INFORMATION AND COMPUTATION **78**, 124–169 (1988)

Algorithms for Determining Relative Star Height and Star Height*

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Let $\mathscr{C} = \{R_1, ..., R_m\}$ be a finite class of regular languages over a finite alphabet Σ. Let $\Delta = \{b_1, ..., b_m\}$ be an alphabet, and δ be the substitution from Δ^* into Σ^* such that $\delta(b_i) = R_i$ for all i $(1 \leqslant i \leqslant m)$. Let R be a regular language over Σ which can be defined from $\mathscr{C}$ by a finite number of applications of the operators union, concatenation, and star. Then there exist regular languages over Δ which can be transformed onto R by δ. The relative star height of R w.r.t. $\mathscr{C}$ is the minimum star height of regular languages over Δ which can be transformed onto R by δ. This paper proves the existence of an algorithm for determining relative star height. This result obviously implies the existence of an algorithm for determining the star height of any regular language.

1. INTRODUCTION

Eggan (1963) introduced the notion of star height, and proved that for any integer $k \geqslant 0$, there exists a regular language of star height k. The problem was left open in his paper to determine the star height of any regular language. Dejean and Schutzenberger (1966) showed that for any integer $k \geqslant 0$, there exists a regular language of star height k over the two letter alphabet. McNaughton (1967) presented an algorithm for determining the loop complexity (i.e., the star height) of any regular language whose syntactic monoid is a group. Cohen (1970, 1971) and Cohen and Brzozowski (1970) investigated many properties of star height some of which provide algorithms for determining the star height of any regular language of certain reset-free type. Hashiguchi and Honda (1979) presented an algorithm for determining the star height of any reset-free language and any reset language. Hashiguchi (1982B) pesented an algorithm for deciding whether or not an arbitrary regular language is of star height one. To obtain this result, it uses the limitedness theorem on finite automata with

* A part of this work was done while the author was visiting Université Pierre et Marie Curie.

0890-5401/88 $3.00

distance functions (in short, D-automata), and the $(\cup, \cdot)$-representation theorem in Hashiguchi (1982A, 1983).

In this paper, we shall introduce the notion of relative star height and show the existence of an algorithm for determining relative star height which obviously implies the existence of an algorithm for determining the star height of any regular language. To prove these results, we again need D-automata and some of ideas which were used for the proof of the limitedness theorem. The main result of this paper is Theorem 5.1, from which an algorithm for determining relative star height follows by induction. Theorem 5.1 will be deduced easily from the main lemma in Section 5. We need many definitions and propositions in Sections 5 and 7 to establish the main lemma whose proof is by induction on $I(w)$ for all words w, where $I(w)$ is defined by the D-automaton $\mathscr{B}$ in Definition 5.14.

This paper consists of seven sections. Section 2 will present preliminaries which include definitions of star height, relative star height, and D-automata, and several preliminary results. Section 3 will present elementary properties of relative star height, and definitions about positions and paths of regular expressions. Section 4 will present algorithms for deciding whether or not the relative star height equals infinity or zero. Section 5 will present Theorem 5.1, the main lemma and an algorithm for determining relative star height. The proof of the main lemma will be presented in the final section. Section 6 will present an algorithm for determining star height.

2. Preliminaries

Σ is a finite alphabet. λ is the null word. $\varnothing$ is the empty set. For $w \in \Sigma^*$, $l(w)$ is the length of w. For a set A, $\# A$ is the cardinality of A.

Definition 2.1. The class $\mathrm{RE}(\Sigma)$ of regular expressions (over Σ) is defined inductively as follows.

(1) λ, $\varnothing$, and $a \in \Sigma$ are in $\mathrm{RE}(\Sigma)$;

(2) If E_1 and E_2 are in $\mathrm{RE}(\Sigma)$, then $E_1 \cup E_2$, $E_1 E_2$, and $(E_1)^*$ are in $\mathrm{RE}(\Sigma)$;

(3) No other expressions are in $\mathrm{RE}(\Sigma)$.

Definition 2.2. For any $E \in \mathrm{RE}(\Sigma)$, the language $|E|$ denoted by E is defined inductively as follows.

(1) $|\lambda| = \{\lambda\}$, $|\varnothing| = \varnothing$ and $|a| = \{a\}$ for $a \in \Sigma$;

(2) $|E_1 \cup E_2| = |E_1| \cup |E_2|$, $\quad |E_1 E_2| = |E_1| \cdot |E_2| = \{vw \in \Sigma^* \mid v \in |E_1|$

and $w \in |E_2|\}$, and $|E^*| = |E|^* = \{\lambda\} \cup \{w_1 w_2 \cdots w_k \mid k \geqslant 1$ and $w_i \in |E|$ for all $i\}$.

DEFINITION 2.3. The star height $h(E)$ of $E \in \mathrm{RE}(\Sigma)$ is defined inductively as follows.

(1) $h(\lambda) = h(\emptyset) = h(a) = 0$ for $a \in \Sigma$;

(2) $h(E_1 \cup E_2) = h(E_1 E_2) = \max\{h(E_1), h(E_2)\}$ and $h(E^*) = 1 + h(E)$.

DEFINITION 2.4. The star height $h(R)$ of a regular language R over Σ is defined by: $h(R) = \min\{h(E) \mid E \in \mathrm{RE}(\Sigma)$ and $|E| = R\}$.

When we consider finite automata instead of regular expressions, the star height of regular expressions corresponds to the cycle rank of finite automata which is defined as follows. A finite automaton $\mathscr{A}$ over the input alphabet Σ is a quintuple, $\langle \Sigma, Q, M, S, F\rangle$, such that Q is the set of states, M is the transition function $M: Q \times (\Sigma \cup \{\lambda\}) \to 2^Q$, and $S, F \subset Q$ are the sets of initial and final states, respectively. M is extended to $M: Q \times \Sigma^* \to 2^Q$ or $M: 2^Q \times 2^{\Sigma^*} \to 2^Q$ in the usual way. The language accepted by $\mathscr{A}$ is denoted by $R(\mathscr{A})$, and $R(\mathscr{A}) = \{w \in \Sigma^* \mid M(S, w) \cap F \neq \emptyset\}$. When S and F are irrelevant to the context, $\mathscr{A}$ is simply denoted by the triple, $\langle \Sigma, Q, M\rangle$. For $p, q \in Q$, p and q are said to be strongly connected iff $q \in M(p, \Sigma^*)$ and $p \in M(q, \Sigma^*)$. $\mathscr{A}$ is said to be strongly connected iff for all $p, q \in Q$, p and q are strongly connected. A subautomaton of $\mathscr{A}$ is a finite automaton $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0\rangle$ such that $Q_0 \subset Q$, and for all $(q, a) \in Q_0 \times (\Sigma \cup \{\lambda\})$, $M_0(q, a) \subset M(q, a) \cap Q_0$. A section of $\mathscr{A}$ is a maximal strongly connected subautomaton of $\mathscr{A}$. A section $\mathscr{S} = \langle \Sigma, Q_0, M_0\rangle$ of $\mathscr{A}$ is said to be trivial iff $\# Q_0 = 1$, and $M_0(Q_0, \Sigma) = \emptyset$. For $Q_0 \subset Q$, $\mathscr{A} - [Q_0]$ is the maximal subautomaton of $\mathscr{A}$ whose set of states is $Q - Q_0$, where if $Q_0 = Q$, then $\mathscr{A} - [Q_0] = \emptyset$, and in the following definition, $r(\emptyset)$ is defined to be zero. $Q(\mathscr{A})$ is the set of states of $\mathscr{A}$.

DEFINITION 2.5. The cycle rank $r(\mathscr{A})$ of a finite automaton $\mathscr{A}$ is defined inductively as follows.

(1) If all sections of $\mathscr{A}$ are trivial, then $r(\mathscr{A}) = 0$;

(2) If $\mathscr{A}$ has a nontrivial section, then

$$r(\mathscr{A}) = \max\{1 + \min\{r(\mathscr{S} - [q]) \mid q \in Q(\mathscr{S})\} \mid \mathscr{S} \text{ is a nontrivial section of } \mathscr{A}\}.$$

LEMMA 2.1. *For any finite automaton* $\mathscr{A}$,

$$r(\mathscr{A}) = \max\{r(\mathscr{S}) \mid \mathscr{S} \textit{ is a section of } \mathscr{A}\}.$$

Eggan (1963) proved the following theorem.

THEOREM 2.1. (Eggan, 1963). *From any finite automaton $\mathscr{A}$, we can effectively construct a regular expression E such that $|E| = R(\mathscr{A})$ and $h(E) = r(\mathscr{A})$, and vice versa.*

Theorem 2.1 gives the following theorem.

THEOREM 2.2. (Eggan, 1963). *For any regular language R,*

$$\begin{aligned} h(R) &= \min\{h(E) \mid E \text{ is a regular expression denoting } R\} \\ &= \min\{r(\mathscr{A}) \mid \mathscr{A} \text{ is a finite automaton accepting } R\}. \end{aligned}$$

By Theorem 2.2, we can regard the star height of a regular language R as the loop complexity of finite automata accepting R, and determining $h(R)$ is equivalent to determining the minimum cycle rank of finite automata accepting R. But regular expressions can be regarded as "one-dimensional expressions," i.e., a sequence of symbols, but finite automata need "multi-dimensional expressions," i.e., those like "graphs." So in our proof, regular expressions are more easily manipulated than finite automata, and we shall investigate properties of regular expressions in the sequel. (But to do this, we need finite automata with distance functions.)

The relative star height is defined as follows. Let $\mathscr{C} = \{R_1, ..., R_m\}$ be a finite class of regular languages over Σ. $\mathscr{C}(\cdot, \cup, *)$ is the closure of $\mathscr{C} \cup \{\{\lambda\}, \varnothing\}$ under the operations union ($\cup$), concatenation ($\cdot$), and star ($*$), that is, the smallest class of regular languages such that (i) $\mathscr{C} \cup \{\{\lambda\}, \varnothing\} \subset \mathscr{C}(\cdot, \cup, *)$, and (ii) if $R, R' \in \mathscr{C}(\cdot, \cup, *)$, then $R \cup R'$, $R \cdot R'$, $R^* \in \mathscr{C}(\cdot, \cup, *)$. Let $\Delta = \{b_1, ..., b_m\}$ be a new alphabet such that $\# \Delta = \# \mathscr{C}$. δ denotes the substitution from Δ^* to 2^{Σ^*} or from 2^{Δ^*} to 2^{Σ^*} which is defined as follows.

(1) $\delta(\lambda) = \{\lambda\}$;

(2) $\delta(b_i) = R_i$ for all i $(1 \leqslant i \leqslant m)$;

(3) For $W = b_{i_1} b_{i_2} \cdots b_{i_n} \in \Delta^+ (b_{i_j} \in \Delta)$,

$$\delta(W) = \delta(b_{i_1})\, \delta(b_{i_2}) \cdots \delta(b_{i_n}) = R_{i_1} R_{i_2} \cdots R_{i_n};$$

(4) For $L \subset \Delta^*$, $\delta(L) = \bigcup_{W \in L} \delta(W)$.

PROPOSITION 2.1. *Let $\mathscr{C}$, Δ, and δ be as above. Then the following hold.*

(1) *For any regular language $L \subset \Delta^*$, $\delta(L) \in \mathscr{C}(\cdot, \cup, *)$;*

(2) *For any $R \in \mathscr{C}(\cdot, \cup, *)$, there exists a regular language $LC\Delta^*$ such that $\delta(L) = R$.*

Proof. (1) If $L \subset \Delta^*$ is regular, then there exist $E \in \mathrm{RE}(\Delta)$ which denotes L, and by induction on the length of E, we can see that $\delta(L) \in \mathscr{C}(\cdot, \cup, *)$. (2). If $R \in \mathscr{C}(\cdot, \cup, *)$, then we can see the assertion by induction on the number of operations which we need for defining R from $\mathscr{C}$. ∎

DEFINITION 2.6. Let $\mathscr{C}$, Δ, and δ be as above. For any regular language $R \subset \Sigma^*$, the relative star height $h_r(R, \mathscr{C})$ of R w.r.t. $\mathscr{C}$ is defined as follows.

(1) If $R \notin \mathscr{C}(\cdot, \cup, *)$, then $h_r(R, \mathscr{C}) = \infty$, where ∞ denotes infinity;

(2) If $R \in \mathscr{C}(\cdot, \cup, *)$, then $h_r(R, \mathscr{C}) = \min\{h(L) \mid L$ is a regular language over Δ and $\delta(L) = R\}$.

Remark 2.1. We note that $h_r(R, \mathscr{C})$ is uniquely determined by R and $\mathscr{C}$ and is irrelevant to our choice of Δ and δ. We also note that there may be many $L \subset \Delta^*$ such that $\delta(L) = R$, and $h_r(R, \mathscr{C})$ is the minimum star height $h(L)$ of languages L such that $\delta(L) = R$.

The following proposition relates star height with relative star height, the proof of which is clear.

PROPOSITION 2.2. *For any regular language* $R \subset \Sigma^*$, $h(R) = h_r(R, \mathscr{C})$, *where* $\mathscr{C} = \{\{a\} \mid a \in \Sigma\}$.

Remark 2.2. Let $R \subset \Sigma^*$ be any regular language. R is said to be limited (or to have the finite power property) due to J. A. Brzozowski iff $R^* = (R \cup \{\lambda\})^k$ for some integer $k \geqslant 1$. Let $\mathscr{C} = \{R\}$. Then $\mathscr{C}(\cdot, \cup, *) \supset \{R^*\} \cup \{(R \cup \{\lambda\})^i \mid i \geqslant 0\}$, and we can see immediately that R is limited iff $h_r(R^*, \mathscr{C}) = 0$. The existence of algorithms for deciding whether or not R is limited was proved by Simon (1978) and Hashiguchi (1979), independently.

EXAMPLE 2.1. Let $\Sigma = \{0, 1\}$, and $\mathscr{C} = \{0^*1, (0^*1)(0^*1)^*\}$. Let $\Delta = \{b_1, b_2\}$ and δ be the substitution from Δ^* into Σ^* such that $\delta(b_1) = 0^*1$ and $\delta(b_2) = (0^*1)(0^*1)^*$. Then $(0^*1)^* \in \mathscr{C}(\cdot, \cup, *)$, and $(0^*1)^* = \delta(b_1^*)$. However, it also holds that $\delta(\lambda \cup b_1 \cup b_2) = \lambda \cup 0^*1 \cup (0^*1)(0^*1)^* = (0^*1)^*$. So $h_r((0^*1)^*, \mathscr{C}) = 0$.

Next we shall present the definition of finite automata with distance functions and the limitedness theorem.

DEFINITION 2.7. A finite automaton $\mathscr{B}$ with a distance function (in short, a D-automaton) over the input alphabet Σ is a sextuple $\langle \Sigma, Q, M, S, F, d \rangle$ such that Q is the finite set of states, $M: Q \times \Sigma \to 2^Q$ is the transition function, $S \subset Q$ and $F \subset Q$ are the sets of initial states and final states respectively, and d is the distance function $d: Q \times \Sigma \times Q \to \{0, 1, \infty\}$. d satisfies the following condition: for any $(q, a, q') \in Q \times \Sigma \times Q$, $d(q, a, q') = \infty$ iff $q' \notin M(q, a)$. d is extended to $d: Q \times \Sigma^* \times Q \to \{0, 1, 2, ...\} \cup \{\infty\}$ and $d: 2^Q \times \Sigma^* \times 2^Q \to \{0, 1, 2, ...\} \cup \{\infty\}$ as follows: for any $q, q' \in Q$, $t, t' \subset Q$, $w \in \Sigma^*$, and $a \in \Sigma$,

(1) $d(q, \lambda, q') = 0$ if $q = q'$; $d(q, \lambda, q') = \infty$ otherwise;

(2) $d(q, wa, q') = \min\{d(q, w, q'') + d(q'', a, q') \mid q'' \in Q\}$;

(3) $d(t, w, t') = \min\{d(q, w, q') \mid q \in t$ and $q' \in t'\}$.

The language accepted by $\mathscr{B}$ is denoted by $R(\mathscr{B})$ and $R(\mathscr{B}) = \{w \in \Sigma^* \mid M(S, w) \cap F \neq \varnothing\}$. $\mathrm{ID}(\mathscr{B})$ is the set of distances associated with words in $R(\mathscr{B})$, that is, $\mathrm{ID}(\mathscr{B}) = \{d(S, w, F) \mid w \in R(\mathscr{B})\}$. $D(\mathscr{B})$ is the upper limit of $\mathrm{ID}(\mathscr{B})$. $\mathscr{B}$ is said to be limited in distance iff $D(\mathscr{B}) < \infty$. M_{d0} is the zero-distance transition function from $Q \times \Sigma^*$ to 2^Q such that for any $q \in Q$ and $w \in \Sigma^*$, $M_{d0}(q, w) = \{q' \in Q \mid d(q, w, q') = 0\}$. $\mathscr{B}$ is said to be 0-deterministic iff for any $q \in Q$ and $w \in \Sigma^*$, $\# M_{d0}(q, w) \leqslant 1$. Two D-automata $\mathscr{B} = \langle \Sigma, Q, M, S, F, d \rangle$ and $\mathscr{B}' = \langle \Sigma, Q', M', S', F', d' \rangle$ are said to be strongly D-equivalent iff for all $w \in \Sigma^*$, $d(S, w, F) = d'(S', w, F')$.

Remark 2.3. In the above definition, we note that for any $q \in Q$, $M(q, \lambda) = \{q\}$ and the following hold.

(1) For any $q, q' \in Q$, $t, t' \subset Q$, and $w \in \Sigma^*$, $d(q, w, q') = \infty$ iff $q' \notin M(q, w)$, and $d(t, w, t') = \infty$ iff $t' \cap M(t, w) = \varnothing$;

(2) M_{d0} is extended to $M_{d0}: 2^Q \times \Sigma^* \to 2^Q$ as follows: $M_{d0}(t, w) = \{q' \in Q \mid d(t, w, q') = 0\}$. For any $q \in Q$, $t \subset Q$, and $v, w \in \Sigma^*$, it holds that $M_{d0}(q, vw) = M_{d0}(M_{d0}(q, v), w)$ and $M_{d0}(t, vw) = M_{d0}(M_{d0}(t, v), w)$;

(3) $\mathscr{B}$ is 0-deterministic iff for any $q \in Q$ and $a \in \Sigma$, $\# M_{d0}(q, a) \leqslant 1$;

(4) If $\mathscr{B}$ and $\mathscr{B}'$ are strongly D-equivalent, then $R(\mathscr{B}) = R(\mathscr{B}')$, $\mathrm{ID}(\mathscr{B}) = \mathrm{ID}(\mathscr{B}')$ and $D(\mathscr{B}) = D(\mathscr{B}')$.

The following three theorems are proved in Hashiguchi (1982A).

THEOREM 2.3. *For any D-automaton $\mathscr{B} = \langle \Sigma, Q, M, S, F, d \rangle$, one can construct a D-automaton $\mathscr{B}' = \langle \Sigma, Q', M', S', F', d' \rangle$ which satisfies the following.*

(1) $S' = \{S\}$;

(2) $Q' \subset 2^Q$;

(3) *$\mathscr{B}'$ is 0-deterministic, and for any $q \in Q'$ and $a \in \Sigma$, $\# \{q' \in Q' \mid d'(q, a, q') = 1\} \leqslant 1$;*

(4) *$\mathscr{B}$ and $\mathscr{B}'$ are strongly D-equivalent.*

Remark 2.4. In Theorem 2.1, for any $w \in \Sigma^*$ and $q \in Q'$, it holds that $\# M'_{d0}(q, w) \leqslant 1$, but $\# \{q' \in Q' \mid 1 \leqslant d'(q, w, q') < \infty\}$ may be greater than one.

LIMITEDNESS THEOREM 2.4. *For any* 0*-deterministic D-automaton* $\mathscr{B}$ *with* $\# S = 1$, $D(\mathscr{B}) < \infty$ *iff* $D(\mathscr{B}) \leqslant n((n+1)^{3n} \cdot 2^{6n^2+1})^n$, *where* $n = \# Q$.

From the above two theorems, the folowing theorem holds.

LIMITENDESS THEOREM 2.5. *For an arbitrary D-automaton* $\mathscr{B}$, $D(\mathscr{B}) < \infty$ *iff* $D(\mathscr{B}) \leqslant n_1((n_1+1)^{3n_1} \cdot 2^{6n_1^2+1})^{n_1}$, *where* $n_1 = 2^{\# Q}$.

Remark 2.5. In Hashiguchi (1986), the upper bound in Theorem 2.4 is improved as follows: for any 0-deterministic D-automaton $\mathscr{B}$, $D(\mathscr{B}) < \infty$ iff $D(\mathscr{B}) \leqslant 2^{4n^3+2n^2+5n}$, where $n = \# Q$.

The rest of this section will present definitions about regular expressions over Δ. However, the same terminology will be also used for regular expressions over Σ in Section 6.

DEFINITION 2.8. The class SRE(Δ) of regular expressions in string form over Δ is defined as follows: for any $E \in \mathrm{RE}(\Delta)$,

(1) if $h(E) = 0$, then $E \in \mathrm{SRE}(\Delta)$ iff $E = W_1 \cup \cdots \cup W_n$ for some $n \geqslant 1$ and $W_i \in \Delta^*$ $(1 \leqslant i \leqslant n)$;

(2) if $h(E) > 0$, then $E \in \mathrm{SRE}(\Delta)$ iff $E = F_1 \cup \cdots \cup F_n$ for some $n \geqslant 1$, and for some $F_i \in \mathrm{SRE}(\Delta)$, $1 \leqslant i \leqslant n$, where for each i, either $F_i = W$ for some $W \in \Delta^*$, or F_i is of the form: $F_i = W_{i_1} H_{i_1}^* W_{i_2} H_{i_2}^* \cdots W_{i_{p_i}} H_{i_{p_i}}^* W_{i_{p_i+1}}$, where $p_i \geqslant 1$, $W_{i_j} \in \Delta^*$ $(1 \leqslant j \leqslant p_i + 1)$, and $H_{i_k} \in \mathrm{SRE}(\Delta)$ $(1 \leqslant k \leqslant p_i)$.

By the distributivity of concatenation over union, the following proposition holds.

PROPOSITION 2.3 (Eggan, 1963). *For any* $E \in \mathrm{RE}(\Delta)$, *there exists* $E' \in \mathrm{SRE}(\Delta)$ *such that* $|E'| = |E|$ *and* $h(E') = h(E)$.

DEFINITION 2.9. Any $E \in \mathrm{RE}(\Delta)$ is in complete string form iff $E \in \mathrm{SRE}(\Delta)$ and either $E = W$ for some $W \in \Delta^*$ or E is of the form: $E = W_1 H_1^* W_2 H_2^* \cdots W_p H_p^* W_{p+1}$, where $p \geqslant 1$, $W_j \in \Delta^*$ $(1 \leqslant j \leqslant p+1)$, and $H_k^* \in \mathrm{SRE}(\Delta)$ $(1 \leqslant k \leqslant p)$. CSRE($\Delta$) denotes the class of regular expressions in complete string form over Δ.

PROPOSITION 2.4. *Any regular expression in* SRE(Δ) *is a union of some regular expressions in* CSRE(Δ).

DEFINITION 2.10. For any $E \in \mathrm{SRE}(\Delta)$, the set $\mathrm{Fact}(E)$ of factors of E is defined inductively as follows.

(1) $\mathrm{Fact}(\lambda)=\{\lambda\}$, $\mathrm{Fact}(\emptyset)=\{\emptyset\}$, and $\mathrm{Fact}(a)=\{a\}$ for $a \in \Delta$;

(2) If $E=E_1E_2$, then $\mathrm{Fact}(E)=\{E\} \cup \{E_0 \mid \text{for some } E_1', E_2' \in \mathrm{CSRE}(\Delta) \text{ with } E=E_1'E_2', E_0 \in \mathrm{Fact}(E_1') \cup \mathrm{Fact}(E_2')\}$;

(3) If $E=E_1^*$, then $\mathrm{Fact}(E)=\{E\} \cup \mathrm{Fact}(E_1)$;

(4) If $E=E_1 \cup E_2$, then $\mathrm{Fact}(E)=\mathrm{Fact}(E_1) \cup \mathrm{Fact}(E_2)$.

