Richard Goldberg Solutions

Nathanael Seen

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Preface

In this book, I would be providing solutions to selected exercises in Richard R. Goldberg's Real Analysis book.

(*As a matter of formatting, I will start each question on a new page.*)

Disclaimer: The solutions I provide are not official solutions, just my own solutions to the exercises.

This book is only available as an electronic copy, and not available in print.

If you happen to spot any mistakes in this book, have suggestions on how to improve this book, or have any other queries, you may reach me at my email.

$\sim$ Nathanael Seen

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Chapter 1

Sets and Functions

1.1 Exercise 1.7 Solutions

Q1.

Ans.

(a) 7

(b) $\pi + 1$

(c) π

Q2.

Ans.

(a) 8

(b) $\pi + 1$

Q5.

Ans.
A only consists one element; $x \in \mathbb{R}$.

Chapter 2

Sequences of Real Numbers

2.1 Exercise 2.1 Solutions

Q5.

Proof.
Let S be a sequence in the set A.

We note that it is given by the function; $f : \mathbb{N} \to A$.

Now, consider an (arbitrary) subsequence S' of S,which has the form $f \circ g$, where $h : \mathbb{N} \to \mathbb{N}$, satisfies $h(n) < h(n+1)$, for all n.

(*Claim:* $(g \circ h)(n) < (g \circ h)(n+1)$, $\forall n \in \mathbb{N}$)

We note that $h(n) < h(n+1)$, and $g(n) < g(n+1)$.

Now, let $b = h(n)$, and $c = h(n+1)$.

Also, $b < c$ since $h(n) < h(n+1)$.

Thus,

$$\begin{aligned} & b < b+1 \leqslant c \text{ (due to composite function } g(h(n))) \\ \implies & g(b) < g(b+1) \leqslant g(c) \\ \implies & g(b) < g(c) \\ \implies & g(h(n)) < g(h(n+1)). \blacksquare \end{aligned}$$

Q6.

Proof.
We note that a subsequence of S has the form; $S \circ N$, where $N : \mathbb{N} \to \mathbb{N}$, and that $N(k) < N(k+1)$, $\forall k \in \mathbb{N}$.

Also, $N(k) = n_k$. Thus, $n_k < n_{k+1}$.

(*Claim:* $n_k \geqslant k$)

Base Case: $k = 1$

Then, it is obvious, that $n_1 \geqslant 1$.

Inductive Case:

Assume that, $n_k \geqslant k$. (WTS: $n_{k+1} \geqslant k+1$)

Consider, n_{k+1}.

Since, $n_k < n_{k+1}$, then,

$$
\begin{aligned}
& k \leqslant n_k < n_{k+1} + 1 \\
\implies & k + 1 < n_{k+1} < n_{k+1} + 1 \\
\implies & k + 1 < n_{k+1}.
\end{aligned}
$$

Thus, by induction, $n_k \geqslant k$ ($\forall k \in \mathbb{N}$), and the theorem is proved.
$\blacksquare$

2.2 Exercise 2.2 Solutions

Q1.

Proof.
Since $(M - s_n)_{n=1}^{\infty}$ converges to $(M - L) \in \mathbb{R}$, and $s_n \leqslant M \implies M - s_n \geqslant 0$ $(\forall n \in \mathbb{N})$, then, $M - L \geqslant 0$, as needed. ■

Q2.

Proof.
Let $\varepsilon > 0$, be given.

Suppose the contary, that $L > M$.

Also, by the hypothesis, $L \leqslant M + \varepsilon$.

Then, there are 2 cases; either $L < M + \varepsilon$ or $L = M + \varepsilon$.

Case A: $L = M + \varepsilon$

Then, $L - M = \varepsilon$, for all $\varepsilon > 0$.

But, this is a contradiction, because $L - M = \varepsilon_a$, for some $\varepsilon_a \in \mathbb{R}$ only.

Case B: $L < M + \varepsilon$

Then,

$$\begin{aligned} & M < L \leqslant M + \varepsilon \\ \Longrightarrow\ & M - \varepsilon < L \leqslant M + \varepsilon \\ \Longrightarrow\ & |L - M| < \varepsilon \ (\forall \varepsilon > 0). \end{aligned}$$

We note that since $L > M$, by our assumption, $|L - M| > 0$.

