MAE 541 Final Project

Using Lyapunov Exponents to Identify Chaos in the Logistic Map and Other 1D Maps

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1 Introduction

Chaos in a dynamical system does not have a single definition. One method to detect chaos is Melnikov's method. If Melnikov's function has a simple zero, then the stable and unstable manifolds intersect trasversally. This implies that the system has a homoclinic tangle and therefore exhibits chaos. One of the most widely accepted definitions of chaos in dynamical systems is given by Devaney in [1]. He posits that for a system to be chaotic, it must have a strong dependence on initial conditions (SDIC), topological transitivity, and dense periodic orbits. Another method to detect chaos in a dynamic system is to use the Lyapunov exponent. Lyapunov exponents examine whether infinitesemally close orbits converge or diverge exponentially fast. This allows one to detect SDIC. A system that has one or more positive Lyapunov exponents is defined to be chaotic [2]. This method of finding chaos will be outlined and used in this project.
A simple map that exhibits chaos is the logisitc map. It is given by:

$$x_{n+1} = f(x_n) = \lambda x_n(1 - x_n) \tag{1}$$

This 1D map has rich behavior for varying values of the parameter λ. This map can be applied to real-world systems, such as population dynamics. This project aims to examine the logisitc map using Lyapunov exponents, ultimately showing that the map has a positive Lyapunov exponent and is therefore chaotic.

2 Exploring the Logistic Map

We can identify the fixed points on the map by plotting $y = f(x_n)$ and $y = x_n$ and looking at the intersections. For $\lambda = 1$, we find Figure 1.

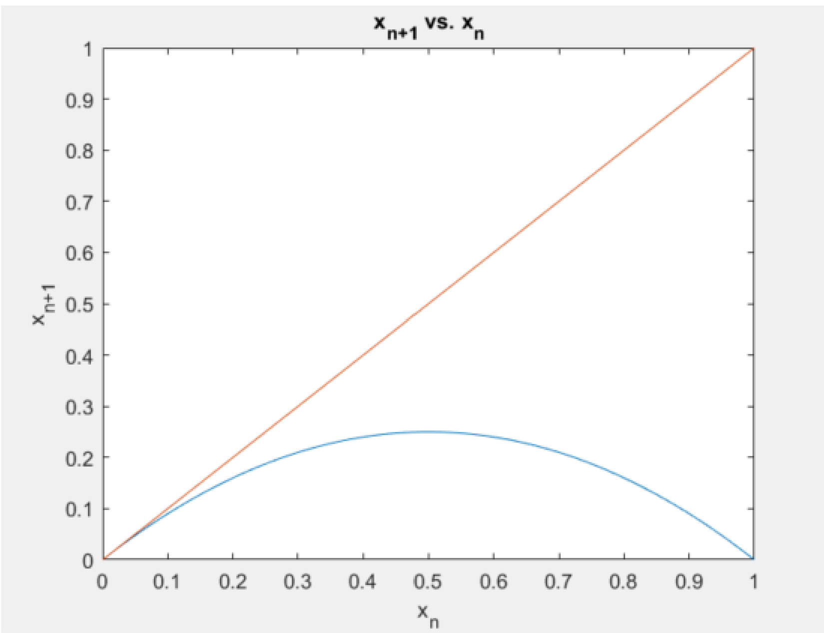

Figure 1: The intersections of this plot give the fixed points for $\lambda = 1$

We can see that the two curves do not intersect. This implies that there are no fixed points of our map. What happens if we increase λ? The case of $\lambda = 4$ is plotted in Figure 2.

