# Estimating a Linear Stochastic Discount Factor

MATH 364 Final Project

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## I. Introduction

Investment pricing must account for both the risk associated with returns and the opportunity cost of the capital needed to fund the investment. Discounting investment payoffs using the time value of money captures the latter. The stochastic discount factor (SDF) improves on this approach, by defining the discount factor as a random variable that can capture the conditional relationship between the investment return and the utility of that return. This SDF is not known, however; it must be estimated using predictors of investment risk. We estimate the SDF as a linear function of company fundamentals and macroeconomic variables. We, then, leverage properties of the factor to extract an implicit estimation of the tangent portfolio. We test an investment strategy using this portfolio on a universe of 65 real estate companies between 2015 and 2023. We find that this portfolio outperforms the universe index.

## II. Existing Research

Research has previously explored several methods of leveraging the SDF to create an investment strategy. Taking a statistical approach, [2] creates a modified version of principal component analysis that takes advantage of SDF theory. This procedure allows PCA factor loadings to vary, conditional on asset characteristics and macroeconomic factors. From these loadings, one can extract a portfolio that maximizes returns conditional on these factors.

Taking a computationally different but theoretically similar approach, [1] uses machine learning to estimate the SDF as a function of several asset-specific and macroeconomic variables. It does so by minimizing the mispricing error of the most mispriced assets. From this estimation, it extracts the implicit weights of the tangent portfolio then shows the superior performance of this portfolio compared to several benchmarks.

## III. Model Formulation

Taking a similar approach to [1], we estimate the SDF as a function of several company-specific and macroeconomic variables. We do so by taking advantage of the relationship between the SDF and minimizing the sum of squared mispricing errors. From this factor, we extract the implied tangent portfolio weights and evaluate that portfolio's performance.

### A. The SDF

The SDF is an extension of the time value of money discount rate. The discount rate,

$$d = \frac{1}{1+r_f},$$

where $r_f$ is the risk-free rate of return over a given period, captures the effects of the time value of money on an asset's price. Using this factor, an asset with deterministic payoffs $c_i,\ 1 \le i \le n$ at the end of each period $i$ should have price

$$p = \sum_{i=1}^{n} c_i d^i.$$

This factor captures the risk-free payoff the investor forgoes by buying the asset.

This factor can be understood more generally to capture difference in the utility of wealth between two states of the world: the present and one step in the future. Because money can grow in time, money today is worth more than money tomorrow, so we get the common inequality $d < 1$.

This factor, however, assumes that the payoff and the utility of wealth are independent (or, even more simplistically, deterministic). The expression above can more generally be expressed as

$$p = \sum_{i=1}^{n} E[c(\omega_i)d(\omega_i)],$$

where the values $c$ and $d$ have become random variables dependent on the state of the world at time $i$, $\omega_i$. This expectation only expands to the deterministic sum above if the payoff and the discount factor are independent.

The payoff and the discount factor are rarely independent, especially in the context of public equities. Consider a catastrophic event in the market: the marginal utility of wealth in this scenario would increase, as it will take less money to buy the same securities; the payoff on a given stock will decrease with the market. Now consider the opposite event: the marginal utility of wealth in this scenario would decrease relatively as more money is now required to buy the same securities, and the payoff of that same stock would increase with the market. Clearly the utility of wealth and the payoff are not independent in these scenarios.

The deterministic equation overlooks significant effects of this dependence. The SDF addresses these shortcomings of a deterministic discount factor. The SDF at time $t$ is the value $M_{t+1}(\omega)$ such that

$$E[M_{t+1} r^e_{i,t+1}] = 0, \tag{1}$$

where $r_{t,i}(\omega)$ is the random variable representing the excess return on an asset $i$ at time $t$. That is, there is no investment with non-zero expected return; all assets are priced correctly.

