The Complexity of Infinite Advice Strings

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Infinite words

Infinite sequences of letters (over a finite alphabet)

$abdabaddcbadbcabdbbcad\cdots$

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Why?

- describe real numbers
- model data streams for online algorithms, unbounded runs...
- relations to logic
- + number theory, physics, biology...

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Rational vs irrational

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$101001000100001000000\cdots$.

Several possible definitions

- smallest "program" defining α? **Kolmogorov complexity**
- number of finite factors in α? **Subword complexity**
- is α computable from β? **Turing degrees**

The (intuitive) complexity of infinite words

Our notion of complexity

- infinite words are used as *advices* ($\simeq$ oracle, definitions later)
  $\mathscr{C}[\alpha]$ the class of "what can be done" with advice α

- α **"simpler" than** β iff $\mathscr{C}[\alpha] \subseteq \mathscr{C}[\beta]$ holds

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Our results: equivalent characterizations

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Basic ingredients

Outline

1. Advice regular languages
2. Automatic structures
3. Advice automatic structures and transductions
4. A new framework: the two-way transductions hierarchy

Advice regular languages

Regular languages with advice [Salomaa, 1968]

Advice automata

Acceptance condition: end the run in an accepting state.

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Advice automata

Acceptance condition: end the run in an accepting state.

$\mathrm{Reg}[\alpha]$ = class of regular languages with advice α (fixed)

Example

$\mathrm{Pref}(\alpha)$ set of finite prefixes of α.

$\alpha = abba\cdots$

$\mathrm{Pref}(\alpha) = \{\varepsilon, a, ab, abb, abba, \ldots\}$

Regular languages with advice

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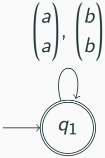

Regular languages with advice

Regular-like properties [Baer and Spanier, 1969]

- $\mathrm{Reg}[\alpha]$ is a boolean algebra;
- $\mathrm{Reg} \subseteq \mathrm{Reg}[\alpha]$
- $\mathrm{Reg} = \mathrm{Reg}[\alpha]$ iff α is ultimately periodic;
- ...

When does $\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$?

From advice power to transductions

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Mealy machine: letter-to-letter deterministic finite transducer.

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When does $\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$?

Mealy machine: letter-to-letter deterministic finite transducer.

Proposition

The following are equivalent:

1. $\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$;
2. there is a Mealy machine transforming β into α.

More or less trivial...

If ω-words were musical instruments

recorder
repertoire

clarinet
repertoire

If ω-words were musical instruments

Intrinsic complexity: $\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$

recorder repertoire

clarinet repertoire

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recorder repertoire

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Explicit transformation: $\alpha \longleftarrow \beta$

"Physical" simplification

Summary table

	LANGUAGES	STRUCTURES
Advice	$\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$	
Logic	★	
Machine	α image of β under a Mealy machine	

Automatic structures

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AutStr = class of automatic structures.

Automatic structures

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can be encoded
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AutStr = class of automatic structures.

Example

$(\mathbb{N}, +) \in$ AutStr

Decidability issues

Every structure in AutStr has a decidable FO theory.

Automatic structures: limitations

The rational group

- $(\mathbb{Q}, +)$ is a simple structure with decidable FO-theory;
- **but**. . .

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- $(\mathbb{Q}, +)$ is a simple structure with decidable FO-theory;
- **but**… $(\mathbb{Q}, +)$ is not an (ω-)automatic structure [Tsankov, 2011];
- **but**…

Automatic structures: limitations

The rational group

- $(\mathbb{Q}, +)$ is a simple structure with decidable FO-theory;
- **but**... $(\mathbb{Q}, +)$ is not an (ω-)automatic structure [Tsankov, 2011];
- **but**... $(\mathbb{Q}, +)$ can be represented with languages from Reg[α] for some α [Kruckman et al., 2012].

Advice automatic structures

$\mathrm{AutStr}[\alpha]$ = class of structures presentable with languages of $\mathrm{Reg}[\alpha]$.

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Decidability issues

If α has a decidable MSO-theory, any structure of $\mathrm{AutStr}[\alpha]$ has a decidable FO-theory.

Advice automatic structures and transductions

Advice strings classification w.r.t. structures

Question

When does $\mathrm{AutStr}[\alpha] \subseteq \mathrm{AutStr}[\beta]$ holds?

