Question 1

Consider a Poisson process with rate λ. Given that there was exactly 1 arrival in the time interval $[0, t]$, what is the probability that the arrival occurred in the interval $[0, s]$, where $0 < s < t$?

(Hints: Apply the definition of conditional probability. Draw a timeline and break it up into 2 disjoint intervals, then use independence. The answer should be a very simple fraction involving s and t.)

Answer: Given that there was exactly 1 arrival in $[0, t]$, the probability that exactly 1 arrival occurred in $[0, s]$ where $0 < s < t$ can be represented as:

$$\begin{aligned}
\mathbb{P}(1 \text{ arrival in } [0,s] \mid 1 \text{ arrival in } [0,t]) &= \frac{\mathbb{P}(1 \text{ arrival in } [0,s] \cap 1 \text{ arrival in } [0,t])}{\mathbb{P}(1 \text{ arrival in } [0,t])} \\
&= \frac{\mathbb{P}(1 \text{ arrival in } [0,s] \cap 0 \text{ arrival in } [s,t])}{\mathbb{P}(1 \text{ arrival in } [0,t])} \\
&= \frac{\mathbb{P}(1 \text{ arrival in } [0,s]) \times \mathbb{P}(0 \text{ arrival in } [s,t])}{\mathbb{P}(1 \text{ arrival in } [0,t])} \\
&= \frac{\mathbb{P}(N(s)=1) \times \mathbb{P}(N(t-s)=0)}{\mathbb{P}(N(t)=1)} \\
&= \frac{\left(\frac{e^{-\lambda(s)}(\lambda s)^1}{1!}\right)\left(\frac{e^{-\lambda(t-s)}(\lambda(t-s))^0}{0!}\right)}{\left(\frac{e^{-\lambda t}(\lambda t)^1}{1!}\right)} \\
&= \frac{e^0(\lambda s)(\lambda(t-s))^0}{(\lambda t)}
\end{aligned}$$

$$\boxed{\mathbb{P}(1 \text{ arrival in } [0,s] \mid 1 \text{ arrival in } [0,t]) = \frac{s}{t}}$$

Question 2

Consider two independent Poisson processes with rates λ_1 and λ_2 respectively, and denote their first arrival times by Y_1 and Y_2 respectively. Suppose that we merge the two processes.
(a) What is the distribution of the arrival time of the second event in the merged process?
(b) Recall that the arrival time of the first event in the merged process is $\min(Y_1, Y_2)$. Is it true that the arrival time of the second event in the merged process always equals $\max(Y_1, Y_2)$?

(Hint: for (a), use the general theory, instead of calculations; be specific with any parameters. For (b), justify your answer.)

(a) **Answer**: Merging the two independent Poisson processes results in another Poisson process (denoted by P) with parameter $(\lambda_1 + \lambda_2)$. The interarrival times in this Poisson Process is exponentially distributed as follows:

$$T_i \sim \text{exponential}(\lambda_1 + \lambda_2)$$

where T_i denotes the interarrival time of the i-th arrival of P. Intuitively, the arrival time of the second event, T, is the sum of the 1st and 2nd interrarival times. Thus:

$$T = T_1 + T_2$$

Since T is a sum of two independent identical exponential distributions of parameter $\lambda_1 + \lambda_2$, it is clear that T has a gamma distribution with parameters $\alpha = 2$ and $\lambda = \lambda_1 + \lambda_2$. Thus the distribution of the second arrival time.

$$\boxed{T \sim \text{gamma}(2, \lambda_1 + \lambda_2)}$$

(b) **Answer**: This is an erroneous conclusion. The arrival time of the second event is given by $T = T_1 + T_2 \neq \max(Y_1, Y_2)$. In short, the second arrival time in the merged process can occur before or after $\max(Y_1, Y_2)$. Consider a scenario where Y_1 is very large compared to Y_2. In this case, it is highly possible for T_2 to occur before $\max(Y_1, Y_2) = Y_1$. Thus, the conclusion provided is false.

Question 3

(A variation on the German tank problem.) Suppose that the enemy has tanks numbered $0, 1, 2, ..., N$, where N is fixed but unknown to you. You observe n of the tanks at random with replacement, and note down their numbers: $X_1, X_2, ..., X_n$. Using the sample mean of the X_i 's, find an unbiased estimator for the total number of tanks.

(Hint: carefully compute $\mathbb{E}(X_i)$ and $\mathbb{E}(\bar{X}_n)$, then relate the answer to the total number of tanks.)

