Day 4: Triangles

Properties of Triangles Review

1. Angles sum to 180 degrees
2. Pythagorean Theorem
3. Law of Sines: $a/\sin A = b/\sin B = c/\sin C$
4. Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$
5. Every triangle can be inscribed in a circle, and it is possible to circumscribe a circle in every triangle (circumcircle and incircle, respectively)
6. Area of a triangle = $bh/2 = ab\sin C/2 = abc/4R$ (circumradius) = rs (inradius times semi-perimeter) = $\sqrt{s(s-a)(s-b)(s-c)} = I + B/2 - 1$, I are inside points and B are boundary points (Pick's Theorem)
7. Incenter is the intersection of all angle bisectors. It is also the center of the incircle.
8. Circumcenter is the intersection of all perpendicular bisectors (segments that bisect the opposite side and create a 90 degree angle). It is also the center of the circumcircle.
9. Special for right triangles: Circumradius bisects hypotenuse, midpoint of hypotenuse is center of circumcircle
10. Centroid is intersection of medians
11. Orthocenter is intersection of altitudes
12. Know what sine and cosine are, and know key identities:

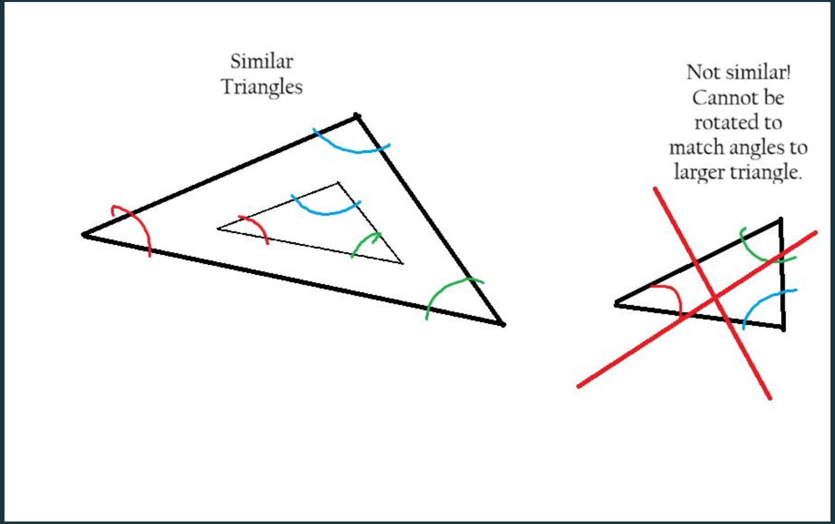
1) $\sin x = \cos(90 - x)$, $(\sin x)^2 + (\cos x)^2 = 1$, $\sin x = \sin(180 - x)$, $\cos(180 - x) = -\cos x$

Similar Triangles Review

Two triangles are similar if they have the same angles and proportional sides. The angles and sides must be in the same order. That is, one triangle can be rotated to have matching angles to the other.

Corresponding sides have the same ratio. If triangles ABC and DEF are similar, AB/DE = BC/EF = AC/DF.

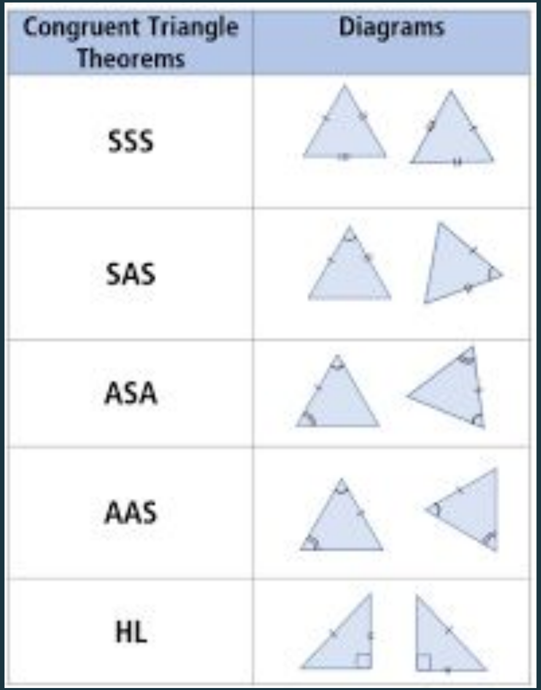
Always be on the lookout for congruent angles! These are often created by parallel lines and transversals.



