Question 1

A discrete random variable Y has the following pmf:

$$\mathbb{P}(Y = -1) = 0.4, \quad \mathbb{P}(Y = 0) = 0.25, \quad \mathbb{P}(Y = 1) = 0.35.$$

Using only a uniform(0, 1) random variable, find a way to simulate Y. (That is, any output of the simulation should be a value of Y, and should follow the above probabilities.) (Hint: in Python, use `random.random()`, and based on the result, use an `if` statement to determine the value of Y. Note that the question is not asking you to estimate (say) $\mathbb{E}(Y)$ through many simulations. Please paste your Python code into your PDF submission.)

Answer

```python
import random

def simulate_y():
    x = random.random()
    if x >= 0.65:
        y = 1
    elif x >= 0.4:
        y = 0
    else:
        y = -1
    return y

# EXTRA: To simulate y
count_1, count_0, count_min1, total = 0, 0, 0, 0
for i in range(100):
    total += 1
    y = simulate_y()
    # print(f"Y = {y}")
    if y == -1:
        count_min1 += 1
    if y == 0:
        count_0 += 1
    elif y == 1:
        count_1 += 1

print(f"percentage(Y = -1) = {100 * count_min1/total} %")
print(f"percentage(Y = 0) = {100 * count_0/total} %")
print(f"percentage(Y = 1) = {100 * count_1/total} %")
```

Listing 1: Python Code to Simulate Y

Question 2

Two fair dice are rolled, and the following events are defined:
E_1 = 'the sum of the two dice is 7',
E_2 = 'the first dice shows a 4',
E_3 = 'the second dice shows a 3'.

Last term, we showed that E_1 and E_2 are independent, that is, $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\,\mathbb{P}(E_2)$. Carefully show that E_1 and E_2 are not conditionally independent given E_3.
(Hints: you do not need to re-prove that E_1 and E_2 are independent; you simply need to show that E_1, E_2, E_3 do not satisfy the definition of conditional independence.)

Answer: Recall that E_1 and E_2 are conditionally independent given E_3 if

$$\mathbb{P}(E_1 \cap E_2 \mid E_3) = \mathbb{P}(E_1 \mid E_3)\,\mathbb{P}(E_2 \mid E_3)$$

To show that E_1 and E_2 are not conditionally independent given E_3, we can simply calculate the LHS and RHS of the above equation and show that the LHS is not equal to the RHS. Let D_1 and D_2 denote the outcome of the first and second dice respectively.

$$\begin{aligned}
\text{LHS} = \mathbb{P}(E_1 \cap E_2 \mid E_3) &= \frac{\mathbb{P}(E_1 \cap E_2 \cap E_3)}{\mathbb{P}(E_3)} \quad \text{by definition of conditional probability.}\\
&= \frac{\mathbb{P}(D_1 = 4 \cap D_2 = 3)}{\mathbb{P}(D_2 = 3)} \quad \text{since } (D_1 = 4 \cap D_2 = 3) \text{ satisfies } (E_1 \cap E_2 \cap E_3).\\
&= \frac{\mathbb{P}(D_1 = 4)\,\mathbb{P}(D_2 = 3)}{\mathbb{P}(D_2 = 3)}\\
&= \mathbb{P}(D_1 = 4)\\
&\boxed{\text{LHS} = \mathbb{P}(E_1 \cap E_2 \mid E_3) = 1/6}
\end{aligned}$$

Now for the right hand side.

