Open Sets

Analysis

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Definition. Let $\langle M, \rho \rangle$, be a metric space. A set $G \subseteq M$ is open (in M), if for every $x \in G$ there exists $r > 0$ such that $B[x, r] \subseteq G$.

Examples.

1. In any metric space; $\langle M, \rho \rangle$, an open ball; $B[a, r]$, is an open set.
   Proof. Let $x \in B[a, r]$. Then, $\rho(x, a) < r$, and set
   $s = r - \rho(x, a) > 0$.
   (WTS: $B[x, s] \subseteq B[a, r]$) Now, take any $y \in B[x, s]$. Then, by the triangle inequality;

   $$\rho(y, a) \leq \rho(y, x) + \rho(x, a) < s + (r - s) = r$$

   Thus, $y \in B[a, r]$, and we are done.
2. In the discrete metric space; $\langle M, d \rangle$, any subset $G \subseteq M$ is open.
   Proof. Take any $x \in G$. (WTS: $B[x, 1] \subseteq G$) Now, for any $y \in B[x, 1]$, $\rho(x, y) < 1$. Thus, $y = x \in G$, and we are done.

Theorem. In any metric space; $\langle M, \rho \rangle$, the empty set; $\varnothing$, and M, are both open (in M).

Proof. There are no $x \in \varnothing$ in the first place. Thus, by vacuity, every $x \in \varnothing$ satisfies the open set requirement, and $\varnothing$ is open.

Now, for M, we can always find an open ball; $B[x, r]$, namely one that only contains x, by choosing a (small enough) $r > 0$. But, by definition of open balls, all balls must be contained in M. Hence, M is open.

Theorem. Let $\mathcal{F}$ be a nonempty family of open sets in M, then $\bigcup\limits_{G \in \mathcal{F}} G$ is also open in M.

Proof. Take any $x \in H = \bigcup\limits_{G \in \mathcal{F}}$. Then, $x \in G$, for some $G \in \mathcal{F}$. Since, G is open, then there exists an $r > 0$ where $B[x, r] \subseteq G \subseteq H$. Thus, H is open.

Theorem. The intersection of finitely many open sets in M; $\bigcap\limits_{i=1}^{m} G_i$ is also open in M.

Proof. Take any $x \in H = \bigcap\limits_{i=1}^{m} G_i$. Then, $x \in G_i$, for all $i \in [1, m]$. Since, each G_i is open, there exists $r_i > 0$ where $B[x, r_i] \subseteq G_i$. Let $r = \min\{r_1, r_2, \ldots, r_m\}$. Then, $B[x, r] \subseteq B[x, r_i] \subseteq G_i$ (for all i) $= H$. Hence, H is open.

Remark. The intersection of infinitely many open sets however might not always be open. Consider $\mathbb{R}$ with the Euclidean metric and $G_n = B[0, \frac{1}{n}] = \left(\frac{-1}{n}, \frac{1}{n}\right)$, which is open for all $n \in \mathbb{N}$. However, $\bigcap\limits_{n=1}^{\infty} G_n = \{0\}$, is clearly not open (as we can never find any open ball of positive radii around 0).

Theorem. Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces, and $f : M_1 \to M_2$ be a function. Then f is continuous (on M_1) if and only if $f^{-1}(G)$ is open in M_1 whenever G is open in M_2 (for any subset $G \subseteq M_1$).

Proof. Suppose that f is continuous (on M_1) and G is open (in M_2). Then, take any $x \in f^{-1}(G) \implies y = f(x) \in G$. Since, G is open, then there exists some $r > 0$, where $B[y, r] \subseteq G$. But, since f is continuous, then (by the open ball characterisation of continuity) $f^{-1}(B[y, r]) = f^{-1}(B[f(x), r]) \supseteq B[x, s]$, for some $s > 0$. Thus, $B[x, s] \subseteq f^{-1}(G)$, and $f^{-1}(G)$ is open, as needed.

Conversely, suppose that $f^{-1}(G)$ is open in M_1 if G is open in M_2. (WTS: f is continuous $\iff \forall x \in M_1$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $f^{-1}(B[f(x), \varepsilon]) \supseteq B[x, \delta]$.) Now, let $x \in M_1$ and $\varepsilon > 0$ be given. Then, $f(x) \in f(M_1) \subseteq M_2$, and consider the open ball; $B[f(x), \varepsilon] \subseteq M_2$, which is open in M_2. Then, by our supposition, $f^{-1}(B[f(x), \varepsilon]) \subseteq M_1$ is also open in M_1. But, $x \in f^{-1}(B[f(x), \varepsilon])$, and since $B[f(x), \varepsilon]$ is open, then there exists a $\delta > 0$ such that $B[x, \delta] \subseteq f^{-1}(B[f(x), \varepsilon])$. But, this is exactly what we wanted for f to be continuous, and we are done.

Remark. Both directions of the previous proof uses the open ball characterisation of continuity. The proof however, gives us yet another characterisation of continuity; this time more generally in terms of open sets!