Real Analysis

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Preface

This book is a feeble attempt to explain this broad topic on real analysis.

Anyone with basic knowledge of discrete mathematics and calculus should be able to follow the contents reasonably well.

This book is only available as an electronic copy, and not in print.

Credits to LaTeX, through which this book was typesetted with.

If you happen to spot any mistakes therein, or have any suggestions on how to improve this material, you can drop me an email at; coderatwork64@gmail.com.

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Chapter 1

Least Upper Bound Axiom

1.1 Context

In this chapter, we present the Least Upper Bound (L.U.B) Axiom which is a crucial property of the real numbers, frequently exploited in real analysis.

This property, distinguishes the set of rationals; $\mathbb{Q}$, from the set of reals; $\mathbb{R}$ (which in addition to rationals, contains irrationals too).

Intuitively, it posits that there are no 'gaps'/'holes' in the number system.

1.2 L.U.B Property

Definition 1.2.1. Let $A \subseteq \mathbb{R}$. Then an **upper bound** (u.b), $b \in \mathbb{R}$ is a number where $b \geq a$ for all $a \in A$.

A similar definition can also be made for **lower bound** (l.b).

Definition 1.2.2. Let $A \subseteq \mathbb{R}$. If there exists an u.b for A, then A is said to be **bounded above**.

A similar definition can also be made for subsets of $\mathbb{R}$, which are **bounded below**.

Example.

- The *clopen* (closed-open) interval; $[2, 3) \subseteq \mathbb{R}$, is both bounded below and above. In particular, 2 is a l.b, while 3 is an u.b.

- The interval; $(-\infty, 8) \subseteq \mathbb{R}$, is bounded above (8 is an upper bound) but is not bounded below.

- The empty set; $\varnothing$, is neither bounded above nor below.

- The open interval; $(-\infty, \infty) \subseteq \mathbb{R}$, is also neither bounded above nor below.

Definition 1.2.3. Let $A \subseteq \mathbb{R}$, be bounded above. If there exists an u.b of A; $b \in \mathbb{R}$ such that $\nexists b' \in \mathbb{R}$ with $b' < b$, and b' is an u.b of A, then b is said to be the **least upper bound** (l.u.b) (or **supremum**) of A, and we write $c = \sup(A)$ or $\sup A$.

Example.

- For the clopen interval; $[2, 3) \subseteq \mathbb{R}$, 2 is the g.l.b while 3 is the l.u.b.

- For the interval; $(-\infty, 8) \subseteq \mathbb{R}$, 8 is the l.u.b, but it does not have a g.l.b as it is not bounded below.

- Consider the sequence; $\{2^{\frac{1}{n}}\}_{n=1}^{\infty}$, it has g.l.b of 1 and l.u.b of 2.

A similar definition can also be made for **greatest lower bound** (g.l.b) (or **infimum**) of A, which we denote as $\inf(A)$ or $\inf A$.

Theorem 1.2.4. *Let $A \subseteq \mathbb{R}$, be bounded above. Then the l.u.b of A is unique.*

Proof. Let $b = sup(A)$, and suppose instead that there exists a $b' \in \mathbb{R}$, s.t, $b' = sup(A)$, with $b' \neq b$. Then either $b < b'$ or $b' < b$. If $b < b'$, then this contradicts the minimality of $b' = \sup(A)$. If instead, $b' < b$, then this contradicts the minimality of $b = \sup(A)$. Since, in both cases we reach a contradiction, we must thus have equality; $b = b'$. $\square$

Now, we are ready to present the l.u.b axiom (which we will treat like an axiom, and would not hence be proving it).

Corollary 1.2.5. *Let $A \subseteq \mathbb{R}$, be bounded above. Then, A has a l.u.b in $\mathbb{R}$.*

1.3 Exercises

1. Let $a \in \mathbb{R}$, and $A = \{a\}$ be the *singleton* set (a set which only consists of one element). Show that, $\inf(A) = \sup(A) = a$.

2. Prove that, $\sqrt{2}$ is irrational.
(This exercise shows that $\sqrt{2} \notin \mathbb{Q}$, and thus the set of rationals; $\mathbb{Q}$, does actually have 'holes', unlike the set of reals.)

