Question 1

An inventory system uses continuous control as follows. Demands arrive one at a time according to a PP(λ) (that means time between demands $\sim \exp(\lambda)$). When a demand arrives, if there are items in inventory, it is immediately satisfied, otherwise the demand is lost. As soon as the inventory level drops to R, an order is placed for K items from a supplier. However, the order is fulfilled after an $\exp(\theta)$ amount of time. Assume that $K > R$. Let $X(t)$ be the number of items in inventory at time t. Model $\{X(t), t \geq 0\}$ as a CTMC with the assumption that $R < X(0) \leq K + R$.

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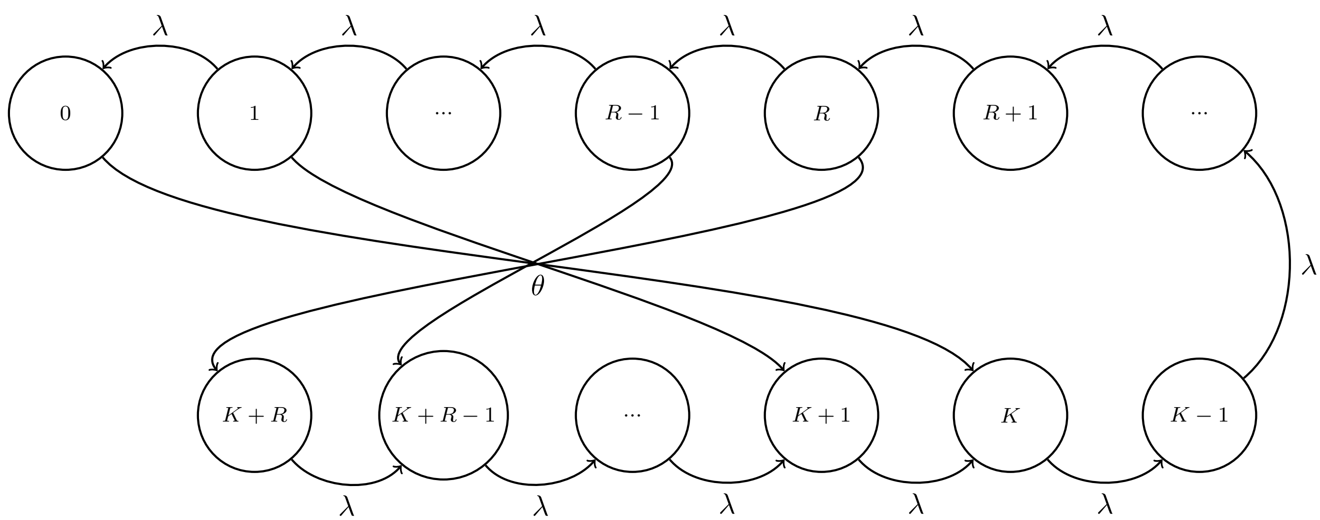
Answer: We begin by defining $X(t)$ as the number of items in inventory at time t. Now, define the state space S as all integers from 0 to $K + R$, based on the assumption that $R < X(0) \leq X + R$.

$$S = \{0, 1, 2, \ldots, K + R\}$$

We now draw the rate diagram (only showing arrows with positive rates). For all states, the rate of transition to a state 1 unit lower is simply given by λ, as mean time between demands follows $\exp(\lambda)$. Meanwhile, for states R and lower, due to reordering, we can transition into a state K units higher with rate given by θ. This follows the fact that the reorder time follows $\exp(\theta)$.



With this, we get the generating matrix Q.

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ R-1 \\ R \\ R+1 \\ \vdots \\ K-1 \\ K \\ K+1 \\ \vdots \\ K+R-1 \\ K+R \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} 0 & 1 & \cdots & R-1 & R & R+1 & \cdots & K-1 & K & K+1 & \cdots & K+R-1 & K+R \end{array} \\
\begin{bmatrix}
-\theta & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \theta & 0 & \cdots & 0 & 0 \\
\lambda & -\theta-\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \theta & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\theta-\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \theta & 0 \\
0 & 0 & \cdots & \lambda & -\theta-\lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \theta \\
0 & 0 & \cdots & 0 & \lambda & -\lambda & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda & -\lambda & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \lambda & -\lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda & -\lambda
\end{bmatrix}
\end{array}$$

To define Q explicitly:

$$Q_{i \to j} = \left\{ \begin{array}{lr} \lambda, & \text{if } j = i - 1 \\ \theta, & \text{if } j = i + K, i \leq R \\ -\theta, & \text{if } j = i = 0 \\ -\theta - \lambda, & \text{if } j = i \leq R \\ -\lambda, & \text{if } j = i > R \\ 0, & \text{otherwise} \end{array} \right\}$$

Question 2

Consider a bus station where customers arrive in PP(λ) fashion. Busses arrive empty one after another in PP(μ) fashion. Each bus has capacity to carry a maximum of k customers. When a bus arrives at the station, if there are x customers waiting then the bus departs instantaneously with $\min(x, k)$ passengers. Let $X(t)$ be the number of customers waiting at the station at time t. Model $\{X(t), t \geq 0\}$ as a CTMC.

