Day 3: Polynomials & Algebra

Algebra Review

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$a^2 - b^2 = (a-b)(a+b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$(a + b)^3 = a^3 + 3a^2b + 3b^2a + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3b^2a - b^3$

New

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Simon's Favorite Factoring Trick

$$xy + kx + jy + jk = (x+j)(y+k)$$

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}\right)$$

$$a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots + a^2b^{n-3} - ab^{n-2} + b^{n-1}\right)$$

NOTE: n must be odd

Sum and Difference of Odd Powers

Polynomials, basics review

Formal definition: A polynomial is an expression made of variables, numbers, and nonnegative integer exponents, combined using only addition, subtraction, and multiplication.

Polynomials can be factored off of their roots:

If a polynomial with leading coefficient a has roots r1, r2, r3, etc, it can be factored as

a(x - r1)(x-r2)(x-r3)...

General Polynomial Form: P(x) = Q(x)*D(x) + R(x), where Q(x) is the quotient, D(x) is the divisor, and R(x) is the remainder. The degree of R(x) must be strictly less than the degree of D(x) or 0.

Remainder Theorem: The remainder when a polynomial P(x) is divided by x - a is P(a). This should be obvious when plugging in x - a for Q(x) and substituting x - a in for x in the General Polynomial Form.

Vieta's Formulas

Vieta's Formulas:

In a polynomial $a_n x^n + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \ldots + a_0$,

the sum of its roots is $-a_{n-1}/a_n$, the sum of the product of pairs of roots is a_{n-2}/a_n, the sum of the product of triples of roots is $-a_{n-3}/a_n$, etc..., the product of all roots is $(-1)^n * a_0/a_n$

In general, the sum of the kth products of roots is $(-1)^k * a_{n-k}/a_n$.

Vieta's Formulas can be understood by expanding root factorization. If we have a polynomial in the form $a_n(x - r1)(x - r2)(x - r3)$.., the coefficients will become what we calculated!

Manipulating Roots

Given any polynomial $P(x) = a_n x^n + a_{n-1}x^{(n-1)} + a_{n-2}x^{(n-2)} + \ldots + a_0$:

The polynomial with the reciprocal roots of $P(x)$ is $a_0 x^n + a_1 x^{(n-1)} + a_2 x^{(n-2)} + \ldots + a_n$ (you just flip the coefficients).

Why? If a is a root of $P(x)$, then $P(a) = 0$. We want a polynomial $Q(1/a) = 0$. This can be found by calculating $P(1/x)$ and multiplying by x^n to get rid of the denominator.

Similarly, the polynomial with the roots of opposite sign of $P(x)$ is $P(-x)$, and the polynomial with roots ‘a’ more than that of $P(x)$ is $P(x-a)$.

Algebraic Manipulations

1. Look for ways you can make smart substitutions
2. Use symmetry
3. Factor and group whenever you can
4. Complete the square
5. If the degree of the equation is big, try getting it down to a quadratic, or factor and look for a condition in the problem like “the value must be an integer” and prime factorize whatever number the equation is equal to.

A great example of manipulating is finding the roots to the equation $(x + i)(x + 2i)(x + 3i)(x + 4i) = 2025$.

We notice symmetry around 2.5 since it’s the median of the first 4 numbers, which are the coefficients of i. Substituting $a - 2.5i$ for x, we get:

$(a - 1.5i)(a - 0.5i)(a + 0.5i)(a + 1.5i) = 2025$

Now we can just use difference of squares! Recall: $(a - b)(a + b) = a^2 - b^2$.

We get $(a^2 + 2.25)(a^2 - 0.25) = 2025$. Now we substitute b in for a^2:

$(b + 2.25)(b - 0.25) =$

$b^2 + 2b - 0.5625 = 2025$

and solve for b like a normal quadratic and work backwards.

A hard problem elegantly solved with our tricks

Problem 8. For a positive integer $k \geq 2$, let α_k, β_k, γ_k be the complex roots (with multiplicity) of the cubic equation $\left(x - \frac{1}{k-1}\right)\left(x - \frac{1}{k}\right)\left(x - \frac{1}{k+1}\right) = \frac{1}{k}$. Determine the value of

$$\sum_{k=2}^{\infty} \frac{\alpha_k \beta_k \gamma_k \cdot (1+\alpha_k) \cdot (1+\beta_k) \cdot (1+\gamma_k)}{k+1}.$$

Practice!

1. Given x and y are distinct nonzero real numbers such that $x + 2/x = y + 2/y$, find xy.

2. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

3. Suppose a and b are real numbers. When the polynomial $x^3 + x^2 + ax + b$ is divided by $x - 1$, the remainder is 4. When the polynomial is divided by $x - 2$, the remainder is 6. What is $b - a$?

4. The roots of $x^3 + 2x^2 - x + 3$ are p, q, and r. What is the value of $(p^2 + 4)(q^2 + 4)(r^2 + 4)$?

Note: two ways you could solve this problem, root manipulations, or Vieta's Formulas!

A lot more practice in the homework assignments.