### B. The SDF and the Tangent Portfolio

The SDF can be linked to the mean-variance efficient tangent portfolio. As [3] shows,

$$M_{t+1} = 1 - F_{t+1},$$

where

$$F_{t+1} = \frac{(\mathbf{w}^T)'\mathbf{r}^e_{t+1}}{\gamma_t},$$

for vector of excess returns $\mathbf{r}^e_t$. That is, $F_{t+1}$ is the return on the mean-variance efficient tangent portfolio scaled by a time-varying coefficient $\frac{1}{\gamma_t}$. As a result,

$$M_{t+1} = 1 - \frac{(\mathbf{w}^T)'\mathbf{r}^e_{t+1}}{\gamma_t}. \quad (2)$$

### C. Estimating the SDF

In practice, the SDF can be estimated simply by using the Equation (1). Historical returns are known, so the SDF can be fit by the problem

$$\min_{\boldsymbol{\theta}} \sum_i \sum_t (M_{t+1}(\boldsymbol{\theta}) r^e_{i,t+1})^2,$$

minimizing the squared mispricing error. This formulation can be made conditional on inputs $\mathbf{x}$ (proxies for $\omega$), making the problem

$$\min_{\boldsymbol{\theta}} \sum_i \sum_t (M_{t+1}(\mathbf{x};\boldsymbol{\theta}) r^e_{i,t+1})^2.$$

This empirical estimation can be modified into an asset management strategy by using the relation in Equation (2). If one wants to invest in the tangent portfolio, then one can learn a model for its weights by defining

$$M_{t+1}(\mathbf{x};\boldsymbol{\theta},\gamma) = 1 - \frac{(\mathbf{w}^T(\mathbf{x};\boldsymbol{\theta}))'\mathbf{r}^e_{t+1}}{\gamma}.$$

The problem is now

$$\min_{\boldsymbol{\theta},\gamma} \sum_i \sum_t (1 - \frac{(\mathbf{w}^T(\mathbf{x};\boldsymbol{\theta}))'\mathbf{r}^e_{t+1}}{\gamma})^2 (r^e_{i,t+1})^2.$$

## IV. Technical Implementation

To improve model robustness, we introduce an L2 regularization term into our objective function. We weight this term against the objective with a constant coefficient $\lambda$. Our final implementation uses this combined objective, addressing the problem

$$\min_{\boldsymbol{\theta},\gamma} \sum_i \sum_t (1 - \frac{(\mathbf{w}^T(\mathbf{x};\boldsymbol{\theta}))'\mathbf{r}^e_{t+1}}{\gamma})^2 (r^e_{i,t+1})^2 + \frac{\lambda}{10}||\boldsymbol{\theta}||^2. \quad (3)$$

Using the dataset provided in class for 65 real estate companies over 97 months from 2015 to 2023, we jointly estimate a model of the SDF and tangent portfolio according to Equation (3). Our dataset includes 3 fundamental variables for each company and 3 macroeconomic factors in addition to each company's excess returns (descriptive statistics are shown in the Appendix in Table II). We normalize each factor before inputting them to the model. We train on values between 2015 until the beginning of 2021. We validate the model using values between the beginning of 2021 and the beginning of 2023.

Our implementation uses gradient descent (in the style of Algorithm 1) to minimize the objective. First, we define factor loadings $\mathbf{b}$ and scaling factor $\gamma$, both initialized randomly. We use these parameters to calculate

$$\boldsymbol{\omega}_t(\mathbf{X};\mathbf{b}) = \mathbf{X_t b}$$

and

$$M_{t+1}(\mathbf{X};\mathbf{b}) = 1 - \frac{\boldsymbol{\omega}'_t(\mathbf{X};\mathbf{b})\mathbf{r}^e_{t+1}}{\gamma},$$

where $\mathbf{X}_{t,i,k}$ is a three-dimensional tensor containing each value of factor $k$ for each company $i$ at time $t$ and $\boldsymbol{\omega}_t$ is an approximator of $\mathbf{w}^T$ and should not be confused with random state $\omega$ discussed above. We use the Adam optimizer to learn these parameters, modifying the parameter update step in Algorithm 1.

