Question 1

Consider a network with N nodes. Assume that each node is connected to every other node. At time 0 a cat resides on node N and a mouse resides on node 1. During one time unit the cat chooses a random node from the remaining $N-1$ nodes and moves to it. The mouse moves in the same way independently of the cat. If the cat and the mouse occupy the same node at any time, the cat promptly eats the mouse. Now let $X_n = 0$ or 1, if the mouse is dead or alive respectively, at time n. Model $X_n, n \geq 0$ as a DTMC.

Answer: To solve this problem, first we define the state space:

$$S = \{0, 1\}$$

where the state is 0 when the mouse is dead and 1 otherwise. Here, we notice that the memoryless and time-invariant properties are held. Firstly, the memoryless property is maintained because the cat or mouse's position at time $n+1$ only depends on their position at time n (but not $n-1, n-2, ...$ Secondly, the probability that both cat and mouse lands in the same node at any time $n \geq 0$ is constant at any stage of the stochastic process. That is, the state transitions can be represented by using just 1 transition probability matrix. As such, this problem can be modelled as a DTMC.

Now, we proceed to calculate P, the state transition probability matrix.

$$P = \begin{bmatrix} \mathbb{P}(X_{n+1} = 0|X_n = 0) & \mathbb{P}(X_{n+1} = 1|X_n = 0) \\ \mathbb{P}(X_{n+1} = 0|X_n = 1) & \mathbb{P}(X_{n+1} = 1|X_n = 1) \end{bmatrix}$$

We calculate each entry. If the mouse is dead at n, it must be dead at $n+1$ too and hence we have:

$$\mathbb{P}(X_{n+1} = 0|X_n = 0) = 1$$
$$\mathbb{P}(X_{n+1} = 1|X_n = 0) = 0$$

Next, if the mouse is alive at n, then it can either die or stay alive at $n+1$. We calculate this probability using enumeration.

$$\begin{aligned} \mathbb{P}(X_{n+1} = 1|X_n = 0) &= \mathbb{P}(\text{Node(Cat)} = \text{Node(Mouse)}) \\ &= \frac{N-2}{(N-1)(N-1)} = \frac{N-2}{(N-1)^2} \\ \mathbb{P}(X_{n+1} = 1|X_n = 1) &= 1 - \mathbb{P}(X_{n+1} = 1|X_n = 0) \\ &= 1 - \frac{N-2}{(N-1)^2} \end{aligned}$$

Thus we have our transition probability matrix.

$$P = \begin{bmatrix} 1 & 0 \\ \frac{N-2}{(N-1)^2} & 1 - \frac{N-2}{(N-1)^2} \end{bmatrix}$$

Question 2

The weather at a resort city is either sunny or rainy. The weather tomorrow depends on the weather today and yesterday as follows: if it was sunny yesterday and today, it will be sunny tomorrow with probability 0.8; if it was rainy yesterday, but sunny today, it will be sunny tomorrow with probability 0.75; if it was sunny yesterday, but rainy today, it will be sunny tomorrow with probability 0.5; if it was rainy yesterday and today, it will be sunny tomorrow with probability 0.4. Define today's state of the system as a pair (weather yesterday, weather today). Model this system as a DTMC.

Answer: Let $S_n = (X_{n-1}, X_n)$ denote the system state at day n, where X_i refers to the weather at the resort city at day i which takes the value of 1 if the weather is sunny and 0 otherwise (or if rainy). The state space of this system is the permutation of sunny and rainy weathers for both X_n and X_{n+1}.

$$S_n = \{(0,0), (0,1), (1,0), (1,1)\}, \quad n \geq 0$$

Here, our problem definition allows this system to be modelled as a DTMC because the weather state in a given time only depends on the previous weather state, and because the time-invariant property holds due to the time-invariant transition probabilities. Hence, we can proceed with calculating the transition probability matrix.

