

# Richard Goldberg Solutions

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July 3, 2020

# Preface

In this book, I would be providing solutions to selected exercises in Richard R. Goldberg's Real Analysis book.

*(As a matter of formatting, I will start each question on a blank page.)*

Disclaimer: The solutions I provide are not official solutions, just my own solutions to the exercises.

This book is only available as an electronic copy, and not available in print.

If you spot any mistakes in this book, have suggestions on how to improve this book, or have any other queries, you may reach me at my email.

~ Nathanael Seen

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# Chapter 1

## Sets and Functions

### 1.1 Exercise 1.7 Solutions

Q1.

Ans.

(a) 7

(b)  $\pi + 1$

(c)  $\pi$

# Chapter 2

## Sequences of Real Numbers

### 2.1 Exercise 2.1 Solutions

Q5.

Proof.

Let  $S$  be a Sequence in the set  $A$ .

We note that it is given by the function;  $f : \mathbb{N} \rightarrow A$ .

Now, consider an (arbitrary) subsequence  $S'$  of  $S$ , which has the form  $f \circ g$ , where  $h : \mathbb{N} \rightarrow \mathbb{N}$ , satisfies  $h(n) < h(n+1)$ , for all  $n$ .

(*Claim:*  $(g \circ h)(n) < (g \circ h)(n+1)$ ,  $\forall n \in \mathbb{N}$ )

We note that  $h(n) < h(n+1)$ , and  $g(n) < g(n+1)$ .

Now, let  $b = h(n)$ , and  $c = h(n+1)$ .

Also,  $b < c$  since  $h(n) < h(n+1)$ .

Thus,

$$\begin{aligned} & b < b+1 \leq c \text{ (due to composite function } g(h(n))) \\ \implies & g(b) < g(b+1) \leq g(c) \\ \implies & g(b) < g(c) \\ \implies & g(h(n)) < g(h(n+1)). \blacksquare \end{aligned}$$

**Q6.**

**Proof.**

We note that a subsequence of  $S$  has the form;  $S \circ N$ , where  $N : \mathbb{N} \rightarrow \mathbb{N}$ , and that  $N(k) < N(k+1)$ ,  $\forall k \in \mathbb{N}$ .

Also,  $N(k) = n_k$ . Thus,  $n_k < n_{k+1}$ .

(*Claim:  $n_k \geq k$* )

Base Case:  $k = 1$

Then, it is obvious, that  $n_1 \geq 1$ .

Inductive Case:

Assume that,  $n_k \geq k$ . (WTS:  $n_{k+1} \geq k+1$ )

Consider,  $n_{k+1}$ .

Since,  $n_k < n_{k+1}$ , then,

$$\begin{aligned} k &\leq n_k < n_{k+1} + 1 \\ \implies k + 1 &< n_{k+1} < n_{k+1} + 1 \\ \implies k + 1 &< n_{k+1}. \end{aligned}$$

Thus, by Induction,  $n_k \geq k$  ( $\forall k \in \mathbb{N}$ ), and the Theorem is proved.

■

## 2.2 Exercise 2.2 Solutions

Q1.

**Proof.**

Since  $(M - s_n)_{n=1}^{\infty}$  converges to  $(M - L) \in \mathbb{R}$ , and  $s_n \leq M \implies M - s_n \geq 0$  ( $\forall n \in \mathbb{N}$ ), then,  $M - L \geq 0$ , as needed. ■

**Q2.**

**Proof.**

Let  $\varepsilon > 0$ , be given.

Suppose the contrary, that  $L > M$ .

Also, by the hypothesis,  $L \leq M + \varepsilon$ .

Thus,

$$\begin{aligned} M < L &\leq M + \varepsilon \text{ (since } \varepsilon > 0) \\ \implies M - \varepsilon &< L \leq M + \varepsilon \\ \implies |L - M| &< \varepsilon \text{ } (\forall \varepsilon > 0). \end{aligned}$$

We note that since  $L > M$ , by our assumption,  $|L - M| > 0$ .

But, in particular, pick an  $\varepsilon_a < |L - M| < \varepsilon_a$ .

Thus,  $\varepsilon_a < \varepsilon_a$ , which is a Contradiction.

Hence, the Theorem is true. ■



Q4(b).

**Proof.**

Let  $\varepsilon > 0$  be given.

(WTS:  $\lim_{n \rightarrow \infty} \frac{2n}{n+3} = 2$ )

Pick,

$$\begin{aligned} N \in \mathbb{N} \ni N &\geq \frac{6}{\varepsilon} - 3 \\ &\iff N \geq \frac{6-3\varepsilon}{\varepsilon} \\ &\iff N\varepsilon \geq 6-3\varepsilon \\ &\iff N\varepsilon + 3\varepsilon \geq 6 \\ &\iff \varepsilon(N+3) \geq 6 \\ &\iff 6 \leq \varepsilon(N+3) \\ &\iff \frac{6}{N+3} \leq \varepsilon \\ &\iff \left| \frac{2N-2(N+3)}{N+3} \right| \leq \varepsilon \\ &\iff \left| \frac{2N}{N+3} - 2 \right| < \varepsilon. \end{aligned}$$

Then,

$$\begin{aligned} &\left| \frac{1}{n} \right| \leq \left| \frac{1}{N} \right| \\ \implies &\left| \frac{1}{n+3} \right| \leq \left| \frac{1}{N+3} \right| \\ \implies &\left| \frac{2n}{n+3} \right| \leq \left| \frac{2N}{N+3} \right| \\ \implies &\left| \frac{2n}{n+3} - 2 \right| \leq \left| \frac{2N}{N+3} - 2 \right| < \varepsilon. \end{aligned}$$

Hence, we have found an  $N$ , where,  $\left| \frac{2n}{n+3} - 2 \right| < \varepsilon$  ( $n \geq N$ ).

This,  $\lim_{n \rightarrow \infty} \frac{2n}{n+3} = 2$ , as needed. ■

**2.3   Exercise 2.3 Solutions**

**2.4   Exercise 2.4 Solutions**

**2.5   Exercise 2.5 Solutions**

**2.6   Exercise 2.6 Solutions**

**2.7   Exercise 2.7 Solutions**