A factor $E' \in \mathrm{Fact}(E)$ is a word factor of E if $E' \in \Delta^*$. A factor $E' \in \mathrm{Fact}(E)$ is a star factor of E if it is a star expression.

EXAMPLE 2.2. Let $\Delta=\{a, b\}$, and $E=ab \cup a(ba \cup b)^*b$. Then $\mathrm{Fact}(E) = \{a, b, ab, ba, (ba \cup b)^*, a(ba \cup b)^*, (ba \cup b)^*b, a(ba \cup b)^*b\}$, $(ba \cup b)^*$ is a star factor of E, and ab is a word factor of E.

DEFINITION 2.11. For any $E \in \mathrm{SRE}(\Delta)$, $v(E)$ is defined by

$$v(E)=\max\{l(W) \mid W \text{ is a word factor of } E\}.$$

3. ELEMENTARY PROPERTIES OF RELATIVE STAR HEIGHT

Throughout this section, $\mathscr{C}=\{R_1, ..., R_m\}$ is a finite class of regular languages over Σ, and $\Delta=\{b_1, ..., b_m\}$ and δ are defined as in Section 2. We first note the following proposition about star height.

PROPOSITION 3.1. *For any regular language $R \subset \Sigma^*$ and any finite language $R_0 \subset \Sigma^*$, $h(R)=h(R \cup R_0)=h(R-R_0)$.*

Proof. Let E and E' be in $\mathrm{RE}(\Sigma)$ such that $|E|=R$, $|E'|=R_0$, $h(E)=h(R)$, and $h(E')=0$. Then $E \cup E' \in \mathrm{RE}(\Sigma)$, and $|E \cup E'|=R \cup R_0$. Thus $h(R \cup R_0) \leqslant h(R)$. Conversely let E be in $\mathrm{RE}(\Sigma)$ such that $|E|=R \cup R_0$, and $h(E)=h(R \cup R_0)$. We note that for any star expression H^*, it holds that $|H^*|=\{\lambda\} \cup |H| \cup |HH^*H|$. From this remark, we can see easily that there exists $E' \in \mathrm{RE}(\Sigma)$ such that $h(E')=h(E)$, $|E'|=|E|$, and $E'=E'' \cup E_0$ for some $E'', E_0 \in \mathrm{RE}(\Sigma)$, where $h(E_0)=0$, $|E''|=R$, and $|E_0|=R_0$. Then $h(R) \leqslant h(E'')=h(R \cup R_0)$. Thus $h(R)=h(R \cup R_0)$. Now we note that $R=(R-R_0) \cup R_0$, and we have $h(R)=h(R-R_0)$. ∎

For relative star height, the corresponding proposition does not hold.

PROPOSITION 3.2. (1) *There exist a regular language R, a finite language R_0, and a finite class $\mathscr{C}$ of regular languages such that $0 \leqslant h_r(R, \mathscr{C}) < \infty$ but $h_r(R \cup R_0, \mathscr{C})=\infty$;*

(2) *For any regular language R, any finite language R_0, and any finite class $\mathscr{C}$ of regular languages, if $R, R \cup R_0 \in \mathscr{C}(\cdot, \cup, *)$, then $h_r(R \cup R_0, \mathscr{C}) \leqslant h_r(R, \mathscr{C})$;*

(3) *There exist a regular language R, a finite language R_0, and a finite class $\mathscr{C}$ of regular languages such that $R, R_0 \in \mathscr{C}(\cdot, \cup, *)$ and $h_r(R \cup R_0, \mathscr{C}) < h_r(R, \mathscr{C})$.*

Proof. (1) Let $\Sigma = \{0\}$, $\mathscr{C} = \{0^*00\}$, $R = 0^*00$, and $R_0 = \{0\}$. Then $h_r(R, \mathscr{C}) = 0$, and $h_r(R \cup R_0, \mathscr{C}) = \infty$. (2) Assume that $R, R \cup R_0 \in \mathscr{C}((\cdot, \cup, *)$. Then there exists $L \subset \Delta^*$ such that $\delta(L) = R \cup R_0$. Then for each $w \in R_0$, there exists $W \in L$ such that $w \in \delta(W)$. So there exists a finite language $L_0 \subset L$ such that $R_0 \subset \delta(L_0)$. Let E and E' be in $\mathrm{RE}(\Delta)$ such that $\delta(|E|) = R$, $h(E) = h_r(R, \mathscr{C})$, $|E'| = L_0$, and $h(E') = 0$. Then $E \cup E' \in \mathrm{RE}(\Delta)$, $\delta(|E \cup E'|) = R \cup R_0$, and $h(E \cup E') = h_r(R, \mathscr{C})$. Thus $h_r(R \cup R_0, \mathscr{C}) \leqslant h_r(R, \mathscr{C})$. (3) We shall present a regular language R, and a finite class $\mathscr{C}$ of regular languages such that $R \in \mathscr{C}(\cdot, \cup, *)$ and $h_r(R \cup \{\lambda\}, \mathscr{C}) < h_r(R, \mathscr{C})$. Let $\Sigma = \{a, b\}$, and R be any regular language over Σ with $h(R) = 2$. Let $\Sigma' = \{a, b, c\}$. Define $R_1, R_2 \in (\Sigma')^*$ as follows: $R_1 = cRc$ and $R_2 = R_1 \cup \{\lambda\}$. Clearly $h(R_1) = h(R_2) = 2$. Let E_1 be in $\mathrm{RE}(\Sigma)$ such that $|E_1| = R_1$ and $h(E_1) = 2$. Define a finite class $\mathscr{C}$ of regular languages as follows: $\mathscr{C} = \{\{a\}, \{b\}, \{c\}\} \cup \{R_2\} \cup \{|H^*| \mid H^* \in \mathrm{Fact}(E_1) \text{ and } h(H^*) = 1\}$. It is clear that $h_r(R_2, \mathscr{C}) = 0$ and $h_r(R_1, \mathscr{C}) \leqslant 1$. We shall prove that $h_r(R, \mathscr{C}) = 1$. Assume the contrary. Then there exists $E_{11} \in \mathrm{RE}(\Delta)$ such that $h(E_{11}) = 0$ and $\delta(|E_{11}|) = R_1$. Since $h(R_1) = 2$, it is clear that $h_r(R_1, \mathscr{C} - \{R_2\}) = 1$. Thus e in Δ appears in E_{11}, where $\delta(e) = R_2$. But then for some $x, y \in (\Sigma')^*$ with $xy \neq \lambda$, it holds that $xR_2y \subset R_1 \subset R_2$. This is a contradiction. ∎

The following proposition may be the relative star height version of Proposition 3.1.

Proposition 3.3. *Let R and $\mathscr{C}$ be as above. Let E be in $\mathrm{RE}(\Delta)$ such that $\delta(|E|) = R$ and $h(E) = h_r(R, \mathscr{C})$. Then for any finite language $L \subset \Delta^*$, $h_r(R, \mathscr{C}) = h_r(\delta(|E| \cup L), \mathscr{C}) = h_r(\delta(|E| - L), \mathscr{C})$.*

Proof. Clearly $h_r(R, \mathscr{C}) \leqslant h_r(\delta(|E| - L), \mathscr{C})$. As in the proof of Proposition 3.1, from E, we can obtain $E_2 \in \mathrm{RE}(\Delta)$ such that $|E_2| = |E| - L$ and $h(E_2) = h(E)$. Thus $h_r(\delta(|E| - L), \mathscr{C}) \leqslant h_r(R, \mathscr{C})$ and $h_r(R, \mathscr{C}) = h_r(\delta(|E| - L), \mathscr{C})$. We also note that $R = \delta((|E| \cup L) - (\overline{|E|} \cap L)$, and we have $h_r(R, \mathscr{C}) = h_r(\delta(|E| \cup L), \mathscr{C})$. ∎

The rest of this section will present definitions about paths of regular expressions in $\mathrm{SRE}(\Delta)$ and two propositions. By Proposition 3.3, we can see that $h_r(R, \mathscr{C}) = h_r(\delta(|E| - \{\lambda\}), \mathscr{C})$, where $h_r(R, \mathscr{C}) = h(E)$. So in the sequel, we consider only regular expressions in SRE(Δ) in which none of λ and $\varnothing$ appears. However, it may hold that $\lambda \in \delta(|E| - \{\lambda\})$.

DEFINITION 3.1. *Let E be in* SRE(Δ).

(1) The length $l(E)$ of E is the number of occurrences of symbols of Δ in E: we do not count the number of occurrences of operation symbols such as union ($\cup$), or star ($*$). For convenience of notation, we define $l(\varnothing)$ and $l(\lambda)$ by $l(\varnothing) = l(\lambda) = 0$;

(2) $p(E)$ is the set of positions of E, that is, $p(E) = \{1, 2, ..., l(E)\}$. Each $i \in p(E)$ corresponds to the ith position of E from the left;

(3) Any $E' \in \mathrm{Fact}(E)$ is said to occur between i and j, $1 \leqslant i \leqslant j \leqslant l(E)$, iff E' occurs in E between position i and position j from the left;

(4) Δ_E is the mapping from $p(E)$ to Δ such that for each $i \in p(E)$, $\Delta_E(i)$ is the symbol in Δ which occurs in E at the ith position from the left.

EXAMPLE 3.1. Let $\Delta = \{a, b\}$ and $E = ab \cup ba^*b$. Then $l(E) = 5$, $p(E) = \{1, 2, 3, 4, 5\}$, $\Delta_E(1) = \Delta_E(4) = a$, and $\Delta_E(2) = \Delta_E(3) = \Delta_E(5) = b$. $ba^* \in \mathrm{Fact}(E)$ occurs in E between 3 and 4.

DEFINITION 3.2. For any $E \in \mathrm{SRE}(\Delta)$, two sets, $\mathrm{ip}(E)$ and $\mathrm{fp}(E)$, of initial and final positions of E, respectively, are defined inductively as follows.

(1) If $E = b$ for $b \in \Delta$, then $\mathrm{ip}(E) = \mathrm{fp}(E) = \{1\}$;

(2) If $E = E_1 E_2$, then $\mathrm{ip}(E) = \mathrm{ip}(E_1)$ and $\mathrm{fp}(E) = \{l(E_1) + i \mid i \in \mathrm{fp}(E_2)\}$;

(3) If $E = E_1 \cup E_2$, then $\mathrm{ip}(E) = \mathrm{ip}(E_1) \cup \{l(E_1) + i \mid i \in \mathrm{ip}(E_2)\}$, and $\mathrm{fp}(E) = \mathrm{fp}(E_1) \cup \{l(E_1) + i \mid i \in \mathrm{fp}(E_2)\}$;

(4) If $E = E_1^*$, then $\mathrm{ip}(E) = \mathrm{ip}(E_1)$ and $\mathrm{fp}(E) = \mathrm{fp}(E_1)$.

EXAMPLE 3.2. Let $\Delta = \{a, b\}$, and $E = ab \cup ab^*a \cup a^*$. Then $\mathrm{ip}(E) = \{1, 3, 6\}$ and $\mathrm{fp}(E) = \{2, 5, 6\}$.

DEFINITION 3.3. Let $E \in \mathrm{SRE}(\Delta)$ and $E' \in \mathrm{Fact}(E)$. Then $i \in p(E)$ is an initial position of E' w.r.t. E (a final position of E' w.r.t. E, respectively) iff for some $j, k \in p(E)$ with $j \leqslant i \leqslant k$, E' occurs in E between j and k, and $i = i_0 + j - 1$ for some $i_0 \in \mathrm{ip}(E')$ ($i = i_0 + j - 1$ for some $i_0 \in \mathrm{fp}(E')$, respectively).

EXAMPLE 3.3. Let $\Delta = \{a, b\}$ and $E = ba \cup aab^*a$. Then 4 is an initial position of ab^* w.r.t. E. It is also an initial position of a w.r.t. E, and is also a final position of aa w.r.t. E.

DEFINITION 3.4. Let $E \in \mathrm{SRE}(\Delta)$. A path P of E is a sequence of integers $(i_1, ..., i_n)$ such that $n \geqslant 1$, $i_j \in p(E)$, $1 \leqslant j \leqslant n$, and for each k, $1 \leqslant k \leqslant n-1$, either $i_{k+1} = i_k + 1$ or $i_{k+1} \neq i_k + 1$ and for some star factor $H^* \in \mathrm{Fact}(E)$, one of the following holds.

(1) i_k and i_{k+1} are a final position of H^* w.r.t. E and an initial position of H^* w.r.t. E, respectively;

(2) $i_k < i_{k+1}$, i_{k+1} is an initial position of H^* w.r.t. E, and for some j, $i_{k+1} \leqslant j \leqslant l(E)$, H^* occurs in E between $i_k + 1$ and j.

The length $l(P)$ of P is $n-1$, and a subpath of P is a subsequence $(i_j, i_{j+1}, ..., i_{j+k})$ for some j, k, $1 \leqslant j \leqslant j+k \leqslant n$. $\mathrm{in}(P)$ and $\mathrm{fn}(P)$ denote i_1 and i_n, respectively. Δ_E is extended to the set of paths of E by: $\Delta_E(i_1, ..., i_n) = \Delta_E(i_1) \cdots \Delta_E(i_n) (\in \Delta^+)$.

EXAMPLE 3.4. Let $\Delta = \{a, b\}$, and $E = ab \cup ab^*a(a \cup (ab \cup ba)^* b)^* a$. Then some examples of paths of E are $(1, 2)$, $(4, 4, 5)$, and $(5, 9, 10, 11, 6, 12)$. Note that by our definition, none of $(3, 5)$ and $(6, 11)$ is a path of E.

$(9, 10, 11)$ is a subpath of $(5, 9, 10, 11, 6)$, and $\Delta_E(5, 9, 10, 11, 6) = ababa$. $\mathrm{in}(5, 9, 10, 11) = 5$, and $\mathrm{fn}(5, 9, 10, 11) = 11$.

DEFINITION 3.5. Let $E \in \mathrm{SRE}(\Delta)$ and $P = (i_1, ..., i_n)$ be a path of E. Then P is a whole path of E iff $\{i_1, ..., i_n\} = \{1, 2, ..., l(E)\}$, $i_1 \in \mathrm{ip}(E)$, and $i_n \in \mathrm{fp}(E)$.

DEFINITION 3.6. For any integer $i \geqslant 1$, the integer $g_0(i)$ is defined inductively as follows, where $m = \#\mathscr{C} = \#\Delta$.

(1) $g_0(1) = 2m$;

(2) For $i > 1$, $g_0(i) = 4 \cdot (i-1) \cdot (g_0(i-1))^2$.

PROPOSITION 3.4. *For any integer $i \geqslant 1$, the following hold.*

(1) $\#\{E \mid E \in \mathrm{SRE}(\Delta) \text{ and } l(E) = i\} \leqslant g_0(i)$;

(2) $\#\{E \mid E \in \mathrm{SRE}(\Delta) \text{ and } l(E) \leqslant i\} \leqslant (g_0(i))^2$.

Proof. (1) The proof is by induction on i. If $i = 1$, then E is of the form: $E = b$ or $E = b^*$ for some $b \in \Delta$. So the assertion follows. Now let $i > 1$. We consider three cases.

Case (1). E is of the form $E = E_1 E_2$, where $1 \leqslant l(E_1)$, $l(E_2) \leqslant i-1$. The cardinality of the set of regular expressions of this form $\leqslant \sum_{j=1}^{i-1} g_0(j) \cdot g_0(i-j) \leqslant (i-1) \cdot (g_0(i-1))^2$.

Case (2). *E is of the form $E=E_1 \cup E_2$*, where $1 \leqslant l(E_1), l(E_2) \leqslant i-1$. The cardinality of the set of regular experessions of this form $\leqslant \sum_{j=1}^{i-1} g_0(j) \cdot g_0(i-j) \leqslant (i-1) \cdot (g_0(i-1))^2$.

Case (3). E is of the form $E=(E_1)^*$, where E_1 is of the form, $E_1=E_{11}E_{12}$ or $E_1=E_{11} \cup E_{12}$, and $l(E_1)=i$. From Cases (1), (2), the cardinality of the set of regular expressions of this form $\leqslant 2 \cdot (i-1) \cdot (g_0(i-1))^2$.

From Cases (1), (2), (3), we can see that the assertions hold.

(2) $\#\{E \mid E \in \mathrm{SRE}(\Delta) \text{ and } l(E) \leqslant i\} \leqslant g_0(1)+g_0(2)+\cdots+g_0(i) \leqslant 1+2+\cdots+g_0(i)=(g_0(i) \cdot (g_0(i)+1))/2 \leqslant (g_0(i))^2$. ▮

Definition 3.7. For any integers $i, j \geqslant 0$, the integer $g_1(i, j)$ is defined inductively as follows.

(1) $g_1(0, j)=j$;

(2) For $i>0$, $g_1(i, j)=j \cdot (g_0(g_1(i-1, j)))^2 \cdot g_1(i-1, j)$.

Proposition 3.5. *For any $E \in \mathrm{SRE}(\Delta)$ and any $w \in \delta(|E|) \cap \Sigma^+$, there exist $E_1 \in \mathrm{CSRE}(\Delta)$ and a path $P_1=(i_1, ..., i_n)$ of E_1 such that $|E_1| \subset |E|$, $h(E_1) \leqslant h(E)$, $l(E_1) \leqslant g_1(h(E), l(E))$, P_1 is a whole path of E_1, $w \in \delta(\Delta_{E_1}(P_1))$, and for some $x_1, ..., x_n \in \Sigma^+$, $w=x_1 \cdots x_n$ and $x_j \in \delta(\Delta_E(i_j))$ for all j, $1 \leqslant j \leqslant n$.*

Proof. The proof is by induction on $h(E)$.

Basis. $h(E)=0$. Then for some $W \in \Delta^+$, $W \in |E|$, and $w \in \delta(W)$. Clearly $l(W) \leqslant l(E)$. Let $W=b_{i_1}b_{i_2}\cdots b_{i_r}$ with $b_{i_j} \in \Delta$, $1 \leqslant j \leqslant r$. There exist a sequence of integers $(j_1, ..., j_n)$ and $x_1, ..., x_n \in \Sigma^+$ such that (i) $1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant r$, (ii) $w=x_1 \cdots x_n$, (iii) $x_k \in \delta(b_{i_{j_k}})$ for all k, $1 \leqslant k \leqslant n$, and (iv) for all $k' \in \{i_1, ..., i_r\} - \{i_{j_1}, i_{j_2}, ..., i_{j_n}\}$, $\lambda \in \delta(b_{k'})$. Put $E_1=b_{i_{j_1}}b_{i_{j_2}}\cdots b_{i_{j_n}}$. Then $|E_1| \subset |W| \subset |E|$, and $P_1=(1, ..., n)$ is a whole path of E_1, and other assertions also hold.

Inductive step. $h(E)>0$. By Proposition 2.4, there exists $E_0 \in \mathrm{CSRE}(\Delta)$ such that $|E_0| \subset |E|$, $w \in \delta(|E_0|)$, $h(E_0) \leqslant h(E)$, and $l(E_0) \leqslant l(E)$. Then E_0 is of the form $E_0=W_1H_1^*W_2H_2^*\cdots W_pH_p^*W_{p+1}$, where $p \geqslant 1$, $W_j \in \Delta^*$, $1 \leqslant j \leqslant p$, and H_k^* are star expressions, $1 \leqslant k \leqslant p$. Since $w \in \delta(|E_0|)$, w has a decomposition $(x_1, y_1, x_2, y_2, ..., x_p, y_p, x_{p+1})$ for which the following hold.

(1) $x_1, y_1, ..., x_p, y_p, x_{p+1} \in \Sigma^*$;

(2) $w=x_1y_1\cdots x_py_px_{p+1}$;

(3) For $j=1, ..., p+1$, $x_j \in \delta(W_j)$;

(4) For $k=1, ..., p$, $y_k \in \delta(|H_k^*|)$.

We define two sets, A and B, as follows.

$A = \{j \mid 1 \leqslant j \leqslant p+1, \text{ and } x_j \neq \lambda\};$

$B = \{j \mid 1 \leqslant j \leqslant p \text{ and } y_j \neq \lambda\}.$

By the basis, we know that for each $j \in A$, there exist $W_{j1} \in \Delta^+$ and a path P_{j1} of W_{j1} for which the assertions hold for x_j. We consider B. Let $j \in B$. By Proposition 2.4, H_j is of the form $H_j = H_{j1} \cup \cdots \cup H_{jk}$ where $k \geqslant 1$ and each H_{jr} is in CSRE(Δ), $1 \leqslant r \leqslant k$. Since $y_j \in \delta(|H_j^*|)$, there exists a decomposition of y_j $(v_1, ..., v_t)$, $t \geqslant 1$, such that $v_r \in \delta(|H_{ju_r}|) \cap \Sigma^+$ for some u_r, $1 \leqslant u_r \leqslant k$. By induction, we know that for each r, $1 \leqslant r \leqslant t$, there exist $H'_r \in$ CSRE(Δ) and a whole path P'_r of H'_r such that $l(H'_r) \leqslant g_1(h(E)-1, l(E))$, and other assertions hold for v_r. Now define the set C as follows: $C = \{H'_r \mid 1 \leqslant r \leqslant t\}$. By Proposition 3.4, $\# C \leqslant (g_0(g_1(h(E)-1, l(E))))^2$. Now define the regular expression H'_j as follows: $H'_j = H'_{10} \cup H'_{20} \cup \cdots \cup H'_{u0}$, where $C = \{H'_{10}, ..., H'_{u0}\}$. We note that $l(H'_j) \leqslant \# C \cdot \max\{l(H') \mid 1 \leqslant r \leqslant t\} \leqslant (g_0(g_1(h(E)-1, l(E))))^2 \cdot g_1(h(E)-1, l(E))$. We also can see that $(H'_j)^*$ satisfies the assertions for y_j with some whole path of $(H'_j)^*$. By concatenating all W_{i1}, $i \in A$, and all $(H'_j)^*$, $j \in B$, in the obvious order, we can obtain the regular expression E_1 and a whole path P_1 of E_1 for which the assertions hold, where we note that the following hold.

$$\begin{aligned} l(E_1) &\leqslant l(E) \cdot \max\{l(H'_j) \mid j \in B\} \\ &\leqslant l(E) \cdot (g_0(g_1(h(E)-1, l(E))))^2 \cdot g_1(h(E)-1, l(E)) \\ &= g_1(h(E), l(E)). \quad \blacksquare \end{aligned}$$

4. Algorithms for Deciding Whether or Not $h_r(R, \mathscr{C})$ Equals Infinity or Zero

Throughout this section, let $R \subset \Sigma^*$ be regular, $\mathscr{C} = \{R_1, ..., R_m\}$ be a finite class of regular languages over Σ, and $\Delta = \{b_1, ..., b_m\}$ and δ be as in Section 2. $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, \{s_0\}, F_0 \rangle$ is the reduced deterministic automaton accepting R, and for each i, $1 \leqslant i \leqslant m$, $\mathscr{A}_i = \langle \Sigma, Q_i, M_i, \{s_i\}, F_i \rangle$ is the reduced deterministic automaton accepting R_i. Without loss of generality, we assume that for all i, j, $0 \leqslant i < j \leqslant m$, $Q_i \cap Q_j = \varnothing$. This section will present two algorithms for deciding whether or not $h_r(R, \mathscr{C}) = \infty$ or 0, respectively. We shall first present an algorithm for deciding whether or not $h_r(R, \mathscr{C}) = \infty$.

Definition 4.1. $\mathscr{A}_\Delta$ is a finite deterministic automaton $\langle \Delta, Q_\Delta, M_\Delta, \{s_\Delta\}, F_\Delta \rangle$ which satisfies the following.

