Black Ninjas in the Dark: Formal Analysis of Population Protocols

Javier Esparza

Joint work with Michael Blondin, Pierre Ganty, Stefan Jaax, Antonín Kučera, Jérôme Leroux, Rupak Majumdar, Philipp J. Meyer, and Chana Weil-Kennedy

Technical University of Munich

European Research Council
Established by the European Commission

Deaf Black Ninjas in the Dark

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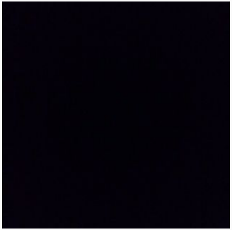

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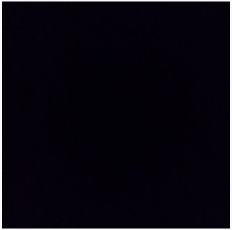
- They must decide **by majority** to attack or not (no attack if tie)
- **How can they conduct the vote?**



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active

don’t attack
active

attack
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- Initially: all ninjas active, estimation = own vote.

Goal of voting protocol:

- eventually all ninjas reach the same estimation, and
- this estimation corresponds to the majority.

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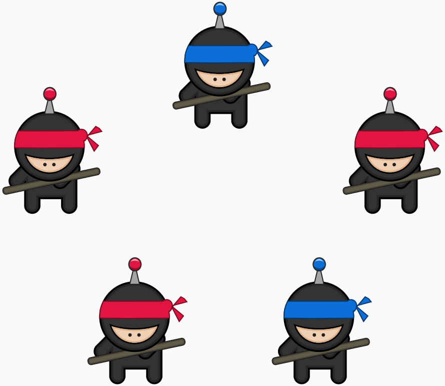
Graphically:

- Initially more red ninjas $\Longrightarrow$ eventually all ninjas red.
- Initially more blue ninjas or tie $\Longrightarrow$ eventually all ninjas blue.

Majority protocol: Are there more red ninjas than blue ninjas?



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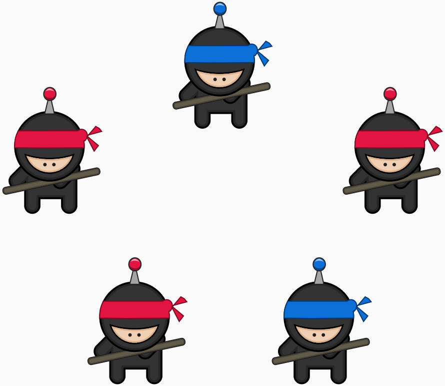
- Active ninjas of opposite colors become passive and blue



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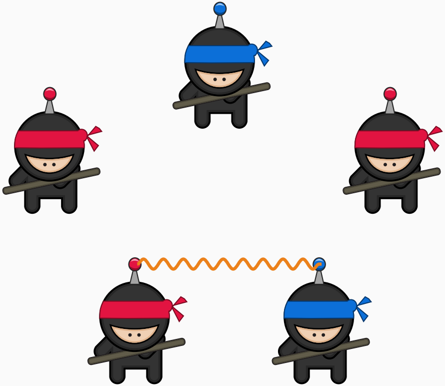

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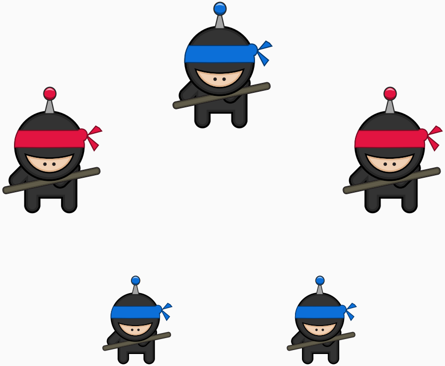

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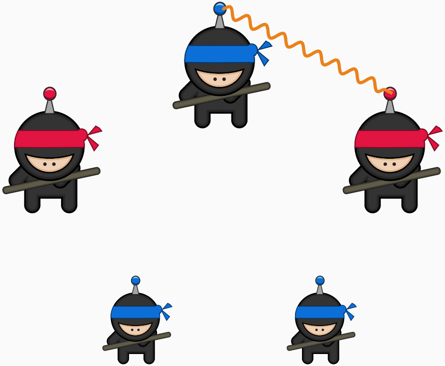

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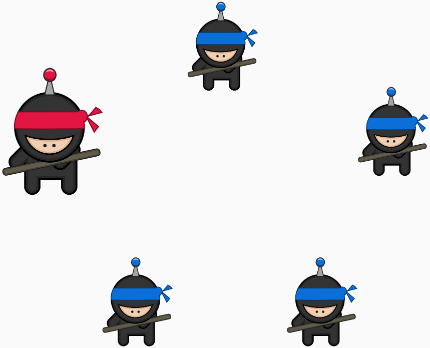

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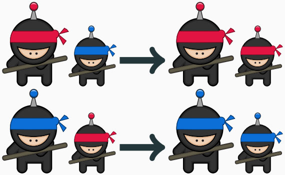

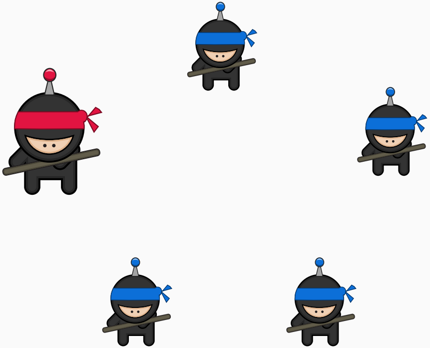

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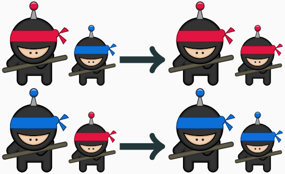

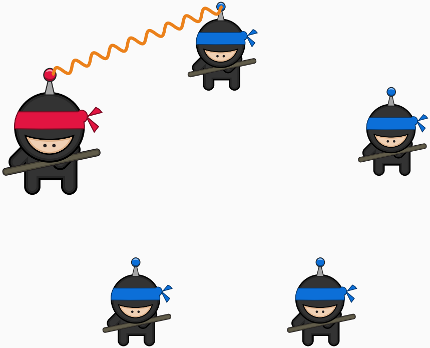

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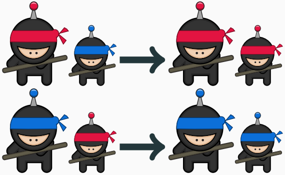

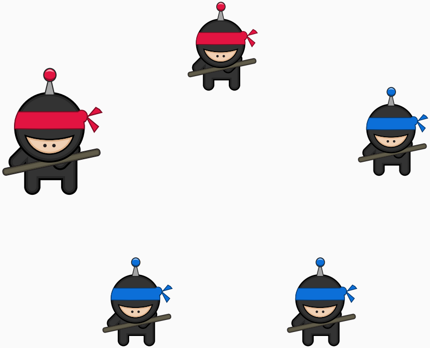

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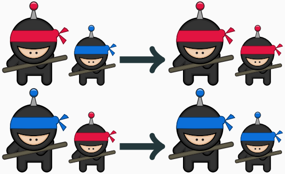

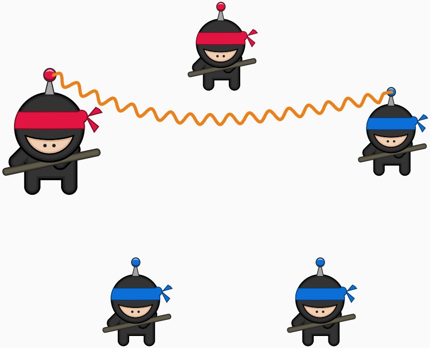

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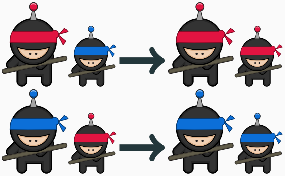

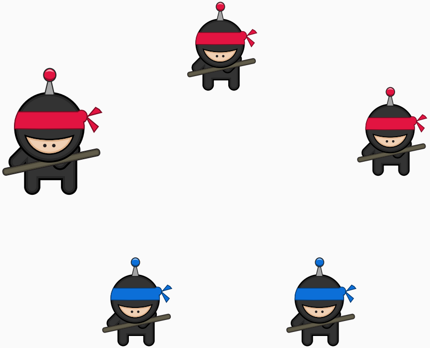

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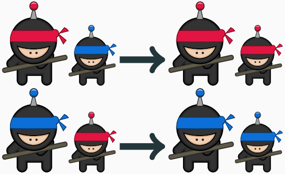

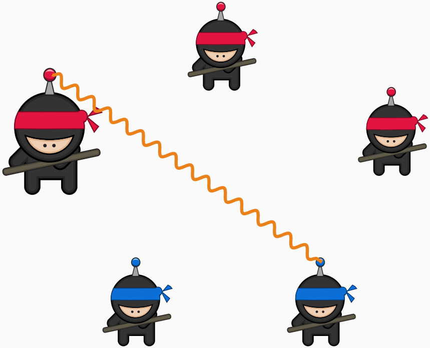

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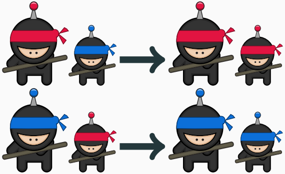

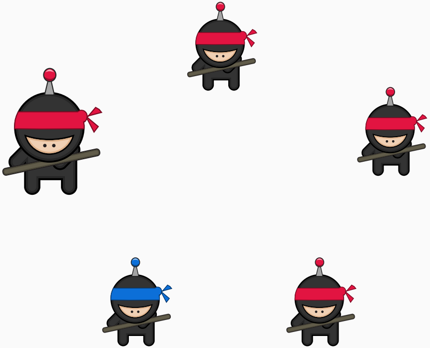

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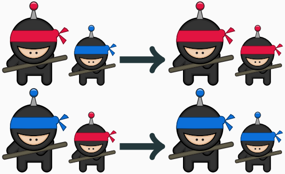

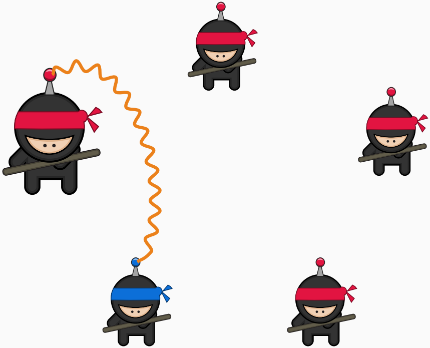

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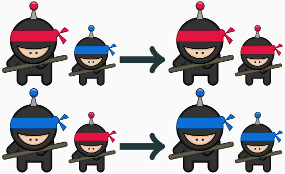

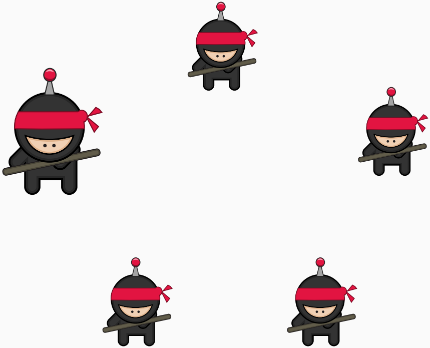

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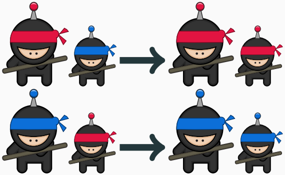
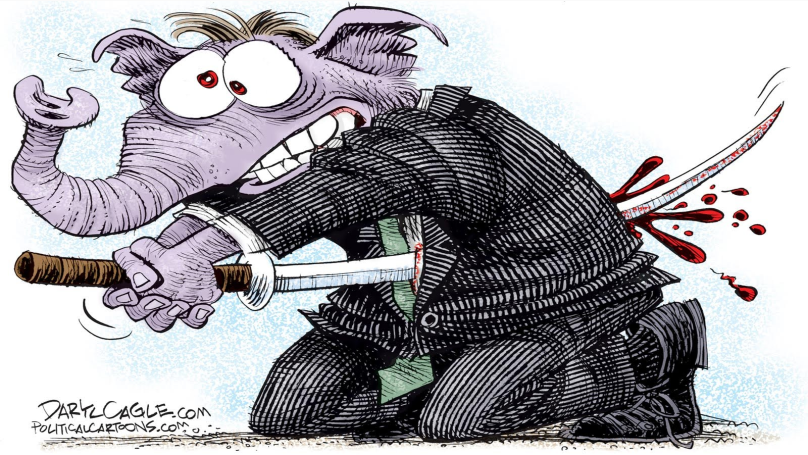


Sad story ...



Sensei II

Majority protocol: Why?