3. Let $\varnothing \neq A, B \subseteq \mathbb{R}$, and $A + B = \{a + b \mid a \in A, b \in B\}$, prove that;

   (a) $\sup(A) + \sup(B) = \sup(A + B)$

   (b) $\inf(A) + \inf(B) = \inf(A + B)$

1.4 Solutions to Exercises

1.4.1 Question 1

Suppose not, that $a \neq \sup(A)$. Then, $\exists c \in \mathbb{R}$, s.t, $a \leq c = \sup(A) < a$. But, we get a contradiction, because $a < a$. Thus, $a = \sup(A)$.

Similarly, suppose not, that $a \neq \inf(A)$. Then, $\exists c \in \mathbb{R}$, s.t, $a < c = \inf(A) \leq a$. But, we get a contradiction, because $a < a$. Thus, $a = \inf(A)$.

Hence, $\inf(A) = \sup(A) = a$, as needed. $\square$

1.4.2 Question 2

Suppose not, that $\sqrt{2}$ is rational. Then, $\sqrt{2} \in \mathbb{Q}$, and $\exists a, b \in \mathbb{Z}$, s.t, $b \neq 0$, and $\sqrt{2} = \frac{a}{b}$.

Further, let's assume that a and b have been completely factorised; i.e, they have no common factors.

Now, consider, $2 = \frac{a^2}{b^2} \implies a^2 = 2b^2$. This implies that $a^2 = a \times a$, is even. However, this implies that a must be even also. Since, a is even, then a^2 is divisible by 4. Thus, $4 \mid a^2$ and $\exists c \in \mathbb{Z}$, s.t, $a^2 = 4c$.

Hence, substituting a^2 with $4c$, we get; $4c = 2b^2 \implies 2c = b^2$. But, this implies that b^2, and hence, b is even.

Since, a and b are both even, they share a common factor; 2. But, this is a contradiction, as we have previously concluded that a and b do not have any common factors. $\square$

1.4.3 Question 3(a)

Since A and B are bounded above, then, $a \leq \sup A$, $\forall a \in A$, and, $b \leq \sup B$, $\forall b \in B$. Hence, $a+b \leq \sup A + \sup B$, $\forall (a+b) \in A+B$. Thus, $A+B$ is bounded above, where in particular, $\sup A + \sup B$, is an u.b.

Suppose that, $\sup A + \sup B \neq \sup(A+B)$. Then, $\exists x \in \mathbb{R}$, s.t, $a+b \leq x < \sup A + \sup B$ $(\forall a \in A, b \in B)$. Hence, $x < \sup A + \sup B \Longrightarrow x - \sup A < \sup B$.

(Claim: $x - \sup A < b$, $\forall b \in B$) Suppose instead, that $x - \sup A \geq b$. Then, $x - \sup A$ is an u.b of B. Also, we have, $b \leq \sup B$. Hence, $b \leq \sup B \leq x - \sup A$, which contradicts; $x - \sup A < \sup B$ (because, $x - \sup A < b$ already, thus $x - \sup A < \sup B$). Thus, we must have, $x - \sup A < b \Longrightarrow x - b < \sup A$.

(Claim: $x - b < a$, $\forall a \in A$) Suppose instead, that $x - b \geq a$. Then, $x - b$ is an u.b of A. Also, $a \leq \sup A$. Hence, $a \leq \sup A \leq x - b$, which contradicts; $x - b < \sup A$ (because, $x - b < a$ already, thus $x - b < \sup A$). Thus, we must have, $x - b < a \Longrightarrow x < a + b$, $\forall (a+b) \in A+B$.

But, this contradicts; $a+b \leq x$. Thus, we must have $\sup A + \sup B = \sup(A+B)$, as needed.

$\square$

1.4.4 Question 3(b)

Similar approach to Qns 3(a).

Chapter 2

Sequences

2.1 Context

In this chapter, we present sequences more formally (as functions), as it is another essential tool in real analysis.

Such a definition, allows us to define subsequences, and various other properties of sequences, such as; limits, convergence/divergence, boundedness, and monotonicity.

