# Question 1

An exam has 2 sections (section A and section B), each with 9 questions, but only 12 questions (in total) needs to be answered. If a student randomly chooses 12 questions to answer, then what is the probability, to 2 decimal places, that:
(a) 6 of the questions come from one section, and 6 from the other?
(b) 5 of the questions come from one section, and 7 from the other?
(Hint: for (b), consider all possible cases.)

(a) Let $E_1$ denote the event that the student chooses 6 questions from each of the section.

$$\begin{aligned}
\mathbb{P}(E_1) &= \mathbb{P}\big((A=6)\cap(B=6)\big) \\
&= \frac{n(\text{ways to choose 6 from A}) \times n(\text{ways to choose 6 from B})}{n(\text{ways to choose 12 from 18 questions})} \\
&= \frac{\binom{9}{6}\times\binom{9}{6}}{\binom{18}{12}} \\
\Aboxed{\mathbb{P}(E_1) &= \frac{84}{221} = 0.38}
\end{aligned}$$

(b) Let $E_2$ denote the event that the student chooses 5 questions from one section, and 7 from the other. In this case, the student can choose 5 questions from A and 7 from B, or the student can also choose 7 questions from A and another 5 from B. $E_2$ comprises both of these events.

$$\begin{aligned}
\mathbb{P}(E_2) &= \mathbb{P}\Big(\big((A=5)\cap(B=7)\big)\cup\big((A=7)\cap(B=5)\big)\Big) \\
&= \mathbb{P}\big((A=5)\cap(B=7)\big) + \mathbb{P}\big((A=7)\cap(B=5)\big) \\
&= \frac{\binom{9}{5}\times\binom{9}{7}}{\binom{18}{12}} + \frac{\binom{9}{7}\times\binom{9}{5}}{\binom{18}{12}} \\
\Aboxed{\mathbb{P}(E_2) &= \frac{108}{221} = 0.49}
\end{aligned}$$

# Question 2

Suppose that 6 hats are randomly returned to 6 men, so that each man gets 1 hat.
(a) What is the probability that at least 1 man receives his own hat?
(b) What is the probability that at least 2 men receive their own hats?
(c) What is the probability that at least 5 men receive their own hats?
(Hint: derangement, with some more enumeration.)

(a) Let $X$ denote the number of men who receives his own hat. The probability that at least 1 man receives his own hat is denoted by $\mathbb{P}\big(X \geq 1\big)$.

$$\mathbb{P}\big(X \geq 1\big) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6)$$

Since $\Sigma_{x=0}^{6}\mathbb{P}(X = x) = 1$ by the axioms of probability, we find that $\mathbb{P}\big(X \geq 1\big) = 1 - \mathbb{P}(X = 0)$, with $\mathbb{P}(X = 0)$ being none other than the probability of a derangement taking place for 6 objects. In other words:

$$\begin{aligned}\mathbb{P}\big(X \geq 1\big) &= 1 - \frac{D_6}{6!} \\ &= 1 - \frac{265}{720} \\ \boxed{\mathbb{P}(X \geq 1) = \frac{91}{144} = 0.632}\end{aligned}$$

(b) The probability that at least 2 men receive their own hats is denoted by $\mathbb{P}\big(X \geq 2\big)$. Similarly as in the previous case, we can use the fact that $\Sigma_{x=0}^{6}\mathbb{P}(X = x) = 1$ to get the following for $\mathbb{P}\big(X \geq 2\big)$.

$$\begin{aligned}\mathbb{P}\big(X \geq 2\big) &= \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) \\ &= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \\ &= 1 - \frac{D_6}{6!} - \frac{n(\text{1 man gets his hat}) \cdot n(\text{5 men get the wrong hat})}{n(\text{6 men get 6 hats})} \\ &= 1 - \frac{265}{720} - \frac{\binom{6}{1}D_5}{6!} \\ &= 1 - \frac{265}{720} - \frac{6 \cdot 44}{720} \\ \boxed{\mathbb{P}\big(X \geq 2\big) = \frac{191}{720} = 0.265}\end{aligned}$$