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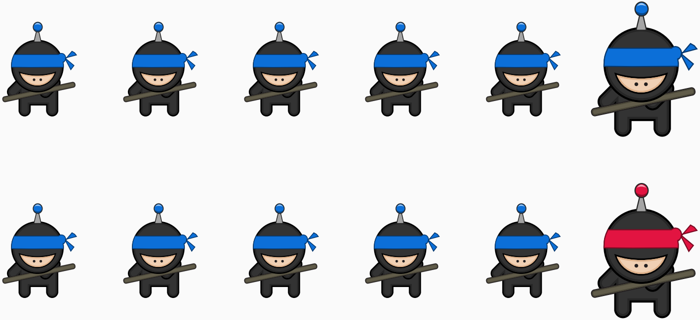

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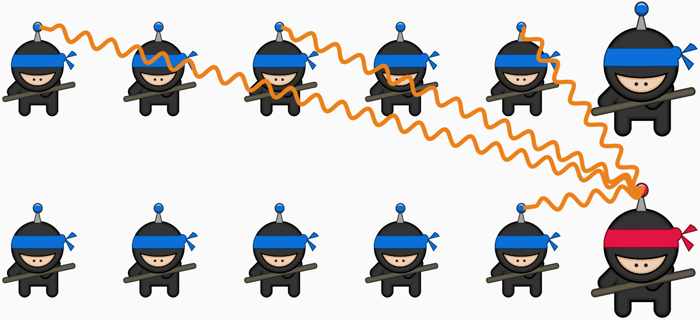

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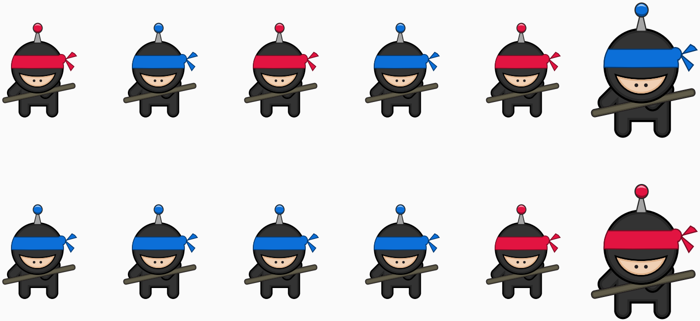

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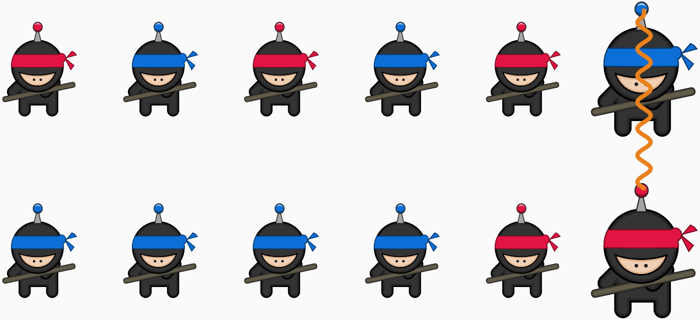

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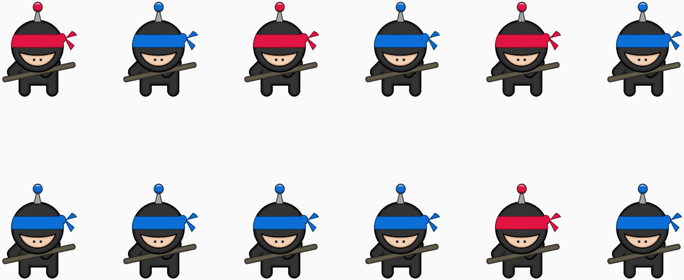

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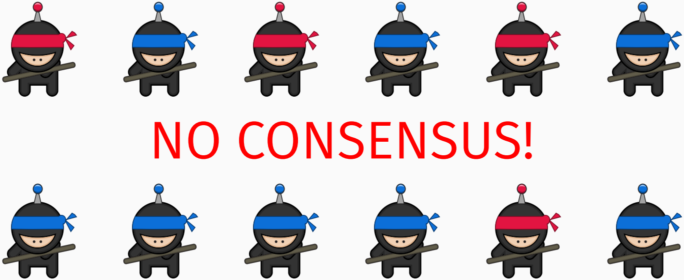

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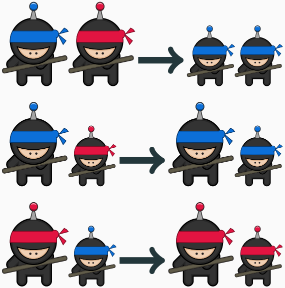
Interaction rules:



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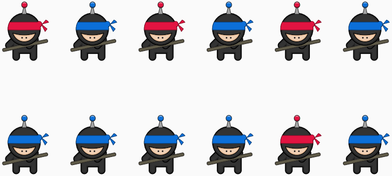

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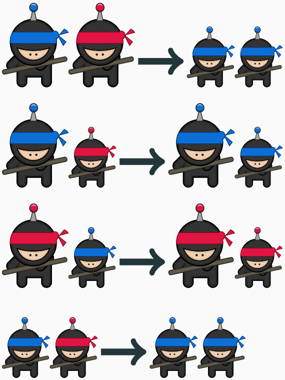

Passive blue ninjas convert passive red ninjas to their color

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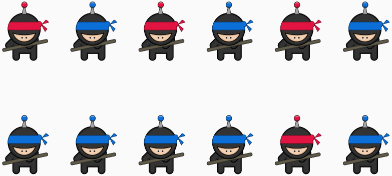

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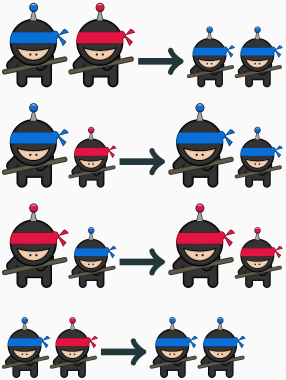

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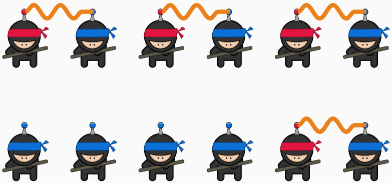

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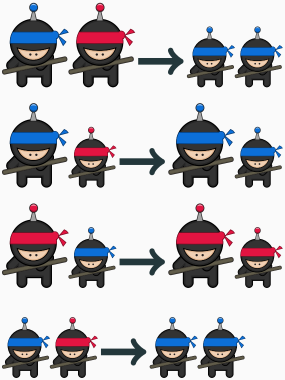

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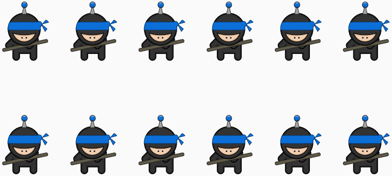

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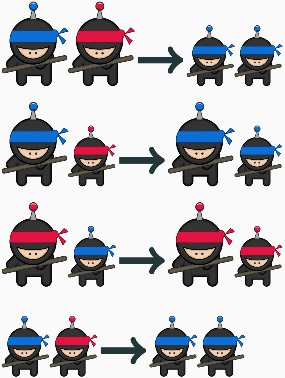

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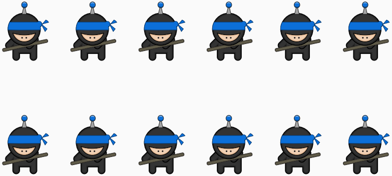
Go!



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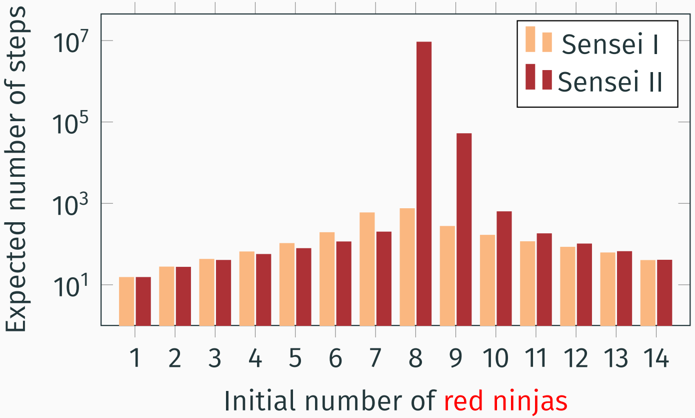


Expected number of steps to stable consensus for a population of 15 ninjas.

Very sad story ...

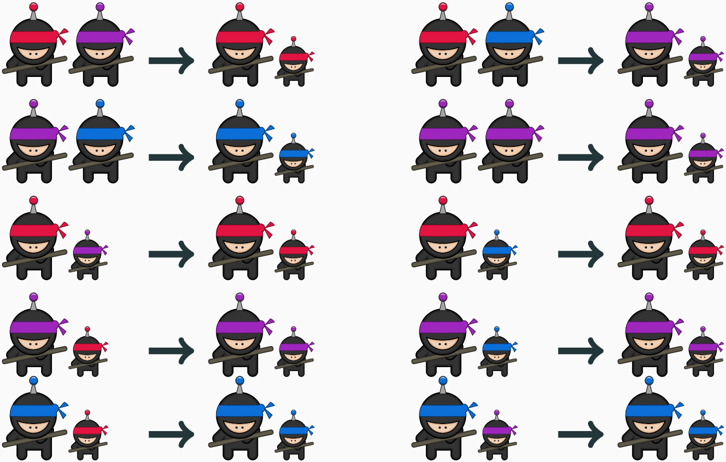
Sensei III

Sensei III's protocol

 = Attack majority  = Don't attack majority  = Tie

Interaction rules:

Go!



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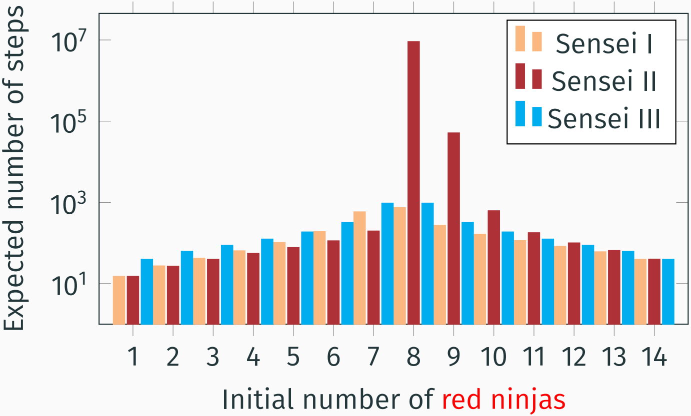

Expected number of steps to stable consensus for a population of 15 ninjas.

Sensei III's questions

Formalization questions:

- What is a protocol?
- When is a protocol "correct"?
- When is a protocol "efficient"?

Sensei III's questions

Verification questions:

- How do I check that my protocol is correct?
- How do I check that my protocol is efficient?

Sensei III's questions

Expressivity questions:

- Are there protocols for other problems?
- How large is the smallest protocol for a problem?
- And the smallest efficient protocol?

Formal model of distributed computation by collections of

identical, finite-state, and mobile agents

like

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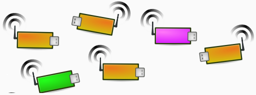

ad-hoc networks of devices

Population protocols

Angluin, Aspnes et al. PODC'04

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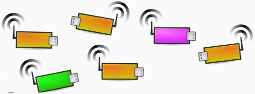

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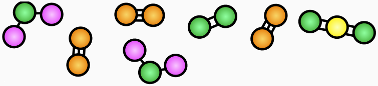

"soups" of molecules

(Chemical Reaction Networks)

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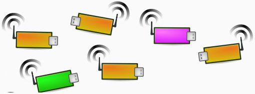

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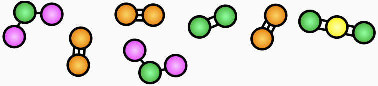

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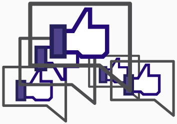

people in social networks

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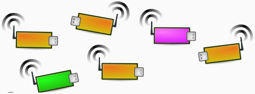

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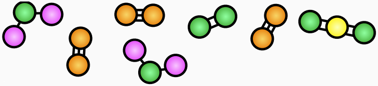

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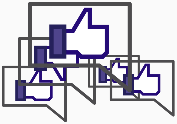
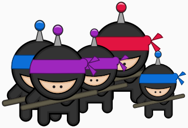


people in social networks



...and ninjas!