But, in particular, pick an $\varepsilon_a < |L - M| < \varepsilon_a$.

Thus, $\varepsilon_a < \varepsilon_a$, which is a contradiction.

Since, in all cases, we reach a contradiction, the theorem is true. ■

Q4(b).

Proof.
Let $\varepsilon > 0$ be given.

(WTS: $\lim_{n\to\infty} \dfrac{2n}{n+3} = 2$)

Pick,

$$\begin{aligned}
N \in \mathbb{N} \ni N &> \frac{6}{\varepsilon} - 3 \\
&\iff N > \frac{6-3\varepsilon}{\varepsilon} \\
&\iff N\varepsilon > 6 - 3\varepsilon \\
&\iff N\varepsilon + 3\varepsilon > 6 \\
&\iff \varepsilon(N+3) > 6 \\
&\iff 6 < \varepsilon(N+3) \\
&\iff \frac{6}{N+3} < \varepsilon \\
&\iff \left|\frac{2N - 2(N+3)}{N+3}\right| < \varepsilon \\
&\iff \left|\frac{2N}{N+3} - 2\right| < \varepsilon.
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\frac{1}{n}\right| \leqslant \left|\frac{1}{N}\right| \\
\implies &\left|\frac{1}{n+3}\right| \leqslant \left|\frac{1}{N+3}\right| \\
\implies &\left|\frac{2n}{n+3}\right| \leqslant \left|\frac{2N}{N+3}\right| \\
\implies &\left|\frac{2n}{n+3} - 2\right| \leqslant \left|\frac{2N}{N+3} - 2\right| < \varepsilon.
\end{aligned}$$

Hence, we have found an N, where, $\left|\dfrac{2n}{n+3} - 2\right| < \varepsilon$ $(n \geqslant N)$.

This proves, $\lim_{n\to\infty} \dfrac{2n}{n+3} = 2$, as needed. ■

2.3 Exercise 2.3 Solutions

Q1.

Proof.
We note that,

$$\begin{aligned} &|a| = |(a-b)+b| \leqslant |a-b| + |b| \\ \implies &|a| - |b| \leqslant |a-b|. \end{aligned}$$

Similarly,

$$\begin{aligned} &|b| = |(b-a)+a| \leqslant |b-a| + |a| \\ \implies &|b| - |a| \leqslant |b-a| = |a-b|. \end{aligned}$$

Since, $||a| - |b||$ is either; $|a| - |b|$ or $|b| - |a|$, then, $||a| - |b|| \leqslant |a-b|$. ■

Let $\varepsilon > 0$, be given.

Since, $(s_n)_{n=1}^{\infty}$ converges to L, $\exists N \in \mathbb{N} \ni |s_n - L| < \varepsilon \ (n \geqslant N)$.

Hence, $\lim\limits_{n\to\infty} |s_n| = |L|$. ■

Q5.

Proof.
Let $\varepsilon > 0$, be given.

Since, $\lim\limits_{m\to\infty} s_{2m} = L$, $\exists N_1 \in \mathbb{N} \ni |s_{2m} - L| < \varepsilon \ (2m \geqslant N_1)$.

Similarly, since, $\lim\limits_{m\to\infty} s_{2m-1} = L$, $\exists N_2 \in \mathbb{N} \ni |s_{2m-1} - L| < \varepsilon \ (2m-1 \geqslant N_2)$.

Set $N = \max\{N_1, N_2\}$, then, in both cases, where m is odd or even, $\exists N \in \mathbb{N} \ni |s_m - L| < \varepsilon \ (m \geqslant N)$.

Hence, $\lim\limits_{m\to\infty} s_m = L$, as needed. ■

2.4 Exercise 2.4 Solutions

Q3.

