Homework 5 (due Thursday October 31)

Quiz 5 Thursday October 31.

Main Topics:

Linear Systems: Direct Methods. Chapter 5 in textbook.

Main Objectives:

Study Gauss elimination method and some of its ramifications.

Main Tools:

Basic linear Algebra. (results and work done in HW 4)

1. Review questions
   a) What is the overall cost in flops of applying a forward or backward solve?
   b) What is the overall cost in flops of decomposing a matrix using LU?
   c) During the course of Gaussian elimination without pivoting a zero pivot has been encountered. Is this matrix singular? Give an example to justify your answer.
   d) State three disadvantages of computing the inverse of a matrix to solve a linear system rather than using the LU decomposition approach.
   e) What is the fundamental difference between error and residual in terms of computability?
   f) Suppose we compute an approximate solution $\tilde{x}$ for $Ax = b$ and get $r = b - A\tilde{x}$ whose norm is very small. Can we conclude in this case that the error $x - \tilde{x}$ must also be small?
   g) The condition number is an important concept, but it is rarely computed exactly. Why?

2. The MATLAB code given in section 5.4 (call it myLsolver) for solving linear systems of equations, using LU decomposition in outer form with partial pivoting, works well if the matrix A is nonsingular to a working precision. But if A is singular, then the exit is not graceful. Implement this code in the computer, and fix this problem by modifying the functions $ainvb$ and plu to include checks so that $ainvb + plu$ will always return as follows:
   (i) In case of a nonsingular system, return silently with the solution $\boldsymbol{x}$.
   (ii) In case of a singular system, display a message regarding the singularity and return with $\boldsymbol{x}$ assigned the value NaN.
   Assume that MATLAB will complain if a division by a number smaller than $eps = 2.22 \times 10^{-16}$ is attempted. (You want to avoid precisely this sort of complaint.)

3. Apply the modified solver (myLsolver) obtained in previous exercise to the following systems. In each case check the difference between the computed solution $\boldsymbol{x}$ and the result of MATLAB's built-in solver $A\backslash b$.

   (a)
$$\begin{aligned} x_1 + x_2 + x_4 &= 4 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ 4x_1 - x_2 - 2x_3 + 2x_4 &= 0 \\ 3x_1 - x_2 - x_3 + x_4 &= -3 \end{aligned}$$

   (b) Same as the previous system, but with the coefficient of x_4 in the last equation set to $a_{4,4} = 2$

   (c)
$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + (1 + 10^{-15})x_2 + 2x_3 &= 2 \\ x_1 + 2x_2 + 2x_3 &= 1 \end{aligned}$$

   (d) Same as the previous system, but with the second equation multiplied by 10^{20}.

(e) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

4.
   (a) Implement in your computer The Cholesky algorithm given on page 116 (with all the loops in there).
   (b) In view of Section 5.4 and the program $ainvb$ we should be able to achieve also the Cholesky decomposition effect more efficiently. Write a code implementing the Cholesky decomposition with only one loop (on k), utilizing outer products.

5. The script in example 5.13 will give different answers each time you run it, because the random number generator starts at different spots. Modify the program, making it have the option to run on the same input when call it again. (Check help *randn* to see how to do this.) Run the program for $n = 200$ and $n = 500$. What are your observations?

6. Write a MATLAB function (call it tri) that solves tridiagonal systems of equations of size n. Assume that no pivoting is needed, but do not assume that the tridiagonal matrix A is symmetric. Your program should expect as input four vectors of size n or $(n-1)$: one right-hand-side b and the three nonzero diagonals of A. It should calculate and return $x = A^{-1}b$ using a Gaussian elimination variant that requires $\mathcal{O}(n)$ flops and consumes no additional space as a function of n. (i.e., in total $5n$ storage locations are required).

7. Apply your program (tri) from previous exercise to the problem described in example 4.17 (in textbook) using the second set of boundary conditions, $v(0) = v'(1) = 0$, for $g(t) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}t\right)$ and $N = 100$. Compare the results to the vector u composed of $u(ih) = \sin\left(\frac{\pi}{2}ih\right)$, $i = 1, \cdots, N$ by recording $\|v - u\|_\infty$.

8. Let $b + \delta b$ be a perturbation of a vector $b \neq 0$, and let x and δx be such that $Ax = b$ and $A(x + \delta x) = b + \delta b$, where A is a given nonsingular matrix. Show that $\frac{\|\delta x\|}{\|x\|} = \kappa(A)\frac{\|\delta b\|}{\|b\|}$

9. Run the MATLAB script of example 5.22 in your textbook, plugging in different values for d. In particular, try $d = 0.25, d = 0.85 \ and \ d = -0.75$. What do you observe? What will happen as $d \downarrow (-1)$?