Closure properties of synchronized relations

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[1] Joint work with D. Figueira and S. Figueira

Synchronized pairs of words over $\mathbb{A}$

Synchronizing pairs of words

A synchronization of $(w_1, w_2) \in \mathbb{A}^{*2}$ is a word over $\{1, 2\} \times \mathbb{A}$ so that the projection on $\mathbb{A}$ of positions labeled i is exactly w_i for $i = 1, 2$.

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Example

$(1,a)(1,b)(2,a)$ and $(1,a)(2,a)(1,b)$ synchronize (ab, a).

Every word $w \in (\{1,2\} \times \mathbb{A})^*$ is a synchronization of a unique pair (w_1, w_2) that we denote $[\![w]\!]$.

$$[\![(1,a)(1,b)(2,a)]\!] = [\![(1,a)(2,a)(1,b)]\!] = (ab, a).$$

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We lift this notion to languages $L \subseteq (\{1,2\} \times \mathbb{A})^*$

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Example

$\mathbb{A} = \{a, b\},\ L = ((1,a)(2,a) \cup (1,a)(2,b) \cup (1,b)(2,a) \cup (1,b)(2,b))^*,$

$$[\![L]\!] = \{(w_1, w_2) \mid |w_1| = |w_2|\}.$$

C-controlled relations

Restrictions on the shape of the projection over $\{1, 2\}$

$\rightsquigarrow$

Infinitely many different classes of relations.

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C-controlled languages

$C \subseteq_{reg} \{1,2\}^*$
-$w \in (\{1,2\} \times \mathbb{A})^*$ is C-controlled if its projection over $\{1,2\}$ lies in C.
-$L \subseteq (\{1,2\} \times \mathbb{A})^*$ is C-controlled if all its words are.

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Examples

-Everything is $\{1,2\}^*$-controlled,
-$(1,a)(1,b)(2,a)$ is 1^*2^*-controlled,
-$(1,a)(2,a)(1,b)$ isn't 1^*2^*-controlled,
-L (previous slide) is $(12)^*$-controlled.

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C-controlled relations

Given $C \subseteq_{reg} \{1,2\}^*$

$$\textsc{Rel}(C) = \{[\![L]\!] \mid L \text{ is reg. and } C\text{-controlled}\}$$

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Examples

-$\textsc{Rel}(1^*2^*) =$REC,
-$\textsc{Rel}((12)^*(1^* \cup 2^*)) =$REG,
-$\textsc{Rel}(\{1,2\}^*) =$RAT.

Closure properties

Closure under union problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\textsc{Rel}(C)$ closed under union?

Closure properties

Closure under intersection problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\textsc{Rel}(C)$ closed under intersection?

Closure properties

Closure under complement problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\mathrm{Rel}(C)$ closed under complement?

Closure properties

Closure under concatenat. problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\textsc{Rel}(C)$ closed under concatenat.?

Closure properties

Closure under Kleene star problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\text{Rel}(C)$ closed under Kleene star?

Closure properties

Closure under union problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\mathrm{Rel}(C)$ closed under union?

– $\mathrm{Rel}(C)$ is closed under union for all $C \subseteq_{reg} \{1,2\}^*$.

Closure properties

Closure under concatenat. problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\mathrm{Rel}(C)$ closed under concatenat.?

- $\mathrm{Rel}(C)$ is closed under union for all $C \subseteq_{reg} \{1,2\}^*$.
- $\mathrm{Rel}(C)$ is closed under concatenation if, and only if $C \cdot C \subseteq_{\mathrm{Rel}} C$.

Closure properties

Closure under Kleene star problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\textsc{Rel}(C)$ closed under Kleene star?

- $\textsc{Rel}(C)$ is closed under union for all $C \subseteq_{reg} \{1,2\}^*$.
- $\textsc{Rel}(C)$ is closed under concatenation if, and only if $C \cdot C \subseteq_{\textsc{Rel}} C$.
- $\textsc{Rel}(C)$ is closed under Kleene star if, and only if $C^* \subseteq_{\textsc{Rel}} C$.

Closure properties

Closure under intersection problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\text{Rel}(C)$ closed under intersection?

