Monadic Second-Order Logic with Arbitrary Monadic Predicates[*]

Nathanaël Fijalkow[1,2] and Charles Paperman[1]

[1] LIAFA, Paris 7
[2] University of Warsaw

Abstract. We study Monadic Second-Order Logic (**MSO**) over finite words, extended with (non-uniform arbitrary) monadic predicates. We show that it defines a class of languages that has algebraic, automata-theoretic and machine-independent characterizations. We consider the *regularity question*: given a language in this class, when is it regular? To answer this, we show a *substitution property* and the existence of a *syntactical predicate*.

We give three applications. The first two are to give very simple proofs that the Straubing Conjecture holds for all fragments of **MSO** with monadic predicates, and that the Crane Beach Conjecture holds for **MSO** with monadic predicates. The third is to show that it is decidable whether a language defined by an **MSO** formula with morphic predicates is regular.

1   Introduction

The Monadic Second-Order Logic (**MSO**) over finite words equipped with the linear ordering on positions is a well-studied and understood logic. It provides a mathematical framework for applications in many areas such as program verification, database and linguistics. In 1962, Büchi [5] proved the decidability of the satisfiability problem for **MSO** formulae.

Uniform Monadic Predicates. In 1966, Elgot and Rabin [9] considered extensions of **MSO** with uniform monadic predicates. For instance, the following formula

$$\forall x, \ \mathbf{a}(x) \iff x \text{ is prime},$$

describes the set of finite words such that the letters a appear exactly in prime positions. The predicate "x is a prime number" is a uniform monadic predicate on the positions, it can be seen as a subset of $\mathbb{N}$.

Elgot and Rabin were interested in the following question: for a uniform monadic predicate $\mathbf{P} \subseteq \mathbb{N}$, is the satisfiability problem of $\mathbf{MSO}[\leq, \mathbf{P}]$ decidable? A series of papers gave tighter conditions on $\mathbf{P}$, culminating to two final answers: in 1984, Semenov [21] gave a characterization of the predicates $\mathbf{P}$ such that $\mathbf{MSO}[\leq, \mathbf{P}]$ is decidable, and in 2006, Rabinovich and Thomas [16, 18] proved that it is equivalent to the predicate $\mathbf{P}$ being effectively profinitely ultimately periodic.

Further questions on uniform monadic predicates have been investigated. For instance, Rabinovich [17] gave a solution to the Church synthesis problem for $\mathbf{MSO}[\leq, \mathbf{P}]$, for a large class of predicates $\mathbf{P}$.

In this paper, we consider non-uniform monadic predicates: such a predicate $\mathbf{P}$ is given, for each length $n \in \mathbb{N}$, by a predicate over the n first positions $\mathbf{P}_n \subseteq \{0, \ldots, n-1\}$. The set $\mathcal{M}$ of these predicates contains the set $\mathcal{M}^{\mathrm{unif}}$ of uniform monadic predicates.

Advice Regular Languages. We say that a language definable in $\mathbf{MSO}[\leq, \mathcal{M}]$ is *advice regular*. Note that no computability assumptions are made on the monadic predicates, so this class contains undecidable languages.

Our first contribution is to give equivalent presentations of this class, which is a Boolean algebra extending the class of regular languages:

1. It has an equivalent automaton model: *automata with advice*.
2. It has an equivalent algebraic model: *one-scan programs*.
3. It has a machine-independent characterization, based on generalizations of Myhill-Nerode equivalence relations.

This extends the equivalence between automata with advice and Myhill-Nerode equivalence relations proved in [13] for the special case of uniform monadic predicates. We will rely on those characterizations to obtain several properties of the advice regular languages. Our main goal is the following regularity question: given an advice regular language L, when is L regular?

To answer this question, we introduce two notions:

- The *substitution property*, which states that if a formula φ together with the predicate $\mathbf{P}$ defines a regular language $L_{\varphi, \mathbf{P}}$, then there exists a regular predicate $\mathbf{Q}$ such that $L_{\varphi, \mathbf{Q}} = L_{\varphi, \mathbf{P}}$.
- The *syntactical predicate* of a language L, which is the "simplest" predicate $\mathbf{P}_L$ such that $L \in \mathbf{MSO}[\leq, \mathbf{P}_L]$.

Our second contribution is to show that the class of advice regular languages has the substitution property, and that an advice regular language L is regular if and only if $\mathbf{P}_L$ is regular.

We apply these results to the case of morphic predicates [6], and obtain the following decidability result: given a language defined by an **MSO** formula with morphic predicates, one can decide whether it is regular.

Motivations from Circuit Complexity. Extending logics with predicates also appears in the context of circuit complexity. Indeed, a descriptive complexity theory initiated by Immermann [10] relates logics and circuits; it shows that a language is recognized by a Boolean circuit of constant depth and unlimited fan-in if and only if it can be described by a first-order formula with predicates (of any arity, so not only monadic ones), *i.e.* $\mathbf{AC^0} = \mathbf{FO}[\mathcal{N}]$.

This correspondence led to the study of two properties, which amount to characterize the regular languages (Straubing Conjecture) and the languages with a neutral letter (Crane Beach Conjecture) in several fragments of $\mathbf{FO}[\mathcal{N}]$. The Straubing Conjecture would, if true, give a deep understanding of many complexity classes inside $\mathbf{NC^1}$. Many cases of this conjecture are still open. On the other side, unfortunately the Crane Beach Conjecture does not hold in general, as shown by Barrington, Immermann, Lautemann, Schweikardt and Thérien [3]. On the positive side, both conjectures hold in the special case of uniform monadic predicates [3, 22].

Our third contribution is to give simple proofs of the both the Straubing and the Crane Beach Conjectures for monadic predicates, relying on our previous characterizations.

Outline. The Section 2 gives characterizations of advice regular languages, in automata-theoretic, algebraic and machine-independent terms. In Section 3, we study the regularity question, and give two different answers: one through the substitution property, and the other through the existence of a syntactical predicate. The last section, Section 4, provides applications of our results: easy proofs that the Straubing and the Crane Beach Conjectures hold for monadic predicates and decidability of the regularity problem for morphic regular languages.

2   Advice Regular Languages

In this section, we introduce the class of advice regular languages and give several characterizations.

Predicates A monadic predicate $\mathbf{P}$ is given by $\mathbf{P} = (\mathbf{P}_n)_{n \in \mathbb{N}}$, where $\mathbf{P}_n \subseteq \{0, \ldots, n-1\}$. Since we mostly deal with monadic predicates, we

often drop the word "monadic". In this definition the predicates are non-uniform: for each length the predicate is different. A predicate $\mathbf{P}$ if uniform is there exists $\mathbf{Q} \subseteq \mathbb{N}$ such that for every n, $\mathbf{P}_n = \mathbf{Q} \cap \{0, \ldots, n-1\}$. In such case we identify $\mathbf{P}$ and $\mathbf{Q}$, and see uniform predicates as subsets of $\mathbb{N}$.