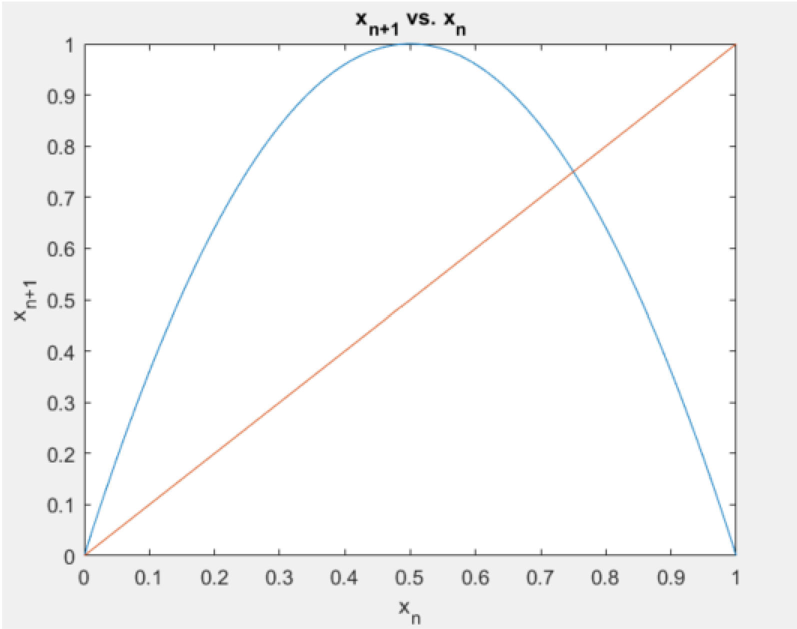

Figure 2: The intersections of this plot give the fixed points for $\lambda = 4$

Here we can see, that around $(x_{n+1}, x_n) = (0.75, 0.75)$, the curves intersect. Since the slope of the blue curve is negative at the intersection, we can conclude that this fixed point is stable. An interesting feature to point out on Figure 2 is that $max(x_{n+1}) = 1$. This divides our domain into two subsections, $x_n < 0.5$, and $x_n > 0.5$. In fact, this case is analogous to a bit shift map.

Now we want to examine the behavior after applying 2 map iterations: $f(f(x_n))$ vs. x_n. Let's look at $\lambda = 3.5$. The plot is shown in Figure 4.

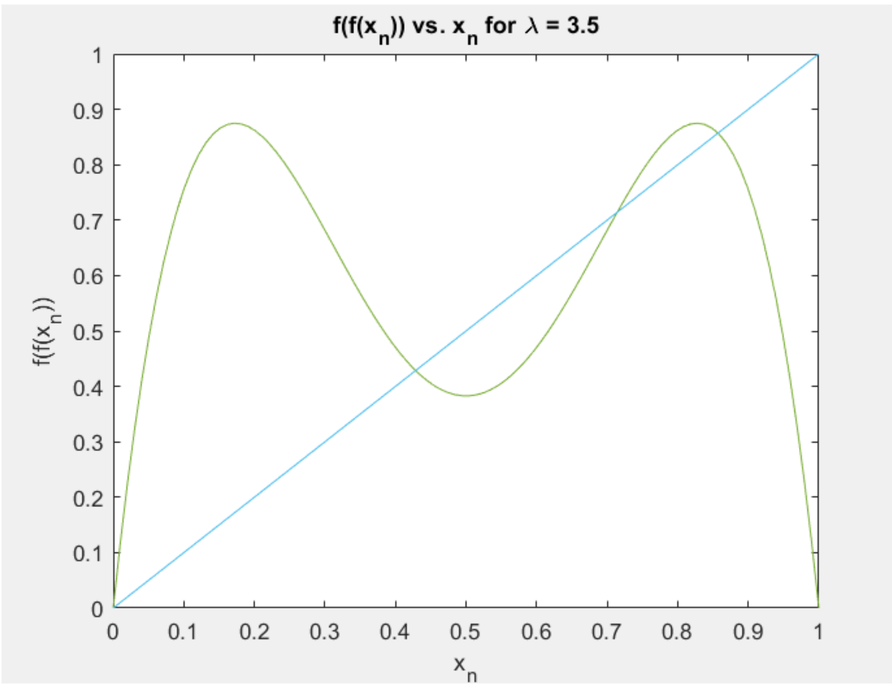

Figure 3: $f(f(x_n))$ vs. x_n, $\lambda = 3.5$, we find 3 fixed points.

The intersections of the two curves on Figure 4 give the fixed points of $f(f(x_n))$. The middle fixed point is also a fixed point of $f(x_n)$. We see that this fixed point has become an unstable fixed point of $f(f(x_n))$, as the slope is greater than zero when crossing the line $f(x_n) = x_n$. The other two fixed points correspond to a period-2 cycle of $f(x_n)$. For high enough λ, we can also find period 3 orbits, or fixed points of the map $f(f(f(x_n)))$. The plot is shown below for $\lambda = 3.9$.