(c) To solve this problem, let us introduce another variable $Y$, which represents the number of men with the wrong hat. The probability that at least 5 men receive their own hats can be denoted by $\mathbb{P}(X \geq 5) = \mathbb{P}(Y \leq 1)$.

$$\begin{aligned}\mathbb{P}(X \geq 5) &= \mathbb{P}(Y \leq 1) = \mathbb{P}(Y = 1) + \mathbb{P}(Y = 0) \\ &= \frac{n(\text{1 man gets the wrong hat})}{n(\text{6 men get 6 hats})} + \frac{n(\text{0 man gets the wrong hat})}{n(\text{6 men get 6 hats})} \\ &= 0 + \frac{\binom{6}{0}}{6!} \\ \boxed{\mathbb{P}(X \geq 5) = \frac{1}{720} = 1.39 \times 10^{-3}}\end{aligned}$$

$\mathbb{P}(Y = 1) = 0$ because when a man (say Jacob) receives the wrong hat, another man whose hat was taken by Jacob would never receive his own hat, thus inevitably having to take a wrong hat too. So in this scenario there is no way for just 1 man to receive a wrong hat.

# Question 3

An experiment involves tossing a fair coin until a T appears for the first time, or until $n$ tosses are completed, whichever happens first. Let $X$ be the number of coin tosses upon the completion of the experiment. For the following parts, please simplify your answers as much as possible.
(a) Find the probability mass function of $X$.
(b) Compute $\mathbb{E}(X)$.
(Hints: for (a), note that $X$ can only be $1, 2, 3, ..., n$, where $n$ is a fixed integer; also, $\mathbb{P}(X = n)$ looks different from the other terms. For (b), you may need to differentiate a finite geometric series – you may of course look up the geometric series formula; there is also a similar example in the slides.)

(a) We can calculate several variations of $\mathbb{P}(X = x)$ for several values of $x$ to determine its PMF. Let $R_i$ and $p$ respectively denote the $i$-th round of the coin toss and the probability of a tail appearing in a single coin toss. For $X = 1, 2, 3,$ and 4:

$$\begin{aligned}
&\mathbb{P}(X = 1) = \mathbb{P}(R_1 = T) = p \\
&\mathbb{P}(X = 2) = \mathbb{P}(R_1 = H) \cdot \mathbb{P}(R_2 = T) = (1 - p)(p) \\
&\mathbb{P}(X = 3) = \mathbb{P}(R_1 = H) \cdot \mathbb{P}(R_2 = H) \cdot \mathbb{P}(R_3 = T) = (1 - p)^2(p) \\
&\mathbb{P}(X = 4) = \mathbb{P}(R_1 = H) \cdot \mathbb{P}(R_2 = H) \cdot \mathbb{P}(R_3 = H) \cdot \mathbb{P}(R_4 = T) = (1 - p)^3(p)
\end{aligned}$$

We can conclude that $X \sim$ geometric($p$). However, this distribution does not apply for $X = n$ since, after the $n$-th toss, the experiment ends regardless whether the coin flip results in a T or H. Thus $\mathbb{P}(X = n) = (1 - p)^{x-1}((p) + (1 - p)) = (1 - p)^{x-1}$ and we derive the following PMF for $X$.

$$f(x) = \left\{ \begin{array}{lr} (1-p)^{x-1}(p), & \text{if } x \in \{1, 2, ..., n-1\} \\ (1-p)^{x-1}, & \text{if } x = n \\ 0, & \text{otherwise} \end{array} \right\}$$

Given $p = 0.5$, we fully derive the PMF of X.

$$f(x) = \left\{ \begin{array}{lr} (0.5)^{x}, & \text{if } x \in \{1, 2, ..., n-1\} \\ (0.5)^{x-1}, & \text{if } x = n \\ 0, & \text{otherwise} \end{array} \right\}$$