Example 1.

- The predicate **first** $= (\{0\})_{n \in \mathbb{N}}$, which is true only on the *first* position, is uniform; we denote it by $\{0\}$.
- The predicate **last** $= (\{n-1\})_{n \in \mathbb{N}}$, which is true only for the *last* position, is not uniform.

For the sake of readability, we often define predicates as $\mathbf{P} = (\mathbf{P}_n)_{n \in \mathbb{N}}$ with $\mathbf{P}_n \subseteq \{0,1\}^n$. In such case we can see $\mathbf{P}$ as a language over $\{0,1\}$, which contains exactly one word for each length. Also, we often define predicates $\mathbf{P} = (\mathbf{P}_n)_{n \in \mathbb{N}}$ with $\mathbf{P}_n \in A^n$ for some finite alphabet A. This is not formally a predicate, but this amounts to define one predicate for each letter in A, and this abuse of notations will prove very convenient. Similarly, any infinite word $w \in A^\omega$ can be seen as a uniform predicate.

Monadic Second-Order Logic The formulae we consider are monadic second-order (**MSO**) formulae, obtained from the following grammar:

$$\varphi \;=\; \mathbf{a}(x) \mid x \leq y \mid P(x) \mid \varphi \wedge \varphi \mid \neg\varphi \mid \exists x,\; \varphi \mid \exists X,\; \varphi$$

Here $x, y, z, \ldots$ are first-order variables, which will be interpreted by positions in the word, and $X, Y, Z, \ldots$ are monadic second-order variables, which will interpreted by sets of positions in the word. We say that $\mathbf{a}$ is a letter symbol, $\leq$ the ordering symbol and $P, Q, \ldots$ are the numerical monadic predicate symbols, often refered to as predicate symbols.

The notation $\varphi(P^1, \ldots, P^\ell, x^1, \ldots, x^n, X^1, \ldots, X^p)$ means that in φ, the predicate symbols are among $P^1, \ldots, P^\ell$, the free first-order variables are among $x^1, \ldots, x^n$ and the free second-order variables are among $X^1, \ldots, X^p$. A formula without free variables is called a sentence.

We use the notation $\overline{P}$ to abbreviate $P^1, \ldots, P^\ell$, and similarly for all objects (variables, predicate symbols, predicates).

We now define the semantics. The letter symbols and the ordering symbol are always interpreted in the same way, as expected. For the predicate symbols, the predicate symbol P is interpreted by a predicate $\mathbf{P}$. Note that P is a syntactic object, while $\mathbf{P}$ is a predicate used as the interpretation of P.

Consider $\varphi(\overline{P}, \overline{x}, \overline{X})$ a formula, u a finite word of length n, $\overline{\mathbf{P}}$ predicates interpreting the predicate symbols from $\overline{P}$, $\overline{\mathbf{x}}$ valuation of the free first-order variables and $\overline{\mathbf{X}}$ valuation of the free second-order variables. We define $u, \overline{\mathbf{P}}, \overline{\mathbf{x}}, \overline{\mathbf{X}} \models \varphi$ by induction as usual, with

$$u, \overline{\mathbf{P}}, \overline{\mathbf{x}}, \overline{\mathbf{X}} \models P(y) \quad \text{if} \quad \mathbf{y} \in \mathbf{P}_n \ .$$

A sentence $\varphi(\overline{P})$ and a tuple of predicates $\overline{\mathbf{P}}$ interpreting the predicate symbols from $\overline{P}$ define a language

$$L_{\varphi, \overline{\mathbf{P}}} = \{ u \in A^* \mid u, \overline{\mathbf{P}} \models \varphi \} \ .$$

Such a language is called advice regular, and the class of advice regular languages is denoted by $\mathbf{MSO}[\leq, \mathcal{M}]$.

Automata with Advice We introduce automata with advice. Unlike classical automata, they have access to two more pieces of information about the word being read: its length and the current position. Both the transitions and the final states can depend on those two pieces of information. For this reason, they are (much) more expressive than classical automata, and recognize undecidable languages.

Formally, a non-deterministic automaton with advice is given by $\mathcal{A} = (Q, q_0, \delta, F)$ where Q is a finite set of states, $q_0 \in Q$ is the initial state, $\delta \subseteq \mathbb{N} \times \mathbb{N} \times Q \times A \times Q$ is the transition relation and $F \subseteq \mathbb{N} \times Q$ is the set of final states. In the deterministic case $\delta : \mathbb{N} \times \mathbb{N} \times Q \times A \to Q$.

A run over a finite word $u = u_0 \cdots u_{n-1} \in A^*$ is a finite word $\rho = q_0 \cdots q_n \in Q^*$ such that for all $i \in \{0, \ldots, n-1\}$, we have $(i, n, q_i, u_i, q_{i+1}) \in \delta$. It is accepting if $(n, q_n) \in F$.

One obtains a uniform model by removing one piece of information in the transition function: the length of the word. This automaton model is strictly weaker, and is (easily proved to be) equivalent to the one introduced in [13], where the automata read at the same time the input word and a fixed word called the advice. However, our definition will be better suited for some technical aspects: for instance, the number of Myhill-Nerode equivalence classes exactly correspond to the number of states in a minimal deterministic automaton.

Example 2. The language $\{a^n b^n c^n \mid n \text{ is a prime number}\}$ is recognized by a (deterministic) automaton with advice. The automaton is represented in figure 1. It has five states, q_a, q_b, q_c, q_F and $\bot$. The initial state

is q_a. The transition function is defined as follows:

$$\begin{aligned}
\delta(i, 3n, q_a, a) &= q_a \quad \text{if } i < n - 1 \\
\delta(n - 1, 3n, q_a, a) &= q_b \\
\delta(i, 3n, q_b, b) &= q_b \quad \text{if } n \le i < 2n - 1 \\
\delta(2n - 1, 3n, q_b, c) &= q_c \\
\delta(i, 3n, q_c, c) &= q_c \quad \text{if } 2n \le i < 3n - 1 \\
\delta(3n - 1, 3n, q_c, c) &= q_F
\end{aligned}$$

All other transitions lead to $\bot$, the sink rejecting state. The set of final states is $F = \{(3n, q_F) \mid n \text{ is a prime number}\}$.
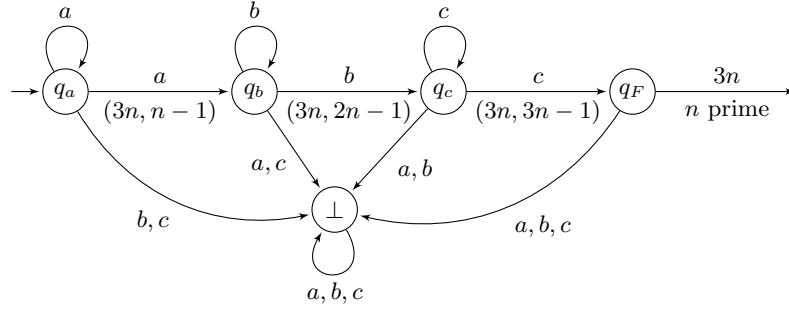

Fig. 1. The automaton for Example 2.