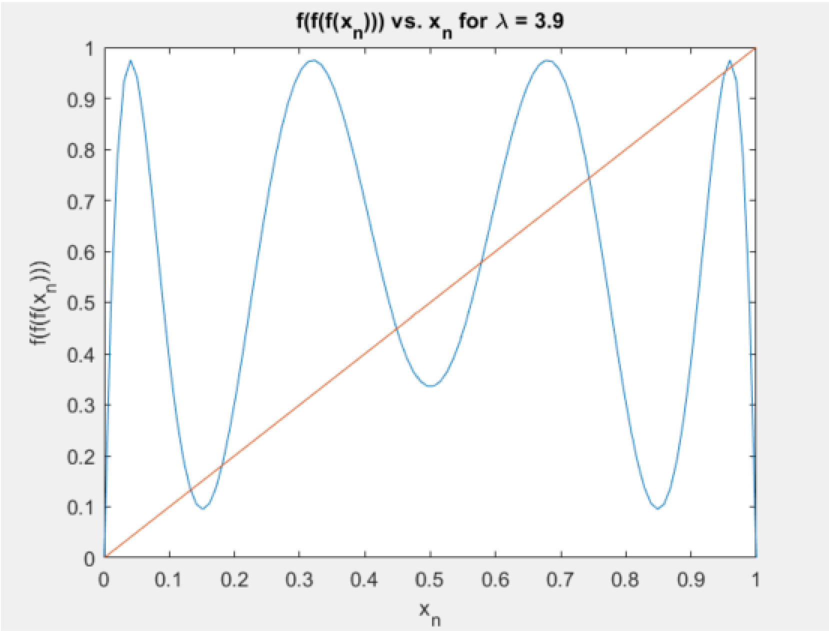

Figure 4: $f(f(f(x_n)))$ vs. x_n, $\lambda = 3.9$, we find 7 fixed points.

Now the fixed points of this map, $f(f(f(x_n)))$ correspond to period-3 orbits. The existence of period-3 orbits imply chaos, so we know the logistic map is indeed chaotic for sufficiently large λ.
Now We want to examine the overall behavior of the map in the range of the λ values we have studied thus far. We can do this by iteratively applying our map to an initial state. When writing a script to plot the behavior after successive iterations, I removed the first half of the iterations to neglect the initial transience of the map. The interesting behavior comes about around $\lambda = 3$, and continues up until $\lambda = 4$, for higher values of which our map blows up, due to the existence of a point such that $max(f(x_n)) > 1$.
The bifurcation diagram in 5 was generated in MATLAB using an initial condition of $x_0 = 0.1$. The map was applied for 200 iterations at every 0.001 increment of λ for the domain $2.000 < \lambda < 3.999$.

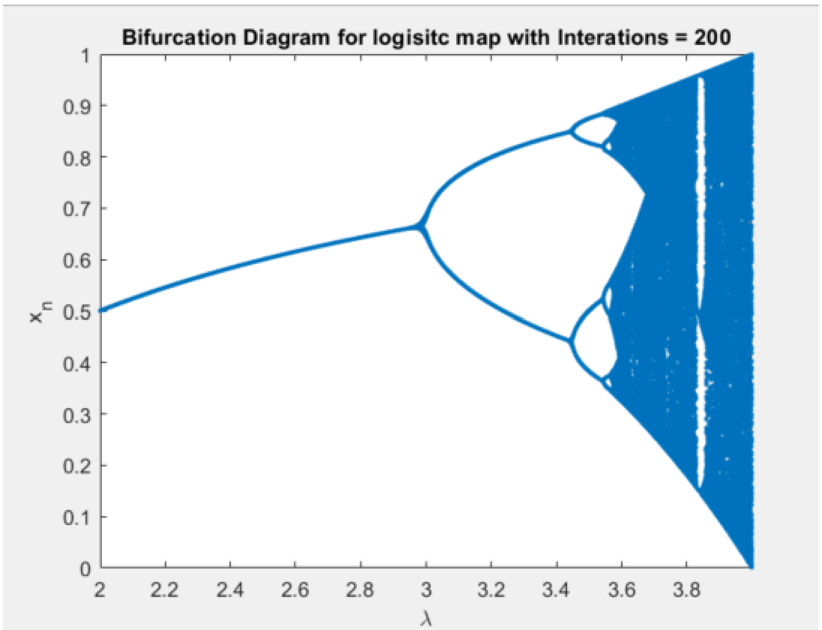

Figure 5: Bifurcation plot of the logistic map, using 200 iterations, with the first 100 removed.

