3️⃣

Assignment 3

Question 1:

Consider the boundary value problem:

$$(a(x)y')' = f(x); 0 \leq x \leq 1$$
$$y(0) = 1, y(1) = 2$$

Where $a(x) = 1 + x^2$, $f(x) = 2 + 6x^2$.

Part A)

Use the energy method to show that the BVP has a unique solution, with that solution being $y_e(x) = 1 + x^2$.

Uniqueness:

1. Let $y_1(x), y_2(x)$ be two solutions. Let $u = y_1 - y_2$.
   a. Then we have $(a(x)y_1')' - (a(x)y_2')' = (a(x)u')' = 0$ and $u(0) = u(1) = 0$.
2. We can multiply by u and integrate: $\int_0^1 u \cdot [a(x)u']' dx$
   a. Written fully: $\int_0^1 u \cdot [(1+x^2)u']' dx$
   b. $\int_0^1 u \cdot [(1+x^2)u']' dx = \int_0^1 u \cdot [u' + x^2u']' dx = \int_0^1 u \cdot [u'' + 2xu' + x^2u''] dx$
3. We can solve this using integration by parts:
   a. $\int_0^1 u \cdot [(1+x^2)u']' dx = u \cdot (1+x^2)u'|_0^1 - \int_0^1 (u')^2(1+x^2) dx$
   b. $= u \cdot (1+x^2)u'|_0^1 - (\int_0^1 (u')^2 dx + \int_0^1 (u'x)^2 dx)$
   c. Because $u(0) = u(1) = 0$ we can reduce this to: $-(\int_0^1 (u')^2 dx + \int_0^1 (u'x)^2 dx)$
   d. $= -(uu'|_0^1 + \int_0^1 (u')^2 dx + \int_0^1 (u'x)^2 dx)$ (via integration by parts on $\int_0^1 (u')^2 dx$)
      i. We've shown that $\int_0^1 (u')^2 dx = 0$
      ii. Since $(u')^2$ is always positive, its integral being 0 on an interval implies it is the 0 function on that interval
   e. So we can again reduce this to $-\int_0^1 (u'x)^2 dx = 0$
4. So we have that $u = 0 \ \forall x \in [0,1]$, and therefore $y_1 = y_2$ and our BVP has a unique solution.

Finding the solution:

1. $[(y'(1+x^2)]' = 2 + 6x^2$
2. $y'(1+x^2) = \int(2+6x^2)dx = 2x + 2x^3 + C = 2x(1+x^2) + C$
3. $y' = 2x + \frac{C}{1+x^2}$
4. $y = \int 2x + \frac{Cx}{1+x^2} dx$
5. $y = x^2 + Ctan^{-1}(x) + D$

6. $y(0)=1, y(1)=2; \therefore C=0, D=1$
7. $y=x^2+1$

Part B)

Let $x_i = ih, i=1,2,...,n$, with $h=\frac{1}{n+1}$ be a discretization of $(0,1)$. Set $x_0=0, x_{n+1}=1$. Use the symmetric differencing formula:

$$\frac{d}{dx}((a(x))\frac{dy}{dx}) \approx \frac{1}{h^2}[a_{i-1/2}y_{i-1} - (a_{i-1/2}+a_{i+1/2})y_i + a_{i+1/2}y_{i+1}]$$

Where $a_{i-1/2} = a(x_i - h/2),\ i=1,2,\ldots,n+1$ To discretize the problem in part A. Obtain the linear system and associated matrix A.

