Three-Point Trilateration

Given three points in a plane:

$$P_1(x_1, y_1), \quad P_2(x_2, y_2), \quad \text{and} \quad P_3(x_3, y_3)$$

and the distances from an unknown point $P(x, y)$ to each of these three points are r_1, r_2, and r_3 respectively, we can set up the following equations based on the Pythagorean theorem:

$$\begin{aligned} A) \quad & (x - x_1)^2 + (y - y_1)^2 = r_1^2 \\ B) \quad & (x - x_2)^2 + (y - y_2)^2 = r_2^2 \\ C) \quad & (x - x_3)^2 + (y - y_3)^2 = r_3^2 \end{aligned}$$

Subtracting the first equation from the other two, we get the linear equations:

$$\begin{aligned} D) \quad & 2(x_2 - x_1)x + 2(y_2 - y_1)y = r_1^2 - r_2^2 + x_2^2 - x_1^2 + y_2^2 - y_1^2 \\ E) \quad & 2(x_3 - x_1)x + 2(y_3 - y_1)y = r_1^2 - r_3^2 + x_3^2 - x_1^2 + y_3^2 - y_1^2 \end{aligned}$$

From D and E, we solve for x and y.

Overconstrained Trilateration

When given more than three points and their distances to an unknown point, the system is overconstrained. This means it may not be possible to find a point that satisfies all the distance equations exactly. Therefore, we use a least squares method to find the point $P(x, y)$ that minimizes the total squared error with respect to all given distances.

Given n points $P_i(x_i, y_i)$ and their distances r_i to the unknown point, we formulate the equations:

$$f_i(x, y) = x^2 + y^2 - 2x_i x - 2y_i y + x_i^2 + y_i^2 - r_i^2$$

The goal is to find x and y that minimize:

$$S(x, y) = \sum_{i=1}^{n} f_i(x, y)^2$$

Using optimization techniques, such as gradient descent, we can minimize $S(x, y)$ to determine the coordinates of point P.