Least Upper Bound Axiom

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1 Context

In this paper, we present the Least Upper Bound (L.U.B) Axiom which is a fundmaental property of the reals heavily exploited in real analysis.

This property is what distinguishes the set of rationals; $\mathbb{Q}$, from the set of reals; $\mathbb{R}$ (which contains irrationals too).

Intuitively, it posits that there are no 'gaps'/'holes' in the number system.

2 L.U.B Property

Definition 2.1. Let $A \subseteq \mathbb{R}$. Then an **upper bound** (u.b), $b \in \mathbb{R}$ is a number where $b \geq a$ for all $a \in A$.

A similar definition can also be made for **lower bound** (l.b).

Definition 2.2. Let $A \subseteq \mathbb{R}$. If there exists an u.b for A, then A is said to be **bounded above**.

A similar definition can also be made for subsets of $\mathbb{R}$, which are **bounded below**.

Example.

- The clopen (closed-open) interval; $(2, 3] \subseteq \mathbb{R}$, is both bounded below and above. In particular, 2 is a l.b, while 3 is an u.b.
- The interval; $(-\infty, 8) \subseteq \mathbb{R}$, is bounded above (8 is an upper bound) but is not bounded below.
- The empty set; $\varnothing$, is neither bounded above nor below.

- The open interval; $(-\infty, \infty) \subseteq \mathbb{R}$, is also neither bounded above nor below.

Definition 2.3. Let $A \subseteq \mathbb{R}$, be bounded above, with B being the set of upper bounds. If there exists a $c \in \mathbb{R}$ such that $c \leq b$ for all $b \in B$, then c is the **least upper bound** (l.u.b) of A, and we write $c = \sup(A)$.

A similar definition can also be made for **greatest lower bound** (g.l.b) of A.

Example.

- For the clopen interval; $(2, 3] \subseteq \mathbb{R}$, 2 is the g.l.b while 3 is the l.u.b.
- For the interval; $(-\infty, 8) \subseteq \mathbb{R}$, 8 is the l.u.b, but it does not have a g.l.b as it is not bounded below.
- Consider the sequence; $\left(2^{\frac{1}{n}}\right)_{n=1}^{\infty}$, it has g.l.b of 1 and l.u.b of 2.

3 Exercises

1. Let $\varnothing \neq A, B \subseteq \mathbb{R}$, prove that $\sup(A) + \sup(B) = \sup(A + B)$, where $A + B = \{a + b \mid a \in A, b \in B\}$.
2. Prove that $\sqrt{2}$ is irrational.
(This exercise shows that $\sqrt{2} \notin \mathbb{Q}$, and thus giving a (counter-example) showing that the set of rationals actually has 'holes', mainly leaving out the **irrationals**. Hence, not all proper non-empty subsets of $\mathbb{Q}$, might necessarily have an l.u.b.)