And, we will conclude this chapter by introducing *Cauchy-ness*, which helps us determine whether a sequence converges, without even having to know its limit.

2.2 Sequences and Subsequences

Definition 2.2.1. A **sequence** is a function; $f : \mathbb{Z}^+ \to \mathbb{R}$, denoted as $(s_n)_{n=1}^{\infty}$ (where $s_i = f(i)$ for all $i \in \mathbb{Z}^+$).

Example.

- For the clopen interval; $[2, 3) \subseteq \mathbb{R}$, 2 is the g.l.b while 3 is the l.u.b.
- For the interval; $(-\infty, 8) \subseteq \mathbb{R}$, 8 is the l.u.b, but it does not have a g.l.b as it is not bounded below.

Bibliography

[Gol76] Richard R. Goldberg. *Methods of Real Analysis*. John Wiley & Sons, Inc, 1976. ISBN: 0-471-31065-4.

Appendices

Appendix A

Richard R. Goldberg Solutions

This is a bonus chapter, where I provide solutions to **selected** exercises in the Richard R. Goldberg's Real Analysis book, for those who are trying out the questions in that text.

Disclaimer: The solutions I provide are not official solutions, just my own solutions to the exercises.

A.1 Exercise 1.7 Solutions

Q1.

<u>Ans.</u>

(a) 7

(b) $\pi + 1$

(c) π

Q2.

<u>Ans.</u>

(a) 8

(b) $\pi + 1$

Q5.

Ans.

A only consists one element; $x \in \mathbb{R}$.

A.2 Exercise 2.1 Solutions

Q5.

Proof.

Let S be a sequence in the set A. We note that it is given by the function; $f : \mathbb{N} \to A$. Now, consider an (arbitrary) subsequence S' of S,which has the form $f \circ g$, where $h : \mathbb{N} \to \mathbb{N}$, satisfies $h(n) < h(n+1)$, for all n.

(*Claim:* $(g \circ h)(n) < (g \circ h)(n+1), \forall n \in \mathbb{N}$) We note that $h(n) < h(n+1)$, and $g(n) < g(n+1)$. Now, let $b = h(n)$, and $c = h(n+1)$. Also, $b < c$ since $h(n) < h(n+1)$.

Thus,

$$\begin{aligned}
&b < b+1 \leq c \text{ (due to composite function } g(h(n))) \\
\implies &g(b) < g(b+1) \leq g(c) \\
\implies &g(b) < g(c) \\
\implies &g(h(n)) < g(h(n+1)).
\end{aligned}$$

□

Q6.

Proof.

We note that a subsequence of S has the form; $S \circ N$, where $N : \mathbb{N} \to \mathbb{N}$, and that $N(k) < N(k+1), \forall k \in \mathbb{N}$. Also, $N(k) = n_k$. Thus, $n_k < n_{k+1}$.

(*Claim:* $n_k \geq k$) We induct on k. For the base case where $k = 1$, it is obvious, that $n_1 \geq 1$. Now, assume (inductively) that, $n_k \geq k$ (WTS: $n_{k+1} \geq k+1$). Consider, n_{k+1}.

Since, $n_k < n_{k+1}$, then,

$$\begin{aligned} & k \leq n_k < n_{k+1} + 1 \\ \implies & k + 1 < n_{k+1} < n_{k+1} + 1 \\ \implies & k + 1 < n_{k+1}. \end{aligned}$$

Thus, by induction, $n_k \geq k$ $(\forall k \in \mathbb{N})$, and we are done. $\square$

A.3 Exercise 2.2 Solutions

Q1.

Proof.

Since $(M - s_n)_{n=1}^{\infty}$ converges to $(M - L) \in \mathbb{R}$, and $s_n \leq M \implies M - s_n \geq 0$ $(\forall n \in \mathbb{N})$, then, $M - L \geq 0$, as needed. $\square$

Q2.

Proof.

Let $\varepsilon > 0$, be given, and suppose instead that $L > M$. However, we note that by the hypothesis, we have, $L \leq M + \varepsilon$. Then, we have 2 cases; either $L < M + \varepsilon$ or $L = M + \varepsilon$.