**Algorithm 1** Gradient Descent Procedure

**Input:**
- observations $\mathbf{X}$
- iterations $N$
- regularization coefficient $\lambda$
- learning rate $\eta$

**Output:** Estimated parameters $\mathbf{b}^{(N)}$

Initialize $\mathbf{b}^{(0)} \sim \mathcal{U}(0,1)^6$
Define $\mathcal{L}(M_{t+1},\mathbf{b}) = \sum_i \sum_t (M_{t+1})^2 (r^e_{i,t+1})^2 + \frac{\lambda}{10}||\mathbf{b}||^2$
Initialize $\gamma^{(0)} \leftarrow 1$
**for** $n = 0$ **to** $N$ **do**
  $\boldsymbol{\omega}^{(n)}_t \leftarrow \mathbf{X_t b}^{(n)}$
  $M^{(n)}_{t+1} \leftarrow 1 - \frac{(\boldsymbol{\omega}^{(n)}_t)'\mathbf{r}^e_{t+1}}{\gamma^{(n)}}$
  $g^{(n)}_\gamma, g^{(n)}_{\mathbf{b}} \leftarrow \nabla_{\gamma,\mathbf{b}}\mathcal{L}(M^{(n)}_{t+1},\mathbf{b})$
  $\gamma^{(n+1)} \leftarrow \gamma^{(n)} - \eta g^{(n)}_\gamma$
  $\mathbf{b}^{(n+1)} \leftarrow \mathbf{b}^{(n)} - \eta g^{(n)}_{\mathbf{b}}$
**end for**
**return** $\mathbf{b}^{(T)}$

## V. Results

We test our optimization using several different gradient descent algorithms (training losses shown in Figure 1 and validation results shown in Figure 2). We find that Adam is the most effective optimizer, converging to a better solution than stochastic gradient descent (although it does so more slowly). This best model achieves a total squared mispricing error of $2.525 \times 10^3$.

To verify our model, we observe the predicted stochastic discount factor over our validation set (Figure 7 in the Appendix). We find that $M_{t+1}$ takes positive values and fluctuates significantly, indicating that our model is outputting a valid SDF that varies with our data. Our estimates of factor loadings can be found in Table I and are shown in Figure 3. The most significant loading is on Extraordinary Items and Discontinued Operations with New House Building Permits

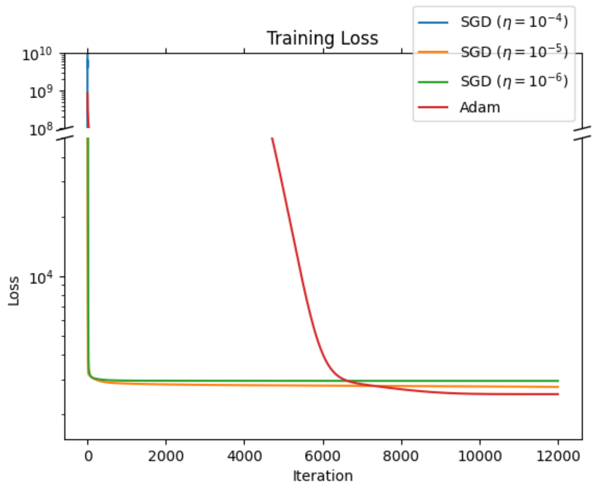


Fig. 1. Adam minimizes the model objective most effectively. SGD with various learning rates either diverge (as in the case of $\eta = 10^{-4}$) or do not efficiently reach the value that Adam does. The best model achieves a total squared mispricing error of $2.525 \times 10^3$.