Answer: To begin, let us compute $\mathbb{E}(X_i)$, which is literally the expectation of the serial number of a single randomly selected German tank. Assuming every tank has an equal probability of getting selected:

$$\begin{aligned}
\mathbb{E}(X_i) &= \sum_{i=0}^{N} X_i \cdot \mathbb{P}(X = i) \\
&= \frac{0}{N+1} + \frac{1}{N+1} + ... + \frac{N}{N+1} \quad \text{(since } \mathbb{P}(X = i) = \frac{1}{N+1} \text{ for every } i\text{)} \\
&= \frac{1}{N+1}\left(\frac{N(N+1)}{2}\right) \quad \text{(by the formula for the sum of arithmetic progression)} \\
&= \frac{N}{2}
\end{aligned}$$

Next, let us compute $\mathbb{E}(\bar{X}_n)$.

$$\begin{aligned}
\mathbb{E}(\bar{X}_n) &= \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{(since } \bar{X}_n \text{ is the average of all } X_i\text{)} \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} \mathbb{E}(X_i)\right) \\
&= \frac{1}{n}\left(n \cdot \frac{N}{2}\right) \\
&= \frac{N}{2}
\end{aligned}$$

We know that there are $N+1$ tanks in total. Thus, we manipulate the right hand side so that it equals $N+1$:

$$\begin{aligned}
\mathbb{E}(\bar{X}_n) &= \frac{N}{2} \\
2 \cdot \mathbb{E}(\bar{X}_n) + 1 &= N + 1 \\
\mathbb{E}(2\bar{X}_n + 1) &= N + 1
\end{aligned}$$

An unbiased estimator for $N+1$ fulfils $\mathbb{E}(\hat{N}+1) - (N+1) = 0$. Thus:

$$\begin{aligned}
\mathbb{E}(\hat{N}+1) - (N+1) &= 0 \\
\mathbb{E}(\hat{N}+1) - \mathbb{E}(2\bar{X}_n + 1) &= 0 \\
\mathbb{E}(\hat{N}+1) &= \mathbb{E}(2\bar{X}_n + 1) \\
\boxed{\hat{N} + 1 = 2\bar{X}_n + 1}
\end{aligned}$$

This is an unbiased estimator for the total number of tanks using the sample mean.

Question 4

(A continuous version of the German tank problem.) Let $X_1, X_2, ..., X_n$ be an i.i.d. random sample drawn from a uniform$(0, \theta)$ distribution. θ is fixed but unknown to you, and the goal here is to estimate θ using $X_{\max}$ (the maximum of the X_i's).
(a) Compute $\mathbb{P}(X_{\max} < x)$, where x is between 0 and θ.
(b) Part (a) gives the cdf of $X_{\max}$; now find the pdf, and use the pdf to find $\mathbb{E}(X_{\max})$.
(c) Use the result of part (b) to construct an unbiased estimator for θ.

(Hints: for (a), if the maximum of the X_i's is $< x$, then each X_i is $< x$; use independence. For (b), differentiation and integration are involved. For (c), use the linearity of expectation.)

(a) **Answer**: The condition $X_{\max} < x$ necessitates that $X_i < x$ for every $i = 1, 2, ..., n$.

Thus:

$$\begin{aligned}
\mathbb{P}(X_{\max} < x) &= \mathbb{P}\Big((X_1 < x) \cap (X_2 < x) \cap ... \cap (X_n < x)\Big) \\
&= \mathbb{P}(X_1 < x) \cdot \mathbb{P}(X_2 < x) \cdot ... \cdot \mathbb{P}(X_n < x) \quad \text{(by independence)} \\
&= \Big(\mathbb{P}(X_1 < x)\Big)^n \quad \text{(since each } X_i \text{ are identical independent random samples)} \\
&= \left(\int_0^x f_{X_1}(x')dx'\right)^n = \left(\int_0^x \frac{1}{\theta}dx'\right)^n = \left[\frac{x'}{\theta}\right]_0^x
\end{aligned}$$

$$\boxed{\mathbb{P}(X_{\max} < x) = \left(\frac{x}{\theta}\right)^n}$$

(b) **Answer**: To find the pdf of $X_{\max}$, we can simply differentiate the cdf with respect to x.

$$\begin{aligned}
f_{X_{\max}}(x) &= \frac{d}{dx}\left(\frac{x}{\theta}\right)^n \\
&= \frac{nx^{n-1}}{\theta^n}
\end{aligned}$$

We can use this to find $\mathbb{E}(X_{\max})$.

