Introductory Advanced 2026

Day 1: Combinatorics & Casework

Permutations, the basics

Permutations tells us the number of ways we can rearrange a set of objects amongst themselves from a larger group n.

Num of ways = $P(n,k) = n!/(n-k)!$

Example: Choosing a president and vice president from a set of 20 people.

There are 20 choices for the president and 19 choices for the vice president, giving us $20 \times 19 = 380$ ways.

The formula also works here, with $n = 20$ and $k = 2$: $P(20, 2) = 20!/(20 - 2)! = 20 \times 19 = 380$.

Permutations with repetition

What if we want to figure out the number of rearrangements of the letters

AAABBCCCDEF

(And we consider repeating letters to be identical objects)?

1. Count the number of letters (11)
2. There are 11 choices for what the first letter can be, 10 choices for what the second letter can be, etc.. giving us 11!
3. However, A,B, and C are being repeated. We divide by the number of ways to rearrange each type of letter.
4. There are 3x2x1 = 3! ways to rearrange the As amongst themselves, 2x1 = 2! ways to rearrange the Bs amongst themselves, and 3x2x1 = 3! ways to rearrange the Cs amongst themselves.
5. The total number of rearrangements is 11!/(3! x 2! x 3!) = 554400.

Permutations with constraints

Now, what if I have a set of 5 fiction and 6 nonfiction books, with the condition that every fiction book must be next to a nonfiction book?

FFFFFNNNNNN

There are 2 possibilities:

FNFNFNFNFN | N <== There are 6! ways to choose which type of nonfiction books to go in their placeholders, and 5! ways to choose which fiction books go in their placeholders. The leftover nonfiction book can go in 6 different spots without violating the problem's condition.

NFNFNFNFNF | N <== Same calculations as for the first case here by symmetry so multiply by 2.

Our final count is 6! x 5! x 6 x 2 = 1036800.

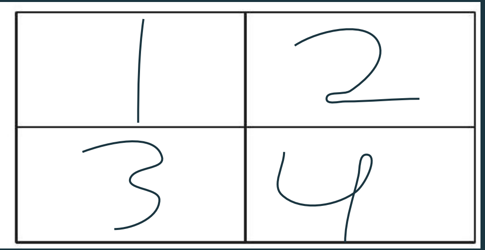
Casework

The technique we used in the last problem is called casework! We broke the constraint down into 2 different possibilities, then just multiplied by 2 because we knew they were symmetric. Casework is very very important, especially in combinatorics. It saves us time when we see things like symmetry, and it gives us a well-defined way of solving a seemingly complicated problem.

To review: Casework is breaking a multi-step problem down into smaller subcases. We calculate the number of ways for each subcase, then add them up to get to our final answer. We'll see this in our next problem.

Apply Casework

2021 Fall AMC 10A Problem 18: A farmer's rectangular field is partitioned into a 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?



Direct application of casework: Notice the symmetry around the crops. Without loss of generality, say the first cell is wheat. That tells us that sections 2 and 3 cannot contain corn. There are 2 cases for cell 2: 1) It is the same as cell 1 or 2) It is different than cell 1.

Case 1: WLOG* cell 2 is also wheat. Then cell 4 can either be wheat, soybean, or potato. If it is also wheat, then cell 3 will have 3 options. If it is not wheat (in two ways), then cell 3 will have 2 options: to either match cell 4 or cells 1 and 2.

Case 2: WLOG cell 2 is not wheat (in two ways), so say cell 2 is soybeans. Then cell 4 has 3 options. This is symmetric to the first case for cell 3.

4(3 + 2x2) + 4x2(3 + 2x2) = 4x3(3 + 2x2) = 12*7 = 84 ways.

WLOG stands for "Without Loss of Generality". It's a fancy way of saying, we'll consider one example, then we will extend our calculation for this example to all possible cases by multiplying or using symmetry.

Combinations, the basics

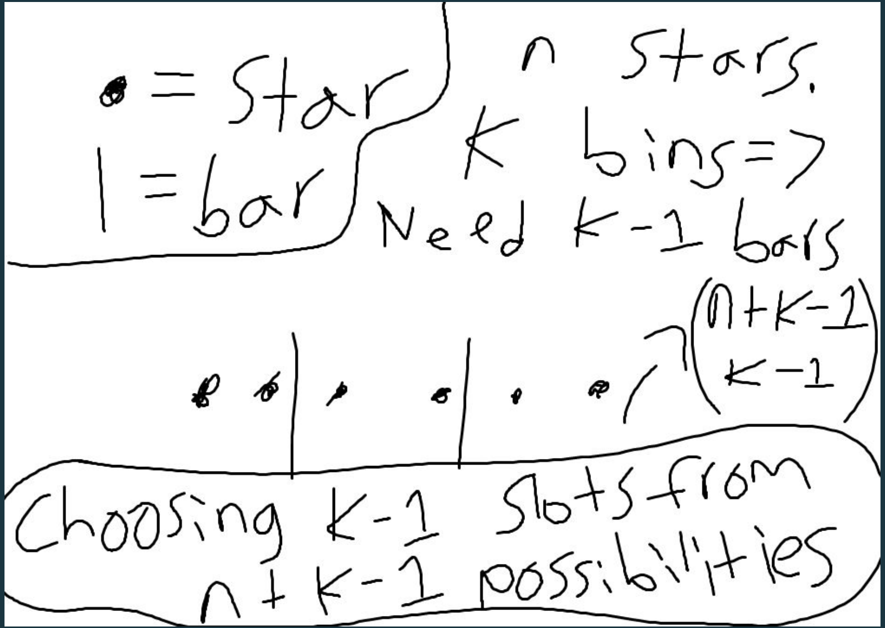
Combinations tells us the number of ways we can select a group of objects from a larger set (here it's selecting k objects from a set of n).

Num of ways = n choose k = n!/(k!(n-k)!)

Note: n choose n - k = n choose k

Stars and Bars: In how many ways can I distribute n objects into k bins?

n + k - 1 choose k - 1, or (n + k - 1)!/((k-1)!(n!))



Permutations vs Combinations

n choose k is usually associated with combinations. It’s the number of ways to CHOOSE objects from a larger set. It doesn’t matter which order you picked the objects in, if they’re picked, then they’re all counted in 1 way. This is why we say in combinations, order DOESN’T matter.

Permutations counts the number of rearrangements of objects. If I have 5 distinct objects selected, combinations would count that as one way, but permutations would count it as 5! = 120 ways (the number of ways to rearrange the objects amongst themselves).

Crucial to understand this difference!!

Practice!

1. In how many ways can 8 students and 3 teachers sit in the front row if, between every pair of teachers, there must be at least 2 students?

2. How many ways can a person place 5 (undistinguishable) 6-sided dice into 4 different containers given exactly one container must have 2 dice?

3. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

4. How many distinct word re-arrangements are possible for the set of letters ‘RIEMANNMATH’?

5. In how many ways can I select 2 people from a set of 2025 if order doesn’t matter? If order does matter?