If it is sunny yesterday and today, it will be sunny tomorrow with probability 0.8. Hence we have:

$$\mathbb{P}(S_{n+1} = (1,1) \mid S_n = (1,1)) = 0.8$$

If it was rainy yesterday, but sunny today, it will be sunny tomorrow with probability 0.75.

$$\mathbb{P}(S_{n+1} = (1,1) \mid S_n = (0,1)) = 0.75$$

If it was sunny yesterday, but rainy today, it will be sunny tomorrow with probability 0.5.

$$\mathbb{P}(S_{n+1} = (0,1) \mid S_n = (1,0)) = 0.5$$

If it was rainy yesterday and today, it will be sunny tomorrow with probability 0.4.

$$\mathbb{P}(S_{n+1} = (0,1) \mid S_n = (0,0)) = 0.4$$

Additionally, some states are impossible given a previous, contradictory state. For example, if $S_n = (0,0)$, then it would be impossible to observe $S_{n+1} = (1,0)$ as we would have contradicting X_n values.

$$\begin{aligned}
\mathbb{P}(S_{n+1} = (1,0) \mid S_n = (0,0)) &= 0 & \qquad \mathbb{P}(S_{n+1} = (1,1) \mid S_n = (0,0)) &= 0 \\
\mathbb{P}(S_{n+1} = (0,0) \mid S_n = (0,1)) &= 0 & \qquad \mathbb{P}(S_{n+1} = (0,1) \mid S_n = (0,1)) &= 0 \\
\mathbb{P}(S_{n+1} = (1,0) \mid S_n = (1,0)) &= 0 & \qquad \mathbb{P}(S_{n+1} = (1,1) \mid S_n = (1,0)) &= 0 \\
\mathbb{P}(S_{n+1} = (0,0) \mid S_n = (1,1)) &= 0 & \qquad \mathbb{P}(S_{n+1} = (0,1) \mid S_n = (1,1)) &= 0
\end{aligned}$$

We can now partially fill our transition probability matrix (represented using a table).

$i \setminus j$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)		0.4	0	0
(0,1)	0	0		0.75
(1,0)		0.5	0	0
(1,1)	0	0		0.8

By the law of total probability, each row must add up to 1. Thus, we deduce the values of the remaining cells.

$i \setminus j$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	0.6	0.4	0	0
(0,1)	0	0	0.25	0.75
(1,0)	0.5	0.5	0	0
(1,1)	0	0	0.2	0.8

Question 3

A computer program initially contains k bugs (in practice k is unknown). Every time the program fails to perform according to specification, a SINGLE bug is discovered. In the process of fixing the bug, the programmer inevitably introduces some other bugs. This process continues until the program performs satisfactorily on a wide range of input data. Again, in practice, one can never be sure that all bugs have been fixed. We model this software development process as follows: when the program is executed for the nth time, it will hit upon a bug with probability p_k if there are k bugs in the program. With probability $(1 - p_k)$, no bug is dicovered on the n-th run and the program is run with a different input. Of course $p_0 = 0$. If a bug is discovered, it is fixed. There is a probability b_i (for $i = 0, 1, 2$) that after fixing the original bug, i new bugs are introduced. Assume that bug discovering and bug introducing processes are independent of the past and each other. Let X_n be the number of bugs in the program just before running it for the nth time. Show that $\{X_n, n \geq 1\}$ is a DTMC.

Answer: To solve this problem, firstly define X_n as the number of bugs encountered just before running the program for the n-th time. Secondly, we define the state space as follows.

$$S = \{0, 1, 2, \dots\}$$

Here, the state space is unbounded since we do not know how many bugs there are in the program. Additionally, this problem can be modelled as a DTMC because the process of bug discovery and introduction are independent, and the next state X_{n+1} only depends on the previous state X_n, making the problem memoryless. In addition, the state transitions are invariant of time.

To find the transition probability matrix, we consider all possible scenarios and their probabilities.

- No bug is discovered with probability $1 - p_k$.
  - The number of bugs remain the same ($X_{n+1} = X_n$).
- A bug is discovered with probability p_k. It is fixed and i new bugs are introduced.
  - 0 new bugs introduced with probability b_0 ($X_{n+1} = X_n - 1$).
  - 1 new bug introduced with probability b_1 ($X_{n+1} = X_n$).
  - 2 new bugs introduced with probability b_2 ($X_{n+1} = X_n + 1$).