We mention another example, that appeared in the context of automatic structures [14]. They show that the structure $(\mathbb{Q}, +)$ is automatic with advice, which amounts to show that the language

$$\{\widehat{x} \sharp \widehat{y} \sharp \widehat{z} \mid z = x + y\},$$

where $\widehat{x}$ denotes the factorial representation of the rational x, is advice regular. A very difficult proof shows that this is not possible without advice [23].

One-scan Programs Programs over monoids were introduced in the context of circuit complexity [1]: Barrington showed that any language in $\mathbf{NC^1}$ can be computed by a program of polynomial length over a non-solvable group. Here we present a simplification introduced in [22], adapted to the context of monadic predicates.

A one-scan program is given by $P = (M, (f_{i,n} : A \to M)_{i,n \in \mathbb{N}}, S)$ where M is a finite monoid and $S \subseteq M$. The function $f_{i,n}$ is used to

compute the effect of the ith letter of an input word of length n. We say that the program P accepts the word $u = u_0 \cdots u_{n-1}$ if

$$f_{0,n}(u_0) \cdots f_{n-1,n}(u_{n-1}) \in S.$$

Note that this echoes the classical definition of recognition by monoids, where a morphism $f : A \to M$ into a finite monoid M recognizes the word $u = u_0 \cdots u_{n-1}$ if $f(u_0) \cdots f(u_{n-1}) \in S$. Here, a one-scan program uses different functions $f_{i,n}$, depending on the position i and the length of the word n.

Example 3. Let U_1 be the monoid over $\{0, 1\}$ equipped with the classical multiplication. Consider the alphabet $A = \{a, b\}$. We define the one-scan program $(U_1, (f_{i,n})_{i,n \in \mathbb{N}}, \{1\})$ as follows. If n is not a prime number, then $f_i : A \to M$ is constant equal to 0. Otherwise, $f_{i,n}(a) = 0$ and $f_{i,n}(b) = 1$. Therefore, a word is accepted if and only if its length is prime and all prime positions are labelled by b.

Myhill-Nerode Equivalence Relations Let $L \subseteq A^*$ and $p \in \mathbb{N}$, we define two equivalence relations:

- $u \sim_L v$ if for all $w \in A^*$, we have $uw \in L \iff vw \in L$,
- $u \sim_{L,p} v$ if for all $w \in A^p$, we have $uw \in L \iff vw \in L$.

The relation $\sim_L$ is called the (classical) Myhill-Nerode equivalence relation, and the second is a coarser relation, which we call the p-Myhill-Nerode equivalence relation.

Recall that $\sim_L$ contains finitely many equivalence classes if and only if L is regular, *i.e.* $L \in \mathbf{MSO}[\leq]$.

Equivalence We state several characterizations of advice regular languages.

Theorem 1 (Advice Regular Languages). *Let L be a language of finite words, the following properties are equivalent:*

(1) $L \in \mathbf{MSO}[\leq, \mathcal{M}]$,
(2) L is recognized by a non-deterministic automaton with advice,
(3) L is recognized by a deterministic automaton with advice,
(4) There exists $K \in \mathbb{N}$ such that for all $i, p \in \mathbb{N}$, the restriction of $\sim_{L,p}$ to words of length i contains at most K equivalence classes.
(5) L is recognized by a one-scan program,

In such case, we say that L is advice regular.

This extends the Myhill-Nerode theorem proposed in [13], which proves the equivalence between (3) and (5) for the special case of uniform predicates.

The proof consists in showing the following:

- the equivalence between 2 and 3 consists in determinizing automata with advice,
- the implication from 1 to 2 requires us to show closure properties of automata with advice under union, projection and complementation,
- the implication from 2 to 1 amounts to write a formula checking for the existence of a run,
- the implication from 3 to 4,
- the implication from 4 to 3, where we construct a deterministic automaton with advice,
- the equivalence between 3 and 5 is syntactical.

Proof. The implication $2 \Rightarrow 3$ consists in determinizing automata with advice, extending the classical powerset construction. Let $\mathcal{A} = (Q, q_0, \delta, F)$ be a non-deterministic automaton with advice, we construct the deterministic automaton with advice $\mathcal{A}' = (Q', \{q_0\}, \delta', F')$, where Q' is the powerset of Q, the set of final states is $F' = \{(n, S) \mid \exists q \in S, (n, q) \in F\}$, and the transition function δ' is defined by:

$$\delta'(i, n, S, a) = \{q' \in Q \mid \exists q \in S, (i, n, q, a, q') \in \delta\} \ .$$

It is easy to see that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent.

The implication $3 \Rightarrow 2$ is immediate from the definitions.

The implication $1 \Rightarrow 2$ requires us to show closure properties of automata with advice under union, projection and complementation. The first two closures are obtained in the exact same way as in the classical case, we do not detail them here; for the third case, we rely on the equivalence between 3 and 4, and complement deterministic (complete) automata with advice by simply exchanging F and its complement in $\mathbb{N} \times Q$.

The implication $2 \Rightarrow 1$ amounts to write a formula checking for the existence of a run. Let $\mathcal{A} = (Q, q_0, \delta, F)$ be a non-deterministic automaton with advice recognizing a language L. Without loss of generality we have $Q = \{0, \dots, \ell - 1\}$ and $q_0 = 0$.

