Question 1

Let $\{X_n, n \geq 0\}$ be a DTMC with state space $\{1,2,3,4,5\}$ and transition probability matrix

$$P = \begin{bmatrix} 0.1 & 0.0 & 0.2 & 0.3 & 0.4 \\ 0.0 & 0.6 & 0.0 & 0.4 & 0.0 \\ 0.2 & 0.0 & 0.0 & 0.4 & 0.4 \\ 0.0 & 0.4 & 0.0 & 0.5 & 0.1 \\ 0.6 & 0.0 & 0.3 & 0.1 & 0.0 \end{bmatrix}$$

Suppose the initial distribution is $a = (0.5 \quad 0 \quad 0 \quad 0 \quad 0.5)$. Compute the following:
(a) $P\{X_2 = j\}$ for $j = 1,2,3,4,5$
(b) $P\{X_2 = 2, X_4 = 5\}$
(c) $P\{X_7 = 3|X_3 = 4\}$

....

(a)

Given initial distribution a, after k steps from time slot n to $n+k$, the final PMF of all states are summarized in row vector $[a \times P^k]_j$.

$$\mathbb{P}\{X_{n+k} = j\} = [a \times P^k]_j$$

By performing the matrix multiplication $a \times P^k$ for $k = 2$, we get:

$$a \times P^2 = [0.205 \quad 0.08 \quad 0.13 \quad 0.325 \quad 0.26]$$

With this, we get $\mathbb{P}\{X_2 = j\}$ for $j = 1,2,3,4,5$.

j	1	2	3	4	5
$\mathbb{P}\{X_2 = j\}$	0.205	0.08	0.13	0.325	0.26

(b)

We observe that

$$\begin{aligned} \mathbb{P}\{X_2 = 2, X_4 = 5\} &= \mathbb{P}\{X_4 = 5|X_2 = 2\} \cdot \mathbb{P}\{X_2 = 2\} \\ &= [P^2]_{2,5} \cdot (0.08) \\ &= 0.04 \cdot 0.08 \\ \boxed{\mathbb{P}\{X_2 = 2, X_4 = 5\} = 0.0032} \end{aligned}$$

(c)

For k step transitions on the matrix k, we can compute the probability as follows:

$$\mathbb{P}\{X_{n+k} = j|X_n = i\} = [P^k]_{ij}$$

Thus, we have

$$\mathbb{P}\{X_7 = 3|X_3 = 4\} = [P^4]_{4,3}$$

Calculating P^4:

$$P^4 = \begin{bmatrix} 0.1565 & 0.2576 & 0.09 & 0.3603 & 0.1356 \\ 0.0312 & 0.4656 & 0.018 & 0.4276 & 0.0576 \\ 0.1494 & 0.2768 & 0.0854 & 0.3644 & 0.124 \\ 0.0564 & 0.4276 & 0.0318 & 0.4145 & 0.0697 \\ 0.1314 & 0.2232 & 0.0909 & 0.3661 & 0.1884 \end{bmatrix}$$

Thus we have:

$$\mathbb{P}\{X_7 = 3|X_3 = 4\} = 0.0318$$

Question 2

Consider a DTMC $\{X_n, n \geq 0\}$ on the state space $\{1, 2, 3, 4\}$ with $X_0 = 1$ and transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.5 & 0.0 & 0.0 & 0.5 \\ 0.5 & 0.0 & 0.0 & 0.5 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{bmatrix}$$

Compute the following:
(a) $P\{X_2 = 4\}$
(b) $P\{X_1 = 2, X_2 = 4, X_3 = 1\}$
(c) $P\{X_7 = 4|X_5 = 2\}$
(d) $E[X_3]$

. . . .

(a)

Since $X_0 = 1$, we have $a = (0.4 \quad 0.3 \quad 0.2 \quad 0.1)$. We can use a to compute

$$\begin{aligned} \mathbb{P}\{X_2 = 4\} &= [a \times P^1]_4 \\ &= [0.42 \quad 0.14 \quad 0.11 \quad 0.33]_4 \end{aligned}$$

$$\boxed{\mathbb{P}\{X_2 = 4\} = 0.33}$$

(b)

$$\begin{aligned} \mathbb{P}\{X_1 = 2, X_2 = 4, X_3 = 1\} &= \mathbb{P}\{X_3 = 1|X_2 = 4, X_1 = 2\} \cdot \mathbb{P}\{X_2 = 4|X_1 = 2\} \cdot \mathbb{P}\{X_1 = 2|X_0 = 1\} \\ &= \mathbb{P}\{X_3 = 1|X_2 = 4\} \cdot \mathbb{P}\{X_2 = 4|X_1 = 2\} \cdot \mathbb{P}\{X_1 = 2|X_0 = 1\} \\ &= 0.1 \cdot 0.5 \cdot 0.3 \end{aligned}$$

$$\boxed{\mathbb{P}\{X_1 = 2, X_2 = 4, X_3 = 1\} = 0.015}$$

(c)