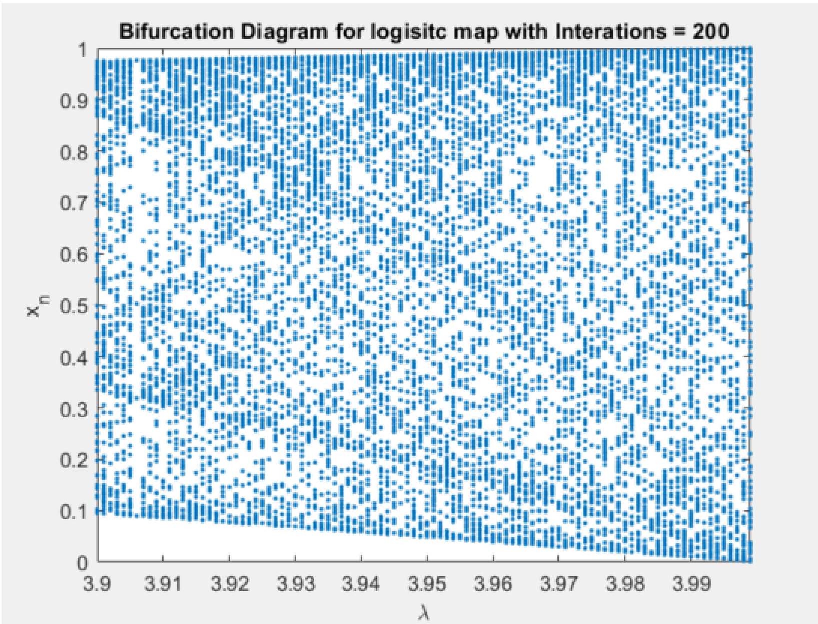

Figure 6: Bifurcation plot of the logistic map, using 200 iterations, zoomed in on the range $3.9 < \lambda < 3.999$.

This classic bifurcation plot of the logistic map in Figure 5 shows period doubling in the range $3 < \lambda < 3.5$. For λ values sufficiently high, the behavior becomes chaotic. This behavior is zoomed in on in Figure 6. Remember, this behavior is all for the same initial condition. In the range $3.9 < \lambda < 3.999$ we can see no overall structures governing the bifurcations, and the behavior looks chaotic. We want to examine this chaotic behavior with an analytical tool called the Lyapunov exponent. Finding a positive Lyapunov coefficient would imply that our system has indeed become chaotic.

3 Lyapunov Exponent

Following the derivation done in [4], define two trajectories infinitesimally close to one another in phase space, $Z(t)$ and $Z_0(t)$. Write

$$\delta Z(t) = Z(t) - Z_0(t) \tag{2}$$

$$\delta Z_0 = Z(0) - Z_0(0) \tag{3}$$

If we can write

$$|\delta Z(t)| \approx e^{\lambda_L t}|\delta Z_0| \tag{4}$$

then we may treat λ_L as the Lyapunov exponent. This simple relation is quite powerful. If $\lambda_L < 0$, the distance between our infinitesimally close orbits in phase space will decay exponentially. If $\lambda_L > 0$, then the distance between the infinitesimally close orbits will grow exponentially, and we will find SDIC; two trajectories that are initially infinitesimally close will have the distance between one another grow exponentially. With a little massaging, we can transform the definition of the Lyapunov exponent into a tool we can use to numerically approximate the Lyapunov exponent.

Assume we are applying this concept to a 1D map:

$$x_{n+1} = f(x_n) \tag{5}$$

Then, take the trajectory starting from x_0, and the trajectory originating from an infinitesimally close initial condition $x_0 + \epsilon$. In terms of equations 2 and 3, we can write

$$|\delta Z(n)| = |f^n(x_0 + \epsilon) - f^n(x_0)| \tag{6}$$

$$|\delta Z(0)| = |f^0(x_0 + \epsilon) - f^0(x_0)| \tag{7}$$

Here we have taken the time coordinate from equations 2-4 as our iteration count, n. The notation of f^n corresponds to our map f applied n times. $|\delta Z(0)|$ is just the separation at our initial time, and is constant. We can assume that it is small and order ϵ.

Our transformed equation becomes

$$|f^n(x_0 + \epsilon) - f^n(x_0)| \approx \epsilon e^{\lambda_L t} \tag{8}$$

Rearrange,

$$e^{\lambda_L t} \approx \frac{|f^n(x_0 + \epsilon) - f^n(x_0)|}{\epsilon} \tag{9}$$

Taking the natural logarithm of both sides

$$\lambda_L t \approx ln \frac{|f^n(x_0 + \epsilon) - f^n(x_0)|}{\epsilon}$$

$$\lambda_L \approx \frac{1}{t} ln \frac{|f^n(x_0 + \epsilon) - f^n(x_0)|}{\epsilon} \tag{10}$$

We arrive at an isolated equation for our Lyapunov exponent. With respect to Lyapunov exponents, we are concerned with the behavior as $n \to \infty$. We are also stipulating that the initial conditions are infinitesimally close, so we want to take $\epsilon \to 0$. Adding these stipulations,

$$\lambda_L \approx \lim_{n\to\infty \epsilon\to 0} \frac{1}{t} ln \frac{|f^n(x_0+\epsilon) - f^n(x_0)|}{\epsilon} \tag{11}$$

The ϵ limit is the definition of the derivative of f, so we find

$$\lambda_L \approx \lim_{n\to\infty} \frac{1}{t} ln |\frac{df^n}{dx}|_{x=x_0}| \tag{12}$$