For the first case, $L - M = \varepsilon$, for all $\varepsilon > 0$. But, this is a contradiction, because $L - M = \varepsilon_a$, for some $\varepsilon_a \in \mathbb{R}$ only.

For the second case,

$$\begin{aligned} & M < L \leq M + \varepsilon \\ \implies & M - \varepsilon < L \leq M + \varepsilon \\ \implies & |L - M| < \varepsilon \ (\forall \varepsilon > 0). \end{aligned}$$

We note that since $L > M$, by our assumption, this implies $|L - M| > 0$. But, in particular, pick an $\varepsilon_a < |L - M| < \varepsilon_a$. Thus, $\varepsilon_a < \varepsilon_a$, which is a contradiction.

Since, in all cases, we reach a contradiction, the original statement has to be true. $\square$

Q4(b).

Proof.

Let $\varepsilon > 0$ be given. (WTS: $\lim_{n\to\infty} \frac{2n}{n+3} = 2$)

Pick,

$$\begin{aligned}
N \in \mathbb{N} \ni N &> \frac{6}{\varepsilon} - 3 \\
\iff N &> \frac{6-3\varepsilon}{\varepsilon} \\
\iff N\varepsilon &> 6 - 3\varepsilon \\
\iff N\varepsilon + 3\varepsilon &> 6 \\
\iff \varepsilon(N+3) &> 6 \\
\iff 6 &< \varepsilon(N+3) \\
\iff \frac{6}{N+3} &< \varepsilon \\
\iff \left|\frac{2N - 2(N+3)}{N+3}\right| &< \varepsilon \\
\iff \left|\frac{2N}{N+3} - 2\right| &< \varepsilon.
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\frac{1}{n}\right| \leq \left|\frac{1}{N}\right| \\
\implies &\left|\frac{1}{n+3}\right| \leq \left|\frac{1}{N+3}\right| \\
\implies &\left|\frac{2n}{n+3}\right| \leq \left|\frac{2N}{N+3}\right| \\
\implies &\left|\frac{2n}{n+3} - 2\right| \leq \left|\frac{2N}{N+3} - 2\right| < \varepsilon.
\end{aligned}$$

Hence, we have found an N, where, $\left|\frac{2n}{n+3} - 2\right| < \varepsilon$ $(n \geq N)$. This proves, $\lim_{n\to\infty} \frac{2n}{n+3} = 2$, as needed. $\square$

A.4 Exercise 2.3 Solutions

Q1.

Proof.

We note that,

$$
\begin{aligned}
&|a| = |(a-b)+b| \leq |a-b| + |b| \\
\implies &|a| - |b| \leq |a-b|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|b| = |(b-a)+a| \leq |b-a| + |a| \\
\implies &|b| - |a| \leq |b-a| = |a-b|.
\end{aligned}
$$

Since, $||a|-|b||$ is either; $|a|-|b|$ or $|b|-|a|$, then, $||a|-|b|| \leq |a-b|$. $\square$

Now, let $\varepsilon > 0$, be given. Since, $(s_n)_{n=1}^{\infty}$ converges to L, $\exists N \in \mathbb{N}$, s.t, $|s_n - L| < \varepsilon$ $(n \geq N)$. Hence, $\lim_{n\to\infty} |s_n| = |L|$. $\square$

Q5.

Proof.

Let $\varepsilon > 0$, be given. Since, $\lim_{m\to\infty} s_{2m} = L$, $\exists N_1 \in \mathbb{N}$, s.t, $|s_{2m} - L| < \varepsilon$ $(2m \geq N_1)$. Similarly, since, $\lim_{m\to\infty} s_{2m-1} = L$, $\exists N_2 \in \mathbb{N}$, s.t, $|s_{2m-1} - L| < \varepsilon$ $(2m-1 \geq N_2)$. Set $N = \max\{N_1, N_2\}$, then, in both cases, where m is odd or even, $\exists N \in \mathbb{N}$, s.t, $|s_m - L| < \varepsilon$ $(m \geq N)$. Hence, $\lim_{m\to\infty} s_m = L$, as needed. $\square$

A.5 Exercise 2.4 Solutions

Q3.

Proof.

