A Proof of the Reconstruction Conjecture

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Abstract

The reconstruction conjecture, proposed by Kelly and Ulam, asks whether a graph is uniquely determined by its multiset of proper induced subgraphs. Despite substantial effort, over the 70 years since its proposal only modest progress has been made towards resolving the problem. In this work, we prove the statement in full generality. The proof techniques are novel, involving the sacrifice of one author's soul in exchange for earthly and divine knowledge.

1 Introduction

[**JF:** So we're actually doing this on Overleaf? I thought you were joking.]
[**M:** Hey, nothing wrong with Overleaf. You know when you keep getting those "server disconnected" syncing errors? I know the guy who came up with that one – big stuff. Anyway yeah, all that matters is that you do this of your own free will and understand what you're giving up yada yada. Go ahead and start writing.]

1.1 Background and History

In 1957, P. J. Kelly, then a graduate student working under Stanisław Ulam, published the first results on the recontruction conjecture, proving the statement for trees. A full statement of the problem was given by Ulam in 1960, although it is thought that he may have originally encountered it in Poland in 1929.

In 2012, the first author learned about the problem via personal communication from Gretchen Anderson. A January blizzard was in the process of burying Ann Arbor in 14 inches of snow – it was perhaps 5 in the afternoon, but the darkness outside made it feel surreally late. He and Anderson were sitting together on the tiled floor of her tiny kitchen, sipping hot chocolate as they played Go Fish and tried to forget about the problem sets they were avoiding.

When they'd first broken out the deck, they had been full of energy. The first author laughed at Anderson as she, mummified by a mass of fuzzy blankets, struggled to drink from her mug or hold her cards without shifting from her comfortable sitting position. Every time Anderson told the first author to "go fish", she would supplement with some snide remark or boastful taunt. But eventually the game had settled into a state of comfortable silence, the half-yawned requests for 5s and jacks accompanied only by the sound of cards being rearranged and the distant banshee cries of the swirling wind outside. This silence was broken as Anderson, shifting her leg, accidentally kicked over the deck, scattering the cards across the floor – the first author (who was losing the round badly) lept to his feet, accusing her with mock indignance of tampering with the deck, and demanding she return it to its original order. She laughed, protesting against the impossible request, and, as up until the end their conversations almost always did, the discussion turned to math.

Proposition 1. Once you mix up a deck of cards, there's no way to unshuffle them back to how they started.

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Proof. This is a consequence of the data processing inequality – once the information has been lost, no local operations can get it back again. □

"You know, it's crazy how little of math humans have figured out," Anderson said once the cards had been returned to a nice neat pile. "I mean, there's statements like the reconstruction conjecture that seem like they should either have simple counterexamples, or be as blatantly obvious as data processing inequality. But nobody has any clue how to even start to prove them."

The first author took the bait. "Ok, what's the reconstruction conjecture?"

The instant he responded, Anderson eagerly tossed aside her shroud of blankets, and began to write with a pen on her cocoa-stained napkin (see Figure 1). Smiling and shaking his head in a mixture of amusement and false exasperation, the first author scooched over to watch over her shoulder as she explained.

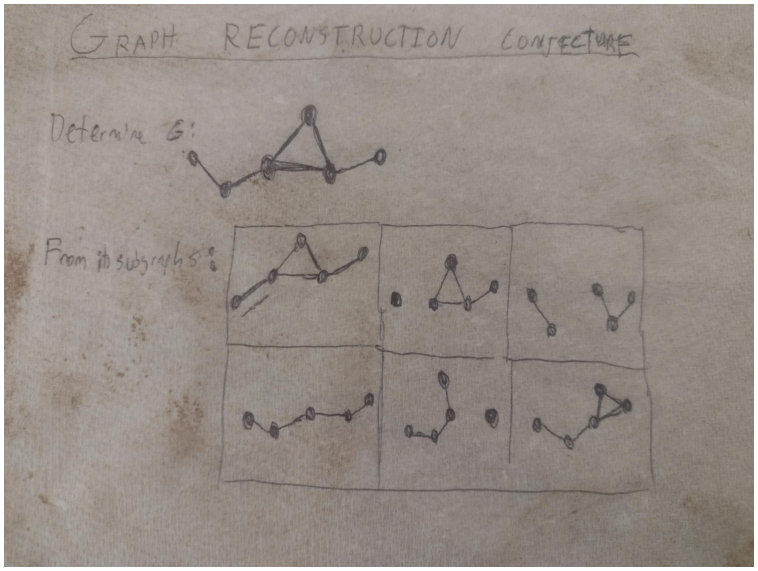

Figure 1: The original napkin Anderson wrote on that evening.

"Suppose I have a graph, I throw away one of its vertices, and I show you what's left. Can you figure out what the original was?"