1. $((a(x))y')' \approx \frac{1}{h^2}[a_{i-1/2}y_{i-1} - (a_{i-1/2}+a_{i+1/2})y_i + a_{i+1/2}y_{i+1}]$
2. This gives us a tridiagonal matrix system in $\mathbb{R}^n$, $AY=F$

$$A = \begin{pmatrix} -(a_{1/2}+a_{3/2}) & a_{3/2} & & & \\ a_{3/2} & -(a_{3/2}+a_{5/2}) & a_{5/2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1/2} \\ & & & a_{n-1/2} & -(a_{n-1/2}+a_{n+1/2}) \end{pmatrix}$$

$$Y = \begin{pmatrix} y_1 & y_2 & \ldots & y_{n-1} & y_n \end{pmatrix}^T$$

$$F = \begin{pmatrix} f_1h^2 - a_{1/2} & h^2f_2 & \ldots & h^2f_{n-1} & h^2f_n - 2a_{n+1/2} \end{pmatrix}^T$$

Part C)

Show that the numerical scheme above is second order accurate; i.e the truncation error satisfies $\tau_h = O(h^2)$.

1. Our truncation error $\tau_h^i = (a(x_i)y(x_i)')' - \frac{1}{h^2}[a_{i-1/2}y_{i-1} - (a_{i-1/2}+a_{i+1/2})y_i + a_{i+1/2}y_{i+1}]$. We can rewrite the RHS as:
2. $a'(x_i)y'(x_i) + a(x_i)y''(x_i) - \frac{1}{h^2}[a_{i-1/2}y_{i-1} - (a_{i-1/2}+a_{i+1/2})y_i + a_{i+1/2}y_{i+1}]$
3. $= a'(x_i)y'(x_i) + a(x_i)y''(x_i) - \frac{1}{h^2}[a_{i-1/2}(y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta)) - (a_{i-1/2} + a_{i+1/2})y_i + a_{i+1/2}(y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta))]$
   a. Taylor expansion of y_{i-1}, y_{i+1}
4. $= a'(x_i)y'(x_i) + a(x_i)y''(x_i) - \frac{1}{h^2}[a_{i-1/2}(-hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta)) + a_{i+1/2}(hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta))]$
   a. cancellation of like terms
5. $= a'(x_i)y'(x_i) + a(x_i)y''(x_i) - \frac{1}{h^2}[(a(x_i) - ha'(x_i) + \frac{h^2}{2}a''(x_i) - \frac{h^3}{6}a'''(x_i) + \frac{h^4}{24}a^{(4)}(\zeta))(-hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta)) + (a(x_i) + ha'(x_i) + \frac{h^2}{2}a''(x_i) + \frac{h^3}{6}a'''(x_i) + \frac{h^4}{24}a^{(4)}(\zeta))(hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta))]$
   a. Taylor expansion of $a_{i-1/2}, a_{i+1/2}$
6. $= a'(x_i)y'(x_i) + a(x_i)y''(x_i) - \frac{1}{h^2}[2(a(x_i) + \frac{h^2}{2}a''(x_i) + \frac{h^4}{24}a^{(4)}(x_i))(\frac{h^2}{2}y''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta)) + (ha'(x_i) + \frac{h^3}{6}a^{'''}(\zeta))(hy'(x_i) + \frac{h^3}{6}y'''(x_i))]$
   a. Grouping together like terms, cancelling negated terms from $a_{i-1/2}$ Taylor expansion
   b. Notice that we have one copy of $a'(x_i)y'(x_i) - a(x_i)y''(x_i)$ inside the negative bracketed expression, can use that to cancel with the positive front terms.

7. $= a'(x_i)y'(x_i) + a(x_i)y''(x_i) - a'(x_i)y'(x_i) - a(x_i)y''(x_i) + [(a''(x_i) + \frac{h^2}{12}a^{(4)}(x_i))(\frac{h^2}{2}y''(x_i) + \frac{h^4}{24}y^{(4)}(\zeta)) + 2a(x_i)\frac{h^2}{12}y^{(4)}(\zeta) + (\frac{h}{6}a'''(\zeta))(\frac{h^3}{6}y'''(x_i)) + \frac{h^2}{6}a'(x_i)y'''(x_i)] = O(h^2)$
    a. Distributing the $\frac{1}{h^2}$ in front of the brackets, bringing out the terms to cancel the front terms, and the rest is all proportionate to $O(h^2)$

Part D)

Show that the associated linear system has a unique solution, i.e that A_h is non singular.

