

Richard Goldberg Solutions

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Preface

In this book, I would be providing solutions to selected exercises in Richard R. Goldberg's Real Analysis book.

(As a matter of formatting, I will start each question on a new page.)

Disclaimer: The solutions I provide are not official solutions, just my own solutions to the exercises.

This book is only available as an electronic copy, and not available in print.

If you happen to spot any mistakes in this book, have suggestions on how to improve this book, or have any other queries, you may reach me at my email.

~ Nathanael Seen

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Chapter 1

Sets and Functions

1.1 Exercise 1.7 Solutions

Q1.

Ans.

(a) 7

(b) $\pi + 1$

(c) π

Chapter 2

Sequences of Real Numbers

2.1 Exercise 2.1 Solutions

Q5.

Proof.

Let S be a Sequence in the set A .

We note that it is given by the function; $f : \mathbb{N} \rightarrow A$.

Now, consider an (arbitrary) subsequence S' of S , which has the form $f \circ g$, where $h : \mathbb{N} \rightarrow \mathbb{N}$, satisfies $h(n) < h(n+1)$, for all n .

(*Claim:* $(g \circ h)(n) < (g \circ h)(n+1)$, $\forall n \in \mathbb{N}$)

We note that $h(n) < h(n+1)$, and $g(n) < g(n+1)$.

Now, let $b = h(n)$, and $c = h(n+1)$.

Also, $b < c$ since $h(n) < h(n+1)$.

Thus,

$$\begin{aligned} & b < b+1 \leq c \text{ (due to composite function } g(h(n))) \\ \implies & g(b) < g(b+1) \leq g(c) \\ \implies & g(b) < g(c) \\ \implies & g(h(n)) < g(h(n+1)). \blacksquare \end{aligned}$$

Q6.

Proof.

We note that a subsequence of S has the form; $S \circ N$, where $N : \mathbb{N} \rightarrow \mathbb{N}$, and that $N(k) < N(k+1)$, $\forall k \in \mathbb{N}$.

Also, $N(k) = n_k$. Thus, $n_k < n_{k+1}$.

(*Claim: $n_k \geq k$*)

Base Case: $k = 1$

Then, it is obvious, that $n_1 \geq 1$.

Inductive Case:

Assume that, $n_k \geq k$. (WTS: $n_{k+1} \geq k+1$)

Consider, n_{k+1} .

Since, $n_k < n_{k+1}$, then,

$$\begin{aligned} k &\leq n_k < n_{k+1} + 1 \\ \implies k + 1 &< n_{k+1} < n_{k+1} + 1 \\ \implies k + 1 &< n_{k+1}. \end{aligned}$$

Thus, by Induction, $n_k \geq k$ ($\forall k \in \mathbb{N}$), and the Theorem is proved.

■

2.2 Exercise 2.2 Solutions

Q1.

Proof.

Since $(M - s_n)_{n=1}^{\infty}$ converges to $(M - L) \in \mathbb{R}$, and $s_n \leq M \implies M - s_n \geq 0$ ($\forall n \in \mathbb{N}$), then, $M - L \geq 0$, as needed. ■

Q2.

Proof.

Let $\varepsilon > 0$, be given.

Suppose the contrary, that $L > M$.

Also, by the hypothesis, $L \leq M + \varepsilon$.

Then, there are 2 Cases; either $L < M + \varepsilon$ or $L = M + \varepsilon$.

Case A: $L = M + \varepsilon$

Then, $L - M = \varepsilon$, for all $\varepsilon > 0$.

But, this is a Contradiction, because $L - M = \varepsilon_a$, for some $\varepsilon_a \in \mathbb{R}$ only.

Case B: $L < M + \varepsilon$

Then,

$$\begin{aligned} M &< L \leq M + \varepsilon \\ \implies M - \varepsilon &< L \leq M + \varepsilon \\ \implies |L - M| &< \varepsilon \ (\forall \varepsilon > 0). \end{aligned}$$

We note that since $L > M$, by our assumption, $|L - M| > 0$.

But, in particular, pick an $\varepsilon_a < |L - M| < \varepsilon_a$.

Thus, $\varepsilon_a < \varepsilon_a$, which is a Contradiction.

Since, in all Cases, we reach a Contradiction, the Theorem is true. ■

Q4(b).

Proof.

Let $\varepsilon > 0$ be given.