Identifying Congruent Triangles Review

There are four main congruence theorems (covered in most middle school geometry classes):

1. Side-Angle-Side
2. Side-Side-Side
3. Angle-Side-Angle
4. Angle-Angle-Side

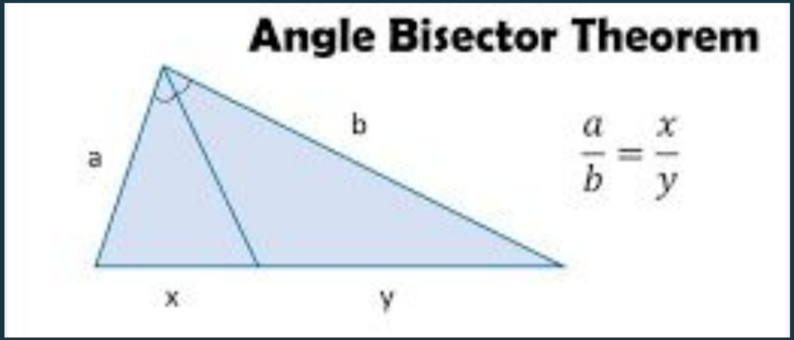


Congruent Triangle Theorems	Diagrams
SSS	
SAS	
ASA	
AAS	
HL	

Angle Bisector Theorem

If a cevian bisects the vertex angle, then the angle bisector theorem applies!

Can you use the Law of Sines to derive the Angle Bisector Theorem on your own?



Angle Bisector Theorem with Law of Sines

$x/(\sin\theta) = a/(\sin\alpha)$

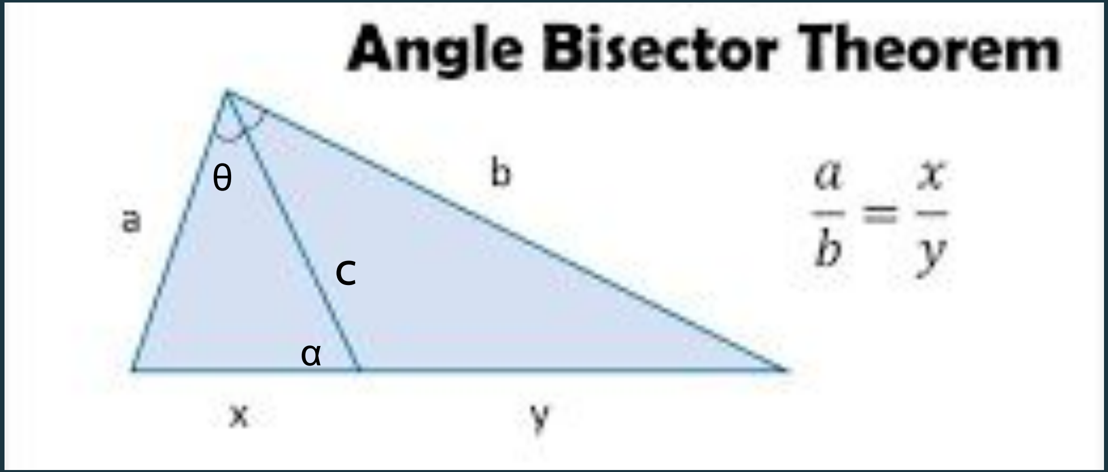
$y/(\sin\theta) = b/(\sin(180 - \alpha)) = b/(\sin\alpha)$

Divide the two equations to get:

$(x/(\sin\theta))/(y/(\sin\theta)) = (a/(\sin\alpha))/(b/(\sin\alpha))$

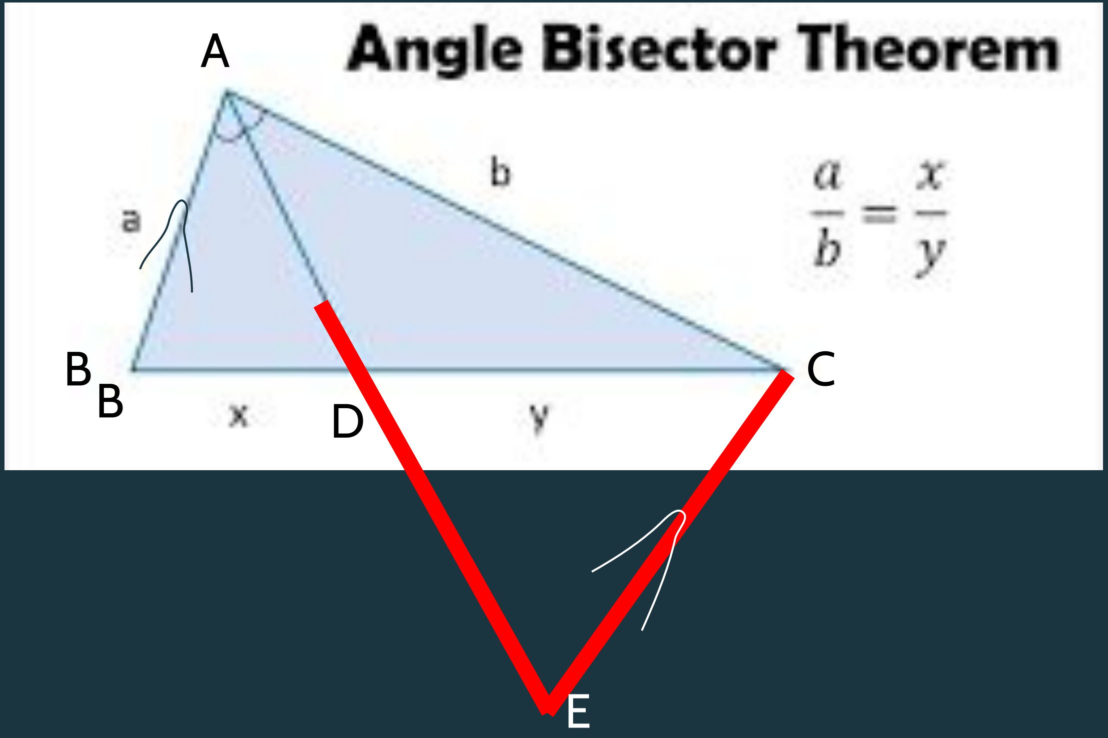
And we are left with:

$x/y = a/b$



Angle Bisector Theorem with similar triangles

Alternatively, construct a line parallel to AB from C. Extend AD to meet this line at point E. Since AB is parallel to CE with transversal BC, angle ABD is equal to angle DCE. Now if we use AE as the transversal, we see that angle BAD is equal to angle DEC. Angles ADB and CDE are equal because they are vertical angles. So all angles of DEC are equal to ABD, so the triangles are similar. Then by similarity ratios, AB/CE = BD/CD = AD/DE. Note that angle CAE = angle BAD = angle DEC, so triangle ACE is isosceles. Then CE = b, and we have AB/b = BD/CD ==> a/b = x/y.



Intro to Coordinate Geometry

Using a coordinate system can be useful in many situations, especially if you forget a formula and need to find the side length of something that would be too hard to do with trigonometry. Use the Distance Formulas and slope intercept form when needed!

Formulas to remember:

$y = mx + b$

Distance between two points

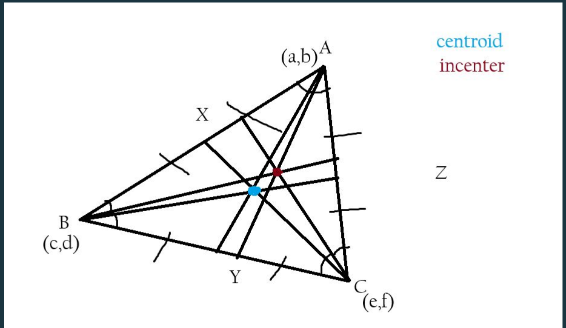
Distance between a point and line

Equation of line perpendicular to another line with slope a has slope -1/a

Coordinates, continued

Say triangle ABC is bounded by the points A(a,b), B(c,d), and C(e,f) with AB having length X, BC having length Y, and AC having length Z.

Then we have:



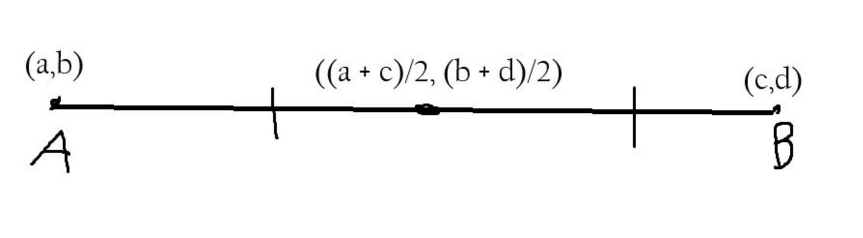
Centroid coordinates:

((a + c + e)/3, (b + d + f)/3)

Incenter coordinates:

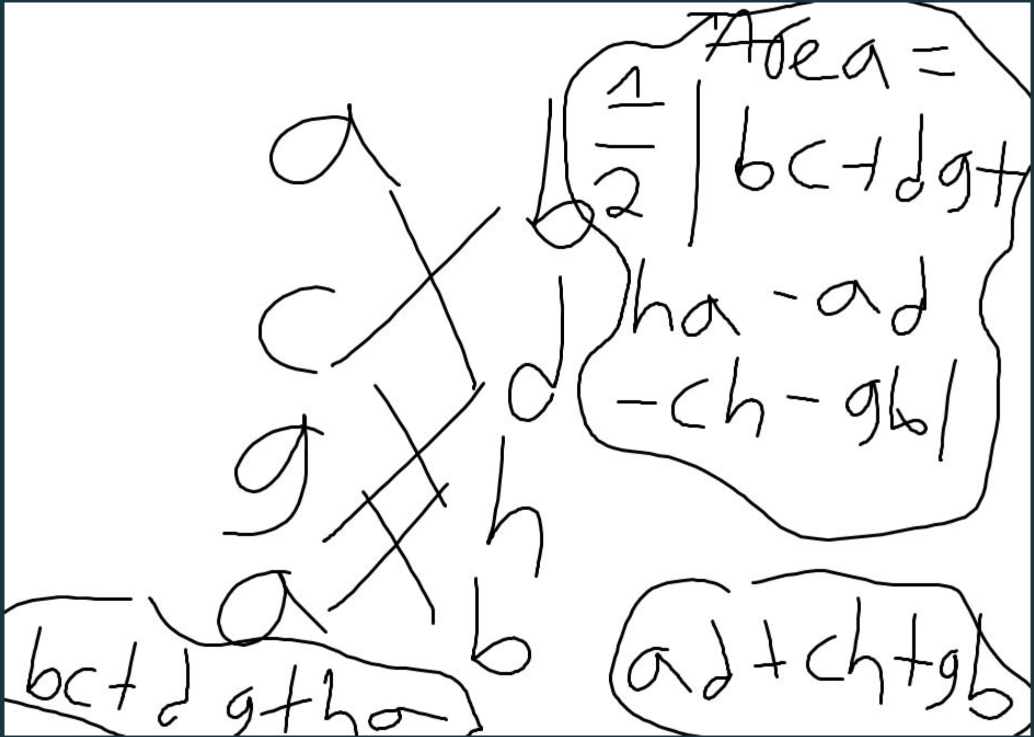
((aY + cZ + eX)/(X + Y + Z), (bY + dZ + fX)/(X + Y + Z))

Also, if you didn't know this already, if we have a line AB with endpoints A(a,b) and B(c,d), the midpoint is simply ((a + b)/2, (c + d)/2).



Shoelace Theorem

Area of a triangle bounded by the points (a,b), (c,d), (g,h) on a coordinate plane:



Practice!

1. Triangle ABC has angle A = 45 degrees and angle C = 60 degrees. The altitude from B meets AC at point D. Given AD = 5, find the area of triangle ABC.
2. Triangles ABC and ACD are inscribed in a circle with center O. Angles BOD and COD are equal. AD and BC meet inside the circle at point X, AC = 6, CX = 2, and AB = 9. Let the length of the diameter be d. Then d^2 can be expressed as m/n, where m and n are relatively prime positive integers. Find m + n.
3. Let $ABCD$ be an isosceles trapezoid having parallel bases AB and CD with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are 2, 3, 4, and 5 starting with the triangle with base CD and moving clockwise. What is the ratio AB/CD?
4. Triangle ABC has vertices A(a,a), B(a,6), and C(10, 6). Triangles ABC shares an incenter with triangle PQR with vertices P(a, 5a), Q(4a, a), and R(x,y). What is (x + y)/a ?
5. Find the area of a quadrilateral inscribed in a circle with side lengths 5, 7, 11, and 13. Hints: Use (sinx)^2 + (cosx)^2 = 1, divide the quadrilateral into two triangles, use Law of Sines and Law of Cosines, the order of the sides doesn’t matter, you will still get the same area.

Day 5: Circles

Properties of Circles Review

Area of a circle is simply pi * r^2.

Chord is a segment inside the circle, tangent is a line touching the circle at exactly one point, secant is a line touching the circle at two points.

Length of a chord: $2r\sin(\theta/2)$, where θ is the central angle subtended by the chord.

Area of segment: Area of sector - Area of triangle formed by radii and chord.

Break a complicated looking shape involving circles into an equilateral triangle and sector to find its area, this strategy works most of the time!