Population protocols: formal model

Angluin, Aspnes et al. PODC'04

- *States*: finite set Q
- *Opinions*: $O : Q \to \{0, 1\}$
- *Initial states*: $I \subseteq Q$
- *Transitions*: $T \subseteq Q^2 \times Q^2$

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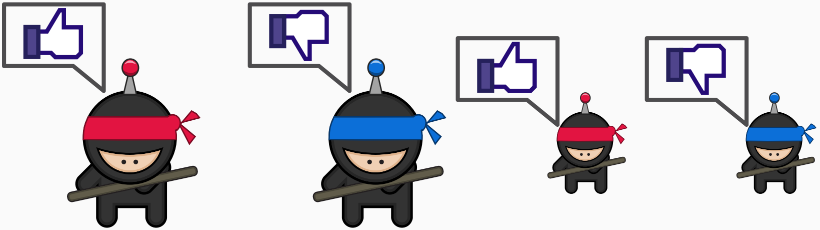

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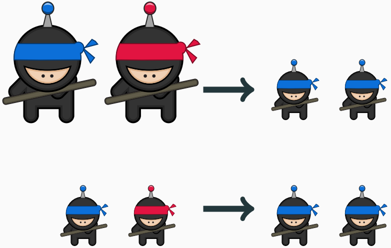

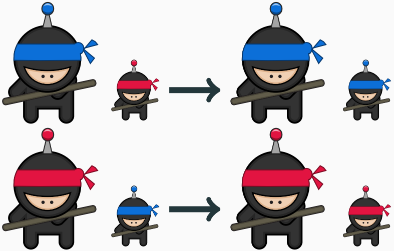

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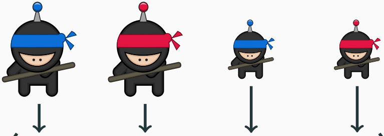

$$(2 \quad 5 \quad 1 \quad 3)$$

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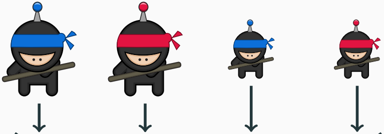
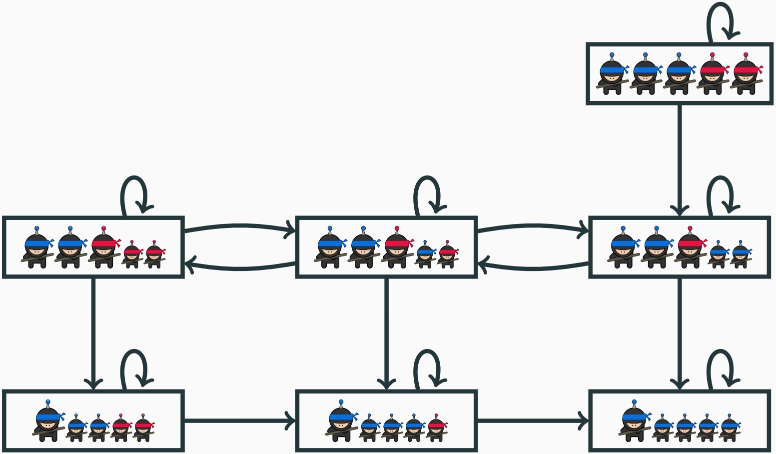


$$(2 \quad 5 \quad 0 \quad 0)$$

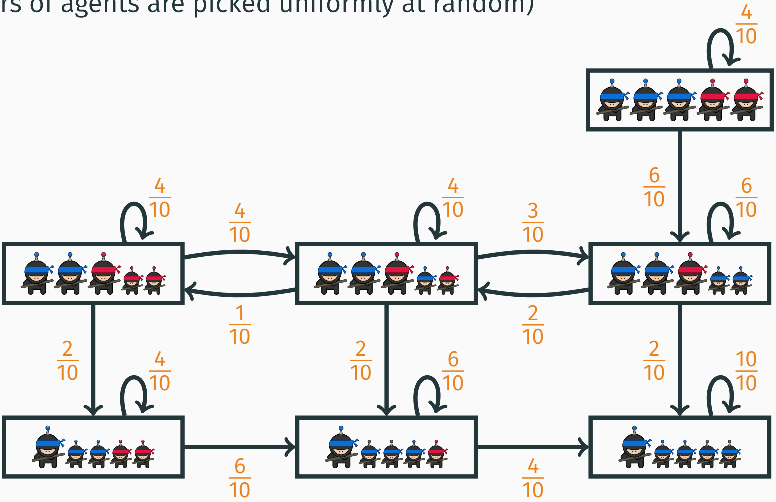
Population protocols: runs

Reachability graph for $(3, 2, 0, 0)$:



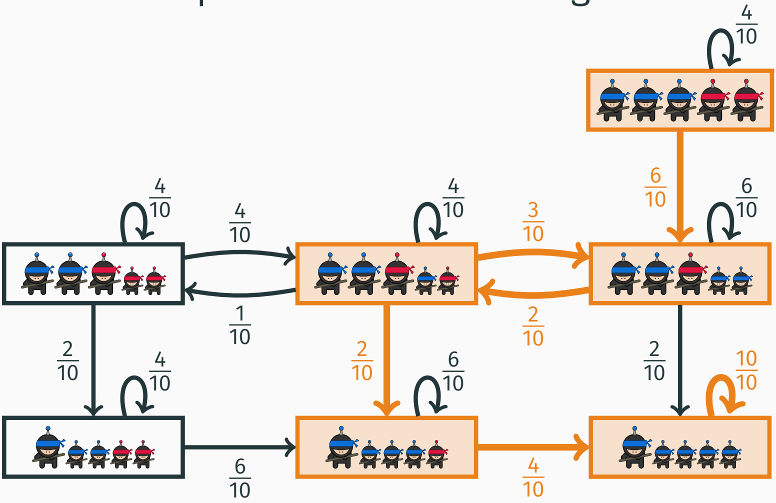
Underlying Markov chain:

(pairs of agents are picked uniformly at random)



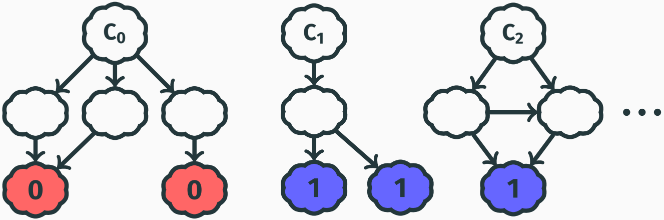
Population protocols: runs

Run : infinite path from initial configuration



Population protocols: computing predicates

Protocol computes φ: InitC $\to \{0, 1\}$:
for every $C \in \text{InitC}$, the runs starting at C
reach **stable consensus** $\varphi(C)$ with probability 1.



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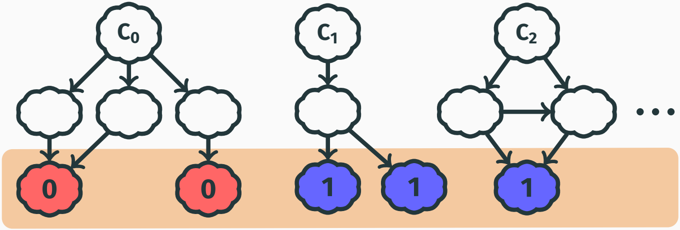

Protocol computes $\varphi(C_0) = 0, \varphi(C_1) = 1, \varphi(C_2) = 1, \ldots$

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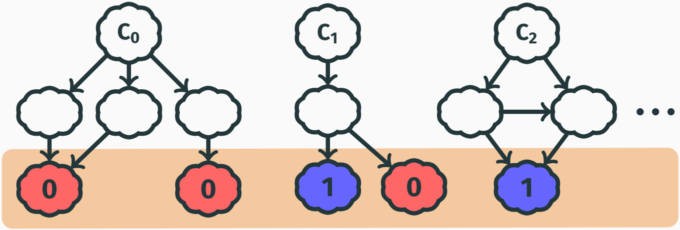

Protocol ill defined for C_1

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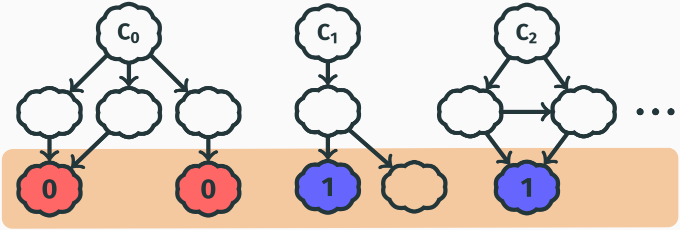

Protocol ill defined for C_1 (Sensei I's problem)

A protocol is **well specified** if it computes some predicate

A protocol is well specified if it computes some predicate

A protocol for a predicate φ is correct if it computes φ (in particular, correct protocols are well specified)

Sensei III's questions

What predicates can we compute?

How fast can we compute them?

How succinctly can we compute them?

How can I check correctness?

How can I check efficiency?

To conclude ...

Expressive power

Angluin, Aspnes, Eisenstat Dist. Comp.'07

Population protocols compute precisely the predicates definable in Presburger arithmetic

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Presburger arithmetic

- Atomic formulas: $a_1 x_1 + \cdots + a_m x_m < b$
- Formulas: Close under boolean operations and quantification
- Formula $F(x_1, \ldots, x_n)$ interpreted over $\mathbb{N}^n$
- Predicate $\varphi \colon \mathbb{N}^n \to \{0, 1\}$ definable in Presburger arithmetic if there is formula $F(x_1, \ldots, x_n)$ s.t. for every $\mathbf{v} \in \mathbb{N}^n$: $\varphi(\mathbf{v}) = 1$ iff $F(\mathbf{v})$ holds .

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Quantifier elimination

Every Presburger formula $F(x_1, \ldots, x_n)$ has an equivalent quantifier-free formula: A boolean combination of threshold and modulo predicates

$$a_1x_1 + \cdots + a_nx_n < b \quad a_1x_1 + \cdots + a_nx_n \equiv b \pmod{c}$$

with coefficients in $\mathbb{Z}$

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Population protocols compute precisely the predicates definable in Presburger arithmetic

Proof:

1) PPs compute all Presburger predicates

Since Presburger arithmetic has quantifier elimination, it suffices to:

- Exhibit PPs for threshold and modulo predicates
- Prove that predicates computable by PPs are closed under negation and conjunction

Threshold predicates: A protocol for $2x - 3y < 5$

A first protocol with infinitely many states

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Initially: x agents, each with 2 euros, and
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Agents must compute if total wealth less than 5 euros

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States: each agent has a number of euros
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Examples: (7, A, $\geq$5), (-2, P, $\geq$5)

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Interactions:

- Two active agents: One agent transfers its money to the other and goes passive; new opinions given by wealth of active agent.

$$(7, \mathrm{A}, \geq 5), (-4, \mathrm{A}, <5) \quad \mapsto \quad (3, \mathrm{A}, <5), (0, \mathrm{P}, <5)$$
$$(2, \mathrm{A}, <5), (4, \mathrm{A}, <5) \quad \mapsto \quad (6, \mathrm{A}, \geq 5), (0, \mathrm{P}, \geq 5)$$

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$$(2, A, <5), (4, A, <5) \mapsto (6, A, \geq 5), (0, P, \geq 5)$$

- One active, one passive agent: Same.
- Two passive agents: Nothing happens.

Threshold predicates: A protocol for $2x - 3y < 5$

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Correctness:

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Final protocol with finitely many states

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Final protocol with finitely many states

Initially: x agents, each with 2 euros, and
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States: agents now can only have -3,-2, ..., 4, 5 euros

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Interactions:

- Two active agents: One agent transfers as much as possible money or debt to the other and goes passive; new opinions given by wealth of active agent.

$$(2, A, <5), (4, A, <5) \mapsto (5, A, \geq 5), (1, P, \geq 5)$$
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Remainder predicates: A protocol for $2x - 3y \equiv 1 \bmod 5$

States: each agent keeps a residue $0, 1, 2, 3, 4$
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has opinion on result ($\equiv$1, $\not\equiv$1)

Examples: (2, A, $\not\equiv$1), (0, P, $\equiv$1)

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Examples: (2, A, $\not\equiv$1), (0, P, $\equiv$1)

Initially:

x active agents, each with residue 2

y active agents, each with residue 2 as well ($-3 \equiv 2 \bmod 5$)

Agents compute total wealth modulo 5

Remainder predicates: A protocol for $2x - 3y \equiv 1 \bmod 5$

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Correctness:

- Residue of total wealth is an invariant
- Eventually only one active agent left (leader).
  Leader has the correct residue and the correct opinion
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Closure under boolean operations

Computable predicates are closed under negation

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Invert the opinions of the states

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Let agents “multitask” to simultaneously compute the two predicates

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Opinion of (q, r): $O(q) \wedge O(r)$

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Angluin, Aspnes, Eisenstat Dist. Comp.'07

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- Much harder!
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- "Constructive" proof by E., Ganty, Leroux, Majumdar Acta Inf.'17
- More on this later!