1. We have that $a_{ij} = 0$ when $|i - j| \geq 2$
2. We have that $a_{ii} = -(a_{i-1/2} + a_{i+1/2})$
3. We have that $a_{i,i-1}, a_{i,i+1} = a_{i-1/2}, a_{i+1/2}$
4. Consider $X^T AX = \Sigma_{i,j=1}^n a_{ij}x_ix_j$
5. $= x_1(a_{3/2}x_2 - (a_{1/2} + a_{3/2})x_1) + \Sigma_{i=2}^{n-1} x_i[a_{i-1/2}x_{i-1} - (a_{i-1/2} + a_{i+1/2})x_i + a_{i+1/2}x_{i+1}] + x_n(x_{n-1}a_{n-1/2} - (a_{n-1/2} + a_{n+1/2})x_n)$
6. $= -a_{1/2}x_1^2 + a_{3/2}(x_2x_1 - x_1^2) + a_{3/2}(x_2x_1 - x_2^2) + a_{5/2}(x_3x_2 - x_2^2) + \cdots + a_{n-3/2}(x_{n-2}x_{n-1} - x_{n-2}^2) + a_{n-1/2}(x_nx_{n-1} - x_n^2) - a_{n+1/2}x_n^2$
7. $= -a_{1/2}x_1^2 - a_{3/2}(-x_2^2 - 2x_1x_2 + x_1^2) - \Sigma_{i=2}^{n-1} a_{i+1/2}(x_i^2 - 2x_ix_{i+1} + x_{i+1}^2) + a_{n-1/2}(2x_{n-1}x_n - x_n^2) - a_{n+1/2}x_n^2$
8. $= -a_{1/2}x_1^2 - \Sigma_{i=2}^{n-1}(a_{i+1/2}(x_i - x_{i+1})^2) - a_{n+1/2}x_n^2 < 0$
9. Therefore, we have that $X^T AX < 0$, $\forall X \in \mathbb{R}^n$
10. Therefore $-A$ is positive definite, so $-A$ is non singular, therefore A must also be non singular.

Part E)

Code:

```
%each in their own file
function ax = a(x)
    ax = 1 + x^2;
end
function fx = f(x)
    fx = 2 + 6*x^2;
end
function yx = y(x)
    yx = 1 + x^2;
end
```

```
function max_err = problem(n)
%n =64;
    h = 1/(n+1);
    A = zeros(n,n);
    x = zeros(n,1);
    F = zeros(n,1);
    for i = 1:n
        x(i) = i*h;
    end
    F(1) = h^2 * f(x(1)) - a(x(1) - h/2);
    F(n) = h^2 * f(x(n)) - 2 * a(x(n) + h/2);
    for i = 2:n-1
        F(i) = h^2 * f(x(i));
    end
```

```
    for i = 1:n
        for j = 1:n
                if (j == i)
                    A(i,j) = -(a(x(i) - h/2) + a(x(i) + h/2));
                end
                if (j == (i - 1))
                    A(i,j) = a(x(i) - h/2);
                end
                if (j == (i + 1))
                    A(i,j) = a(x(i) + h/2);
                end
        end
    end

    Y_h = A\F;
    Y_e = zeros(n,1);
    for i = 1:n
        Y_e(i) = y(x(i));
    end

    err = abs(Y_e - Y_h);
    max_err = max(err);
end
```

With output:

```
>> problem(32)

ans =

   4.3324e-05

>> problem(64)

ans =

   1.1174e-05

>> problem(128)

ans =

   2.8371e-06
```

And effective rates:

```
>> log(problem(64)/problem(128))/log(2)

ans =

    1.9776

>> log(problem(32)/problem(64))/log(2)

ans =

    1.9551
```

```
>> log(problem(128)/problem(256))/log(2)

ans =

    1.9888
```