For k step transitions on the matrix k, we can compute the probability as follows:

$$\mathbb{P}\{X_{n+k} = j|X_n = i\} = [P^k]_{ij}$$

Thus, we have:

$$\mathbb{P}\{X_7 = 4|X_5 = 2\} = [P^2]_{2,4}$$

Computing P^2:

$$P^2 = \begin{bmatrix} 0.42 & 0.14 & 0.11 & 0.33 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.33 & 0.11 & 0.14 & 0.42 \end{bmatrix}$$

Therefore:

$$\boxed{\mathbb{P}\{X_7 = 4 | X_5 = 2\} = 0.25}$$

(d)

Utilizing the equation for expectation:

$$\mathbb{E}[X_3] = \sum_{j=1}^{4} j \cdot \mathbb{P}\{X_3 = j\}$$

We compute the pmf of $\{X_3 = j\}$.

$$\begin{aligned} \mathbb{P}\{X_3 = j\} &= \mathbb{P}\{X_3 = j | X_0 = 1\} = [P^3]_{1,j} \\ &= [0.3260 \quad 0.1920 \quad 0.1830 \quad 0.2990]_j \end{aligned}$$

Thus the expectation of X_3.

$$\begin{aligned} &\mathbb{E}[X_3] = 1(0.3260) + 2(0.1920) + 3(0.1830) + 4(0.2990) \\ &\boxed{\mathbb{E}[X_3] = 2.455} \end{aligned}$$

Question 3

Consider a LAN (local area network) using a token ring. There is a data buffer that can hold at most 3 messages at a time. The token bus (that goes around a ring) arrives at this data buffer once every micro second. When a token bus arrives, it picks up a message from the data buffer if it can find one and leaves the data buffer. If there are no messages in the data buffer, the token bus leaves immediately. (FYI, the token bus takes this message and sends it to the network somehow). Between two token bus arrivals, new messages can arrive at the data buffer from a computer. It is known that with probability 0.4, 0.4 or 0.2, the number of new messages arriving at the data buffer in that micro second is 0, 1 or 2 respectively. Let Xn represent the number of messages in the data buffer immediately after the n th token bus departure. Clearly, $\{X_n, n \geq 0\}$ is a DTMC with state space $S = \{0, 1, 2\}$ and transition probability matrix:

$$P = \begin{bmatrix} 0.8 & 0.2 & - \\ 0.4 & 0.4 & - \\ 0 & 0.4 & - \end{bmatrix}$$

(a) Fill up the blanks in the above P matrix
(b) If I know that after the third token bus departure there were 2 messages left in the data buffer, then what is the probability that there will be no messages in the data buffer after the fifth token bus departure?
(c) What are the long-run probabilities that there will be zero, one and two messages left in the data buffer soon after the token bus departs.

....

(a)

Each row in a transition probability matrix must add up to 1. Hence we have P.

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

(b)

We are interested in finding the probability of $\{X_5 = 0|X_3 = 2\}$. We simply use the k-step transition equation.

$$\begin{aligned} \mathbb{P}\{X_{n+k} = j|X_n = i\} &= [P^k]_{ij} \\ \mathbb{P}\{X_5 = 0|X_3 = 2\} &= [P^2]_{2,0} \end{aligned}$$

Calculating P^2, we get:

$$P^2 = \begin{bmatrix} 0.72 & 0.24 & 0.04 \\ 0.48 & 0.32 & 0.08 \\ 0.16 & 0.16 & 0.08 \end{bmatrix}$$

$$\boxed{\mathbb{P}\{X_5 = 0|X_3 = 2\} = 0.16}$$

(c)

To find the long-run probabilities, we can perform a steady state analysis to find the limiting distribution of this DTMC. Based on P, we observe that this DTMC has a finite space and satisfy the irreducible and aperiodic properties. This DTMC is irreducible because $[P^2]_{ij} > 0, \forall i, j$ as seen in part (b). Further, this DTMC is aperiodic because for some state $i \in S$, such as $i = 1$, we see that $[P]_{ii} > 0$.