With this, we define our transition probability matrix P_{ij}, where entry ij represents the probability of transitioning from state i to j. First, we consider the trivial case where $i = j = 0$. Here, since no bugs are introduced, no bugs can be discovered either and thus $P_{0,0} = 1$. There are now three sub-cases possible for $i > 0$.

1. ($X_{n+1} = X_n - 1$). The number of bugs reduces by 1 if a bug is discovered and fixed without introducing new bugs during the debugging process. This takes place with probability $p_i \cdot b_0$.
2. ($X_{n+1} = X_n$). The number of bugs remains the same if either no bugs are encountered $(1 - p_i)$ or a bug is discovered and fixed but a new bug appears after the initial bug was fixed $(p_i \cdot b_1)$. Thus this takes place with probability $p_i \cdot b_0 + (1 - p_i)$.

3. ($X_{n+1} = X_n + 1$). The number of bugs increases by 1 if a bug is discovered and fixed but two new bugs appear after the initial bug was fixed. This takes place with probability $p_i \cdot b_2$.

Hence, we have our transition probability matrix P_{ij} for $i, j \geq 0$.

$$P_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } i = j = 0 \\ p_i \cdot b_0 & \text{if } i = j + 1 \\ p_i \cdot b_1 + (1 - p_i) & \text{if } i = j \neq 0 \\ p_i \cdot b_2 & \text{if } i = j - 1 \\ 0 & \text{otherwise} \end{array} \right\}$$

We display the matrix as follows.

$$P_{ij} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ p_1 b_0 & (1-p_1) + p_1 b_1 & p_1 b_2 & 0 & 0 & \cdots \\ 0 & p_2 b_0 & (1-p_2) + p_2 b_1 & p_2 b_2 & 0 & \cdots \\ 0 & 0 & p_3 b_0 & (1-p_3) + p_3 b_1 & p_3 b_2 & \cdots \\ 0 & 0 & 0 & p_4 b_0 & (1-p_4) + p_4 b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Question 4

There is an infinite supply of light bulbs, and Z_i is the lifetime of the i-th light bulb. $\{Z_i, i \geq 1\}$ is a sequence of i.i.d. discrete random variables with

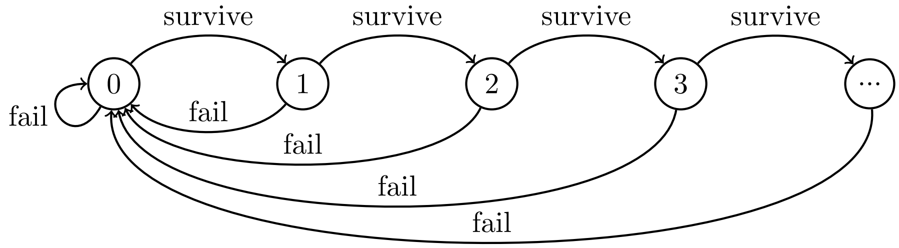
$$\mathbb{P}(Z_i = k) = p_k \quad k = 1, 2, \ldots$$

where $p_k \geq 0$, $\sum_{k=1}^{\infty} = 1$. At time 0 the first light bulb is turned on. It fails at time Z_1. Then it is replaced by the second light bulb which fails at time $Z_1 + Z_2$ and so on. Now let X_n be the age of the light bulb that is on at time n. Note that $X_n = 0$ if a failure has taken place at time n. Show that $\{X_n, n \geq 0\}$ is a DTMC and display its transition probability matrix.