$$\begin{aligned}
\text{RHS} = \mathbb{P}(E_1 \mid E_3)\,\mathbb{P}(E_2 \mid E_3) &= \frac{\mathbb{P}(E_1 \cap E_3)}{\mathbb{P}(E_3)}\,\frac{\mathbb{P}(E_2 \cap E_3)}{\mathbb{P}(E_3)}\\
&= \frac{\mathbb{P}(D_1 = 4 \cap D_2 = 3)}{\mathbb{P}(D_2 = 3)}\,\frac{\mathbb{P}(D_1 = 4 \cap D_2 = 3)}{\mathbb{P}(D_2 = 3)}\\
&\quad \text{because } (D_1 = 4 \cap D_2 = 3) \text{ satisfies } (E_1 \cap E_3) \text{ and } (D_1 = 4 \cap D_2 = 3) \text{ satisfies } (E_2 \cap E_3).\\
&= \left(\frac{\mathbb{P}(D_1 = 4)\,\mathbb{P}(D_2 = 3)}{\mathbb{P}(D_2 = 3)}\right)^2\\
&= \left(\frac{(1/6)(1/6)}{(1/6)}\right)^2\\
&\boxed{\text{RHS} = \mathbb{P}(E_1 \mid E_3)\,\mathbb{P}(E_2 \mid E_3) = 1/36}
\end{aligned}$$

Therefore $\mathbb{P}(E_1 \cap E_2 \mid E_3) \neq \mathbb{P}(E_1 \mid E_3)\,\mathbb{P}(E_2 \mid E_3)$ and it is shown that E_1 and E_2 are not conditionally independent given E_3.

Question 3

An unloaded, 4-sided dice is rolled twice. Let X be the maximum number obtained from the two rolls, and Y be the minimum number from the two rolls. (The dice can show the numbers 1, 2, 3, or 4, each with probability 1/4. As an example, if the two rolls are 1 followed by 3, then $X = 3$ and $Y = 1$.)

(a) Find the joint pmf of X and Y.
(b) Find the conditional pmf of X given that $Y = 2$.

(Hint: part (a) is similar to a problem from Modelling Uncertainty. Please express your answers in table form, and please use fractions in lowest terms.)

(a) **Answer**: Let D_1 and D_2 represent the outcome of the first and second dice respectively. By means of brute force, we can create a table that covers all possible combinations of D_1 and D_2 and compute the corresponding values of X and Y.

$\mathbf{D_1}$	$\mathbf{D_2}$	**X**	**Y**
1	1	1	1
1	2	2	1
1	3	3	1
1	4	4	1
2	1	2	1
2	2	2	2
2	3	3	2
2	4	4	2
3	1	3	1
3	2	3	2
3	3	3	3
3	4	4	3
4	1	4	1
4	2	4	2
4	3	4	3
4	4	4	4

We can now compute the percentage occurrences (i.e., the probability) of each (X, Y) combination and thereby the joint pmf of X and Y.

X\Y	1	2	3	4	f_X
1	1/16	0	0	0	1/16
2	1/8	1/16	0	0	3/16
3	1/8	1/8	1/16	0	5/16
4	1/8	1/8	1/8	1/16	7/16
f_Y	7/16	5/16	3/16	1/16	1

(b) **Answer**: By definition, the conditional pmf of X given that $Y = 2$ is:

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$
$$f_{X|Y}(x|2) = \frac{f(x, 2)}{f_Y(2)}$$

We can simply divide $f(x,2)$ with $f_Y(2)$ for each x to compute $f_{X|Y}(x|2)$.

$f_{X\|Y}(x\|2)$	0	0.2	0.4	0.4
x	1	2	3	4

Question 4

Let Z be a standard normal random variable, and let $X = Z^2$.
(a) Find the pdf of X, and show that X is a gamma random variable.
(b) Hence, show that the sum of squares of two independent standard normal random variables is an exponential random variable.
(Hints: the pdf of X can be taken from Modelling Uncertainty Week 8; the value of $\Gamma(1/2)$ can be found in an extra question; MGF can be used for part (b).)

(a) **Answer**: Since $X = Z^2$, X can only take positive probability for $x > 0$.