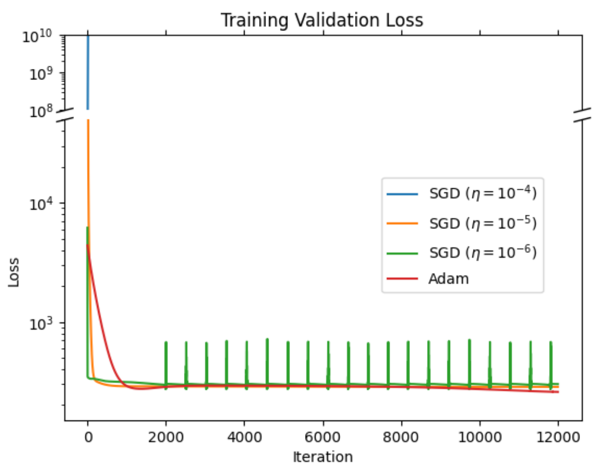


Fig. 2. Adam shows the best results on the validation set as well. SGD with $\eta = 10^{-4}$ still diverges. Interestingly, SGD with $\eta = 10^{-6}$ has an oscillating loss with periodic spikes.

and Net Cash Flows from Operating Activities having near negligible loadings.

TABLE I
FACTOR LOADINGS

| Variable | Factor Loading |
|---|---|
| `st_rev` | $-1.87 \times 10^{-2}$ |
| `oancfy` | $-3.20 \times 10^{-4}$ |
| `xoprq` | $-6.51 \times 10^{-2}$ |
| `FEDFUNDS` | $-2.04 \times 10^{-3}$ |
| `PERMITS` | $5.15 \times 10^{-5}$ |
| `UEMPLT5` | $2.67 \times 10^{-3}$ |

Using the learned parameters $\mathbf{b}$ from our model, we backtest a portfolio with weights given by $\boldsymbol{\omega}_t$ out-of-sample. Early in the test period, the portfolio invests significantly, though not fully. Later in the test period, the portfolio has nearly full short exposure. Contrary to what the factor loadings might suggest, this trend is likely due to changes in the Federal Funds Rate and New House Building Permits Issued. Although these variables' loadings are small, the change in their normalized values are reciprocally large in magnitude. These trends likely drove the change in investment over time. Figure 8 in the Appendix shows the full portfolio investment over the validation period, and Figure 9 in the Appendix shows the average trends in each factor over the same period. Together, these figures illustrate the coinciding similarities between these trends.

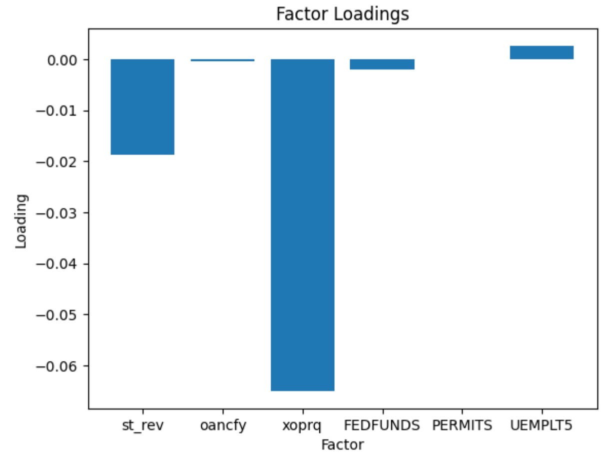


Fig. 3. Extraordinary Items and Discontinued Operations and Standardized Revenue have significant loadings. New House Building Permits and Net Cash Flows from Operating Activities have near negligible loadings.

The portfolio returns outperform a risk-free investment over the validation period and achieve a Sharpe Ratio of 0.33 (Figure 4). We also test a portfolio hedged against sector exposure by shorting an index of our universe (Figure 5). This portfolio performs similarly on total period return basis, though it has signifcantly more drawdown and a lower Sharpe Ratio of 0.22.

In Figure 6, we compare our portfolio to our universe index portfolios (equal weighted and market value weighted) and the risk-free portfolio. Except in the beginning of our test period, our portfolio shows a significant inverse relationship with the sector indices. As a result, it is very successful at capturing the significant losses affecting the whole sector in the final portion of the test period. This performance aligns with the trends from Figure 8. The portfolio's shift from long to short coincides with the reversal of the index trend.

Interestingly, the strategy underperforms the index during most of the remaining test period. Both indices underperform our portfolio on a risk-adjusted basis, with the market value index and equal weighted index both having Sharpe Ratios if $-0.074$.