Apply the chain rule

$$\lambda_L \approx \lim_{n\to\infty} \frac{1}{t} ln |\frac{df}{dx}|_{x=x_{t-1}} \cdot \frac{df}{dx}|_{x=x_{t-2}} \cdots \frac{df}{dx}|_{x=x_0}| \tag{13}$$

Using logarithmic identities, we finally we obtain the relation

$$\lambda_L \approx \lim_{n\to\infty} \frac{1}{t} \sum_{i=0}^{n-1} ln |\frac{df}{dx}|_{x=x_i}| \tag{14}$$

This relation is a tool we can use to compute the Lyapunov exponent numerically. This quite simple relation can give us major insight into the stretching or compressing of the distance between trajectories propagated forward in n. The result is simply a time average of $ln(\frac{df}{dx}|_{x=x_i})$. If the derivative of our system $\frac{df}{dx}$ is available analytically, this is quite simple to implement, and will be done so here. It is important to note, that for arbitrary dimensional systems, a simple relation for the Lyapunov exponent is not possible. For arbitrary systems of n dimensions, a more complex method would have to be used to determine the Lyapunov exponent. One such method is Wolf's algorithm, for determining the first Lyapunov exponent [2].

4 Numerical Simulation

The code to calculate the Lyapunov exponent for the logistic map was implemented in MATLAB. In order to reduce the effects of transience, the first 20 iterations at each λ value were dropped, following the work done in [3]. The Lyapunov exponent calculated at various λ_L values is shown in Figure 7.

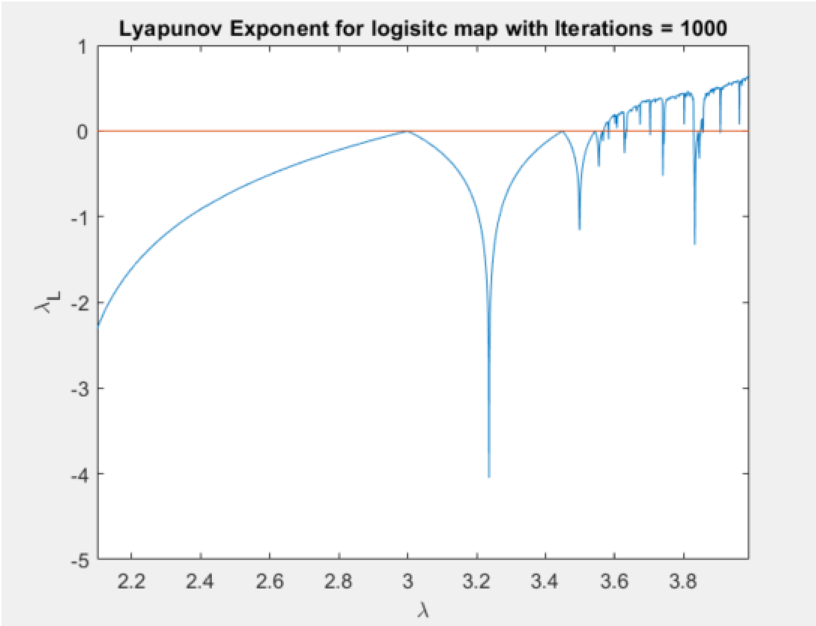

Figure 7: Lyapunov exponent vs. λ for the logistic map.

The plot clearly shows that as λ increases, we find a positive Lyapunov exponent. This means that our system becomes chaotic for sufficiently large λ, which is consistent with our findings from earlier. One benefit of using the Lyapunov exponent to detect chaos is that we can pinpoint the location that the system becomes chaotic. In our system, we see that the Lyapunov exponent approaches 0 once near $\lambda = 3$ and once near $\lambda = 3.4$, but does not cross into the upper-half plane, indicating chaos. According to the algorithm, which calculated Lyapunov exponents with a λ refinement of 0.001, the system becomes chaotic when λ reaches 3.570. Once we pass this value of λ, we find that the Lyapunov exponent does fall back into the lower-half plane for certain values of λ. This is consistent with the regions on the bifurcation plots presented previously that appear to regain non-chaotic behavior. Now that the algorithm has been created in MATLAB, applying this method to other 1D maps should be simple, and will be presented in the following section.