Let $\overline{X}$ be a ℓ-tuple of monadic second-order variables. We first need to express that $\overline{\mathbf{X}}$ partitions the set of all positions of the word. This is

$$\exists \overline{X}, \begin{cases} \chi(\overline{X}) \\ \wedge \; \forall x, \quad \mathrm{first}(x) \implies x \in X_0 \\ \wedge \; \forall x, \forall y, \quad y = x + 1 \wedge \bigwedge_{(q,a,q')} \quad T^{q,a,q'}(x) \wedge x \in X_q \wedge a(x) \\ \quad \implies y \in X_{q'} \\ \wedge \; \forall x, \quad \mathrm{last}(x) \implies \bigvee_q \quad x \in X_q \wedge F^q(x) \;. \end{cases}$$

Fig. 2. Formula checking for the existence of an accepting run.

easily expressed by the following formula, denoted $\chi(\overline{X})$:

$$\forall x, \; \left(\bigvee_i x \in X_i \right) \; \wedge \; \bigvee_i \left(x \in X_i \to \bigwedge_{j \neq i} x \notin X_j \right) \;.$$

For each $q \in Q$, we define the predicates $\mathbf{T}^{q,a,q'}$ by $\mathbf{T}_n^{q,a,q'} = \{i \in \mathbb{N} \mid \delta(i, n, q, a) = q'\}$ and $\mathbf{F}^q$ by $\mathbf{F}^q = \{n \in \mathbb{N} \mid (n, q) \in F\}$.

The **MSO** formula φ in figure 2 checks for the existence of an accepting run, and uses the predicate symbols $T^{q,a,q'}$ and F^q. We have $L_{\varphi, \{\mathbf{T}^{q,a,q'}, \mathbf{F}^q\}} = L$.

For the implication $3 \Rightarrow 4$, let $\mathcal{A}$ be a deterministic automaton with advice. Let $n = i + p$, and consider the mapping $t_{i,n} : A^i \to Q$ defined for $u = u_0 \cdots u_{i-1}$ by

$$t_{i,n}(u) = \delta(i - 1, n, \delta(i - 2, n, \cdots \delta(0, n, q_0, u_0) \cdots, u_{i-2}), u_{i-1}) \;.$$

In words, $t_{i,n}(u)$ is the state reached by $\mathcal{A}$ while reading u assuming that the total word will be of length n. We argue that $t_{i,n}(u) = t_{i,n}(v)$ implies $u \sim_{L,p} v$: indeed, for $w \in A^p$, after reading u or v, the automaton $\mathcal{A}$ is in the same state, so it will either accept both uv and vw or reject both. Note that $t_{i,n}$ can have at most $|Q|$ different values. Consequently, the restriction of $\sim_{L,p}$ to words of length i contains at most $|Q|$ equivalence classes.

We now prove the implication $4 \Rightarrow 3$, by constructing a deterministic automaton with advice. Its set of states is $Q = \{1, \ldots, K\}$. To each word u and length $n \geq 0$ we associate $\lfloor u \rfloor_n \in Q$ such that if u and v both have length i, then $u \sim_{L,n-i} v$ if and only if $\lfloor u \rfloor_n = \lfloor v \rfloor_n$. We set $\lfloor \varepsilon \rfloor_n = 1$ for all n; the initial state is 1. The transition function is defined by $\delta(i, n, \lfloor u \rfloor_n, a) = \lfloor ua \rfloor_n$, for *some* u of length i. (This is well defined:

if both u and v have length i and $\lfloor u \rfloor_n = \lfloor v \rfloor_n$, then $u \sim_{L,n-i} v$, so $ua \sim_{L,n-i-1} va$, thus $\lfloor ua \rfloor_n = \lfloor va \rfloor_n$.) The set of final states is

$$F = \{(n, \lfloor u \rfloor_0) \mid u \in A^n \cap L\}.$$

We argue that this automaton recognizes L: whenever it reads $u = u_0 \cdots u_{n-1}$, the corresponding run is

$$\rho = \lfloor \varepsilon \rfloor_n \lfloor u_0 \rfloor_n \lfloor u_0 u_1 \rfloor_n \cdots \lfloor u_0 \cdots u_{n-1} \rfloor_n,$$

which is accepting if and only if $u \in L$.

The implication $3 \Rightarrow 5$ is syntactical. Consider a deterministic automaton with advice $\mathcal{A} = (Q, q_0, \delta, F)$ recognizing a language L. We define M to be the monoid of applications from Q to Q, with the composition as associative internal law. Define

$$f_{i,n} : \begin{cases} A \to M \\ a \mapsto (q \mapsto \delta(i, n, q, a)) \end{cases},$$

and $S = \{\phi \in M \mid \phi(q_0) \in F\}$. The one-scan program $(M, (f_{i,n})_{i,n \in \mathbb{N}}, S)$ recognizes L.

The converse implication $5 \Rightarrow 3$ is also syntactical. Consider a the one-scan program $(M, (f_{i,n})_{i,n \in \mathbb{N}}, S)$ recognizing a language L. Define the deterministic automaton with advice $\mathcal{A} = (M, 1, \delta, F)$ where 1 is the neutral element of M, the transition function δ is defined by $\delta(i, n, m, a) = m \cdot f_{i,n}(a)$, and $F = \{(n, m) \mid m \in S\}$. The automaton $\mathcal{A}$ recognizes the language L. □

3 The Regularity Question

In this section, we address the following question: given an advice regular language, when is it regular? We answer this question in two different ways: first by showing a substitution property, and second by proving the existence of a syntactical predicate.

Note that the regularity question is not a decision problem, as advice regular languages are not finitely presentable, so we can only provide (non-effective) characterizations of regular languages inside the advice regular languages.

In the next section, we will show how these two notions answer the regularity question: first by proving that the Straubing property holds in this case, and second by proving the decidability of the regularity problem for morphic regular languages.

3.1 Regular Predicates

We start by defining the class of (monadic) *regular predicates*.

The predicates $\{c\}$ and $\mathbf{last} - c = (\{n - 1 - c\})_{n \in \mathbb{N}}$ for a given $c \in \mathbb{N}$ are called *local predicates*. The predicates $\{x \mid x \equiv r \bmod q\}$ and $\mathbf{last} \equiv r \bmod q$ for given $q, r \in \mathbb{N}$ are called *modular predicates*.

Theorem 2 ([15, 22]). *Let* $\mathbf{P} = (\mathbf{P}_n)_{n \in \mathbb{N}}$ *be a predicate. The following properties are equivalent:*

1. *There exists a formula* $\varphi(x) \in \mathbf{MSO}[\leq]$ *over the one-letter alphabet* $\{a\}$ *such that* $\mathbf{P}_n = \{\mathbf{x} \in \{0, n - 1\} \mid a^n, \mathbf{x} \models \varphi(x)\}$.
2. *The predicate* $\mathbf{P}$ *is a boolean combination of local and modular predicates.*
3. *The language* $\mathbf{P} \subseteq A^*$ *is regular.*

In such case, we say that $\mathbf{P}$ *is regular. We denote by* $\mathcal{R}eg_1$ *the class of regular (monadic) predicates.*

3.2 A Substitution Property

In this subsection, we prove a substitution property for $\mathbf{MSO}[\leq, \mathcal{M}]$.