Proof.
First, note that,

$$
\begin{aligned}
&\sqrt{n+1}-\sqrt{n} \\
&= \frac{\left(\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)-\sqrt{n}\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} \\
&= \frac{(n+1)+\sqrt{n(n+1)}-\sqrt{n(n+1)}-n}{\sqrt{n+1}+\sqrt{n}} \\
&= \frac{1}{\sqrt{n+1}+\sqrt{n}} \\
&\leqslant \frac{1}{2\sqrt{n}}.
\end{aligned}
$$

Now, let, $\varepsilon > 0$, be given.

Pick,

$$
\begin{aligned}
N \in \mathbb{N} \ni N &> \left(\frac{1}{2\varepsilon}\right)^2 \\
&\implies \left(\frac{1}{2\varepsilon}\right)^2 < N \\
&\iff \frac{1}{2\varepsilon} < \sqrt{N} \\
&\implies 1 < \varepsilon\left(2\sqrt{N}\right) \\
&\iff \frac{1}{2\sqrt{N}} < \varepsilon \\
&\implies \left|\frac{1}{2\sqrt{N}} - 0\right| < \varepsilon.
\end{aligned}
$$

Then, for $n \geqslant N$,

$$\begin{aligned}
&\frac{1}{n} \leqslant \frac{1}{N} \\
\iff &\frac{1}{2\sqrt{n}} \leqslant \frac{1}{2\sqrt{N}} \\
\iff &\left|\frac{1}{2\sqrt{n}} - 0\right| \leqslant \left|\frac{1}{2\sqrt{N}} - 0\right| < \varepsilon \\
\iff &\left|\frac{1}{2\sqrt{n}} - 0\right| < \varepsilon \\
\iff &\left|\left(\sqrt{n+1} - \sqrt{n}\right) - 0\right| < \varepsilon.
\end{aligned}$$

Thus, $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$. ■

Q5.

<u>**Proof.**</u>
Let $\varepsilon > 0$, be given.

Since, $(s_n)_{n=1}^{\infty}$ converges to 0, $\exists N \in \mathbb{N} \ni |s_n - 0| = |s_n| < \varepsilon \ (n \geqslant N)$.

But,

$$\begin{aligned} |s_n| &= |1| \cdot |s_n| \\ &= |(-1)^n| \cdot |s_n| \\ &= |(-1)^n s_n| \\ &< \varepsilon \ (n \geqslant N). \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} (-1)^n s_n = 0.$ ■

Q6.

Proof.
Let $M > 0$, be given.

We note that, $(s_n)_{n=1}^{\infty}$ converges to some $L \neq 0$.

However, assume the contrary, that instead, $(s_n)_{n=1}^{\infty}$ does not oscillate.

Then, either, it diverges to infinity, or minus infinity, and there are 2 cases.

Case A: $((-1)^n s_n)_{n=1}^{\infty}$ diverges to infinity

Then, $\exists N \in \mathbb{N} \ni (-1)^n s_n \geqslant M \ (n \geqslant N)$.

Here again, there are 2 sub-cases; $(-1)^n = -1$ or $(-1)^n = 1$.

Case A1: $(-1)^n = -1$

Then, $-s_n \geqslant M \Longrightarrow s_n \leqslant -M \ (n \geqslant N)$.

Hence, $(s_n)_{n=1}^{\infty}$ diverges to minus infinity.

But, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Case A2: $(-1)^n = 1$

Then, $s_n \geqslant M \ (n \geqslant N)$.

Thus, $(s_n)_{n=1}^{\infty}$ diverges to infinity.

But, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Case B: $((-1)^n s_n)_{n=1}^{\infty}$ diverges to minus infinity

Thus, $\exists N \in \mathbb{N} \ni (-1)^n s_n \leqslant -M \ (n \geqslant N)$.

Similarly, as with Case A, there are 2 sub-cases; $(-1)^n = -1$ or $(-1)^n = 1$.

Case B1: $(-1)^n = -1$

Then, $-s_n \leqslant -M \Longrightarrow s_n \geqslant M \ (n \geqslant N)$.

Hence, $(s_n)_{n=1}^{\infty}$ diverges to infinity.

But again, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Case B2: $(-1)^n = 1$

Then, $s_n \leqslant -M$, and $(s_n)_{n=1}^{\infty}$ diverges to minus infinity.

