Assignment 1

ELEC 446 Autonomous Mobile Robotics

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1 Question 1

In a practice problem, you derived the kinematic equations of motion for a four-wheeled steering mobile robot, as shown in Figure 1 and where $\mathbf{q} = (x, y, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$. The robot's kinematic equations of motion are,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{l}\tan\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1}$$

where $v_1 \in \mathbb{R}$ is the vehicle's speed [m/s] at location (x, y) (in the direction of θ) and $v_2 = \dot{\phi} \in \mathbb{R}$ commands the steering rate [rad/s]. Let $l = 1.5$ m (e.g., similar to a small car).

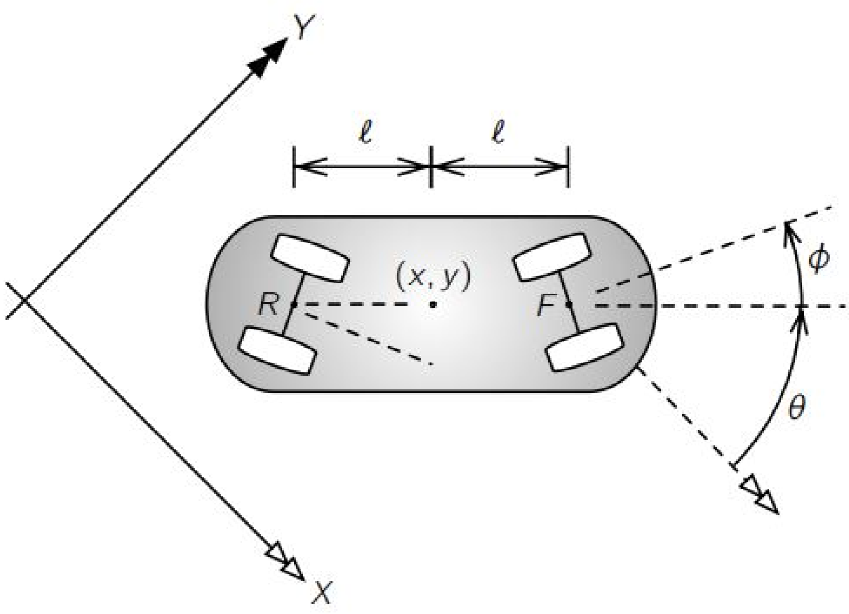

Figure 1: A four-wheeled steering mobile robot with chosen coordinates $\mathbf{q} = (x, y, \theta, \phi)$. This choice of coordinates is not unique. Here ϕ is the steering angle constrained to the domain $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(a) [2 marks]

Suppose that, instead of v_1, we want the input to the model (i.e., to our robot) to be the speed at the front wheel $v_F \in \mathbb{R}$ (i.e., in the direction of $\theta + \phi$), where $v_1 = v_F \cos\phi$. Rewrite the kinematic equations of motion in terms of v_F and v_2.

The system of equations describing the mobile robot with respect to the original inputs v_1 and v_2 is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{l}\tan\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \cos\theta \\ v_1 \sin\theta \\ v_1 \frac{1}{l}\tan\phi \\ v_2 \end{bmatrix}$$

Substituting our new input with the relation $v_1 = v_F \cos\phi$,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} v_F \cos\phi \cos\theta \\ v_F \cos\phi \sin\theta \\ v_F \cos\phi \frac{1}{l}\tan\phi \\ v_2 \end{bmatrix}$$

We can factor our new set of inputs (v_F, v_2) for our new kinematic equations of motion:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & 0 \\ \cos\phi\sin\theta & 0 \\ \frac{1}{l}\cos\phi\tan\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_F \\ v_2 \end{bmatrix}$$

Finally we can simplify some of the trigonometry operations knowing $\tan x = \frac{\sin x}{\cos x}$,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & 0 \\ \cos\phi\sin\theta & 0 \\ \frac{1}{l}\sin\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_F \\ v_2 \end{bmatrix} \tag{2}$$

(b) [8 marks]

Use approximate linearization to design an autonomous mobile robot controller to find (v_F, v_2) that makes the point (x, y) at the centre of the robot in Figure 1 follow a straight-line trajectory moving at a constant speed $v_d > 0$ in the direction θ_d. In other words,

$$x_d(t) = x_d(0) + v_d t \cos\theta_d, \tag{3}$$

$$y_d(t) = y_d(0) + v_d t \sin\theta_d, \tag{4}$$

Check that your setup is controllable and explain how you would choose your controller gains.