$$\begin{aligned}
\mathbb{E}(X_{\max}) &= \int_{-\infty}^{\infty} x f_{X_{\max}}(x)dx \\
&= \int_0^\theta \frac{(x)(nx^{n-1})}{\theta^n}dx \\
&= \int_0^\theta \frac{nx^n}{\theta^n}dx \\
&= \left[\left(\frac{n}{n+1}\right)\left(\frac{x^{n+1}}{\theta^n}\right)\right]_0^\theta
\end{aligned}$$

$$\boxed{\mathbb{E}(X_{\max}) = \frac{n\theta}{n+1}}$$

(c) **Answer**: To find an unbiased estimator for θ, we can reformulate the equation found in (b) such that θ is explicitly defined.

$$\begin{aligned}
\mathbb{E}(X_{\max}) &= \frac{n\theta}{n+1} \\
\left(\frac{n+1}{n}\right)\mathbb{E}(X_{\max}) &= \theta \\
\mathbb{E}\left(\left(\frac{n+1}{n}\right)X_{\max}\right) &= \theta \quad \text{(by the linearity of expectation)}
\end{aligned}$$

An unbiased estimator is one that fulfils $\mathbb{E}(\hat{\theta}) - \theta = 0$. Thus:

$$\begin{aligned}
\mathbb{E}(\hat{\theta}) - \theta &= 0 \\
\mathbb{E}(\hat{\theta}) - \mathbb{E}\left(\left(\frac{n+1}{n}\right)X_{\max}\right) &= 0 \\
\mathbb{E}(\hat{\theta}) &= \mathbb{E}\left(\left(\frac{n+1}{n}\right)X_{\max}\right) \\
&\boxed{\hat{\theta} = \left(\frac{n+1}{n}\right)X_{\max}}
\end{aligned}$$

Question 5

Let $X \sim \text{uniform}(\theta_1, \theta_2)$. Use the method of moments to estimate θ_1 and θ_2.

(Hint: $\mathbb{E}(X)$ is simple, and the formula for $\text{Var}(X)$ can be found in the slides or computed by hand.)

Answer: Recall the expectation and variance of a uniform(θ_1, θ_2) distribution: $\mathbb{E}(X) = (\theta_1 + \theta_2)/2$ and $\text{Var}(X) = (\theta_2 - \theta_1)^2/12$. By the method of moments:

$$\mathbb{E}(X) = \frac{\theta_1 + \theta_2}{2} = \bar{x} \qquad\qquad \text{Var}(X) = \frac{(\theta_2 - \theta_1)^2}{12} = s_x^2$$

This gives $\theta_2 = 2\bar{x} - \theta_1$. Substituting θ_2 into the equation of variance gives:

$$\begin{aligned}
s_x^2 &= \frac{(2\bar{x} - \theta_1 - \theta_1)^2}{12} \\
&= \frac{(\bar{x} - \theta_1)^2}{3} \\
3s_x^2 &= (\bar{x} - \theta_1)^2 \\
\sqrt{3}s_x &= \bar{x} - \theta_1 \\
&\boxed{\hat{\theta}_1 = \bar{x} - \sqrt{3}s_x}
\end{aligned}$$

Substituting θ_1 into $\theta_2 = 2\bar{x} - \theta_1$ gives:

$$\begin{aligned}
\theta_2 &= 2\bar{x} - (\bar{x} - \sqrt{3}s_x) \\
&\boxed{\hat{\theta}_2 = \bar{x} + \sqrt{3}s_x}
\end{aligned}$$

Hence the estimators for θ_1 and θ_2 by the method of moments.

Question 6

Suppose that the waiting time (in minutes) in a queue is exponentially distributed with an unknown parameter θ, and that from a random sample of 4 waiting times, you observe that

$x_1 = 1.3$, $x_2 = 0.6$, $x_3 = 0.3$, and $x_4 = 0.8$.
(a) Find the likelihood function.
(b) If your prior for θ has a gamma$(5, 2)$ distribution, then find the posterior distribution, up to a constant.
(c) Show that the posterior in (b) is also gamma distributed, and find its mean.

(Hint: recall from Week 3 that the gamma pdf is $f(\theta) = C\theta^{\alpha-1}e^{-\lambda\theta}$.)

