Question 2

[Skip as it is about transient analysis of CTMC]. For the inventory problem (#1) in Homework 3, if $K = 4, R = 2, \lambda = 2$ per day and $1/\theta$ is 1 day, write down the Q matrix. What is the distribution of the number of items in inventory at $t = 5$ days, given that $X(0) = 5$?

Continuing with the above problem, pick a large t to obtain the long-run probabilities of the number of items in inventory. For 3 different values of R, i.e. to $R = 1, R = 2$ and $R = 3$, obtain the long-run probabilities. For the 3 cases, obtain (a) the average number of items in inventory in steady state, (b) the proportion of time there is nothing in inventory.

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Answer: Given $K = 4, \lambda = 2$, and $\theta = 1$, the Q matrix is given by:

$$
Q = \begin{array}{cc}
 & \begin{array}{ccccccccccccccc} 0 & 1 & \cdots & R-1 & R & R+1 & \cdots & K-1 & K & K+1 & \cdots & K+R-1 & K+R \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \vdots \\ R-1 \\ R \\ R+1 \\ \vdots \\ K-1 \\ K \\ K+1 \\ \vdots \\ K+R-1 \\ K+R \end{array} &
\left[\begin{array}{ccccccccccccccc}
-\theta & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \theta & 0 & \cdots & 0 & 0 \\
\lambda & -\theta-\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \theta & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\theta-\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \theta & 0 \\
0 & 0 & \cdots & \lambda & -\theta-\lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \theta \\
0 & 0 & \cdots & 0 & \lambda & -\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda & -\lambda & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \lambda & -\lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda & -\lambda
\end{array}\right]
\end{array}
$$

Case 1 (R = 1)

For $R = 1$, Q becomes:

$$
Q = \begin{array}{cc}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{cccccc}
-1 & 0 & 0 & 0 & 1 & 0 \\
2 & -3 & 0 & 0 & 0 & 1 \\
0 & 2 & -2 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 2 & -2 & 0 \\
0 & 0 & 0 & 0 & 2 & -2
\end{array}\right]
\end{array}
$$

The long run probabilities can be found using the matrix multiplication $\pi Q = \vec{0}$ where we retrieve the following simultaneous equations:

$$
\begin{aligned}
-\pi_0 + 2\pi_1 = 0 \quad &\rightarrow \quad \pi_1 = \frac{1}{2}\pi_0 \\
-3\pi_1 + 2\pi_2 = 0 \quad &\rightarrow \quad \pi_2 = \frac{3}{4}\pi_0 \\
-2\pi_2 + 2\pi_3 = 0 \quad &\rightarrow \quad \pi_3 = \frac{3}{4}\pi_0 \\
-2\pi_3 + 2\pi_4 = 0 \quad &\rightarrow \quad \pi_4 = \frac{3}{4}\pi_0 \\
\pi_0 - 2\pi_4 + 2\pi_5 = 0 \quad &\rightarrow \quad \pi_5 = \frac{1}{4}\pi_0 \\
\pi_1 - 2\pi_5 = 0 &
\end{aligned}
$$

Solving the simultaneous equations using $\sum_{i=0}^{5} \pi_i = 1$, we get:

$$1 = \pi_0\Big(1 + \frac{1}{2} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4}\Big)$$
$$\pi = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}$$

Case 2 (R = 2)

For $R = 2$, Q becomes:

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{array}$$

Using $\pi Q = 0$, we retrieve the following simultaneous equations:

$$\begin{aligned}
-\pi_0 + 2\pi_1 = 0 \quad &\rightarrow \quad \pi_1 = \frac{1}{2}\pi_0 \\
-3\pi_1 + 2\pi_2 = 0 \quad &\rightarrow \quad \pi_2 = \frac{3}{4}\pi_0 \\
-3\pi_2 + 2\pi_3 = 0 \quad &\rightarrow \quad \pi_3 = \frac{9}{8}\pi_0 \\
-2\pi_3 + 2\pi_4 = 0 \quad &\rightarrow \quad \pi_4 = \frac{9}{8}\pi_0 \\
\pi_0 - 2\pi_4 + 2\pi_5 = 0 \quad &\rightarrow \quad \pi_5 = \frac{5}{8}\pi_0 \\
\pi_1 - 2\pi_5 + 2\pi_6 = 0 \quad & \\
\pi_2 - 2\pi_6 = 0 \quad &\rightarrow \quad \pi_6 = \frac{3}{8}\pi_0
\end{aligned}$$

