On the number of types in sparse graphs

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Part 1: Sparsity

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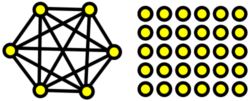

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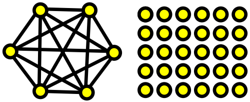

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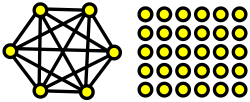

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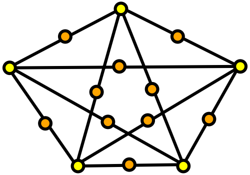

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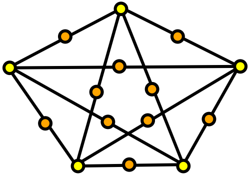

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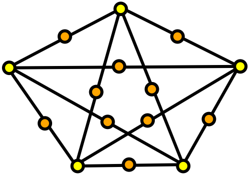

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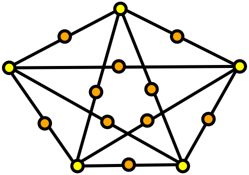
  - If we are looking for a structurally robust notion of sparsity, morally this example should be dense.



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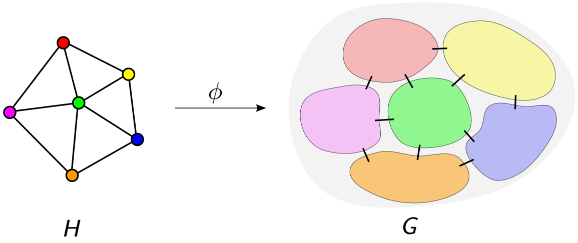

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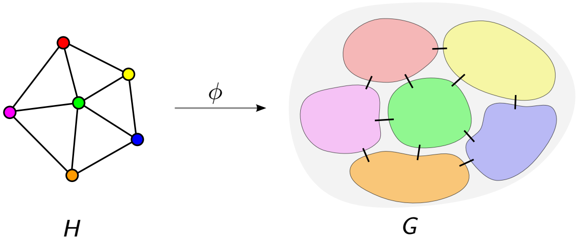

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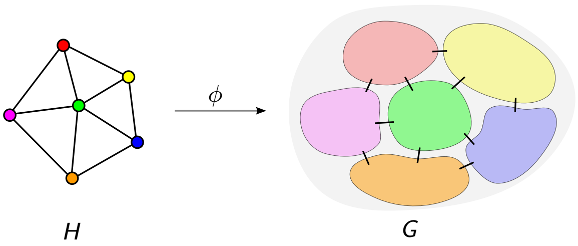

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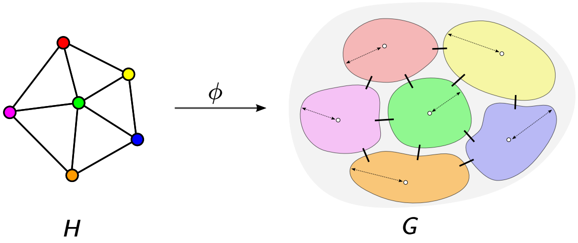

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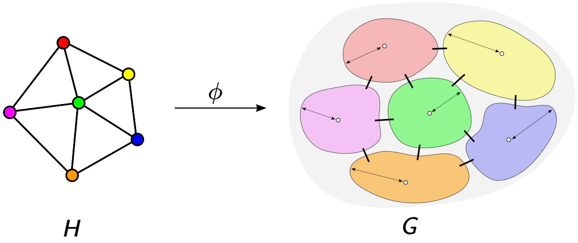

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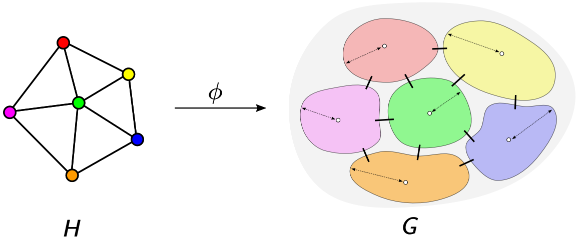
- **Idea**: Replace **subgraphs** with **shallow minors** in the definition.



