Richard Goldberg Solutions

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Preface

In this book, I would be providing solutions to selected exercises in Richard R. Goldberg's Real Analysis book.

(*As a matter of formatting, I will start each question on a new page.*)

Disclaimer: The solutions I provide are not official solutions, just my own solutions to the exercises.

This book is only available as an electronic copy, and not available in print.

If you happen to spot any mistakes in this book, have suggestions on how to improve this book, or have any other queries, you may reach me at my email.

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Chapter 1

Sets and Functions

1.1 Exercise 1.7 Solutions

Q1.

Ans.

(a) 7

(b) $\pi + 1$

(c) π

Chapter 2

Sequences of Real Numbers

2.1 Exercise 2.1 Solutions

Q5.

Proof.
Let S be a Sequence in the set A.

We note that it is given by the function; $f : \mathbb{N} \to A$.

Now, consider an (arbitrary) subsequence S' of S,which has the form $f \circ g$, where $h : \mathbb{N} \to \mathbb{N}$, satisfies $h(n) < h(n+1)$, for all n.

(*Claim:* $(g \circ h)(n) < (g \circ h)(n+1)$, $\forall n \in \mathbb{N}$)

We note that $h(n) < h(n+1)$, and $g(n) < g(n+1)$.

Now, let $b = h(n)$, and $c = h(n+1)$.

Also, $b < c$ since $h(n) < h(n+1)$.

Thus,

$$\begin{aligned} & b < b+1 \leqslant c \text{ (due to composite function } g(h(n))) \\ \implies & g(b) < g(b+1) \leqslant g(c) \\ \implies & g(b) < g(c) \\ \implies & g(h(n)) < g(h(n+1)). \blacksquare \end{aligned}$$

Q6.

Proof.
We note that a subsequence of S has the form; $S \circ N$, where $N : \mathbb{N} \to \mathbb{N}$, and that $N(k) < N(k+1)$, $\forall k \in \mathbb{N}$.

Also, $N(k) = n_k$. Thus, $n_k < n_{k+1}$.

(*Claim:* $n_k \geqslant k$)

Base Case: $k = 1$

Then, it is obvious, that $n_1 \geqslant 1$.

Inductive Case:

Assume that, $n_k \geqslant k$. (WTS: $n_{k+1} \geqslant k+1$)

Consider, n_{k+1}.

Since, $n_k < n_{k+1}$, then,

$$
\begin{aligned}
& k \leqslant n_k < n_{k+1} + 1 \\
\implies & k + 1 < n_{k+1} < n_{k+1} + 1 \\
\implies & k + 1 < n_{k+1}.
\end{aligned}
$$

Thus, by Induction, $n_k \geqslant k$ ($\forall k \in \mathbb{N}$), and the Theorem is proved.
■

2.2 Exercise 2.2 Solutions

Q1.

Proof.
Since $(M - s_n)_{n=1}^{\infty}$ converges to $(M - L) \in \mathbb{R}$, and $s_n \leqslant M \implies M - s_n \geqslant 0$ $(\forall n \in \mathbb{N})$, then, $M - L \geqslant 0$, as needed. $\blacksquare$

Q2.

Proof.
Let $\varepsilon > 0$, be given.

Suppose the contary, that $L > M$.

Also, by the hypothesis, $L \leqslant M + \varepsilon$.

Thus,

$$\begin{aligned} & M < L \leqslant M + \varepsilon \text{ (since } \varepsilon > 0) \\ \implies & M - \varepsilon < L \leqslant M + \varepsilon \\ \implies & |L - M| < \varepsilon \ (\forall \varepsilon > 0). \end{aligned}$$

We note that since $L > M$, by our assumption, $|L - M| > 0$.

But, in particular, pick an $\varepsilon_a < |L - M| < \varepsilon_a$.

Thus, $\varepsilon_a < \varepsilon_a$, which is a Contradiction.

Hence, the Theorem is true. ■

Q4(b).

Proof.

Let $\varepsilon > 0$ be given.

(WTS: $\lim_{n\to\infty} \frac{2n}{n+3} = 2$)

Pick,

$$\begin{aligned} N \in \mathbb{N} \ni N &\geqslant \frac{6}{\varepsilon} - 3 \\ &\iff N \geqslant \frac{6-3\varepsilon}{\varepsilon} \\ &\iff N\varepsilon \geqslant 6 - 3\varepsilon \\ &\iff N\varepsilon + 3\varepsilon \geqslant 6 \\ &\iff \varepsilon(N+3) \geqslant 6 \\ &\iff 6 \leqslant \varepsilon(N+3) \\ &\iff \frac{6}{N+3} \leqslant \varepsilon \\ &\iff \left|\frac{2N - 2(N+3)}{N+3}\right| \leqslant \varepsilon \\ &\iff \left|\frac{2N}{N+3} - 2\right| < \varepsilon. \end{aligned}$$

Then,

$$\begin{aligned} &\left|\frac{1}{n}\right| \leqslant \left|\frac{1}{N}\right| \\ \implies &\left|\frac{1}{n+3}\right| \leqslant \left|\frac{1}{N+3}\right| \\ \implies &\left|\frac{2n}{n+3}\right| \leqslant \left|\frac{2N}{N+3}\right| \\ \implies &\left|\frac{2n}{n+3} - 2\right| \leqslant \left|\frac{2N}{N+3} - 2\right| < \varepsilon. \end{aligned}$$

Hence, we have found an N, where, $\left|\frac{2n}{n+3} - 2\right| < \varepsilon \ (n \geqslant N)$.

This proves, $\lim_{n\to\infty} \frac{2n}{n+3} = 2$, as needed. ■

2.3 Exercise 2.3 Solutions

Q1.

Proof.

We note that,

$$
\begin{aligned}
&|a| = |(a-b)+b| \leqslant |a-b| + |b| \\
\implies &|a| - |b| \leqslant |a-b|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|b| = |(b-a)+a| \leqslant |b-a| + |a| \\
\implies &|b| - |a| \leqslant |b-a| = |a-b|.
\end{aligned}
$$

Since, $||a| - |b||$ is either; $|a| - |b|$ or $|b| - |a|$, then, $||a| - |b|| \leqslant |a-b|$. ■

Let $\varepsilon > 0$, be given.

Since, $(s_n)_{n=1}^{\infty}$ converges to L, $\exists N \in \mathbb{N} \ni |s_n - L| < \varepsilon \ (n \geqslant N)$.

Hence, $\lim\limits_{n\to\infty} |s_n| = |L|$. ■

Q5.

Proof.
Let $\varepsilon > 0$, be given.

Since, $\lim_{m\to\infty} s_{2m} = L$, $\exists N_1 \in \mathbb{N} \ni |s_{2m} - L| < \varepsilon$ $(2m \geqslant N_1)$.

Similarly, since, $\lim_{m\to\infty} s_{2m-1} = L$, $\exists N_2 \in \mathbb{N} \ni |s_{2m-1} - L| < \varepsilon$ $(2m-1 \geqslant N_2)$.

Set $N = \max\{N_1, N_2\}$, then, in both cases, where m is odd or even, $\exists N \in \mathbb{N} \ni |s_m - L| < \varepsilon$ $(m \geqslant N)$.

Hence, $\lim_{m\to\infty} s_m = L$, as needed. $\blacksquare$

2.4 Exercise 2.4 Solutions

2.5 Exercise 2.5 Solutions

2.6 Exercise 2.6 Solutions

2.7 Exercise 2.7 Solutions