(WTS: $\lim_{n \rightarrow \infty} \frac{2n}{n+3} = 2$)

Pick,

$$\begin{aligned} N \in \mathbb{N} \ni N &> \frac{6}{\varepsilon} - 3 \\ \iff N &> \frac{6-3\varepsilon}{\varepsilon} \\ \iff N\varepsilon &> 6-3\varepsilon \\ \iff N\varepsilon + 3\varepsilon &> 6 \\ \iff \varepsilon(N+3) &> 6 \\ \iff 6 < \varepsilon(N+3) \\ \iff \frac{6}{N+3} &< \varepsilon \\ \iff \left| \frac{2N-2(N+3)}{N+3} \right| &< \varepsilon \\ \iff \left| \frac{2N}{N+3} - 2 \right| &< \varepsilon. \end{aligned}$$

Then,

$$\begin{aligned} &\left| \frac{1}{n} \right| \leq \left| \frac{1}{N} \right| \\ \implies \left| \frac{1}{n+3} \right| &\leq \left| \frac{1}{N+3} \right| \\ \implies \left| \frac{2n}{n+3} \right| &\leq \left| \frac{2N}{N+3} \right| \\ \implies \left| \frac{2n}{n+3} - 2 \right| &\leq \left| \frac{2N}{N+3} - 2 \right| < \varepsilon. \end{aligned}$$

Hence, we have found an N , where, $\left| \frac{2n}{n+3} - 2 \right| < \varepsilon$ ($n \geq N$).

This proves, $\lim_{n \rightarrow \infty} \frac{2n}{n+3} = 2$, as needed. ■

2.3 Exercise 2.3 Solutions

Q1.

Proof.

We note that,

$$\begin{aligned} |a| &= |(a - b) + b| \leq |a - b| + |b| \\ \implies |a| - |b| &\leq |a - b|. \end{aligned}$$

Similarly,

$$\begin{aligned} |b| &= |(b - a) + a| \leq |b - a| + |a| \\ \implies |b| - |a| &\leq |b - a| = |a - b|. \end{aligned}$$

Since, $||a| - |b||$ is either; $|a| - |b|$ or $|b| - |a|$, then, $||a| - |b|| \leq |a - b|$. ■

Let $\varepsilon > 0$, be given.

Since, $(s_n)_{n=1}^{\infty}$ converges to L , $\exists N \in \mathbb{N} \ni |s_n - L| < \varepsilon$ ($n \geq N$).

Hence, $\lim_{n \rightarrow \infty} |s_n| = |L|$. ■

Q5.

Proof.

Let $\varepsilon > 0$, be given.

Since, $\lim_{m \rightarrow \infty} s_{2m} = L$, $\exists N_1 \in \mathbb{N} \ni |s_{2m} - L| < \varepsilon$ ($2m \geq N_1$).

Similarly, since, $\lim_{m \rightarrow \infty} s_{2m-1} = L$, $\exists N_2 \in \mathbb{N} \ni |s_{2m-1} - L| < \varepsilon$ ($2m - 1 \geq N_2$).

Set $N = \max\{N_1, N_2\}$, then, in both cases, where m is odd or even, $\exists N \in \mathbb{N} \ni |s_m - L| < \varepsilon$ ($m \geq N$).

Hence, $\lim_{m \rightarrow \infty} s_m = L$, as needed. ■

2.4 Exercise 2.4 Solutions

Q3.

Proof.

First, note that,

$$\begin{aligned} & \sqrt{n+1} - \sqrt{n} \\ = & \frac{(\sqrt{n+1})(\sqrt{n+1} + \sqrt{n}) - \sqrt{n}(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\ = & \frac{(n+1) + \sqrt{n(n+1)} - \sqrt{n(n+1)} - n}{\sqrt{n+1} + \sqrt{n}} \\ = & \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ \leq & \frac{1}{2\sqrt{n}}. \end{aligned}$$

Now, let, $\varepsilon > 0$, be given.

Pick,

$$\begin{aligned} N \in \mathbb{N} \ni N &> \left(\frac{1}{2\varepsilon}\right)^2 \\ \implies \left(\frac{1}{2\varepsilon}\right)^2 &< N \\ \iff \frac{1}{2\varepsilon} &< \sqrt{N} \\ \implies 1 &< \varepsilon(2\sqrt{N}) \\ \iff \frac{1}{2\sqrt{N}} &< \varepsilon \\ \implies \left| \frac{1}{2\sqrt{N}} - 0 \right| &< \varepsilon. \end{aligned}$$

Then, for $n \geq N$,

$$\begin{aligned} & \frac{1}{n} \leq \frac{1}{N} \\ \Leftrightarrow & \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} \\ \Leftrightarrow & \left| \frac{1}{2\sqrt{n}} - 0 \right| \leq \left| \frac{1}{2\sqrt{N}} - 0 \right| < \varepsilon \\ \Leftrightarrow & \left| \frac{1}{2\sqrt{n}} - 0 \right| < \varepsilon \\ \Leftrightarrow & \left| \left(\sqrt{n+1} - \sqrt{n} \right) - 0 \right| < \varepsilon. \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = 0$. ■

Q5.