5 Simulating Other 1D Maps

cos(λ x) map

The next map that we will study is given by

$$x_{n+1} = cos^2(\lambda x_{n-1}) \tag{15}$$

The bifurcation diagram for this map is plotted in Figure 8

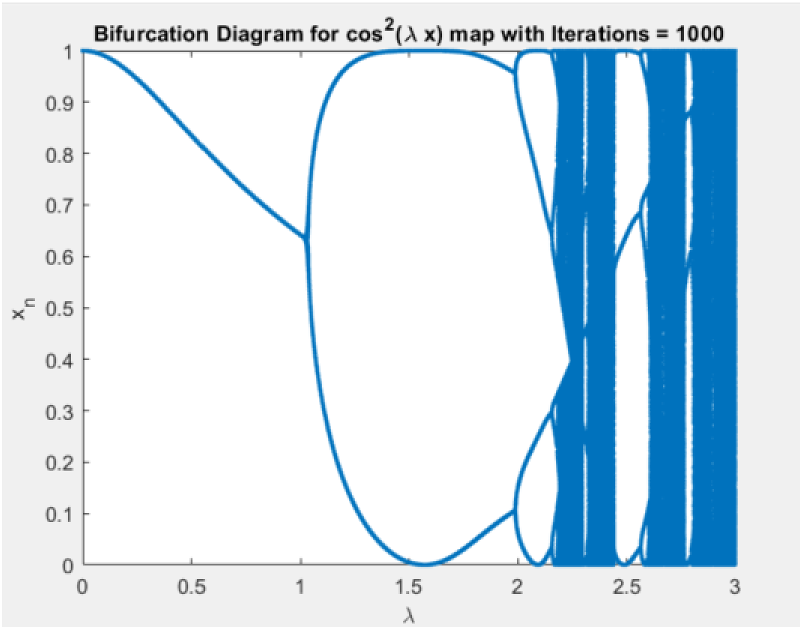

Figure 8: Bifurcation diagram for the map under study.

The first 250 iterations were not plotted to discard initial transience. Like before, we see some transition to chaos in the region $2 < \lambda < 2.5$. Using the Lyapunov exponent, we can find the point that the system becomes chaotic. The Lyapunov exponent is plotted below, in Figure 9.

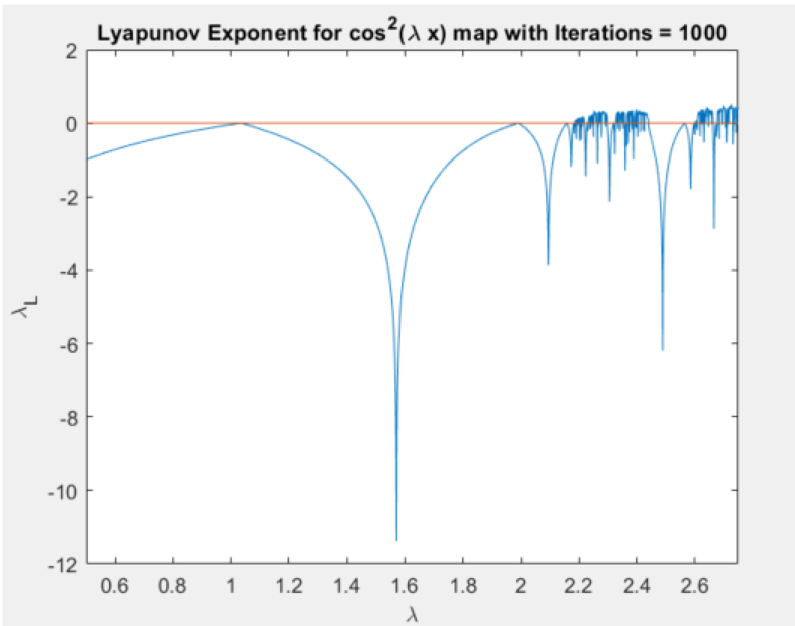

Figure 9: Lyapunov exponent vs. λ for the map under study.

As expected, we see that the Lyapunov exponent becomes positive in the region we identified earlier. The numerical solver finds that at $\lambda = 2.187$, the Lyapunov exponent crosses into the upper-half plane. Similar to the logistic map, we see that after the Lyapunov exponent crosses into the upper-half plane, it sometimes crosses back into the lower-half plane as λ increases. This creates regions that are not chaotic, which we can directly observe from the bifurcation plot in Figure 8.

Gauss iterated map

The Gauss iterated map is given by

$$x_{n+1} = \exp(-\alpha x_n^2) + \beta \tag{16}$$

We look to plot the bifurcation diagrams for the Gauss iterated map for slices of constant α.