- $\text{Rel}(C)$ is closed under union for all $C \subseteq_{reg} \{1,2\}^*$.
- $\text{Rel}(C)$ is closed under concatenation if, and only if $C \cdot C \subseteq_{\text{Rel}} C$.
- $\text{Rel}(C)$ is closed under Kleene star if, and only if $C^* \subseteq_{\text{Rel}} C$.
- Closure under intersection is more difficult.

Closure properties

Closure under complement problem	
Input:	$C \subseteq_{reg} \{1,2\}^*$
Question:	Is $\mathrm{Rel}(C)$ closed under complement?

- $\mathrm{Rel}(C)$ is closed under union for all $C \subseteq_{reg} \{1,2\}^*$.
- $\mathrm{Rel}(C)$ is closed under concatenation if, and only if $C \cdot C \subseteq_{\mathrm{Rel}} C$.
- $\mathrm{Rel}(C)$ is closed under Kleene star if, and only if $C^* \subseteq_{\mathrm{Rel}} C$.
- Closure under intersection is more difficult.
- Closure under complement follows from closure under intersection.

Positive examples

$\textsc{Rel}(1^*2^*)$ is closed under intersection.

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Parikh-image

-$u \in \{1,2\}^*$, $\pi(u) = (|u|_1, |u|_2)$.

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$C \subseteq \{1,2\}^*$ is

- Parikh-injective if
$$\forall u \neq v \in C, \pi(u) \neq \pi(v).$$
- Parikh-surjective if
$$\pi(C) = \mathbb{N}^2.$$
- Parikh-bijective if it is both.

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$\mathrm{REL}(1^*2^*)$ is closed under intersection.

$\mathrm{REL}(C)$ is closed under intersection $\forall C$ such that $\exists D$ Parikh-injective with $C =_{\mathrm{REL}} D$.

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-$\text{Rel}((12)^*1^*2^*)$ is closed under intersection;
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$\mathrm{Rel}(C)$ is closed under intersection $\forall C$ such that $\exists D, X$ Parikh-injective with $X \subseteq_{\mathrm{Rel}} 1^*2^*$ and $C =_{\mathrm{Rel}} D \cup X$.

Negative examples

$\textsc{Rel}(1^*(12)^*2^*)$ is not closed under intersection.

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$\textsc{Rel}((12)^*1^* \cup 1^*(12)^*)$ is not closed under intersection.

Characterization

Characterization of closure under intersection

Given $C \subseteq_{reg} \{1,2\}^*$, the following are equivalent:

1. $\textsc{Rel}(C)$ is closed under intersection;
2. for all $R, S \in \textsc{Rel}(C)$, $R \cap S \in \mathrm{RAT}$;
3. there exist Parikh-injective languages $X, D \subseteq_{reg} \{1,2\}^*$ such that $X \subseteq_{\textsc{Rel}} 1^*2^*$ and $C =_{\textsc{Rel}} D \cup X$;
4. there exists a Parikh-injective language $D \subseteq_{reg} \{1,2\}^*$ such that $C \subseteq_{\textsc{Rel}} D$.

Examples

$\text{REL}(1^*2^*)$ is closed under complement.

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$\text{REL}(C)$ is closed under complement for every C such that there exists a Parikh-bijective D with $C =_{\text{REL}} D$.

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Characterization of closure under complement

Given $C \subseteq_{reg} \{1,2\}^*$, the following are equivalent:

1. $\textsc{Rel}(C)$ is closed under complement;
2. there exists a Parikh-bijective language $D \subseteq_{reg} \{1,2\}^*$ such that $C =_{\textsc{Rel}} D$.

Summary of results

The closure under intersection (resp. complement, concatenation, Kleene star) problem is decidable.

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Classes that are closed under complement have decidable containment and universality problems.

For $C \subseteq_{reg} \{1,2\}^*$, if $\textsc{Rel}(C)$ is closed under intersection, for all $R, S \in \textsc{Rel}(C)$, we can effectively construct a language $L \subseteq_{reg} (\{1,2\} \times \mathbb{A})^*$ such that $[\![L]\!] = R \cap S$. Similarly for complement, concatenation and Kleene star.

Future work

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– Extended CRPQ's?

Thanks for your attention!