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \le x) = \mathbb{P}(Z^2 \le x) \\ &= \mathbb{P}(-\sqrt{x} \le Z \le \sqrt{x}) \\ &= \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) \end{aligned}$$

Taking the derivative of F_X with respect to x gives us the pdf of X.

$$\begin{aligned} f_X(x) &= \frac{1}{2\sqrt{x}}\phi(\sqrt{x}) + \frac{1}{2\sqrt{x}}\phi(-\sqrt{x}) \\ &= \frac{1}{\sqrt{2\pi x}}e^{-x^2/2} + \frac{1}{\sqrt{2\pi x}}e^{-x^2/2} \quad \text{because } \phi(x) = \frac{1}{2\sqrt{x}}e^{-x^2} \\ &= \frac{1}{\sqrt{2\pi x}}e^{-x^2/2} \quad \text{for } x > 0 \end{aligned}$$

Thus the pdf of X.

$$f_X(x) = \left\{ \begin{array}{ll} \dfrac{1}{\sqrt{2\pi x}}e^{-x^2/2}, & \text{if } x > 0 \\ \\ 0, & \text{if } x \le 0 \end{array} \right\}$$

It can be shown that X also follows the distribution of a Gamma random variable. A random variable $Y \sim \text{Gamma}(\alpha, \lambda)$ has the following probability density function.

$$f_Y(y) = \left\{ \begin{array}{ll} \dfrac{\lambda^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\lambda y}, & \text{if } y \ge 0 \\ \\ 0, & \text{if } y < 0 \end{array} \right\}$$

In the case of $X = Z^2$, it can be shown that $X \sim \text{gamma}(\alpha, \lambda)$ with $\alpha = 1/2$ and $\lambda = 1/2$.

$$f_X(x) = \left\{ \begin{array}{ll} \dfrac{(1/2)^{1/2}\ (x)^{-1/2}\ (e)^{-x^2/2}}{\Gamma(1/2)} & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{array} \right\}$$

$\Gamma(1/2)$ can be computed following the definition of the Gamma function.

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}\ e^{-x} dx \quad \text{ where } \alpha > 0 \\ \Gamma(1/2) &= \int_0^\infty \ x^{-1/2}\ e^{-x} dx \end{aligned}$$

Here, we can perform a clever substitution with $x = t^2/2$ and $dx = tdt$, giving us:

$$\begin{aligned} \Gamma(1/2) &= \int_{t_1}^{t_2} \left(\frac{t^2}{2}\right)^{-1/2} e^{-t^2/2}\ tdt \\ &= \int_{t_1}^{t_2} \sqrt{2} e^{-t^2/2} dt \\ &= 2\sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \quad (t_1 = 0 \text{ since } x = 0 \text{ and } t_2 = \infty \text{ since } x = \infty) \end{aligned}$$

Conveniently, $\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ is simply the pdf of the standard normal variable (i.e. $\phi(t)$). We can substitute this into the equation, allowing us to conveniently compute the above integral.

$$\begin{aligned} \Gamma(1/2) &= 2\sqrt{\pi}\ \int_0^\infty \phi(t) dt \\ &= 2\sqrt{\pi}(1/2) \text{ (since } \int_0^\infty \ \phi(t)dt \text{ is half the area of the standard normal curve.)} \\ &= \sqrt{\pi} \end{aligned}$$

So $\Gamma(1/2) = \sqrt{\pi}$ and we can substitute this into the pdf of X to show that X is indeed a gamma random variable.

$$\begin{aligned} f_X(x) &= \frac{(1/2)^{1/2}\ (x)^{-1/2}\ (e)^{-x^2/2}}{\sqrt{\pi}} \quad \text{ for } x > 0 \\ &= \frac{1}{\sqrt{2\pi x}} e^{-x^2/2} \quad \text{ for } x > 0 \end{aligned}$$

Note that $f_X(x)$ is only defined for $x > 0$ (when $x = 0$, $f_X = 1/0$ which is an undefined result; so we set $f_X(0) = 0$). Thus, we have the pdf of X for $x \in \mathbb{R}$.

$$f_X(x) = \left\{ \begin{array}{ll} \dfrac{1}{\sqrt{2\pi x}} e^{-x^2/2}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{array} \right\}$$

We have therefore shown that the pdf of X can be derived from the pdf of a Gamma random variable. This means that X is indeed a Gamma random variable with parameters $\alpha = 1/2$ and $\lambda = 1/2$.