Answer: First, we define X_n as the age of the active lightbulb at time n. $X_n = 0$ if a lightbulb has failed at time n, since a new light bulb with age 0 would be installed to replace the failed lightbulb. Secondly, we define the state space as

$$S = \{0, 1, 2, \ldots\}$$

The state space is unbounded since the light bulb can theoretically last forever. Here, the Markov properties hold true because the state of the lightbulb only depends on its previous state, and because the transition probabilities remain the same at any point in time. Thus, this problem can be modelled as a DTMC. Now, we can draw the following state diagram. Everytime a lightbulb fails, we return to state 0. However, if a lightbulb doesn't fail, we enter a state larger by 1 unit than the previous state.



Generally, given $X_n = k \geq 0$ (the lightbulb has lasted k time units so far), then either $X_{n+1} = 0$ (the lightbulb fails at the next time period) or $X_{n+1} = k + 1$ (the lightbulb continues running at the next time period). We need to utilise conditional probability to analyse the probability of these state transitions.

- Given $X_n = k$, the probability of $X_{n+1} = 0$ is:

$$\begin{aligned}\mathbb{P}(X_{n+1} = 0 | X_n = k) &= \frac{\mathbb{P}((X_{n+1} = 0) \cap (X_n = k))}{\mathbb{P}(X_n = k)} \\ &= \frac{\mathbb{P}(Z_i = k+1)}{\mathbb{P}(Z_i > k)} \\ &= \frac{p_{k+1}}{\sum_{i=k+1}^{\infty} p_i}\end{aligned}$$

- On the other hand, by the axioms of probability, given $X_n = 0$, the probability of $X_{n+1} = k + 1$ is:

$$\begin{aligned}\mathbb{P}(X_{n+1} = k+1 | X_n = k) &= 1 - \mathbb{P}(X_{n+1} = 0 | X_n = k) \\ &= 1 - \frac{p_{k+1}}{\sum_{i=k+1}^{\infty} p_i}\end{aligned}$$

Here, notice that for $k = 0$, the probability equation simplifies into:

$$\begin{aligned}\mathbb{P}(X_{n+1} = 0 | X_n = 0) = p_1 \\ \mathbb{P}(X_{n+1} = 1 | X_n = 0) = 1 - p_1\end{aligned}$$

With this, we can now formulate our transition probability matrix.

$$P_{ij} = \left\{ \begin{array}{lr} \dfrac{p_{i+1}}{\sum_{m=i+1}^{\infty} p_m} & \text{if } j = 0 \\ 1 - \dfrac{p_{i+1}}{\sum_{m=i+1}^{\infty} p_m} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{array} \right\}$$

We can also display it as follows.

$$P_{ij} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} p_1 & 1-p_1 & 0 & 0 & 0 & \cdots \\ \frac{p_2}{\sum_{i=2}^{\infty} p_i} & 0 & 1-\frac{p_2}{\sum_{i=2}^{\infty} p_i} & 0 & 0 & \cdots \\ \frac{p_3}{\sum_{i=3}^{\infty} p_i} & 0 & 0 & 1-\frac{p_3}{\sum_{i=3}^{\infty} p_i} & 0 & \cdots \\ \frac{p_4}{\sum_{i=4}^{\infty} p_i} & 0 & 0 & 0 & 1-\frac{p_4}{\sum_{i=4}^{\infty} p_i} & \cdots \\ \frac{p_5}{\sum_{i=5}^{\infty} p_i} & 0 & 0 & 0 & 0 & 1-\frac{p_5}{\sum_{i=5}^{\infty} p_i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Question 5

A shuttle bus with infinite capacity stops at bus stops numbered $0, 1, 2, ...,$ on its infinite route. Let Y_n be the number of people who board the bus at stop n. Assume that $\{Y_n, n \geq 0\}$ is an i.i.d. sequence of random variables with common p.m.f.

$$p_k = \mathbb{P}(Y_n = k), \forall k = 0, 1, 2, \ldots$$

Every passenger who is on the bus when the bus arrives at stop n has a probability p of alighting at that stop. The passengers behave independently of each other. Let X_n be the number of passengers on the bus when it leaves the n-th stop. Show that $\{X_n, n \geq 0\}$ is a DTMC. Display its transition probability matrix.

