OPTIMAL CONTROL

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Contents

1 Optimal Control Introduction

1.1 Deterministic Finite-Dimensional Continuous-Time Problem

$$\inf_{u\in\mathcal{U}} J(u;t_0,t_f,X_0) \equiv K(t_f,X_f) + \int_{t_0}^{t_f} L(s,X_s,u_s)\,ds \tag{1}$$

$$dX_t \equiv f(t,X_t,u_t)\,dt \quad X_0 \in \mathbb{R}^m \tag{2}$$

$$\psi(t,X_t,u_t) = 0 \in \mathbb{R}^l, \quad \forall t \in [t_0,t_f] \tag{3}$$

$$\phi(t_f,X_f,u_t) \leq 0 \in \mathbb{R}^k, \quad \forall t \in [t_0,t_f] \tag{4}$$

1.2 Definitions

Summary. Fundamental definitions

1. Metric Space: (M,d)
2. Inner Product Induced Metric: $(M,\langle\cdot,\cdot\rangle)$
3. Topology: $\mathcal{T} \equiv (A_i)$
4. Open & Closed Sets: A, A^c
5. Open & Closed Balls: $B(x,\epsilon;d), \bar{B}(x,\epsilon;d)$
6. Metric Topology: $\mathcal{T}(M)$
7. Set Closure: $\bar{A}$
8. Set Interior: A°
9. Open Neighborhood: $A \subseteq M$

Definition 1.1: Metric Space
Defining

- (M,d): a metric space
- M: a set with topology induced by d
- $d : M \times M- > [0,\infty)$

Then

1. d(x,y) = d(y,x) ∀x,y ∈M — Symmetric
2. d(x,x) = 0 ∀x ∈M
3. d(x,y) > 0, ∀x,y ∈M, x ≠y — Non-Negative
4. d(x,y) ≤d(x,z) + d(z,y) ∀x,y,z ∈M — Triangle Inequality

Definition 1.2: Inner Product Induced Metric
Defining

- $(M, \langle\cdot,\cdot\rangle)$: an inner product space
- M: a vector space
- $\langle\cdot,\cdot\rangle$: an inner product

This induces the metric

$$d(x,y) = |x-y| \equiv \langle x-y, x-y\rangle^{1/2}, \quad \forall x,y \in M \tag{5}$$

Definition 1.3: Topology
Defining

- $\mathcal{T} \equiv (A_i)$: a collection of subsets of M

$\mathcal{T}$ forms a topology for M if the following hold

1. $M, \emptyset \in \mathcal{T}$
2. If $(E_i) \subseteq \mathcal{T}$ is a countable collection
3. If $(E_i) \subseteq \mathcal{T}$ is a finite collection

Definition 1.4: Open & Closed Sets

- Open Set: elements of a topology (i.e., $A \in \mathcal{T}$)
- Closed Set: complement of an open set (i.e., A^c)

Definition 1.5: Open & Closed Balls
Defining

- (M, d): a metric space
- $\epsilon > 0$: Radius
- $x \in M$: Center

Open and closed balls are defined as

1. Open Ball: $B(x, \epsilon; d) \equiv \{y \in M | d(x,y) < \epsilon\}$
2. Closed Ball: $\bar{B}(x, \epsilon; d) \equiv \{y \in M | d(x,y) \leq \epsilon\}$

Definition 1.6: Metric Topology
Given

- (M, d): a metric space

The metric can induce a topology by considering a collection of open balls.

Example 1.6.1*:* Borel Topology

- standard topology on $\mathbb{R}^m$
- all open balls centered at rational numbers $\mathbb{Q}$
- radius is positive rational

Definition 1.7: Set Closure
Given

- $A \subseteq M$: a subset of a metric space
- D_i: collection of all closed sets that contain A
- $\bar{A} \supseteq A$: the closure of A

Closure is defined as

$$\bar{A} \equiv \bigcap_i D_i \tag{6}$$

Remark. In Borel topology, isolated points are closed

Definition 1.8: Set Interior
Given

- $A \subseteq M$: a subset of a metric space
- E_i: collection of all open sets that contain A
- $A^\circ \subseteq A$: the interior of A

The interior is defined as

$$A^\circ \equiv \bigcup_i E_i \tag{7}$$

Definition 1.9: Open Neighborhood
Given

- $x \in M$: a point in a metric space
- $\mathcal{T}(M)$: the topology of M

$A \subseteq M$ is an open neighborhood of x if

- $x \in A$
- $A \in \mathcal{T}(M)$

Remark. The neighborhood is implied to be small, with motivation from metric topology implying that A looks like a small open ball centered at x.

2 Parameter Optimization Conditions

2.1 Defining Optimality

Summary. Defining local and global minimum on a metric space for a cost function

1. Local minimum
2. Global minimum
3. Local minimum; $(\mathbb{R}, d)$: Local minimum in standard 1D metric space
4. Global minimum; $(\mathbb{R}, d)$: Global minimum in standard 1D metric space
5. Extremum

2.2 Unconstrained Smooth Parameter Optimization

Summary. Deriving optimiality conditions in Euclidean space $\mathbb{R}^n$
Definitions:

1. Continuously Bounded Differentiable Function
2. Compact Set
3. Stationary Point
4. Unit Sphere
5. Hessian Matrix

Theorems:

1. Heine-Borel Property
2. Weierstrass Extreme Value
3. First Order Necessary Condition; $(\mathbb{R}, d)$
4. Taylor's Formula; Using Lagrange Form of the Mean-Value of the Remainder
5. Second Order Necessary Condition; $(\mathbb{R}, d)$
6. Second Order Sufficient Condition; $(\mathbb{R}, d)$
7. First Order Necessary Condition; $(\mathbb{R}^n, d)$