Thus, a limiting distribution π exists and we can find it by multiplying $\pi = \pi P$.

$$\begin{aligned} \pi_0 &= 0.8\pi_0 + 0.4\pi_1 \\ \pi_1 &= 0.2\pi_0 + 0.4\pi_1 + 0.4\pi_2 \\ \pi_2 &= 0.2\pi_1 + 0.6\pi_2 \end{aligned}$$

From $\pi_0 = 0.8\pi_0 + 0.4\pi_1$, we get:

$$\pi_1 = 0.5\pi_0$$

Plugging this into $\pi_1 = 0.2\pi_0 + 0.4\pi_1 + 0.4\pi_2$, we get:

$$\begin{aligned} 0.5\pi_0 &= 0.2\pi_0 + 0.4(0.5\pi_0) + 0.4\pi_2 \\ \pi_2 &= 0.25\pi_0 \end{aligned}$$

Since $\sum \pi_i = 0$:

$$\begin{aligned} 1 &= \pi_0 + \pi_1 + \pi_2 \\ &= \pi_0 + 0.5\pi_0 + 0.25\pi_0 \end{aligned}$$

$$\boxed{\pi_0 = 4/7 \quad \pi_1 = 2/7 \quad \pi_2 = 1/7}$$

Question 4

Consider a DTMC with state space $\{1, 2, 3\}$ and transition probabilities

$$P = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

Compute the steady-state probabilities $\pi = (\pi_1 \;\; \pi_2 \;\; \pi_3)$ ONLY by solving for $\pi = \pi P$ and $\sum \pi = 1$. Then compute P^k for a large k and compare against π.

. . . .

Answer: Performing the matrix multiplication

$$\pi = \pi P$$
$$(\pi_1 \;\; \pi_2 \;\; \pi_3) = (\pi_1 \;\; \pi_2 \;\; \pi_3)P$$

yields the following simultaneous equations:

$$\begin{aligned} \pi_1 &= 0.6\pi_1 + 0.2\pi_3 &\longrightarrow\quad& \pi_1 = 0.5\pi_3 \\ \pi_2 &= 0.4\pi_1 + 0.5\pi_2 &\longrightarrow\quad& \pi_2 = 0.8\pi_1 \\ \pi_3 &= 0.5\pi_2 + 0.8\pi_3 &\longrightarrow\quad& \pi_3 = 2.5\pi_2 \end{aligned}$$

Plugging the above into $\sum \pi = 1$, we get:

$$\begin{aligned} 1 &= \pi_1 + \pi_2 + \pi_3 \\ &= \pi_1 + 0.8\pi_1 + 2.5\pi_2 \\ &= \pi_1 + 0.8\pi_1 + 2.5(0.8\pi_1) \\ &= 3.8\pi_1 \\ \pi_1 &= 1/3.8 \longrightarrow \boxed{\pi_1 = 0.263157895} \\ \pi_2 &= 0.8/3.8 \longrightarrow \boxed{\pi_2 = 0.210526315789} \\ \pi_3 &= 2.5(0.8/3.8) \longrightarrow \boxed{\pi_3 = 0.526315789474} \end{aligned}$$

Thus $\pi = (0.263157895 \;\; 0.210526315789 \;\; 0.526315789474)$. Using MATLAB, computing P_k for a large k, say $k = 10^{10}$, gives:

```
>> P = [0.6 0.4 0; 0 0.5 0.5; 0.2 0 0.8]

P =

    0.6000    0.4000         0
         0    0.5000    0.5000
    0.2000         0    0.8000

>> P^(10^10)

ans =

    0.2632    0.2105    0.5263
    0.2632    0.2105    0.5263
    0.2632    0.2105    0.5263
```

Question 5

Let $\{X_n, n \geq 0\}$ be a DTMC with state space $\{0, 1, 2, \ldots\}$ and the following transition probabilities:

$$p_{ij} = \left\{ \begin{array}{lr} \frac{1}{i+2} & \text{if } 0 \leq j \leq i+1, \text{ and } i \geq 0 \\ 0 & \text{otherwise} \end{array} \right\}$$