Proof.

Let $\varepsilon > 0$, be given.

Since, $(s_n)_{n=1}^{\infty}$ converges to 0, $\exists N \in \mathbb{N} \ni |s_n - 0| = |s_n| < \varepsilon$ ($n \geq N$).

But, $|s_n| = |1| \cdot |s_n| = |(-1)^n| \cdot |s_n| = |(-1)^n s_n| < \varepsilon$ ($n \geq N$).

Thus, $\lim_{n \rightarrow \infty} (-1)^n s_n = 0$. ■

Q6.

Proof.

Let $M > 0$, be given.

We note that, $(s_n)_{n=1}^{\infty}$ converges to some $L \neq 0$.

However, assume the contrary, that instead, $(s_n)_{n=1}^{\infty}$ does not oscillate.

Then, either, it diverges to infinity, or minus infinity, and there are 2 Cases.

Case A: $((-1)^n s_n)_{n=1}^{\infty}$ diverges to infinity

Then, $\exists N \in \mathbb{N} \ni (-1)^n s_n \geq M \ (n \geq N)$.

Here again, there are 2 sub-Cases; $(-1)^n = -1$ or $(-1)^n = 1$.

Case A1: $(-1)^n = -1$

Then, $-s_n \geq M \implies s_n \leq -M \ (n \geq N)$.

Hence, $(s_n)_{n=1}^{\infty}$ diverges to minus infinity.

But, from our hypothesis, it converges to some $L \neq 0$.
Contradiction.

Case A2: $(-1)^n = 1$

Then, $s_n \geq M \ (n \geq N)$.

Thus, $(s_n)_{n=1}^{\infty}$ diverges to infinity.

But, from our hypothesis, it converges to some $L \neq 0$.
Contradiction.

Case B: $((-1)^n s_n)_{n=1}^{\infty}$ diverges to minus infinity

Thus, $\exists N \in \mathbb{N} \ni (-1)^n s_n \leq -M \ (n \geq N)$.

Similarly, as with Case A, there are 2 sub-Cases; $(-1)^n = -1$ or $(-1)^n = 1$.

Case B1: $(-1)^n = -1$

Then, $-s_n \leq -M \implies s_n \geq M \ (n \geq N)$.

Hence, $(s_n)_{n=1}^{\infty}$ diverges to infinity.

But again, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Case B2: $(-1)^n = 1$

Then, $s_n \leq -M$, and $(s_n)_{n=1}^{\infty}$ diverges to minus infinity.

But again, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Since, in all Cases, we reach a Contradiction, the theorem is true, and hence, $(s_n)_{n=1}^{\infty}$ indeed oscillates. ■

2.5 Exercise 2.5 Solutions

Q3.

Proof.

Let $\varepsilon > 0$, be given.

Also, assume the contrary, that $(s_n)_{n=1}^{\infty}$ is bounded.

Then, $\exists M \in \mathbb{R} \ni |s_{n_1}| \leq M \ (\forall n_1 \in \mathbb{N})$.

(In particular, let's claim that $\lim_{n \rightarrow \infty} \left(\frac{s_n}{n}\right) = 0$.)

Pick,

$$\begin{aligned} N \in \mathbb{N} \ni N &> \frac{M}{\varepsilon} \\ \implies N\varepsilon &> M \\ \iff N\varepsilon &> s_{n_1} \\ \iff s_{n_1} &< N\varepsilon \\ \iff \frac{s_{n_1}}{N} &< \varepsilon \\ \iff \left| \frac{s_{n_1}}{N} \right| &< \varepsilon \ (\forall n_1 \in \mathbb{N}). \end{aligned}$$

Thus, for $n \geq N$,

$$\begin{aligned} \left| \frac{1}{n} \right| &\leq \left| \frac{1}{N} \right| \\ \implies \left| \frac{s_{n_1}}{n} \right| &\leq \left| \frac{s_{n_1}}{N} \right| < \varepsilon \ (\forall n_1 \in \mathbb{N}). \end{aligned}$$

Hence, we have found an N , where, $\left| \frac{s_{n_1}}{n} - 0 \right| < \varepsilon$ ($n \geq N$, and $\forall n_1 \in \mathbb{N}$).

Thus, $\lim_{n \rightarrow \infty} \frac{s_n}{n} = 0$, as needed. ■