Solving the simultaneous equations using $\sum_{i=0}^{6} \pi_i = 1$, we get:

$$1 = \pi_0\Big(1 + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \frac{9}{8} + \frac{5}{8} + \frac{3}{8}\Big)$$
$$\pi = \begin{bmatrix} \frac{2}{11} & \frac{1}{11} & \frac{3}{22} & \frac{9}{44} & \frac{9}{44} & \frac{5}{44} & \frac{3}{44} \end{bmatrix}$$

Case 3 (R = 3)

For $R = 3$, Q becomes:

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{array}$$

Using $\pi Q = 0$, we retrieve the following simultaneous equations:

$$\begin{aligned}
-\pi_0 + 2\pi_1 = 0 &\quad\rightarrow\quad \pi_1 = \frac{1}{2}\pi_0 & \pi_0 - 2\pi_4 + 2\pi_5 = 0 &\quad\rightarrow\quad \pi_5 = \frac{19}{16}\pi_0 \\
-3\pi_1 + 2\pi_2 = 0 &\quad\rightarrow\quad \pi_2 = \frac{3}{4}\pi_0 & \pi_1 - 2\pi_5 + 2\pi_6 = 0 &\quad\rightarrow\quad \pi_6 = \frac{15}{16}\pi_0 \\
-3\pi_2 + 2\pi_3 = 0 &\quad\rightarrow\quad \pi_3 = \frac{9}{8}\pi_0 & \pi_2 - 2\pi_6 + 2\pi_7 = 0 & \\
-3\pi_3 + 2\pi_4 = 0 &\quad\rightarrow\quad \pi_4 = \frac{27}{16}\pi_0 & \pi_3 - 2\pi_7 = 0 &\quad\rightarrow\quad \pi_7 = \frac{9}{16}\pi_0
\end{aligned}$$

Solving the simultaneous equations using $\sum_{i=0}^{7} \pi_i = 1$, we get:

$$\begin{aligned}
1 &= \pi_0\Big(1 + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \frac{19}{16} + \frac{15}{16} + \frac{9}{16}\Big) \\
\pi &= \begin{bmatrix} \frac{4}{31} & \frac{2}{31} & \frac{3}{31} & \frac{9}{62} & \frac{27}{124} & \frac{19}{124} & \frac{15}{124} & \frac{9}{124} \end{bmatrix}
\end{aligned}$$

Average Steady State Inventory Level

Recall the long-term probabilities, as found in the previous section.

$$\begin{aligned}
\pi_{R=1} &= \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix} \\
\pi_{R=2} &= \begin{bmatrix} \frac{2}{11} & \frac{1}{11} & \frac{3}{22} & \frac{9}{44} & \frac{9}{44} & \frac{5}{44} & \frac{3}{44} \end{bmatrix} \\
\pi_{R=3} &= \begin{bmatrix} \frac{4}{31} & \frac{2}{31} & \frac{3}{31} & \frac{9}{62} & \frac{27}{124} & \frac{19}{124} & \frac{15}{124} & \frac{9}{124} \end{bmatrix}
\end{aligned}$$

To calculate the average inventory level at steady state, we simply use the formula for expectation:

$$\mathbb{E}[\text{Inventory}] = \mathbb{E}[X] = \sum_{i=0}^{R} \pi_i \cdot X_i$$