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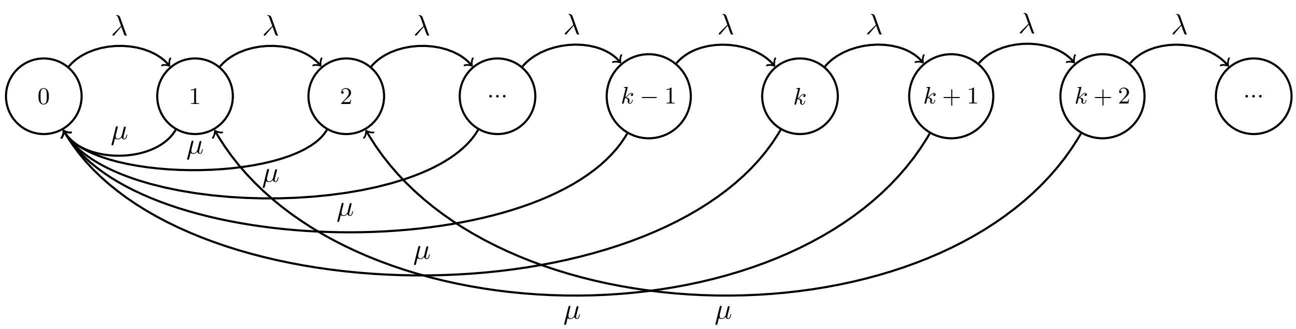
Answer: Let $X(t)$ be the number of customers waiting at the station at time t. As there is no limit to the number of customers at the station, we have the infinite state space $S = \{0, 1, 2, \ldots\}$.

Each customer arrives following a poisson process with rate λ, whereas each bus arrives following a poisson process with rate μ. Also, each bus can only carry k people at once. This means that if $X(t) = i \leq k$ right before the bus arrives, $X(t) = 0$ right after the bus leaves. Meanwhile, if $X(t) = i > k$ right before the bus arrives, then $X(t) = i - k$ right after the bus leaves. Based on this, we get the following rate diagram.



Hence our matrix generator Q.

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k-1 \\ k \\ k+1 \\ k+2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & k-1 & k & k+1 & \ldots \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & \ldots & 0 & 0 & 0 & \ldots \\ \mu & -\mu-\lambda & \lambda & \ldots & 0 & 0 & 0 & \ldots \\ \mu & 0 & -\mu-\lambda & \ldots & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots \\ \mu & 0 & 0 & \ldots & -\mu-\lambda & \lambda & 0 & \ldots \\ \mu & 0 & 0 & \ldots & 0 & -\mu-\lambda & \lambda & \ldots \\ 0 & \mu & 0 & \ldots & 0 & 0 & -\mu-\lambda & \ldots \\ 0 & 0 & \mu & \ldots & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

$$Q_{i\to j} = \left\{ \begin{array}{lr} \lambda, & \text{if } j = i+1 \\ \mu, & \text{if } j = 0, 1 \le i \le k \\ \mu, & \text{if } j = i-k \ge 1 \\ -\lambda, & \text{if } i = j = 0 \\ -\mu - \lambda, & \text{if } i = j > 0 \\ 0, & \text{otherwise} \end{array} \right\}$$

Question 3

Consider a postoffice where there is a seperate line for passport applications. The postoffice adopts the following policy: no one is appointed to serve the customers if there are fewer than N customers waiting for their passport application. As soon as there are N customers waiting, the post office sends a staff member to process the applications one by one. Notice that while processing, new customers could join the line. The staff member continues processing until all customers are served and the line is empty, at which time the staff member leaves and returns only when there are N customers in the line. Customers arrive for passport applications according to PP(λ) and processing time for each application is $\sim \exp(\mu)$. Let $X(t)$ be the number of customers in the system at time t, and $Y(t)$ is one or zero depending on whether or not a staff member is assigned to the line at time t. Model $\{(X(t), Y(t)), t \ge 0\}$ as a bivariate CTMC.