But again, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Since, in all cases, we reach a contradiction, the theorem is true, and hence, $(s_n)_{n=1}^{\infty}$ indeed oscillates. ■

2.5 Exercise 2.5 Solutions

Q3.

Proof.
Let $\varepsilon > 0$, be given.

Also, assume the contrary, that $(s_n)_{n=1}^{\infty}$ is bounded.

Then, $\exists M \in \mathbb{R} \ni |s_{n_1}| \leqslant M \ (\forall n_1 \in \mathbb{N})$.

(In particular, let's claim that $\lim\limits_{n\to\infty} \left(\dfrac{s_n}{n}\right) = 0$.)

Pick,

$$
\begin{aligned}
N \in \mathbb{N} \ni N &> \frac{M}{\varepsilon} \\
\implies N\varepsilon &> M \\
\iff N\varepsilon &> s_{n_1} \\
\iff s_{n_1} &< N\varepsilon \\
\iff \frac{s_{n_1}}{N} &< \varepsilon \\
\iff \left|\frac{s_{n_1}}{N}\right| &< \varepsilon \ (\forall n_1 \in \mathbb{N}).
\end{aligned}
$$

Thus, for $n \geqslant N$,

$$
\begin{aligned}
&\left|\frac{1}{n}\right| \leqslant \left|\frac{1}{N}\right| \\
\implies &\left|\frac{s_{n_1}}{n}\right| \leqslant \left|\frac{s_{n_1}}{N}\right| < \varepsilon \ (\forall n_1 \in \mathbb{N}).
\end{aligned}
$$

Hence, we have found an N, where, $\left|\dfrac{s_{n_1}}{n} - 0\right| < \varepsilon$ ($n \geqslant N$, and $\forall n_1 \in \mathbb{N}$).

Thus, $\lim\limits_{n\to\infty} \dfrac{s_n}{n} = 0$, as needed. ■

Q4.

<u>**Proof.**</u>
Let $\varepsilon > 0$, be given.

Since, the sequence $(s_n)_{n=1}^{\infty}$ is bounded, $\exists M \in \mathbb{R} \ni |s_n| \leqslant M$ $(\forall n \in \mathbb{N})$ $\implies -M \leqslant s_n \leqslant M$ $(\forall n \in \mathbb{N})$.

Now, we note that within the closed interval $[-M, M]$, the number of terms in $(s_n)_{n=1}^{\infty}$ is (countably) infinite.

Also, the closed interval has length $2M > 0$.

Now, we could divide this interval into 2 parts, of length; $2M - \varepsilon$ and ε respectively, such that each part has N_1 and N_2 terms, and the total number of terms is $N = N_1 + N_2$.

(Here, we note that $\varepsilon < 2M$.)

Since N is countably infinite, we can infer that either both N_1 and N_2 are countably infinite, or that one of them is.

In either case, one of them is (guaranteed) to be countably infinite, and hence we obtain a set $J \subseteq \mathbb{R}$ of finite arbitrary length, which contains infinite elements.

Now, for the case of any $\varepsilon \geqslant 2M$, we can always box our closed interval $[-M, M]$, with that $\varepsilon > 0$.

Thus, for any $\varepsilon \geqslant 2M$, all infinite elements of the sequence $(s_n)_{n=1}^{\infty}$ is present there. ■

2.6 Exercise 2.6 Solutions

Q3

Proof.
Since $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are nondecreasing and bounded (above), they are convergent to some $L_1 = \lim_{n\to\infty} s_n$, and $L_2 = \lim_{n\to\infty} t_n$.

Also, we note that $L_1 = \sup\{s_1, s_2, \ldots\}$, and $L_2 = \sup\{t_1, t_2, \ldots\}$.

Assume, the contrary, that $L_1 > L_2$, instead.

Then, $L_1 - \varepsilon = L_2$, is not an u.b of $\{s_1, s_2, ...\}$ (for $\varepsilon > 0$).

Thus, $\exists k \ni L_2 < s_k$.

But, $L_2 \geqslant t_n$ $(\forall n \in \mathbb{N})$, and in particular, $L_2 \geqslant t_k$.

Thus, $t_k < L_2 < s_k \Longrightarrow t_k < s_k$.