(b) **Answer**: The sum of squares of two independent standard normal random variable is given by $Z_1^2 + Z_2^2 = X_1 + X_2$ ($X = Z^2$ as given by the problem). By Theorem 2 of Moment Generating Functions:

$$M_{Z_1^2+Z_2^2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

Recall that for $X \sim \text{gamma}(\alpha, \lambda)$, $M_X(t) = \lambda^\alpha(\lambda - t)^{-\alpha}$. Therefore, given that $X1, X2 \sim \text{gamma}(1/2, 1/2)$:

$$\begin{aligned} M_{Z_1^2+Z_2^2}(t) &= \left((1/2)^{1/2}(1/2-t)^{-1/2}\right)^2 \\ &= \frac{1/2}{1/2-t} \end{aligned}$$

Recall that the moment generating function of an exponential random variable with parameter λ is given by:

$$M_{\exp}(t) = \frac{\lambda}{\lambda - t}$$

When $\lambda = 1/2$, $M_{\exp}(t) = M_{Z_1^2+Z_2^2}(t)$ and by the Uniqueness Theorem of Moment Generating Functions, we can deduce that $Z_1^2 + Z_2^2 \sim \text{exponential}(1/2)$.

$$\begin{aligned} M_{\exp}(t) &= \frac{1/2}{\lambda - 1/2} \\ M_{\exp}(t) &= M_X(t) \\ &\boxed{Z_1^2 + Z_2^2 \sim \text{exponential}(1/2)} \end{aligned}$$

Therefore, we have shown that the sum of squares of two independent normal random variables is an exponential random variable.

Question 5

A fair coin is repeatedly tossed until three consecutive H's appear for the first time (at which point the tossing stops). Using the law of total expectation, find the expected number of coin tosses. (Hints: for example, a valid sequence of 8 tosses might be HTHHTHHH. In order to apply the law, consider the following 4 ways for any sequence to start: T, HT, HHT, HHH.)

Answer: To begin, let us define the following events:

- A: The first toss is a T (i.e. we have 'T').

- B: The first toss is a H but the second is a T (i.e. we have 'HT').
- C: The first and second tosses are a H but the third is a T (i.e. we have 'HHT').
- D: The first three tosses are a H (i.e. we have 'HHH').

We realize that if A, B, or C take place, the experiment continues and we keep on trying to reach three consecutive H's. However, if D takes place, the experiment ends here because we have reached three consecutive H's. Therefore, A, B, C, and D cover all the possible outcomes exhaustively and we can use the Law of Total Expectation to compute X, the number of coin tosses until the experiment ends.

$$\begin{aligned}\mathbb{E}(X) &= \sum_{\text{all y}} \mathbb{E}(X|Y=y)\mathbb{P}(Y=y) \\ &= \mathbb{E}(X|A)\mathbb{P}(A) + \mathbb{E}(X|B)\mathbb{P}(B) + \mathbb{E}(X|C)\mathbb{P}(C) + \mathbb{E}(X|D)\mathbb{P}(D)\end{aligned}$$

Interestingly enough, $\mathbb{E}(X|A) = 1 + \mathbb{E}(X)$ because it takes 1 toss to get T for the first time (A is the event that the first toss is a T) and it takes another $\mathbb{E}(X)$ tosses for us to finally reach three consecutive H's afterwards. Essentially, after getting a T, we have to start over "from scratch" to reach three consecutive H's.

Similarly, in the case of B, it takes two tosses to get 'HT' and another $\mathbb{E}(X)$ tosses to get to three consecutive H's. Therefore, $\mathbb{E}(X|B) = 2 + \mathbb{E}(X)$ and (by the same logic) $\mathbb{E}(X|C) = 3 + \mathbb{E}(X)$ and as such:

$$\begin{aligned}\mathbb{E}(X) &= (1+\mathbb{E}(X))\mathbb{P}(A) + (2+\mathbb{E}(X))\mathbb{P}(B) + (3+\mathbb{E}(X))\mathbb{P}(C) + \mathbb{E}(X|D)\mathbb{P}(D) \\ &= (1+\mathbb{E}(X))\mathbb{P}(\text{T}) + (2+\mathbb{E}(X))\mathbb{P}(\text{HT}) + (3+\mathbb{E}(X))\mathbb{P}(\text{HHT}) + \mathbb{E}(X|D)\mathbb{P}(\text{HHH}) \\ &= (1+\mathbb{E}(X))(0.5) + (2+\mathbb{E}(X))(0.5\cdot 0.5) + (3+\mathbb{E}(X))(0.5\cdot 0.5\cdot 0.5) \\ &\quad + (\mathbb{E}(X|D))(0.5\cdot 0.5\cdot 0.5)\end{aligned}$$