So we can conclude that the method indeed converges $O(h^2)$

Question 2

Part A)

```
function solveODEWithBeta(beta)
    z1p = [];
    s0 = -1;
    eps1 = 1e-12;
    error = [];
    S = s0;
    for I = 1:100
        [X,Y] = ode23s(@ydotp, [0, 1], [0; s0; 0; 1]);
        y1 = Y(end, 1);
        y1p = Y(end, 3);
        z1p = [z1p; y1p];
        s1 = s0 - (y1 - beta) / y1p;
        error = [error; abs(s0 - s1)];
        if(abs(s0 - s1) < eps1), break, end
        s0 = s1;
        S = [S; s1];
    end


    [X,Y] = ode23s(@ydot, [0,1], [0, s0]);
    figure(1), clf
    plot(X, Y(:,1), 'b', 'linewidth', 2), hold on
    title('Solution of the ODE')
end
```

I used this code and a binary search over the values of β, so tested with $\beta = 3.5, 3.75, 3.875, 3.934, 3.969, 3.992, 3.993, \ldots 3.999$. The first value for β which diverged was 3.999, the solution does not diverge at 3.998, therefore the critical point is 3.999.

Part B)

```
function solveODEWithBetaBisection(beta)
    function y1_minus_beta = shootingFunction(s)
        [X,Y] = ode23s(@ydotp, [0, 1], [0; s; 0; 1]);
        y1 = Y(end, 1);
        y1_minus_beta = y1 - beta;
    end

    s0_initial_guess = [-10, 10];
```

```
    s0 = fzero(@shootingFunction, s0_initial_guess);

    [X,Y] = ode23s(@ydot, [0, 1], [0, s0]);
    figure(1), clf
    plot(X, Y(:,1), 'r', 'linewidth', 2), hold on
    title('Solution of the ODE')
end
```

This code uses fzero to find a solution for a given β, and it does work on 3.999, unlike Newton's method.

Solving with ode45 like so:

```
function solveODEWithSolver(beta)
    [X,Y] = ode45(@ydotp, [0,0.999], [0; 1.236; 0; 1]);
    plot(X, Y(:, 1), 'r')
end
```

Shows that result is very sensitive to our initial guess $v'(0)$, where even slightly varying it can significantly change the result. This suggests that the problem is ill conditioned. This would adversely effect Newton's method, as it relies on the partial derivative to iterate and make its next guess. So if the estimates of the derivatives are misleading or vary greatly, it may struggle to converge. Bisection on the other hand is guaranteed to converge as long as the interval has a root.

Question 3

Bonus Question:
Consider tridiagonal matrix, in $\mathbb{R}^{n\times n}$,

$$A = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & c & \ddots & \ddots & \\ & & \ddots & a & b \\ & & & c & a \end{bmatrix}.$$

with a, b, c are fixed constants, $b, c > 0$. Find the eigenvalues and eigenvectors of A. For which values of a this matrix is non singular.

Eigenpairs:

1. Let $Av = \lambda v,\ v \in \mathbb{R}^n$.
2. From the structure of A: $(Av)_i = \lambda v_i = cv_{i-1} + av_i + bv_{i+1},\ v_0 = 0, v_{n+1} = 0$
3. $det(A - \lambda I) = 0 = p(\lambda)$ (characteristic polynomial)
4. Eigenvalues of A_n ($n \times n$ version of A) are the roots of p_n (char. poly. of A_n).
5. We can see trivially that $p_1 = (a - \lambda), p_2 = (a - \lambda)^2 - bc$
6. If expand the first row of A_n we can see the recurrence: $p_n = (a - \lambda)p_{n-1} - bcp_{n-2}$
7. It is also easy to see that we can let $p_0(\lambda) = 1$ while maintaining this recurrence relation.
8. In order to think about this in terms of a difference equation, let's sub in $x^i = p_i(\lambda)$