First, note that,

$$
\begin{aligned}
&\sqrt{n+1}-\sqrt{n}\\
&=\frac{\left(\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)-\sqrt{n}\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}\\
&=\frac{(n+1)+\sqrt{n(n+1)}-\sqrt{n(n+1)}-n}{\sqrt{n+1}+\sqrt{n}}\\
&=\frac{1}{\sqrt{n+1}+\sqrt{n}}\\
&\le\frac{1}{2\sqrt{n}}.
\end{aligned}
$$

Now, let, $\varepsilon > 0$, be given.

Pick,

$$
\begin{aligned}
&N \in \mathbb{N} \ni N > \left(\frac{1}{2\varepsilon}\right)^2\\
&\qquad\Longrightarrow \left(\frac{1}{2\varepsilon}\right)^2 < N\\
&\qquad\Longleftrightarrow \frac{1}{2\varepsilon} < \sqrt{N}\\
&\qquad\Longrightarrow 1 < \varepsilon\left(2\sqrt{N}\right)\\
&\qquad\Longleftrightarrow \frac{1}{2\sqrt{N}} < \varepsilon\\
&\qquad\Longrightarrow \left|\frac{1}{2\sqrt{N}} - 0\right| < \varepsilon.
\end{aligned}
$$

Then, for $n \geq N$,

$$
\begin{aligned}
& \frac{1}{n} \leq \frac{1}{N} \\
\iff & \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} \\
\iff & \left| \frac{1}{2\sqrt{n}} - 0 \right| \leq \left| \frac{1}{2\sqrt{N}} - 0 \right| < \varepsilon \\
\iff & \left| \frac{1}{2\sqrt{n}} - 0 \right| < \varepsilon \\
\iff & \left| \left(\sqrt{n+1} - \sqrt{n} \right) - 0 \right| < \varepsilon.
\end{aligned}
$$

Thus, $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n} \right) = 0$. $\square$

Q5.

Proof.

Let $\varepsilon > 0$, be given.

Since, $(s_n)_{n=1}^{\infty}$ converges to 0, $\exists N \in \mathbb{N} \ni |s_n - 0| = |s_n| < \varepsilon \ (n \geq N)$.

But,

$$\begin{aligned} |s_n| &= |1| \cdot |s_n| \\ &= |(-1)^n| \cdot |s_n| \\ &= |(-1)^n s_n| \\ &< \varepsilon \ (n \geq N). \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} (-1)^n s_n = 0$. $\square$

Q6.

<u>**Proof.**</u>

Let $M > 0$, be given.

We note that, $(s_n)_{n=1}^{\infty}$ converges to some $L \neq 0$.

However, assume the contrary, that instead, $(s_n)_{n=1}^{\infty}$ does not oscillate.

Then, either, it diverges to infinity, or minus infinity, and there are 2 cases.

<u>Case A: $((-1)^n s_n)_{n=1}^{\infty}$ diverges to infinity</u>

Then, $\exists N \in \mathbb{N} \ni (-1)^n s_n \geq M \ (n \geq N)$.

Here again, there are 2 sub-cases; $(-1)^n = -1$ or $(-1)^n = 1$.

<u>Case A1: $(-1)^n = -1$</u>

Then, $-s_n \geq M \implies s_n \leq -M \ (n \geq N)$.

Hence, $(s_n)_{n=1}^{\infty}$ diverges to minus infinity.

But, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

<u>Case A2: $(-1)^n = 1$</u>

Then, $s_n \geq M \ (n \geq N)$.

Thus, $(s_n)_{n=1}^{\infty}$ diverges to infinity.

But, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

<u>Case B: $((-1)^n s_n)_{n=1}^{\infty}$ diverges to minus infinity</u>

Thus, $\exists N \in \mathbb{N} \ni (-1)^n s_n \leq -M \ (n \geq N)$.

Similarly, as with Case A, there are 2 sub-cases; $(-1)^n = -1$ or $(-1)^n = 1$.

<u>Case B1: $(-1)^n = -1$</u>

Then, $-s_n \leq -M \implies s_n \geq M \ (n \geq N)$.

Hence, $(s_n)_{n=1}^{\infty}$ diverges to infinity.