(1) $Q_\Delta = \{t \subset Q \mid \text{for some } W \in \Delta^*, t = M_0(s_0, \delta(W))\}$;

(2) For any $t \subset Q_\Delta$ and $b \in \Delta$, $M_\Delta(t, b) = M_0(t, \delta(b))$;

(3) $s_\Delta = \{s_0\}$, and $F_\Delta = \{t \in Q_\Delta \mid t \subset F_0\}$.

Remark 4.1. For any $t \subset Q$ and any regular language $R' \subset \Sigma^*$, we can effectively obtain $M_0(t, R')$ as in Remark 3.1 of Hashiguchi (1983). Since Δ and Q_Δ are finite, we can effectively construct $\mathscr{A}_\Delta$ from $\mathscr{A}_0$ and $\mathscr{C}$.

The following theorem was proved in Hashiguchi (1983).

THEOREM 4.1. *$R \in \mathscr{C}(\cdot, \cup, *)$ iff $\delta(R(\mathscr{A}_\Delta)) = R$.*

The following algorithm holds.

ALGORITHM 4.1. *$h_r(R, \mathscr{C}) < \infty$ iff $\delta(R(\mathscr{A}_\Delta)) = R$.*

Next we shall present an algorithm for deciding whether or not $h_r(R, \mathscr{C}) = 0$.

DEFINITION 4.2. $\mathscr{B}_0 = \langle \Sigma, Q', M', S', \{q_f\}, d'\rangle$ is a D-automaton which satisfies the following.

(1) $Q' = \{q_f\} \cup \{(t, q) \mid t = M_0(s_0, \delta(W))$ for some $W \in \Delta^*$ and $q \in Q_1 \cup \cdots \cup Q_m\}$, where q_f is a new state;

(2) $S' = \{(\{s_0\}, s_i) \mid i = 1, ..., m\}$ if $\lambda \notin R$ and $S' = \{q_f\} \cup \{(\{s_0\}, s_i) \mid i = 1, ..., m\}$ otherwise;

(3) For any $(t, q) \in Q'$ with $q \in Q_i$, $1 \leqslant i \leqslant m$, and $a \in \Sigma$, the following (3.1)–(3.4) hold.

(3.1) If $M_i(q, a) = \varnothing$, then $M'((t, q), a) = \varnothing$;

(3.2) If $M_i(q, a) \neq \varnothing$ and $M_i(q, a) \notin F_i$, then $M'((t, q), a) = \{(t, M_i(q, a))\}$ and $d'((t, q), a, (t, M_i(q, a))) = 0$;

(3.3) If $M'(q, a) \in F_i$ and $M'(t, R_i) - F \neq \varnothing$, then $M'((t, q), a) = \{(t, M_i(q, a))\} \cup \{(M_0(t, R_i), s_j) \mid j = 1, ..., m\}$, $d'((t, q), a, (t, M_i(q, a))) = 0$, and $d'((t, q), a, (M_0(t, R_i), s_j)) = 1$ for $j = 1, ..., m$;

(3.4) If $M_i(q, a) \in F_i$ and $M(t, R_i) \subset F$, then $M'((t, q), a) = \{q_f\} \cup \{(t, M_i(q, a))\} \cup \{(M_0(t, R_i), s_j) \mid j = 1, ..., m\}$, $d'((t, q), a, (t, M_i(q, a))) = 0$, and $d'((t, q), a, q_f) = d'((t, q), a, (M_0(t, R_i), s_j)) = 1$ for $j = 1, ..., m$;

(4) $M'(q_f, a) = \varnothing$ for all $a \in \Sigma$.

The following theorem is a direct consequence of Theorem 6.1 in Hashiguchi (1983).

THEOREM 4.2. *$h_r(R, \mathscr{C}) = 0$ iff $R = R(\mathscr{B}_0)$ and $D(\mathscr{B}_0) < \infty$.*

From this theorem and Theorem 2.5, we have the following algorithm.

ALGORITHM 4.2. *From* $\mathscr{A}_0$ *and* $\mathscr{A}_i$, $1 \leqslant i \leqslant m$, *we construct the D-automaton* $\mathscr{B}_0$. *For any* $q, q' \in Q'$, *we obtain the regular language,* $R_0(q, q') = \{w \in \Sigma^* \mid d'(q, w, q') = 0\}$, *and obtain the finite class of regular languages* $L(\mathscr{B}_0)$ *as follows:* $L(\mathscr{B}_0) = \{R' \mid R' = R_0(s', q_f)$ *for some* $s' \in S\} \cup \{R'_1 \cdot \{a_1\} \cdot R'_2 \cdot \{a_2\} \cdots \{a_n\} \cdot R'_{n+1} \mid$, (1) $n \leqslant n_0 \cdot (n_0+1)^{3(n_0)^2} \cdot 2^{n_0(6(n_0)^2+1}$, *where* $n_0 = 2^{\# Q'}$, (2) $a_i \in \Sigma$ *for* $i = 1, ..., n$, (3) *for some* $q_{j1}, q_{j2} \in Q'$, $1 \leqslant j \leqslant n+1$, $q_{11} \in S$, $q_{n+1,2} = q_f$, $R_j = R_0(q_{j1}, q_{j2})$, $1 \leqslant j \leqslant n+1$, *and* (4) $d'(q_{i2}, a_i, q_{i+1,1}) = 1$ *for* $i = 1, ..., n\}$. *Then it holds that* $h_r(R, \mathscr{C}) = 0$ *iff* $R = \bigcup_{R' \in L(\mathscr{B}_0)} R'$.

Theorems 2.5, 4.2 also provide the following algorithm.

ALGORITHM 4.3. $h_r(R, \mathscr{C}) = 0$ *iff* $R = \delta(W_1 \cup \cdots \cup W_n)$ *for some* $W_i \in \Delta^*$ *with* $l(W_i) \leqslant n_0(n_0+1)^{3(n_0)^2} \cdot 2^{n_0(6(n_0)^2+1)}$, $1 \leqslant i \leqslant n$, *where* $n_0 = 2^{n_1}$ *and* $n_1 = 2^{\# Q_0} \cdot (\# Q_1 + \cdots + \# Q_m)$.

5. AN ALGORITHM FOR DETERMINING RELATIVE STAR HEIGHT

Throughout this section, $R, \mathscr{C} = \{R_1, ..., R_m\}$, $\Delta = \{b_1, ..., b_m\}$, δ, $\mathscr{A}_0$, and $\mathscr{A}_i$, $1 \leqslant i \leqslant m$, are as in Section 4. Let $\mathscr{M}$ be the syntactic monoid of R, and α be the canonical morphism from Σ^* onto $\mathscr{M}$. Thus for all $v, w \in \Sigma^*$, $\alpha(v) = \alpha(w)$ iff (for all $x, y \in \Sigma^*$, $xvy \in R$ iff $xwy \in R$). α is extended to $\alpha \colon 2^{\Sigma^*} \to 2^{\mathscr{M}}$ in the usual way. So $R = \alpha^{-1}(\mathscr{M}_0)$ for some $\mathscr{M}_0 \subset \mathscr{M}$. This section will present the main lemma, Theorem 5.1, and an algorithm for determining relative star height. We need many definitions. In the sequel, we consider only regular expressions in which none of λ and $\varnothing$ appears. But for brevity of description, we adopt the following notation: $E = E_1 \cup E_2E_3E_4 \cup E_5$ implies that E is a regular expression in which none of λ and $\varnothing$ appears, and very often, our main concern is for E_3, and E is one of the following form: (1) $E = E_1 \cup E_2E_3E_4 \cup E_5$, (2) $E = E_2E_3E_4 \cup E_5$, (3) $E = E_1 \cup E_2E_3E_4$, (4) $E = E_1 \cup E_3E_4 \cup E_5$, (5) $E = E_3E_4 \cup E_5, ...,$ or $E = E_3$, etc. In the sequel E with subscripts denotes any regular expression in $\mathrm{SRE}(\Delta)$, H^* with subscripts denotes any star expression in $\mathrm{SRE}(\Delta)$, W with subscripts denotes any word in Δ^*, and b with subscripts denotes any symbol in Δ.

DEFINITION 5.1. For any $E \in \mathrm{SRE}(\Delta)$ and any $i \in p(E)$, $\beta(E, i)$ is the nonnegative integer which is defined inductively as follows.

(1) If $E = E_1 \cup E_2bE_3 \cup E_4$ and $i = l(E_1) + l(E_2) + 1$, then $\beta(E, i) = 0$;

(2) If $E = E_1 \cup E_2H^*E_3 \cup E_4$ and $l(E_1) + l(E_2) + 1 \leqslant i \leqslant l(E_1) + l(E_2) + l(H)$, then $\beta(E, i) = 1 + \beta(H, i - l(E_1) - l(E_2))$.

DEFINITION 5.2. For any $E \in \mathrm{SRE}(\Delta)$, $i \in p(E)$ with $\beta(E, i) \geqslant 1$, and $j \in \{1, 2, ..., \beta(E, i)\}$, the jth star factor of E containing i is defined inductively as follows.

(1) If $E = E_1 \cup E_2 H^* E_3 \cup E_4$, $l(E_1) + l(E_2) + 1 \leqslant i \leqslant l(E_1) + l(E_2) + l(H)$, and $\beta(E, i) = 1$, then the first star factor of E containing i is H^*;

(2) If $E = E_1 \cup E_2 H^* E_3 \cup E_4$, $l(E_1) + l(E_2) + 1 \leqslant i \leqslant l(E_1) + l(E_2) + l(H)$, and $\beta(E, i) \geqslant 2$, then the $\beta(E, i)$th star factor of E containing i is H^*, and for $j \in \{1, ..., \beta(E, i) - 1\}$, the jth star factor of E containing i is the jth star factor of H^* containing $i - (l(E_1) + l(E_2))$.

EXAMPLE 5.1. Let $\Delta = \{a, b\}$ and $E = ba \cup a^*((ab \cup a)^* \cup b)^* a$. Then $\beta(E, 1) = \beta(E, 2) = \beta(E, 8) = 0$, $\beta(E, 3) = \beta(E, 7) = 1$, and $\beta(E, 4) = \beta(E, 5) = \beta(E, 6) = 2$. The first star factor of E containing 3 is a^*, the first star factor of E containing 5 is $(ab \cup a)^*$, the first star factor of E containing 7 is $((ab \cup a)^* \cup b)^*$, and the second star factor of E containing 4 is $((ab \cup a)^* \cup b)^*$.

DEFINITION 5.3. For any $E \in \mathrm{SRE}(\Delta)$, $i \in p(E)$, and $j \in \{0, 1, ..., \beta(E, i)\}$, $\gamma(E, i, j)$ is defined inductively as follows.

(1) If $E = E_1 \cup E_2 b E_3 \cup E_4$ and $i = l(E_1) + l(E_2) + 1$, then $j = 0$, and $\gamma(E, i, 0) = (\alpha(\delta(|E_2|)), b)$;

(2) If $E = E_1 \cup E_2 H^* E_3 \cup E_4$ and $l(E_1) + l(E_2) + 1 \leqslant i \leqslant l(E_1) + l(E_2) + l(H)$, then for $j \in \{0, 1, ..., \beta(E, i) - 1\}$, $\gamma(E, i, j) = \gamma(H, i - l(E_1) - l(E_2), j)$, and $\gamma(E, i, \beta(E, i)) = (\alpha(\delta(|E_2|)), h(H^*), \alpha(\delta(|H^*|)))$.

DEFINITION 5.4. For any $E \in \mathrm{SRE}(\Delta)$ and $i \in p(E)$, $\zeta(E, i)$ is defined by $\zeta(E, i) = \{(j, \gamma(E, i, j)) \mid 0 \leqslant j \leqslant \beta(E, i)\}$.

EXAMPLE 5.2. Let $\Delta = \{a, b\}$ and $E = a \cup ba^*(a \cup (ba \cup a)^*)^*$. Then $\beta(E, 1) = 0$, $\gamma(E, 1, 0) = (\alpha(\lambda), a)$, $\beta(E, 3) = 1$, $\gamma(E, 3, 0) = (\alpha(\lambda), a)$, $\gamma(E, 3, 1) = (\alpha(\delta(b)), 1, \alpha(\delta(a^*)))$, and $\zeta(E, 3) = \{(0, \gamma(E, 3, 0)), (1, \gamma(E, 3, 1))\}$, etc.

From now to Theorem 5.1 below, we consider the case where $h_r(R, \mathscr{C}) < \infty$.

DEFINITION 5.5. E_0 is a regular expression in $\mathrm{SRE}(\Delta)$ such that $h(E_0) = h_r(R, \mathscr{C})$, and $R = \delta(|E_0|)$.

Remark 5.1. There may exist many regular expressions E_0 such that $h(E_0) = h_r(R, \mathscr{C})$ and $R = \delta(|E_0|)$, but in the sequel, we consider some fixed E_0. We also note that at this time, we do not know $h(E_0)$ exactly, but we can obtain an upper bound of $h(E_0)$ which is the cycle rank of the finite automaton $\mathscr{A}_\Delta$ in Definition 4.1 by Eggan's theorem.

DEFINITION 5.6. $\xi(E_0)$ is the set of $E \in \mathrm{CSRE}(\Delta)$ such that $\delta(|E|) \subset R$, $h(E) \leqslant h(E_0)$, and $l(E) \leqslant g_1(h(E_0), l(E_0))$, where $g_1(h(E_0), l(E_0))$ is the integer defined in Definition 3.7.

By Proposition 3.5, the following proposition holds.

PROPOSITION 5.1. *For any $w \in R \cap \Sigma^+$, there exist $E \in \xi(E_0)$ and a path $P = (i_1, ..., i_n)$ of E such that (1) $n \geqslant 1$, (2) P is a whole path of E, (3) $w \in \delta(\Delta_E(P))$, and (4) for some $x_i, ..., x_n \in \Sigma^+$, $w = x_1 \cdots x_n$ and $x_j \in \delta(\Delta_E(i_j))$ for all j.*

DEFINITION 5.7. $\eta(\xi(E_0), R)$ is the set $\{(\zeta(E, i), q) \mid E \in \xi(E_0),\ i \in p(E)$ and $q \in Q_j$, where $\Delta_E(i) = b_j,\ 1 \leqslant j \leqslant m\}$.

LEMMA 5.1. *$\# \eta(\xi(E_0), R) < g_3$, where $g_3 = \binom{g_2}{h(E_0)+1} \cdot \#(Q_1 \cup \cdots \cup Q_m)$, and $g_2 = (4^{\# \mathscr{M}} + 1) \cdot (h(E_0) + 1)$.*

Proof. Let us first consider the set $A = \{\gamma(E, i, j) \mid E \in \xi(E_0), i \in p(E)$ and $0 \leqslant j \leqslant \beta(E, i)\}$. It is clear that $\# A \leqslant 2^{\# \mathscr{M}} \cdot (h(E_0) + 1) \cdot 2^{\# \mathscr{M}} \leqslant 4^{\# \mathscr{M}} \cdot (h(E_0) + 1)$. Then we can see that for any $E \in \xi(E_0)$ and $i \in p(E)$, $\zeta(E, i)$ is a set of some ordered k elements, $1 \leqslant k \leqslant h(E_0) + 1$, from the set A. However, this order is unique by each set. Thus $\# \{\zeta(E, i) \mid E \in \xi(E_0)$ and $i \in p(E)\} \leqslant \binom{\# A}{1} + \binom{\# A}{2} + \cdots + \binom{\# A}{h(E_0)+1} \leqslant \binom{\# A + h(E_0)}{h(E_0)+1} \leqslant \binom{(4^{\# \mathscr{M}}+1) \cdot (h(E_0)+1)}{h(E_0)+1}$.
Now the assertion is clear. ∎

DEFINITION 5.8. $F\eta(\xi(E_0, R)$ is the set of functions from $\eta(\xi(E_0), R)$ to the set of subsets of $\eta(\xi(E_0), R) \times \{0, 1, 2, 3\}$.

The following lemma is clear from Lemma 5.1.

LEMMA 5.2. *$\# F\eta(\xi(E_0), R) < g_4$, where $g_4 = (2^{4 g_3})^{g_3} = 16^{(g_3)^2}$.*

DEFINITION 5.9. For any $f_1, f_2 \in F\eta(\xi(E_0), R)$, the function $f_2 \cdot f_1 \in F\eta(\xi(E_0), R)$ is defined as follows: for any $(u, q) \in \eta(\xi(E_0), R)$, $f_2 \cdot f_1(u, q) = \{(u'', q'', c) \mid$ there exist $(u', q', c') \in f_1(u, q)$ and $c'' \in \{0, 1, 2, 3\}$ such that $(u'', q'', c'') \in f_2(u', q')$, and $c = \max\{c', c''\}\}$.

PROPOSITION 5.2. *For any $f_1, f_2, f_3 \in F\eta(\xi(E_0), R)$, $f_3 \cdot (f_2 \cdot f_1) = (f_3 \cdot f_2) \cdot f_1$.*

Proof. For any $(u_0, q_0) \in \eta(\xi(E_0), R)$ and any $(u_3, q_3, c) \in \eta(\xi(E_0), R) \times \{0, 1, 2, 3\}$, it holds that $(u_3, q_3, c) \in (f_3 \cdot (f_2 \cdot f_1))(u_0, q_0)$ iff there exist $(u_1, q_1, c_1), (u_2, q_2, c_2) \in \eta(\xi(E_0), R) \times \{0, 1, 2, 3\}$ and $c_3 \in \{0, 1, 2, 3\}$ such that $(u_1, q_1, c_1) \in f_1(u_0, q_0)$, $(u_2, q_2, c_2) \in f_2(u_1, q_1)$, $(u_3, q_3, c_3) \in f_3(u_2, q_2)$ and $c = \max\{c_1, c_2, c_3\}$ iff $(u_3, q_3, c) \in ((f_3 \cdot f_2) \cdot f_1)(u_0, q_0)$. ∎

DEFINITION 5.10. Let $E \in \mathrm{SRE}(\Delta)$, $P=(i_1, ..., i_n)$ be a path of E, and $w \in \Sigma^*$. For any $q \in Q_{j0}$, $q' \in Q_{j_1}$, and $c \in \{0, 1, 2, 3\}$, where $\Delta_E(i_1)=b_{j_0}$ and $\Delta_E(i_n)=b_{j_1}$, $1 \leqslant j_0, j_1 \leqslant m$, P is said to strictly spell w with (q, q', c) iff one of the following (1)–(3) hold.

(1) $w=\lambda$, $n=1$, $q=q'$, and $c=0$;

(2) $w \in \Sigma^+$, $n=1$, $c=0$, and $q'=M_{j0}(q, w)$;

(3) $w \in \Sigma^+$, $n \geqslant 2$, and there exist $v_1, ..., v_{n-1} \in \Sigma^+$ and $v_n \in \Sigma^*$ for which the following (3.1)–(3.5) hold.

(3.1) $w=v_1 \cdots v_{n-1} v_n$;

(3.2) $M_{j0}(q, v_1) \in F_{j0}$, $M_{j_1}(s_{j_1}, v_n)=q'$, and $v_k \in \delta(\Delta_E(i_k))$ for all k, $2 \leqslant k \leqslant n-1$;

(3.3) $c=1$ iff the minimum factor of E containing P is a word over Δ;

(3.4) $c=2$ iff the minimum factor of E containing P is not a word over Δ, and P is contained in some star factor of E;

(3.5) $c=3$ iff the minimum factor of E contaiing P is not a word over Δ, and P is not contained in any star factor of E.

DEFINITION 5.11. For any $w \in \Sigma^*$, f_w is the function in $F\eta(\xi(E_0), R)$ such that for any $(u, q) \in \eta(\xi(E_0), R)$, $f_w(u, q)=\{(u', q', c) \mid$ for some $E \in \xi(E_0)$ and a path P of E, (1) $\zeta(E, \mathrm{in}(P))=u$ and $\zeta(E, \mathrm{fn}(P))=u'$, (2) $q \in Q_{j0}$ and $q' \in Q_{j_1}$, where $\Delta_E(\mathrm{in}(P))=b_{j0}$ and $\Delta_E(\mathrm{fn}(P))=b_{j_1}$, $1 \leqslant j_0, j_1 \leqslant m$, and (3) P strictly spells w with $(q, q', c)\}$.

PROPOSITION 5.3. (1) For any $(u, q) \in \eta(\xi(E_0), R)$, $f_\lambda(u, q)=\{(u, q, 0)\}$; (2) For any $f \in F\eta(\xi(E_0), R)$, $f \cdot f_\lambda = f_\lambda \cdot f = f$.

DEFINITION 5.12. For any $v_1, ..., v_n \in \Sigma^*$, $f_{(v_1, ..., v_n)}$ is the function in $F\eta(\xi(E_0), R)$ which is defined by: $f_{(v_1, ..., v_n)} = f_{v_n} \cdot f_{v_{n-1}} \cdots f_{v_1}$.

DEFINITION 5.13. $\mathrm{Seq}(\Sigma^*)$ is the set of sequences of words over Σ $(v_1, ..., v_n)$ such that $n \geqslant 1$ and $v_i \in \Sigma^*$ for all i. For any $(x_1, ..., x_i)$, $(y_1, ..., y_j) \in \mathrm{Seq}(\Sigma^*)$, $(x_1, ..., x_i) \cdot (y_1, ..., y_j)$ is the sequence $(x_1, ..., x_i, y_1, ..., y_j)$. For any $i \geqslant 1$, $(x_1, ..., x_n)^1 = (x_1, ..., x_n)$, and $(x_1, ..., x_n)^i = (x_1, ..., x_n)^{i-1} \cdot (x_1, ..., x_n)$ for $i \geqslant 2$.

For presenting the main lemma, we need the following D-automaton $\mathscr{B} = \langle \Sigma, Q, M, S, F, d \rangle$ in which S and F are not specified since they are irrelevant for us.

DEFINITION 5.14. $\mathscr{B} = \langle \Sigma, Q, M, S, F, d \rangle$ is a D-automaton which is defined as follows.

(1) $Q=\{(\alpha(\delta(Wb)),q) \mid W\in\Delta^*,\ b\in\Delta,\ \delta(b)=R_j,\ 1\leqslant j\leqslant m,\ q\in Q_j,$ and for some $W_0, W_1\in\Delta^*$, $\delta(W_0WbW_1)\subset R\}$;

(2) For any $(t,q)\in Q$ with $t=\alpha(\delta(Wb_i))$, $1\leqslant i\leqslant m$, and $q\in Q_i$, and $a\in\Sigma$, the following (2.1)–(2.3) hold.

(2.1) If $M_i(q,a)=\varnothing$, then $M((t,q),a)=\varnothing$;

(2.2) If $M_i(q,a)\neq\varnothing$ and $M_i(q,a)\notin F_i$, then $M((t,q),a)=\{(t,M_i(q,a))\}$ and $d((t,q),a,(t,M_i(q,a)))=0$;

(2.3) If $M_i(q,a)\neq\varnothing$ and $M_i(q,a)\in F_i$, then $M((t,q),a)=\{(t,M_i(q,a))\}\cup\{(\alpha(\delta(Wb_ib_j)),s_j) \mid 1\leqslant j\leqslant m$, and for some $W_0, W_1\in\Delta^*$, $\delta(W_0Wb_ib_jW_1)\subset R\}$, $d((t,q),a,(t,M_i(q,a)))=0$, and $d((t,q),a,(\alpha(\delta(Wb_ib_j)),s_j))=1$ for each j, $1\leqslant j\leqslant m$, which satisfies the above condition.

DEFINITION 5.15. For any $w\in\Sigma^*$, M_{d0w} is the function from Q to 2^Q such that for any $q\in Q$, $M_{d0w}(q)=\{q'\in Q \mid d(q,w,q')=0\}$.

Remark 5.2. Since $\mathscr{B}$ is 0-deterministic, for any $w\in\Sigma^*$ and $q\in Q$, it holds that $\#M_{d0w}(q)\leqslant 1$. When $M_{d0w}(q)=\{q'\}$, we write $M_{d0w}(q)=q'$.

The proof of the following proposition is clear by definition.