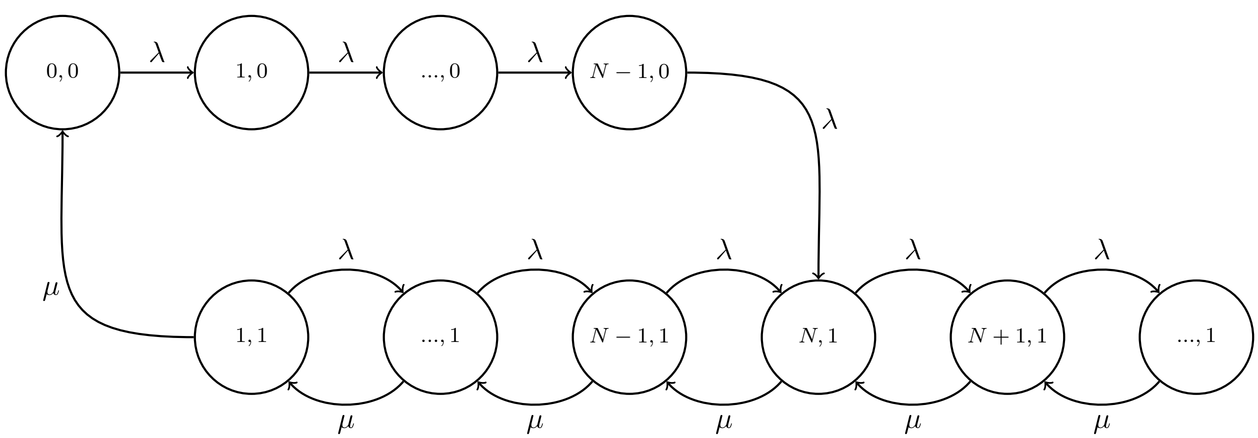
Answer: We begin by defining the state of the system at t. We have $X(t)$, which is the number of customers queueing at the postoffice, and $Y(t)$, a boolean variable indicating whether or not a staff is currently working. $X(t)$ and $Y(t)$ form the dual state $(X(t), Y(t))$ at a given time t.

$$Y(t) = \left\{ \begin{array}{lr} 1, & \text{if staff is active} \\ 0, & \text{otherwise} \end{array} \right\}$$

The dual state space is given as follows:

$$S = \{(0,0), (1,0), \ldots, (N-1,0), (0,1), (1,1), \ldots, (N,1), (N+1,1), \ldots\}$$

Now we can draw the state diagram.



With this, we can define our generator matrix as follows.

$$Q_{i\to j} = \left\{ \begin{array}{lr} \mu, & \text{if } i=(a,1), j=(a-1,1), a>1 \\ \mu, & \text{if } i=(1,1), j=(0,0) \\ \lambda, & \text{if } i=(a,b), j=(a+1,b), a<N-1, b\in\{0,1\} \\ \lambda, & \text{if } i=(N-1,0), j=(N,1) \\ -\lambda, & \text{if } i=j=(a,0), a<N \\ -\mu-\lambda, & \text{if } i=j=(a,1), a\geq 0 \\ 0, & \text{otherwise} \end{array} \right\}$$

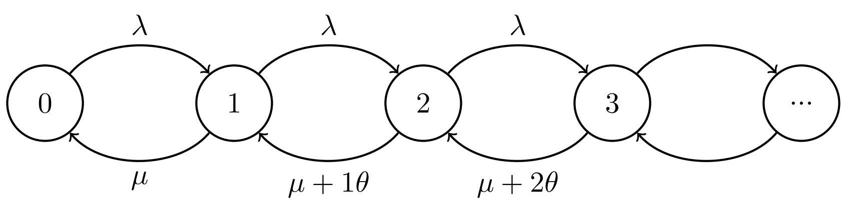
$$Q = \begin{array}{c} \\ (0,0) \\ (1,0) \\ (2,0) \\ \vdots \\ (N-2,0) \\ (N-1,0) \\ (1,1) \\ (2,1) \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} (0,0) & (1,0) & (2,0) & \cdots & (N-2,0) & (N-1,0) & (1,1) & (2,1) & \cdots \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & 0 & 0 & \cdots \\ \mu & 0 & 0 & \cdots & 0 & 0 & -\mu-\lambda & \lambda & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu & -\mu-\lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Question 4

Consider a queue with infinite waiting space and a single server. Customers are served according to first-come-first-served basis. Every customer has a "patience time", i.e. if he has to wait for service initiation longer than the patience time, he will leave the queue without service. Assume that patience times of all arriving customers are i.i.d. $\exp(\theta)$ random variables. Let the arrival process by PP(λ) and the service times be i.i.d. $\exp(\mu)$. Let $X(t)$ be the number of customers in the system at time t. Model $\{X(t), t \geq 0\}$ as a CTMC.