Compute the steady state probabilities of X_n as $n \to \infty$

Answer: First, we visualize the transition probability matrix.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Now, we perform the matrix multiplication $\pi = \pi P$. It is clear that:

$$\begin{aligned}
\pi_0 &= \frac{\pi_0}{2} + \frac{\pi_1}{3} + \frac{\pi_2}{4} + \frac{\pi_3}{5} + \frac{\pi_4}{6} + \ldots \\
\pi_1 &= \frac{\pi_0}{2} + \frac{\pi_1}{3} + \frac{\pi_2}{4} + \frac{\pi_3}{5} + \frac{\pi_4}{6} + \ldots \\
\pi_2 &= \frac{\pi_1}{3} + \frac{\pi_2}{4} + \frac{\pi_3}{5} + \frac{\pi_4}{6} + \ldots \\
\pi_3 &= \frac{\pi_2}{4} + \frac{\pi_3}{5} + \frac{\pi_4}{6} + \ldots \\
\pi_4 &= \frac{\pi_3}{5} + \frac{\pi_4}{6} + \ldots \\
\cdots &= \cdots + \cdots + \cdots
\end{aligned}$$

We observe the following recursive pattern.

$$\begin{aligned}
\pi_2 &= \pi_1 - \frac{\pi_0}{2} = \frac{\pi_0}{2} \\
\pi_3 &= \pi_2 - \frac{\pi_1}{3} = \frac{\pi_0}{6} \\
\pi_4 &= \pi_3 - \frac{\pi_2}{4} = \frac{\pi_0}{12} \\
\pi_5 &= \pi_4 - \frac{\pi_3}{5} = \frac{\pi_0}{24} \\
\ldots &= \ldots
\end{aligned}$$

For all $i \geq 0$, we can describe π_i as:

$$\pi_i = \pi_{i-1}\frac{\pi_{i-2}}{i} = \frac{\pi_0}{(i)!}$$

Given $\sum_{i=0}^{\infty} \pi_i = 1$, we get:

$$\begin{aligned}
1 &= \sum_{i=0}^{\infty} \pi_i \\
&= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \ldots \\
&= \frac{\pi_0}{0!} + \frac{\pi_0}{1!} + \frac{\pi_0}{2!} + \frac{\pi_0}{3!} + \ldots \\
&= \pi_0 \left(\sum_{i=0}^{\infty} \frac{1}{i!} \right)
\end{aligned}$$

By analysing the Maclaurin series for $f(x) = e^x$ and setting $x = 1$, we get:

$$\begin{aligned}
f(x) &= \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \\
e^x &= \frac{e^0}{0!} + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots \quad (\text{since } f^a(x) = e^x \text{ for } f(x) = e^x, a \geq 1) \\
e^1 &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \\
e &= \sum_{i}^{\infty} \frac{1}{i!}
\end{aligned}$$

Thus, we get π_0 and, with it, the rest of π_i:

$$\begin{aligned}
1 &= \pi_0 \left(\sum_{i=0}^{\infty} \frac{1}{i!} \right) = \pi_0 e \\
\pi_0 &= e^{-1} \\
\therefore \ & \boxed{\pi_i = \frac{e^{-1}}{i!},\ \forall i = 0, 1, \ldots}
\end{aligned}$$

Question 6

A machine produces two items per day. The probability that an item is nondefective is p. Successive items are independent. Defective items are thrown away instantly. The demand is one item per day which occurs at the end of a day. Any demand that cannot be satisfied immediately is lost. Let X_n be the number of items in storage at the beginning of the n-th day (before production and demand of that day). Using the DTMC $\{X_n, n \geq 0\}$, compute the steady state probabilities of the number of items in storage at the beginning of a day.

. . . .

Answer: First, we define the transition probability matrix. Let D denote the number of defects in any given day's production. With p as the probability that an item is nondefective, the PMF of D is given by:

$$\begin{aligned}
\mathbb{P}(D = 0) &= p^2 \\
\mathbb{P}(D = 1) &= 2p(1 - p) \\
\mathbb{P}(D = 2) &= (1 - p)^2
\end{aligned}$$

For any given state $i > 0$, there are three possible transitions.

1. $X_{n+1} = X_n + 1$. This happens when $D = 0$ and thus $\mathbb{P}(X_{n+1} = i + 1|X_n = i) = p^2$.
2. $X_{n+1} = X_n$. This happens when $D = 1$ and thus $\mathbb{P}(X_{n+1} = i|X_n = i) = 2p(1 - p)$.
3. $X_{n+1} = X_n - 1$. This happens when $D = 2$ and thus $\mathbb{P}(X_{n+1} = i - 1|X_n = i) = (1 - p)^2$.

A special case takes places when $i = 0$, since the inventory levels cannot be negative even when $D = 2$. There are only two possible transitions.