Beyond Presburger predicates

- PPs can only compute predicates in DSPACE($\log \log n$)

  Meaning: if n agents can compute $\varphi(n)$ then there is a deterministic TM that on input the unary encoding of n computes $\varphi(n)$ using $\log \log n$ space

  Proof: Presburger languages are regular

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For all three: n agents can simulate a NCM with counters bounded by n^c, and so an NTM using logspace in n

Beyond Presburger predicates

Increasing the expressive power to NSPACE($n \log n$) or NSPACE(n^2):

- Community protocols — Guerraoui, Ruppert ICALP'09
    - Agents have unique identities (integers)
    - Agents can store a fixed number of identities in registers
    - Agents can only compare identities according to $<$
    - New states depends on old states and register contents

Beyond Presburger predicates

Increasing the expressive power to NSPACE($n \log n$) or NSPACE(n^2):

- Community protocols — Guerraoui, Ruppert ICALP'09
  - Agents have unique identities (integers)
  - Agents can store a fixed number of identities in registers
  - Agents can only compare identities according to $<$
  - New states depends on old states and register contents
- Mediated protocols — Michail et al. ICALP'09, TCS'11

  No identities, but channels have state.

Sensei III's questions

What predicates can we compute?

How fast can we compute them?

How succinctly can we compute them?

How can I check correctness?

How can I check efficiency?

To conclude ...

Efficiency

Efficiency measured by the expected number of interactions until stable consensus: $Inter(n)$

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Expected parallel time to consensus depends on the concurrency model

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Expected parallel time to consensus depends on the concurrency model

Most popular model:

- A communication channel for every pair of agents
- For each pair, number of communications follows a Poisson distribution with given rate
- Important advantage of the model: expected parallel time $\mathit{Time}(n)$ satisfies

$$\mathit{Time}(n) = \mathit{Inter}(n)/n$$

Efficiency

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Can a leader be elected in $O(\log^{O(1)}(n))$ time ?

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Doty and Soloveichik, DISC '15 and Dist. Comp. '18

Electing a leader takes $\Omega(n)$ time.

Efficiency

Alistarh *et al.* consider families $\{\mathcal{P}_n\}_{n=1}^{\infty}$ of protocols, where $\mathcal{P}_n$ is the protocol used for inputs with n agents.

Alistarh et al. PODC '15

There is a uniform family of protocols with $O(n)$ that computes majority (without ties) in $O(\log^{O(1)}(n))$ time.

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Alistarh et al. SODA '18

There exists a uniform family of protocols with $O(\log^2 n)$ states that computes majority in $\mathcal{O}(\log^{O(1)}(n))$ time.

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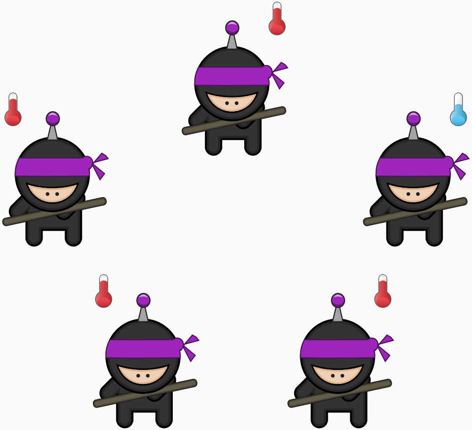
Succinctness–An Example

Protocol for: Are there at least 4 sick ninjas?



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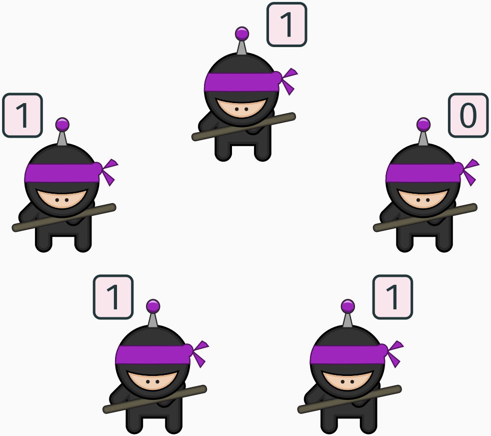

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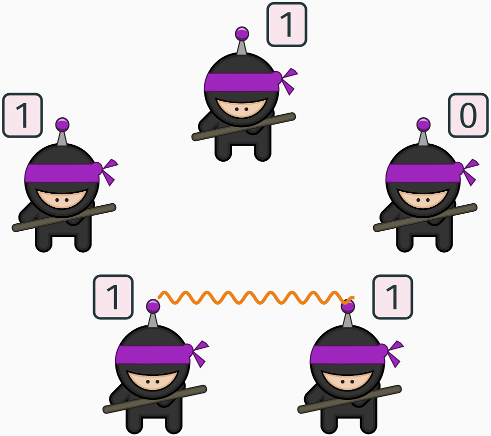

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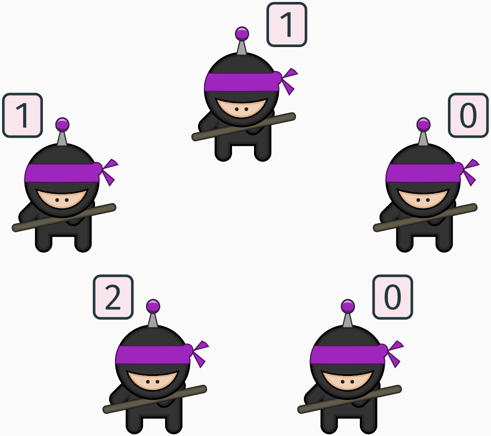

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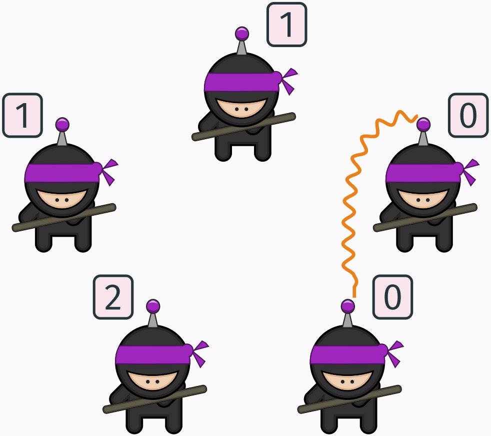

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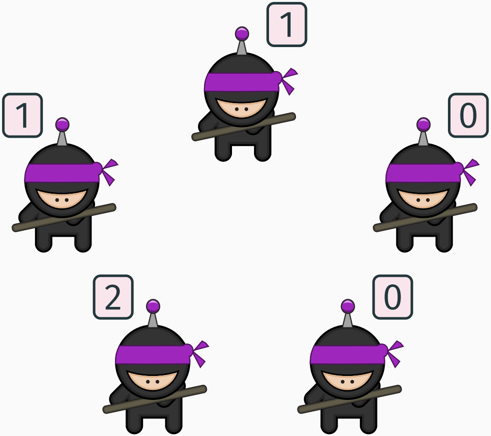

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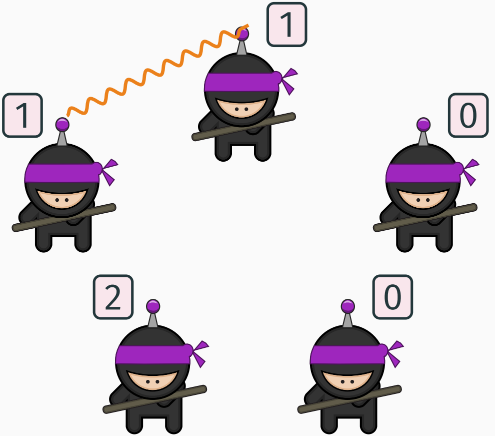

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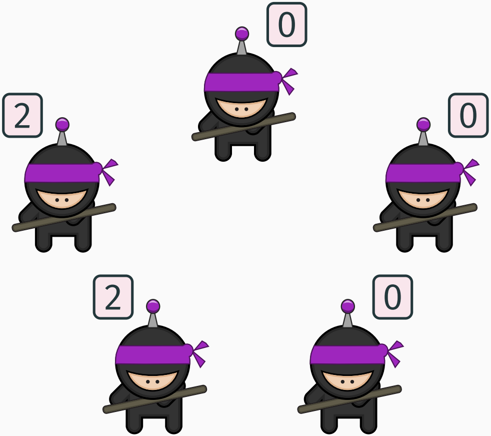

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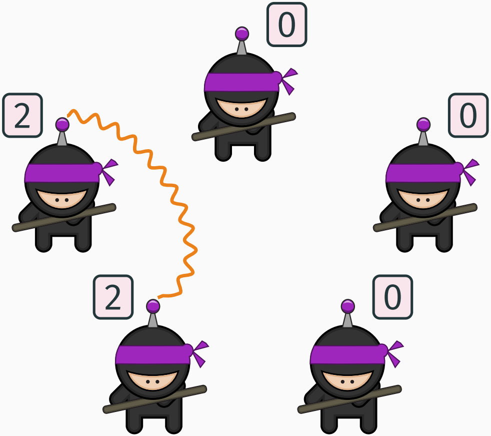

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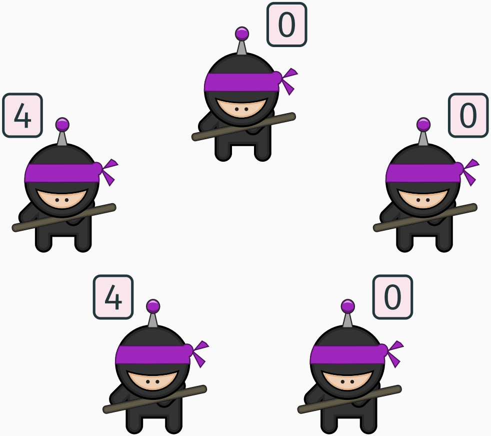

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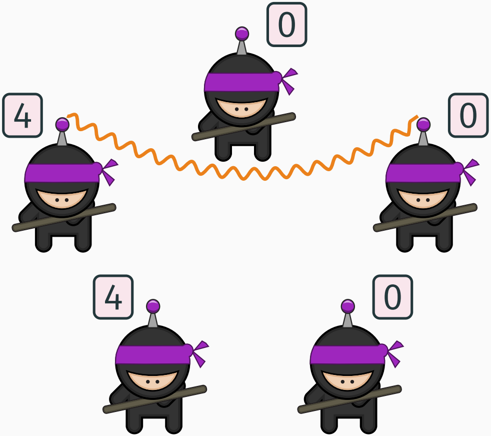

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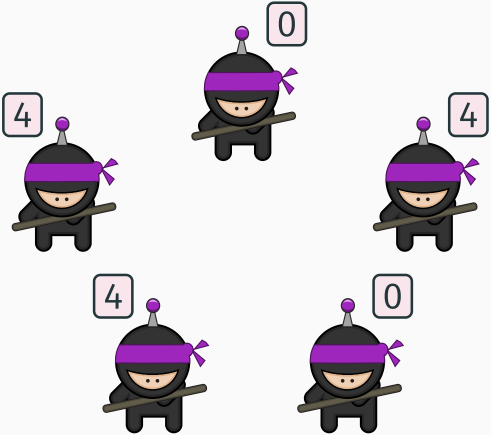

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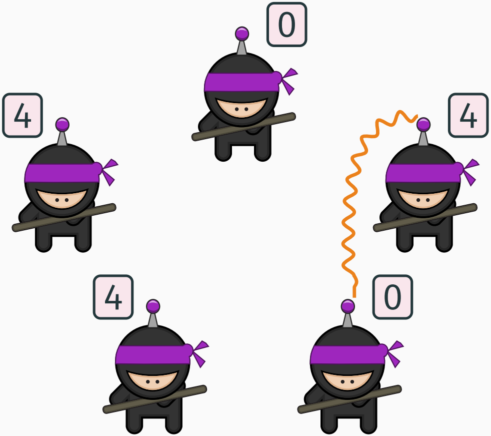

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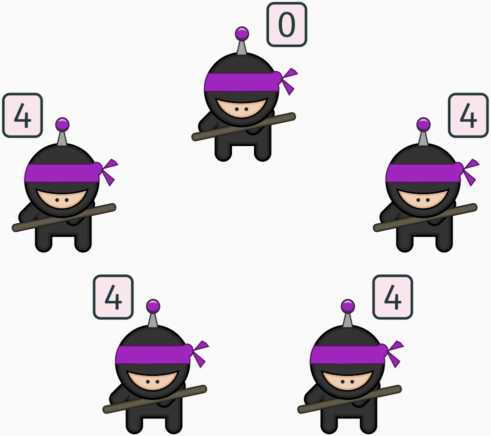

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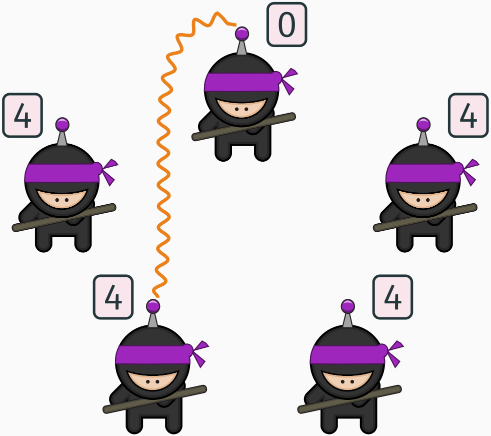

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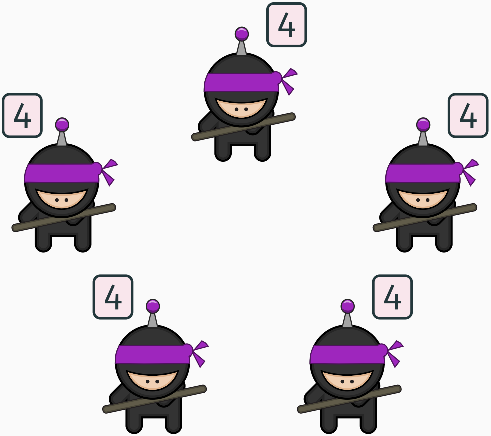

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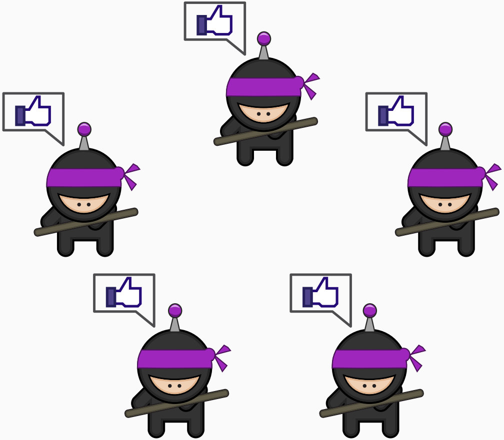

Sensei III's questions: Succinctness–An Example

Protocol for: Are there at least 2^ℓ sick ninjas?