(b) To calculate $\mathbb{E}(X)$, we simply use the fact that $\mathbb{E}(X) = \sum_{i=1}^{n} x_i f(x_i)$.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{x=1}^{n-1} \Big( x(0.5)^x \Big) + (n)(0.5)^{n-1} \\
&= 0.5 \times \sum_{x=1}^{n-1} \Big( x(0.5)^{x-1} \Big) + (n)(0.5)^{n-1}
\end{aligned}$$

Since $\sum_{x=1}^{n-1}(x)(r)^{x-1} = \frac{d}{dr}\left[\sum_{x=0}^{n-1}(r)^x\right] = \frac{d}{dr}\left[\frac{r^n - 1}{r-1}\right]$, and with $r = 0.5$, we get:

$$\begin{aligned}
\mathbb{E}(X) &= 0.5 \times \frac{d}{dr}\left[\frac{r^n-1}{r-1}\right] + (n)(0.5)^{n-1} \\
&= 0.5\times\left[\frac{nr^{n-1}}{r-1} - \frac{r^n-1}{(r-1)^2}\right] + (n)(0.5)^{n-1} \\
&= 0.5\times\left[\frac{n(0.5)^{n-1}}{-0.5} - \frac{(0.5)^n-1}{(-0.5)^2}\right] + (n)(0.5)^{n-1} \\
&= -(n)(0.5)^{n-1} - (0.5)^{n-1} + (0.5)^{-1} + (n)(0.5)^{n-1}
\end{aligned}$$

$$\boxed{\mathbb{E}(X) = 2 - (2)^{1-n}}$$

# Question 4

The weekly production of a factory has mean 50 tons and standard deviation 5 tons. With help from Chebyshev's inequality, give a lower bound for the probability that the weekly production is between 41 and 61 tons (inclusive).
(Hints: a picture would help, and you would need to manipulate some inequalities; a correct bound will not be 'sharp' or 'tight'.)

We want to calculate $\mathbb{P}(41 \leq X \leq 61)$ by using Chebyshev's inequality:

$$\mathbb{P}(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

We can utilize $\mathbb{P}(X \leq x) = 1 - \mathbb{P}(X \geq x)$ to get the following form of Chebyshev's inequality.

$$\begin{aligned}
1 - \mathbb{P}(|Y-\mu| \leq k\sigma) &= \mathbb{P}(|Y-\mu| \geq k\sigma) \\
1 - \mathbb{P}(|Y-\mu| \leq k\sigma) &\leq \frac{1}{k^2} \\
\mathbb{P}(|Y-\mu| \leq k\sigma) &\geq 1 - \frac{1}{k^2}
\end{aligned}$$

We are looking for the *lower* bound for the probability that the weekly production is between 41 and 61 tons. We could either select (i) $k\sigma = |50 - 41| = 9$ or (ii) $k\sigma = |50 - 61| = 11$ to insert into the inequality. We choose to select (i) because it generates the lower bound. By letting $Y$ as the weekly production of the factory (in tons), $\mu = 50$ (given), and $\sigma = 5$ (given), we get:

$$\mathbb{P}(|Y - 50| \leq 9) \geq 1 - \frac{1}{(9/5)^2}$$

$$\boxed{\mathbb{P}(|Y - 50| \leq 9) \geq \frac{56}{81}}$$

In fact, we can try calculating for (ii) too.

$$\mathbb{P}(|Y-50| \leq 11) \geq 1 - \frac{1}{(11/5)^2}$$
$$\mathbb{P}(|Y-50| \leq 11) \geq \frac{96}{121}$$

This, however, gives us the upper bound rather than the lower bound. In other words:

$$\text{upper bound: } \mathbb{P}(41 \leq X \leq 61) \geq \mathbb{P}(|Y-50| \leq 11)$$
$$\text{lower bound: } \mathbb{P}(41 \leq X \leq 61) \leq \mathbb{P}(|Y-50| \leq 9)$$

