1978). Symmetry and causality properties of physical fields. Proc. Nat. Acad. Sci. USA. 75, pp. 1609-11.
6. J. F. Nicoll and I. E. Segal (1978). Statistical scrutiny of the phenomenological redshift-distance square law. Ann. Phys. 113, pp. 1-28.
7. J. L. Tits. Lectures at College de France, 1978-79; see also Les espace isotropes de la relativite, Colloque sur la theorie de la relativite, 1959, pp. 107-119, Centre Belge Rech. Math., 1960.
8. E. Vinberg (1963). The theory of homogeneous convex cones. Trudy Moskov. Nat. Obsc. 12, pp. 303-358.
9. D. P. Woody and P. L. Richards (1979). Spectrum of the cosmic background radiation. Phys. Rev. Lett. 42, pp. 925-929.