1.0.1 Coordinates

Using Equation (3) and (4) let's set up a the desired trajectory,

$$\begin{cases} x_d(t) = & x_d(0) + v_d t \cos\theta_d \\ y_d(t) = & y_d(0) + v_d t \sin\theta_d \\ \theta_d(t) = & \theta_d \\ \phi_d(t) = & 0 \end{cases}$$

with dynamics

$$\begin{cases} \dot{x}_d(t) = & v_d \cos\theta_d \\ \dot{y}_d(t) = & v_d \sin\theta_d \\ \dot{\theta}_d(t) = & 0 \\ \dot{\phi}_d(t) = & 0 \end{cases}$$

Note we may substitute our desired velocity v_d with the desired input $v_{F,d}$ knowing $v_d = v_{F,d} \cos\phi_d$ (and $\phi_d = 0$).

$$\begin{cases} \dot{x}_d(t) = & v_{F,d} \cos\theta_d \\ \dot{y}_d(t) = & v_{F,d} \sin\theta_d \\ \dot{\theta}_d(t) = & 0 \\ \dot{\phi}_d(t) = & 0 \end{cases} \tag{5}$$

Next we'll write our system in coordinates attached to the moving point and call this frame z;

$$z = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\theta} \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \\ \phi_d - \phi \end{bmatrix} \tag{6}$$

and

$$\tilde{u} = \begin{bmatrix} \tilde{v}_F \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} x_d - v_F \\ y_d - v_2 \\ \theta_d - \theta \end{bmatrix} \tag{7}$$

Our dynamics in this new coordinate system are,

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \\ \dot{\tilde{\theta}} \\ \dot{\tilde{\phi}} \end{bmatrix} = \begin{bmatrix} \dot{x}_d - \dot{x} \\ \dot{y}_d - \dot{y} \\ \dot{\theta}_d - \dot{\theta} \\ \dot{\phi}_d - \dot{\phi} \end{bmatrix} \tag{8}$$

We can substitute the dynamics systems (2) and (5) into (8) and write in terms of our $(\tilde{\cdot})$ and desired coordinates.

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \\ \dot{\tilde{\theta}} \\ \dot{\tilde{\phi}} \end{bmatrix} = \begin{bmatrix} v_F \cos\phi\cos\theta - v_{F,d}\cos\theta_d \\ v_F\cos\phi\sin\theta - v_{F,d}\sin\theta_d \\ v_F\frac{1}{l}\sin\phi \\ v_2 \end{bmatrix}$$

writing in terms of our $(\tilde{\cdot})$ and desired coordinates

$$\tilde{f}(z,\tilde{u}) = \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \\ \dot{\tilde{\theta}} \\ \dot{\tilde{\phi}} \end{bmatrix} = \begin{bmatrix} (v_{F,d} - \tilde{v}_F)\cos\phi\cos\theta - v_{F,d}\cos\theta_d \\ (v_{F,d} - \tilde{v}_F)\cos\phi\sin\theta - v_{F,d}\sin\theta_d \\ (v_{F,d} - \tilde{v}_F)\frac{1}{l}\sin\phi \\ (v_{2,d} - \tilde{v}_2) \end{bmatrix} \tag{9}$$

1.0.2 Linearization

Next we calculate the Jacobian of $\tilde{f}(z,\tilde{u})$ with respect to z and u.

$$\frac{\partial\tilde{f}(z,\tilde{u})}{\partial z} = \begin{bmatrix} 0 & 0 & -(v_{F,d}+\tilde{v}_F)\cos(\phi_d+\tilde{\phi})\sin(\theta_d+\tilde{\theta}) & -(v_{F,d}+\tilde{v}_F)\sin(\phi_d+\tilde{\phi})\cos(\theta_d+\tilde{\theta}) \\ 0 & 0 & (v_{F,d}+\tilde{v}_F)\cos(\phi_d+\tilde{\phi})\cos(\theta_d+\tilde{\theta}) & -(v_{F,d}+\tilde{v}_F)\sin(\phi_d+\tilde{\phi})\sin(\theta_d+\tilde{\theta}) \\ 0 & 0 & 0 & (v_{F,d}+\tilde{v}_F)\frac{1}{l}\cos(\phi_d+\tilde{\phi}) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\frac{\partial\tilde{f}(z,\tilde{u})}{\partial\tilde{u}} = \begin{bmatrix} \cos(\phi_d+\tilde{\phi})\cos(\theta_d+\tilde{\theta}) & 0 \\ \cos(\phi_d+\tilde{\phi})\sin(\theta_d+\tilde{\theta}) & 0 \\ \frac{1}{l}\sin\phi_d & 0 \\ 0 & 1 \end{bmatrix}$$