(a) **Answer**: To begin, let X denote the queue waiting time in minutes. As given in the problem, $X \sim$ exponential(θ) with $f_X(\theta) = \lambda e^{-\lambda\theta}$ for $\theta \geq 0$ and $f_X(\theta) = 0$ elsewhere. To compute the likelihood function, recall that $L(\theta) = f(x_1|\theta)f(x_2|\theta)...f(x_n|\theta)$. Given $x_1 = 1.3$, $x_2 = 0.6$, $x_3 = 0.3$, and $x_4 = 0.8$, we have:

$$\begin{aligned} L(\theta) &= (\theta e^{-1.3\theta})(\theta e^{-0.6\theta})(\theta e^{-0.3\theta})(\theta e^{-0.8\theta}) \\ &= \theta^4 e^{-\theta(1.3+0.6+0.3+0.8)} \end{aligned}$$

$$\boxed{L(\theta) = \theta^4 e^{-3\theta}}$$

(b) **Answer**: Given a prior distribution $g(\theta)$ and likelihood function $L(\theta) = f(x_1, ..., x_n)$, the resulting posterior distribution $h(\theta|x_1, ..., x_n)$ can be computed using the following equation:

$$h(\theta|x_1, ..., x_n) = C_1 L(\theta)g(\theta) \quad \text{(where } C_1 \text{ is a constant)}$$

Since the prior for θ has a gamma$(5, 2)$ distribution, we can equate $g(\theta)$ to the pdf of a gamma distribution.

$$\begin{aligned} g(\theta) &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta} \quad \text{(with } \alpha = 5 \text{ and } \lambda = 2\text{).} \\ &= \frac{2^5}{\Gamma(5)}\theta^{5-1}e^{-2\theta} \\ &= \frac{4}{3}\theta^4 e^{-2\theta} \end{aligned}$$

Using this, we get the posterior distribution for θ:

$$h(\theta|x_1, ..., x_n) = C_1(\theta^4 e^{-3\theta})((4/3)\theta^4 e^{-2\theta})$$

$$\boxed{h(\theta|x_1, ..., x_n) = C_2\ \theta^8\ e^{-5\theta}} \quad \text{(where } C_2 \text{ is a constant)}$$

(c) **Answer**: As suggested in the hint, the pdf of a gamma(α, λ) distribution is given by $f(\theta) = C\ \theta^{\alpha-1}e^{-\lambda\theta}$. Thus, it can be seen obviously that the posterior is a gamma distribution with parameters $\alpha = 9$ and $\lambda = 5$.

$$\begin{aligned} f_{\text{gamma}(\alpha,\lambda)}(\theta) &= C\theta^{\alpha-1}e^{-\lambda\theta} \\ f_{\text{gamma}(9,5)}(\theta) &= C\theta^{9-1}e^{-5\theta} \\ &= C\theta^8 e^{-5\theta} \end{aligned}$$

$$\boxed{h(\theta|x_1, ..., x_n) = f_{\text{gamma}(9,5)}} \quad (\therefore \text{ The posterior is gamma distributed.})$$

The mean of a gamma(α, λ) distribution is given by α/λ. Thus:

$$\mathbb{E}(h(\theta|x_1, ..., x_n)) = \alpha/\lambda$$
$$\boxed{\mathbb{E}(h(\theta|x_1, ..., x_n)) = 9/5}$$

Question 7

Using integration, prove that the expectation of a beta(a, b) random variable is $a/(a + b)$. (Hints: use the fact that the total area under any pdf is 1; also, recall that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$

Answer: Recall the pdf of a beta distribution $X \sim$ beta(a, b):

$$f(x) = \left\{ \begin{array}{lr} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, & \text{if } 0 \leq x \leq 1 \\ \\ 0, & \text{otherwise} \end{array} \right\}$$

Using this pdf, we can compute the expectation of a beta random variable.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} dx$$

Since $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, we can use $\Gamma(a) = \Gamma(a + 1)/a$ and $\Gamma(a + b) = \Gamma(a + b + 1)/(a + b)$ to get:

$$\begin{aligned} \mathbb{E}(X) &= \int_0^1 \frac{\Gamma(a+b+1)/(a+b)}{(\Gamma(a+1)/a)\Gamma(b)} x^{(a+1)-1}(1-x)^{b-1} dx \\ &= \frac{a}{a+b} \int_0^1 \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b)} x^{(a+1)-1}(1-x)^{b-1} dx \\ &= \frac{a}{a+b} \int_0^1 f_B(x) dx \quad \text{(where B} \sim \text{beta } (a+1, b)) \end{aligned}$$

As suggested by the hint, the total area under any pdf is 1. Thus, $\int_0^1 f_B(x)dx = 1$ as the integral covers the whole area under the pdf of B. Note that the pdf of B is only defined for $0 \leq x \leq 1$; elsewhere, the pdf is 0 and this explains the why integration lower and upper bounds are 0 and 1 respectively. Therefore:

$$\boxed{\mathbb{E}(X) = \frac{a}{a+b}}$$