But, we have from the hypothesis that $s_n \leqslant t_n$ $(\forall n \in \mathbb{N})$.

Since, we have a contradiction, the theorem is proved. ■

Q6

Proof.
(WTS: $(s_n)_{n=1}^{\infty}$ is convergent $\Longleftrightarrow$ $(s_n)_{n=1}^{\infty}$ is bounded below and monotonic nonincreasing.)

We note that $s_1 = \dfrac{1}{2}$, $s_2 = s_1 \cdot \dfrac{3}{4} < s_1$, $s_3 = s_2 \cdot \dfrac{5}{6} < s_2$, and in general, $s_k = s_{k-1} \cdot \dfrac{2k-1}{2k} < s_{k-1}$ (since, $\dfrac{2k-1}{2k} < 1$).

Hence, the sequence $(s_n)_{n=1}^{\infty}$ is nonincreasing; $\dfrac{1}{2} = s_1 > s_2 > s_3 > \cdots > s_{k-1} > s_k > s_{k+1} > \cdots$.

However, we also note that since $s_1 = \dfrac{1}{2}$, and that, $0 < \dfrac{2k-1}{2k} < 1$, since, $k > 0$, the sequence is bounded below by 0.

Thus, $(s_n)_{n=1}^{\infty}$ is convergent.

Also, we note that $\inf\{s_1, s_2, \ldots s_k, s_{k+1}, \ldots\} \leqslant \dfrac{1}{2}$, as $s_1 = \dfrac{1}{2}$. ■

Q10(a)

Proof.

We note that, $t_1 = 1$, $t_2 = t_1 + \dfrac{1}{1!}$, $t_3 = t_2 + \dfrac{1}{2!}$, and in general, $t_k = t_{k-1} + \dfrac{1}{k!}$ $> t_{k-1}$, since $\dfrac{1}{k!} > 0$, and $t_1 = 1$.

Hence, the sequence $(t_n)_{n=1}^{\infty}$ is nondecreasing, where, $t_1 < t_2 < \cdots < t_{k-1}$ $< t_k < t_{k+1} < \cdots$. ■

Q10(b)

Proof.
(WTS: $(t_n)_{n=1}^{\infty}$ is bounded above)

First, note that,

$$\begin{aligned}
t_n &= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \\
&= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdots n} \\
&\leqslant 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\
&= 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \\
&< 1 + \frac{1}{1-\frac{1}{2}} \\
&= 3
\end{aligned}$$

Hence, $(t_n)_{n=1}^{\infty}$ is bounded above, where in particular, 3 is an upper bound.

Also, we note that $s_n \leqslant t_n$ $(\forall n \in \mathbb{N})$, from the proof of 2.6C $\left[(s_n)_{n=1}^{\infty} = \left(\left(1+\frac{1}{n}\right)^n\right)_{n=1}^{\infty}\right]$.

Since, that is the case, from Q3, $\lim\limits_{n\to\infty} s_n \leqslant \lim\limits_{n\to\infty} t_n$, as $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are nondecreasing, bounded sequences too. ■

2.7 Exercise 2.7 Solutions

Q4

Proof.
Since, $s_{n+1} < xs_n < s_n$, as $0 < x < 1$, and $s_n > 0$, the sequence is monotonically nonincreasing; $s_1 > s_2 > \cdots > s_k > s_{k+1} > \cdots$.

Also, since the sequence $(s_n)_{n=1}^{\infty}$ only contains positive terms, it is bounded below.

In particular, 0 is a lower bound.

Hence, $(s_n)_{n=1}^{\infty}$ converges, to some $L \in \mathbb{R}$.

Now, $s_{n+1} < xs_n$, and taking limits; $\lim_{n\to\infty} s_{n+1} \leqslant \lim_{n\to\infty} xs_n = x \lim_{n\to\infty} s_n$.

Also, $L \leqslant xL$, since $(s_{n+1})_{n=1}^{\infty}$ is a subsequence of $(s_n)_{n=1}^{\infty}$.

Thus, $xL - L \geqslant 0 \implies L(x-1) \geqslant 0$.

But, since $0 < x < 1$, $L = 0$. ■