$$\begin{aligned}
\mathbb{E}[\text{Inventory}, R = 1] &= 0 \times \frac{1}{4} + 1 \times \frac{1}{8} + 2 \times \frac{3}{16} + 3 \times \frac{3}{16} + 4 \times \frac{3}{16} + 5 \times \frac{1}{16} \\
&= 2.125 \\
\mathbb{E}[\text{Inventory}, R = 2] &= 0 \times \frac{2}{11} + 1 \times \frac{1}{11} + 2 \times \frac{3}{22} + 3 \times \frac{9}{44} + 4 \times \frac{9}{44} + 5 \times \frac{5}{44} + 6 \times \frac{3}{44} \\
&= 2.7\overline{72} \\
\mathbb{E}[\text{Inventory}, R = 3] &= 0 \times \frac{4}{31} + 1 \times \frac{2}{31} + 2 \times \frac{3}{31} + 3 \times \frac{9}{62} + 4 \times \frac{27}{124} + 5 \times \frac{19}{124} + 6 \times \frac{15}{124} + 7 \times \frac{9}{124} \\
&= 3.565
\end{aligned}$$

Proportion of Time of Empty Inventory

For each scenario, the proportion of time when the inventory is empty is simply π_0. Thus, $\pi_0 = 0.25$, $\pi_0 = 0.\overline{18}$, and $\pi_0 = 0.129$ for $R = 1$, $R = 2$, and $R = 3$ respectively.

Question 3

$\{X(t), t \geq 0\}$ is a CTMC with state space $S = \{0, 1, 2\}$ and infinitesmall generator matrix:

$$Q = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$

Compute p_0, p_1, and p_2, the limiting distributions of the CTMC by solving for the balance equations.

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Answer: We utilise the following equations.

$$\pi Q = 0$$
$$\sum_{i \in S} \pi_i = 1$$

$$-3\pi_0 + 2\pi_1 + \pi_2 = 0 \tag{1}$$
$$2\pi_0 - 4\pi_1 + \pi_2 = 0 \tag{2}$$
$$\pi_0 + 2\pi_1 - 2\pi_2 = 0 \tag{3}$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \tag{4}$$

Subtracting equation 2 from 1 yields:

$$-5\pi_0 + 6\pi_1 = 0 \tag{5}$$
$$\pi_1 = \frac{5}{6}\pi_0 \tag{6}$$

Substituting 6 into 3 and 4:

$$\pi_0 + \frac{5}{3}\pi_0 - 2\pi_2 = 0$$
$$\pi_0 + \frac{5}{6}\pi_0 + \pi_2 = 1$$

Combining the above two equations, we eliminate π_2 and find π_0, π_1, and π_2.

$$3\pi_0 + \frac{10}{3}\pi_0 = 2 \quad \rightarrow \quad \pi_0 = \frac{6}{19}$$
$$\pi = \begin{bmatrix} \frac{6}{19} & \frac{5}{19} & \frac{8}{19} \end{bmatrix}$$

Question 4

A barbershop has 2 servers (i.e. hair stylists) and no waiting room. Arrivals to the shop are PP(λ). Server i (for i = 1, 2) takes exp(μ_i) amount of time to serve a customer. Let $(X_1(t), X_2(t))$ be the state of the system at time t, where $X_i(t)$ is the number of customers at server i. Assume that if there are more than one servers free, the arriving customer picks one of them at random with equal probabilities. Model the system as a CTMC.

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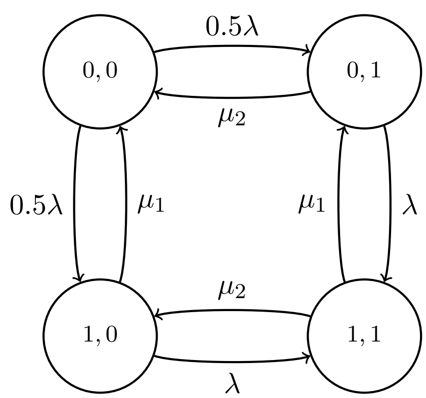
Answer: Let $(X_1(t), X_2(t))$ denote the state of the system at time t, where $X_i(t)$ is the number of customers at server i. Given that $X_i(t) \in \{0, 1\}$, our state space is simply $S = \{(0,0),(0,1),(1,0),(1,1)\}$.