- Each ninja is in a state of $\{0, 1, \ldots, 2^\ell - 1, 2^\ell\}$
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- $(m, n) \mapsto (m + n, 0)$ if $m + n < 2^\ell$
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- $(2^\ell, n) \mapsto (2^\ell, 2^\ell)$
- Can be generalized to non-powers of 2

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Just gave a protocol for **X ≥ c** with $\mathcal{O}(\log c)$ states.

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Not for every $\mathbf{c}$ ...

Blondin, E., Jaax STACS'18

There exist infinitely many $\mathbf{c}$ such that every protocol for $\mathbf{X} \geq \mathbf{c}$ has at least $(\log \mathbf{c})^{1/4}$ states

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...but for some $\mathbf{c}$, if we allow leaders:

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For infinitely many $\mathbf{c}$ there is a protocol with two leaders and $\mathcal{O}(\log \log \mathbf{c})$ states that computes $\mathbf{X} \geq \mathbf{c}$

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Proof:

Succinctness

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For infinitely many **c** there is a protocol with two leaders and $\mathcal{O}(\log\log \mathbf{c})$ states that computes **X ≥ c**

Proof:

- **Mayr and Meyer '82**: For every n there is a commutative semigroup presentation and two elements s, t such that the shortest word α leading from s to t (i.e., $t = s\,\alpha$) has length $|\alpha| \geq 2^{2^n}$

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- Construct a protocol that "simulates" derivations in the semigroup

Succinctness

$\mathcal{O}(\log \log c)$ without leaders?

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Checking correctness

Protocols can become complex, even for B ≥ R:

Fast and Exact Majority in Population Protocols

Dan Alistarh
Microsoft Research

Rati Gelashvili*
MIT

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$$weight(x) = \begin{cases} |x| & \text{if } x \in StrongStates \text{ or } x \in WeakStates; \\ 1 & \text{if } x \in IntermediateStates. \end{cases}$$

$$sgn(x) = \begin{cases} 1 & \text{if } x \in \{+0, 1_d, \ldots, 1_1, 3, 5, \ldots, m\}; \\ -1 & \text{otherwise.} \end{cases}$$

$value(x) = sgn(x) \cdot weight(x)$

`/* Functions for rounding state interactions */`

$\phi(x) = -1_1$ if $x = -1$; 1_1 if $x = 1$; x, otherwise

$R_\downarrow(k) = \phi(k$ if k odd integer, $k-1$ if k even)

$R_\uparrow(k) = \phi(k$ if k odd integer, $k+1$ if k even)

$$Shift\text{-}to\text{-}Zero(x) = \begin{cases} -1_{j+1} & \text{if } x = -1_j \text{ for some index } j < d \\ 1_{j+1} & \text{if } x = 1_j \text{ for some index } j < d \\ x & \text{otherwise.} \end{cases}$$

$$Sign\text{-}to\text{-}Zero(x) = \begin{cases} +0 & \text{if } sgn(x) > 0 \\ -0 & \text{oherwise.} \end{cases}$$

procedure update$\langle x, y \rangle$
  if $(weight(x) > 0$ *and* $weight(y) > 1)$ ***or*** $(weight(y) > 0$ *and* $weight(x) > 1)$ **then**
    $x' \leftarrow R_\downarrow\left(\frac{value(x)+value(y)}{2}\right)$ and $y' \leftarrow R_\uparrow\left(\frac{value(x)+value(y)}{2}\right)$
  else if $weight(x) \cdot weight(y) = 0$ *and* $value(x) + value(y) > 0$ **then**
    if $weight(x) \neq 0$ **then** $x' \leftarrow Shift\text{-}to\text{-}Zero(x)$ and $y' \leftarrow Sign\text{-}to\text{-}Zero(x)$
    else $y' \leftarrow Shift\text{-}to\text{-}Zero(y)$ and $x' \leftarrow Sign\text{-}to\text{-}Zero(y)$
  else if $(x \in \{-1_d, +1_d\}$ *and* $weight(y) = 1$ *and* $sgn(x) \neq sgn(y))$ ***or***
      $(y \in \{-1_d, +1_d\}$ *and* $weight(x) = 1$ *and* $sgn(y) \neq sgn(x))$ **then**
    $x' \leftarrow -0$ and $y' \leftarrow +0$
  else
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    if $weight(x) \neq 0$ **then** $x' \leftarrow Shift\text{-}to\text{-}Zero(x)$ and $y' \leftarrow Sign\text{-}to\text{-}Zero(x)$
    else $y' \leftarrow Shift\text{-}to\text{-}Zero(y)$ and $x' \leftarrow Sign\text{-}to\text{-}Zero(y)$
  else if $(x \in \{-1_d, +1_d\}$ *and* $weight(y) = 1$ *and* $sgn(x) \neq sgn(y))$ ***or***
      $(y \in \{-1_d, +1_d\}$ *and* $weight(x) = 1$ *and* $sgn(y) \neq sgn(x))$ **then**
    $x' \leftarrow -0$ and $y' \leftarrow +0$
  else
    $x' \leftarrow Shift\text{-}to\text{-}Zero(x)$ and $y' \leftarrow Shift\text{-}to\text{-}Zero(y)$

How can we verify correctness automatically?

Checking correctness—Early days

Model checkers:

- `PAT`: model checker with global fairness

  (Sun, Liu, Song Dong and Pang CAV'09)

- `bp-ver`: graph exploration

  (Chatzigiannakis, Michail and Spirakis SSS'10)

- Conversion to counter machines + `PRISM`/`Spin`

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Only for populations of fixed size!

Checking correctness—Early days

Theorem provers:

- Verification with the interactive theorem prover **Coq**

(Deng and Monin TASE'09)

Checking correctness—Early days

Theorem provers:

- Verification with the interactive theorem prover **Coq**

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Not automatic!

Checking correctness—Early days

Theorem provers:

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(Deng and Monin TASE'09)

Challenge: verifying automatically
<u>all</u> sizes

Checking correctness—Decidability

E., Ganty, Leroux, Majumdar   Acta Inf.’17

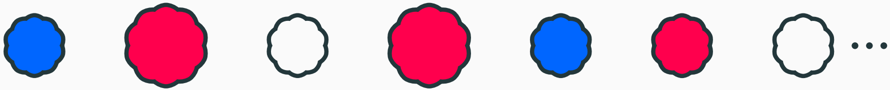
It is decidable if a population protocol computes a given (Presburger) predicate.

Checking correctness—Decidability

Acta Inf.'17

Initial configurations, colored with intended result

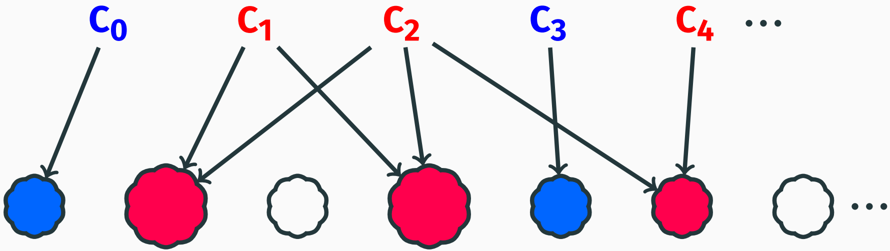
$C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad \cdots$



Bottom configurations, colored if consensus

Checking correctness—Decidability

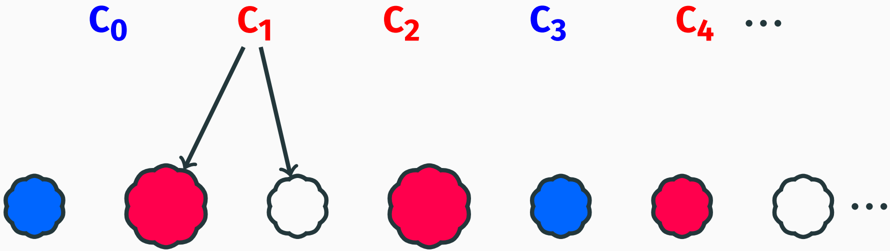
Initial configurations, colored with intended result



Bottom configurations, colored if consensus

Correct protocol

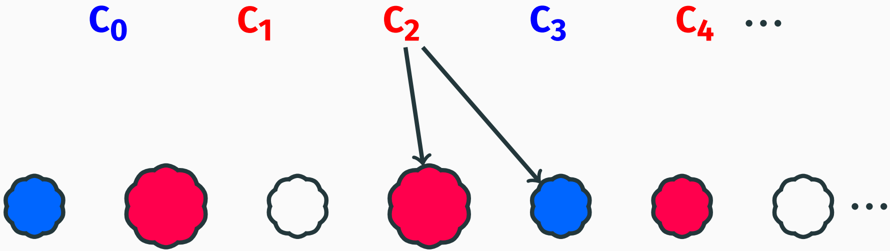
Initial configurations, colored with intended result



Bottom configurations, colored if consensus

Incorrect protocol: sometimes no result for C_1

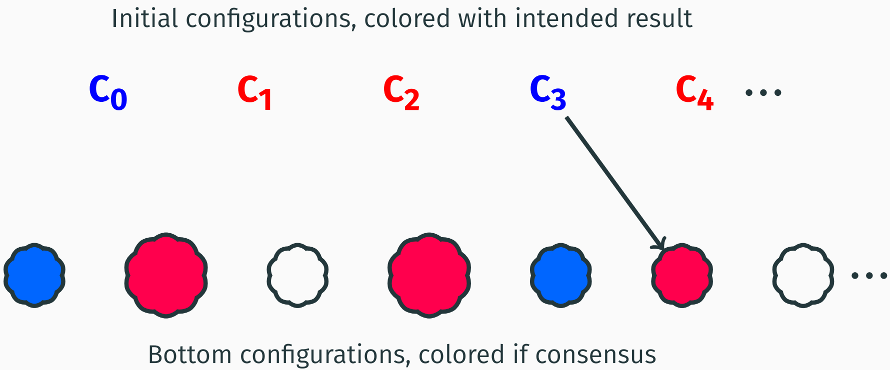
Initial configurations, colored with intended result



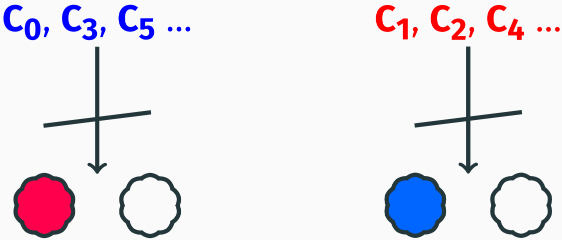
Bottom configurations, colored if consensus

Incorrect protocol: sometimes wrong for C_2

Checking correctness—Decidability



Incorrect protocol: always wrong result for C_3



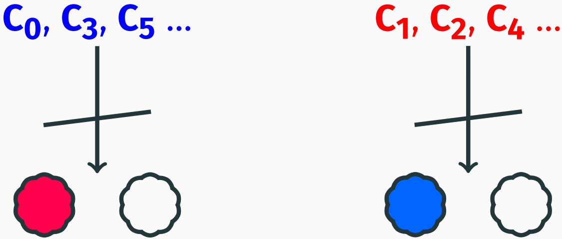
A protocol correctly computes the given predicate iff:

- no white or red SCCs reachable from blue initial configurations
- no white or blue SCCs reachable from red initial configurations

Correctness reduced to reachability question between infinite sets of configurations

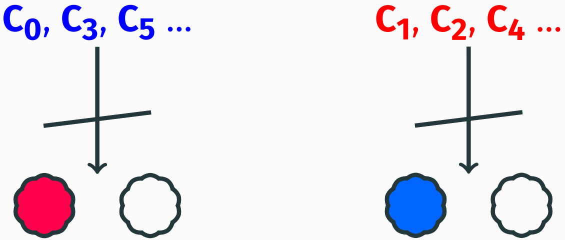
Checking correctness—Decidability

Acta Inf.'17



Define: sets $\mathbf{I}$, $\mathbf{I_1}$ and $\mathbf{I_0}$ of initial configurations
sets $\mathbf{B}$, $\mathbf{B_1}$ and $\mathbf{B_0}$ of bottom configurations

Checking correctness—Decidability



Define: sets $\mathbf{I}$, $\mathbf{I}_1$ and $\mathbf{I}_0$ of initial configurations
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We study the shape of these infinite sets

Basic notions: Presburger arithmetic

Presburger arithmetic

- Atomic formulas: $a_1x_1 + \cdots + a_nx_n < b$
- Formulas: Close under boolean operations and quantification
- Formula $F(x_1, \ldots, x_m)$ with free variables $x_1, \ldots, x_m$ interpreted over $\mathbb{N}^m$
- Set of models of $F(x_1, \ldots, x_n)$ is effectively semilinear

Semilinear sets

- Subsets of $\mathbb{N}^m$ for fixed m
- Finite unions of linear sets
- A linear set is determined by a root $\mathbf{r}$ and a set $\mathbf{p_1}, \ldots, \mathbf{p_m}$ of periods.
- A vector $\mathbf{v} \in \mathbb{N}^m$ belongs to the linear set iff there are $\lambda_1, \ldots, \lambda_m \in \mathbb{N}$ such that

$$\mathbf{v} = \mathbf{r} + \lambda_1 \mathbf{p_1} + \cdots \lambda_m \mathbf{p_m}$$

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Presburger sets = Semilinear sets

Init, Init_1 and Init_0 are Presburger definable

- **Init** is the set of configurations having
    - arbitrarily many agents in initial states, and
    - zero agents in non-initial states

$\Rightarrow$ **Init** is Presburger definable

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  $\Rightarrow$ **Init** is Presburger definable

- **Init_1** is the set of initial configurations satisfying the given predicate φ, and φ is Presburger definable

  $\Rightarrow$ **Init_1** is Presburger definable

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  $\Rightarrow$ $\mathbf{Init_1}$ is Presburger definable

- $\mathbf{Init_0}$ is the set of initial configurations that do not satisfy φ, and φ is Presburger definable

  $\Rightarrow$ $\mathbf{Init_0}$ is Presburger definable

Bottom, Bottom_1 and Bottom_0 are Presburger definable

The mutual reachability relation $\xleftrightarrow{*}$ on the configurations of a given protocol is defined by:

$$C \xleftrightarrow{*} C' \ \text{ iff } \ C \xrightarrow{*} C' \text{ and } C' \xrightarrow{*} C$$

Bottom, Bottom_1 and Bottom_0 are Presburger definable

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$$C \overset{*}{\longleftrightarrow} C' \;\text{ iff }\; C \overset{*}{\longrightarrow} C' \text{ and } C' \overset{*}{\longrightarrow} C$$

$\overset{*}{\longleftrightarrow}$ is a congruence:

- $\overset{*}{\longleftrightarrow}$ is an equivalence relation
- $C_1 \overset{*}{\longleftrightarrow} C_1' \;\wedge\; C_2 \overset{*}{\longleftrightarrow} C_2' \;\Rightarrow\; C_1 + C_2 \overset{*}{\longleftrightarrow} C_1' + C_2'$

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Eilenberg and Schützenberger '69, Hirshfeld '94

Congruences over $\mathbb{N}^n$ are semilinear subsets of $\mathbb{N}^{2n}$

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Congruences over $\mathbb{N}^n$ are semilinear subsets of $\mathbb{N}^{2n}$

Corollary

$\xleftrightarrow{*}$ is semilinear, and so definable in Presburger arithmetic

Bottom, Bottom_1 and Bottom_0 are Presburger definable

Eilenberg and Schützenberger '69, Hirshfeld '94

Congruences over $\mathbb{N}^n$ are semilinear subsets of $\mathbb{N}^{2n}$

Proof.

Given a congruence R over $\mathbb{N}^n$ and $(u, v) \in R$, let $\mathit{MinOff}_{u,v}$ be the minimal offsets of (u, v): $(s, t) \in \mathit{MinOff}_{u,v}$ iff $(u + s, v + t) \in R$ and (s, t) minimal

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Claim 1: $\mathit{MinOff}_{u,v}$ is finite (Dickson's lemma)

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Claim 3: $R = \bigcup\{L_{u,v} \mid (u, v) \text{ is minimal }\}$

Claim 4: R has finitely many minimal pairs. $\square$

Bottom, Bottom_1 and Bottom_0 are Presburger definable

A configuration C is a bottom configuration iff for every configuration C', C'':

$$\left(C \xleftrightarrow{*} C' \ \wedge \ C' \longrightarrow C'' \right) \ \Rightarrow \ C \xleftrightarrow{*} C''$$

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So the predicate $Bottom(C)$ is captured by the formula:

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Since both $\overset{*}{\longleftrightarrow}$ and $\longrightarrow$ (one step!) are Presburger definable:

Proposition

Bottom, **Bottom$_1$** and **Bottom$_0$** are Presburger definable

Bottom, Bottom_1 and Bottom_0 are effectively Presburger definable

Leroux '11, Acta.Inf. '17

The mutual reachability relation of a population protocol is effectively Presburger definable

Proof:

1) Prove it first for globally cyclic protocols in which mutual reachability and reachability coincide

2) Show: for every protocol there is a globally cyclic protocol with the same mutual reachability relation

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Leroux '11, Acta.Inf. '17

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Corollary

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From correctness to Petri net reachability

Recall that a protocol correctly computes a predicate iff:

> $\mathbf{Bottom} \setminus \mathbf{Bottom_0}$ is not reachable from $\mathbf{Init_1}$, and
> $\mathbf{Bottom} \setminus \mathbf{Bottom_1}$ is not reachable from $\mathbf{Init_0}$.

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We have reduced the correctness problem to:

Given: A population protocol $\mathcal{P}$
(Eff.) Presburger sets $\mathcal{C}, \mathcal{C}'$ of configurations of $\mathcal{P}$

Decide: Is some configuration of $\mathcal{C}'$ reachable from some configuration of $\mathcal{C}$?

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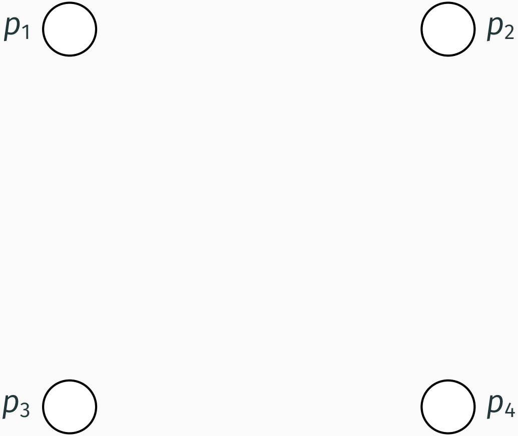
Now we reduce this to the reachability problem for Petri nets:

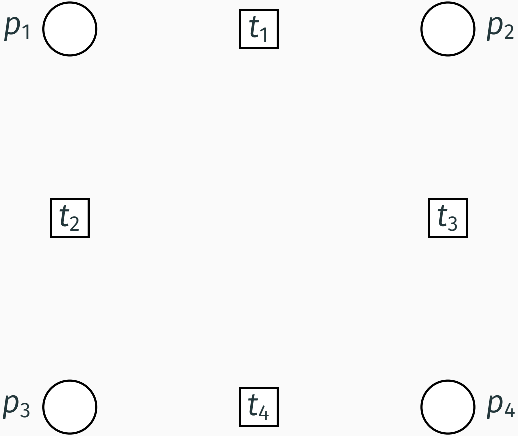
Given: Two markings M, M' of a Petri net

Decide: Is M' reachable from M ?

Petri nets



Petri nets



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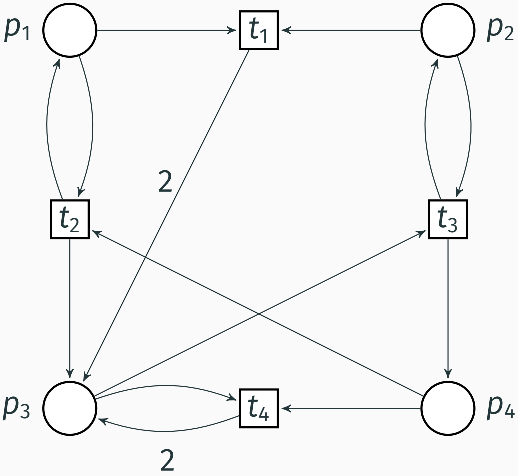
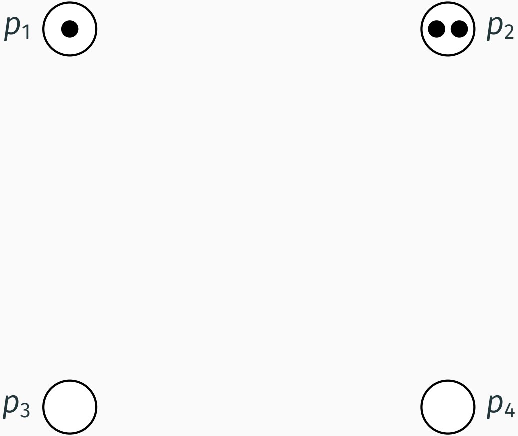
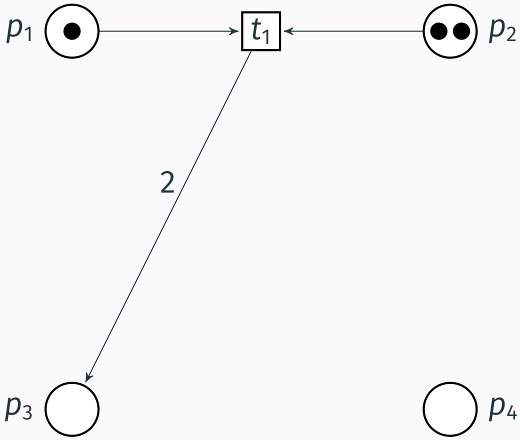


Markings



Firing rule



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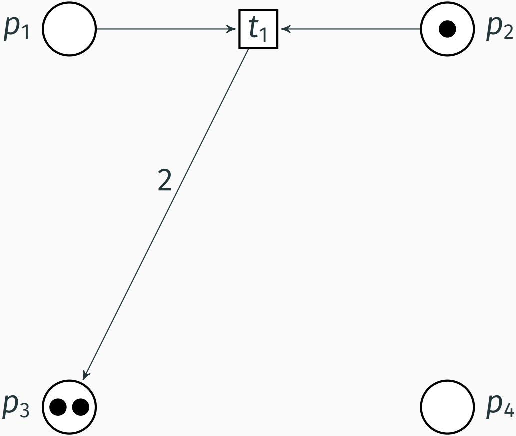
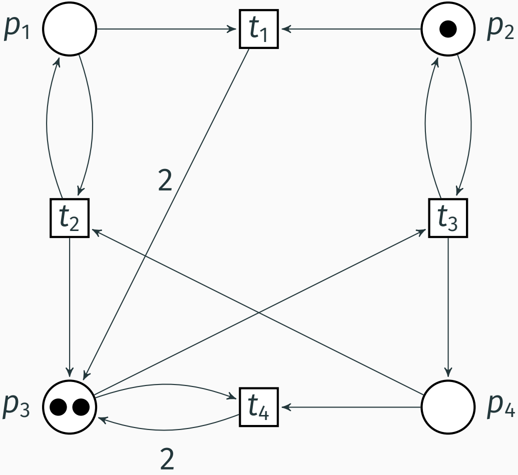


Petri nets



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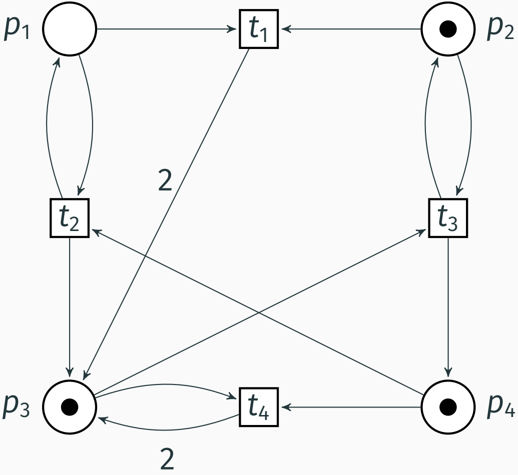