and evaluate them at the origin of our coordinate system

$$A = \left.\frac{\partial \tilde{f}(z, \tilde{u})}{\partial z}\right|_{z=\bar{\mathbf{0}}, \tilde{u}=\bar{\mathbf{0}}} = \begin{bmatrix} 0 & 0 & -v_{F,d}\cos(\phi_d)\sin(\theta_d) & -v_{F,d}\sin(\phi_d)\cos(\theta_d) \\ 0 & 0 & v_{F,d}\cos(\phi_d)\cos(\theta_d) & -(v_{F,d})\sin(\phi_d)\sin(\theta_d) \\ 0 & 0 & 0 & v_{F,d}\frac{1}{l}\cos\phi_d \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$B = \left.\frac{\partial \tilde{f}(z, \tilde{u})}{\partial \tilde{u}}\right|_{z=\bar{\mathbf{0}}, \tilde{u}=\bar{\mathbf{0}}} = \begin{bmatrix} \cos(\phi_d)\cos(\theta_d) & 0 \\ \cos(\phi_d)\sin(\theta_d) & 0 \\ \frac{1}{l}\sin\phi_d & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$

1.0.3 Controllability

To assess the controllability of the system we define the matrix M

$$M = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

First we compute the powers of A

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 0 & 0 & -v_{F,d}\cos\phi_d\sin\theta_d & -v_{F,d}\sin\phi_d\cos\theta_d \\ 0 & 0 & -v_{F,d}\cos\phi_d\cos\theta_d & -v_{F,d}\sin\phi_d\sin\theta_d \\ 0 & 0 & 0 & v_{F,d}\frac{1}{l}\cos\phi_d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & -v_{F,d}^2\frac{1}{l}\cos^2(\phi_d)\cos\theta_d \\ 0 & 0 & 0 & -v_{F,d}^2\frac{1}{l}\cos^2(\phi_d)\cos\theta_d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Followed by their products of B,

$$BA^0 = \begin{bmatrix} \cos(\phi_d)\cos(\theta_d) & 0 \\ \cos(\phi_d)\sin(\theta_d) & 0 \\ \frac{1}{l}\sin\phi_d & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA^1 = \begin{bmatrix} -v_{F,d}\cos\phi_d\sin\theta_d\frac{1}{l}\sin\phi_d & -v_{F,d}\sin\phi_d\cos\theta_d \\ -v_{F,d}\cos\phi_d\cos\theta_d\frac{1}{l}\sin\phi_d & -v_{F,d}\sin\phi_d\sin\theta_d \\ 0 & v_{F,d}\frac{1}{l}\cos\phi_d \\ 0 & 0 \end{bmatrix}$$

$$BA^2 = \begin{bmatrix} 0 & -v_{F,d}^2\frac{1}{l}\cos^2(\phi_d)\sin\theta_d \\ 0 & -v_{F,d}^2\frac{1}{l}\cos^2(\phi_d)\cos\theta_d \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus:

$$M = \begin{bmatrix} \cos\phi_d\cos\theta_d & 0 & -v_{F,d}\cos\phi_d\sin\theta_d\frac{\sin\phi_d}{l} & -v_{F,d}\sin\phi_d\cos\theta_d & 0 & -v_{F,d}^2\frac{\cos^2\phi_d\sin\theta_d}{l} & 0 & 0 \\ \cos\phi_d\sin\theta_d & 0 & -v_{F,d}\cos\phi_d\cos\theta_d\frac{\sin\phi_d}{l} & -v_{F,d}\sin\phi_d\sin\theta_d & 0 & -v_{F,d}^2\frac{\cos^2\phi_d\cos\theta_d}{l} & 0 & 0 \\ \frac{\sin\phi_d}{l} & 0 & 0 & v_{F,d}\frac{\cos\phi_d}{l} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We want to determine whether the matrix $M \in \mathbb{R}^{4\times 8}$ is full rank, which is equivalent to checking whether the associated system is controllable (i.e., $\mathrm{rank}(M) = n$, the number of states). Since M has $n = 4$ rows, it is full row rank if and only if its rows are linearly independent. By inspection, row 4 of M is $[0, 1, 0, 0, 0, 0, 0, 0]$, which is clearly independent of the other rows. Columns 2, 5, 7, and 8 contain zeros in the first three rows and therefore do not contribute to the linear independence of these rows. Consequently, to determine the rank of the first three rows, we can focus on a 3×3 submatrix formed from columns $\{1, 4, 6\}$:

$$\mathbb{M} = M_{[1:3,\{1,4,6\}]} = \begin{bmatrix} \cos\phi_d\cos\theta_d & -v_{F,d}\sin\phi_d\cos\theta_d & -v_{F,d}^2\frac{\cos^2\phi_d\sin\theta_d}{l} \\ \cos\phi_d\sin\theta_d & -v_{F,d}\sin\phi_d\sin\theta_d & -v_{F,d}^2\frac{\cos^2\phi_d\cos\theta_d}{l} \\ \frac{\sin\phi_d}{l} & \frac{v_{F,d}\cos\phi_d}{l} & 0 \end{bmatrix}.$$

If the determinant of this submatrix is nonzero, then these three rows are linearly independent. Together with row 4, this shows that M has full row rank ($\mathrm{rank}(M) = 4$) for generic parameter values.

We calculate the determinant of $\mathbb{M}$ we use the standard formula for a 3×3 matrix

$$\det(\mathbb{M}) = \mathbb{M}_{31}(\mathbb{M}_{12}\mathbb{M}_{23} - \mathbb{M}_{13}\mathbb{M}_{22}) - \mathbb{M}_{32}(\mathbb{M}_{11}\mathbb{M}_{23} - \mathbb{M}_{13}\mathbb{M}_{21}) + \mathbb{M}_{33}(\mathbb{M}_{11}\mathbb{M}_{22} - \mathbb{M}_{12}\mathbb{M}_{21})$$

since $\mathbb{M}_{33}$ is zero, this simplifies to

$$\det(\mathbb{M}) = \mathbb{M}_{31}(\mathbb{M}_{12}\mathbb{M}_{23} - \mathbb{M}_{13}\mathbb{M}_{22}) - \mathbb{M}_{32}(\mathbb{M}_{11}\mathbb{M}_{23} - \mathbb{M}_{13}\mathbb{M}_{21})$$

We can then evaluate the determinant.

$$\begin{aligned}\det(\mathbb{M}) = &\frac{\sin\phi_d}{l}(-v_{F,d}\sin\phi_d\cos\theta_d \cdot -v_{F,d}^2\frac{\cos^2\phi_d\cos\theta_d}{l} - -v_{F,d}^2\frac{\cos^2\phi_d\sin\theta_d}{l}\cdot -v_{F,d}\sin\phi_d\sin\theta_d)\\ &-\frac{v_{F,d}\cos\phi_d}{l}(\cos\phi_d\cos\theta_d \cdot -v_{F,d}^2\frac{\cos^2\phi_d\cos\theta_d}{l} - -v_{F,d}^2\frac{\cos^2\phi_d\sin\theta_d}{l}\cdot\cos\phi_d\sin\theta_d)\end{aligned}$$

and simplifies to

$$\det(\mathbb{M}) = v_{F,d}^3\frac{1}{l^2}\cos^2\phi_d(\cos^2\theta_d - \sin^2\theta_d) \tag{12}$$

Equation (12) shows the determinant is nonzero and therefore M has full rank and thus the system is controllable provided that:

$$v_{F,d} \neq 0, \quad l \neq 0, \quad \cos\phi_d \neq 0, \quad \cos^2\theta_d \neq \sin^2\theta_d.$$

1.0.4 Controller Design

We design a controller of the form

$$\tilde{u} = -K\tilde{x}$$

From the Theorem of Linear Stability discussed in class, want to choose K such that the matrix $(A - BK)$ has eigen values λ_i in the left hand complex plane (i.e. $\text{Re}(\lambda_i) < 0$). In doing so we insure the linear system $\dot{\tilde{x}} = (A - BK)\tilde{x}$ will be asymptotically stable. The location of these poles within the LHCP will affect the behaviour of the system. For a steady response we prefer small magnitude poles lying on the real access. Pole placement in this assignment was done by trial and error, in the end I selected an arbitrary set of poles $p = \{-2.0, -1.5, -0.5, -0.5\}$, that showed good performance.