PROPOSITION 5.4. *For any* $W, W'\in\Delta^+$, $q\in Q_i$, $q'\in Q_j$, $1\leqslant i,j\leqslant m$, *and* $w\in\Sigma^*$, *if* $(\alpha(\delta(W)),q)$, $(\alpha(\delta(W')),q')\in Q$, *then* $d((\alpha(\delta(W)),q),w,(\alpha(\delta(W')),q'))=0$ *iff* $i=j$, $q'=M_i(q,w)$ *and for some* $W_0\in\Delta^*$, $\alpha(\delta(W))=\alpha(\delta(W'))=\alpha(\delta(W_0b_i))$.

DEFINITION 5.16. For any $w\in\Sigma^*$, the integer $I(w)$ is defined by, $I(w)=\#(Q-M_{d0}(Q,w))$.

PROPOSITION 5.5. $I(\lambda)=0$ *and for any* $w\in\Sigma^*$, $0\leqslant I(w)\leqslant \#Q$.

LEMMA 5.3. *For any* $x,y\in\Sigma^*$, $I(x), I(y)\leqslant I(xy)$.

Proof. We note the following.

(1) $M_{d0}(Q,xy)=M_{d0}(M_{d0}(Q,x),y)$. Thus $\#M_{d0}(Q,xy)\leqslant\#M_{d0}(Q,x)$.

(2) $M_{d0}(Q,xy)=M_{d0}(M_{d0}(Q,x),y)\subset M_{d0}(Q,y)$. Thus $\#M_{d0}(Q,xy)\leqslant\#M_{d0}(Q,y)$. ∎

PROPOSITION 5.6. *For any* $x,y,z\in\Sigma^*$, $I(y)\leqslant I(xyz)$.

Proof. By the lemma, $I(y)\leqslant I(xy)\leqslant I(xyz)$. ∎

PROPOSITION 5.7. *For any* $x, y \in \Sigma^*$, *if* $I(xy) = I(y)$, *then* $M_{d0}(Q, xy) = M_{d0}(Q, y)$.

Proof. Note that $M_{d0}(Q, xy) = M_{d0}(M_{d0}(Q, x), y) \subset M_{d0}(Q, y)$. Thus if $I(xy) = I(y)$, then $M_{d0}(Q, xy) = M_{d0}(Q, y)$. ∎

The following proposition consitutes the base for the inductive proof of the main lemma in Section 7.

PROPOSITION 5.8. *If* $I(w) = 0$ *for* $w \in \Sigma^*$, *then for any* $q \in Q$ *and* $q' \in M(q, w)$, $d(q, w, q') \leqslant \# Q$.

Proof. Assume that $I(w) = 0$. By Proposition 5.6, for any prefix x of w, $I(x) = 0$, i.e., $M_{d0}(Q, x) = Q$. Thus for any $q \in Q$, there exists $p \in Q$ such that $q = M_{d0}(p, x)$. Now let $q \in Q$, $q' \in M(q, w)$ and assume that $d(q, w, q') \geqslant \# Q + 1$. Then there exist $x_1, ..., x_{n+1} \in \Sigma^+$ and $q_0, q_1, ..., q_{n+1} \in Q$ such that $n = \# Q$, $w = x_1 \cdots x_{n+1}$, $q_0 = q$, $q_{n+1} = q'$, $d(q_0, x_1, q_1) + d(q_1, x_2, q_2) + \cdots + d(q_n, x_{n+1}, q_{n+1}) = d(q, w, q')$, and for each i, $0 \leqslant i \leqslant n$, $0 < d(q_i, x_{i+1}, q_{i+1})$. From the above observation, for each q_i, $1 \leqslant i \leqslant n+1$, there exists $p_i \in Q$ such that $q_i = M_{d0}(p_i, x_1 \cdots x_i)$. Since $n = \# Q$, there exist i, j, $1 \leqslant i < j \leqslant n+1$, such that $p_i = p_j$. But then $q_j = M_{d0}(p_j, x_1 \cdots x_j) = M_{d0}(p_i, x_1 \cdots x_j) = M_{d0}(M_{d0}(p_i, x_1 \cdots x_i), x_{i+1} \cdots x_j) = M_{d0}(q_i, x_{i+1} \cdots x_j)$, and $d(q_i, x_{i+1} \cdots x_j, q_j) = 0$, a contradiction. ∎

DEFINITION 5.17. $\equiv$ is the equivalence relation over $\mathrm{Seq}(\Sigma^*)$ such that for any $(x_1, ..., x_i)$, $(y_1, ..., y_j) \in \mathrm{Seq}(\Sigma^*)$, $(x_1, ..., x_i) \equiv (y_1, ..., y_j)$ iff $f_{(x_1, ..., x_i)} = f_{(y_1, ..., y_j)}$, $M_{d0 x_1 \cdots x_i} = M_{d0 y_1 \cdots y_j}$, *and* $\alpha(x_1 \cdots x_i) = \alpha(y_1 \cdots y_j)$.

PROPOSITION 5.9. $\equiv$ *is a congruence relation.*

Proof. Let $(v_1, ..., v_{i_1})$, $(w_1, ..., w_{i_2})$, $(x_1, ..., x_{i_3})$, $(y_1, ..., y_{i_4}) \in \mathrm{Seq}(\Sigma^*)$, and assume that $(v_1, ..., v_{i_1}) \equiv (w_1, ..., w_{i_2})$ and $(x_1, ..., x_{i_3}) \equiv (y_1, ..., y_{i_4})$. Then $f_{(v_1, ..., v_{i_1}, x_1, ..., x_{i_3})} = f_{x_{i_3}} \cdots f_{x_{i_1}} \cdot f_{v_{i_1}} \cdots f_{v_1} = f_{y_{i_4}} \cdots f_{y_1} \cdot f_{w_{i_2}} \cdots f_{w_1}$, $M_{d0 v_1 \cdots v_{i_1} x_1 \cdots x_{i_3}} = M_{d0 w_1 \cdots w_{i_2} y_1 \cdots y_{i_4}}$, and $\alpha(v_1 \cdots v_{i_1} x_1 \cdots x_{i_3}) = \alpha(w_1 \cdots w_{i_2} y_1 \cdots y_{i_4})$. ∎

The following lemma is clear by Lemma 5.2 and definition.

LEMMA 5.4. $\# \Sigma^* / \equiv \; < g_5$, *where* $g_5 = g_4 \cdot (\# Q + 1)^{\# Q} \cdot \# \mathscr{M}$.

PROPOSITION 5.10. *For any* $x_1, x_2, ..., x_{g_6} \in \Sigma^*$, *where* $g_6 = (g_5)^2$, *there exist* i, j, $1 \leqslant i < j < g_6$, *such that* $(x_1, ..., x_i) \equiv (x_1, ..., x_j)$, *and* $(x_{i+1}, ..., x_{g_6}) \equiv (x_{j+1}, ..., x_{g_6})$.

Proof. From Lemma 5.4, we can see that there exist $i_1, i_2, ..., i_{g_5}$, $1 \leqslant i_1 < i_2 < \cdots < i_{g_5} < g_6$, such that $(x_1, ..., x_{i_1}) \equiv (x_1, ..., x_{i_k})$ for all

k, $1 \leqslant k \leqslant g_5$. Now we consider the set of sequences of words $\{(x_{i_k+1}, ..., x_{g_6}) \mid 1 \leqslant k \leqslant g_5\}$. There exist n_0, n_1, $1 \leqslant n_0 < n_1 \leqslant g_5$, such that $(x_{i_{n_0}+1}, ..., x_{g_6}) \equiv (x_{i_{n_1}+1}, ..., x_{g_6})$. ∎

DEFINITION 5.18. (1) The class of WE$(+, \Sigma)$ of (word, $+$)-expressions over Σ is the smallest class of expressions which satisfies the following.

(i) $a \in \Sigma$ is in WE$(+, \Sigma)$;

(ii) If E_1 and E_2 are in WE$(+, \Sigma)$, then so are $E_1 E_2$ and $(E_1)^+$.

(2) For any $E \in$ WE$(+, \Sigma)$, $|E|$ denotes the regular language over Σ *which is defined by* $|a| = \{a\}$ for $a \in \Sigma$, $|E_1 E_2| = |E_1| \cdot |E_2|$, and $|E^+| = |E|^* - \{\lambda\}$.

DEFINITION 5.19. For any $E \in$ WE$(+, \Sigma)$ and $k \geqslant 1$, $E(k)$ denotes the word over Σ which is obtained from E by replacing each occurrence of $+$ with k.

EXAMPLE 5.3. Let $\Sigma = \{0, 1\}$, and $E = 0(11)^+(0^+1)^+0$. Then $|E| = |011(11)^*(00^*1)(00^*1)^*0|$, and for each $k \geqslant 1$, $E(k) = 0(11)^k\,(0^k1)^k0$.

PROPOSITION 5.11. *For any* $E_1, E_2 \in$ WE$(+, \Sigma)$, *and* $k \geqslant 1$, $E_1 E_2(k) = E_1(k)\, E_2(k)$.

DEFINITION 5.20. For any $w \in \Sigma^+$ with $I(w) > 0$, $I(w)$-decomposition of w, $D(w, I(w))$, is a sequence $(x_1, a_1, x_2, a_2, ..., x_n, a_n, x_{n+1})$ such that (1) $w = x_1 a_1 \cdots x_n a_n x_{n+1}$, (2) for each i, $1 \leqslant i \leqslant n+1$, $x_i \in \Sigma^*$, and $I(x_i) < I(w)$, and (3) for each j, $1 \leqslant j \leqslant n$, $a_j \in \Sigma$ and $I(x_j a_j) = I(w)$. The length of $D(w, I(w))$ is n.

PROPOSITION 5.12. *For any* $w \in \Sigma^+$ *with* $I(w) > 0$, $D(w, I(w))$ *is unique and its length is positive.*

DEFINITION 5.21. For any $w \in \Sigma^*$, Dec(w) is the set of decompositions of w defined by Dec$(w) = \{(x_1, ..., x_n) \mid n \geqslant 1,\ x_i \in \Sigma^*$ for all i, and $w = x_1 \cdots x_n\}$.

DEFINITION 5.22. For any $w \in \Sigma^*$, $S(w)$ and $S(w, g_6)$ are two decompositions of w in Dec(w) which are defined as follows.

(1) If $I(w) = 0$, then $S(w) = S(w, g_6) = (w)$;

(2) If $I(w) > 0$, then let $D(w, I(w)) = (x_1, a_1, ..., x_n, a_n, x_{n+1})$. Then $S(w) = S(x_1) \cdot (a_1) \cdots S(x_n) \cdot (a_n) \cdot S(x_{n+1})$, where $\cdot$ is the operation defined in Definition 5.13, and $S(w, g_6)$ is defined as follows.

(2.1) If $n<g_6$, then $S(w, g_6)=(w)$;

(2.2) If $kg_6 \leqslant n<(k+1) g_6$ for some $k \geqslant 1$, then $S(w, g_6)=(w_1, w_2, \ldots, w_{k+1})$, where $w=w_1 \cdots w_{k+1}$ and for each i, $1 \leqslant i \leqslant k$, $w_i=x_{(i-1)g_6+1} a_{(i-1)g_6+1} \cdots x_{ig_6} a_{ig_6}$.

Proposition 5.13. *For any $w \in \Sigma^+$ with $I(w)>0$, the following holds. Let $D(w, I(w))=(x_1, a_1, \ldots, x_n, a_n, x_{n+1})$. Then for any i, j, $1 \leqslant i \leqslant j \leqslant n$, $S(x_i a_i \cdots x_j a_j)=S(x_i) \cdot (a_i) \cdots S(x_j) \cdot (a_j)$.*

Definition 5.23. k_0 is the integer defined by, $k_0=\max\{l(E)+1, \#Q+1 \mid E \in \xi(E_0)\}$.

Definition 5.24. For any $w \in \Sigma^*$, the sequence $S(w, g_6, \equiv, k_0)$, the (word, +)-expression E_w, and the sequence $S(E_w, k_0)$ are defined inductively as follows.

(1) If $I(w)=0$, then $S(w, g_6, \equiv, k_0)=(w)$, $E_w=w$, and $S(E_w, k_0)=(w)$;

(2) If $I(w)>0$, then let $D(w, I(w))=(w_1, a_1, \ldots, x_n, a_n, x_{n+1})$.

(2.1) If $n<g_6$, then $S(w, g_6, \equiv, k_0)=(w)$, $E_w=E_{x_1} a_1 E_{x_2} a_2 \cdots E_{x_n} a_n E_{x_{n+1}}$, and $S(E_w, k_0)=(E_{x_1}(k_0), a_1, E_{x_2}(k_0), a_2, \ldots, E_{x_n}(k_0), a_n, E_{x_{n+1}}(k_0))$;

(2.2) If $n=g_6$, then let i, j be the least integers such that $1 \leqslant i< j \leqslant g_6$, $(E_{x_1}(k_0), a_1, E_{x_2}(k_0), a_2, \ldots, E_{x_i}(k_0), a_i) \equiv (E_{x_1}(k_0), a_1, \ldots, E_{x_j}(k_0), a_j)$, and $(E_{x_{i+1}}(k_0), a_{i+1}, \ldots, E_{x_{g_6}}(k_0), a_{g_6}) \equiv (E_{x_{j+1}}(k_0), a_{j+1}, \ldots, E_{x_{g_6}}(k_0), a_{g_6})$. (By Proposition 5.10, such i and j exist.) Then $S(w, g_6, \equiv, k_0)=((i, j), (x_1 a_1 \cdots x_i a_i, x_{i+1} a_{i+1} \cdots x_j a_j, x_{j+1} a_{j+1} \cdots x_{g_6} a_{g_6}))$, $E_w=E_{x_1} a_1 \cdots E_{x_i} a_i (E_{x_{i+1}} a_{i+1} \cdots E_{x_j} a_j)^+ E_{x_{j+1}} a_{j+1} \cdots E_{x_n} a_n E_{x_{n+1}}$, and $S(E_w, k_0)=(E_{x_1}(k_0), a_1, \ldots, E_{x_i}(k_0), a_i) \cdot (E_{x_{i+1}}(k_0), a_{i+1}, \ldots, E_{x_j}(k_0), a_j)^{k_0} \cdot (E_{x_{j+1}}(k_0), a_{j+1}, \ldots, E_{x_n}(k_0), a_n, E_{x_{n+1}}(k_0))$.

(2.3) If $n>g_6$, then let $S(w, g_6)=(w_1, \ldots, w_r)$. Then $S(w, g_6, \equiv, k_0) =(S(w_1, g_6, \equiv, k_0), S(w_2, g_6, \equiv, k_0), \ldots, S(w_r, g_6, \equiv, k_0))$, $E_w=E_{w_1} E_{w_2} \cdots E_{w_r}$, and $S(E_w, k_0)=S(E_{w_1}, k_0) \cdot S(E_{w_2}, k_0) \cdots S(E_{w_r}, k_0)$.

Proposition 5.14. *For any $w \in \Sigma^*$, $S(w)$, $S(w, g_6) \in \operatorname{Dec}(w)$, and $S(E_w, k_0) \in \operatorname{Dec}(E_w(k_0))$.*

Before presenting the main lemma, Theorem 5.1, and an algorithm for determining relative star height, we shall present several properties of the congruence relation $\equiv$, and functions in $F\eta(\xi(E_0), R)$, some of which we need in the final section.

Notation. For any $f_1, f_2 \in F\eta(\xi(E_0), R)$, $f_1 \supset f_2$ denotes that for any $(u, q) \in \eta(\xi(E_0), R)$, $f_1(u, q) \supset f_2(u, q)$.

Proposition 5.15. *For any $x_1, \ldots, x_n \in \Sigma^*$, $f_{(x_1, \ldots, x_n)} \supset f_{x_1 \cdots x_n}$.*

Proof. The proof is by induction on n. If $n=1$, then the assertion is trivial. Let $n>1$. If one of $x_1 \cdots x_{n-1}$ or x_n is λ, then the assertion is clear by induction and Proposition 5.3. Assume that $x_1 \cdots x_{n-1}$, $x_n \neq \lambda$. Let $(u, q) \in \eta(\xi(E_0), R)$, and consider any $(u', q', c) \in f_{x_1 \cdots x_n}(u, q)$. There exist $E \in \xi(E_0)$, and a path $P=(i_1, ..., i_k)$ of E such that $\zeta(E, i_1)=u$, $\zeta(E, i_k)=u'$, and P strictly spells $x_1 \cdots x_n$ with (q, q', c). Then there exist r, $1 \leqslant r \leqslant k$, $c_1, c_2 \in \{0, 1, 2, 3\}$, and $q'' \in Q_j$, $1 \leqslant j \leqslant m$, such that (1) $\Delta_E(i_r)=b_j$, (2) the path $P_1=(i_1, ..., i_r)$ strictly spells $x_1 \cdots x_{n-1}$ with (q, q'', c_1), and (3) the path $P_2=(i_r, ..., i_k)$ strictly spells x_n with (q'', q', c_2). Moreover the following hold.

(4) $c=0$ iff $k=1$ iff $c_1=c_2=0$;

(5) $c=1$ iff $k>1$, $c_1, c_2 \leqslant 1$, and at least one of c_1 and c_2 equals one;

(6) $c=2$ iff $k>1$, $c_1, c_2 \leqslant 2$, and at least one of c_1 and c_2 equals two;

(7) $c=3$ iff $k>1$ and at least one of c_1 and c_2 equals three.

Thus $c=\max\{c_1, c_2\}$. Now put $u_1=\zeta(E, i_r)$. Then $(u_1, q'', c_1) \in f_{x_1 \cdots x_{n-1}}(u, q)$, and $(u', q', c_2) \in f_{x_n}(u_1, q'')$. By induction, $(u_1, q'', c_1) \in f_{(x_1, ..., x_{n-1})}(u, q)$, and by definition, $(u', q', c) \in f_{(x_1, ..., x_n)}(u, q)$. ∎

PROPOSITION 5.16. *For any $w \in \Sigma^*$, the following hold.*

(1) $f_{S(w)} \supset f_{S(E_w, k_0)} \supset f_{E_w(k_0)}$;

(2) *If* $I(w)>0$, $D(w, I(w))=(x_1, a_1, ..., x_n, a_n, x_{n+1})$, $n=g_6$ *and* $x_{n+1}=\lambda$, *then for any* $i \geqslant 1$, *the following hold, where* $S(w, g_6, \equiv, k_0)=((r_0, r_1), (w_1, w_2, w_3))$.

(2.1) $f_{S(w_1)}, f_{S(w_1 w_2)} \supset f_{S(E_{w_1}, k_0)} = f_{S(E_{w_1}, k_0) \cdot (S(E_{w_2}, k_0))^i} \supset f_{E_{w_1}(k_0)}$, $f_{E_{w_1}(k_0)(E_{w_2}(k_0))^i}$;

(2.2) $f_{S(w_3)}, f_{S(w_2 w_3)} \supset f_{S(E_{w_3}, k_0)} = f_{(S(S_{w_2}, k_0))^i \cdot S(E_{w_3}, k_0)} \supset f_{E_{w_3}(k_0)}$, $f_{E_{w_2}(k_0)(E_{w_2}(k_0))^i}$;

Proof. The proof is by induction on $I(w)$. If $I(w)=0$, then $E_w=w$, $S(E_w, k_0)=(w)$, and the assertion is trivial. Let $I(w)>0$, and $D(w, I(w))=(x_1, a_1, ..., x_n, a_n, x_{n+1})$. We shall first prove (2). So assume that $n=g_6$ and $x_{n+1}=\lambda$. By induction, it holds that $f_{S(x_k)} \supset f_{E_{x_k}(k_0)}$ for all k, $1 \leqslant k \leqslant n$. By definition, $S(E_{w_1}, k_0)=(E_{x_1}(k_0), a_1, ..., E_{x_{r_0}}(k_0), a_{r_0})$, $S(E_{w_2}, k_0)=(E_{x_{r_0+1}}(k_0), a_{r_0+1}, ..., E_{x_{r_1}}(k_0), a_{r_1})$ and $S(E_{w_3}, k_0)=(E_{x_{r_1+1}}(k_0), a_{r_1+1}, ..., E_{x_n}(k_0), a_n)$. Now we can see easily from the above argument and Propositions 5.2, 5.15 that for all $i \geqslant 1$,

$$f_{S(w_1)}=f_{S(x_1) \cdot (a_1) \cdots S(x_{r_0}) \cdot (a_{r_0})} \supset f_{(E_{x_1}(k_0), a_1, ..., E_{x_{r_0}}(k_0), a_{r_0})}$$
$$=f_{S(E_{w_1}, k_0)}=f_{S(E_{w_1}, k_0) \cdot (S(E_{w_2}, k_0))^i} \supset f_{E_{w_1}(k_0)}, f_{E_{w_1}(k_0)(E_{w_2}(k_0))^i}.$$

Similarly (2.2) holds. Now we shall prove (1). We consider three cases.

Case (1): $n<g_6$. By induction, it holds that $f_{S(x_k)} \supset f_{E_{x_k}(k_0)}$ for all k, $1 \leqslant k \leqslant n+1$. Since $S(E_w, k_0)=(E_{x_1}(k_0), a_1, ..., E_{x_n}(k_0), a_n, E_{x_{n+1}}(k_0))$ and $S(w)=S(x_1)\cdot(a_1)\cdots S(x_n)\cdot(a_n)\cdot S(x_{n+1})$, the assertion holds.

Case (2): $n=g_6$. Let $S(w, g_6, \equiv, k_0)=((r_0, r_1), (w_1, w_2, w_3))$. From the above proof of (2), we can see that $f_{S(x_1 a_1 \cdots x_n a_n)} \supset f_{S(E_{w_1},k_0)\cdot(S(E_{w_2},k_0))^{k_0}\cdot S(E_{w_3},k_0)} = f_{S(E_{w_1},k_0)\cdot S(E_{w_2},k_0)\cdot S(E_{w_3},k_0)} \supset f_{E_{w_1 w_2 w_3}(k_0)}$. So it holds that $f_{S(w)} = f_{S(x_1 a_1 \cdots x_n a_n)\cdot S(x_{n+1})} \supset f_{S(E_{w_1},k_0)\cdot(S(E_{w_2},k_0))^{k_0}\cdot S(E_{w_3},k_0)\cdot(E_{x_{n+1}}(k_0))} = f_{S(E_w,k_0)} \supset f_{E_w(k_0)}$.

Case (3): $n>g_6$. Let $S(w, g_6)=(w_1, ..., w_k)$. From Cases (1), (2), we have $f_{S(w_i)} \supset f_{S(E_{w_i},k_0)} \supset f_{E_{w_i}(k_0)}$ for all i, $1 \leqslant i \leqslant k$. Thus it holds that $f_{S(w)}=f_{S(w_1)\cdot S(w_2)\cdots S(w_k)} \supset f_{S(E_{w_1},k_0)\cdot S(E_{w_2},k_0)\cdots S(E_{w_k},k_0)} = f_{S(E_w,k_0)} \supset f_{E_w(k_0)}$. ∎

PROPOSITION 5.17. *For any $v, w \in \Sigma^*$ and any $k \geqslant 1$, the following* (1) *and* (2) *hold.*

(1) $\alpha(w)=\alpha(E_w(k))$, $I(w)=I(E_w(k))$, *and* $M_{d0w}=M_{d0E_w(k)}$;

(2) *If* $M_{d0v}=M_{d0vw}$, *then* $M_{d0v}=M_{d0vw^k}$.