Theorem 3. *For all sentences* $\varphi(\overline{P})$ *in* $\mathbf{MSO}[\leq, \mathcal{M}]$ *and predicates* $\overline{\mathbf{P}} \in \mathcal{M}$ *such that* $L_{\varphi, \overline{\mathbf{P}}}$ *is regular, there exist* $\overline{\mathbf{Q}} \in \mathcal{R}eg_1$ *such that* $L_{\varphi, \overline{\mathbf{Q}}} = L_{\varphi, \overline{\mathbf{P}}}$.

The main idea of the proof is that among all predicates $\overline{\mathbf{Q}}$ such that $L_{\varphi, \overline{\mathbf{P}}} = L_{\varphi, \overline{\mathbf{Q}}}$, there is a minimal one with respect to a lexicographic ordering, which can be defined by an $\mathbf{MSO}$ formula. The key technical point is given by the following lemma, which can be understood as a regular choice function.

Lemma 1 (Regular Choice Lemma). *Let* M *be a regular language such that for all* $k \in \mathbb{N}$, *there exists a word* $w \in M$ *of length* k. *Then there exists* $M' \subseteq M$ *a regular language such that for all* $k \in \mathbb{N}$, *there exists exactly one word* $w \in M'$ *of length* k.

Proof. Without loss of generality, we assume that the alphabet of M is $A = \{0, \ldots, \ell - 1\}$, naturally ordered. We denote by $\preceq$ the lexicographic ordering on A^*.

Let ψ be an $\mathbf{MSO}$ formula defining M. The objective is to define an $\mathbf{MSO}$ formula Ψ such that w satisfies Ψ if and only if w is minimal among the words of its length to satisfy ψ with respect to $\preceq$. The language defined by this formula satisfies the desired properties.

First, let $\overline{X}$ be a ℓ-tuple of monadic second-order variables. We say that $\overline{\mathbf{X}} \in (\{0,1\}^\ell)^k$ represents the word $v \in A^k$ if $\overline{\mathbf{X}}$ partitions the set of all positions and for all $j \in \{0, \ldots, k-1\}$, we have $v_j = i$ if and only if $i \in \mathbf{X}_j$.

The formula expressing that $\overline{\mathbf{X}}$ partitions the set of all positions is denoted by $\chi(\overline{X})$ (see the proof of Theorem 1 for the definition of this formula).

We obtain a formula $\varphi(\overline{X})$ from ψ by syntactically replacing in ψ each letter predicate $\mathbf{i}(x)$ by $x \in X_i$. The following property holds: for all $\overline{\mathbf{X}} \in (\{0,1\}^\ell)^k$, if $\overline{\mathbf{X}}$ represents the word v, then $\overline{\mathbf{X}} \models \varphi(\overline{X})$ is equivalent to $v \models \psi$.

Now, we define a formula $\theta(\overline{X})$ such that for all words $w \in A^k$ and $\overline{\mathbf{X}} \in (\{0,1\}^\ell)^k$, if $\overline{\mathbf{X}}$ represents a word v, then $w, \overline{\mathbf{X}} \models \theta(\overline{X})$ if and only if $w \preceq v$. The formula $\theta(\overline{X})$ is a disjunction of two formulae, the first stating that $\overline{\mathbf{X}}$ represents v and $w \prec v$:

$$\exists x, \left(\bigvee_{i<j} \mathbf{i}(x) \wedge x \in X_j \right) \wedge \left(\forall y,\ y < x \to \left(\bigvee_i \mathbf{i}(y) \wedge y \in X_i \right) \right)$$

and the second stating that $\overline{\mathbf{X}}$ represents w:

$$\forall x,\ x \in X_i \iff \mathbf{i}(x) .$$

The **MSO** formula Ψ that selects the minimal word in M is given by:

$$\psi \ \wedge \ \forall \overline{X},\ \big(\chi(\overline{X}) \wedge \varphi(\overline{X}) \big) \to \theta(\overline{X}) .$$

This concludes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now prove Theorem 3 relying on Lemma 1.

Proof. Consider $\varphi(\overline{P})$ a sentence and $\overline{\mathbf{P}}$ predicates such that $L_{\varphi, \overline{\mathbf{P}}}$ is regular, denote it by L. Let θ be an **MSO** formula defining L.

Consider the language $M = \{\overline{\mathbf{X}} \in (\{0,1\}^\ell)^* \mid L_{\varphi, \overline{\mathbf{X}}} = L\}$. We fist argue that M is regular. Define the formula ψ in **MSO**[$\leq$] over the alphabet $A \times \{0,1\}^\ell$ as $\varphi(\overline{X}) \iff \theta$, it has ℓ free second-order variables and defines the language $\{w \otimes \overline{\mathbf{X}} \mid\ w, \overline{\mathbf{X}} \models \varphi(\overline{X}) \iff \theta\}$, where the notation $w \otimes \overline{X}$ denotes the word over $A \times \{0,1\}^\ell$ whose first component is w and second component is $\overline{X}$. The language M is obtained from the latter by a universal quantification on the first component, so it is regular.

Also, it is clear that for all $k \in \mathbb{N}$, there exists a word in M of length k, namely $\overline{\mathbf{P}}_k$. Thus Lemma 1 applies, so there exists $M' \subseteq M$ a regular

language so that for all $k \in \mathbb{N}$, there exists a unique word in M' of length k, which we denote by $\overline{\mathbf{Q}}_k$. Thanks to Theorem 2, this gives a tuple of regular predicates $\overline{\mathbf{Q}}$ such that $L_{\varphi,\overline{\mathbf{Q}}} = L_{\varphi,\overline{\mathbf{P}}}$. $\qquad\qquad\square$

3.3 The Syntactical Predicate

In this subsection, we define the notion of syntactical predicate for an advice regular language. The word "syntactical" here should be understood in the following sense: the syntactical predicate $\mathbf{P}_L$ of L is the most regular predicate that describes the language L. In particular, we will prove that L is regular if and only if $\mathbf{P}_L$ is regular.

Let L be an advice regular language. We define the predicate $\mathbf{P}_L = (\mathbf{P}_{L,n})_{n \in \mathbb{N}}$. Thanks to Theorem 1, there exists $K \in \mathbb{N}$ such that for all $i, p \in \mathbb{N}$, the restriction of $\sim_{L,p}$ to words of length i contains at most K equivalence classes. Denote $Q = \{1, \ldots, K\}$ and $\Sigma = (Q \times A \to Q) \uplus Q$, where $Q \times A \to Q$ is the set of (partial) functions from $Q \times A$ to Q. We define $\mathbf{P}_{L,n} \in \Sigma^n$.