Since $\mathbb{E}(X|D) = 3$,

$$\begin{aligned}\mathbb{E}(X) &= (7/8)\mathbb{E}(X) + (1)(0.5) + (2)(0.5)^2 + (3)(0.5)^3 + (3)(0.5)^3 \\ &= 8(0.5 + 2\cdot 0.5^2 + 3\cdot 2\cdot 0.5^3) \\ &\boxed{\mathbb{E}(X) = 14}\end{aligned}$$

Question 6

Suppose that in a given year, a client files a claim on her insurance policy with probability 0.2, and at most one claim may be filed. If a claim is filed, then the amount paid is exponentially distributed with mean \$1000. (If no claim is filed, then nothing is paid.)

(a) Using the law of total expectation, find the mean of the amount paid.
(b) Using the law of total variance, find the variance of the amount paid.

(Hints: let X be the amount paid, and Y be the number of claims filed. For (b), carefully write down the pmf table for the random variables E(X—Y) and Var(X—Y); with the latter, think about how the exponential variance is related to its mean. As Y is discrete, no integration is required for this question.)

(a) **Answer**: Let X be the amount paid and Y be the number of claims filed. As there can only be at most 1 claim filed, either $Y = 0$ or $Y = 1$. If no claim is filed, then nothing is paid; that is, $\mathbb{E}(X|Y = 0) = 0$. However, if a claim is indeed filed, then the mean amount of payment is \$1000 as given in the question. In other words: $\mathbb{E}(X|Y = 1) = 1000$. By using the Law of Total Expectation, we can compute $\mathbb{E}(X)$ (i.e. the mean of the amount paid).

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{\text{all y}} \mathbb{E}(X|Y = y)\mathbb{P}(Y = y) \\
&= \mathbb{E}(X|Y = 0)\mathbb{P}(Y = 0) + \mathbb{E}(X|Y = 1) \cdot \mathbb{P}(Y = 1) \\
&= \mathbb{E}(X|Y = 0)(1 - \mathbb{P}(Y = 1)) + \mathbb{E}(X|Y = 1) \cdot \mathbb{P}(Y = 1) \\
&= (0)(0.8) + (1000)(0.2)
\end{aligned}$$

$$\boxed{\mathbb{E}(X) = \$200}$$

(b) **Answer**: Following the hint, we compute the pmf table for $\mathbb{E}(X|Y)$ and $\mathrm{Var}(X|Y)$.

Y	$\mathbb{E}(X\|Y = y)$	$\mathbb{P}(Y = y)$
0	0	0.8
1	1000	0.2

Table 1: pmf table of $\mathbb{E}(X|Y = y)$

Y	$\mathrm{Var}(X\|Y = y)$	$\mathbb{P}(Y = y)$
0	0	0.8
1	1000000	0.2

Table 2: pmf table of $\mathbb{E}(X|Y = y)$

An exponential random variable $W \sim \text{exponential}(\lambda)$ has mean $\mathbb{E}(W) = 1/\lambda$ and variance $\mathrm{Var}(W) = 1/\lambda^2 = \mathbb{E}(W)^2$. Therefore, since X is an exponential random variable for Y=1, we get $\mathrm{Var}(X|Y = 1) = 1000000$ (as seen in table 2). Meanwhile, $\mathrm{Var}(X|Y = 0) = 0$ as there is no payment made.