. . . .

Answer: We begin by defining the state $X(t)$ as the number of customers in the system. Given an infinite waiting space, we have an infinite state space $S = \{0, 1, 2, \ldots\}$. Customers arrive following a poisson process with rate λ. Meanwhile, customers leave either when they run out of patience (where each queueing customer's patience time follows an i.i.d. $\exp(\theta)$ random variable) or when the server finishes serving the previous customer (where each service time follows an i.i.d. $\exp(\mu)$ random variable). The rate depends on the current number of queueing customers. If there is currently 1 customer queueing ($X(t) = 2$), the rate of transition to state 1 is $\mu + \theta$. If, however, there are 2 customers queueing ($X(t) = 3$), the rate of transition to state 2 is $\mu + 2\theta$, as now there are 2 people who are likely to lose their patience. Similarly, given 3 queueing customers, the rate of transition to state 3 is $\mu + 3\theta$. Hence we have our rate diagram.



With this, we can define our generator matrix as follows.

$$Q_{ij} = \left\{ \begin{array}{lr} \lambda, & \text{if } i = j - 1 \\ -\lambda, & \text{if } i = j = 0 \\ \mu + (i-1)\theta, & \text{if } i = j + 1 \\ -(\mu + (i-1)\theta + \lambda), & \text{if } i = j > 0 \\ 0, & \text{otherwise} \end{array} \right\}$$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots \\ 0 & \mu+\theta & -(\mu+\theta+\lambda) & \lambda & \cdots \\ 0 & 0 & \mu+2\theta & -(\mu+2\theta+\lambda) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

Question 5

Consider the *Unslotted Aloha* system where messages arrive according to PP(λ). As soon as a message arrives, it attempts transmission. The message transmission times are exponentially distributed with mean $1/\mu$ units of time. If no other message tries to transmit during the transmission time of this message, the transmission is successful. If any other message tries to transmit during this transmission, a collision results and all transmissions are terminated instantly. All messages involved in a collision are called backlogged and are forced to retransmit. All backlogged messages wait for an exponential amount of time (with mean $1/\theta$) before starting retransmission. Let $X(t)$ denote the number of backlogged messages at time t and $Y(t)$ be a binary variable that denotes whether or not a message is under transmission at time t. Model the process $\{(X(t), Y(t)), t \geq 0\}$ as a CTMC by drawing the rate diagram.

. . . .

Answer: We first begin by defining the dual state $(X(t), Y(t))$. $X(t)$ is the number of backlogged messages, spanning from $0, 1, 2, \ldots$. Meanwhile, $Y(t)$ is a boolean variable indicating whether or not a transmission is currently happening.

$$Y(t) = \left\{ \begin{array}{ll} 1, & \text{if a transmission is currently happening} \\ 0, & \text{otherwise} \end{array} \right\}$$

With this, we have our state space.

$$S = \{(0,0), (1,0), (2,0), \ldots, (0,1), (1,1), (2,1), \ldots\}$$

Suppose we begin with 0 backlogged messages and 0 transmissions. Thus, we are currently at state $(0,0)$. We can transition into a new state only when a new message arrives. This message is immediately transmitted and we enter state $(0,1)$. This takes place with rate λ, which is the rate of arrival for new messages.

From state $(0,1)$, we can now transition into two new states. Either a new message arrives and interrupts the current transmission, and we transition into state $(2,0)$; or the current transmission is successful and we transition into state $(0,0)$. The former takes place with rate λ whereas the latter takes place with rate μ, which is the rate at which a transmission is completed.

The pattern continues. From state $(2,0)$, we can transition into either $(2,1)$ when a new message arrives or $(1,1)$ if a backlogged message now attempts retransmission. Again, the former takes place at rate λ whereas the latter at rate μ.