Let $i, n \in \mathbb{N}$. Among all words of length i, we denote by $u_1^{i,n}, u_2^{i,n}, \ldots$ the lexicographically minimal representants of the equivalence classes of $\sim_{L,n-i}$, enumerated in the lexicographic order:

$$u_1^{i,n} <_{\text{lex}} u_2^{i,n} <_{\text{lex}} u_3^{i,n} <_{\text{lex}} \ldots \qquad (1)$$

In other words, $u_\ell^{i,n}$ is minimal with respect to the lexicographic order $<_{\text{lex}}$ among all words of length i in its equivalence class for $\sim_{L,n-i}$. Thanks to Theorem 1, there are at most K such words for each $i, n \in \mathbb{N}$.

We define $\mathbf{P}_{L,n}(i)$ (the ith letter of $\mathbf{P}_{L,n}$) by:

$$\mathbf{P}_{L,n}(i)(\ell, a) = k \quad \text{if} \quad u_\ell^{i,n} \cdot a \sim_{L,n-i-1} u_k^{i+1,n} \text{ , for } i < n \qquad (2)$$

$$\mathbf{P}_{L,n}(n-1)(\ell) \quad \text{if} \quad u_\ell^{n,n} \in L \text{ .} \qquad (3)$$

Intuitively, the predicate $\mathbf{P}_L$ describes the transition function with respect to the equivalence relations $\sim_{L,p}$. We now give an example.

Example 4. Consider the language $L = (ab)^* + (ba)^*b$. We represent $\mathbf{P}_{L,4}$ in figure 3. Each circle represents an equivalence class with respect to $\sim_{L,4}$, inside words of a given length. For instance, there are three equivalence classes for words of length 3: a^3, aba and bab. Note that these three words are the minimal representants of their equivalence classes with respect to the lexicographic order. For the last position (here 3), the equivalence class of $(ab)^2$ (which is actually reduced to $(ab)^2$ itself) is darker since it belongs to the language L.
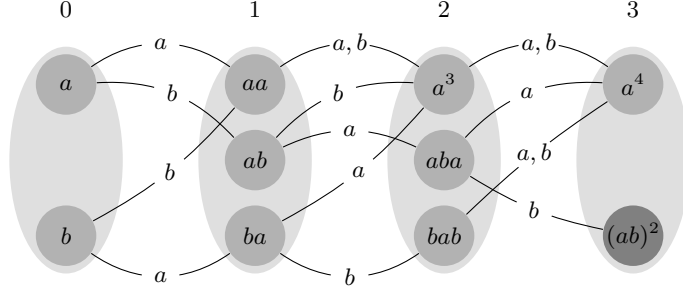
Fig. 3. The predicate $\mathbf{P}_L$ (here $\mathbf{P}_{L,4}$) for $L = (ab)^* + (ba)^*b$.

We state the main property of the predicate $\mathbf{P}_L$.

Theorem 4. *Let L be an advice regular language. Then L is regular if and only if $\mathbf{P}_L$ is regular.*

The proof is split in two lemmas, giving each direction.

Lemma 2. *Let L be an advice regular language. Then $L \in \mathbf{MSO}[\leq, \mathbf{P}_L]$.*

Proof. From the definition of $\mathbf{P}_L$, it is easy to see that $u = u_0 \cdots u_{n-1}$ belongs to L if and only if there exists $X : \{0, \ldots, n\} \to Q$ such that:

$$
\begin{aligned}
& X(0) = \ell \iff u_0 = a \wedge \mathbf{P}_{L,n}(0)(1, a) = \ell \\
\wedge \ & \forall i < n, \ X(i+1) = k \iff X(i) = \ell \wedge u_i = a \wedge \mathbf{P}_{L,n}(i)(\ell, a) = k \\
\wedge \ & \qquad\qquad X(n) = \ell \implies \mathbf{P}_{L,n}(n-1)(\ell) \ .
\end{aligned}
$$

This can be written down as an **MSO** formula with the predicate $\mathbf{P}_L$.
□

Lemma 3. *Let L be an advice regular language defined with the predicates $\overline{\mathbf{P}}$. Then $\mathbf{P}_L \in \mathbf{MSO}[\leq, \overline{\mathbf{P}}]$.*

Proof. Assume that L is defined by an **MSO** formula θ with the predicates $\overline{\mathbf{P}}$. Then the three equations (1), (2) and (3) defining $\mathbf{P}_L$ can be written down as an **MSO** formula with the predicates $\overline{\mathbf{P}}$.

To this end, we represent words as monadic second-order variables as in the proof of Lemma 1. Let $\ell = |\Sigma|$, a ℓ-tuple $\overline{X}$ of monadic second-order variables represents the word $v = v_0 \cdots v_{k-1} \in A^k$ if $\overline{\mathbf{X}}$ partitions the set of all positions up to position k, and for all $j \in \{0, \ldots, k-1\}$, we have $v_j = a$ if and only if $i \in \mathbf{X}_a$.

Denote by $\chi(\overline{X}, z)$ the **MSO** formula expressing that $\overline{\mathbf{X}}$ partitions the set of all positions up to position k. Similarly, denote by $\Xi(\overline{X}, z)$ the **MSO** formula expressing that $\overline{\mathbf{X}}$ partitions the set of all positions from the position $k + 1$.

The formulae for (1) and (3) make use of the formulae $\chi(\overline{X}, z)$ and θ. We omit them as they are easy to write down, and focus on (2).

The first step is to construct a formula $\varphi(\overline{X}, \overline{Y}, z)$ such that if w has length n, $\overline{\mathbf{X}}$ represents u and $\overline{\mathbf{Y}}$ represents v both of length i, then $w, \overline{\mathbf{X}}, \overline{\mathbf{Y}}, i \models \varphi(\overline{X}, \overline{Y}, z)$ if and only if $u \sim_{L, n-i} v$. Define $\varphi(\overline{X}, \overline{Y}, z)$ as:

$$\forall \overline{Z}, \; \Xi(\overline{Z}, z) \implies (\phi(\overline{X}, z, \overline{Z}) \iff \phi(\overline{Y}, z, \overline{Z})) \,,$$

where the formula $\phi(\overline{X}, z, \overline{Z})$ is obtained from θ by syntactically replacing in θ each letter predicate $\mathbf{i}(x)$ by $(x \leq z \wedge x \in X_i) \vee (x > z \wedge x \in Z_i)$.

