

# Metric Spaces

## Analysis

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**Definition.** Let  $M$  be a nonempty set. A metric (on the set  $M$ ) is a function;  $\rho : M \times M \rightarrow [0, \infty)$ , such that;

1.  $\rho(x, x) = 0$  for all  $x \in M$
2.  $\rho(x, y) > 0$  for all  $x, y \in M$  where  $x \neq y$
3.  $\rho(x, y) = \rho(y, x)$  for all  $x, y \in M$
4.  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$  for all  $x, y, z \in M$

The pair  $\langle M, \rho \rangle$  is called a metric space.

## Examples.

1. Consider  $\mathbb{R}$  given the absolute-value metric;  $\rho(x, y) = |x - y|$ , for all  $x, y \in \mathbb{R}$ . Clearly, all four properties of a metric are satisfied.
2. Let  $M$  be any non-empty set endowed with the discrete metric;

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

We denote the metric space;  $\langle M, d \rangle$ , by  $M_d$ .

3. Consider  $\mathbb{R}^n$  given the Euclidean metric;  $\rho_2(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ , for all  $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ . The first three properties can be easily verified, we will focus on verifying the fourth one (for the rest of this handout).
4. Consider  $\mathbb{R}^n$  given the  $\infty$ -metric (or sup-metric);  
 $\rho_\infty(x, y) = \max \{|x_i - y_i| : 1 \leq i \leq n\}$ , for all  $x, y \in \mathbb{R}^n$ . Clearly, all four properties of a metric are satisfied.

**Proposition.** (Cauchy-Schwarz inequality) For any  $x_i, y_i \in \mathbb{R}$

$$\sum_{i=1}^n |x_i y_i| \leq \sqrt{\sum_{i=1}^n |x_i|^2} \cdot \sqrt{\sum_{i=1}^n |y_i|^2}$$

*Proof.* Let  $X = \sqrt{\sum_{i=1}^n |x_i|^2}$  and  $Y = \sqrt{\sum_{i=1}^n |y_i|^2}$ .

If  $X = 0$  or  $Y = 0$ , then WLOG (Without Loss of Generality) suppose that  $X = 0$ . Then, this implies that  $|x_i| = 0$ , for all  $1 \leq i \leq n$ , since  $|x_i| \geq 0$ .

Thus,

$$\begin{aligned}\sum_{i=1}^n |x_i y_i| &= |x_1 y_1| + |x_2 y_2| + \cdots + |x_n y_n| \\&= |x_1| |y_1| + |x_2| |y_2| + \cdots + |x_n| |y_n| = 0 \\&= \sqrt{\sum_{i=1}^n |x_i|^2} \cdot \sqrt{\sum_{i=1}^n |y_i|^2} = A \cdot B = 0 \cdot B = 0\end{aligned}$$

*Proof (cont.)* Suppose now that  $X, Y > 0$ .

Observe that, for fixed  $1 \leq j \leq n$ ,

$$\begin{aligned} \frac{|x_j||y_j| + |x_j||y_j|}{\sqrt{\sum_{i=1}^n |x_i|^2 \cdot \sum_{i=1}^n |y_i|^2}} &\leq \frac{(\sum_{i=1}^n |y_i|^2) |x_j|^2 + (\sum_{i=1}^n |x_i|^2) |y_j|^2}{\sum_{i=1}^n |x_i|^2 \cdot \sum_{i=1}^n |y_i|^2} \\ \iff \frac{2|x_j||y_j|}{XY} &\leq \frac{Y^2|x_j|^2 + X^2|y_j|^2}{X^2Y^2} \\ \iff \frac{|x_j||y_j|}{XY} &\leq \frac{1}{2} \left[ \frac{|x_j|^2}{X^2} + \frac{|y_j|^2}{Y^2} \right] = \frac{1}{2} \left[ \left( \frac{|x_j|}{X} \right)^2 + \left( \frac{|y_j|}{Y} \right)^2 \right] \end{aligned}$$

Taking summation from 1 to  $n$ , we have;

$$\begin{aligned} \frac{1}{XY} \sum_{j=1}^n |x_j y_j| &\leq \frac{1}{2} \sum_{j=1}^n \left[ \left( \frac{|x_j|}{X} \right)^2 + \left( \frac{|y_j|}{Y} \right)^2 \right] \\ &= \frac{1}{2} \left[ \frac{\sum_{j=1}^n |x_j|^2}{X^2} + \frac{\sum_{j=1}^n |y_j|^2}{Y^2} \right] \\ &= \frac{1}{2} \left[ \frac{X^2}{X^2} + \frac{Y^2}{Y^2} \right] = \frac{1}{2}(2) = 1 \end{aligned}$$

*Proof (cont.)* Hence,

$$\begin{aligned} \frac{1}{XY} \sum_{j=1}^n |x_j y_j| &\leq 1 \\ \iff \sum_{j=1}^n |x_j y_j| &\leq XY = \sqrt{\sum_{i=1}^n |x_i|^2} \cdot \sqrt{\sum_{i=1}^n |y_i|^2} \end{aligned}$$

And we are done.

**Proposition.** (Minkowski's inequality) For any  $x_i, y_i \in \mathbb{R}$

$$\sqrt{\sum_{i=1}^n |x_i + y_i|^2} \leq \sqrt{\sum_{i=1}^n |x_i|^2} + \sqrt{\sum_{i=1}^n |y_i|^2}$$

*Proof.* Let  $X = \sqrt{\sum_{i=1}^n |x_i|^2}$  and  $Y = \sqrt{\sum_{i=1}^n |y_i|^2}$ , as per last proposition.

Then,

$$\begin{aligned}\sum_{i=1}^n |x_i + y_i|^2 &= \sum_{i=1}^n [|x_i|^2 + 2|x_i y_i| + |y_i|^2] \\&= \sum_{i=1}^n |x_i|^2 + 2 \sum_{i=1}^n |x_i y_i| + \sum_{i=1}^n |y_i|^2 \\&\leq X^2 + 2XY + Y^2 \quad (\text{by the } \underline{\text{Cauchy-Schwarz}} \text{ inequality}) \\&= (X + Y)^2\end{aligned}$$

*Proof (cont.)* Taking square roots, we have;

$$\begin{aligned}\sqrt{\sum_{i=1}^n |x_i + y_i|^2} &\leq \sqrt{(X + Y)^2} \\&= X + Y \\&= \sqrt{\sum_{i=1}^n |x_i|^2} + \sqrt{\sum_{i=1}^n |y_i|^2}\end{aligned}$$

And we are done.

We are now ready to verify the fourth property (triangle inequality) for the Euclidean metric;

$$\begin{aligned}\rho_2(x, y) &= \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \sqrt{\sum_{i=1}^n |x_i - y_i|^2} \\&= \sqrt{\sum_{i=1}^n |(x_i - z_i) + (z_i - y_i)|^2} \\&\leq \sqrt{\sum_{i=1}^n |x_i - z_i|^2} + \sqrt{\sum_{i=1}^n |z_i - y_i|^2} \\&\quad (\text{by the } \underline{\text{Minkowski's}} \text{ inequality}) \\&= \rho_2(x, z) + \rho_2(y, z)\end{aligned}$$

And we are done.