From PPs to Petri nets

Population protocols	Petri nets
State	Place

From PPs to Petri nets

Population protocols	Petri nets
State	Place
Interaction $(q_1, q_2) \mapsto (q'_1, q'_2)$	Transition with input places q_1, q_2 output places q'_1, q'_2

From PPs to Petri nets

Population protocols	Petri nets
State	Place
Interaction $(q_1, q_2) \mapsto (q_1', q_2')$	Transition with input places q_1, q_2 output places q_1', q_2'
PP-scheme	Net without marking

From PPs to Petri nets

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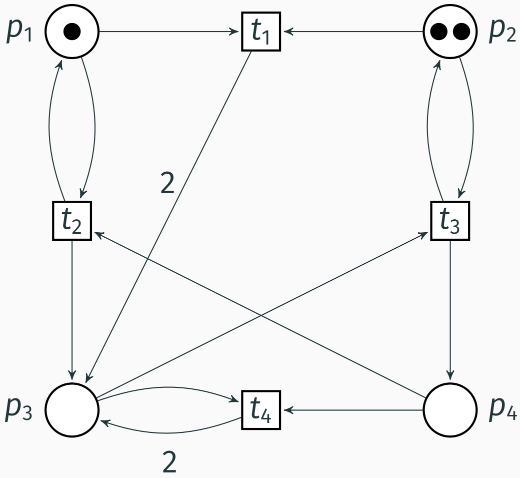
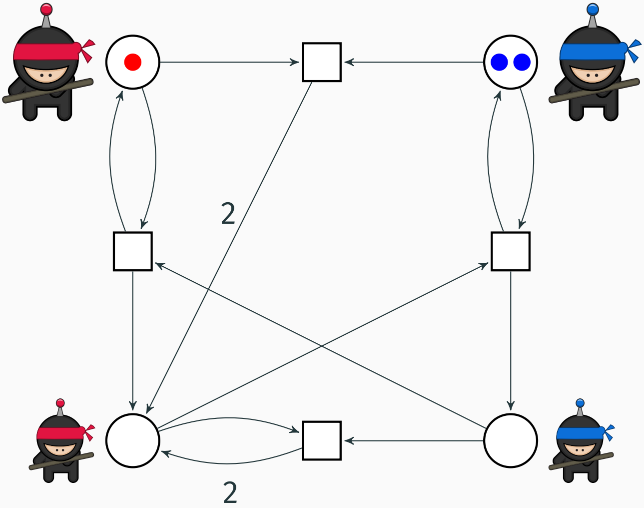
Interaction $(q_1, q_2) \mapsto (q_1', q_2')$	Transition with input places q_1, q_2 output places q_1', q_2'
PP-scheme	Net without marking
Configuration	Marking
PP	Net + infinite family of initial markings

Petri net of the slow majority protocol



Petri net of the majority protocol



From PPs to Petri nets

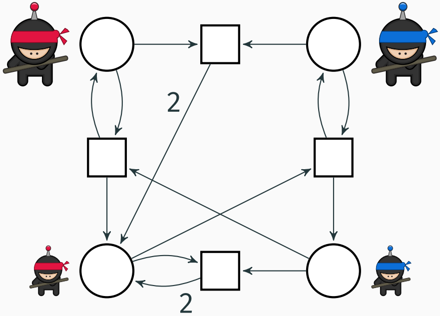
Every population protocol yields a net (without marking)

Not every net corresponds to a protocol!

- Protocol transitions neither create nor destroy tokens
- In particular, Petri nets derived from protocols are bounded for every initial marking

Petri nets “more general” than population protocols in this sense

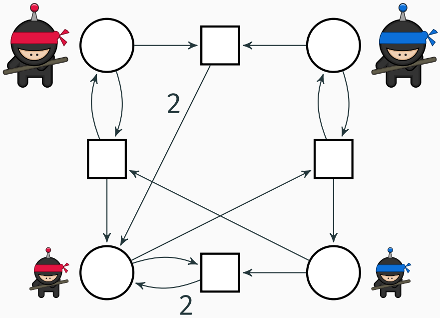
Additional power of Petri nets

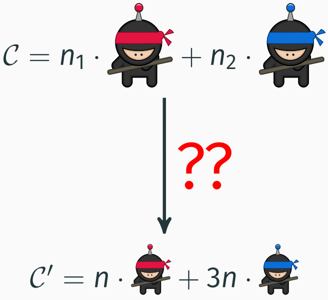


$\mathcal{C} = n_1 \cdot$  $+ n_2 \cdot$ 

$\mathcal{C}' = n \cdot$  $+ 3n \cdot$ 

Additional power of Petri nets





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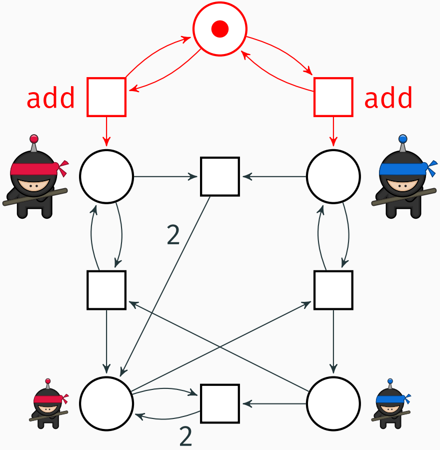

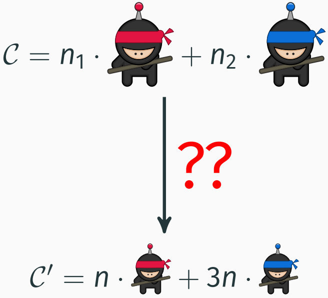

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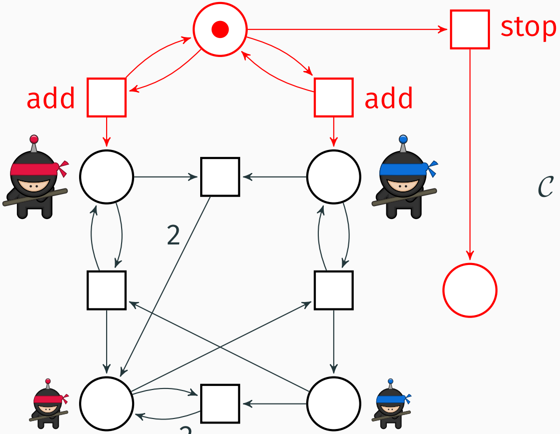

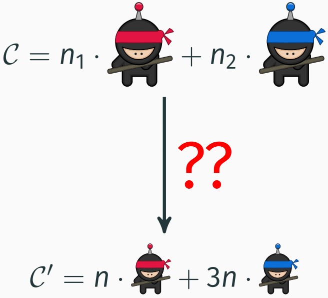

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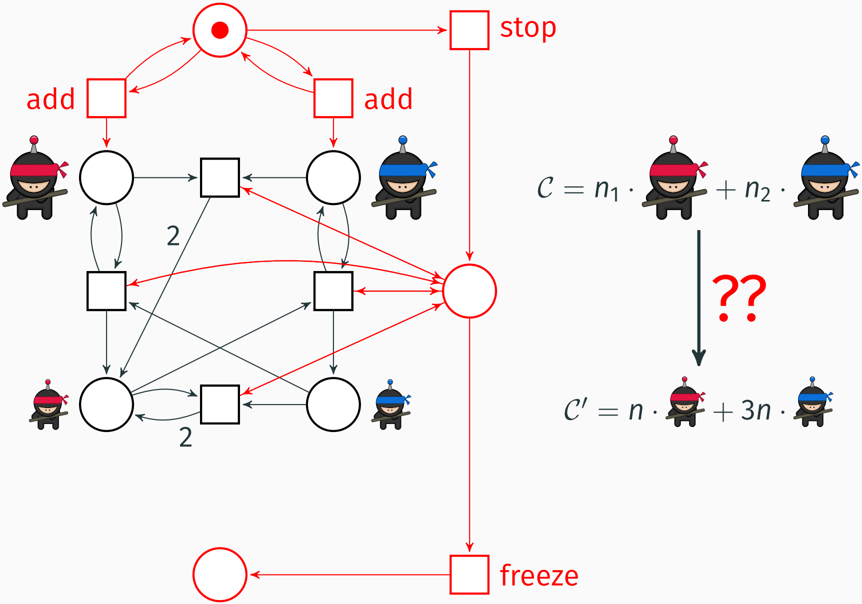
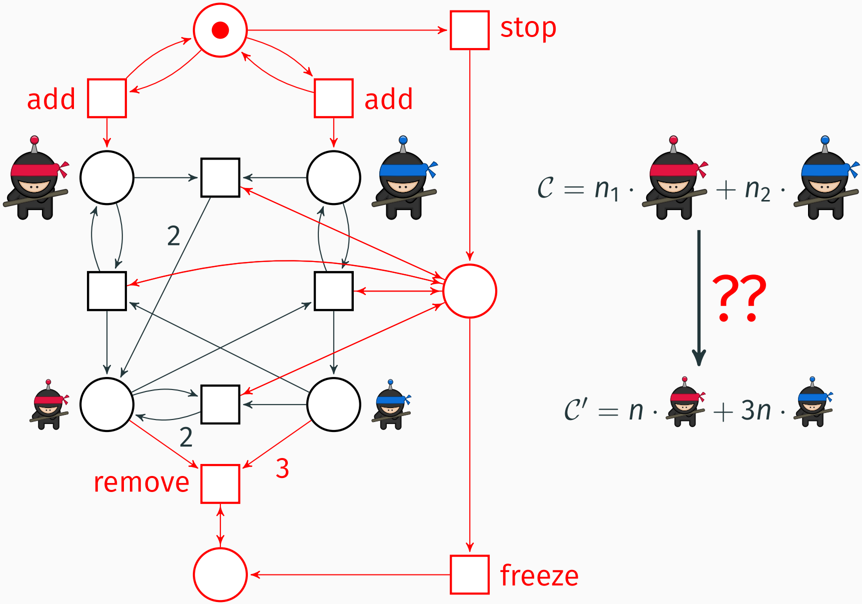


Additional power of Petri nets



Checking correctness—Feasibility

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- Stage graphs for $b = 0$ and $b = 1$, describing "milestones" from the initial configurations for which the output should be b

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- Most protocols are naturally analyzed in “stages”: “milestones” until the protocol reaches consensus 0 or 1, depending on the input
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Stage Graphs

- Stage graphs for $b = 0$ and $b = 1$, describing “milestones” from the initial configurations for which the output should be b
- SMT-based semi-algorithm for the automatic construction of stage graphs

Stage graphs: preliminaries

Let T be the set of transitions of a protocol

- A pattern is a language $P \subseteq T^*$ of the form

  $$w_1^* \, w_2^* \cdots w_n^*$$

  for some $w_1, \ldots, w_n \in T^*$

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  for some $w_1, \ldots, w_n \in T^*$

Let $\mathcal{C}$ be a set of configurations.

- $\text{pre}_P(\mathcal{C})$ denotes the set of configurations C such that

$$C \xrightarrow{w} C'$$

  for some $C' \in \mathcal{C}$ and some $w \in P$.