She drew an example on the napkin. The first author looked at her quizzically – he had no idea where she was going with this.

"Of course not – it's like the deck of cards, the information is just gone. But what if I showed you *all* the different ways to delete a vertex? Maybe, even though they're each missing something, if you stare hard enough at all different fragments that remain there's some way of piecing back together the original."

"... is there?"

She raised her eyebrows and paused for a moment, barely suppressing a grin. The first author held his breath involuntarily.

"Nobody knows."

1.2 Previous Work

In 2013, Anderson et al wrote a sprawling paper analyzing reconstruction of ***perfect graphs*** (i.e. graphs where clique number equals chromatic number) [AFS13]. The paper grew out of over a year's worth of

late nights in Anderson's kitchen, where she, Faust, and occasionally a third undergrad they managed to drag along, would convene every evening for a witches' sabbath. Anderson would sit cross-legged on the counter, absent-mindedly picking fluff from her thick wool sweater, as they went back and forth, excitedly proposing new ideas, finding flaws in old ones, and leaving faint smudges of red and blue and black and purple on a whiteboard too old to ever fully erase. In the end, it turned out that most of their results had already been shown, far more succinctly, in section 4 of a 20-year-old journal article – so they never tried to publish anywhere. And it's probably for the best, because their write-up was by any account a total mess, full of photos of hand-sketched figures and thickets of indecipherable lemma statements. For a while, Faust was devastated that they'd been wasting their efforts re-deriving already known results. But even after Anderson had read those older proofs, for all of their surgical elegance, she continued to maintain that theirs had been the more beautiful.

In 2015, Anderson and Faust, now working alongside their mutual advisor at UW, managed to publish their first paper. They defined a class of ***joyful*** graphs, and proved that all such graphs require a large number of subgraphs to reconstruct [AFK15]. Although the result was somewhat niche, the paper was written meticulously, full of wheels and sunflowers and stars that Faust had spent hours drawing at his computer while Anderson perched behind him, providing unsolicited commentary and trying to distract him with a card trick she could never quite get right. The day they got the acceptance notice, they went back to Faust's apartment, gathered together his roommates and, holding mugs of hot chocolate to the air, swore before them a solemn oath of battle: one day, together, they would defeat the reconstruction conjecture completely. And over the next couple years, though their victories won only modest ground, they continued to fight this battle side-by-side.

In 2016, Anderson and Faust identified a statement about order ideals that would imply that all posets are reconstructible [AF16]. The following year, they finished a sweeping 300-page tour-de-force, bounding the behaviour of reconstruction numbers in graphs formed from joins or unions of two smaller graphs [AF17].

In 2018, Faust and Faust studied reconstruction in the unitary Cayley graphs of rings [FF18a]. Later that year, they showed that it's always possible to reconstruct a graph if it's known to have a perfect matching [FF18b]. In 2019, in a sequence of surprising works, they showed that by searching for ***stable sets*** in a so-called ***happy family*** of subgraphs, the minimal set of proper subgraphs needed for reconstruction could be found in quasi-polynomial time [FF19a; FF19b; FF19c].

When, in 2023, Faust published his paper on reconstructing graphs with large holes [Fau23a], the graph theory community was relieved to see him picking himself up again. But as he continued to churn out a furious stream of incomprehensibly dense and error-filled results [Fau23b; Fau24a], survey papers that read more like the garbled ranting of a fever patient [Fau24b; Fau24c], and a frankly impressive number of flailing attempts to prove the reconstruction conjecture in a single stroke [Fau24d; Fau24e], this relief quickly turned to concern. His old advisor wrote to him "Please Johann, you have to let go. She wouldn't want to see you like this" [Kel24]. He never responded, but he thought to himself: "she would have wanted us to **know**".

1.3 This Paper

Three hours ago, the first author was walking back alone to his hotel from the day's session of the Gretchen Faust Memorial Conference. Kela had given a beautiful talk in tribute, and as he walked the first author felt himself once again soaked in a sea of ill-defined emotions, staining the corners of his mind a dark brown until the memories seemed all to bleed into each other. He longed desperately for a way to cut through all the fuzziness that coated his thoughts, to scrub away the messy residues and see clearly enough to finally *understand*. But all he had was the same old mind he'd always had – locked in the same loops of guessing and second-guessing, going through the whole deck twice every time he tried to find the card he was thinking of. What must she think of him, looking down from up there? With all the knowledge of the divine, how pathetic must she find it to watch him continue to lose the same battle to the same problem?