We also test the effect of factor loading regularization on our portfolio. Figure 10 shows the Sharpe Ratios generated by portfolios based on parameters fit with different regularization

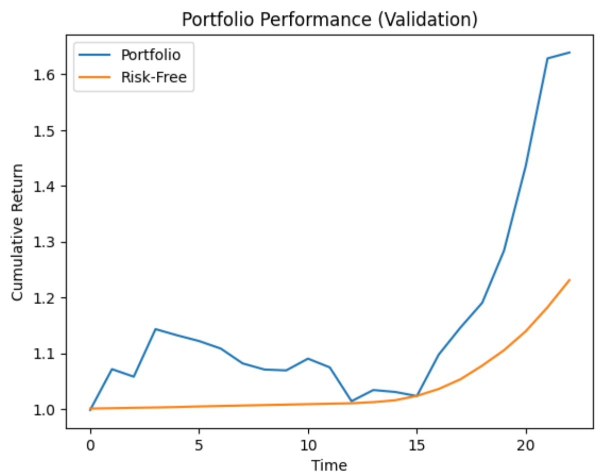


Fig. 4. Our estimate of the tangent portfolio outperforms a risk-free investment with Sharpe Ratio $0.33$.

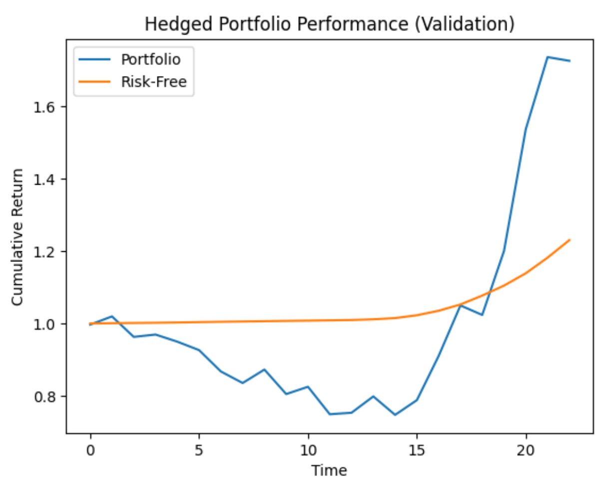


Fig. 5. Our estimate of the tangent portfolio hedged against sector indices has more drawdown and a lower Sharpe Ratio ($0.22$) than our unhedged portfolio.

weights $\lambda$. This curve shows a general trend with average Sharpe Ratio decreasing as $\lambda$ increases, though portfolios calculated with low regularization have significantly wider ranges of Sharpe Ratios (comparing locally in $\lambda$). The lack of monotonicity in this curve is particularly interesting. This suggests that small changes in regularization cause significant changes in the factor weights, which have rather unpredictable effects on performance in our test period. This could be a result of failing to capture returns during outlier events in the test period, suggesting that our model might be overfit.

Although we can observe a general relationship between the strategy's Sharpe Ratio and the value of $\lambda$, the lack of monotonicity in the curve makes it very difficult to use these results to choose an optimal $\lambda$. For example, if we choose $\lambda = 45$, we find a Sharpe Ratio of approximately $0.32$. Changing this $\lambda$ by just $1$ in either direction, however, results in a Sharpe Ratio decrease of more than $.1$ units.

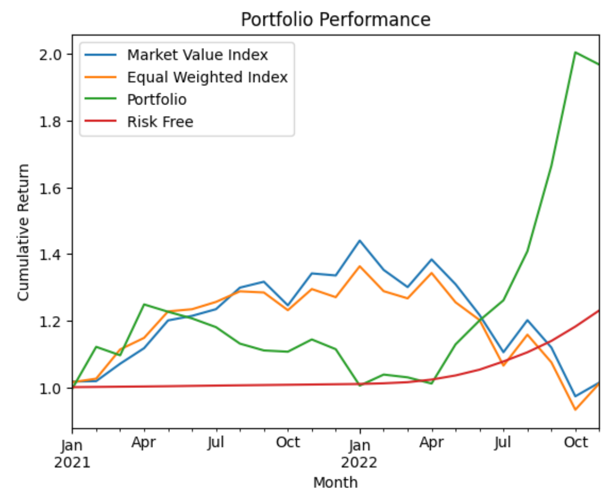


Fig. 6. Our estimate of the tangent portfolio outperforms the risk-free investment and sector indices over out test period.