The second step is to construct a finite number of formulae $\gamma_\ell(\overline{X}, z)$ for $\ell \in Q$ such that if w has length n and $\overline{\mathbf{X}}$ represents u of length i, then $w, \overline{\mathbf{X}}, i \models \gamma_\ell(\overline{X}, z)$ if and only if there are exactly $\ell - 1$ words of length i that are (i) pairwise not equivalent with respect to $\sim_{L, n-i}$, (ii) not equivalent to u with respect to $\sim_{L, n-i}$, and (iii) smaller than u with respect to the lexicographic order.

We can now put the pieces together and give a formula for (2):

$$\forall z, \forall \overline{X}, \forall \overline{Y}, \bigwedge_{a \in A, \; \ell, k \in Q} z \in X_a \wedge \chi(\overline{X}, z) \wedge \chi(\overline{Y}, z) \implies$$
$$(\ell, a, k)(z) \iff (\gamma_\ell(\overline{X}, z) \wedge \gamma_k(\overline{Y}, z)) \,.$$

It follows that $\mathbf{P}_L$ is definable in $\mathbf{MSO}[\leq, \overline{\mathbf{P}}]$. □

4 Applications

In this section we show several consequences of Theorem 1 (characterization of the advice regular languages), Theorem 3 (a substitution property for advice regular languages) and Theorem 4 (a syntactical predicate for advice regular predicates).

The first two applications are about two conjectures, the Straubing and the Crane Beach Conjectures, introduced in the context of circuit complexity. We first explain the motivations for these two conjectures, and show very simple proofs of both of them in the special case of monadic predicates.

The third application shows that one can determine, given an **MSO** formula with morphic predicates, whether it defines a regular language.

4.1 A Descriptive Complexity for Circuit Complexity Classes

We first quickly define some circuit complexity classes. The most important here is $\mathbf{AC^0}$, the class of languages defined by boolean circuits of bounded depth and polynomial size, and its subclass $\mathbf{LAC^0}$ where the circuits have linear size. From $\mathbf{AC^0}$, adding the modular gates gives rise to $\mathbf{ACC}$. Finally, the class of languages defined by boolean circuits of logarithmic depth, polynomial size and fan-in 2 is denoted by $\mathbf{NC^1}$. Separating $\mathbf{ACC}$ from $\mathbf{NC^1}$ remains a long-standing open problem.

One approach to better understand these classes is through descriptive complexity theory, giving a perfect correspondence between circuit complexity classes and logical formalisms. Unlike what we did so far, the logical formalisms involved in this descriptive complexity theory use predicates of any arity (we focused on predicates of arity one). A k-ary predicate $\mathbf{P}$ is given by $(\mathbf{P}_n)_{n \in \mathbb{N}}$, where $\mathbf{P}_n \subseteq \{0, \ldots, n-1\}^k$. We denote by $\mathcal{N}$ the class of all predicates, and by $\mathcal{R}eg$ the class of regular predicates as defined in [22].

We use the classical classes of formulae: $\mathbf{FO}$ (first-order quantifiers), $\mathbf{FO + MOD}$ (first-order and modular quantifiers: $\exists^{r,q} x, \varphi(x)$ reads "the number of x satisfying $\varphi(x)$ is equal to r mod q"), $\mathbf{FO^2}$ (first-order with at most two variables) and $\mathcal{B}\Sigma_k$ (at most $k-1$ alternations of $\exists$ and $\forall$ quantifiers).

Theorem 5 ([1, 4, 7, 12]).

(1) $\mathbf{AC^0} = \mathbf{FO}[\mathcal{N}]$,
(2) $\mathbf{LAC^0} = \mathbf{FO^2}[\mathcal{N}]$,
(3) $\mathbf{ACC} = (\mathbf{FO + MOD})[\mathcal{N}]$.

Two conjectures have been formulated on the logical side, which aim at clarifying the relations between different circuit complexity classes. They have been stated and studied in special cases, we extrapolate them here to all fragments. We first need to give an abstract notion of (logical) fragment. We fix the universal signature, containing infinitely many predicates symbols for each arity. Let $\mathbf{F}$ be a class of formulae over this signature and $\mathcal{P}$ a class of predicates, describing the fragment $\mathbf{F}[\mathcal{P}]$ by:

$$\mathbf{F}[\mathcal{P}] = \{L_{\varphi, \overline{\mathbf{P}}} \mid \varphi \in \mathbf{F} \wedge \overline{\mathbf{P}} \in \mathcal{P}\} .$$

The first property, called the Straubing property, characterizes the regular languages (denoted by $\mathbf{REG}$) inside a larger fragment.

Definition 1 (Straubing Property). $\mathbf{F}[\mathcal{P}]$ *has the Straubing property if: all regular languages definable in* $\mathbf{F}[\mathcal{P}]$ *are also definable in* $\mathbf{F}[\mathcal{P}\cap\mathcal{R}eg]$.

In equation:
$$\mathbf{F}[\mathcal{P}] \cap \mathbf{REG} = \mathbf{F}[\mathcal{P} \cap \mathcal{R}eg] \ .$$

This statement appears for the first time in [2], where it is proved that $\mathbf{FO}[\mathcal{N}]$ has the Straubing property, relying on lower bounds for $\mathbf{AC^0}$ and an algebraic characterisation of $\mathbf{FO}[\mathcal{R}eg]$. Following this result, Straubing conjectures in [22] that $(\mathbf{FO}+\mathbf{MOD})[\mathcal{N}]$ and $\mathcal{B}\mathbf{\Sigma}_k[\mathcal{N}]$ have the Straubing property for $k \geq 1$. Recently, this conjecture has been extended to $\mathbf{FO}^2[\mathcal{N}]$ (see [12]). If true, it would imply the separation of $\mathbf{ACC}$ from $\mathbf{NC^1}$, and for the $\mathbf{FO}^2$ case, tight lower bounds on the addition of two integers in binary.

It has been shown that several fragments have the Straubing property, as for instance, $\mathbf{\Sigma_1}[\mathcal{N}]$, $\mathbf{FO}[\leq,\mathcal{M}^{\mathrm{unif}}]$ and $(\mathbf{FO}+\mathbf{MOD})[\leq,\mathcal{M}^{\mathrm{unif}}]$ (in [22]). In this paper, we give a simple proof that all fragments $\mathbf{F}[\leq,\mathcal{M}]$ have the Straubing property.

The second property, called the Crane Beach property, characterizes the languages having a neutral letter, and has been proposed by Thérien for the special case of first-order logic.

Definition 2 (Neutral letter). *A language* L *has a neutral letter* $e \in A$ *if for all words* u, v, *we have* $uv \in L$ *if and only if* $uev \in L$.