9. $x^n - (a-\lambda)x^{n-1} - bcx^{n-2} = 0$
10. → $x^{n-2}[x^2 - (a-\lambda)x - bc] = 0$
11. → $x_\pm = \frac{(a-\lambda)\pm\sqrt{(a-\lambda)^2-4bc}}{2}$
12. The roots can be equal ($x_+ = x_-$) or distinct.
13. Case 1: Repeated Roots
    a. $x_+ = x_- = \frac{a-\lambda}{2}$
    b. → $p_n(\lambda) = c_1x_+^n + nc_2x_-^n$
        i. $p_0(\lambda) = 1 = c_1$
        ii. $p_1(\lambda) = (a-\lambda) = (c_1+c_2)x_+^1$ → $c_1 + c_2 = 2$
    c. → $c_1 = c_2 = 1$
    d. → $p_n(\lambda) = (n+1)x_+^n = (\frac{a-\lambda}{2})^n(n+1)$
    e. Therefore all our eigenvalues are a
    f. → $b = c = 0$, but this contradicts the problem statement $b, c > 0$, so this case is not possible, and we have that our roots are distinct.
14. Case 2: Distinct roots
    a. $x_+ \neq x_-$
    b. $p_n(\lambda) = c_1x_+^n + c_2x_-^n$
        i. $c_1 + c_2 = 1$
        ii. $c_1x_+ + c_2x_- = a - \lambda$
        iii. → $c_2 = 1 - c_1$
        iv. → $c_1x_+ + (1-c_1)x_- = a - \lambda$
        v. → $c_1 = \frac{(a-\lambda)-x_-}{x_+-x_-}$, $c_2 = 1 - \frac{(a-\lambda)-x_-}{x_+-x_-}$
        vi. This is fine since in the case of distinct roots, $x_+ - x_- \neq 0$
    c. → $p_n(\lambda) = x_+^n\frac{(a-\lambda)-x_-}{x_+-x_-} + x_-^n\frac{x_+-(a-\lambda)}{x_+-x_-}$
    d. Since we've defined $x_\pm$ as roots on line 11, we can rewrite this as:
    e. $p_n(\lambda) = x_+^n\frac{x_++x_--x_-}{x_+-x_-} + x_-^n\frac{-x_-+x_+-x_+}{x_+-x_-} = \frac{x_+^{n+1}-x_-^{n+1}}{x_+-x_-}$
    f. So we have that the eigenvalues λ satisfy $x_+^{n+1} = x_-^{n+1}$
    g. We can see that $x_+x_- = \frac{1}{4}[([a-\lambda] + \sqrt{[a-\lambda]^2-4bc})([a-\lambda] - \sqrt{[a-\lambda]^2-4bc})] = bc$ (using the definition from line 11)
    h. → $\frac{x_+}{x_-} = \frac{x_+^2}{x_+x_-} = \frac{x_+^2}{bc} = (\frac{x_+}{\sqrt{bc}})^2$
    i. Then using the property from 14f, we get that $(\frac{x_+}{\sqrt{bc}})^{2n+2} = 1$, so we have $2n+2$ roots going around the unit circle
    j. $\frac{x_{+k}}{\sqrt{bc}} = e^{\frac{k\pi i}{2n+2}}$, $k = 0, 1, \ldots, 2n-1$
    k. → $x_{+k} = \sqrt{bc}e^{\frac{k\pi i}{2n+2}}$, $x_{-k} = \frac{bc}{x_{+k}} = \sqrt{bc}e^{-\frac{k\pi i}{2n+2}}$
    l. Using the identity $e^{i\theta} + e^{-i\theta} = 2cos\theta$:
    m. $x_{+k} + x_{-k} = 2\sqrt{bc}cos(\frac{k\pi}{2n+2}) = (a - \lambda_k)$
    n. → $\lambda_k = a - 2\sqrt{bc}cos(\frac{k\pi}{2n+2})$

Non Singularity

A matrix is non singular when all eigenvalues are positive. Since we know from above that our eigenvalues are $a - 2\sqrt{bc}cos(\frac{k\pi}{2n+2})$, the matrix is non singular when $a > 2\sqrt{bc}$.