(c) [2 marks]

Neatly sketch a block diagram of your robot control system design. Clearly label each of the blocks and the signals that connect them.

A simple block diagram of the control system is presented in Figure 2 Note in the diagram the state variables are referred to as $x = \begin{bmatrix} x & y & \theta & \phi \end{bmatrix}^T$ as convention. Elsewhere in the document we have used other variables to represent the full state to avoid confusion between the full state and x component.

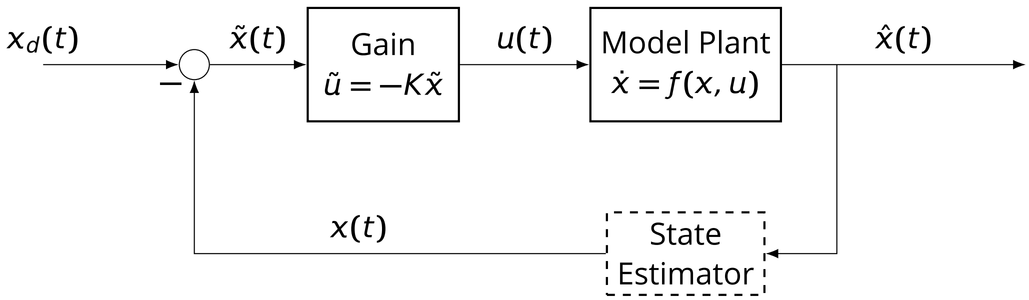

Figure 2: Control block diagram of the vehicle.

(d) [8 marks]

Simulate this mobile robot and your controller design by using a computer (e.g., in Python or whatever language you prefer). Use the following parameters in your simulation:

- Robot's initial pose $\mathbf{q}(0) = (0, 0, \frac{3\pi}{2}, 0)$ and the trajectory start $(x_d(0), y_d(0)) = (0, 0)$;
- Trajectory speed $v_d = 20$ km/h and direction $\theta_d = \frac{\pi}{4}$ for all $t \geq 0$.

Choose appropriate controller gains to get what you think is reasonable robot behaviour. Submit your code and plots showing the robot's trajectory, control inputs, and tracking error.

Simulation of this system used a python environment adapted from Dr. Joshua Marshall's agv-examples repository [1]. Source code can be found attach in the assignment submission as well as my personal autonomous_mobile_robotics repository.