First, trying $\alpha = 5$, we find the bifurcation diagram in Figure 12

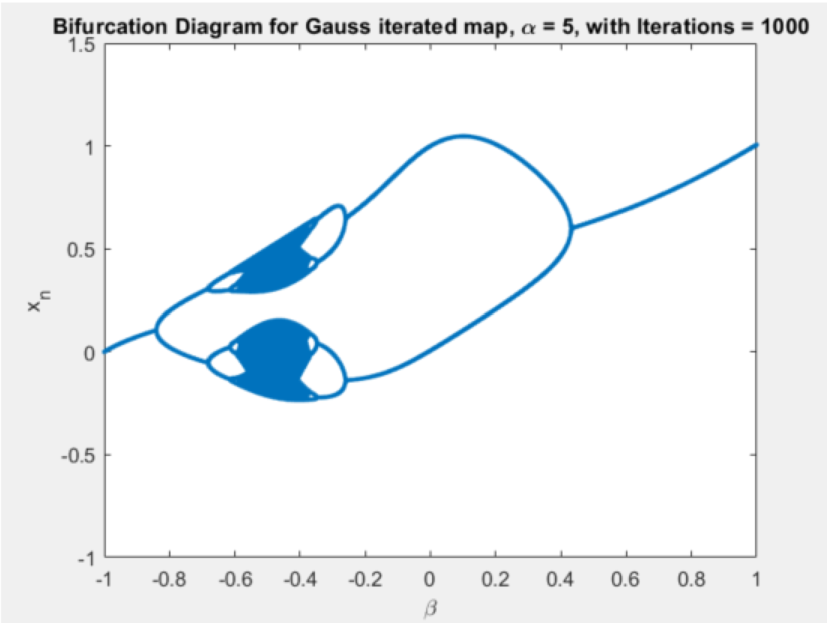

Figure 10: Bifurcation diagram for Gauss iterated map, $\alpha = 5$.

As β increases from -1, we guess that the system becomes chaotic in roughly the region $-0.65 < \beta < -0.35$, and then is not chaotic for increasing β. As before, we will use the Lyapunov exponent to pinpoint where the system becomes chaotic. The calculated values are plotted below.

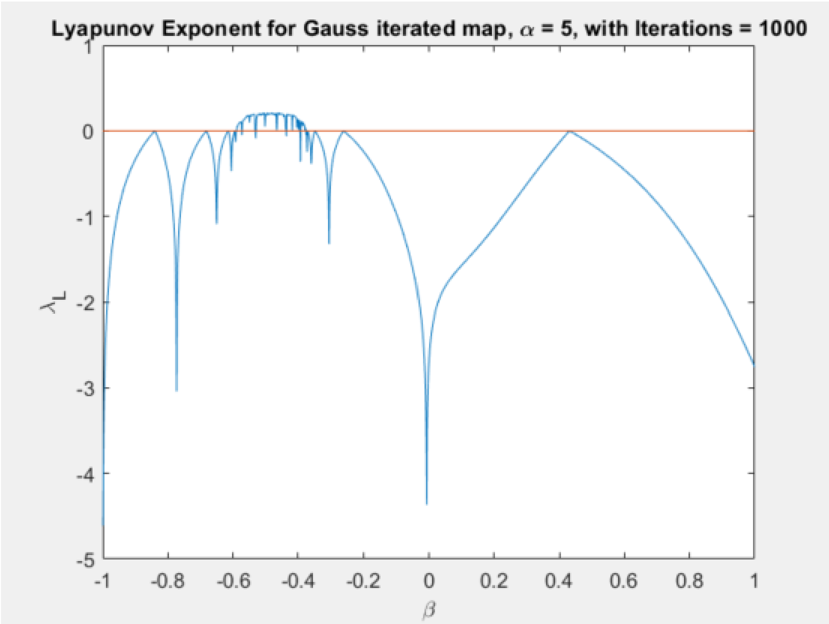

Figure 11: Lyapunov exponent vs. λ for the Gauss iterated map, $\alpha = 5$.

According to the solver, as β increases, the map becomes chaotic at $\beta = -0.589$, and finally crosses into the lower-half plane for the last time at $\beta = -0.377$.

As a final simulation, we will apply the method to the Gauss iterated map for $\alpha = 7$. The bifurcation diagram is plotted below.