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Hierarchy of sparsity

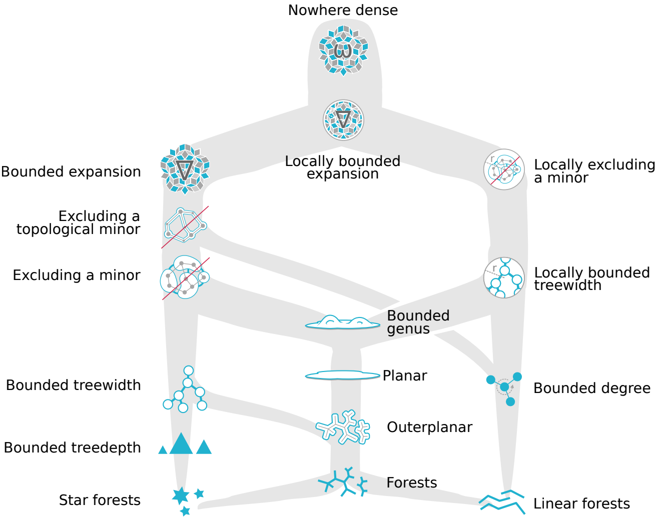

Figure by Felix Reidl

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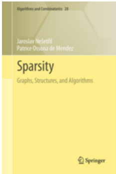

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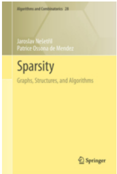

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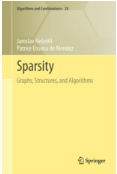

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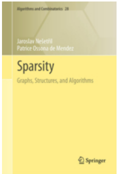

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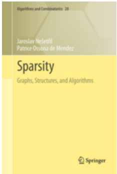

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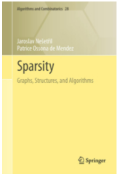

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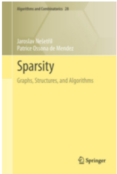

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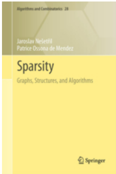

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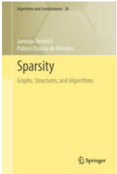

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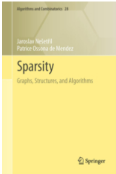

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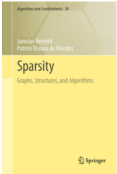

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  - Nowhere denseness delimits tractability for many basic problems.

Model checking on sparse graphs

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- FPT algorithms for structures whose Gaifman graphs have bounded degree, are planar, H-minor-free, ...

FO model-checking dichotomy

Theorem [Grohe et al., Dvořák et al.]

Let $\mathcal{C}$ be a monotone graph class (closed under taking subgraphs). Then:

- If $\mathcal{C}$ is nowhere dense, then FO model-checking can be done in time $f(\varphi) \cdot n^{1+\varepsilon}$ on structures with Gaifman graphs from $\mathcal{C}$, for any $\varepsilon > 0$.
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- Provides a natural barrier for locality-based methods.

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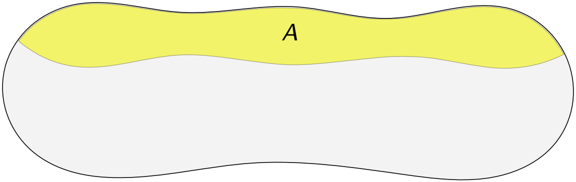

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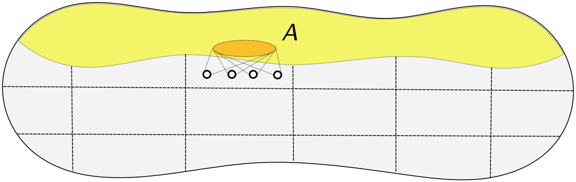

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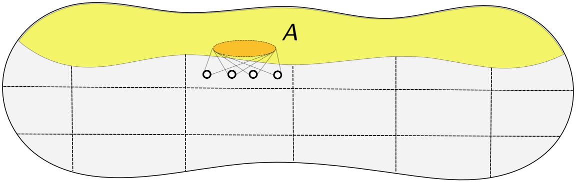

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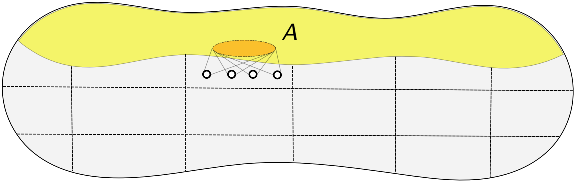
  - In general, even $2^{|A|}$.