Now to calculate $\mathrm{Var}(X)$, recall the Law of Total Variance:

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|Y)) + \mathrm{Var}(\mathbb{E}(X|Y))$$

Since $\mathrm{Var}(\mathbb{E}(X|Y)) = \mathbb{E}(\mathbb{E}(X|Y)^2) - [\mathbb{E}(\mathbb{E}(X|Y))]^2$:

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}(\mathrm{Var}(X|Y)) + \mathbb{E}(\mathbb{E}(X|Y)^2)) - [\mathbb{E}(\mathbb{E}(X|Y))]^2 \\ &= \mathbb{E}(\mathrm{Var}(X|Y)) + \mathbb{E}(\mathbb{E}(X^2|Y)) - [\mathbb{E}(X)]^2 \\ &= \mathbb{E}(\mathrm{Var}(X|Y)) + \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\ &= \mathbb{E}(\mathrm{Var}(X|Y)) + [0.2(1000)^2 + 0.8(0)^2] - [200]^2\end{aligned}$$

We can now calculate $\mathbb{E}(\mathrm{Var}(X|Y))$ using the formula for expectation $\mathbb{E}(X) = \sum_{\text{all i}} x_i p(x_i)$ and observing the pmf table we have written down earlier.

$$\begin{aligned}\mathbb{E}(\mathrm{Var}(X|Y)) &= \mathrm{Var}(X|Y=0)\mathbb{P}(Y=0) + \mathrm{Var}(X|Y=1)\mathbb{P}(Y=1) \\ &= 0.8(0) + 0.2(1000000) \\ &= 200000\end{aligned}$$

Plugging this into the previous equation of variance gives us the final answer.

$$\begin{aligned}&\mathrm{Var}(X) = [200000] + [0.2(1000)^2 + 0.8(0)^2] - [200]^2 \\ &\boxed{\mathrm{Var}(X) = 360000}\end{aligned}$$

Question 7

Let X and Y have a bivariate normal distribution, with

$$\mu_X = 2, \quad \sigma_X = 1, \quad \mu_Y = -1, \quad \sigma_Y = 2, \quad \rho = -1/2$$

(a) Compute $\mathbb{P}(2X + 3Y \geq 6)$.
(b) Compute $\mathbb{P}(X < 1|Y = 2)$.

(Hint: for part (a), carefully compute the $\mathbb{E}$ and Var of $2X + 3Y$; for part (b), find the conditional pdf of $X|Y = 2$ first. Please express your answers to 3 decimal places.)

(a) **Answer**: Let $S = 2X + 3Y$. By Theorem 1 of Bivariate Normal Distributions, since X and Y have a bivariate normal distribution, and since S is a linear combination of X and Y, it can be said that S is also normally distributed for some μ_S and σ_S^2 (i.e. $S \sim \mathcal{N}(\mu_S, \sigma_S^2)$). Finding μ_S and σ_S^2 will swiftly allow us to find $\mathbb{P}(2X+3Y \geq 6)$. Thus, by using the property of linear combination for expectation:

$$\begin{aligned}\mu_S &= \mathbb{E}(S) \\ &= \mathbb{E}(2X + 3Y) \\ &= 2\mathbb{E}(X) + 3\mathbb{E}(Y) \\ &= 2\mu_X + 3\mu_Y \\ &= 2(2) + 3(-1) \\ &\boxed{\mu_S = 1}\end{aligned}$$

Similarly, we can apply the property of linear combination on variance.

$$\begin{aligned}
\mu_S^2 &= \text{Var}(S) \\
&= \text{Var}(2X + 3Y) \\
&= 2^2\text{Var}(X) + 3^2\text{Var}(Y) + 2(2)(3)\text{Cov}(X, Y) \\
&= 4\sigma_X^2 + 9\sigma_Y^2 + 12\rho\, \sigma_X\, \sigma_Y \quad (\text{since } \rho_{X,Y} = \tfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}) \\
&= 4(1^2) + 9(2^2) + 12(-1/2)(1)(2) \\
&= 4 + 36 - 12
\end{aligned}$$

$$\boxed{\mu_S^2 = 28}$$

Therefore, we can conclude that $S \sim \mathcal{N}(1, 28)$. Based on this information, we can use the standard normal distribution to calculate $\mathbb{P}(2X + 3Y \geq 6)$.