## VI. Conclusion

Using a linear model, we estimate the stochastic discount factor as a function of several macroeconomic and company fundamental variables. Taking advantage of the relationship between the SDF and the mean-variance efficient tangent portfolio, we define an implied investment strategy. To examine our estimation, we backtest this strategy, finding that it outperforms universe indices on a risk-adjusted basis.

## References


[1] Luyang Chen, Markus Pelger, and Jason Zhu. Deep learning in asset pricing. *Management Science*, 70(2):714–750, 2024.

[2] Bryan T. Kelly, Seth Pruitt, and Yinan Su. Characteristics are covariances: A unified model of risk and return. *Journal of Financial Economics*, 134(3):501–524, 2019.

[3] Henry Schellhorn and Tianmin Kong. *Machine Learning for Asset Management and Pricing*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2024.

## APPENDIX

TABLE II
TRAINING DATA DESCRIPTIVE STATISTICS

| Variable | Description | Mean | Std. Dev. | Min | Max | Median |
|---|---|---|---|---|---|---|
| st_rev | Standardized annual revenue | -0.0057 | 0.3496 | -21.5 | 2.99 | -0.0015 |
| oancfy | Net cash flow from operating activities (annual, $m) | 290.0 | 394.5 | -164.0 | 3808 | 158.8 |
| xoprq | Extraordinary items & discontinued operations (quarterly, $m) | 191.3 | 216.7 | 1.39 | 1391 | 120.4 |
| FEDFUNDS | Effective federal funds rate | 0.00983 | 0.00825 | 0.0005 | 0.0242 | 0.00725 |
| PERMITS | New housing building permits (thousands) | 660.9 | 84.7 | 523 | 900 | 641 |
| UEMPLT5 | Persons unemployed (longer than 5 weeks) (thousands) | 2501 | 1432 | 1849 | 14263 | 2294 |
| excess_return | Stock excess return over risk-free rate | -0.00784 | 0.0941 | -0.725 | 2.973 | -0.00956 |

*Data Source*: Given in class by Henry Schellhorn and Theeraphat Pothisawang

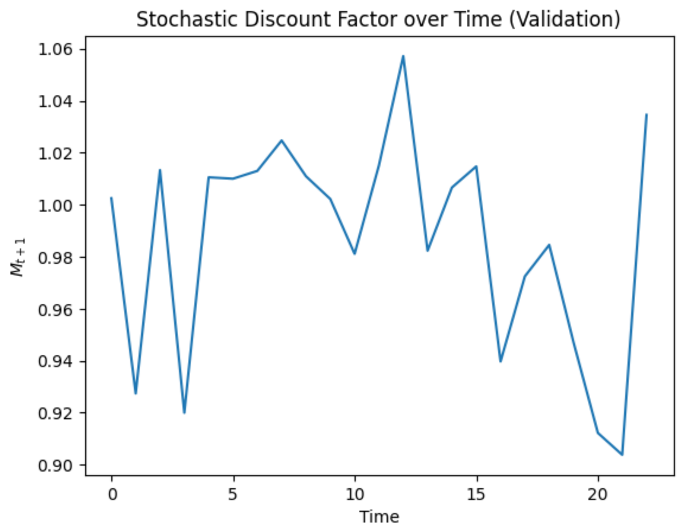


Fig. 7. $M_{t+1}$ takes positive values and fluctuates significantly, indicating that our model is outputting a valid SDF that varies with our data.