Proof. (2) is clear by definition. We shall prove (1) by induction on $I(w)$. If $I(w)=0$, then $E_w(k)=w$, and the assertion is trivial. Let $I(w)>0$, and $D(w, I(w))=(x_1, a_1, ..., x_n, a_n, x_{n+1})$. By induction, $\alpha(x_i)=\alpha(E_{x_i}(k))$, $I(x_i)=I(E_{x_i}(k))$, and $M_{d0x_i}=M_{d0E_{x_i}(k)}$ for all i, $1 \leqslant i \leqslant n+1$. Now we consider the case where $n=g_6$ and $x_{n+1}=\lambda$. (The proof for other cases is similar.) Let $S(w, g_6, \equiv, k_0)=((r_0, r_1), (w_1, w_2, w_3))$. It is easy to see that $\alpha(w_i)=\alpha(E_{w_i}(k))$, $I(w_i)=I(E_{w_i}(k))$ and $M_{d0w_i}=M_{d0E_{w_i}(k)}$ for all i, $1 \leqslant i \leqslant 3$. By definition, $E_w(k)=E_{w_1}(k)(E_{w_2}(k))^k E_{w_3}(k)$. Now the assertions are clear by definition and (2). ∎

DEFINITION 5.25. For any $i \in \{0, 1, ..., \#Q\}$, the integer $o_1(i)$ is defined inductively as follows.

(1) $o_1(0)=\#Q$;

(2) For $i>0$, $o_1(i)=4\cdot(\#Q+2)\cdot g_6\cdot(o_1(i-1)+1)$.

PROPOSITION 5.18. $o_1(\#Q) \leqslant 8^{\#Q}\cdot(\#Q+2)^{\#Q}\cdot(g_6)^{\#Q}\cdot\#Q \leqslant (g_6+\#Q+2)^{4\cdot\#Q}$.

DEFINITION 5.26. For any $i \in \{0, 1, ..., \#Q\}$, and $j \in \{1, 2, ..., g_3\}$, two integers $o_2(i)$ and $g_7(i,j)$ are defined inductively as follows.

(1) $o_2(0)=k_0\cdot o_1(1)$;

(2) For $i \in \{0, 1, ..., \#Q\}$ and $j \in \{1, 2, ..., g_3\}$,

(2.1) $g_7(i, 1) = ((g_0(n_1))^2 + 2)^{3\cdot(h(E_0)+1)}$, where $n_1 = 4^{g_6+4}\cdot(h(E_0)+1)^{g_6+2}\cdot o_2(i-1)$;

(2.2) For $2 \leqslant j \leqslant g_3$, $g_7(i,j) = (4\cdot(h(E_0)+1)\cdot(n_2+2)\cdot g_7(i-1, j-1))^{3(h(E_0)+1)}$, where $n_2 = (g_0(n_3))^2$ and $n_3 = 4\cdot(h(E_0)+1)\cdot g_7(i-1, j-1)$;

(2.3) For $i > 0$, $o_2(i) = 4^{k_0+1}\cdot(h(E_0)+1)^{k_0+1}\cdot g_7(i-1, g_3)$.

PROPOSITION 5.19. *For any* $i \in \{0, 1, ..., \#Q\}$, $o_2(i) \geqslant o_1(i+1)$.

Proof. The proof is by induction on i. When $i = 0$, the assertion is clear. For $i > 0$, $o_2(i) \geqslant g_7(i-1, 1) \geqslant ((4^{g_6+4}\cdot(h(E_0)+1)^{g_6+2}\cdot o_2(i-1))^2 + 2)^{3\cdot(h(E_0)+1)} \geqslant ((4^{g_6+4}\cdot(h(E_0)+1)^{g_6+2}\cdot o_1(i))^2 + 2)^{3\cdot(h(E_0)+1)} \geqslant o_1(i+1)$, where we note that $g_6 \geqslant \#Q + 2$ from Proposition 5.10. ▮

DEFINITION 5.27. Let E be in SRE(Δ) and $P = (i_1, ..., i_n)$ be a path of E. Then P is a path of degree zero iff either $n = 1$ or $n \geqslant 2$ and for any j, $1 \leqslant j \leqslant n-1$, $i_{j+1} = i_j + 1$ and none of symbols (,), and $\cup$ occurs in E between positions i_j and i_{j+1}. $v_0(P)$ is defined by: $v_0(P) = \max\{l(P') + 1 \mid P'$ is a subpath of degree zero of $P\}$.

EXAMPLE 5.4. Let $\Delta = \{a, b\}$, and $E = ba \cup ab^*a(aba \cup b(ab \cup a)^*)^*\, ba$. Some examples of paths of degree zero are (1, 2), (4), and (6, 7). None of the paths (3, 4) and (9, 10) is of degree zero. $v_0(1, 2) = 2$, $v_0(6, 7, 8) = 3$, and $v_0(9, 12) = 1$, etc.

Now we shall present the main lemma whose proof will be presented in the final section.

MAIN LEMMA. *Let* $w \in \Sigma^*$, $E \in \xi(E_0)$, $(u_0, q_0), (u_1, q_1) \in \eta(\xi(E_0), R)$, $c \in \{0, 1, 2, 3\}$, *and* P_1 *be a path of E. Assume that the following* (1) *and* (2) *hold.*

(1) $\zeta(E, \mathrm{in}(P_1)) = u_0$ *and* $\zeta(E, \mathrm{fn}(P_1)) = u_1$;

(2) P_1 *strictly spells* $E_w(k_0)$ *with* (q_0, q_1, c).

Then there exist $E' \in \mathrm{CSRE}(\Delta)$ *and a path* P_2 *of* E' *for which the following* (3)–(8) *hold.*

(3) $h(E') \leqslant h(E)$ *and* $\delta(|E'|) \subset R$;

(4) $\zeta(E', \mathrm{in}(P_2)) = u_0$ *and* $\zeta(E', \mathrm{fn}(P_2)) = u_1$;

(5) P_2 *strictly spells* w *with* (q_0, q_1, c);

(6) *If* $\mathrm{in}(P_1) \in \mathrm{ip}(E)$, *then* $\mathrm{in}(P_2) \in \mathrm{ip}(E')$, *and if* $\mathrm{fn}(P_1) \in \mathrm{fp}(E)$, *then* $\mathrm{fn}(P_2) \in \mathrm{fp}(E')$;

(7) $v_0(P_2) \leqslant o_1(l(w))$;

(8) $l(E') \leqslant o_2(l(w))$.

PROPOSITION 5.20. *Let* $w \in R$, $E \in \mathrm{CSRE}(\Delta)$, *and* P *be a path of* E. *Assume that* (1) $\delta(|E|) \subset R$, (2) $\mathrm{in}(P) \in \mathrm{ip}(E)$, *and* $\mathrm{fn}(P) \in \mathrm{fp}(E)$, *and* (3) P *strictly spells* w *with* (s_i, q, c), *where* $\Delta_E(\mathrm{in}(P)) = b_i$, $\Delta_E(\mathrm{fn}(P)) = b_j$ *and* $q \in F_j$, $1 \leqslant i, j \leqslant m$. *Then there exists* $E' \in \mathrm{CSRE}(\Delta)$ *such that* (4) $\delta(|E'|) \subset R$, (5) $w \in \delta(|E'|)$, (6) $h(E') \leqslant h(E)$, (7) $v(E') = v_0(P)$, *and* (8) $l(E') \leqslant l(E)$.

Proof. We consider two cases.

Case (1): $h(E) = 0$. Then $E \in \Delta^+$, $\Delta_E(P) = E$, and assertion is clear.

Case (2): $h(E) > 0$. Then E is of the form $E = W_1 H_1^* W_2 H_2^* \cdots W_n H_n^* W_{n+1}$. Since P is a path of E and (2) and (3) hold, there exist $x_1, ..., x_{n+1} \in \Sigma^*$ and $y_1, ..., y_n \in \Sigma^+$ such that (i) $w = x_1 y_1 \cdots x_n y_n x_{n+1}$, (ii) $x_i \in \delta(W_i)$ and $x_i = \lambda$ iff $W_i = \lambda$ for all i, $1 \leqslant i \leqslant n+1$, and (3) $y_j \in \delta(|H_j^*|) \cap \Sigma^+$ for all j, $1 \leqslant j \leqslant n$. For each H_j^* and y_j, $1 \leqslant j \leqslant n$, we can obtain the corresponding star expression $(H_j')^*$ and a path P_j of $(H_j')^*$ as in the proof of Proposition 3.5 such that P_j is a whole path of $(H_j')^*$, and P_j strictly spells y_j, where it also holds that $v(H_j') = v_0(P_j)$, $h(H_j') \leqslant h(H_j)$, $|H_j'| \subset |H_j|$, and $l(H_j') \leqslant l(H_j)$. Now we put $E' = W_1(H_1')^* W_2(H_2')^* \cdots W_n(H_n')^* W_{n+1}$. Then the assertion is clear. ∎

From the main lemma and Proposition 5.20, we have the following theorem.

THEOREM 5.1. *Let* R, $\mathscr{C}$, Δ, δ, $\mathscr{B}$, *and* g_6 *be as above, and assume that* $h_r(R, \mathscr{C}) < \infty$. *Then there exists* $E \in \mathrm{SRE}(\Delta)$ *for which the following* (1)–(3) *hold.*

(1) $\delta(|E|) = R$;

(2) $h(E) = h_r(R, \mathscr{C})$;

(3) $v(E) \leqslant o_1(\#Q) \leqslant (g_6 + \#Q + 2)^{4 \cdot \#Q}$.

Proof. Assume that $h_r(R, \mathscr{C}) < \infty$. Then there exists $E_0 \in \mathrm{SRE}(\Delta)$ such that $\delta(|E_0|) = R$ and $h_r(R, \mathscr{C}) = h(E_0)$. $\xi(E_0)$ is defined as in Definition 5.6. Consider any $w \in R \cap \Sigma^+$. Since $\alpha(w) = \alpha(E_w(k_0))$, $E_w(k_0) \in R$. By Proposition 5.1, there exist $E \in \xi(E_0)$ and a path $P = (i_1, ..., i_n)$ of E such that (i) P is a whole path of E, (ii) $E_w(k_0) \in \delta(\Delta_E(P))$, and (iii) for some $x_1, ..., x_n \in \Sigma^+$, $w = x_1 \cdots x_n$ and $x_j \in \delta(\Delta_E(i_j))$ for all j, $1 \leqslant j \leqslant n$. By the main lemma and Proposition 5.20, we can see that there exists $E(w) \in \mathrm{CSRE}(\Delta)$ such that $\delta(|E(w)|) \subset R$, $w \in \delta(|E(w)|)$, $h(E(w)) \leqslant h(E_0)$, $v(E(w)) \leqslant o_1(I(w)) \leqslant o_1(\#Q)$, and $l(E(w)) \leqslant o_2(I(w)) \leqslant o_2(\#Q)$. By Proposition 3.4, the set $\{E(w) \mid w \in R \cap \Sigma^+\}$ is finite. Now we put $E' = E_{10} \cup E_1 \cup \cdots \cup E_r$, where $E_{10} = \lambda$ if $\lambda \in R$, $E_{10} = \varnothing$ if $\lambda \notin R$, and $\{E_1, ..., E_r\} = \{E(w) \mid w \in R \cap \Sigma^+\}$. Then $h(E') = h(E_0) = h_r(R, \mathscr{C})$, $\delta(|E'|) = R$, and $v(E') \leqslant o_1(\#Q) \leqslant (g_6 + \#Q + 2)^{4 \cdot \#Q}$. ∎

DEFINITION 5.28. $g_8(R, \mathscr{C})$ denotes the integer obtained from $(g_6 + \# Q + 2)^{4 \cdot \# Q}$ by replacing each occurrence of $h(E_0)$ in g_6 with $r(\mathscr{A}_\Delta)$.

From Algorithms 4.1, 4.2 and Theorem 5.1, we obtain the following algorithm for determining relative star height.

ALGORITHM 5.1. *Let R, $\mathscr{C}$, Δ, and δ be as above.*

1. *By Algorithm* 4.1, *decide whether or not $h_r(R, \mathscr{C}) = \infty$. If $h_r(R, \mathscr{C}) < \infty$, then proceed to* 2.

2. *By Algorthm* 4.2, *decide whether or not $h_r(R, \mathscr{C}) = 0$. If $h_r(R, \mathscr{C}) > 0$, then proceed to* 3.

3. *Calculate $g_8(R, \mathscr{C})$ and obtain the following finite class $\mathscr{C}_1$ of regular languages over Σ: $\mathscr{C}_1 = \mathscr{C} \cup \{\delta(|(W_1 \cdots W_n)^*|) | \ n \geqslant 1, \ W_i \in \Delta^+$ and $l(W_i) \leqslant g_8(R, \mathscr{C})$ for all $i\}$.*

Recursively, determine $h_r(R, \mathscr{C}_1)$. Then put $h_r(R, \mathscr{C}) = h_r(R, \mathscr{C}_1) + 1$.

Proof of the correctness of Algorithm 5.1. When $h_r(R, \mathscr{C}) = \infty$ or $h_r(R, \mathscr{C}) = 0$, the algorithm is clearly correct. Assume that $0 < h_r(R, \mathscr{C}) < \infty$. By definition, $\mathscr{C}_1$ is clearly finite, and we can effectively obtain $\mathscr{C}_1$ from R and $\mathscr{C}$. Thus it will suffice to show that $h_r(R, \mathscr{C}) = h_r(\mathrm{R}, \mathscr{C}_1) + 1$. Let $\mathscr{C}_1 = \mathscr{C} \cup \mathscr{C}'$, where $\mathscr{C} = \{R_1, ..., R_m\}$, $\mathscr{C}' = \{\delta(|(W_1 \cup \cdots \cup W_n)^*|) | \ n \geqslant 1$, $W_i \in \Delta^+$ and $l(W_i) \leqslant g_8(R, \mathscr{C})$ for all $i\} = \{|H_1^*|, ..., |H_e^*|\}$, $\Delta_1 = \Delta \cup \Delta'$, where $\Delta = \{b_1, ..., b_m\}$, $\Delta' = \{c_1, ..., c_e\}$, and δ_1 is the substitution from $2^{\Delta_1^*}$ to 2^{Σ^*} such that $\delta_1(b_i) = R_i$, $1 \leqslant i \leqslant m$, and $\delta_1(c_j) = |H_j^*|$, $1 \leqslant j \leqslant e$.

We note that for each $|H_j^*|$, $1 \leqslant j \leqslant e$, there exists $K_j^* \in \mathrm{SRE}(\Delta)$ such that $h(K_j^*) = 1$, $v(K_j^*) \leqslant g_8(R, \mathscr{C})$, and $\delta(|K_j^*|) = |H_j^*|$.

Proof of $h_r(R, \mathscr{C}_1) + 1 \geqslant h_r(R, \mathscr{C})$. Since $\mathscr{C}_1 \supset \mathscr{C}$, it is clear that $h_r(R, \mathscr{C}_1) < \infty$. There exists $E_1 \in \mathrm{SRE}(\Delta_1)$ such that $h(E_1) = h_r(R, \mathscr{C}_1)$ and $\delta_1(|E_1|) = R$. From E_1, we obtain $E_0 \in \mathrm{SRE}(\Delta)$ by replacing each occurrence of c_j in E_1, $1 \leqslant j \leqslant e$, with the corresponding K_j^*. Then it is clear that $\delta(|E_0|) = R$, and $h(E_0) \leqslant h(E_1) + 1$. Thus $h_r(R, \mathscr{C}) \leqslant h(E_0) \leqslant h_r(R, \mathscr{C}_1) + 1$.

Proof of $h_r(R, \mathscr{C}) \geqslant h_r(R, \mathscr{C}_1) + 1$. There exists $E_0 \in \mathrm{SRE}(\Delta)$ such that $\delta(|E_0|) = R$, and $h(E_0) = h_r(R, \mathscr{C})$. By Theorem 5.1, we may assume that $v(E_0) \leqslant (g_6 + \# Q + 2)^{4 \cdot \# Q} \leqslant g_8(R, \mathscr{C})$. Consider any star factor K^* of E_0 with $h(K^*) = 1$. Clearly $\delta(|K^*|) \in \mathscr{C}'$. So for some $c \in \Delta'$, $\delta_1(c) = \delta(|K^*|)$. By replacing each occurrence of K^* in E_0 with the corresponding $c \in \Delta'$, we obtain $E_1 \in \mathrm{SRE}(\Delta_1)$ such that $\delta_1(|E_1|) = R$ and $h(E_1) \leqslant h(E_0) - 1$. Thus $h_r(R, \mathscr{C}) = h(E_0) \geqslant h(E_1) + 1 \geqslant h_r(R, \mathscr{C}_1) + 1$. ▮

Theorem 5.1 also gives an upper bound of the minimum length of $E \in \mathrm{SRE}(\Delta)$ such that $\delta(|E|) = R$ and $h(E) = h_r(R, \mathscr{C})$.

DEFINITION 5.29. Let R, $\mathscr{C}$, Δ, and δ be as above. For each $i \in \{0, 1, ..., r(\mathscr{A}_\Delta)\}$, we define the finite class $\mathscr{C}_i$ of regular languages over Σ and two integers $g_9(i)$ and $g_{10}(i)$ inductively as follows.

(1) For $i = 0$,

(1.1) $\mathscr{C}_0 = \mathscr{C}$;

(1.2) $g_9(0) = 1$;

(1.3) $g_{10}(0) = g_8(R, \mathscr{C}) \cdot (\#\mathscr{C} + 1)^{g_8(R,\mathscr{C})+1}$;

(2) For $i > 0$,

(2.1) $\mathscr{C}_i = \mathscr{C}_{i-1} \cup \{(L_1 \cup \cdots \cup L_n)^* \mid n \geqslant 1$ and $L_j \in \mathscr{C}_{i-1} \cup (\mathscr{C}_{i-1})^2 \cup \cdots \cup (\mathscr{C}_{i-1})^{g_8(R,\mathscr{C}_{i-1})}$ for all $j\}$;

(2.2) $g_9(i) = g_9(i-1) \cdot g_8(R, \mathscr{C}_{i-1}) \cdot (\#\mathscr{C}_{i-1} + 1)^{g_8(R,\mathscr{C}_{i-1})+1}$;

(2.3) $g_{10}(i) = g_9(i) \cdot g_8(R, \mathscr{C}_i) \cdot (\#\mathscr{C}_i + 1)^{g_8(R,\mathscr{C}_i)+1}$.

PROPOSITION 5.21. *For any* $i \in \{0, 1, ..., r(\mathscr{A}_\Delta) - 1\}$, $g_9(i+1) = g_{10}(i)$.

THEOREM 5.2. *Let* R, $\mathscr{C}$, Δ, *and* δ *be as above, and assume that* $h_r(R, \mathscr{C}) < \infty$. *Then there exists* $E \in \mathrm{SRE}(\Delta)$ *such that* $\delta(|E|) = R$, $h(E) = h_r(R, \mathscr{C})$, *and* $l(E) \leqslant g_{10}(r(\mathscr{A}_\Delta))$.

Proof. By Theorem 5.1, Definition 5.28, and Algorithm 5.1, we can see that $R \in \mathscr{C}_{r(\mathscr{A}_\Delta)}(\cup, \cdot)$, where for any finite class $\mathscr{C}'$ of regular languages over Σ, $\mathscr{C}'(\cup, \cdot)$ is the closure of $\mathscr{C}' \cup \{\{\lambda\}, \varnothing\}$ under the operations, union $(\cup)$ and concatenation $(\cdot)$. Thus it will suffice to prove the following (1) and (2) by induction on $i \in \{0, 1, ..., r(\mathscr{A}_\Delta)\}$.

(1) For any $L \in \mathscr{C}_i$, there exists $E \in \mathrm{SRE}(\Delta)$ such that $\delta(|E|) = L$, $h(E) \leqslant i$ and $l(E) \leqslant g_9(i)$;

(2) For any $L \in \mathscr{C}_i(\cup, \cdot)$, there exists $E \in \mathrm{SRE}(\Delta)$ such that $\delta(|E|) = L$, $h(E) \leqslant i$, and $l(E) \leqslant g_{10}(i)$.

Proof of (1) *and* (2): *Base.* $i = 0$. (1) is obvious. (2) Assume that $L \in \mathscr{C}_0(\cup, \cdot)$. By Theorem 5.1, there exists $E \in \mathrm{SRE}(\Delta)$ such that $\delta(|E|) = L$, and E is of the form: $E = W_1 \cup \cdots \cup W_n$ for some $n \geqslant 1$ and $W_i \in \Delta^*$, where $l(W_i) \leqslant g_8(R, \mathscr{C})$ for all i. Then

$$\begin{aligned} l(E) &= l(W_1) + \cdots + l(W_n) \\ &\leqslant (\#\mathscr{C}) + 2 \cdot (\#\mathscr{C})^2 + \cdots + g_8(R, \mathscr{C}) \cdot (\#\mathscr{C})^{g_8(R,\mathscr{C})} \\ &\leqslant g_8(R, \mathscr{C}) \cdot (\#\mathscr{C} + (\#\mathscr{C})^2 + \cdots + (\#\mathscr{C})^{g_8(R,\mathscr{C})+1}, \\ &\leqslant g_8(R, \mathscr{C}) \cdot (\#\mathscr{C} + 1)^{g_8(R,\mathscr{C})+1} \\ &= g_{10}(0), \end{aligned}$$

where we note that for any $j \geqslant 1$, $\#\{W \mid W \in \Delta^j\} = (\#\mathscr{C})^j$.

Inductive step. $i>0$. For each $j \in \{0, 1, ..., i\}$, let Δ_j be an alphabet such that $\#\Delta_j = \#\mathscr{C}_j$ and δ_j be the corresponding substitution from $2^{\Delta_j^*}$ to 2^{Σ^*}. Thus $\Delta_0 = \Delta$ and $\delta_0 = \delta$.

(1) Assume that $L \in \mathscr{C}_i$. Then either $L \in \mathscr{C}_{i-1}$ or there exists $E_1 \in \mathrm{SRE}(\Delta_{i-1})$ such that $\delta_{i-1}(|E_1|) = L$, and E_1 is of the form $E_1 = (W_1 \cup \cdots \cup W_n)^*$ for some $n \geqslant 1$, and $W_k \in \Delta_{i-1}^+$, where $l(W_k) \leqslant g_8(R, \mathscr{C}_{i-1})$ for all k. From E_1, we obtain $E \in \mathrm{SRE}(\Delta)$ by replacing each $c \in \Delta_{i-1}$ with the corresponding regular expression $E_c \in \mathrm{SRE}(\Delta)$, where by induction $l(E_c) \leqslant g_9(i-1)$. When $L \in \mathscr{C}_{i-1}$, then the assertion holds by induction. Otherwise we have

$$\begin{aligned} l(E) &\leqslant g_9(i-1)\cdot(l(W_1) + \cdots + l(W_n)) \\ &\leqslant g_9(i-1)\cdot g_8(R, \mathscr{C}_{i-1})\cdot(\#\mathscr{C}_{i-1}+1)^{g_8(R,\mathscr{C}_{i-1})+1}, \end{aligned}$$

where the last inequality follows as in the base.

(2) Assume that $L \in \mathscr{C}_i(\cup, \cdot)$. By Theorem 5.1, there exists $E_1 \in \mathrm{SRE}(\Delta_i)$ such that $\delta_i(|E_1|) = L$, and E_1 is of the form $E_1 = W_1 \cup \cdots \cup W_n$ for some $n \geqslant 1$ and $W_k \in \Delta_i$, where $l(W_k) \leqslant g_8(R, \mathscr{C}_i)$ for all k. As in (1), we can see that there exists $E \in \mathrm{SRE}(\Delta)$ such that $\delta(|E|) = L$, $h(E) \leqslant i$ and $l(E) \leqslant g_9(i)\cdot(l(W_1) + \cdots + l(W_n)) \leqslant g_9(i)\cdot g_8(R, \mathscr{C}_i)\cdot(\#\mathscr{C}_i + 1)^{g_8(R,\mathscr{C}_i)+1} = g_{10}(i)$. ∎

Remark 5.3. Theorem 5.2 gives an alternative algorithm for determining relative star height since we can construct all finitely many regular expressions E over Δ such that $l(E) \leqslant g_{10}(r(\mathscr{A}_\Delta))$ as in the proof of Proposition 3.4, decide whether or not $\delta(|E|) = R$, and determine $h(E)$. However, Algorithm 5.1 is clearly more systematic than this algorithm.

Remark 5.4. In Algorithm 5.1, we can also obtain $E \in \mathrm{SRE}(\Delta)$ such that $\delta(|E|) = R$ and $h(E) = h_r(R, \mathscr{C})$. This can be seen as in the proof of Theorem 5.2.