- Given state (0,0), an arriving customer will choose server 1 or 2 with equal probability. As such, we perform Poisson Process splitting and the rates of transition to state (0,1) or (1,0) are 0.5λ respectively.
- Given state (0,1), either server 2 finishes its service with rate μ_2 or another customer arrives and gets served by server 1 with rate λ.
- Given state (1,0), either server 1 finishes its service with rate μ_1 or another customer arrives and gets served by server 2 with rate λ.
- Given state (1,1), either server 1 finishes its service with rate μ_1 or server 2 finishes its service with rate μ_2.

Thus our rate diagram.



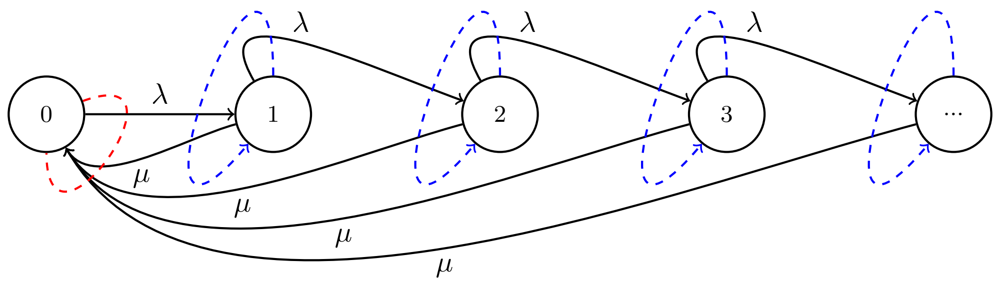
$$Q = \begin{array}{c} \\ (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \end{array} \\ \begin{bmatrix} -\lambda & \lambda/2 & \lambda/2 & 0 \\ \mu_1 & -\lambda-\mu_1 & 0 & \lambda \\ \mu_2 & 0 & \lambda-\mu_2 & \lambda \\ 0 & \mu_2 & \mu_1 & -\mu_2-\mu_1 \end{bmatrix} \end{array}$$

Question 6

For the bus problem (#2) in Homework 3, let $k = \infty$, that means the bus would be able to let in all the passengers that are waiting when it arrives. Draw the rate diagram for the CTMC and obtain the steady state distribution of the CTMC.

Answer: Let $X(t)$ be the number of customers waiting at the station at time t. As there is no limit to the number of customers at the station, we have the infinite state space $S = \{0, 1, 2, \dots\}$.

Each customer arrives following a poisson process with rate λ, whereas each bus arrives following a poisson process with rate μ. Whereas previously each bus can only carry a finite k people at once, now k is infinite, meaning we will always transition to state $X(t) = 0$ anytime the bus leaves, regardless of our previous state. Based on this, we get the following rate diagram.



Based on the above, we perform two types of cut (red and blue) and obtain the following steady state equations:

$$\lambda\pi_0 = \mu(\pi_1 + \pi_2 + \pi_3 + \dots)$$
$$(\lambda+\mu)\pi_1 = \mu\pi_0 \quad \rightarrow \quad \pi_1 = \left(\frac{\mu}{\mu+\lambda}\right)\pi_0$$
$$(\lambda+\mu)\pi_2 = \mu\pi_1 \quad \rightarrow \quad \pi_2 = \left(\frac{\mu}{\mu+\lambda}\right)^2\pi_0$$
$$(\lambda+\mu)\pi_3 = \mu\pi_2 \quad \rightarrow \quad \pi_3 = \left(\frac{\mu}{\mu+\lambda}\right)^3\pi_0$$

This gives:

$$\frac{\lambda}{\mu}\pi_0 = \pi_1 + \pi_2 + \pi_3 + \dots$$
$$\pi_i = \left(\frac{\mu}{\mu+\lambda}\right)\pi_{i-1} = \left(\frac{\mu}{\mu+\lambda}\right)^i\pi_0$$

Plugging π_0 into $\sum_{i=0} \infty \pi_i$, we get:

$$1 = \pi_0 + \pi_1 + \pi_2 + \dots = \pi_0 + \frac{\lambda}{\mu}\pi_0$$
$$\pi_0 = \frac{\mu}{\mu+\lambda}$$

Thus, we have:

$$\boxed{\pi_i = \left(\frac{\mu}{\mu+\lambda}\right)^{i+1} \quad \forall i \in S \dots}$$