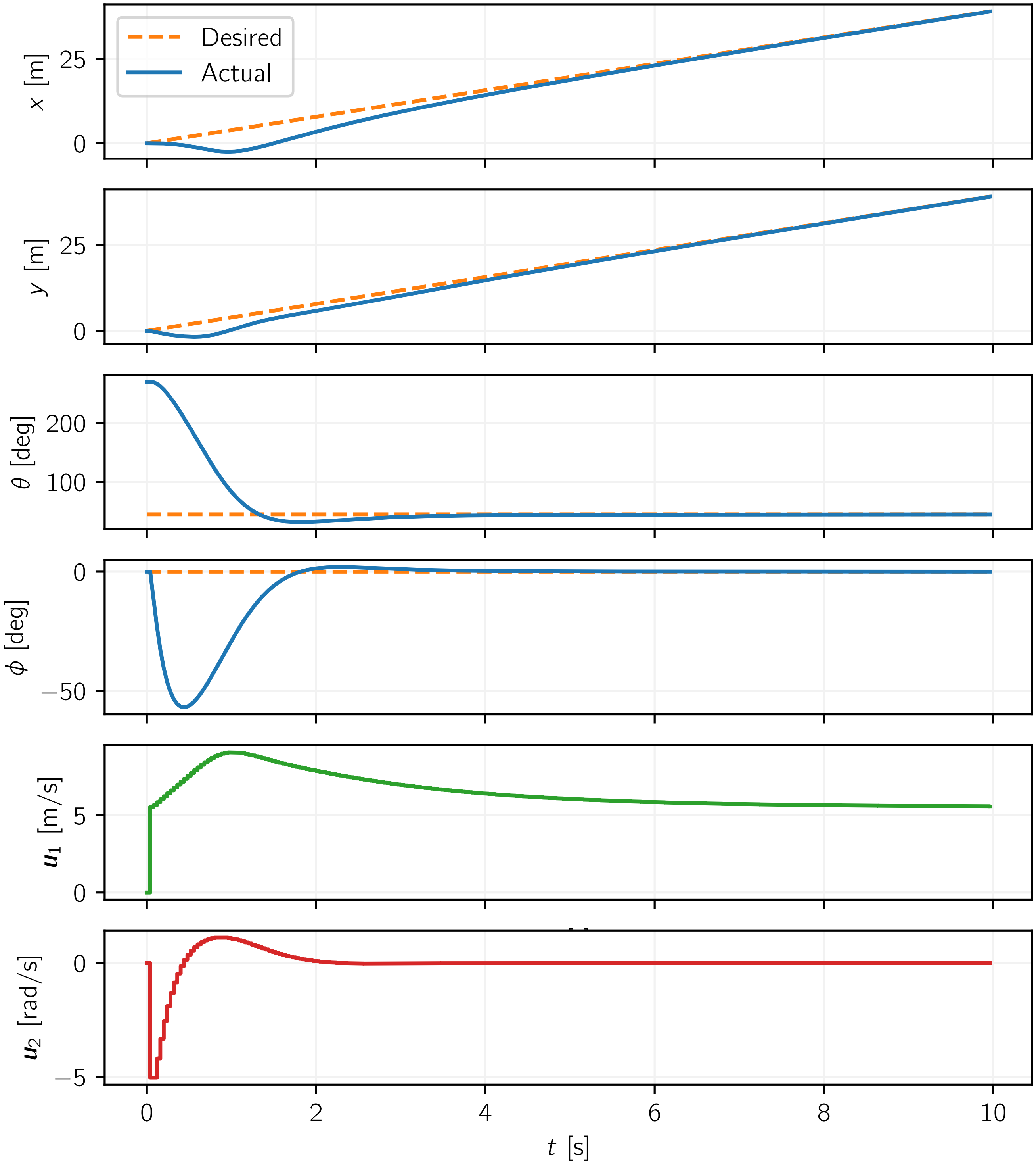

Figure 3: System state and input variables overtime compared to the desired trajectory.