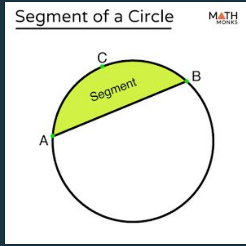
The perpendicular bisector of any chord passes through the center

Graphing: $(x-h)^2+(y-k)^2 = r^2$, where (h,k) is the center of the circle and r is its radius.

Arcs

Angles that subtend the same arc are equal! This very useful principle is often the key trick in circle problems.

Any inscribed angle subtending the diameter of the circle must be 90 degrees, and any 90 degree inscribed angle must subtend a diameter.



Area of sector: $(\theta/360) * \pi r^2$

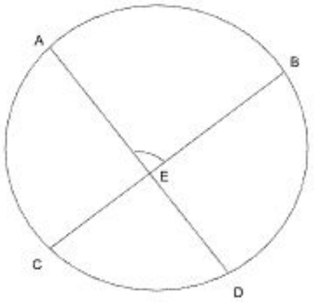
Major arcs: Central angle is greater than 180 degrees

Minor arcs: Central angle is less than 180 degrees

A segment is a sector subtracted by the triangle formed by the radii and the line connecting the two points the radii touch the circle at.

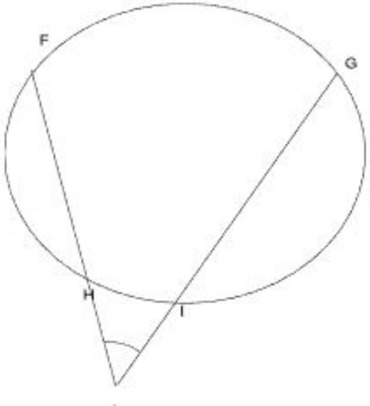
The measure of the arc is twice the measure of the inscribed angle.

Power of A Point



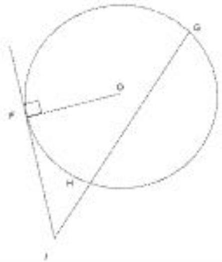
Power of A Point and Circles

1. $\angle AEB$ = (arc AB + arc CD)/2
2. AE x ED = BE X EC



1. JH X JF = JI X JG
2. JF^2 = JH X HG

1. $\angle FJG$ = (arc FG - arc HI)/2



Practice part 1! (segments and angles)

1. Two chords, AB and CD, intersect at a point P inside a circle. If AP = 4, PB = 6, and CP = 3, find the length of PD.
2. A tangent segment PT is drawn from an external point P to a circle, and a secant line through P intersects the circle at points A and B. If the tangent PT = 12 and the external part of the secant PA = 8, find the length of the entire secant segment PB and the length of the chord AB.
3. Point P is outside a circle with radius r = 5. A secant line from P passes through the center of the circle, creating an external segment of length 4. A second secant line from P is bisected by the circle (meaning the length of the external segment is equal to the length of the chord). Find the total length of this second secant.
4. In circle O, angle ABC is an inscribed angle that intercepts arc AC. If the measure of arc AC is 84 degrees, what is the measure of angle ABC? If a central angle AOC were drawn to the same arc, what would its measure be?
5. A quadrilateral ABCD is inscribed in a circle. If diagonal AC is a diameter of the circle and angle BCA = 35 degrees, find the measures of angle ABC, angle BAC, and angle ADC.
6. Two chords EF and GH intersect at point X inside a circle. The measure of arc EG = 40 degrees and the measure of arc FH = 110 degrees. A) Find the measure of the vertical angles angle EXG and angle FXH. B) If an inscribed angle angle EHF is drawn, find its measure and explain its relationship to arc EF.

Practice part 2! (areas)

1. Let ABCD be a square with side length 1. From each vertex of the square, a quarter circle is drawn inside the square with radius 1, such that each quarter circle touches two of the other vertices. Find the area of the intersection of all four quarter circles.
2. An equilateral triangle ABC is inscribed in a circle of radius R. Three smaller circles are then drawn such that each circle is tangent to two sides of the triangle and tangent to the larger circumscribed circle. Find the total area of these three smaller circles in terms of R.
3. Consider a right-angled triangle ABC with the right angle at C. Semicircles are constructed with diameters AC, BC, and AB. The two smaller semicircles (on sides AC and BC) are drawn outward, while the largest semicircle (on the hypotenuse AB) is drawn such that it passes through point C, overlapping the triangle. Prove that the sum of the areas of the two "lunes" (the crescent-shaped regions outside the large semicircle but inside the smaller ones) is exactly equal to the area of triangle ABC.