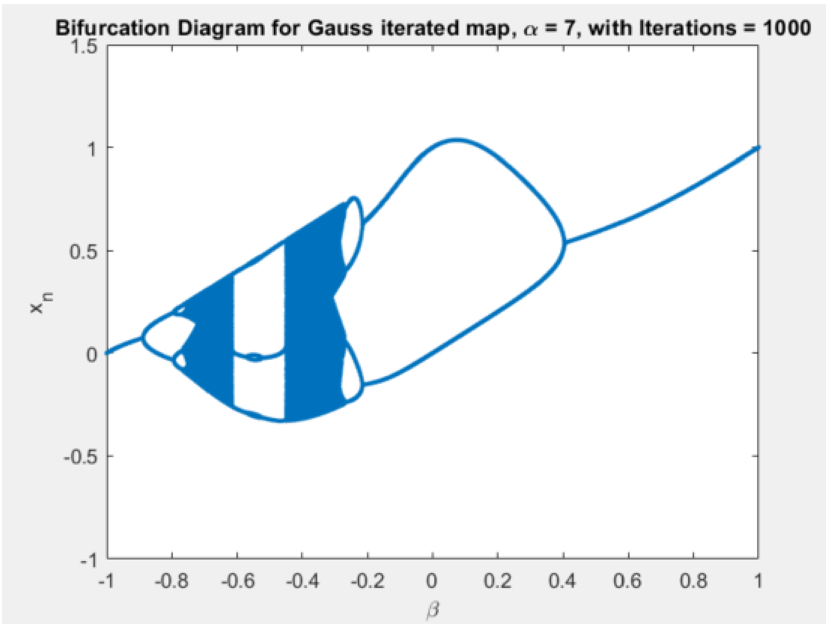

Figure 12: Bifurcation diagram for Gauss iterated map, $\alpha = 7$.

For this value of α, we seem to have two separate regions of chaos, which we will now pinpoint using the Lyapunov exponent.

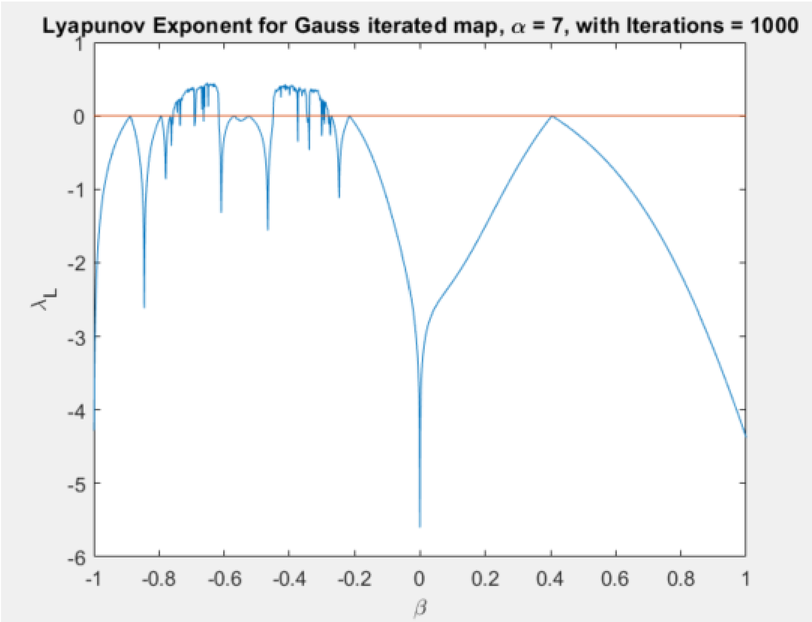

Figure 13: Lyapunov exponent vs. λ for the Gauss iterated map, $\alpha = 7$.

In Figure 13, we can see that the Lyapunov exponent reaches the upper-half plane in the two regions where we predicted chaos. The solver finds that the β values where chaos is located are: $-0.756 \leq \beta \leq -0.617$ and $-0.449 \leq \beta \leq -0.282$. This is not to say that the regions remain chaotic, as the Lyapunov exponent falls into the lower-half plane within both regions.

6 Conclusion

Through this project the logistic map as well as other 1D maps were studied. To start the logisitic map was studied; its fixed points and periodic orbits were found. A bifurcation diagram was plotted, and chaos seemed to occur for sufficiently large λ. The Lyapunov exponent was defined, and an approximation for a 1D map was derived. A solver was then created to approximate the Lyapunov exponent for varying parameters. It was shown that all of the maps studied had chaotic regions using the Lyapunov exponent due to the fact that it had values in the upper-half plane. This method proves useful to determining where chaos begins and ends with respect to varying a parameter. This method is not as simple in higher dimensions and arbitrary dynamics, though, and more sophisticated algorithms must be used to compute the Lyapunov exponent in these cases.

References

[1] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, "On Devaney's definition of chaos," The American Mathematical Monthly, vol. 99, no. 4, pp. 332–334, 1992.

[2] Alan Wolf, Jack B. Swift, Harry L. Swinney, John A. Vastano, "Determining Lyapunov exponents from a time series," Physica D: Nonlinear Phenomena, Volume 16, Issue 3, 1985, Pages 285-317.

[3] Peter Young. The Logistic Map. Personal Collection of Peter Young. http://physics.ucsc.edu/ peter/242/logistic.pdf

[4] Sayama, H., & Open SUNY Textbooks,. (2015). Introduction to the modeling and analysis of complex systems