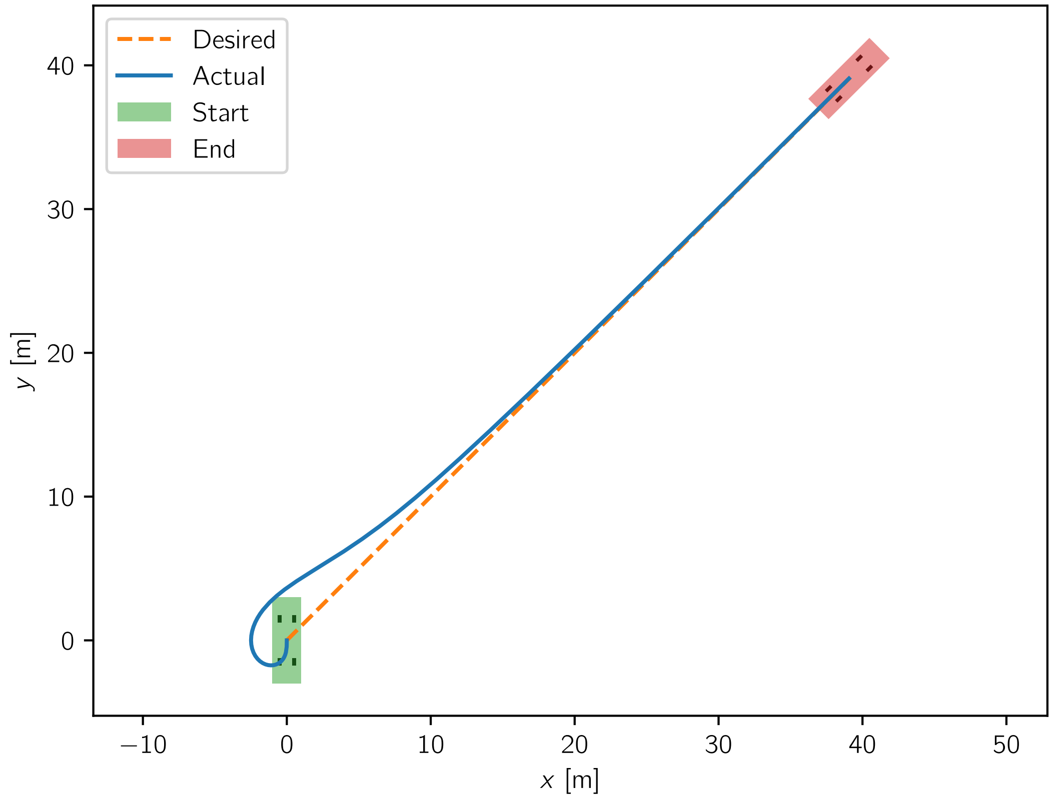

Figure 4: Planar view of the vehicle's trajectory in x and y.

(e) [2 marks BONUS]

Show that your controller design is only locally stabilizing by providing a (simulation) counterexample where the robot's trajectory diverges from the desired trajectory. Note: This is a bonus question, but you cannot get more than 100 % on this assignment.

Linear approximations are generally effective for systems when $\tilde{q}$ is small. So in this example we choose the unstable initial conditions far from the desired trajectory

$$q_{0,\text{unstable}} = \begin{bmatrix} 60 & 0 & \frac{3.5\pi}{2} & 0 \end{bmatrix}^T$$

Figures 5 and 6 show the response of the system to these initial conditions. Note in Figure 5 around 8 seconds once the input $u_2 = \dot{\phi}$ deviates from its desired state it introduces unstable oscillations to the system.