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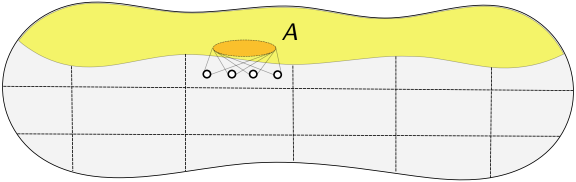

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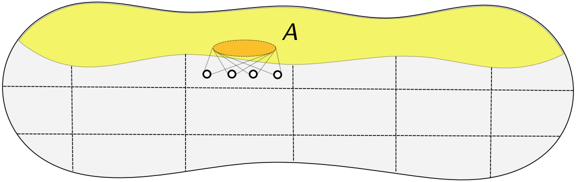

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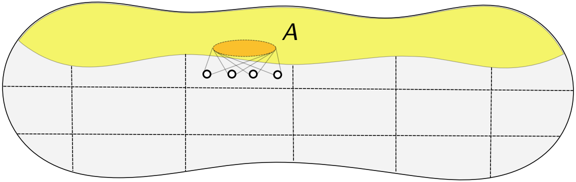
- **Now**: Proof for $r = 1$ and bounded expansion classes.



Proof for $r = 1$

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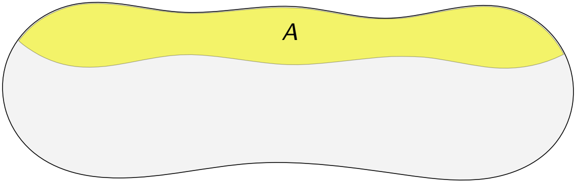

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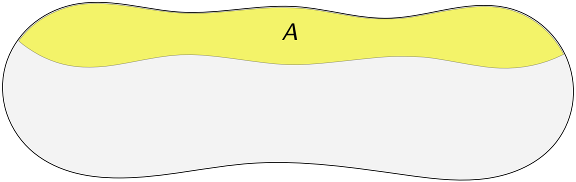

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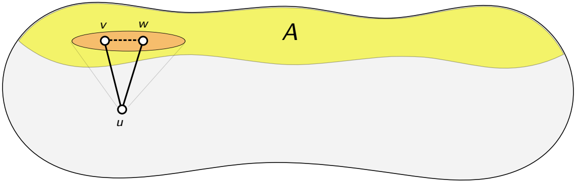

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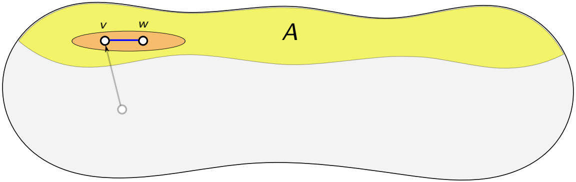

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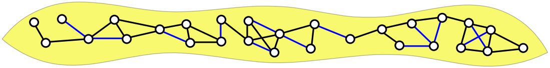

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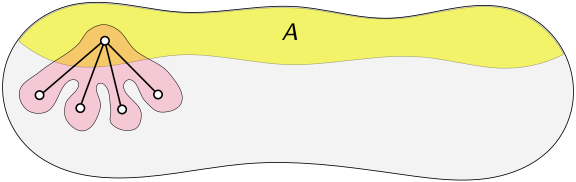

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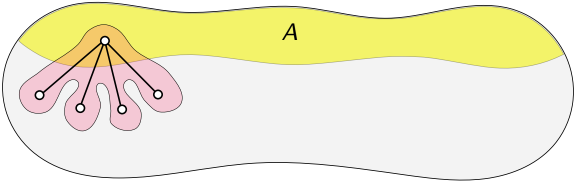

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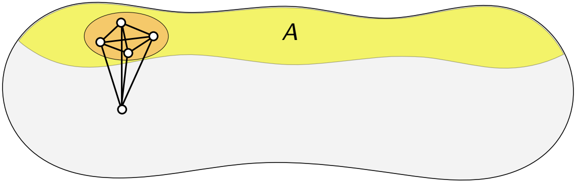