Remark 5.5. From the main lemma, it is clear that $h_r(R, \mathscr{C}) = 0$ iff $R = \bigcup_{W \in L} \delta(W)$, where $L = \{W \in \Delta^* \mid \delta(W) \subset R$ and $l(W) \leqslant g_8(R, \mathscr{C})\}$. Moreover in Definition 5.14, if we define the set S of initial sets and the set F of final sets of $\mathscr{B}$ as $S = \{(\alpha(\delta(b_i)), s_i) \mid 1 \leqslant i \leqslant m\}$ and $F = \{(\alpha(\delta(Wb_i)), q) \mid W \in \Delta^*,\ \delta(Wb_i) \subset R,\ 1 \leqslant i \leqslant m$, and $q \in F_i\}$, then we can see easily that $h_r(R, \mathscr{C}) < \infty$ iff $R - \{\lambda\} = R(\mathscr{B}) - \{\lambda\}$. Thus from the main lemma, we can obtain an algorithm for determining relative star height without depending on Algorithms 4.1, 4.2, where we note that if $h_r(R, \mathscr{C}) < \infty$, then we can obtain $E \in \mathrm{SRE}(\Delta)$ with $\delta(|E|) = R$, and an upper bound $h(E)$ of $h_r(R, \mathscr{C})$.

6. Algorithms for Determining Star Height

This section will present two algorithms for determining star height. Throughout this section, let $R \subset \Sigma^*$ be regular, $\mathscr{A}_0 = \langle \Sigma, Q_0, M_0, S_0, F_0 \rangle$ be the deterministic reduced automaton accepting R, and $\mathscr{M}$ be the syntactic monoid of R. The following theorem was proved in Hashiguchi (1982B).

THEOREM 6.1. *Let R and $\mathscr{M}$ be as above. Then there exists $E \in \mathrm{SRE}(\Sigma)$ such that $|E| = R$, $h(E) = h(R)$, and $v(E) \leqslant 16 \cdot n \cdot (n+2) \cdot (h(R) \cdot n \cdot (n+2)+1)$,* where $n = \#\mathscr{M}$.

We have the following algorithm for determining star height whose effectiveness and correctness are clear from Theorem 6.1 and Algorithm 5.1.

ALGORITHM 6.1. *Let R, $\mathscr{A}_0$, and $\mathscr{M}$ be as above.*

1. *If R is finite, then put $h(R) = 0$. Otherwise proceed to 2.*
2. *Calculate the integer $g = 16 \cdot n \cdot (n+2) \cdot (r(\mathscr{A}_0) \cdot n \cdot (n+2)+1)$, where $n = \#\mathscr{M}$. Obtain the finite class $\mathscr{C}$ of regular languages over Σ as follows: $\mathscr{C} = \{\{a\} \mid a \in \Sigma\} \cup \{|(w_1 \cup \cdots \cup w_p)^*| \mid p \geqslant 1,\ w_i \in \Sigma^+$ and $l(w_i) \leqslant g$ for all $i\}$. By Algorithm 5.1, determine $h_r(R, \mathscr{C})$. Put $h(R) = h_r(R, \mathscr{C}) + 1$.*

We also have the following algorithm for determining star height. But Algorithm 6.1 may be more systematic than the following.

ALGORITHM 6.2. *Let R be as above. By Algorithm 5.1, determine $h_r(R, \mathscr{C})$, where $\mathscr{C} = \{\{a\} \mid a \in \Sigma\}$. Put $h(R) = h_r(R, \mathscr{C})$.*

7. Proof of the Main Lemma

This section will present the proof of the main lemma. The notations and the definitions denote the same nottions as those in Section 5. We shall present a method for constructing E' and P_2 in the main lemma from E and P_1 by induction on $l(w)$. To do this, we need four more propositions (Propositions 7.1–7.4). The proof of the main lemma will begin after the end of the proof of Proposition 7.4.

DEFINITION 7.1. Let $E \in \mathrm{SRE}(\Delta)$ and $P_1 = (i_1, ..., i_n)$ and $P_2 = (j_1, ..., j_r)$ be two paths of E such that $i_n = j_1$. Then $P_1 \cdot P_2$ denotes the path $(i_1, ..., i_n, j_2, ..., j_r)$.

DEFINITION 7.2. The symbol $\circ$ denotes the binary operation over Δ^+ such that for any $W_1, W_2 \in \Delta^+$,

(1) $W_1 \circ W_2 = W_{10} b W_{20}$ *if* $W_1 = W_{10} b$ and $W_2 = b W_{20}$ for some $b \in \Delta$ and $W_{10}, W_{20} \in \Delta^*$;

(2) $W_1 \circ W_2$ is undefined otherwise.

PROPOSITION 7.1. *Let* $v, w \in \Sigma^*$, (u_0, q_0), (u_1, q_1), $(u_2, q_2) \in \eta(\xi(E_0), R)$, $c_1, c_2 \in \{0, 1, 2, 3\}$, *and assume that there exist* $E_1, E_2 \in \mathrm{CSRE}(\Delta)$ *and two paths* P_1 *and* P_2 *of* E_1 *and* E_2, *respectively, such that* (1) $\delta(|E_1|)$, $\delta(|E_2|) \subset R$, (2) $\zeta(E_1, \mathrm{in}(P_1)) = u_0$, $\zeta(E_1, \mathrm{fn}(P_1)) = \zeta(E_2, \mathrm{in}(P_2)) = u_1$ *and* $\zeta(E_2, \mathrm{fn}(P_2)) = u_2$, (3) P_1 *strictly spells* v *with* (q_0, q_1, c_1), *and* (4) P_2 *strictly spells* w *with* (q_1, q_2, c_2). *Then there exist* $E_3 \in \mathrm{CSRE}(\Delta)$ *and a path* P_3 *of* E_3 *such that* (5) $\delta(|E_3|) \subset R$, (6) $h(E_3) = \max\{h(E_1), h(E_2)\}$, (7) $\zeta(E_3, \mathrm{in}(P_3)) = u_0$ *and* $\zeta(E_3, \mathrm{fn}(P_3)) = u_2$, (8) $l(P_3) = l(P_1) + l(P_2)$, (9) P_3 *strictly spells* vw *with* $(q_0, q_2, \max\{c_1, c_2\})$, (10) $\nu_0(P_3) \leqslant \nu_0(P_1) + \nu_0(P_2)$, *and* (11) $l(E_3) \leqslant 2 \cdot (\max\{h(E_1), h(E_2)\} + 1) \cdot (l(E_1) + l(E_2)) \leqslant 4 \cdot (\max\{h(E_1), h(E_2)\} + 1) \cdot \max\{l(E_1), l(E_2)\}$.

Proof. Assume that the conditions hold. We consider eight cases.

Case (1): $c_1 = 0$. In this case, $l(P_1) = 0$, and we put $E_3 = E_2$ and $P_3 = P_2$. Then $l(P_3) = l(P_1) + l(P_2)$, and (5)–(8) clearly hold. Moreover P_3 strictly spells vw with (q_0, q_2, c_2), and (10)–(11) hold.

Case (2): $c_2 = 0$. In this case, $l(P_2) = 0$, and we put $E_3 = E_1$ and $P_3 = P_1$. As in Case (1), the assertions are clear.

Case (3): $c_1 = 1$ and $c_2 \geqslant 1$. We consider two subcases.

Case (3.1): $\beta(E_1, \mathrm{fn}(P_1)) = 0$. Then E_1 and E_2 are of the form $E_1 = E_{10} b E_{11}$ and $E_2 = E_{20} b E_{21}$, where $b \in \Delta$, $\mathrm{fn}(P_1) = l(E_{10} b)$, and $\mathrm{in}(P_2) = l(E_{20} b)$. Then we put $E_3 = E_{10} b E_{21}$ and $P_3 = P_1 \cdot P'_2$, where $P_2 = (i_1, \ldots, i_n)$, $P'_2 = (j_1, \ldots, j_n)$, and $j_k = i_k - l(E_{20.}) + l(E_{10})$ for all k, $1 \leqslant k \leqslant n$. Then the assertions are clear.

Case (3.2): $\beta(E_1, \mathrm{fn}(P_1)) \geqslant 1$. Then E_1 and E_2 are of the form $E_1 = E_{10} H_1^* E_{12}$ and $E_2 = E_{20} H_2^* E_{22}$, where $l(E_{10}) + 1 \leqslant \mathrm{fn}(P_1) \leqslant l(E_{10} H_1^*)$, and $l(E_{20}) + 1 \leqslant \mathrm{in}(P_2) \leqslant l(E_{20} H_2^*)$. Let H_{10}^* and H_{20}^* be the first star factors of E_1 and E_2 which contain positions $\mathrm{fn}(P_1)$ and $\mathrm{in}(P_2)$, respectively. Then H_{10}^* and H_{20}^* are of the form $H_{10}^* = (E_{110} \cup E_{111} b E_{112} \cup E_{113})^*$ and $H_{20}^* = (E_{210} \cup E_{211} b E_{212} \cup E_{213})^*$, where each b occurs at positions $\mathrm{fn}(P_1)$ and $\mathrm{in}(P_2)$, respectively. Since $\zeta(E_1, \mathrm{fn}(P_1)) = \zeta(E_2, \mathrm{in}(P_2))$, $\alpha(\delta(|E_{111}|)) = \alpha(\delta(|E_{211}|))$. Since $c_1 = 1$, $l(E_{111} b) \geqslant l(P_1) + 1$. Define H_{30}^* as follows: $H_{30}^* = (E_{111} b E_{212} \cup H_{20})^*$. Since $\alpha(\delta(|E_{111} b E_{212}|)) = \alpha(\delta(|E_{211} b E_{212}|))$, $\alpha(\delta(|H_{30}^*|)) = \alpha(\delta(|H_{20}^*|))$. We replace H_{20}^* with H_{30}^* in E_2 to obtain E_3. Then it is easy to see that there exists a path P_3 in E_3 for which the assertions hold.

Case (4): $c_1 \geqslant 1$ and $c_2 = 1$. In this case, if $\beta(E_1, \mathrm{fn}(P_1)) = 0$, then we can prove the assertions as in Case (3.1), and if $\beta(E_1, \text{fn}(P_1)) \geqslant 1$, then we can prove the assertions as in Case (3.2).

Case (5): $c_1 = c_2 = 2$. Let H_1^* and H_2^* be the maximum star factors of E_1 and E_2 that contain P_1 and P_2, respectively. Since $c_1 = c_2 = 2$, such H_1^* and H_2^* exist. Let H_1^* and H_2^* be the nth star factors of E_1 and E_2 containing $\text{fn}(P_1)$ and $\text{in}(P_2)$, respectively. Thus $n = \beta(E_1, \text{fn}(P_1)) = \beta(E_2, \text{in}(P_2))$ since $\zeta(E_1, \text{fn}(P_1)) = \zeta(E_2, \text{in}(P_2))$. For each k, $1 \leqslant k \leqslant n$, we define the star expression G_k^* inductively as follows.

(i) Let G_{11}^* and G_{12}^* be the first star factors of E_1 and E_2 containing $\text{fn}(P_1)$ and $\text{in}(P_2)$, respectively. G_{11}^* and G_{12}^* are of the form $G_{11}^* = (E_{111} \cup E_{112} b E_{113} \cup E_{114})^*$ and $G_{12}^* = (E_{121} \cup E_{122} b E_{123} \cup E_{124})^*$, where each b corresponds to positions $\text{fn}(P_1)$ and $\text{in}(P_2)$ in E_1 and E_2, respectively. Since $\zeta(E_1, \text{fn}(P_1)) = \zeta(E_2, \text{in}(P_2))$, it follows that $\alpha(\delta(|E_{112}|)) = \alpha(\delta(|E_{122}|))$, $\alpha(\delta(|G_{11}^*|)) = \alpha(\delta(|G_{12}^*|))$, and $h(G_{11}^*) = h(G_{12}^*)$. Define G_1^* as follows: $G_1^* = (G_{11} \cup G_{12} \cup E_{112} b E_{123})^*$. Then it holds that $h(G_1^*) = h(G_{11}^*)$, $|G_{11}^*| \cup |G_{12}^*| \subset |G_1^*|$, $\alpha(\delta(|G_1^*|)) = \alpha(\delta(G_{11}^*|))$, and $l(G_1^*) \leqslant 2 \cdot (l(E_1) + l(E_2))$.

(ii) Let G_{k1}^* and G_{k2}^* be the kth star factors of E_1 and E_2 containing $\text{fn}(P_1)$ and $\text{in}(P_2)$, respectively, where $1 < k \leqslant n$. Then G_{k1}^* and G_{k2}^* are of the form $G^*{}_{k1} = (E_{k11} \cup E_{k12} G_{k-11}^* E_{k13} \cup E_{k14})^*$, and $G_{k2}^* = (E_{k21} \cup E_{k22} G_{k-12}^* E_{k23} \cup E_{k24})^*$, where G_{k-11}^* and G_{k-12}^* are the $(k-1)$th star factors of E_1 and E_2 containing $\text{fn}(P_1)$ and $\text{in}(P_2)$, respectively. Define G_k^* as follows: $G_k^* = (G_{k1} \cup G_{k2} \cup E_{k12} G_{k-1}^* E_{k23})^*$. Then it holds that $h(G_k^*) = h(G_{k1}^*)$, $|G_{k1}^*| \cup |G_{k2}^*| \subset |G_k^*|$, $\alpha(\delta(|G_k^*|)) = \alpha(\delta(|G_{k1}^*|))$, and $l(G_k^*) \leqslant 2 \cdot (l(E_1) + l(E_2)) + l(G_{k-1}^*)$. By induction on k, we can see that $l(G_k^*) \leqslant 2 \cdot k \cdot (l(E_1) + l(E_2))$. So we defined $G_1^*, ..., G_n^*$. Now in E_1, we replace H_1^* with G_n^* to obtain E_3. Then we can see easily that $l(E_3) \leqslant l(E_1) + l(G_n^*) \leqslant l(E_1) + 2 \cdot h(E_1) \cdot (l(E_1) + l(E_2)) \leqslant 2 \cdot (h(E_1) + 1) \cdot (l(E_1) + l(E_2))$, and there exists a path P_3 in E_3 for which the other assertions hold.

Case (6): $c_1 = 2$ and $c_2 = 3$. Let H_1^* be the maximum star factor of E_1 that contains P_1. Such H_1^* exists because $c_1 = 2$. Then E_1 is of the form $E_1 = E_{11} H_1^* E_{12}$. Since $\zeta(E_1, \text{fn}(P_1)) = \zeta(E_2, \text{in}(P_2))$, E_2 is of the form $E_2 = E_{21} H_2^* E_{22}$, where H_2^* is the maximum star factor of E_2 that contains $\text{in}(P_2)$. Since $c_2 = 3$, $\text{fn}(P_2) \geqslant l(E_{21} H_2^*) + 1$. Let $P_2 = (i_1, ..., i_n)$. There exist n_0, $1 \leqslant n_0 \leqslant n-1$, $j, k \in \{1, ..., m\}$, $q' \in Q_j$, $w_1, w_2 \in \Sigma^*$, and $a \in \Sigma$ such that (i) $w = w_1 a w_2$, (ii) i_{n_0} is a final position of H_2^* w.r.t. E_2, (iii) i_{n_0+1} is an initial position of E_{22} w.r.t. E_2, (iv) $\Delta_{E_2}(i_{n_0}) = b_j$ and $\Delta_{E_2}(i_{n_0+1}) = b_k$, (v) the path $P_{21} = (i_1, ..., i_{n_0})$ strictly spells w_1 with $(q_1, q', 2)$, (vi) $M_j(q', a) \in F_j$, and (vii) the path $P_{22} = (i_{n_0+1}, ..., i_n)$ strictly spells w_2 with (s_k, q_2, c'), where $c' \leqslant 3$. As in Case (5), from H_1^* and H_2^*, we can construct $G_{n_1}^*$, where $n_1 = \beta(E_1, \text{fn}(P_1)) = \beta(E_2, \text{in}(P_2))$. Then we define E_3 as follows: $E_3 = E_{11} G_{n_1}^* E_{22}$. Then $l(E_3) \leqslant l(E_1) + l(E_2) + l(G_{n_1}^*) \leqslant 2 \cdot (h(E_1) + 1) \cdot (l(E_1) +$

$l(E_2))$, where the last inequality follows from Case (5). Now it is easy to see that there exists a path P_3 in E_3 for which the assertions hold.

Case (7): $c_1=3$ and $c_2=2$. As in Case (6), we can prove the assertions.

Case (8): $c_1=c_2=3$. We consider two subcases.

Case (8.1): $\beta(E_1, \text{fn}(P_1))=0$. Then E_1 and E_2 are of the form $E_1=E_{11}bE_{12}$ and $E_2=E_{21}bE_{22}$, where $\text{fn}(P_1)=l(E_{11}b)$ and $\text{in}(P_2)=l(E_{21}b)$. We define E_3 as follows: $E_3=E_{11}bE_{22}$. The assertions are clear.

Case (8.2): $\beta(E_1, \text{fn}(P_1))\geqslant 1$. In this case, E_1 and E_2 are of the form $E_1=E_{11}H_1^*E_{12}$ and $E_2=E_{21}H_2^*E_{22}$, where H_1^* and H_2^* are the maximum star factors of E_1 and E_2 that contain $\text{fn}(P_1)$ and $\text{in}(P_2)$, respectively. As in Case (6), we can construct G_n^* from H_1^* and H_2^*, where $n=\beta(E_1, \text{fn}(P_1))$, and define E_3 as follows: $E_3=E_{11}G_n^*E_{22}$. The assertions hold as in Case (6). ∎

PROPOSITION 7.2. *Let* $n\geqslant 1$, $w_1, ..., w_n \in \Sigma^+$, (u_0, q_0), $(u_1, q_1), ..., (u_n, q_n)\in\eta(\xi(E_0), R)$, $c_1, ..., c_n\in\{2, 3\}$, *and assume that there exist* $E_1, ..., E_n\in \text{CSRE}(\Delta)$ *and n paths* $P_1, ..., P_n$ *of* $E_1, ...,$ *and* E_n, *respectively, such that for all i,* $1\leqslant i\leqslant n$, *the following hold.*

(1) $\delta(|E_i|)\subset R$;

(2) $\zeta(E_i, \text{in}(P_i))=u_{i-1}$ *and* $\zeta(E_i, \text{fn}(P_i))=u_i$;

(3) P_i *strictly spells* w_i *with* (q_{i-1}, q_i, c_i).

Then there exist $E\in\text{CSRE}(\Delta)$ *and a path P of E for which the following hold.*

(4) $\delta(|E|)\subset R$, *and* $h(E)\leqslant\max\{h(E_i)\mid 1\leqslant i\leqslant n\}$;

(5) $\zeta(E, \text{in}(P))=u_0$ *and* $\zeta(E, \text{fn}(P))=u_n$;

(6) P *strictly spells* $w_1\cdots w_n$ *with* $(q_0, q_n, \max\{c_i\mid 1\leqslant i\leqslant n\})$;

(7) $v_0(P)\leqslant 2\cdot\max\{v_0(P_i)\mid 1\leqslant i\leqslant n\}$;

(8) $l(E)\leqslant 4^{n-1}\cdot(\max\{h(E_i)\mid 1\leqslant i\leqslant n\}+1)^{n-1}\cdot\max\{l(E_i)\mid 1\leqslant i\leqslant n\}$.

Proof. The proof is by induction on n. When $n=1$, the assertions are trivial, and when $n=2$, the asertions are clear from Proposition 7.1. Let $n\geqslant 3$. By the inductive hypothesis, there exist $E'\in\text{CSRE}(\Delta)$ and a path P' of E' for which the assertions hold for $w_1\cdots w_{n-1}$. Now consider the maximal word factor $W\in\Delta^+$ of E' that contains $\text{fn}(P')$. Then W is of the form $W=W_1bW_2$, where b corresponds to position $\text{fn}(P')$. Because $c_{n-1}\geqslant 2$, and by construction, it holds that $l(W_1b)\leqslant v_0(P_{n-1})$. Thus, as in the proof of Proposition 7.1, we can construct from E', P', E_n, and P_n, a regular expression $E \in \mathrm{CSRE}(\Delta)$ and a path P of E for which (4)–(7) hold. Moreover by induction and Proposition 7.1, we have

$$\begin{aligned} l(E) &\leqslant 4 \cdot (\max\{h(E_i) \mid 1 \leqslant i \leqslant n\} + 1) \\ &\quad \cdot \max\{l(E_n), 4^{n-2} \cdot (\max\{h(E_j) \mid 1 \leqslant j \leqslant n-1\} + 1)^{n-2} \\ &\quad \cdot \max\{l(E_j) \mid 1 \leqslant j \leqslant n-1\}\} \\ &\leqslant 4^{n-1} \cdot (\max\{h(E_i) \mid 1 \leqslant i \leqslant n\} + 1)^{n-1} \\ &\quad \cdot \max\{l(E_i) \mid 1 \leqslant i \leqslant n\}. \quad \blacksquare \end{aligned}$$

PROPOSITION 7.3. *Let* $n \geqslant 1$, $w_{10}, w_{11}, w_{20}, w_{21}, ..., w_{n0}, w_{n1} \in \Sigma^+$, $c_{10}, c_{11}, ..., c_{n0}, c_{n1} \in \{0, 1\}$, *and* (u_{10}, q_{10}), (u_{11}, q_{11}), (u_{20}, q_{20}), (u_{21}, q_{21}), ..., (u_{n0}, q_{n0}), (u_{n1}, q_{n1}), $(u_{n+10}, q_{n+10}) \in \eta(\xi(E_0), R)$. *Assume that for each* i, $1 \leqslant i \leqslant n$, *there exist* $E_{i0}, E_{i1} \in \mathrm{CSRE}(\Delta)$ *and two paths* P_{i0} *and* P_{i1} *of* E_{i0} *and* E_{i1}, *respectively, such that* (1) $\delta(|E_{i0}|)$, $\delta(|E_{i1}|) \subset R$, (2) $\zeta(E_{i0}, \mathrm{in}(P_{i0})) = u_{i0}$, $\zeta(E_{i0}, \mathrm{fn}(P_{i0})) = \zeta(E_{i1}, \mathrm{in}(P_{i1})) = u_{i1}$, *and* $\zeta(E_{i1}, \mathrm{fn}(P_{i1})) = u_{i+10}$, *and* (3) P_{i0} *strictly spells* w_{i0} *with* (q_{i0}, q_{i1}, c_{i0}) *and* P_{i1} *strictly spells* w_{i1} *with* $(q_{i1}, q_{i+10}, c_{i1})$. *Moreover assume that the following* (4) *holds.*

(4) *For each* i, $1 \leqslant i \leqslant n$, $(\alpha(\delta(\Delta_{E_{10}}(P_{10}) \circ \Delta_{E_{11}}(P_{11}) \circ \cdots \circ \Delta_{E_{i0}}(P_{i0}))),$ $q_{i1}) \in M_{d0}(Q, w_{10}w_{11}w_{20}w_{21} \cdots w_{i0})$.

Then there exist $E \in \mathrm{CSRE}(\Delta)$ *and a path* P *of degree zero of* E *such that the following* (5)–(9) *hold.*

(5) $\delta(|E|) \subset R$ *and* $h(E) \leqslant \max\{h(E_{10}), h(E_{n1})\}$;

(6) $\zeta(E, \mathrm{in}(P)) = u_{10}$ *and* $\zeta(E, \mathrm{fn}(P)) = u_{n+10}$;

(7) $l(P) < 2\ (\#Q + 1) \cdot \max\{l(P_{i0}) + 1, l(P_{i1}) + 1 \mid 1 \leqslant i \leqslant n\}$;

(8) P *strictly spells* $w = w_{10}w_{11}w_{20}w_{21} \cdots w_{n0}w_{n1}$ *with* $(q_{10}, q_{n+10},$ $\max\{c_{i0}, c_{i1} \mid 1 \leqslant i \leqslant n\})$;

(9) $l(E) \leqslant l(E_{10}) + l(E_{n1}) + l(P) + 1$.