Advice strings classification w.r.t. structures

Question

When does $\mathrm{AutStr}[\alpha] \subseteq \mathrm{AutStr}[\beta]$ holds?

- more difficult & possibly more interesting than $\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$
- a "level of abstraction" higher

MSO-transductions: “regular” transformations of words

MSO-transductions on infinite words

$$0 \longrightarrow 1 \longrightarrow 1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots$$

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MSO-transductions on infinite words

1. make k copies of the word;

$$0 \longrightarrow 1 \longrightarrow 1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots$$

$$\begin{array}{cccccccc} 0 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \end{array}$$

MSO-transductions: "regular" transformations of words

MSO-transductions on infinite words

1. make k copies of the word;
2. remove/relabel the vertices in an MSO-definable way;

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\end{array}$$

MSO-transductions on infinite words

1. make k copies of the word;
2. remove/relabel the vertices in an MSO-definable way;
3. add new edges in an MSO-definable way.

$$0 \longrightarrow 1 \longrightarrow 1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots$$

$$\begin{tikzcd}
 & 1 \arrow[r] & 1 \arrow[d] & & 1 \arrow[dr] & & 1 \arrow[r] & \cdots \\
0 \arrow[ur] & & 1 \arrow[r] & 0 \arrow[ur] & & 0 \arrow[ur] & & \cdots
\end{tikzcd}$$

MSO-transductions

Example: reverse factor

There is an MSO-transduction transforming $\alpha := w_1 \# w_2 \# \cdots \in (\Gamma^* \#)^\omega$ into $\widetilde{\alpha} := \widetilde{w_1} \# \widetilde{w_2} \# \cdots$ (mirror images).

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$$\begin{tikzcd}
\alpha & a \arrow[r] & b \arrow[r] & \# \arrow[r] & b \arrow[r] & a \arrow[r] & a \arrow[r] & \# \arrow[r] & \cdots \\
\widetilde{\alpha} & a \arrow[rr, bend right] & b \arrow[l] & \# \arrow[rrr, bend left] & b \arrow[rrr, bend right] & a \arrow[l] & a \arrow[l] & \# \arrow[r, bend left] & \cdots
\end{tikzcd}$$

Automatic structures vs MSO-transductions

Theorem

The following are equivalent:

1. $\mathsf{AutStr}[\alpha] \subseteq \mathsf{AutStr}[\beta]$;
2. there is an MSO-transduction transforming β into α.

(based on [Colcombet and Löding, 2007])

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(based on [Colcombet and Löding, 2007])

+ also holds for variants of $\mathrm{AutStr}[\alpha]$: $\mathrm{AutStr}^{\infty}[\alpha]$, $\omega\mathrm{AutStr}[\alpha]$.

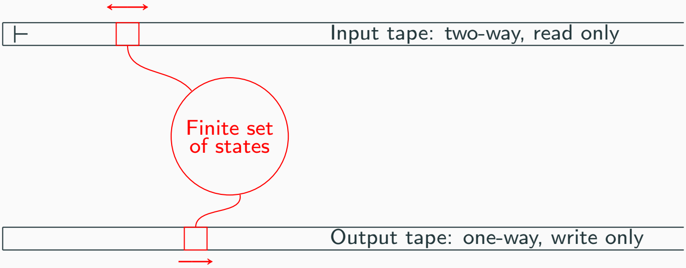
Summary table

	LANGUAGES	STRUCTURES
Advice	$\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$	$\mathrm{AutStr}[\alpha] \subseteq \mathrm{AutStr}[\beta]$ $\mathrm{AutStr}^{\infty}[\alpha] \subseteq \mathrm{AutStr}^{\infty}[\beta]$ $\omega\mathrm{AutStr}[\alpha] \subseteq \omega\mathrm{AutStr}[\beta]$
Logic	★	α image of β under an MSO-transduction
Machine	α image of β under a Mealy machine	

An equivalent computation model for MSO-transductions?