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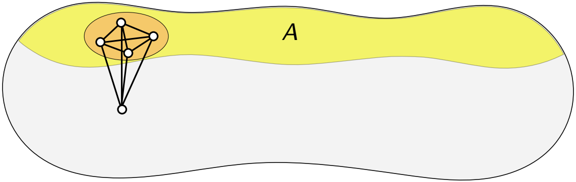

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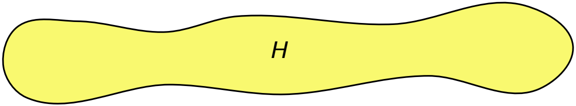

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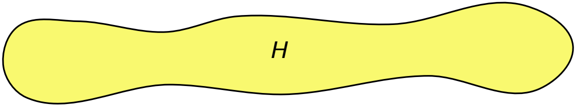

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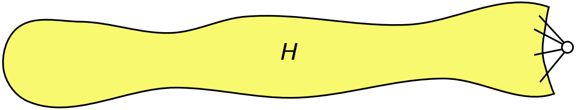

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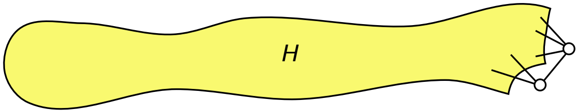

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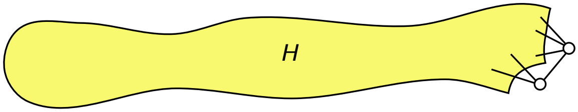

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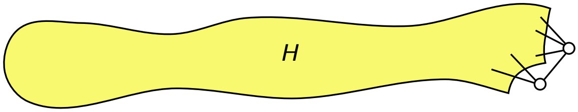

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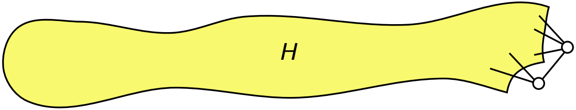

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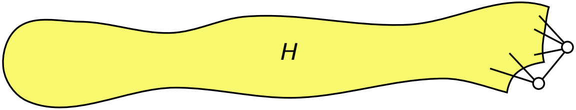

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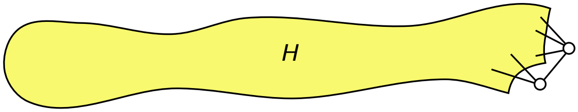

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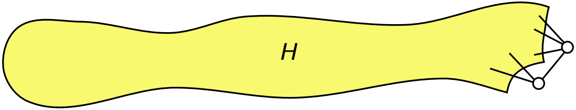

Part 2: Stability and types

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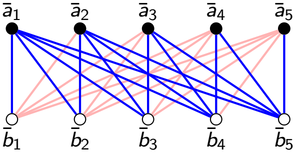

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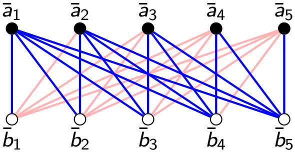

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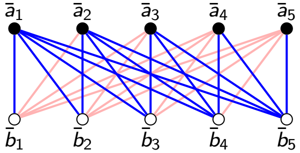

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- **Intuition**: One cannot define arbitrary long linear orders.

Stability theory

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- **Motivation**: Question about the number of **types**.

Types in model theory

- Let Σ be a signature, let $\mathbb{M}$ be a Σ-structure, and let $A \subseteq \mathbb{M}$.

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For an FO formula $\delta(\bar{x}, \bar{y})$ and $\bar{a} \in \mathbb{M}^{\bar{x}}$, the δ-**type** of $\bar{a}$ on A is

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Theorem

A theory $\mathbb{T}$ is stable if and only if for some infinite cardinal κ, for every model $\mathbb{M}$ of $\mathbb{T}$ and set $A \subseteq \mathbb{M}$ with $|A| \geqslant \kappa$, the number of types over A has the same cardinality as A.