$$\begin{aligned}
\mathbb{P}(2X + 3Y \geq 6) &= \mathbb{P}(S \geq 6) \\
&= \mathbb{P}\left(Z \geq \frac{6-1}{\sqrt{28}}\right) \\
&= \mathbb{P}(Z \geq 0.9449111825) \\
&= \mathbb{P}(Z \leq -0.9449111825)
\end{aligned}$$

$$\boxed{\mathbb{P}(2X + 3Y \geq 6) = 0.172}$$

(b) **Answer**: By Theorem 4 of Bivariate Normal Distributions, since X and Y have a bivariate normal distribution, then $X|Y = 2$ is also distributed normally for some $\mu_{X|Y=2}$ and $\sigma^2_{X|Y=2}$. That is, $(X|Y = 2) \sim \mathcal{N}(\mu_{X|Y=2}, \sigma^2_{X|Y=2})$.

To find these values, let us compute the conditional pdf of $X|Y = 2$ first.

$$\begin{aligned}
f_{X|Y}(X|Y = 2) &= \frac{f(x, 2)}{f_Y(2)} \\
&= c_1 f(x, 2) \\
&= c_1 \left(\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\right) \exp\left(\frac{-1}{2(1-p^2)} Q(x, 2)\right) \\
&= c_2 \exp\left(\frac{-1}{2(1-p^2)} Q(x, 2)\right)
\end{aligned}$$

Calculating $Q(x, 2)$, and plugging in $\mu_X = 2,\ \sigma_X = 1,\ \mu_Y = -1,\ \sigma_Y = 2,\ \rho = -1/2$:

$$\begin{aligned}
Q(x, 2) &= \left(\frac{x - \mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x - \mu_X}{\sigma_X}\right)\left(\frac{2 - \mu_Y}{\sigma_Y}\right) + \left(\frac{2 - \mu_Y}{\sigma_Y}\right)^2 \\
&= \left(\frac{x-2}{1}\right)^2 - 2(-1/2)\left(\frac{x-2}{1}\right)\left(\frac{2-(-1)}{2}\right) + \left(\frac{2-(-1)}{2}\right)^2 \\
&= (x-2)^2 + 3/2(x-2) + (3/2)^2
\end{aligned}$$

Substituting $Q(x,2)$ gives us:

$$f_{X|Y}(X|Y=2) = c_2 \exp\left(\frac{-1}{2(1-p^2)}\left((x-2)^2 + 3/2(x-2) + (3/2)^2\right)\right)$$

By means of completing the square:

$$\begin{aligned}
f_{X|Y}(X|Y=2) &= c_2 \exp\left(\frac{-1}{2(1-p^2)}\left((x-2)^2 + 2(3/4)(x-2) + (3/2)^2\right)\right) \\
&= c_2 \exp\left(\frac{-1}{2(1-(-1/2)^2)}\left((x-2+3/4)^2 + (3/2)^2 - (3/4)^2\right)\right) \\
&= c_2 \exp\left(-\frac{2}{3}\left(\left(x-\frac{5}{4}\right)^2 + \frac{27}{16}\right)\right) \\
&= c_2 \exp\left(-\left(\frac{2}{3}\right)\left(\frac{27}{16}\right)\right) \exp\left(-\frac{2}{3}\left(x-\frac{5}{4}\right)^2\right) \\
&= c_3 \exp\left(\frac{(x-5/4)^2}{2(3/4)}\right) \quad \text{with } c_3 \in \mathbb{R}
\end{aligned}$$

Comparing this result with the pdf of a standard normal distribution:

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

we can see that $\mu_{X|Y=2} = 5/4$ and $\sigma^2_{X|Y=2} = 3/4$. In other words, $(X|Y=2) \sim \mathcal{N}(5/4, 3/4)$. With this in our hands, we can calculate $\mathbb{P}(X < 1|Y = 2)$ using the standard normal distribution.

$$\begin{aligned}
\mathbb{P}(X < 1|Y=2) &= \mathbb{P}\left(Z < \frac{1-5/4}{\sqrt{3/4}}\right) \\
&= \mathbb{P}(Z < -0.2886751346) \\
&= 0.3859
\end{aligned}$$

$$\boxed{\mathbb{P}(X < 1|Y=2) \approx 0.386}$$