Answer: Let X_n denote the number of passengers at the bus when it leaves the n-th stop. At every stop, the bus takes in Y_n new passengers where Y_n for $n \geq 0$ is an i.i.d. sequence of random variables with common p.m.f. $p_k = \mathbb{P}(Y_n = k), \forall k = 0, 1, 2, \ldots$. The bus can also lose passengers at every stop. Each individual has a probability p of alighting the bus (independent of other passengers). Given this, we can come up with an equation for X_{n+1}:

$$X_{n+1} = X_n + Y_{n+1} - Z_n$$

where Z_n is the number of passengers alighting at stop n, given by:

$$Z_n = \sum_{i=0}^{X_n} W_i$$

Here, W_i is a Bernoulli random variable with value 1 if a person alights from the bus (with probability p) and 0 otherwise. Notice that Z_n follows a binomial distribution as it is a sum of i.i.d. Bernoulli random variables. Thus, we have:

$$Z_n \sim \text{Binomial}(X_n, p)$$
$$\mathbb{P}(Z_n = z) = \binom{X_n}{z} p^z (1-p)^{X_n - z}$$

Therefore, the probability of transitioning from state $X_n = i$ to $X_{n+1} = j$ can only be modelled by considering the random variables Y_{n+1} and Z_n. We have to utilise the law of total probability to compute $\mathbb{P}(X_{n+1} = j | X_n = i)$.

$$\begin{aligned}
\mathbb{P}(X_{n+1} = j | X_n = i) &= \sum_{k=0}^{\infty} \mathbb{P}\big((X_{n+1} = j | X_n = i) | Y_{n+1} = k\big) \cdot \mathbb{P}(Y_{n+1} = k | X_n = i) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(X_{n+1} = j \mid X_n = i, Y_{n+1} = k) \cdot \mathbb{P}(Y_{n+1} = k) \quad (Y_n,\ X_n \text{ are independent}) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(X_{n+1} = j \mid X_n = i, Y_{n+1} = k) \cdot p_k
\end{aligned}$$

$\mathbb{P}(X_{n+1} = j \mid X_n = i, Y_{n+1} = k)$ is simply the probability of $X_{n+1} = j$ given $X_n = i$ and $Y_{n+1} = k$. Since $X_{n+1} = X_n + Y_{n+1} - Z_n$, this takes place when $Z_n = i + k - j$. Thus, $\mathbb{P}(X_{n+1} = j \mid X_n = i, Y_{n+1} = k) = \mathbb{P}(Z_n = i + k - j)$, and since Z_n follows a binomial distribution, we get:

$$\begin{aligned}
\mathbb{P}(X_{n+1} = j \mid X_n = i, Y_{n+1} = k) &= \mathbb{P}(Z_n = i + k - j) \\
&= \binom{i}{i+k-j} p^{i+k-j} \cdot (1-p)^{i-(i+k-j)} \\
&= \binom{i}{i+k-j} p^{i+k-j} \cdot (1-p)^{j-k}
\end{aligned}$$

Here, the equation is only valid for certain values of i, j, and k. By analyzing the combination function as shown below:

$$\binom{i}{i+k-j} = \frac{(i)!}{(i+k-j)!(j-k)!}$$

it is evident that we require $i \geq 0$, $i+k-j \geq 0$, and $j-k \geq 0$ for the equation to hold true. These three conditions take place only when $k \leq j \leq i+k$ ($i \geq 0$ is always true by the problem definition). As such, we get:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, Y_{n+1} = k) = \left\{ \begin{array}{lr} \binom{i}{i+k-j} p^{i+k-j} \cdot (1-p)^{j-k} & \text{if } k \leq j \leq i+k \\ 0 & \text{otherwise} \end{array} \right\}$$

Returning to the previous equation, we have:

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \sum_{k=0}^{\infty} \mathbb{P}(X_{n+1} = j \mid X_n = i, Y_{n+1} = k) \cdot p_k$$

Now, we need to analyze the bounds of the summation function. Previously, we discovered that $i \geq 0$, $i+k-j \geq 0$, and $j-k \geq 0$. Rearranging these inequalities in terms of k gives:

$$j - i \leq k \leq j$$

That is, for $k < j-i$ and $k > j$, we simply have $\mathbb{P}(X_{n+1} = j | X_n = i, Y_{n+1} = k) = 0$. Of course, we also require $k \geq 0$ at all times, since k is the number of people entering the bus and must therefore be non-negative. This simplifies our transition probability equation into the following.