Part 3: Sparsity and types

Sparsity and stability

Theorem [Adler and Adler, after Podewski and Ziegler]

Let $\mathcal{C}$ be a monotone class of finite graphs.
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Theorem [P, Siebertz, Toruńczyk]

Let $\mathcal{C}$ be a class of finite Σ-structures, $\mathbb{M} \in \mathcal{C}$, $A \subseteq \mathbb{M}$, and $\delta(\bar{x}, \bar{y}) \in \mathrm{FO}$.

- If $\mathcal{C}$ has bounded expansion, then the number of δ-types over A is at most $c|A|^{|\bar{x}|}$, where c depends on $\mathcal{C}$ and δ.
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- We now sketch the proof for graph classes of bounded expansion and $|\bar{x}|, |\bar{y}| = 1$.

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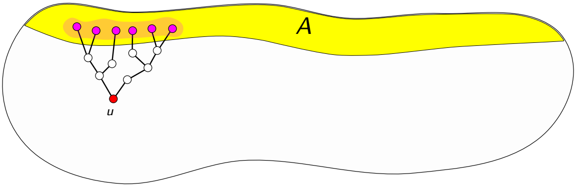

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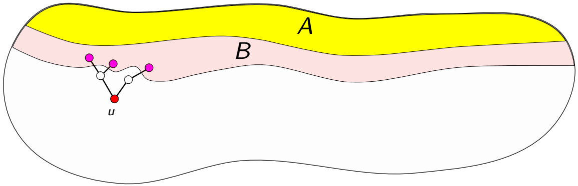
- **Fact**: There exists $B \supseteq A$ such that
  - $|B| \leqslant c|A|$ and
  - every r-projection of $u \notin B$ onto B has size $\leqslant c$.



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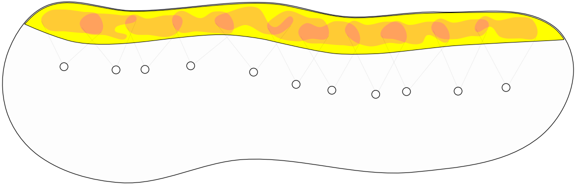

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- **Cor**: We may assume that r-projections onto A are of constant size.
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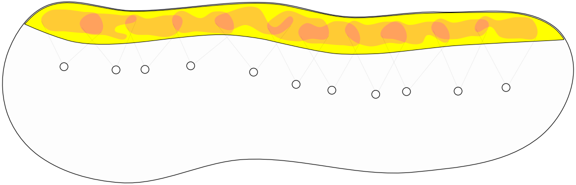
- **Ergo**: It suffices do bound the number of types for each possible distance-r projection by a constant.



Locality

Feferman-Vaught Lemma

For every formula $\delta(x, y) \in \mathsf{FO}(\Sigma)$ there exists $r \in \mathbb{N}$ and a finite set of formulas Δ such that the following holds.
Let G be a graph, u be a vertex, and A and S be subsets of vertices such that every path of length $\leqslant r$ from u to A passes through S. Then

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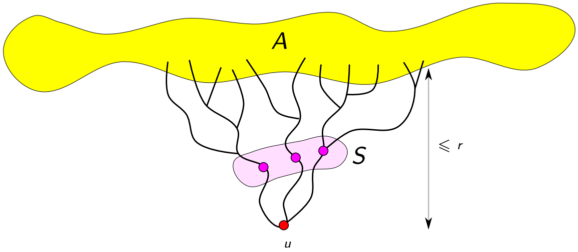

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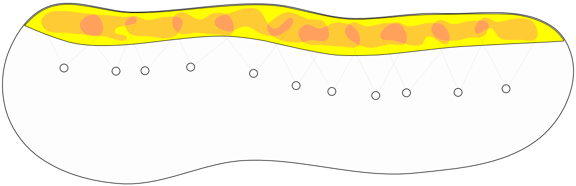

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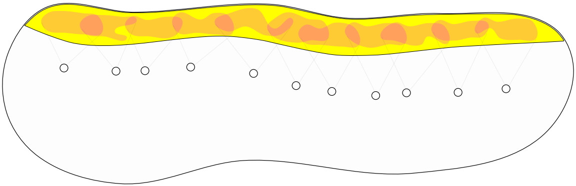

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- **Thank you for your attention!**