Matrix functions

Nathan Rousselot
Mathematical Engineering Department
KU Leuven

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1 Introduction

In this document we introduce the notion of matrix functions. Say we have a function $f : \mathbb{C} \rightarrow \mathbb{C}$, then we can define the function f on a matrix $\mathbf{A}$ as follows: $f : \mathbb{C}^{n\times n} \rightarrow \mathbb{C}^{n\times n}$. In the following, we will first provide some theretical background on matrix function. The, we will describe the numerical issues coming with manipulating matrix functions. Finally, we will provide algorithm for efficient computation of matrix fuctions.

2 Theoretical background

Polynomial functions

Let $p : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial function of degree d:

$$p(t) = \sum_{k=0}^{d} c_k t^k \tag{1}$$

Then, considering a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, and posing $\mathbf{A}^0 = I_n$, we can define the polynomial function $p : \mathbb{C}^{n\times n} \rightarrow \mathbb{C}^{n\times n}$ on a matrix $\mathbf{A}$ as follows:

$$p(\mathbf{A}) = \sum_{k=0}^{d} c_k \mathbf{A}^k \tag{2}$$

Rational functions

Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function of the form:

$$f(t) := \frac{p(t)}{q(t)} \tag{3}$$

It is not immediate how one would approach this function with a matrix. We want to define

$$f(\mathbf{A}) := q(\mathbf{A})^{-1} p(\mathbf{A}) \tag{4}$$