Proof. If $c_{i0} = c_{i1} = 0$ fo all i, $1 \leqslant i \leqslant n$, then $l(P_{i0}) = l(P_{i1}) = 0$ for all i, $1 \leqslant i \leqslant n$, and P_{10} strictly spells w with $(q_{10}, q_{n+10}, 0)$, and the assertions hold. So we assume that $\max\{c_{i0}, c_{i1} \mid 1 \leqslant i \leqslant n\} = 1$. Now for each i, $1 \leqslant i \leqslant n$, we put $t_i = \alpha(\delta(\Delta_{E_{10}}(P_{10}) \circ \Delta_{E_{11}}(P_{11}) \circ \cdots \circ \Delta_{E_{i0}}(P_{i0})))$. We define the sequence of integers, $j_1, j_2, ..., j_{n_0}$, $1 = j_1 < j_2 < \cdots < j_{n_0} \leqslant n$, inductively as follows.

(10) $j_1 = 1$;

(11) For $i > 1$, let k_{i-1} be the largest integer such that $j_{i-1} \leqslant k_{i-1} \leqslant n$, and $d((t_{j_{i-1}}, q_{j_{i-1}1}), \quad w_{j_{i-1}1}w_{j_{i-1}+10}w_{j_{i-1}+11} \cdots w_{k_{i-1}0},$ $(t_{k_{i-1}}, q_{k_{i-1}1})) = 0$, where $k_{i-1} = j_{i-1}$ if such $k_{i-1} > j_{i-1}$ does not exist. If

$k_{i-1}=n$, then $n_0=i-1$, and the procedure ends. Otherwise put $j_i=k_{i-1}+1$, and repeat the procedure. The following claim holds.

CLAIM. $n_0 \leqslant \# Q$.

Proof. Assume that $n_0 > \# Q$. By (4), for each j_i, $1 \leqslant i \leqslant n_0$, there exists $(t'_i, q'_i) \in Q$ such that $(t_{j_i}, q_{j_i 1}) = M_{d0}((t'_i, q'_i), w_{10} w_{11} \cdots w_{j_i 0})$. Since $n_0 > \# Q$, there exist i_0, i_1, $1 \leqslant i_0 < i_1 \leqslant n_0$, such that $(t'_{i_0}, q'_{i_0}) = (t'_{i_1}, q'_{i_1})$. But then $(t_{j_{i_1}}, q_{j_{i_1}1}) = M_{d0}((t'_{i_1}, q'_{i_1}), w_{10} w_{11} \cdots w_{j_{i_1}0}) = M_{d0}(M_{d0}((t'_{i_0}, q'_{i_0}), w_{10} w_{11} \cdots w_{j_{i_0}0}), w_{j_{i_0}1} \cdots w_{j_{i_1}0}) = M_{d0}((t_{j_{i_0}}, q_{j_{i_0}1}), w_{j_{i_0}1} \cdots w_{j_{i_1}0})$, which is a contradiction to $d((t_{j_{i_0}}, q_{j_{i_0}1}), w_{j_{i_0}1} \cdots w_{j_{i_1}0}, (t_{j_{i_1}}, q_{j_{i_1}1})) > 0$. ∎

Proof of Proposition 7.3 (Continued). From the sequence of integers, $j_1, ..., j_{n_0}$, we define the word $W \in \Delta^+$ as follows: $W = \Delta_{E_{10}}(P_{10}) \circ \Delta_{E_{j_2-1,1}}(P_{j_2-1,1}) \circ \Delta_{E_{j_2 0}}(P_{j_2 0}) \circ \cdots \circ \Delta_{E_{j_{n_0}-1,1}}(P_{j_{n_0}-1,1}) \circ \Delta_{E_{j_{n_0}0}}(P_{j_{n_0}0})$.

Let $W = b_{k_1} b_{k_2} \cdots b_{k_{n_1}}$, where $n_1 \geqslant 1$ and $b_{k_i} \in \Delta$, $1 \leqslant i \leqslant n_1$. We can see easily that the following (12)–(14) hold.

(12) $\alpha(\delta(W)) = \alpha(\delta(\Delta_{E_{10}}(P_{10}) \circ \Delta_{E_{11}}(P_{11}) \circ \Delta_{E_{20}}(P_{20}) \circ \Delta_{E_{21}}(P_{21}) \circ \cdots \circ \Delta_{E_{n0}}(P_{n0})))$;

(13) The path $P' = (1, 2, ..., n_1)$ of W strictly spells $w' = w_{10} w_{11} w_{20} w_{21} \cdots w_{n0}$ with (q_{10}, q_{n1}, c') for some $c' \in \{0, 1\}$;

(14) $$n_1 \leqslant 2 \cdot n_0 \cdot \max\{l(P_{i0}) + 1, l(P_{i1}) + 1 \mid 1 \leqslant i \leqslant n\}$$
$$\leqslant 2 \cdot \# Q \cdot \max\{l(P_{i0}) + 1, l(P_{i1}) + 1 \mid 1 \leqslant i \leqslant n\}.$$

Now we consider two cases.

Case (1): $\beta(E_{10}, \mathrm{in}(P_{10})) = 0$. Since $c_{i0}, c_{i1} \leqslant 1$ for all i, $1 \leqslant i \leqslant n$, it follows that $\beta(E_{j_{n_0}0}, \mathrm{fn}(P_{j_{n_0}0})) = 0$. So E_{10} and $E_{j_{n_0}0}$ are of the form $E_{10} = E_{100} b E_{101}$ and $E_{j_{n_0}0} = E_{j_{n_0}00} b' E_{j_{n_0}01}$, where b and b' correspond to positions $\mathrm{in}(P_{10})$ and $\mathrm{fn}(P_{j_{n_0}0})$, respectively. We define E' as follows $E' = E_{100} W E_{j_{n_0}01}$. Then $h(E') \leqslant \max\{h(E_{10}), h(E_{j_{n_0}0})\}$. For each i, $1 \leqslant i \leqslant n_0$, $E_{j_i 0}$ is of the form $E_{j_i 0} = E_{j_i 00} b'_i E_{j_i 01}$, where b'_i corresponds to position $\mathrm{fn}(P_{j_i 0})$. By induction on i, $1 \leqslant i \leqslant n_0$, we can see that $\alpha(\delta(|E_{j_i 00} b'_i|)) = \alpha(\delta(|E_{100} \Delta_{E_{10}}(P_{10}) \circ \Delta_{E_{j_2-1,1}}(P_{j_2-1,1}) \circ \Delta_{E_{j_2 0}}(P_{j_2}) \circ \cdots \circ \Delta_{E_{j_i-1,1}}(P_{j_i-1,1}) \circ \Delta_{E_{j_i 0}}(P_{j_i 0})|))$.

Thus $\alpha(\delta(|E_{100} W|)) = \alpha(\delta(|E_{j_{n_0}00} b'|))$. So $\delta(|E'|) \subset R$. Moreover the following hold.

(15) The path $P'' = (l(E_{100}) + 1, l(E_{100}) + 2, ..., l(E_{100}) + n_1)$ strictly spells $w' = w_{10} w_{11} \cdots w_{n0}$ with (q_{10}, q_{n1}, c');

(16) $\zeta(E', \mathrm{in}(P'')) = u_{10}$ and $\zeta(E', \mathrm{fn}(P'')) = u_{n1}$.

Since $\beta(E_{n1}, \mathrm{in}(P_{n1})) = \beta(E_{n0}, \mathrm{fn}(P_{n0})) = 0$, E_{n1} is of the form

$E_{n1} = E_{n10} b' E_{n11}$, where b' corresponds to position $\text{in}(P_{n1})$. Now we define E as follows: $E = E_{100} W E_{n11}$. Then $h(E) \leqslant \max\{h(E_{10}), h(E_{n1})\}$. As in the above proof of $\delta(|E'|) \subset R$, we can see that $\delta(|E|) \subset R$. Moreover the following hold.

(17) The path $P = (l(E_{100}) + 1, l(E_{100}) + 2, ..., l(E_{100}) + n_1 + l(P_{n1}))$ strictly spells w with $(q_{10}, q_{n+10}, 1)$;

(18) $\zeta(E, \text{in}(P)) = u_{10}$ and $\zeta(E, fn(P)) = u_{n+10}$;

(19)

$$\begin{aligned} l(P) &\leqslant n_1 + l(P_{n1}) \\ &\leqslant 2 \cdot \#Q \cdot \max\{l(P_{i0}) + 1, l(P_{i1}) + 1 \mid 1 \leqslant i \leqslant n\} + l(P_{n1}) \\ &\leqslant 2 \cdot (\#Q + 1) \cdot \max\{l(P_{i0}) + 1, l(P_{i1}) + 1 \mid 1 \leqslant i \leqslant n\}; \end{aligned}$$

(20) $l(E) \leqslant l(E_{10}) + l(E_{n1}) + n_1$.

Case (2): $\beta(E_{10}, \text{in}(P_{10})) \geqslant 1$. In this case, E_{10} and E_{n1} are of the form $E_{10} = E_{100} H_1^* E_{101}$ and $E_{n1} = E_{n10} H_2^* E_{n11}$, where H_1^* and H_2^* are the maximum star factors of E_{10} and E_{n1} that contain P_{10} and P_{n1}, respectively. Let H_{10}^* and H_{20}^* be the minimum star factors of E_{10} and E_{n1} that contain P_{10} and P_{n1}, respectively. Then H_{10}^* and H_{20}^* are contained in H_1^* and H_2^*, respectively, and are of the form $H_{10}^* = (E_{100} \cup E_{101} b E_{102} \cup E_{103})^*$ and $H_{20}^* = (E_{200} \cup E_{201} b' E_{202} \cup E_{203})^*$, where b and b' correspond to positions $\text{in}(P_{01})$ and $\text{in}(P_{n1})$, respectively. Now define H_{30}^* as follows: $H_{30}^* = (E_{101} W E_{202})^*$. In E_{10}, we replace H_{10}^* with H_{30}^* to obtain E. As in Case (1), we can see that the assertions hold. ∎

PROPOSITION 7.4. *Let* $n_0, n_1 \geqslant 1$, $(u, q) \in \eta(\zeta(E_0), R)$, $E_1, ..., E_{n_0} \in \text{CSRE}(\Delta)$, $w_1, ..., w_{n_1} \in \Sigma^+$, *and for each* i, $1 \leqslant i \leqslant n_1$, P_i *be a path of some* E_{j_i}, $1 \leqslant j_i \leqslant n_0$. *Assume that for each* i, $1 \leqslant i \leqslant n_1$, (1) $\delta(|E_{j_i}|) \subset R$, (2) $\zeta(E_{j_i}, \text{in}(P_i)) = \zeta(E_{j_i}, \text{fn}(P_i)) = u$, *and* (3) P_i *strictly spells* w_i *with* $(q, q, 2)$. *Then there exists* $E \in \text{CSRE}(\Delta)$ *such that* (4) $h(E) = h(E_{j_1})$, (5) $\delta(|E|) \subset R$, (6) $l(E) \leqslant (n_0 + 2)^{2 \cdot h(E_{j_1})} \cdot (\max\{l(E_i) \mid 1 \leqslant i \leqslant n_0\})^{2 \cdot h(E_{j_1})}$, *and* (7) *for each* $w \in |(w_1 \cup \cdots \cup w_{n_1})^+|$, *there exists a path* P_w *of* E *which satisfies the following* (7.1)–(7.3).

(7.1) $\zeta(E, \text{in}(P_w)) = \zeta(E, \text{fn}(P_w)) = u$;

(7.2) P_w *strictly spells* w *with* $(q, q, 2)$;

(7.3) $v_0(P_w) \leqslant 2 \cdot \max\{v_0(P_i) \mid 1 \leqslant i \leqslant n_1\}$.

Proof. By (2) and (3), it holds that for each i, $1 \leqslant i \leqslant n_1$, $\beta(E_{j_i}, \text{in}(P_i)) = \beta(E_{j_i}, \text{fn}(P_i)) = \beta(E_{j_1}, \text{in}(P_1)) = \beta(E_{j_1}, \text{fn}(P_1)) \geqslant 1$. We put $n_2 = \beta(E_{j_1}, \text{in}(P_1))$. For each i, $1 \leqslant i \leqslant n_1$, and for each k, $1 \leqslant k \leqslant n_2$, let

H^*_{ik0} and H^*_{ik1} be the kth star factors of E_{j_i} that contain positions $\mathrm{in}(P_i)$ and $\mathrm{fn}(P_i)$, respectively. As in the proof of Case (5) of Proposition 7.1, for each k, $1 \leqslant k \leqslant n_2$, we define the star expression G^*_k inductively as follows.

(i) $k=1$. For each i, $1 \leqslant i \leqslant n_1$, H_{i10} and H_{i11} are of the form $H_{i10} = E_{i100} \cup E_{i101} b E_{i102} \cup E_{i103}$, and $H_{i11} = E_{i110} \cup E_{i111} b E_{i112} \cup E_{i113}$, where each b corresponds to positions $\mathrm{in}(P_i)$ and $\mathrm{fn}(P_i)$ in E_{j_i}, respectively. We define G^*_1 as follows:

$$G^*_1 = \left(\bigcup_{i=1}^{n_1} (H_{i10} \cup H_{i11}) \cup \left(\bigcup_{\substack{1 \leqslant i_0, i_1 \leqslant n_1 \\ 0 \leqslant i_2, i_3 \leqslant 1}} E_{i_0 1 i_2 1} b E_{i_1 1 i_3 2} \right) \right)^*.$$

We can see easily that $\alpha(\delta(|G^*_1|)) = \alpha(\delta(|H^*_{i10}|)) = \alpha(\delta(|H^*_{i11}|))$ for all i, $1 \leqslant i \leqslant n_1$, and $|(\bigcup_{i=1}^{n_1} (H^*_{i10} \cup H^*_{i11})^*)^*| \subset |G^*_1|$. Moreover the sum of the lengths of distinct H^*_{i10} and H^*_{i11} is not greater than $\sum_{i=1}^{n_0} l(E_i)$. So it holds that $l(G^*_1) \leqslant (\sum_{i=1}^{n_0} l(E_i)) + (\sum_{i=1}^{n_0} l(E_i))^2 \leqslant (n_0 + 1)^2 \cdot \max\{(l(E_i))^2 \mid 1 \leqslant i \leqslant n_0\}$.

(ii) $1 < k \leqslant n_2$. For each i, $1 \leqslant i \leqslant n_1$, H^*_{ik0} and H^*_{ik1} are of the form $H^*_{ik0} = (E_{ik00} \cup E_{ik01} H^*_{ik-10} E_{ik02} \cup E_{ik03})^*$, and $H^*_{ik1} = (E_{ik10} \cup E_{ik11} H^*_{ik-11} E_{ik12} \cup E_{ik13})^*$, where positions $\mathrm{in}(P_i)$ and $\mathrm{fn}(P_i)$ are contained in H^*_{ik-10} and H^*_{ik-11}, respectively. We define G^*_k as follows:

$$G^*_k = \left(\bigcup_{i=1}^{n_1} (H_{ik0} \cup H_{ik1}) \cup \left(\bigcup_{\substack{1 \leqslant i_0 \cdot i_1 \leqslant n_1 \\ 0 \leqslant i_2 \cdot i_3 \leqslant 1}} E_{i_0 k i_2 1} G^*_{k-1} E_{i_1 k i_3 2} \right) \right)^*.$$

We can see easily that $\alpha(\delta(|G^*_k|)) = \alpha(\delta(|H^*_{ik0}|)) = \alpha(\delta(|H^*_{ik1}))$ for all i, $1 \leqslant i \leqslant n_1$, and $|(\bigcup_{i=1}^{n_1} (H^*_{ik0} \cup H^*_{ik1}))^*| \subset |G^*_k|$. Moreover it holds that

$$\begin{aligned}
l(G^*_k) &\leqslant \left(\sum_{i=1}^{n_0} l(E_i) \right) + \left(\sum_{i=1}^{n_0} (l(E_i) + l(G^*_{k-1})) \right) \cdot \left(\sum_{i=1}^{n_0} l(E_i) \right) \\
&\leqslant \left(\sum_{i=1}^{n_0} l(E_i) \right)^2 + (n_0 \cdot l(G^*_{k-1}) + 1) \cdot \left(\sum_{i=1}^{n_0} l(E_i) \right) \\
&\leqslant n_0^2 \cdot \max\{(l(E_i))^2 \mid 1 \leqslant i \leqslant n_0\} + (n_0 \cdot l(G^*_{k-1}) + 1) \cdot n_0 \\
&\quad \cdot \max\{l(E_i) \mid 1 \leqslant i \leqslant n_0\} \\
&\leqslant n_0^2 \cdot \max\{(l(E_i))^2 \mid 1 \leqslant i \leqslant n_0\} \cdot (l(G^*_{k-1}) + 2) \\
&\leqslant n_0^2 \cdot \max\{(l(E_i))^2 \mid 1 \leqslant i \leqslant n_0\} \\
&\quad \cdot ((n_0 + 1)^{2 \cdot k - 2} \cdot (\max\{l(E_i) \mid 1 \leqslant i \leqslant n_0\})^{2 \cdot k - 2} + 2) \\
&\leqslant (n_0 + 1)^{2 \cdot k} \cdot (\max\{l(E_i) \mid 1 \leqslant i \leqslant n_0\})^{2 \cdot k},
\end{aligned}$$

where by induction, $l(G^*_{k-1}) \leqslant (n_0+1)^{2\cdot k-2} \cdot (\max\{l(E_i) \mid 1 \leqslant i \leqslant n_0)^{2\cdot k-2}$, and it holds that $(n_0+1)^{2\cdot k} = (n_0+1)^{2\cdot k-2} \cdot (n_0+1)^2 \geqslant n_0^2 \cdot (n_0+1)^{2\cdot k-2} + 2\cdot n_0^2$. Inductively we define $G^*_{n_2}$. Now E_{j_1} is of the form $E_{j_1} = E_{j_10} H^*_{1n_20} E_{j_11}$, where $H^*_{1n_20}$ is the n_2th factor of E_{j_1} that contains position in(P_1). From (3), $H^*_{1n_20} = H^*_{1n_21}$, and $H^*_{1n_20}$ contains P_1. We define $E \in \mathrm{CSRE}(\Delta)$ as follows: $E = E_{j_10} G^*_{n_2} E_{j_1}$. Then it is not difficult to see that $h(E) = h(E_{j_1})$, $\delta(|E|) \subset R$, and for each $w \in |(w_1 \cup \cdots \cup w_{n_1})^+|$, there exists a path P_w of E such that positions in (P_w) and fn(P_w) are contained in the star factor $G^*_{n_2}$ of E, and (7.1)–(7.3) hold. Moreover $l(E) \leqslant l(E_{j_1}) + l(G^*_{n_2}) \leqslant l(E_{j_1}) + (n_0+1)^{2n_2} \cdot (\max\{l(E_i) \mid 1 \leqslant i \leqslant n_0\})^{2n_2} \leqslant (n_0+2)^{2n_2} \cdot (\max\{l(E_i) \mid 1 \leqslant i \leqslant n_0\})^{2n_2} \leqslant (n_0+2)^{2\cdot h(E_{j_1})} \cdot (\max\{l(E_i) \mid 1 \leqslant i \leqslant n_0\})^{2\cdot h(E_{j_1})}$. ∎

Now we shall present the proof of the main lemma.

Proof of Main Lemma. The proof is by induction on $I(w)$.

Basis. $I(w) = 0$. In this case, $E_w(k_0) = w$, and the assertions are clear by Propositions 5.1, 5.8.

Inductive step. $I(w) > 0$. Let $D(w, I(w)) = (x_1, a_1, ..., x_n, a_n, x_{n+1})$, $E \in \xi(E_0)$, (u_0, q_0), $(u_1, q_1) \in \eta(\xi(E_0), R)$, $c \in \{0, 1, 2, 3\}$, and P_1 be a path of E. Assume that (1) and (2) in the main lemma hold. We consider five cases.

Case (1): $n < g_6$. In this case, $E_w(k_0) = E_{x_1}(k_0)\ a_1 \cdots E_{x_n}(k_0)\ a_n E_{x_{n+1}}(k_0)$. There exist $(2\cdot n+1)$ paths, $P_{10}, P_{11}, ..., P_{n0}, P_{n1}, P_{n+10}$ of E, $(u_{10}, q_{10}), (u_{11}, q_{11}), ..., (u_{n+10}, q_{n+10}), (u_{n+11}, q_{n+11}) \in \eta(\xi(E_0), R)$, and $c_{10}, c_{11}, ..., c_{n0}, c_{n1}, c_{n+10} \in \{0, 1, 2, 3\}$ for which the following (9)–(12) hold.

(9) $P_1 = P_{10} \cdot P_{11} \cdots P_{n0} \cdot P_{n1} \cdot P_{n+10}$;

(10) $(u_{10}, q_{10}) = (u_0, q_0)$, and $(u_{n+11}, q_{n+11}) = (u_1, q_1)$;

(11) For each j, $1 \leqslant j \leqslant n$, $\zeta(E, \mathrm{in}(P_{j0})) = u_{j0}$, $\zeta(E, \mathrm{fn}(P_{j0})) = u_{j1}$, and the following (11.1) and (11.2) hold:

(11.1) P_{j0} strictly spells $E_{x_j}(k_0)$ with (q_{j0}, q_{j1}, c_{j0});

(11.2) P_{j1} strictly spells a_j with $(q_{j1}, q_{j+10}, c_{j1})$;

(12) $\zeta(E, \mathrm{in}(P_{n+10})) = u_{n+10}$, $\zeta(E, \mathrm{fn}(P_{n+10})) = u_{n+11}$, and P_{n+10} strictly spells $E_{x_{n+1}}(k_0)$ with $(q_{n+10}, q_{n+11}, c_{n+10})$.

Note that (9) and (11) imply that for each j, $1 \leqslant j \leqslant n$, $\zeta(E, \mathrm{in}(P_{j1})) = u_{j1}$ and $\zeta(E, \mathrm{fn}(P_{j1})) = u_{j+10}$. By induction, for each j, $1 \leqslant j \leqslant n+1$, there exist $E'_j \in \mathrm{CSRE}(\Delta)$ and a path P'_j of E'_j for which (3)–(8) hold for x_j. It is also clear that $c = \max\{c_{j0}, c_{j1}, c_{n+10} \mid 1 \leqslant j \leqslant n\}$. As in the proof of Cases (1)–(4) in Proposition 7.1, we can see that for each j, $1 \leqslant j \leqslant n$, from E'_j, P'_j, E, and P_{j1}, we can obtain $E''_j \in \mathrm{CSRE}(\Delta)$ and a path P''_j of E''_j such that P_j'' spells $x_j a_j$ with $(q_{j0}, q_{j+10}, \max\{c_{j0}, c_{j1}\})$, and other assertions and the following hold: $l(E_j'') \leqslant l(E_j') + l(E) \leqslant 2 \cdot o_2(I(w)-1)$. From Proposition 7.1 and by induction on n, there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) hold for w. Moreover we can see easily that the following hold.

(13) $v_0(P_2) \leqslant (n+1) \cdot (o_1(I(w)-1)+1) \leqslant g_6 \cdot (o_1(I(w)-1)+1)$;

(14) $l(E') \leqslant 4^n \cdot (h(E_0)+1)^n \cdot 2 \cdot o_2(I(w)-1)$,

where the last inequality follows from Propositions 7.1, 7.2 and by induction on n.