He was so preoccupied that it took him some time to realize he was being followed. A small, collarless black poodle stepped gingerly down the street, peering into gutters and slinking around telephone poles, but always remaining about ten feet behind him. As he turned a corner towards his hotel, the dog

happened to turn the same way. It wasn't until he'd walked through the door into the hotel lobby and stepped into the elevator that he finally looked down by his feet and came face to face with the second author. As the elevator rose, the first author watched the figure of the dog twist itself into the figure of a man, tall and neatly dressed, with a faint smirk on his face.

"I've come to make a deal."

2 Preliminaries

Definition 1. We call a subgraph induced by deleting a single vertex from a graph G a ***card***. The set of all cards forms the ***deck*** of G.

Definition 2. The ***soul*** is one's overall essence; a structure that connects millions of individual thoughts, memories, hopes, and experiences into the single web that is a person.

"For the low low price of one soul, I can offer you whatever strikes your fancy. Do you want your own personal guide to the pleasures and novelties of this world? I can take you to hell and back – and then some."

"I want her back."

"Sorry, no can do on the resurrections. We had an incident with some Helen of Troy stuff a while back, and it's been a hard no from the boss ever since. Anything else though – fame, power, a giant hamster wheel and a giant hamster to go with it, you name it. There'll be hell to pay – and we've got deep pockets."

"Then I want to know. I want to hear her prove reconstruction."

Definition 3. The ***reconstruction number*** of a graph is the number of cards needed to uniquely recover that graph after it's lost. The reconstruction conjecture holds that the reconstruction number is always defined.

Definition 4. The ***devil*** is the purest form of evil; the embodiment of all things hollow and empty. What he takes he never gives back.

"Do you understand what it will cost you?"

"I understand."

3 Methods

[**M:** Ok, all you have to do now is prove that you are truly willing to make this deal. Once we've got that, I can get in contact with your missus upstairs and we can see what she's got to tell you about these graphs of yours. I've got to say, not the most exiting satanic pact I've ever seen – but hey, it's your soul.]

[**M:** Come now, let's get typing. You know what they say about idle hands.]

Lemma 1. The reconstruction number can be as high as $\frac{2(n-1)}{3}$. It's believed that this is tight – that the answer to the question "how much of yourself do you have to lose before there's no way back?" is 1/3. But all we know is the upper bound.

Proof. Let $r = \frac{(n-1)}{3}$, and consider the graphs $K_r \cup K_r \cup K_{r+1}$ and $K_{r-1} \cup K_{r+1} \cup K_{r+1}$. In either case, the card $K_{r-1} \cup K_r \cup K_{r+1}$ appears at least $2r$ times in the deck. □

Lemma 2. The following are equivalent:

- The clumpy dregs at the bottom of a really rich cup of hot chocolate, when there's too much cocoa powder to dissolve and the liquid gradually becomes so thick and syrupy that you have to hold your head back, tilt the mug upside down, and wait for it to ooze down the sides.

- The soft warmth of an old alpaca wool sweater against your neck, its fuzzy surface dashed with the remnants of nicks from dry erase markers that never washed out, and a faint dusty smell like a UMich lecture hall.
- Heartbreak.

Proof. The night we swore our battle oath, she stayed close by my side, uncharacteristically quiet throughout the festivities. She laughed along when my roommates errupted into cheers at my St. Crispin's Day speech, but it sounded forced, as if her attention was focused elsewhere, listening to the wind outside our window.

When, eventually, the band of brothers had dispersed and we were alone in my kitchen, we remained in silence for some time. She leaned against the counter, staring fixedly ahead at a point on the wall by the refrigerator, slowly curling and uncurling her toes. I absent-mindedly washed the dishes, scrubbing chocolate residue from the bottom of a mug with a paper towel. After what felt like an impossibly long time, she broke the silence.

"The other news we were waiting for came back today, too. It doesn't look good."

I felt suddenly very cold. My arms and legs shivered slightly, and without thinking I slumped down to the kitchen floor, my knees bent and my back pressed against the sink cabinet. She sat down beside me, wrapping her arms around my neck. Her voice was almost inaudible.

"They say I have about five years in expectation, but the variance is large – we could easily have ten, if we're lucky."

All I could get out was, stupidly, "what do we do now?"

She gave a faint smile. "The same as we were always going to do. We huddle up through the winter. We play Go Fish. We prove some cool theorems."

"But what does it all mean, now that we already know how it ends?"

"We always knew how it ends."

□

Proposition 2. I, Johann Faust, do willingly and without coercion surrender ownership of my soul to the demon Mephistopheles.

Proof. The proof follows from Lemma 1 and Lemma 2. Far, far more than 1/3 is gone. I have no use for what remains; you are free to take it. Just let me hear from her – let her show me the end of the fight we couldn't win in time. □

4 Main Results

Theorem 1. All graphs are reconstructible.

Proof. [**GF:** The proof is left to the reader as an exercise. The battle is worth it, Johann.] □

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