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \left\{ \begin{array}{lr} \displaystyle\sum_{k=j-i}^{j} \binom{i}{i+k-j} \left(p^{i+k-j} \cdot (1-p)^{j-k} \right) \cdot p_k & \text{if } j \geq i \\ \displaystyle\sum_{k=0}^{j} \binom{i}{i+k-j} \left(p^{i+k-j} \cdot (1-p)^{j-k} \right) \cdot p_k & \text{if } j < i \end{array} \right\}$$

Firstly, the upper bound of the summation operator is always j, as we require $k \leq j$. Secondly, the lower bound of the summation operator depends on the values of i and j.

1. If $j \geq i$, then $j - i \geq 0$ and as such we set the lower bound of the operator to be $k = j - i$, knowing that doing so would ensure $k \geq 0$ and $k \geq j - i$ at the same time.
2. On the other hand, if $j < i$, then $k \geq 0$ is not necessarily maintained due to $k = j - i < 0$ being possible in some cases. Thus, we set the lower bound to be $k = 0$, ensuring that both $k \geq 0$ and $k \geq j - i$ are fulfilled concurrently.

The final transition probability matrix is displayed in the next page.

With this, we have now shown that $\{X_n, n \geq 0\}$ is a DTMC. The problem is a valid DTMC since X_{n+1} only depends on X_n, making it memoryless, and since a single transition probability matrix is sufficient to describe the state transitions invariant of time.

With that, we have our final transition probability matrix and display it as follows.

$$P_{ij} = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \left[\begin{array}{cccccc} p_0 & p_1 & p_2 & p_3 & p_4 & \cdots \\ \phi p_0 & (1-\phi)p_0 + \phi p_1 & (1-\phi)p_1 + \phi p_2 & (1-\phi)p_2 + \phi p_3 & (1-\phi)p_3 + \phi p_4 & \cdots \\ \phi^2 p_0 & 2\phi(1-\phi)p_0 + \phi^2 p_1 & (1-\phi)^2 p_0 + 2\phi(1-\phi)p_1 + \phi^2 p_2 & (1-\phi)^2 p_1 + 2\phi(1-\phi)p_2 + \phi^2 p_3 & (1-\phi)^2 p_2 + 2\phi(1-\phi)p_3 + \phi^2 p_4 & \cdots \\ \phi^3 p_0 & 3\phi^2(1-\phi)p_0 + \phi^3 p_1 & 3\phi(1-\phi)^2 p_0 + 3\phi^2(1-\phi)p_1 + \phi^3 p_2 & (1-\phi)^3 p_0 + 3\phi(1-\phi)^2 p_1 + 3\phi^2(1-\phi)p_2 + \phi^3 p_3 & (1-\phi)^3 p_1 + 3\phi(1-\phi)^2 p_2 + 3\phi^2(1-\phi)p_3 + \phi^3 p_4 & \cdots \\ \phi^4 p_0 & 4\phi^3(1-\phi)p_0 + \phi^4 p_1 & 6\phi^2(1-\phi)^2 p_0 + 4\phi^3(1-\phi)p_1 + \phi^4 p_2 & 4\phi(1-\phi)^3 p_0 + 6\phi^2(1-\phi)^2 p_1 + 4\phi^3(1-\phi)p_2 + \phi^4 p_3 & (1-\phi)^4 p_0 + 4\phi(1-\phi)^3 p_1 + 6\phi^2(1-\phi)^2 p_2 + 4\phi^3(1-\phi)p_3 + \phi^4 p_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] \end{array}$$

Here, p is replaced by ϕ to display the matrix clearly.