Metric Spaces

Analysis

Nathanael Seen

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Definition. Let M be a nonempty set. A metric (on the set M) is a function; $\rho : M \times M \to [0, \infty)$, such that;

1. $\rho(x, x) = 0$ for all $x \in M$
2. $\rho(x, y) > 0$ for all $x, y \in M$ where $x \neq y$
3. $\rho(x, y) = \rho(y, x)$ for all $x, y \in M$
4. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in M$

The pair $\langle M, \rho \rangle$ is called a metric space.

Examples.

1. Consider $\mathbb{R}$ given the absolute-value metric; $\rho(x, y) = |x - y|$, for all $x, y \in \mathbb{R}$. Clearly, all four properties of a metric are satisfied.
2. Let M be any non-empty set endowed with the discrete metric;

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

   We denote the metric space; $\langle M, d\rangle$, by M_d.

3. Consider $\mathbb{R}^n$ given the Euclidean metric; $\rho_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, for all $x = (x_1, x_2, \ldots x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The first three properties can be easily verified, we will focus on verifying the fourth one (for the rest of this handout).
4. Consider $\mathbb{R}^n$ given the ∞-metric (or sup-metric); $\rho_\infty(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$, for all $x, y \in \mathbb{R}^n$. Clearly, all four properties of a metric are satisfied.

Proposition. (Cauchy-Schwarz inequality) For any $x_i, y_i \in \mathbb{R}$

$$\sum_{i=1}^{n} |x_i y_i| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |y_i|^2}$$

Proof. Let $X = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ and $Y = \sqrt{\sum_{i=1}^{n} |y_i|^2}$.

If $X = 0$ or $Y = 0$, then WLOG (Without Loss of Generality) suppose that $X = 0$. Then, this implies that $|x_i| = 0$, for all $1 \leq i \leq n$, since $|x_i| \geq 0$.

Thus,

$$\begin{aligned}\sum_{i=1}^{n} |x_i y_i| &= |x_1 y_1| + |x_2 y_2| + \cdots + |x_n y_n| \\ &= |x_1||y_1| + |x_2||y_2| + \cdots + |x_n||y_n| = 0 \\ &= \sqrt{\sum_{i=1}^{n} |x_i|^2} \cdot \sqrt{\sum_{i=1}^{n} |y_i|^2} = A \cdot B = 0 \cdot B = 0\end{aligned}$$

Proof (cont.) Suppose now that $X, Y > 0$.

Observe that, for fixed $1 \leq j \leq n$,

$$\frac{|x_j||y_j| + |x_j||y_j|}{\sqrt{\sum_{i=1}^n |x_i|^2 \cdot \sum_{i=1}^n |y_i|^2}} \leq \frac{\left(\sum_{i=1}^n |y_i|^2\right)|x_j|^2 + \left(\sum_{i=1}^n |x_i|^2\right)|y_j|^2}{\sum_{i=1}^n |x_i|^2 \cdot \sum_{i=1}^n |y_i|^2}$$

$$\iff \frac{2|x_j||y_j|}{XY} \leq \frac{Y^2|x_j|^2 + X^2|y_j|^2}{X^2Y^2}$$

$$\iff \frac{|x_j||y_j|}{AB} \leq \frac{1}{2}\left[\frac{|x_j|^2}{X^2} + \frac{|y_j|^2}{Y^2}\right] = \frac{1}{2}\left[\left(\frac{|x_j|}{X}\right)^2 + \left(\frac{|y_j|}{Y}\right)^2\right]$$

Taking summation from 1 to n, we have;

$$\frac{1}{XY}\sum_{j=1}^n |x_j y_j| \leq \frac{1}{2}\sum_{j=1}^n \left[\left(\frac{|x_j|}{X}\right)^2 + \left(\frac{|y_j|}{Y}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{\sum_{j=1}^n |x_j|^2}{X^2} + \frac{\sum_{j=1}^n |y_j|^2}{Y^2}\right]$$

$$= \frac{1}{2}\left[\frac{X^2}{X^2} + \frac{Y^2}{Y^2}\right] = \frac{1}{2}(2) = 1$$

Proof (cont.) Hence,

$$\frac{1}{XY}\sum_{j=1}^{n}|x_j y_j| \leq 1$$

$$\iff \sum_{j=1}^{n}|x_j y_j| \leq XY = \sqrt{\sum_{i=1}^{n}|x_i|^2} \cdot \sqrt{\sum_{i=1}^{n}|y_i|^2}$$

And we are done.

Proposition. (Minkowski's inequality) For any $x_i, y_i \in \mathbb{R}$

$$\sqrt{\sum_{i=1}^{n} |x_i + y_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i|^2}$$

Proof. Let $X = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ and $Y = \sqrt{\sum_{i=1}^{n} |y_i|^2}$, as per last proposition.

Then,

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^2 &= \sum_{i=1}^{n} \left[|x_i|^2 + 2|x_i y_i| + |y_i|^2 \right] \\
&= \sum_{i=1}^{n} |x_i|^2 + 2 \sum_{i=1}^{n} |x_i y_i| + \sum_{i=1}^{n} |y_i|^2 \\
&\leq X^2 + 2XY + Y^2 \text{ (by the Cauchy-Schwarz inequality)} \\
&= (X + Y)^2
\end{aligned}$$

Proof (cont.) Taking square roots, we have;

$$\begin{aligned}
\sqrt{\sum_{i=1}^{n}|x_i + y_i|^2} &\leq \sqrt{(X+Y)^2} \\
&= X + Y \\
&= \sqrt{\sum_{i=1}^{n}|x_i|^2} + \sqrt{\sum_{i=1}^{n}|y_i|^2}
\end{aligned}$$

And we are done.

We are now ready to verify the fourth property (triangle inequality) for the Euclidean metric;

$$\begin{aligned}
\rho_2(x,y) &= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}|x_i - y_i|^2} \\
&= \sqrt{\sum_{i=1}^{n}|(x_i - z_i) + (z_i - y_i)|^2} \\
&\leq \sqrt{\sum_{i=1}^{n}|x_i - z_i|^2} + \sqrt{\sum_{i=1}^{n}|z_i - y_i|^2} \\
&\quad \text{(by the \underline{Minkowski's} inequality)} \\
&= \rho_2(x,z) + \rho_2(y,z)
\end{aligned}$$

And we are done.