Case (2): $n = g_6$ and $x_{n+1} = \lambda$. Let $S(w, g_6, \equiv, k_0) = ((i_0, i_1), (w_1, w_2, w_3))$. Then

$$\begin{aligned} E_w(k_0) &= E_{x_1}(k_0)\, a_1 \cdots E_{x_{i_0}(k0)}\ a_{i_0}(E_{x_{i_0}+1}(k_0)\, a_{i_0+1} \cdots E_{x_{i_1}}(k_0)\, a_{i_1})^{k_0} \\ &\quad \cdot E_{x_{i_1}+1}(k_0)\, a_{i_1+1} \cdots E_{x_n}(k_0)\, a_n \\ &= E_{w_1}(k_0)(E_{w_2}(k_0))^{k_0} E_{w_3}(k_0). \end{aligned}$$

There exist (k_0+2) paths, $P_{10}, P_{20}, \ldots, P_{k_0+20}$, of E, (u_{01}, q_{01}), $(u_{11}, q_{11}), \ldots, (u_{k_0+21}, q_{k_0+21}) \in \eta(\xi(E_0), R)$, and $c_1, c_2, \ldots, c_{k_0+2} \in \{0, 1, 2, 3\}$ for which the following hold.

(15) $P_1 = P_{10} \cdot P_{20} \cdots P_{k_0+20}$;

(16) $(u_{01}, q_{01}) = (u_0, q_0)$ and $(u_{k_0+21}, q_{k_0+21}) = (u_1, q_1)$;

(17) For each j, $1 \leqslant j \leqslant k_0+2$, the following (17.1) and (17.2) hold.

(17.1) $\zeta(E, \mathrm{in}(P_{j0})) = u_{j-11}$ and $\zeta(E, \mathrm{fn}(P_{j0})) = u_{j1}$;

(17.2) P_{j0} strictly spells w_j' with (q_{j-11}, q_{j1}, c_j), where $w_1' = E_{w_1}(k_0)$, $w_{k_0+2}' = E_{w_3}(k_0)$, and for each k, $2 \leqslant k \leqslant k_0+1$, $w_k' = E_{w_2}(k_0)$.

It also holds that $c = \max\{c_j \mid 1 \leqslant j \leqslant k_0+2\}$. We consider two subcases.

Case (2.1): $c \leqslant 1$. Then P_1 is a path of degree zero of E. Since $l(E) < k_0$, there exist j_0, $2 \leqslant j_0 \leqslant k_0+1$, such that $c_{j_0} = 0$, that is, $l(P_{j_00}) = 0$, $u_{j_0-11} = u_{j_01}$, and $q_{j_01} = M_{j_1}(q_{j_0-11}, E_{w_2}(k_0))$, where $\Delta(P_{j_00}) = b_{j_1}$, and $q_{j_01}, q_{j_0-11} \in Q_{j_1}$, $1 \leqslant j_1 \leqslant m$. By definition of $S(w, g_6, \equiv, k_0)$, it holds that $(u_{j_01}, q_{j_01}, c_1') \in f_{(E_{x_1}(k_0), a_1, \ldots, E_{x_{i_1}}(k_0), a_{i_1})}(u_0, q_0)$, where $c_1' = \max\{c_1, c_2, \ldots, c_{j_0}\}$. For each j, $1 \leqslant j \leqslant i_1$, we can apply the inductive hypothesis. So as in Case (1), we can see that there exist $E_1' \in \mathrm{CSRE}(\Delta)$ and a path P_1' of E_1' for which the following hold.

(18) $h(E_1') \leqslant h(E)$ and $\delta(|E_1'|) \subset R$;

(19) $\zeta(E_1', \mathrm{in}(P_1')) = u_0$ and $\zeta(E_1', \mathrm{fn}(P_1')) = u_{j_01}$;

(20) P_1' strictly spells $w_1 w_2$ with (q_0, q_{j_01}, c_1');

(21) $v_0(P'_1) \leqslant g_6 \cdot (o_1(I(w)-1)+1)$;

(22) $l(E'_1) \leqslant 4^{g_6} \cdot (h(E_0)+1)^{g_6} \cdot 2 \cdot o_2(I(w)-1)$.

Because $c_{j_0}=0$ and $M_{d0w_2}=M_{d0E_{w_2}(k_0)}$ by Proposition 5.17, it holds that

(23) $d((\alpha(\delta(\Delta_{E'_1}(P'_1))), q_{j_0-1 1}), w_2, (\alpha(\delta(\Delta_{E'_1}(P'_1))), q_{j_0 1}))=0$.

Since $I(w_2)=I(w)$, it follows by Proposition 5.7 that

(24) $(\alpha(\delta(\Delta_{E'_1}(P'_1))), q_{j_0 1}) \in M_{d0}(Q, w_1 w_2)$.

By definition of $S(w, g_6, \equiv, k_0)$, it holds that $(u_1, q_1, c'_2) \in f_{(E_{x_{i_1+1}}(k_0), a_{i_1+1}, \ldots, E_{x_n}(k_0), a_n)}(u_{j_0 1}, q_{j_0 1})$ where $c'_2=\max\{c_{j_0+1}, \ldots, c_n\}$. As above, we can see that there exist $E'_2 \in \mathrm{CSRE}(\Delta)$ and a path P'_2 of E'_2 for which the following hold.

(25) $h(E'_2) \leqslant h(E)$ and $\delta(|E'_2|) \subset R$;

(26) $\zeta(E'_2, \mathrm{in}(P'_2))=u_{j_0 1}$ and $\zeta(E'_2, \mathrm{fn}(P'_2))=u_1$;

(27) P'_2 strictly spells w_3 with $(q_{j_0 1}, q_1, c'_2)$;

(28) $v_0(P'_2) \leqslant g_6 \cdot (o_1(I(w)-1)+1)$;

(29) $l(E'_2) \leqslant 4^{g_6} \cdot (h(E_0)+1)^{g_6} \cdot 2 \cdot o_2(I(w)-1)$.

From E'_1, E'_2, P'_1, and P'_2, we can construct $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w, where (31) follows from Proposition 7.1.

(30) $v_0(P_2) \leqslant 2 \cdot g_6(o_1(I(w)-1)+1)$;

(31) $l(E') \leqslant 4^{g_6+1} \cdot (h(E_0)+1)^{g_6+1} \cdot 2 \cdot o_2(I(w)-1)$.

Case (2.2): $c \geqslant 2$. In this case P_1 is not a path of degree zero. We consider two paths, P_{10} and $P'_2=P_{20} \cdots P_{k_0+20}$. Then the following hold.

(32) P_{10} strictly spells $E_{w_1}(k_0)$ with (q_0, q_{11}, c_1);

(33) P'_2 strictly spells $(E_{w_2}(k_0))^{k_0} E_{w_3}(k_0)$ with (q_{11}, q_1, c'_2) for some $c'_2 \in \{0, 1, 2, 3\}$.

It also holds that $c=\max\{c_1, c'_2\}$. Thus at least one of c_1 and c'_2 is greater than one. By definition of $S(w, g_6, \equiv, k_0)$, we can see that $(u_1, q_1, c'_2) \in f_{(E_{x_{i_0+1}}(k_0), a_{i_0+1}, \ldots, E_{x_n}(k_0), a_n)}(\zeta(E, \mathrm{in}(P'_2)), q_{11})$. From Cases (1), (2.1) and Proposition 7.1, we can see that there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w.

(34) $v_0(P_2) \leqslant 2 \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(35) $l(E') \leqslant 4^{g_6+1} \cdot (h(E_0)+1)^{g_6+1} \cdot 2 \cdot o_2(I(w)-1)$.

Case (3): $n=g_6$ and $x_{n+1} \neq \lambda$. In this case,

(36) $E_w(k_0)=E_{w_1}(k_0)(E_{w_2}(k_0))^{k_0} \cdot E_{w_3}(k_0)\, E_{x_{n+1}}(k_0)$.

From Cases (1), (2), it easy to see that there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following (37), (38) hold for w.

(37) $\nu_0(P_2) \leqslant 3 \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(38) $l(E') \leqslant 4^{g_6+2} \cdot (h(E_0)+1)^{g_6+2} \cdot 2 \cdot o_2(I(w)-1)$.

Case (4): $n = kg_6$ for some $k \geqslant 2$ and $x_{n+1} = \lambda$. In this case, $E_w(k_0)$ is of the form $E_w(k_0) = E_{w_1}(k_0)\, E_{w_2}(k_0) \cdots E_{w_k}(k_0)$, where $S(w, g_6) = (w_1, w_2, ..., w_k, \lambda)$. There exist k paths $P_{10}, ..., P_{k0}$ of E, $(u_{01}, q_{01}), ..., (u_{k1}, q_{k1}) \in \eta(\xi(E_0), R)$, and $c_1, ..., c_k \in \{0, 1, 2, 3\}$ for which the following hold.

(39) $P_1 = P_{10} \cdots P_{k0}$;

(40) $(u_{01}, q_{01}) = (u_0, q_0)$ and $(u_{k1}, q_{k1}) = (u_1, q_1)$;

(41) $c = \max\{c_i \mid 1 \leqslant i \leqslant k\}$;

(42) For each i, $1 \leqslant i \leqslant k$, the following (42.1), (42.2) hold.

(42.1) $\zeta(E, \mathrm{in}(P_{i0})) = u_{i-11}$ and $\zeta(E, \mathrm{fn}(P_{i0})) = u_{i1}$;

(42.2) P_{i0} strictly spells $E_{w_i}(k_0)$ with (q_{i-11}, q_{i1}, c_i).

From Case (3), it holds that for each i, $1 \leqslant i \leqslant k$, there exist $E'_i \in \mathrm{CSRE}(\Delta)$ and a path P_{2i} for which the following hold.

(43) $h(E'_i) \leqslant h(E)$ and $\delta(|E'_i|) \subset R$;

(44) $\zeta(E'_i, \mathrm{in}(P_{2i})) = u_{i-11}$ and $\zeta(E'_i, \mathrm{fn}(P_{2i})) = u_{i1}$;

(45) $\nu_0(P_{2i}) \leqslant 2 \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(46) $l(E') \leqslant 4^{g_6+1} \cdot (h(E_0)+1)^{g_6+1} \cdot 2 \cdot o_2(I(w)-1)$;

(47) P_{2i} strictly spells w_i with (q_{i-11}, q_{i1}, c_i).

We consider three subcases.

Case (4.1): $c \leqslant 1$. If $c = 0$, then the assertions are trivial. Let $c = 1$. From Case (2.1), we can see that for each i, $1 \leqslant i \leqslant k$, there exist $(u_{i2}, q_{i2}) \in \eta(\xi(E_0), R)$, two paths P_{2i0} and P_{2i1} of E'_i, and $c_{i0}, c_{i1} \in \{0, 1\}$ for which the following hold, where $S(w_i, g_6, \equiv, k_0) = ((j_{i0}, j_{i1}), (w_{i1}, w_{i2}, w_{i3}))$.

(48) $P_{2i} = P_{2i0} \cdot P_{2i1}$;

(49) P_{2i0} strictly spells $w_{i1} w_{i2}$ with $(q_{i-11}, q_{i2}, c_{i0})$;

(50) P_{2i1} strictly spells w_{i3} with (q_{i2}, q_{i1}, c_{i1});

(51) $c_i = \max\{c_{i0}, c_{i1}\}$;

(52) From (24), $(\alpha(\delta(\Delta_{E'_i}(P_{2i0}))), q_{i2}) \in M_{d0}(Q, w_{i1} w_{i2})$.

By definition of $\mathscr{B}$, E'_i and by induction on i, this implies that

(53) $(\alpha(\delta(\Delta_{E'_1}(P_{21}) \circ \Delta_{E'_2}(P_{22}) \circ \cdots \circ \Delta_{E'_{i-1}}(P_{2i-1}) \circ \Delta_{E'_i}(P_{2i0}))), q_{i2}) \in M_{d0}(Q, w_{i1} w_{i2})$.

Since $I(w)=I(w_{i1}w_{i2})$, it follows from Proposition 5.7 that for each i, $1\leqslant i\leqslant k$, there exist $(t_i, q_{i3})\in Q$ such that $M_{d0}((t_i, q_{i3}), w_1w_2\cdots w_{i-1}w_{i1}w_{i2}) = (\alpha(\delta(\Delta_{E'_1}(P_{21})\circ\Delta_{E'_2}(P_{22})\circ\cdots\circ\Delta_{E'_{i-1}}(P_{2i-1})\circ\Delta_{E'_i}(P_{2i0}))), q_{i2})$.
Then by Proposition 7.3, there exist $E'\in\mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w.

(54)

$$\begin{aligned}v_0(P_2)&\leqslant 2\cdot(\#Q+1)\cdot\max\{v_0(P_{2i})\mid 1\leqslant i\leqslant k\}\\&\leqslant 2\cdot(\#Q+1)\cdot g_6\cdot(o_1(I(w)-1)+1);\end{aligned}$$

(55)

$$\begin{aligned}l(E')&\leqslant l(E'_1)+l(E'_k)+v_0(P_2)\\&\leqslant 2\cdot 4^{g_6+1}\cdot(h(E_0)+1)^{g_6+1}\cdot 2\cdot o_2(I(w)-1)+o_2(I(w)-1)\\&\leqslant 4^{g_6+3}(h(E_0)+1)^{g_6+1}\cdot o_2(I(w)-1),\end{aligned}$$

where the last two inequalities follow from Proposition 5.19.

Case (4.2): $c=2$. Let $i_1, i_2, ..., i_{n0}$ be integers such that (i) $1\leqslant i_1 < i_2<\cdots<i_{n0}\leqslant k$, (ii) for each j, $1\leqslant j\leqslant n_0$, $c_{i_j}=2$, and (iii) for each $j\in\{1, ..., k\}-\{i_1, ..., i_{n0}\}$, $c_j\leqslant 1$. Then we have the following decomposition of w, $(w'_1, ..., w'_{n0}, w'_{n0+1})\in\mathrm{Dec}(w)$, where $w'_1=w_1\cdots w_{i_1}$, $w'_{n0+1}=w_{n0+1}\cdots w_k$, and for each j, $2\leqslant j\leqslant n_0$, $w'_j=w_{i_{j-1}+1}\cdots w_{i_j}$. Note that w'_{n0+1} may be λ. From Cases (4.1), (2.1) and Proposition 7.1, we can see that for each j, $1\leqslant j\leqslant n_0+1$, there exist $E'_j\in\mathrm{CSRE}(\Delta)$, a path P_{2j} of E'_j, and $(u_{j10}, q_{j10})\in\eta(\xi(E_0), R)$ for which the following hold.

(56) $h(E'_j)\leqslant h(E)$ and $\delta(|E'_j|)\subset R$;

(57) $\zeta(E'_j, \mathrm{in}(P_{2j}))=u_{j-110}$ and $\zeta(E'_j, \mathrm{fn}(P_{2j}))=u_{j10}$, where $u_{010}=u_0$, and $u_{n0+110}=u_1$;

(58) P_{2j} strictly spells w'_j with $(q_{j-110}, q_{j10}, 2)$, where $q_{010}=q_0$ and $q_{n0+110}=q_1$;

(59)

$$\begin{aligned}v_0(P_{2j})&\leqslant 2\cdot g_6\cdot(o_1(I(w)-1)+1)+2\cdot(\#Q+1)\cdot g_6\cdot(o_1(I(w)-1)+1)\\&\leqslant 2\cdot g_6\cdot(\#Q+2)\cdot(o_1(I(w)-1)+1);\end{aligned}$$

(60) $l(E'_j)\leqslant 4^{g_6+4}\cdot(h(E_0)+1)^{g_6+2}\cdot o_2(I(w)-1)$.

Now we define the set $B(w)$ as follows: $B(w)=\{(u_{j10}, q_{j10})\mid 0\leqslant j\leqslant n_0+1\}$. By induction on $\#B(w)$, we shall prove the following claim.

CLAIM. *There exist $E'\in\mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which* (3)–(6) *and the following hold for w.*

(61) $v_0(P_2) \leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(62) $l(E') \leqslant g_7(I(w)-1, \#B(w))$.

Proof of the Claim. When $n_0 = 1$, the assertions are clear. So assume that $n_0 > 1$.

Basis: $\#B(w) = 1$. We first note that by Proposition 3.4 and (60), the following hold.

(63) $\#\{E'_j \mid 1 \leqslant j \leqslant n_0 + 1\} \leqslant (g_0(n_1))^2$, where $n_1 = 4^{g_6+4} \cdot (h(E_0)+1)^{g_6+2} \cdot o_2(I(w)-1)$.

By Proposition 7.4, there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P'_1 of E' for which the following hold.

(64) $h(E') \leqslant h(E)$ and $\delta(|E'|) \subset R$;

(65) $\zeta(E', \mathrm{in}(P'_1)) = u_0 = \zeta(E', \mathrm{fn}(P'_1)) = u_1$;

(66) P'_1 strictly spells $w = w'_1 \cdots w'_{n_0+1}$ with $(q_0, q_1 = q_0, 2)$;

(67)

$$\begin{aligned} v_0(P'_1) &\leqslant 2 \cdot \max\{v_0(P_{2j}) \mid 1 \leqslant j \leqslant n_0\} \\ &\leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1); \end{aligned}$$

(68)

$$\begin{aligned} l(E') &\leqslant ((g_0(n_1))^2+2)^{2 \cdot h(E_0)} \cdot (\max\{l(E'_j) \mid 1 \leqslant j \leqslant n_0+1\})^{2 \cdot h(E_0)} \\ &\leqslant ((g_0(n_1))^2+2)^{2 \cdot h(E_0)} \cdot (n_1)^{2 \cdot h(E_0)} \\ &\leqslant g_7(I(w)-1, 1). \end{aligned}$$

So the assertions hold.

In ductive step $\#B(w) > 1$. There exist integers $r_0, r_1, \ldots, r_{n_2}$ for which the following hold.

(69) $O = r_0 < r_1 < \cdots < r_{n_2} \leqslant n_0 + 1$;

(70) For each j, $1 \leqslant j \leqslant n_2$, $(u_{r_j 10}, q_{r_j 10}) = (u_0, q_0)$, and for each $j \in \{1, \ldots, n_0+1\} - \{r_1, \ldots, r_{n_2}\}$, $(u_{r_j - 10}, q_{r_j 10}) = (u_0, q_0)$.

If $n_2 = 0$, then by induction, there exist $E'' \in \mathrm{CSRE}(\Delta)$ and a path P'_2 of E'' for which the assertions hold for $w'_2 w'_3 \cdots w'_{n_0+1}$. Then by Proposition 7.1, there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w.

(71)

$$\begin{aligned} v_0(P_2) &\leqslant 2 \cdot \max\{v_0(P_{2j}) \mid 1 \leqslant j \leqslant n_0+1\} \\ &\leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1); \end{aligned}$$

(72) $l(E') \leqslant 4 \cdot (h(E_0)+1) \cdot g_7(I(w)-1,\ \#B(w)-1)$.

Now assume that $n_2 > 0$. Then for each j, $1 \leqslant j \leqslant n_2$, there exist $E_j'' \in \mathrm{CSRE}(\Delta)$ and a path P_{2j}' of E_j' for which the assertions hold for $w_j'' = w_{r_{j-1}+1}' \cdots w_{r_j}'$. By the inductive hypothesis, there exist E_{n_2+1}'' and a path P_{2n_2+1}' of E_{n_2+1}'' for which the assertions hold for $w_{n_2+1}'' = w_{r_{n_2}+1}' \cdots w_{n_0+1}'$, where $w_{n_2+1}'' = \lambda$ if $r_{n_2} = n_0 + 1$. By Proposition 3.4 and (72), $\#\{E_j'' \mid 1 \leqslant j \leqslant n_2\} \leqslant (g_0(n_3))^2$, where $n_3 = 4 \cdot (h(E_0)+1) \cdot g_7((I(w)-1,\ \#B(w)-1)$. By Proposition 7.4, there exist $E'' \in \mathrm{CSRE}(\Delta)$ and a path P_2' of E'' for which (3)–(6) and the following hold for $w_1'' \cdots w_{n_2}''$.

(73) $v_0(P_2') \leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(74) $l(E'') \leqslant ((n_4+2) \cdot 4 \cdot (h(E_0)+1) \cdot g_7(I(w)-1,\ \#B(w)-1))^{2 \cdot h(E_0)}$, where $n_4 = (g_0(n_3))^2$.

By Proposition 7.1, there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w.

(75) $v_0(P_2) \leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(76)

$$\begin{aligned} l(E') &\leqslant 4 \cdot (h(E_0)+1) \cdot (n_4+2)^{2 \cdot h(E_0)} \\ &\quad \cdot (4 \cdot (h(E_0)+1) \cdot g_7(I(w)-1,\ \#B(w)-1))^{2 \cdot h(E_0)} \\ &\leqslant (4 \cdot (h(E_0)+1) \cdot (n_4+2) \cdot g_7(I(w)-1,\ \#B(w)-1))^{3 \cdot h(E_0)} \\ &\leqslant g_7(I(w)-1,\ \#B(w)). \end{aligned}$$

This completes the proof of the claim. ∎

Proof of the Main Lemma (Continued). By Lemma 5.1, $\#B(w) \leqslant \#\eta(\xi(E_0), R) < g_3$. By the claim, there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w.

(77) $v_0(P_2) \leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(78) $l(E') \leqslant g_7(I(w)-1, g_3)$.

Case (4.3): $c = 3$. Let $i_1, ..., i_{n_5}$ be integers such that (i) $1 \leqslant i_1 < i_2 < \cdots < i_{n_5} \leqslant k$, (ii) for each j, $1 \leqslant j \leqslant n_5$, $c_{i_j} = 3$, and (iii) for each $j \in \{1, ..., k\} - \{i_1, ..., i_{n_5}\}$, $c_j \leqslant 2$. If $n_5 = 1$, then from Cases (4.1), (4.2), and Proposition 7.1, there exist $E'' \in \mathrm{CSRE}(\Delta)$ and a path P_2' of E'' for which (3)–(6) and the following hold for $w_1 \cdots w_{i_{n_5}}$.

(79) $v_0(P_2') \leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(80) $l(E') \leqslant 4 \cdot (h(E_0)+1) \cdot g_7(I(w)-1, g_3)$.

By Proposition 7.1, there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w.

(81) $v_0(P_2) \leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(82) $l(E') \leqslant 4^2 \cdot (h(E_0)+1)^2 \cdot g_7(I(w)-1, g_3)$.

Now assume that $n_5 > 1$. Since $E \in \xi(E_0)$, $n_5 \leqslant l(E) < k_0$. By Proposition 7.2, there exist $E' \in \mathrm{CSRE}(\Delta)$ and P_2 of E' for which (3)–(6), (81) and the following hold for w.

(83)

$$\begin{aligned} l(E') &\leqslant 4^{k_0-2} \cdot 4^2 \cdot (h(E_0)+1)^2 \cdot g_7(I(w)-1, g_3) \cdot (h(E_0))^{k_0-2} \\ &\leqslant 4^{k_0} \cdot (h(E_0)+1)^{k_0} \cdot g_7(I(w)-1, g_3). \end{aligned}$$

Case (5): $k \cdot g_6 < n < (k+1) \cdot g_6$ or $n = k \cdot g_6$ and $x_{n+1} \neq \lambda$ for some $k \geqslant 2$. Then $S(w, g_6) = (w_1, ..., w_{k+1})$, where either $I(w_{k+1}) < I(w)$ or the length of $D(w, I(w))$ is smaller than g_6. From Case (4) and Proposition 7.1, there exist $E' \in \mathrm{CSRE}(\Delta)$ and a path P_2 of E' for which (3)–(6) and the following hold for w.

(84) $v_0(P_2) \leqslant 4 \cdot (\#Q+2) \cdot g_6 \cdot (o_1(I(w)-1)+1)$;

(85) $l(E') \leqslant 4^{k_0+1} \cdot (h(E_0)+1)^{k_0+1} \cdot g_7(I(w)-1, g_3) = o_2(I(w))$.

This completes the proof of the main lemma. ∎

Acknowledgments

I thank Christin Choffrut, Bruno Courcelle, Jean-Paul Mascle, Jacques Sakarovitch, Helmut Seidl, Imre Simon, and Andreas Weber for many discussions and criticisms on the topics of star height and D-automata during my stay at Université Paris 6 in 1986, from which many ideas in this paper come. I also express my special thanks to Jaques Sakaravitch for inviting me to LITP, Université Pierre et Marie Curie.

Received June 3, 1987

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Printed in Belgium