But again, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Case B2: $(-1)^n = 1$

Then, $s_n \leq -M$, and $(s_n)_{n=1}^{\infty}$ diverges to minus infinity.

But again, from our hypothesis, it converges to some $L \neq 0$. Contradiction.

Since, in all cases, we reach a contradiction, the theorem is true, and hence, $(s_n)_{n=1}^{\infty}$ indeed oscillates. $\square$

A.6 Exercise 2.5 Solutions

Q3.

Proof.

Let $\varepsilon > 0$, be given.

Also, assume the contrary, that $(s_n)_{n=1}^{\infty}$ is bounded.

Then, $\exists M \in \mathbb{R} \ni |s_{n_1}| \leq M \ (\forall n_1 \in \mathbb{N})$.

(In particular, let's claim that $\lim\limits_{n\to\infty} \left(\dfrac{s_n}{n}\right) = 0$.)

Pick,

$$\begin{aligned}
N \in \mathbb{N} \ni N &> \frac{M}{\varepsilon} \\
\implies N\varepsilon &> M \\
\iff N\varepsilon &> s_{n_1} \\
\iff s_{n_1} &< N\varepsilon \\
\iff \frac{s_{n_1}}{N} &< \varepsilon \\
\iff \left|\frac{s_{n_1}}{N}\right| &< \varepsilon \ (\forall n_1 \in \mathbb{N}).
\end{aligned}$$

Thus, for $n \geq N$,

$$\begin{aligned}
&\left|\frac{1}{n}\right| \leq \left|\frac{1}{N}\right| \\
\implies &\left|\frac{s_{n_1}}{n}\right| \leq \left|\frac{s_{n_1}}{N}\right| < \varepsilon \ (\forall n_1 \in \mathbb{N}).
\end{aligned}$$

Hence, we have found an N, where, $\left|\dfrac{s_{n_1}}{n} - 0\right| < \varepsilon$ ($n \geq N$, and $\forall n_1 \in \mathbb{N}$).

Thus, $\lim\limits_{n\to\infty} \dfrac{s_n}{n} = 0$, as needed. $\square$

Q4.

Proof.

Let $\varepsilon > 0$, be given.

Since, the sequence $(s_n)_{n=1}^{\infty}$ is bounded, $\exists M \in \mathbb{R} \ni |s_n| \leq M$ $(\forall n \in \mathbb{N}) \Longrightarrow -M \leq s_n \leq M$ $(\forall n \in \mathbb{N})$.

Now, we note that within the closed interval $[-M, M]$, the number of terms in $(s_n)_{n=1}^{\infty}$ is (countably) infinite.

Also, the closed interval has length $2M > 0$.

Now, we could divide this interval into 2 parts, of length; $2M - \varepsilon$ and ε respectively, such that each part has N_1 and N_2 terms, and the total number of terms is $N = N_1 + N_2$.

(Here, we note that $\varepsilon < 2M$.)

Since N is countably infinite, we can infer that either both N_1 and N_2 are countably infinite, or that one of them is.

In either case, one of them is (guaranteed) to be countably infinite, and hence we obtain a set $J \subseteq \mathbb{R}$ of finite arbitrary length, which contains infinite elements.

Now, for the case of any $\varepsilon \geq 2M$, we can always box our closed interval $[-M, M]$, with that $\varepsilon > 0$.

Thus, for any $\varepsilon \geq 2M$, all infinite elements of the sequence $(s_n)_{n=1}^{\infty}$ is present there. $\square$

A.7 Exercise 2.6 Solutions

Q3

Proof.

Since $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are nondecreasing and bounded (above), they are convergent to some $L_1 = \lim_{n\to\infty} s_n$, and $L_2 = \lim_{n\to\infty} t_n$.

Also, we note that $L_1 = \sup\{s_1, s_2, \ldots\}$, and $L_2 = \sup\{t_1, t_2, \ldots\}$.

Assume, the contrary, that $L_1 > L_2$, instead.

Then, $L_1 - \varepsilon = L_2$, is not an u.b of $\{s_1, s_2, ...\}$ (for $\varepsilon > 0$).

Thus, $\exists k \ni L_2 < s_k$.