Two-way transducers

General idea



Two-way transducers

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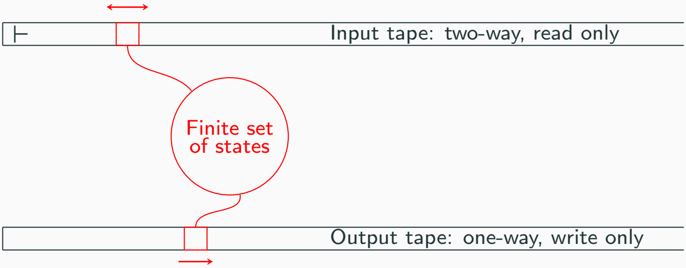

Example: reverse factor

There is a two-way finite transducer computing $\widetilde{\alpha} = \widetilde{w_1} \# \widetilde{w_2} \# \cdots$ from $\alpha = w_1 \# w_2 \# \cdots$.

MSO-transductions vs two-way transducers

Theorem: finite words [Engelfriet and Hoogeboom, 2001]

Over finite words, functions definable by MSO-transductions are exactly functions realized by two-way transducers.

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Theorem: infinite words [Alur et al., 2012]

Over infinite words, functions definable by MSO-transductions are exactly functions realized by two-way transducers with ω-regular lookaround (that read their whole input string).

Question: Can we avoid the lookaround for fixed-input transformations?

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Theorem

There is an MSO-transduction transforming β into α iff there is a two-way transducer transforming β into α.

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Logic	★	α image of β under an MSO-transduction
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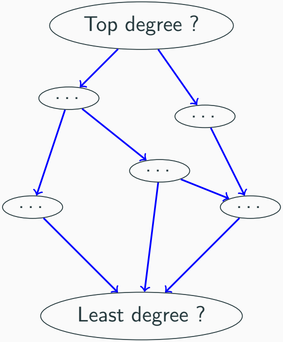
$+$ results for *variants* of the definition: languages over infinite words, etc.

A new framework: the two-way transductions hierarchy

Degrees of infinite words

Fix a preorder over infinite words

- classes of equally complex words: **degrees**
- properties of the poset of degrees. Do least/top degree, upper/lower bounds... exist? Is the hierarchy dense?



Two-way transducibility degrees

Our preorder

$\alpha \preccurlyeq_{\mathsf{2WFT}} \beta$ if α image of β under a two-way transducer.

$\rightarrow$ Motivation: meaning in terms of structures. *Unexplored !*

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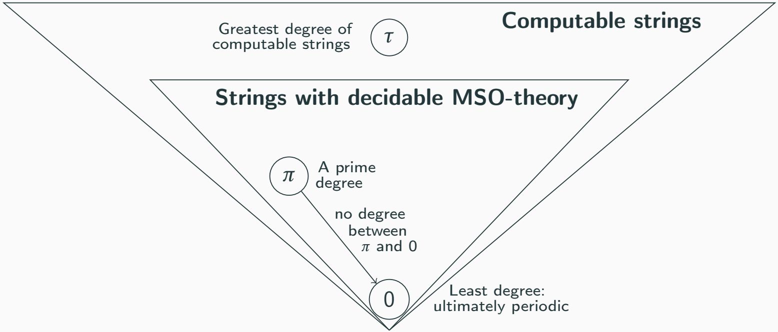
A similar work: [Endrullis et al., 2015]

$\alpha \preccurlyeq_{\mathsf{1WFT}} \beta$ if α image of β under a *one*-way transducer.

→ Motivation: meaning in terms of combinatorics. *Recent !*

Exploring the two-way transductions hierarchy



$\pi = 101001000100001\cdots$

Results

Least degree, prime degree, subhierarchies...

Discussion and outlook

Languages vs transductions

Example: over finite words

- Deterministic, or non-deterministic, or two-way (Boustrophedon) automata recognize the **same languages**.

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The author (along with many other people) has come recently to the conclusion that the functions computed by the various machines are more important - or at least more basic - than the sets accepted by these devices.
Dana Scott [Scott, 1967].

Conclusion

Towards a general theory?

Draw an non-trivial link between accepted languages (via presentable structures) and transductions.

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Open questions:

- the two-way transduction hierarchy
- words with an MSO-decidable theory ?
- specific advice automatic structures: groups, etc ?

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Towards a general theory?

Draw an non-trivial link between accepted languages (via presentable structures) and transductions.

Open questions:

- the two-way transduction hierarchy $\longleftarrow$ work (slowly) in progress
- words with an MSO-decidable theory ?
- specific advice automatic structures: groups, etc ?

Thank you !