Homework 2 (due Tuesday September 10)

Quiz 2 Thursday September 12.

Main Topics:

Numerical algorithms and errors. Chapters 1 and 2 in textbook.

Main Objectives:

Classify the type of errors that occur in numerical algorithms. Discuss round off errors.

Main Tools:

Basic calculus and numbers representation.

Chapter 1 Questions

1. Review questions
   a) Give an example where relative error is a more suitable measure than absolute error, and another example where the absolute error measure is more suitable.
   b) State a major difference between the nature of roundoff errors and discretization errors.
   c) Explain the differences between accuracy, efficiency, and robustness as criteria for evaluating an algorithm.
   d) Distinguish between problem conditioning and algorithm stability.
   e) Explain briefly why accumulation of roundoff errors is inevitable when arithmetic operations are performed in a floating point system. Under which circumstances is it tolerable in numerical computations?
2. Approximate the derivative of $f(x) = e^{-3x}$ at $x = 0$. By $f'(0) \approx \frac{f(h)-f(0)}{h}$. Calculate and plot (use the loglog command) the absolute error of this approximation for values of h that approach zero. Comment and explain your results.

3. Consider the problem presented in example 1.6 in your textbook. There it was presented a numerically unstable procedure for carrying out and integration.
   a) Derive a formula for approximately computing these integrals based on evaluating y_{n-1} given y_n.
   b) Show that for any value $\varepsilon > 0$ and positive integer n_0, there exists an integer $n_1 \geq n_0$ such that taking $y_{n1} = 0$ as a starting value will produce integral evaluations y_n with an absolute error smaller than ε for all $0 < n \leq n_0$.
   c) Explain why your algorithm is stable.
   d) Write a MATLAB function that computes the value of y_{20} within an absolute error of at most 10^{-5}. Explain how you choose n_1 in this case.

Chapter 2 Questions

4. Review Questions
   a) A general floating point system is characterized by four values (β, t, L, U). Explain in a few brief sentences the meaning and importance of each of these parameters.
   b) Write down the floating point representation of a given number x in a decimal system with $t = 4$, using (i) chopping and (ii) rounding.
   c) Define rounding unit (or machine precision) and explain its importance.
   d) Define underflow and overflow. Why is the later more damaging than the former.?
   e) What is cancellation error? Give an example of an application where it arises in a natural way.

5. Write a MATLAB program that receives as input a number x and a parameter n and returns x rounded to n decimal digits. Write your program so that it can handle an array as input, returning an array of the same size in this case.

6. Show that $\ln\left(x - \sqrt{x^2 - 1}\right) = -\ln\left(x + \sqrt{x^2 - 1}\right)$. Which of the two formulas is more suitable for numerical computation? Explain why and provide a numerical example in which the difference in accuracy is evident.

7. Consider the linear system
$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad with\ a, b > 0; a \neq b.$$

   a) If $a \approx b$, what is the numerical difficulty in solving this linear system.
   b) Suggest a numerically stable formula for computing $z = x + y$ given a and b.
   c) Determine whether the following statement is true or false, and explain why:
   "When $a \approx b$, the problem of solving the linear system is ill conditioned, but the problem of computing $x + y$ is not ill conditioned."

8. Write a MATLAB program that
   a) Sums up $\frac{1}{n}$ for $n = 1,2,\cdots,10000$;
   b) Rounds each number $\frac{1}{n}$ to 5 decimal digits and then sums them up in 5-digit decimal arithmetic for $n = 1,2,\cdots,10000$;
   c) Sums up the same rounded numbers (in 5-digit decimal arithmetic) in reverse order, i.e., for $n = 10000,\cdots,2,1$

   Compare the three results and explain your observations. You can use exercise 5.