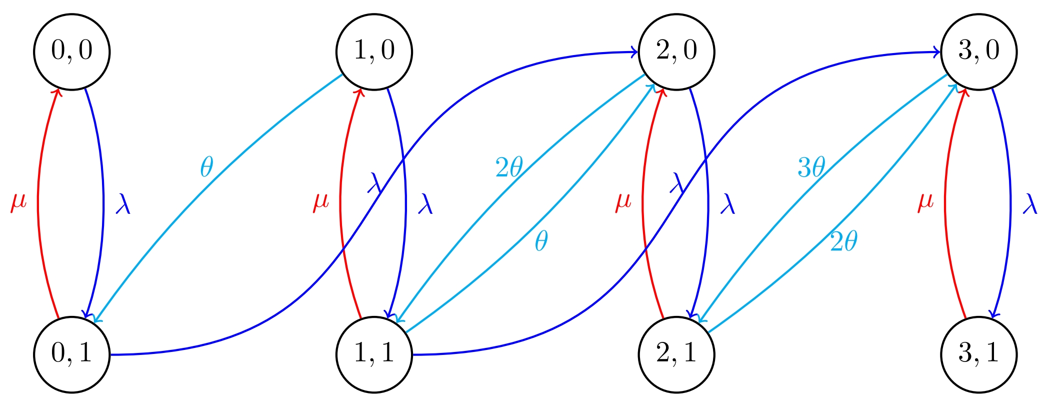
From state $(2,1)$, either:

1. a new message arrives and interrupts the current transmission, and we enter state (4,0);
2. a backlogged message attempts retransmission, interrupts the current transmission, and we enter state (3,0); or
3. the current transmission succeeds and we enter state $(2,0)$.

By continuing this analysis, we can generalise the following cases for the entire state space.

- $[Y(t) = 0]$ There is currently no transmission occuring.
  - $[X(t) = 0]$ There are currently no backlogged messages.
    - A new message arrives $[\lambda]$ and we enter state $(0, 1)$.
  - $[X(t) = x > 0]$ There are some backlogged messages.
    - A new message arrives $[\lambda]$ and we enter state $(x, 1)$.
    - A backlogged message attempts retransmission $[x\theta]$ and we enter state $(x - 1, 1)$.
- $[Y(t) = 1]$ There is an ongoing transmission.
  - $[X(t) = 0]$ There are currently no backlogged messages.
    - A new message arrives $[\lambda]$ and we enter state $(2, 0)$.
    - The current transmission succeeeds $[\mu]$ and we enter state $(0, 0)$.
  - $[X(t) = x > 0]$ There are some backlogged messages.
    - A new message arrives $[\lambda]$ and we enter state $(x + 2, 0)$.
    - A backlogged message attempts retransmission $[x\theta]$ and we enter state $(x + 1, 0)$.
    - The current transmission succeeeds $[\mu]$ and we enter state $(x, 0)$.

Notice that the transition rate of backlogged messages attempting retransmission depends on the current number of backlogged messages. Given x backlogged messages at present, then the rate of transition to a corresponding state is given by $x\theta$. Thus, we have our rate diagram (only showing until node (3,1), but the state space extends to infinity).



We can define Q explicitly as follows.

$$Q_{i\to j} = \left\{ \begin{array}{ll} \mu, & \text{if } i=(a,1), j=(a,0),\ a\in\{0,1,\dots\} \\ \lambda, & \text{if } i=(a,0), j=(a,1),\ a\in\{0,1,\dots\} \\ \lambda, & \text{if } i=(a,1), j=(a+2,0),\ a\in\{0,1,\dots\} \\ a\theta, & \text{if } i=(a,0), j=(a-1,1),\ a\in\{1,2,\dots\} \\ a\theta, & \text{if } i=(a,1), j=(a+1,0),\ a\in\{0,1,\dots\} \\ -\lambda, & \text{if } i=j=(0,0) \\ -a\theta-\mu-\lambda, & \text{if } i=j=(a,1)\ a\in\{0,1,\dots\} \\ -a\theta-\lambda, & \text{if } i=j=(a,0)\ a\in\{0,1,\dots\} \\ 0, & \text{otherwise} \end{array} \right\}$$

Displaying Q in matrix form:

$$Q = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \\ (2,0) \\ (2,1) \\ (3,0) \\ (3,1) \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} (0,0) & (0,1) & (1,0) & (1,1) & (2,0) & (2,1) & (3,0) & (3,1) & \cdots \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu & -\mu-\lambda & 0 & 0 & \lambda & 0 & 0 & 0 & \cdots \\ 0 & \theta & -\theta-\lambda & \lambda & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \mu & -\theta-\mu-\lambda & \theta & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & 2\theta & -2\theta-\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mu & -2\theta-\mu-\lambda & 2\theta & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 3\theta & -3\theta-\lambda & \lambda & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & -3\theta-\mu-\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$