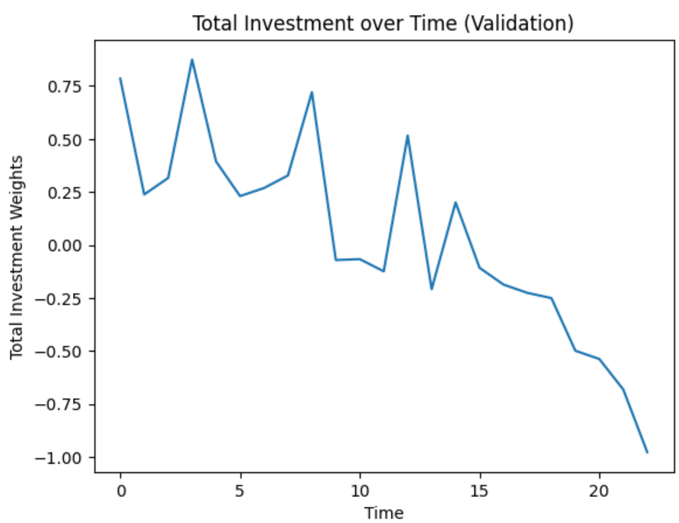


Fig. 8. Early in the test period, the portfolio invests significantly, though not fully. Later in the test period, the portfolio has nearly full short exposure.

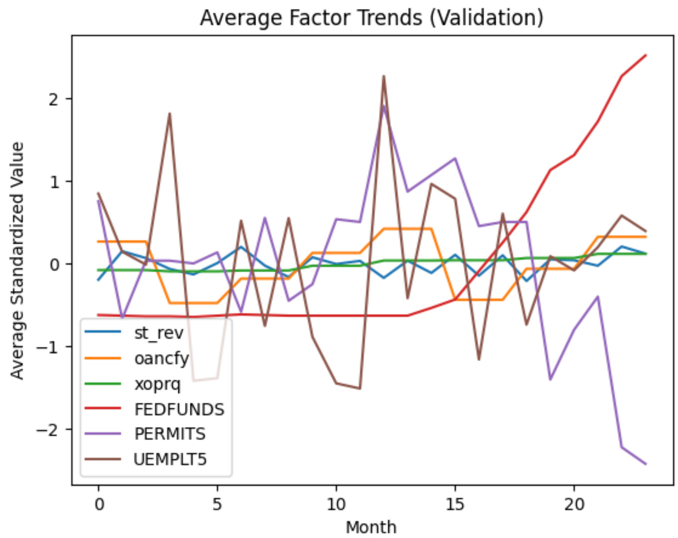


Fig. 9. The directions of the movements in FEDFUND and PERMITS, together with their factor loadings, strongly matches the trends seen in Figure 8.

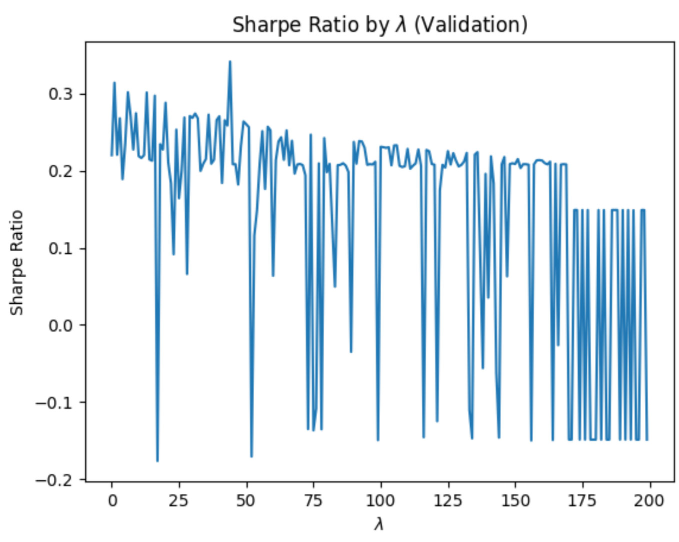


Fig. 10. On average, Sharpe Ratio decreases as regularization weight $\lambda$ increases. Furthermore, the lack of monotonicity in this curve suggests that our model might be overfit.