Thus the lower bound of $\mathbb{P}(41 \leq X \leq 61)$:

$$\boxed{\mathbb{P}(41 \leq X \leq 61) \geq \frac{56}{81}}$$

# Question 5

Let $X \sim \text{negbin}(n,p)$. $X = Y_1 + Y_2 + ... + Y_n$ where $Y_i \sim \text{geometric}(p)$. Since each $Y_i$ is independent of each other, we can use the MGF of a geometric RV (as found during Week 2 Cohort 2 Exercise (b)) and Theorem 2 of Moment Generating Functions to get the following:

$$\begin{aligned} M_{Y_i}(t) &= \frac{pe^t}{1-e^t(1-p)} \\ M_X(t) &= M_{Y_1}(t) + M_{Y_2}(t) + ... + M_{Y_n}(t) \\ &= (M_{Y_1}(t))^n \\ &= \left(\frac{pe^t}{1-e^t(1-p)}\right)^n \end{aligned}$$

Using this moment generating function, we will calculate $\mathbb{E}(X)$.

$$\mathbb{E}(X^n) = M_X^{(n)}(t)$$
$$\mathbb{E}(X) = M_X^{'}(0)$$

$$\begin{aligned} \mathbb{E}(X^{'}) &= n \cdot \left(\frac{pe^t}{1-e^t(1-p)}\right)^{n-1} \left(\frac{pe^t}{1-e^t(1-p)} + \frac{-pe^t}{(1-e^t(1-p))^2}(p-1)(e^t)\right) \\ &= n \cdot \left(\frac{pe^0}{1-e^0(1-p)}\right)^{n-1} \left(\frac{pe^0}{1-e^0(1-p)} + \frac{-pe^0}{(1-e^0(1-p))^2}(p-1)(e^0)\right) \\ &= n\left(\frac{p}{p}\right)^{n-1}\left(\frac{p}{p} + \frac{1-p}{p}\right) \\ &\boxed{\mathbb{E}(X) = n/p} \end{aligned}$$

# Question 6

Let X be a continuous random variable with probability density function

$$f(x) = \frac{1}{2}e^{-|x|}, \text{for all } x \in \mathbb{R}$$

(a) Find the moment generating function of $X$.
(b) Using the MGF, compute $\text{Var}(X)$.
(Hints: the MGF here is only defined on $-1 < t < 1$; due to the absolute value sign, you would need to split an integral into two parts.)

(a) First, let us split $f(x)$ into a piecewise function.

$$f(x) = \begin{Bmatrix} \frac{1}{2}e^{-x}, & \text{if } x \geq 0 \\ \\ \frac{1}{2}e^{x}, & \text{if } x < 0 \end{Bmatrix}$$

Using this new definition of $f(x)$, we can conveniently calculate $M_X(t)$ as such:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x)dx \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{tx}e^{x}dx + \frac{1}{2}\int_{0}^{\infty} e^{tx}e^{-x}dx \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{x(t+1)}dx + \frac{1}{2}\int_{0}^{\infty} e^{x(t-1)}dx \\ &= \frac{1}{2}\left[\frac{e^{x(t+1)}}{t+1}\right]_{x=-\infty}^{x=0} + \frac{1}{2}\left[\frac{e^{x(t-1)}}{t-1}\right]_{x=0}^{x=\infty} \end{aligned}$$