Definition 3 (Crane Beach Property). $\mathbf{F}[\mathcal{P}]$ *has the Crane Beach property if: all languages having a neutral letter definable in* $\mathbf{F}[\mathcal{P}]$ *are definable in* $\mathbf{F}[\leq]$.

Unfortunately, the Crane Beach property does not hold in general.

Theorem 6 ([3, 20]). *There exists a non-regular language having a neutral letter definable in* $\mathbf{FO}[\mathcal{N}]$.

A deeper understanding of the Crane Beach property specialized to first-order logic can be found in [3]. In particular, it has been shown that $\mathbf{FO}[\leq,\mathcal{M}^{\mathrm{unif}}]$ has the Crane Beach property. In this paper, we give a simple proof that $\mathbf{MSO}[\leq,\mathcal{M}]$ has the Crane Beach Property.

4.2 The Straubing Conjecture for Advice Regular Languages

Theorem 7. *All fragments* $\mathbf{F}[\leq,\mathcal{M}]$ *have the Straubing property.*

This is actually a straightforward corollary of Theorem 3.

Proof. Let $\varphi \in \mathbf{F}$ such that $L_{\varphi,\overline{\mathbf{P}}}$ with $\overline{\mathbf{P}} \in \mathcal{M}$ is regular. Thanks to Theorem 3, there exist $\overline{\mathbf{Q}} \in \mathcal{R}eg_1$ such that $L_{\varphi,\overline{\mathbf{Q}}} = L_{\varphi,\overline{\mathbf{P}}}$. This concludes. $\qquad\square$

We state a corollary of Theorem 7.

Corollary 1. *For all $k \geq 1$, $\mathcal{B}\boldsymbol{\Sigma}_k[\leq, \mathcal{M}]$ has the Straubing property.*

This result is, to the best of our knowledge, the first intermediary result towards a proof of the Straubing Conjecture for $\mathcal{B}\boldsymbol{\Sigma}_k[\mathcal{N}]$.

4.3 The Crane Beach Conjecture for Advice Regular Languages

In this subsection, we show that the Crane Beach Conjecture holds for advice regular languages.

Theorem 8. $\mathbf{MSO}[\leq, \mathcal{M}]$ *has the Crane Beach property.*

The proof is a simple corollary of Theorem 1.

Proof. Recall that a language over finite words L has a neutral letter $e \in A$ if for all words u and v, we have $uv \in L$ if and only if $uev \in L$. In other words, $u \sim_L ue$.

Let L an advice regular language, thanks to Theorem 1, there exists $K \in \mathbb{N}$ such that for all $i, p \in \mathbb{N}$, the restriction of $\sim_{L,p}$ to words of length i contains at most K equivalence classes.

We argue that $\sim_L$ contains at most K equivalence classes (without both restrictions to words of a given length). Indeed, assume towards contradiction that there are $K+1$ words that are pairwise non-equivalent with respect to $\sim_L$. By iterating the equivalence $u \sim_L ue$, we obtain $K+1$ words of the same length (the maximal length of the $K+1$ original words), which are still pairwise non-equivalent with respect to $\sim_L$. For two non-equivalent words u, v, there exist a third word w witnessing the non-equivalence: $uw \in L$ but $vw \notin L$ or the other way around. Again by padding with the neutral letter e, we obtain non-equivalence witnesses for each pair of the $K+1$ words of the same length (the maximal length of the $\frac{(K+1)(K+2)}{2}$ witnesses). Hence we have $K+1$ words of the same length which are not equivalent with respect to $\sim_{L,p}$ for the same p, a contradiction.

It follows that L is regular, *i.e* $L \in \mathbf{MSO}[\leq]$. $\qquad\square$

4.4 Morphic Regular Languages

In this subsection, we apply Theorem 4 to the case of morphic predicates, and obtain the following result: given an **MSO** formula with morphic predicates, it is decidable whether it defines a regular language.

The class of morphic predicates was first introduced by Thue in the context of combinatorics on words, giving rise to the HD0L systems. Formally, let A, B be two finite alphabets, $\sigma : A^* \to A^*$ a morphism, $a \in A$ a letter such that $\sigma(a) = a \cdot u$ for some $u \in A^+$ and $\varphi : A^* \to B^*$ a morphism. This defines the sequence of words $\varphi(a), \varphi(\sigma(a)), \varphi(\sigma^2(a)), \dots$, which converges to a finite or infinite word. An infinite word obtained in this way is said morphic.

We see morphic words as predicates, and denote by HD0L the class of morphic predicates. We say that a language is morphic regular if it is definable in **MSO**$[\leq, \mathrm{HD0L}]$.

Theorem 9. *The following problem is decidable: given L a morphic regular language, is L regular? Furthermore, if L is regular, then we can construct a finite automaton for L.*

The proof of this theorem goes in two steps:

- first, we reduce the regularity problem for a morphic regular language L to deciding the ultimate periodicity of $\mathbf{P}_L$,
- second, we show that $\mathbf{P}_L$ is morphic.

Hence we rely on the following result: given a morphic word, it is decidable whether it is ultimately periodic. The decidability of this problem was conjectured 30 years ago and proved recently and simultaneously by Durand and Mitrofanov [8, 11].

The first step is a direct application of Theorem 4. For the second step, observe that thanks to Lemma 3, we have $\mathbf{P}_L \in \mathbf{MSO}[\leq, \mathrm{HD0L}]$. We conclude with the following lemma, which follows from the characterization of morphic words as being those automatically presentable with the lexicographic ordering [19].

Lemma 4. HD0L *is closed under* **MSO***-interpretations, i.e. if $\mathbf{P}$ is an infinite word such that $\mathbf{P} \in \mathbf{MSO}[\leq, \mathrm{HD0L}]$, then $\mathbf{P} \in \mathrm{HD0L}$.*

Furthermore, all constructions in this proof are effective, and if $\mathbf{P}_L$ is ultimately periodic, then one can compute the threshold and the period, and derive from them a finite automaton for L.

We obtain a new proof of the decidability of **MSO**$[\leq, \mathrm{HD0L}]$ (already proved in [6] in a much simpler way).

Corollary 2. *The satisfiability problem for* **MSO**$[\leq, \mathrm{HD0L}]$ *is decidable.*

Indeed, from a language in **MSO**$[\leq, \mathrm{HD0L}]$, determine whether it is regular, and: if it is regular, then determine whether it is empty by looking at the (effectively constructed) finite automaton recognizing it, and if it is not regular, then it is non-empty (since the empty language is regular).

Acknowledgments

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