Continuity

Analysis

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Definition. Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces, and $f : M_1 \to M_2$ be a function. Then f is continuous at a point $a \in M_1$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that;

$$\rho_2(f(x), f(a)) < \varepsilon \text{ for } \underline{\text{all}}\ x \in M_1 \text{ where } \rho_1(x, a) < \delta$$

Remark. Intuitively, a function is continuous (at a point) if it can be drawn in one stroke (pass that point) without you having to lifting your pen from the paper. However, as seen above, much more is needed to formalise this notion of continuity, to make it more precise and rigorous. Because continuity is such an central notion in analysis, we will be seeing many different characterisation (or reformulations) of it.

Terminology. Notice, how continuity is a point property. Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces, and $f : M_1 \to M_2$ be a function. Then we say that f is continuous on M_1 if it is continuous at every point $a \in M_1$.

Definition. Let $\langle M, \rho \rangle$ be a metric space, $a \in M$, and $r > 0$. Then, an open ball; $B[a, r]$ (ball centered at a with radius r), is the set;

$$B[a, r] = \{x \in M \mid \rho(x, a) < r\}$$

Theorem. Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces, and $f : M_1 \to M_2$ be a function. Then f is continuous at a point $a \in M_1$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B[a, \delta] \implies f(x) \in B[f(a), \varepsilon]$.

Proof. Suppose that f is continuous at point $a \in M_1$. Then, by definition of continuity, for any $\varepsilon > 0$, there exists $\delta > 0$ such that;

$$\begin{aligned} & \rho_2(f(x), f(a)) < \varepsilon \text{ for } \underline{\text{all}}\ x \in M_1 \text{ where } \rho_1(x, a) < \delta \\ \iff & f(x) \in B[a, \varepsilon] \text{ for } \underline{\text{all}}\ x \in B[a, \delta] \text{ (by definition of open balls)} \\ \iff & x \in B[a, \delta] \implies f(x) \in B[f(a), \varepsilon] \text{ (easy simplication)} \end{aligned}$$

Remark. The above theorem gives us another characterisation of continuity; in terms of open balls.

Theorem. Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces, and $f : M_1 \to M_2$ be a function. Then f is continuous at a point $a \in M_1$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $B[a, \delta] \subseteq f^{-1}(B[f(a), \varepsilon])$.

Proof. Suppose that f is continuous at point $a \in M_1$. Then, by the previous (open ball) characterisation of continuity, for any $\varepsilon > 0$, there exists $\delta > 0$ such that;

$$\begin{aligned} x \in B[a, \delta] \Longrightarrow & f(x) \in B[f(a), \varepsilon] \\ & \Longleftrightarrow f^{-1}(f(x)) \in f^{-1}(B[f(a), \varepsilon]) \\ & \Longleftrightarrow x \in f^{-1}(B[f(a), \varepsilon]) \end{aligned}$$

Thus, $B[a, \delta] \subseteq f^{-1}(B[f(a), \varepsilon])$, and we are done.

Remark. The above theorem gives us yet another characterisation of continuity; this time in terms of the image/inverse image of open balls. This is perhaps the most prevalent characterisation of continuity.

Theorem. Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$ be metric spaces, $f : M_1 \to M_2$ be a function, and $a \in M_1$ be a cluster point. Then f is continuous at point a <u>if and only if</u> $\lim\limits_{x \to a} f(x) = f(a)$.

Proof. Since $\lim\limits_{x \to a} f(x) = f(a)$, then by definition of limits, for any $\varepsilon > 0$, there exists $\delta > 0$ such that;

$$\rho_2(f(x), f(a)) < \varepsilon \text{ for } \underline{\text{all}}\ x \in M_1 \text{ where } 0 < \rho_1(x, a) < \delta$$

As $\rho_1(x, a) > 0$, this implies that $x \neq a$. However, if $x = a$, then $\rho_2(f(x), f(a)) = \rho_2(f(a), f(a)) = 0 < \varepsilon$, for any (arbitrary) ε. Thus, we can remove the $\rho_1(x, a) > 0$ constraint. But, this is exactly the first characterisation of continuity, and we are done.

Remark. Continuity can also be characterised in terms of limits.

Theorem. Let $\langle M_1, \rho_1\rangle$, $\langle M_2, \rho_2\rangle$ be metric spaces, $f : M_1 \to M_2$ be a function, and $L \in M_2$. Then f is continuous at a point $a \in M_1$ if and only if for any sequence $(x_n)_{n=1}^{\infty}$ in M_1 converging to a, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(a)$ in M_2.

Proof. Since f is continuous at point $a \in M_1$, then by the previous (limit) characterisation of continuity, $\lim_{x \to a} f(x) = f(a)$, and thus for any $\varepsilon > 0$, there exists $\delta > 0$ such that;

$$\rho_2(f(x), f(a)) < \varepsilon \text{ for } \underline{\text{all}}\ x \in M_1 \text{ where } 0 < \rho_1(x, a) < \delta$$

But, this is equivalent to, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that;

$$\rho_2(f(x_n), f(a)) < \varepsilon \text{ for } \underline{\text{all}}\ n \geq N \text{ where } (x_n)_{n=1}^{\infty} \text{ is any sequence in } M_1$$

Since $(x_n)_{n=1}^{\infty}$ was arbitrary, we are done.

Remark. Finally, continuity can also be characterised in terms of sequences (sequentially).

Theorem. Let $\langle M_1, \rho_1 \rangle$, $\langle M_2, \rho_2 \rangle$, $\langle M_3, \rho_3 \rangle$ be metric spaces, and $f : M_1 \to M_2$, $g : M_2 \to M_3$ be functions. If f is continuous at a point $a \in M_1$ and g is continuous at a point $f(a) \in M_2$, then $g \circ f : M_1 \to M_3$ is continuous at point a.

Proof. Let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ be sequences in M_1 and M_2, respectively.

Since f is continuous at point $a \in M_1$, then by the previous (sequential) characterisation of continuity, if $(x_n)_{n=1}^{\infty}$ converges to point $a \in M_1$, $(f(x_n))_{n=1}^{\infty}$ converges to $f(a) \in M_2$.

Since g is continuous at point $f(a) \in M_2$, then again by the sequential characterisation of continuity, if $(y_n)_{n=1}^{\infty}$ converges to point $f(a) \in M_2$ then $(g(y_n))_{n=1}^{\infty}$ converges to $g(f(a)) \in M_3$.

But, set $(y_n)_{n=1}^{\infty} = (f(x_n))_{n=1}^{\infty}$, and thus if $(x_n)_{n=1}^{\infty}$ converges to $a \in M_1$, then this implies that $g(f(x_n))_{n=1}^{\infty}$ converges to $g(f(a)) \in M_3$. Hence, $g \circ f : M_1 \to M_3$ is continuous at a, and we are done.

Remark. Continuity is preserved under function composition.