For $-1 < t < 1$, we have $e^{x(t-1)} = 0$ for $x = \infty$. Thus:

$$\begin{aligned} M_X(t) &= \frac{1}{2}\left[\frac{1-0}{t+1}\right] + \frac{1}{2}\left[\frac{0-1}{t-1}\right] \\ &= \frac{1}{2}\left(\frac{(t-1)-(t+1)}{(t+1)(t-1)}\right) \\ &= \frac{1}{2}\left(\frac{-2}{t^2-1}\right) \end{aligned}$$

$$\boxed{M_X(t) = \frac{1}{1-t^2}}$$

(b) Since $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$, we can use $M_X(t)$ to calculate $\mathbb{E}(X^2)$ and $\mathbb{E}(X)$ and hence $\text{Var}(X)$. First, we begin by calculating $\mathbb{E}(X)$.

$$\begin{aligned}
\mathbb{E}(X^n) &= M_X^{(n)}(t) \\
\mathbb{E}(X) &= M_X'(0) \\
&= \frac{d}{dt}\left(1-t^2\right)^{-1}\Bigg|_{t=0} \\
&= \frac{2t}{(1-t^2)^2}\Bigg|_{t=0} \\
&= \frac{2(0)}{(1-(0)^2)^2} \\
\Aboxed{\mathbb{E}(X) &= 0}
\end{aligned}$$

Now to calculate $\mathbb{E}(X^2)$:

$$\begin{aligned}
\mathbb{E}(X^2) &= M_X''(0) \\
&= \frac{d}{dt}M_X'(t)\Bigg|_{t=0} \\
&= \left(\frac{2}{(1-t^2)^2} + \frac{8t^2}{(1-t)^3}\right)\Bigg|_{t=0} \\
&= \frac{2}{1^2} + 0 \\
\Aboxed{\mathbb{E}(X^2) &= 2}
\end{aligned}$$

Hence the variance of $X$.

$$\begin{aligned}
\text{Var}(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
&= 2 - 0^2 \\
\Aboxed{\text{Var}(X) &= 2}
\end{aligned}$$

# Question 7

The aim here is to give an upper bound for the tail probability of a standard normal random variable $Z$. Firstly, observe that for any $t > 0$, we have

$$\mathbb{P}(Z \geq t) = \mathbb{P}(tZ \geq t^2) = \mathbb{P}(e^{tZ} \geq e^{t^2})$$

(a) Apply Markov's inequality to the above equation, then show that

$$\mathbb{P}(Z \geq t) \leq e^{-t^2/2}$$

(b) Hence, give an upper bound for $\mathbb{P}(Z \leq -7)$ (without using a table or software).
(Hint: for (a), an MGF is involved.)

(a) By Markhov's inequality:

$$\mathbb{P}(e^{tz} \geq e^{t^2}) \leq \frac{\mathbb{E}(e^{tz})}{e^{t^2}}$$

By the definition of the moment generating function, $\mathbb{E}(e^{tz}) = M_Z(t)$. Conveniently, $M_Z(t) = e^{t^2/2}$ as derived in Week 2 Cohort 2 Example 3 and thus:

$$\mathbb{P}(e^{tz} \geq e^{t^2}) \leq \frac{M_Z(t)}{e^{t^2}}$$
$$\mathbb{P}(e^{tz} \geq e^{t^2}) \leq \frac{e^{t^2/2}}{e^{t^2}}$$
$$\mathbb{P}(e^{tz} \geq e^{t^2}) \leq e^{-t^2/2}$$

Since $\mathbb{P}(Z \geq t) = \mathbb{P}(e^{tZ} \geq e^{t2})$:

$$\boxed{\mathbb{P}(Z \geq t) \leq e^{-t^2/2}}$$

(b) The symmetrical nature of the standard normal variable means that $\mathbb{P}(Z \geq t) = \mathbb{P}(Z \leq -t)$. Thus, we can rearrange the inequality found in (a) to find the upper bound of $\mathbb{P}(Z \leq -7)$.

$$\mathbb{P}(Z \leq -t) \leq e^{-t^2/2}$$
$$\mathbb{P}(Z \leq -7) \leq e^{-(-7)^2/2}$$
$$\boxed{\mathbb{P}(Z \leq -7) \leq e^{-49/2}}$$