Stage graphs

A stage graph for a given protocol, a given predicate, and a given $b \in \{0, 1\}$ is a finite DAG satisfying:

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2. Every b-initial configuration of the protocol belongs to some initial stage
3. For every non-terminal stage $\mathcal{C}$ with children $\mathcal{C}_1, \ldots, \mathcal{C}_n$ there is a pattern P such that $\mathcal{C} \subseteq \mathrm{pre}_P(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n)$

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4. Every configuration of a terminal stage is a b-consensus

Example of stage graphs

Majority protocol $\quad (\mathbf{R} \overset{?}{>} \mathbf{B})$

$$t_1: \quad \mathbf{B}, \mathbf{R} \quad \mapsto \quad \mathbf{b}, \mathbf{b} \qquad\qquad t_3: \quad \mathbf{R}, \mathbf{b} \quad \mapsto \quad \mathbf{R}, \mathbf{r}$$

$$t_2: \quad \mathbf{B}, \mathbf{r} \quad \mapsto \quad \mathbf{B}, \mathbf{b} \qquad\qquad t_4: \quad \mathbf{b}, \mathbf{r} \quad \mapsto \quad \mathbf{b}, \mathbf{b}$$

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Stage graph for $b = 0$

$$\begin{tikzcd}
 & \mathbf{R} \leq \mathbf{B} \arrow[dl, "t_1^*"'] \arrow[dr, "t_1^*"] & \\
\mathbf{R} = 0, \mathbf{B} > 0 \arrow[d, "t_2^*"'] & & \mathbf{R} = 0, \mathbf{B} = 0, \mathbf{b} > 0 \arrow[d, "t_4^*"'] \\
\underline{\mathbf{R} = 0, \mathbf{r} = 0}, \mathbf{B} > 0 & & \underline{\mathbf{R} = 0, \mathbf{r} = 0}, \mathbf{B} = 0
\end{tikzcd}$$

Example of stage graphs

Majority protocol $(\mathbf{R} \overset{?}{>} \mathbf{B})$

$$
\begin{array}{llcll}
t_1: & \mathbf{B}, \mathbf{R} & \mapsto & \mathbf{b}, \mathbf{b} & \quad t_3: \ \mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r} \\
t_2: & \mathbf{B}, \mathbf{r} & \mapsto & \mathbf{B}, \mathbf{b} & \quad t_4: \ \mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}
\end{array}
$$

Stage graph for $b = 0$

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\begin{tikzcd}
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\underline{\mathbf{R} = 0, \mathbf{r} = 0}, \mathbf{B} > 0 & & \underline{\mathbf{R} = 0, \mathbf{r} = 0}, \mathbf{B} = 0
\end{tikzcd}
$$

Stage graph for $b = 1$

$$
\begin{tikzcd}
\mathbf{R} > \mathbf{B} \arrow[d, "t_1^*"'] \\
\mathbf{R} > 0, \mathbf{B} = 0 \arrow[d, "t_3^*"'] \\
\mathbf{R} > 0, \underline{\mathbf{B} = 0, \mathbf{b} = 0}
\end{tikzcd}
$$

Stage graphs: soundness, completeness, decidabilty

Soundness

If a protocol has stage graphs for a predicate φ and both 0 and 1, then the protocol computes φ.

Proof.

Easy.
Show that executions "go down" the stage graph w.p.1 till they get "trapped" in a bottom stage. $\square$

Stage graphs: soundness, completeness, decidabilty

A Presburger stage graph is a stage graph whose nodes are Presburger sets.

Completeness **Acta Inf. 2017**

If a protocol computes φ, then it has Presburger stage graphs for φ and both 0 and 1.

Proof.

Difficult.

Initial stage: Inductive Presburger "envelope" of the *b*-initial configurations.

Final stage: set of all configurations that are a *b*-stable consensus. $\square$

Stage graphs: soundness, completeness, decidabilty

Decidability **Acta Inf. 2017**

It is decidable if a given DAG of Presburger sets is a Presburger stage graph for a given b.

Proof.

Follows from properties of Presburger sets:

- Emptiness is decidable
- Effectively closed under boolean operations
- Effectively closed under pre, post, $\mathrm{pre}_P(\_)$ and $\mathrm{post}_P(\_)$

□

Stage graphs: soundness, completeness, decidabilty

Alternative algorithm for decidability of correctness:

- Two semi-decision algorithms
- For non-correctness: enumerate all initial configurations and check convergence to the right value
- For correctness: enumerate all DAGs of Presburger sets, and check if they are Presburger stage graphs for 0 or for 1

Automatic construction of stage graphs

A state q is abandoned at a configuration C if no configuration reachable from C populates q

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Observation:

In manual proofs, milestones towards consensus correspond to abandoning places

Stable consensus b is reached by abandoning all states with output $1 - b$

Automation strategy:

Given a stage $\mathcal{C}$, give efficient semi-algorithms for identifying children stages $\mathcal{C}_1, \ldots, \mathcal{C}_n$ with stricty more abandoned states than $\mathcal{C}$.

Stage representations

A stage representation is a triple

$$\mathcal{S} = (\mathit{Entry}, \mathit{Dead}, \mathit{Abandoned})$$

where

- *Entry* is a Presburger formula
- *Dead* is a set of transitions
- *Abandoned* is a set of states

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where

- *Entry* is a Presburger formula
- *Dead* is a set of transitions
- *Abandoned* is a set of states

$\mathcal{S}$ represents the set of configurations

- satisfying *Entry*,
- at which all transitions of *Dead* are dead, and
- at which all states of *Abandoned* are abandoned

Computing the children of a node

Input: $\mathcal{S} = (\mathit{Entry}, \mathit{Dead}, \mathit{Abandoned})$

$U := \mathit{EvDead}(\mathcal{S})$;

$\mathit{Entry}' := \mathit{DeadAt}(U, \mathcal{S})$;

$\mathcal{S}' := (\mathit{Entry}', \mathit{Dead} \cup U, \mathit{Abandoned})$;

if $\mathit{Split}(\mathcal{S}')$ fails **then abort**

else $\mathcal{S}_1, \ldots, \mathcal{S}_n := \mathit{Split}(\mathcal{S}')$

Output: $\mathcal{S}_1, \ldots, \mathcal{S}_n$

Computing the children of a node

Input: $\mathcal{S} = (\mathit{Entry}, \mathit{Dead}, \mathit{Abandoned})$

$U := \mathit{EvDead}(\mathcal{S})$;

Returns set of transitions that die w.p.1 from *any* configuration of $\mathcal{S}$

$\mathit{Entry}' := \mathit{DeadAt}(U, \mathcal{S})$;

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Overapproximates the configurations reachable from $\mathcal{S}$ at which U is dead

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New stage, but no progress yet because same *Abandoned* set

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Attempt to split $\mathcal{S}'$ into stages with more abandoned states

Output: $\mathcal{S}_1, \ldots, \mathcal{S}_n$

Implementing EvDead($\mathcal{S}$): Kirchoff's equations

Transition $t \longrightarrow$ offset $\Delta(t)$

Examples:
$$t\colon\ q_1, q_2 \mapsto q_2, q_3 \quad\longrightarrow\quad \Delta(t) = (-1, 0, 1)$$
$$t\colon\ q_1, q_2 \mapsto q_3, q_3 \quad\longrightarrow\quad \Delta(t) = (-1, -1, 2)$$

We have: if $C \xrightarrow{t} C'$ then $C' = C + \Delta(t)$

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Transition sequence $w = t_1 \ldots t_n$

$$\longrightarrow \text{ offset } \Delta(w) = \sum_{i=1}^{n} \Delta(t_i) = \sum_{t \in T} \#(t, w) \cdot \Delta(t)$$

Example: if $C \xrightarrow{t_1 t_2 t_1} C'$ then $C' = C + 2 \cdot \Delta(t_1) + Delta(t_2)$

We have: if $C \xrightarrow{w} C'$ then $C' = C + \Delta(w)$

Implementing EvDead($\mathcal{S}$): Kirchoff's equations

If transition u can occur infinitely often from C

$\implies$ there is $C \xrightarrow{w} C$ with $\#(u, w) \geq 1$

$\implies$ there is w with $\Delta(w) = 0$ and $\#(u, w) \geq 1$

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$\Longrightarrow$ there are coefficients n_t for every $t \in T$ with

$$\sum_{t \in T} n_t \cdot \Delta(t) = 0 \quad \text{and} \quad n_u \geq 1$$

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Kirchoff's equations unsatisfiable

$\Longrightarrow$ t cannot occur infinitely often from any configuration

Implementing EvDead($\mathcal{S}$): Layers

A layer of a protocol is a set L of transitions such that for every configuration C (reachable or not):

- all executions from C containing only transitions of L are finite
- if all transitions of L are disabled at C, then they cannot be re-enabled by any sequence $w \in (T \setminus L)^*$.

If L is a layer, then form any configuration all transitions of L eventually die

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If L is a layer, then form any configuration all transitions of L eventually die

There exists a set of integer linear constraints whose solutions correspond to the possible layers of the protocol $\longrightarrow$ finding a layer is in NP

Implementing DeadAt$(U, \mathcal{S})$

Recall: $\mathit{DeadAt}(U, \mathcal{S})$ overapproximates the configurations reachable from $\mathcal{S}$ at which U is dead

Computable as intersection of:

- overapproximation of the configurations reachable from $\mathcal{S}$
  (overapproximation is necessary)
- overapproximation of the configurations at which U is dead
  (overapproximation is optional)

Implementing DeadAt$(U, \mathcal{S})$

Set of configurations reachable from $\mathcal{S}$

Overapproximated by set of configurations satisfying automatically computed linear invariants of the form

$$\sum_{q \in Q} a_q \cdot C(q) \geq b$$

Implementing DeadAt$(U, \mathcal{S})$

Set of configurations at which U is dead

$Dead(U)$: configurations at which U is dead

$En(U)$: configurations enabling some transition of U.

$$Dead(U) = \overline{\mathrm{pre}^*(En(U))}$$

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Observation: $\text{pre}^*(En(U))$ is upward-closed

$Dead(U)$ is downward-closed

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$\Rightarrow$ both are semilinear

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Proposition

$\mathrm{pre}^*(En(U))$ has finitely many minimal elements, and they can be computed using a symbolic backward reachability algorithm.

Implementing **DeadAt**$(U, \mathcal{S})$

Set of configurations at which U is dead

$\mathit{Dead}(U)$: configurations at which U is dead

$\mathit{En}(U)$: configurations enabling some transition of U.

$$\mathit{Dead}(U) = \overline{\mathrm{pre}^*(\mathit{En}(U))}$$

Proposition

$\mathrm{pre}^*(\mathit{En}(U))$ has finitely many minimal elements, and they can be computed using a symbolic backward reachability algorithm.

$\Rightarrow$ both are effectively semilinear

Some experimental results (a bit outdated ...)

Intel Core i7-4810MQ CPU and 16 GB of RAM.

Protocol	Predicate	$\|Q\|$	$\|T\|$	Time[s]
Majority[1]	$x \geq y$	4	4	0.1
Approx. Majority[2]	Not well-specified	3	4	0.1
Broadcast[3]	$x_1 \vee \ldots \vee x_n$	2	1	0.1
Threshold[4]	$\Sigma_i \alpha_i x_i < c$	76	2148	2375.9
Remainder[5]	$\Sigma_i \alpha_i x_i \bmod 70 = 1$	72	2555	3176.5
Sick ninjas[6]	$x \geq 50$	51	1275	181.6
Sick ninjas[7]	$x \geq 325$	326	649	3470.8
Poly-log sick ninjas	$x \geq 8 \cdot 10^4$	66	244	12.79

[1] Draief et al., 2012 [2] Angluin et al., 2007 [3] Clément et al., 2011
[4][5] Angluin et al., 2006 [6] Chatzigiannakis et al., 2010 [7] Clément et al., 2011

Sensei III's questions

What predicates can we compute?

How fast can we compute them?

How succinctly can we compute them?

How can I check correctness?

How can I check efficiency?

To conclude …

Peregrine: a tool for population protocols Blondin, E., Jaax CAV'18

Peregrine: Haskell + Microsoft Z3 + JavaScript

`peregrine.model.in.tum.de`

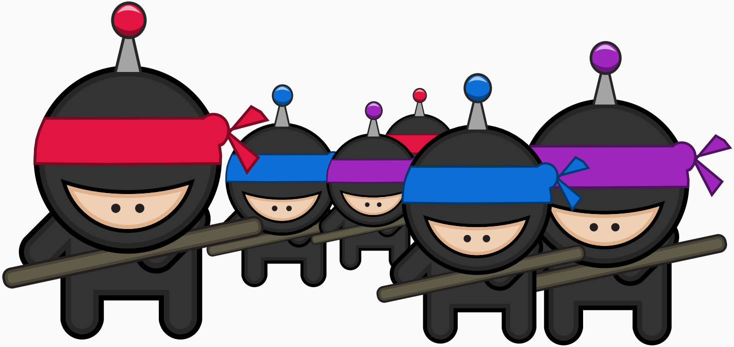
- Design of protocols
- Manual and automatic simulation
- Statistics of properties such as termination time
- Automatic verification of correctness
- More to come!

Population protocols are a great model to study fundamental questions of distributed computation:

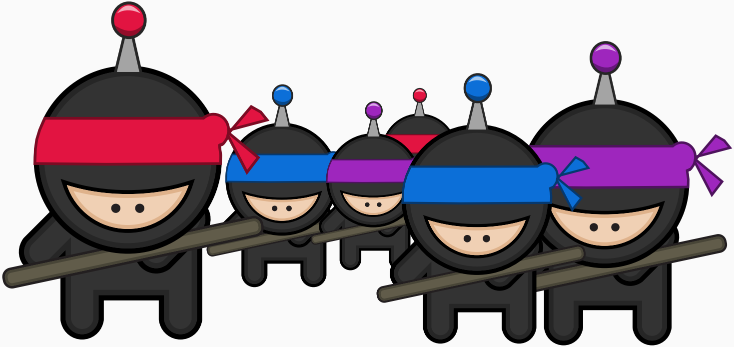
- Power of anonymous computation
- Network-independent algorithms
- Role of leaders
- Emergent behaviour and its limits

…and of formal verification:

- Verification of stochastic parameterized systems (parameterization, liveness under fairness)
- Automatic synthesis of parameterized systems



THANK YOU!



Go!

THANK YOU!