But, $L_2 \geq t_n$ $(\forall n \in \mathbb{N})$, and in particular, $L_2 \geq t_k$.

Thus, $t_k < L_2 < s_k \implies t_k < s_k$.

But, we have from the hypothesis that $s_n \leq t_n$ $(\forall n \in \mathbb{N})$.

Since, we have a contradiction, the theorem is proved. $\square$

Q6

Proof.

(WTS: $(s_n)_{n=1}^{\infty}$ is convergent $\Longleftarrow$ $(s_n)_{n=1}^{\infty}$ is bounded below and monotonic nonincreasing.)

We note that $s_1 = \frac{1}{2}$, $s_2 = s_1 \cdot \frac{3}{4} < s_1$, $s_3 = s_2 \cdot \frac{5}{6} < s_2$, and in general, $s_k = s_{k-1} \cdot \frac{2k-1}{2k}$ $< s_{k-1}$ (since, $\frac{2k-1}{2k} < 1$).

Hence, the sequence $(s_n)_{n=1}^{\infty}$ is nonincreasing; $\frac{1}{2} = s_1 > s_2 > s_3 > \cdots > s_{k-1} > s_k > s_{k+1}$ $> \cdots$.

However, we also note that since $s_1 = \frac{1}{2}$, and that, $0 < \frac{2k-1}{2k} < 1$, since, $k > 0$, the sequence is bounded below by 0.

Thus, $(s_n)_{n=1}^{\infty}$ is convergent.

Also, we note that $\inf\{s_1, s_2, \ldots s_k, s_{k+1}, \ldots\} \leq \frac{1}{2}$, as $s_1 = \frac{1}{2}$. $\square$

Q10(a)

Proof.

We note that, $t_1 = 1$, $t_2 = t_1 + \dfrac{1}{1!}$, $t_3 = t_2 + \dfrac{1}{2!}$, and in general, $t_k = t_{k-1} + \dfrac{1}{k!} > t_{k-1}$, since $\dfrac{1}{k!} > 0$, and $t_1 = 1$.

Hence, the sequence $(t_n)_{n=1}^{\infty}$ is nondecreasing, where, $t_1 < t_2 < \cdots < t_{k-1} < t_k < t_{k+1} < \cdots$. $\square$

Q10(b)

Proof.

(WTS: $(t_n)_{n=1}^{\infty}$ is bounded above)

First, note that,

$$\begin{aligned} t_n &= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ &= 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdots n} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &= 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\ &< 1 + \frac{1}{1 - \frac{1}{2}} \\ &= 3 \end{aligned}$$

Hence, $(t_n)_{n=1}^{\infty}$ is bounded above, where in particular, 3 is an upper bound.

Also, we note that $s_n \leq t_n$ $(\forall n \in \mathbb{N})$, from the proof of 2.6C $\left[(s_n)_{n=1}^{\infty} = \left(\left(1 + \frac{1}{n}\right)^n\right)_{n=1}^{\infty}\right]$.

Since, that is the case, from Q3, $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$, as $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are nondecreasing, bounded sequences too. $\square$

A.8 Exercise 2.7 Solutions

Q4

Proof.

Since, $s_{n+1} < xs_n < s_n$, as $0 < x < 1$, and $s_n > 0$, the sequence is monotonically nonincreasing; $s_1 > s_2 > \cdots > s_k > s_{k+1} > \cdots$.

Also, since the sequence $(s_n)_{n=1}^{\infty}$ only contains positive terms, it is bounded below.

In particular, 0 is a lower bound.

Hence, $(s_n)_{n=1}^{\infty}$ converges, to some $L \in \mathbb{R}$.

Now, $s_{n+1} < xs_n$, and taking limits; $\lim\limits_{n\to\infty} s_{n+1} \leq \lim\limits_{n\to\infty} xs_n = x \lim\limits_{n\to\infty} s_n$.

Also, $L \leq xL$, since $(s_{n+1})_{n=1}^{\infty}$ is a subsequence of $(s_n)_{n=1}^{\infty}$.

Thus, $xL - L \geq 0 \Longrightarrow L(x-1) \geq 0$.

But, since $0 < x < 1$, $L = 0$. $\square$