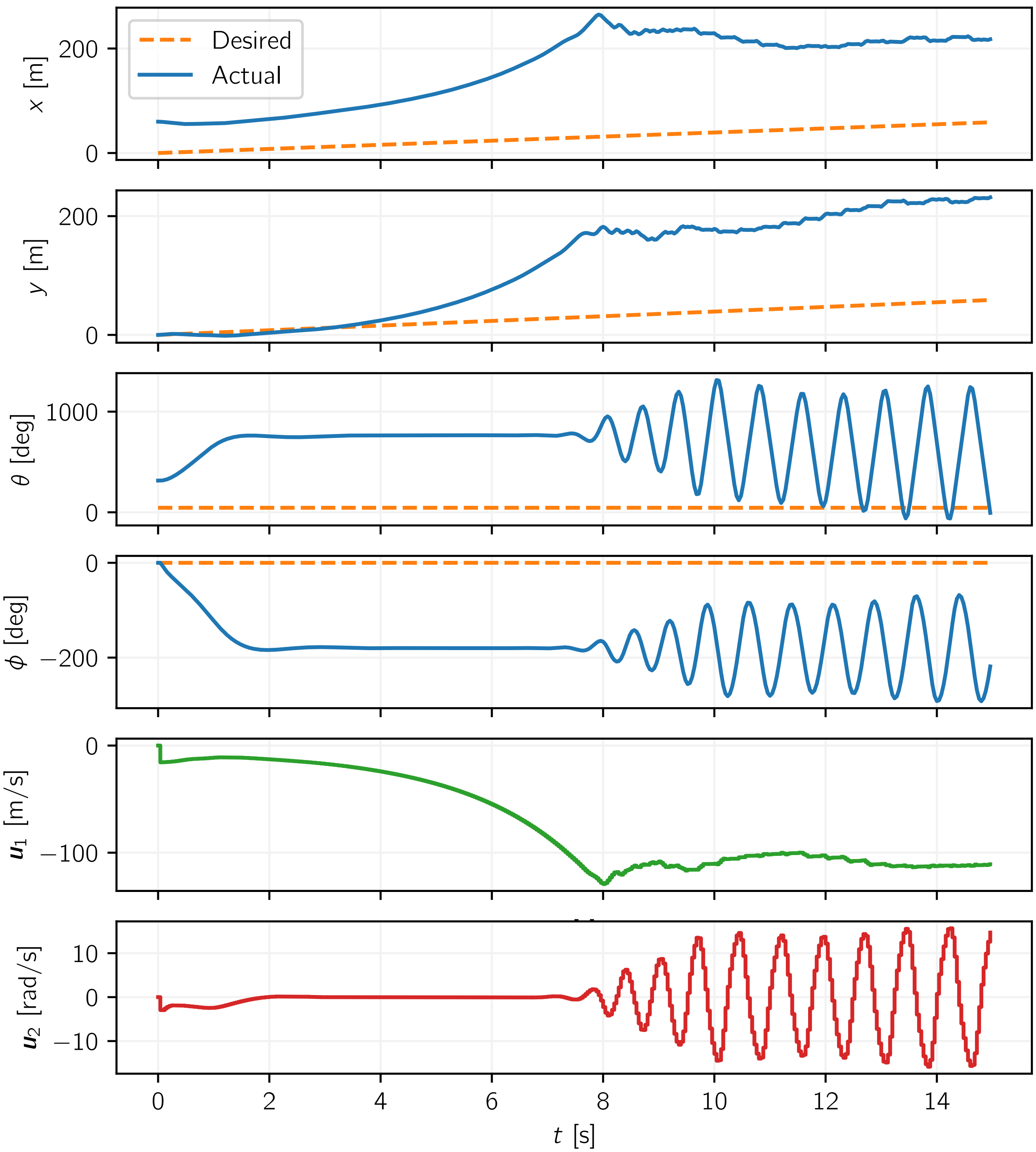

Figure 5: State and input variables of the unstable system overtime compared to the desired trajectory.

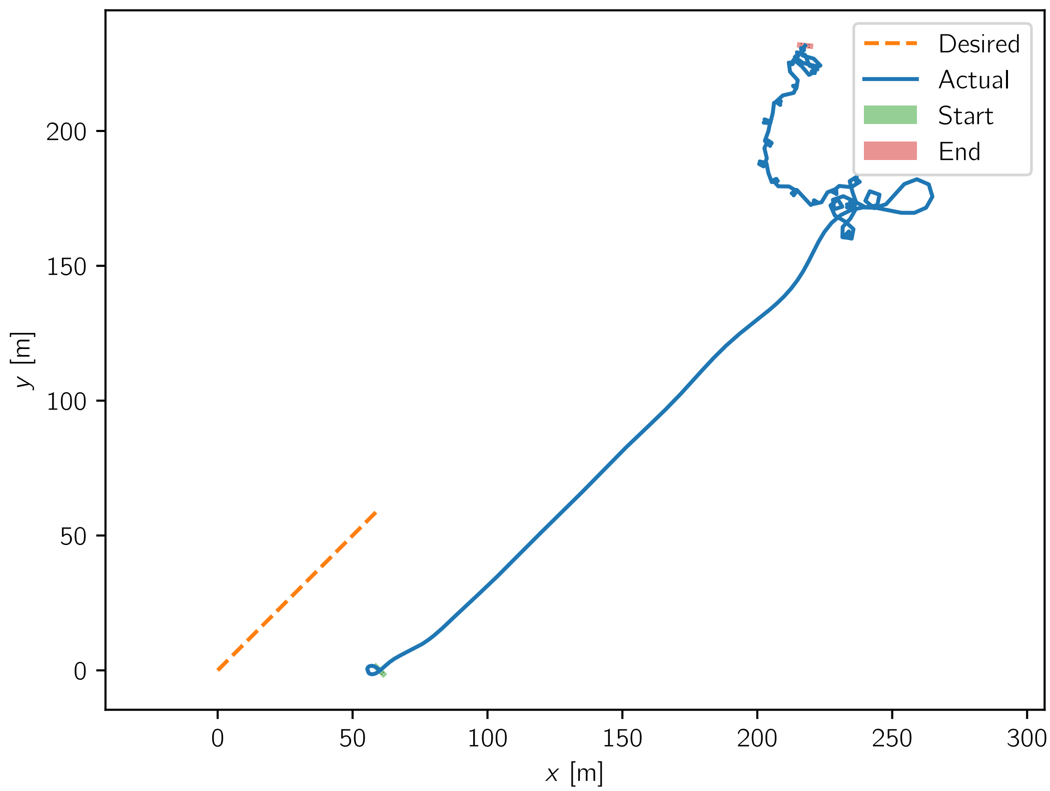

Figure 6: Planar view of the vehicle's trajectory in x and y in unstable conditions.

(f) [0 marks FOR FUN]

Try different trajectories!

Unfortunately, this has not been tested at time of submission but checkout the autonomous_mobile_robotics repo to see if I got there.

References

[1] J. A. Marshall, "Autonomous ground vehicle navigation and control simulation examples in python, v0.4.4." `https://github.com/botprof/agv-examples`, 2023.