1. $X_{n+1} = X_n + 1$. This happens when $D = 0$ and thus $\mathbb{P}(X_{n+1} = i + 1|X_n = i) = p^2$.
2. $X_{n+1} = X_n$. This happens when $D = 1$ or $D = 2$, and since it is the only other possible scenario, $\mathbb{P}(X_{n+1} = i|X_n = i) = 1 - p^2$.

Hence the probability transition matrix P.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{bmatrix}
1 - p^2 & p^2 & 0 & 0 & 0 & \cdots \\
(1 - p)^2 & 2p(1 - p) & p^2 & 0 & 0 & \cdots \\
0 & (1 - p)^2 & 2p(1 - p) & p^2 & 0 & \cdots \\
0 & 0 & (1 - p)^2 & 2p(1 - p) & p^2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{array}$$

Performing steady state analysis:

$$\begin{aligned}
\pi &= \pi P \\
\pi_0 &= \pi_0(1 - p^2) + \pi_1(1 - p)^2 \\
\pi_1 &= \pi_0 p^2 + \pi_1 2p(1 - p) + \pi_2(1 - p)^2 \\
\pi_2 &= \pi_1 p^2 + \pi_2 2p(1 - p) + \pi_3(1 - p)^2 \\
\pi_3 &= \pi_2 p^2 + \pi_3 2p(1 - p) + \pi_4(1 - p)^2 \\
\pi_4 &= \pi_3 p^2 + \pi_4 2p(1 - p) + \pi_5(1 - p)^2 \\
\ldots &= \ldots
\end{aligned}$$

From $\pi_0 = \pi_0(1-p^2) + \pi_1(1-p)^2$, we get:

$$\begin{aligned}\pi_0 &= \pi_0(1-p^2) + \pi_1(1-p)^2 \\ \pi_1 &= \pi_0 \frac{p^2}{(1-p)^2}\end{aligned}$$

From $\pi_1 = \pi_0 p^2 + \pi_1 2p(1-p) + \pi_2(1-p)^2$, we get:

$$\begin{aligned}\pi_1 &= \pi_0 p^2 + \pi_1 2p(1-p) + \pi_2(1-p)^2 \\ \pi_0 \frac{p^2}{(1-p)^2} &= \pi_0 p^2 + \pi_0 \frac{p^2}{(1-p)^2} 2p(1-p) + \pi_2(1-p)^2 \\ \pi_2 &= \pi_0 \frac{p^4}{(1-p)^4}\end{aligned}$$

From $\pi_2 = \pi_1 p^2 + \pi_2 2p(1-p) + \pi_3(1-p)^2$, we get:

$$\begin{aligned}\pi_2 &= \pi_1 p^2 + \pi_2 2p(1-p) + \pi_3(1-p)^2 \\ \pi_0 \frac{p^4}{(1-p)^4} &= \pi_1 p^2 + \pi_0 \frac{p^4}{(1-p)^4} 2p(1-p) + \pi_3(1-p)^2 \\ \pi_3 &= \pi_0 \frac{p^6}{(1-p)^6}\end{aligned}$$

Since the equations for π_3, π_4, and so on are similar in structure, we can expect this pattern to continue. As such, we deduce a formula for π_i based on π_0:

$$\pi_i = \pi_0 \left(\frac{p^2}{(1-p)^2} \right)^i$$

Given $\sum_{i=0}^{\infty} \pi_i = 1$, we get:

$$1 = \sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \left(\frac{p^2}{(1-p)^2} \right)^i = \pi_0 \left(\frac{1}{1 - \frac{p^2}{(1-p)^2}} \right) \quad (\text{for } p^2/(1-p)^2 < 1)$$

Here, we make a crucial assumption that $p^2/(1-p)^2 < 1$. This assumption ensures that the sum of the infinite geometric series does not diverge to infinity. It also ensures that $p < 1/2$, which further implies that this DTMC is positive recurrent and, with it, is ergodic and has a limiting distribution which we can find. Thus, using the formula for the sum of geometric series, we attain:

$$\begin{aligned}\pi_0 &= 1 - \frac{p^2}{(1-p)^2} \\ &= \frac{p^2 - 2p + 1 - p^2}{(1-p)^2}\end{aligned}$$

$$\boxed{\pi_0 = \frac{1-2p}{(1-p)^2}}$$

We now get π_i for $i \geq 0$.

$$\pi_i = \frac{1-2p}{(1-p)^2} \left(\frac{p^2}{(1-p)^2} \right)^i$$

$$\boxed{\pi_i = \frac{(1-2p)p^{2i}}{(1-p)^{(2+2i)}}, \quad \forall i \in S, p < 1/2}$$