OPTIMAL CONTROL

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Contents

1 Optimal Control Introduction

1.1 Deterministic Finite-Dimensional Continuous-Time Problem

$$\inf_{u \in \mathcal{U}} J(u; t_0, t_f, X_0) \equiv K(t_f, X_f) + \int_{t_0}^{t_f} L(s, X_s, u_s)\, ds \tag{1}$$

$$dX_t \equiv f(t, X_t, u_t)\, dt \quad X_0 \in \mathbb{R}^m \tag{2}$$

$$\psi(t, X_t, u_t) = 0 \in \mathbb{R}^l, \quad \forall t \in [t_0, t_f] \tag{3}$$

$$\phi(t_f, X_f, u_t) \leq 0 \in \mathbb{R}^k, \quad \forall t \in [t_0, t_f] \tag{4}$$

1.2 Definitions

Summary. Fundamental definitions

1. Metric Space: (M, d)
2. Inner Product Induced Metric: $(M, \langle \cdot, \cdot \rangle)$
3. Topology: $\mathcal{T} \equiv (A_i)$
4. Open & Closed Sets: A, A^c
5. Open & Closed Balls: $B(x, \epsilon; d), \bar{B}(x, \epsilon; d)$
6. Metric Topology: $\mathcal{T}(M)$
7. Set Closure: $\bar{A}$
8. Set Interior: A°
9. Open Neighborhood: $A \subseteq M$

Definition 1.1 (Metric Space)**:**
Defining

- (M, d): a metric space
- M: a set with topology induced by d
- $d : M \times M- > [0, \infty)$

Then

1. d(x,y) = d(y,x) ∀x,y ∈M — Symmetric
2. d(x,x) = 0 ∀x ∈M
3. d(x,y) > 0, ∀x,y ∈M, x ≠y — Non-Negative
4. d(x,y) ≤d(x,z) + d(z,y) ∀x,y,z ∈M — Triangle Inequality

Definition 1.2 (Inner Product Induced Metric)**:**
Defining

- $(M, \langle\cdot,\cdot\rangle)$: an inner product space
- M: a vector space
- $\langle\cdot,\cdot\rangle$: an inner product

This induces the metric

$$d(x, y) = |x - y| \equiv \langle x - y, x - y\rangle^{1/2}, \quad \forall x, y \in M \tag{5}$$

Definition 1.3 (Topology)**:**
Defining

- $\mathcal{T} \equiv (A_i)$: a collection of subsets of M

$\mathcal{T}$ forms a topology for M if the following hold

1. $M, \emptyset \in \mathcal{T}$
2. If $(E_i) \subseteq \mathcal{T}$ is a countable collection
3. If $(E_i) \subseteq \mathcal{T}$ is a finite collection

Definition 1.4 (Open & Closed Sets)**:**

- Open Set: elements of a topology (i.e., $A \in \mathcal{T}$)
- Closed Set: complement of an open set (i.e., A^c)

Definition 1.5 (Open & Closed Balls)**:**
Defining

- (M, d): a metric space
- $\epsilon > 0$: Radius
- $x \in M$: Center

Open and closed balls are defined as

1. Open Ball: $B(x, \epsilon; d) \equiv \{y \in M | d(x, y) < \epsilon\}$
2. Closed Ball: $\bar{B}(x, \epsilon; d) \equiv \{y \in M | d(x, y) \leq \epsilon\}$

Definition 1.6 (Metric Topology)**:**
Given

- (M, d): a metric space

The metric can induce a topology by considering a collection of open balls.

Example 1.0.1 (Borel Topology)*:*

- standard topology on $\mathbb{R}^m$
- all open balls centered at rational numbers $\mathbb{Q}$
- radius is positive rational

Definition 1.7 (Set Closure)**:**
Given

- $A \subseteq M$: a subset of a metric space
- D_i: collection of all closed sets that contain A
- $\bar{A} \supseteq A$: the closure of A

Closure is defined as

$$\bar{A} \equiv \bigcap_i D_i \tag{6}$$

Remark. In Borel topology, isolated points are closed

Definition 1.8 (Set Interior)**:**
Given

- $A \subseteq M$: a subset of a metric space
- E_i: collection of all open sets that contain A
- $A^\circ \subseteq A$: the interior of A

The interior is defined as

$$A^\circ \equiv \bigcup_i E_i \tag{7}$$

Definition 1.9 (Open Neighborhood)**:**
Given

- $x \in M$: a point in a metric space
- $\mathcal{T}(M)$: the topology of M

$A \subseteq M$ is an open neighborhood of x if

- $x \in A$
- $A \in \mathcal{T}(M)$

Remark. The neighborhood is implied to be small, with motivation from metric topology implying that A looks like a small open ball centered at x.

2 Parameter Optimization Conditions

2.1 Defining Optimality

Summary. Defining local and global minimum on a metric space for a cost function

1. Local minimum
2. Global minimum
3. Local minimum; $(\mathbb{R}, d)$: Local minimum in standard 1D metric space
4. Global minimum; $(\mathbb{R}, d)$: Global minimum in standard 1D metric space
5. Extremum

Definition 2.1 (Local Minimum)**:**
Given

- (M, d): a metric space
- $\mathcal{T}$: a topology on M induced by the metric d
- $f : M \to \mathbb{R}$: a cost function

$x^* \in M$ is a local minimum of f if $\exists$ neighborhood A of x^* such that

$$f(x^*) \leq f(x) \quad \forall x \in A \tag{8}$$

Remark. x^* is a strict local minimum if

$$f(x^*) < f(x) \quad \forall x \in A \backslash x^* \tag{9}$$

Definition 2.2 (Global Minimum)**:**
Given

- (M, d): a metric space
- $\mathcal{T}$: a topology on M induced by the metric d
- $f : M \to \mathbb{R}$: a cost function

$x^* \in M$ is a global minimum of f if

$$f(x^*) \leq f(x) \quad \forall x \in M \tag{10}$$

Definition 2.3 (Local Minimum; $(\mathbb{R}, d)$)**:**
$x^* \in \mathbb{R}$ is a local minimum of $f : \mathbb{R} \to \mathbb{R}$ if $\exists \epsilon > 0$ such that

$$f(x^*) \leq f(x) \quad \forall x \in (x^* - \epsilon, x^* + \epsilon) = B(x^*, \epsilon) \tag{11}$$

x^* is a strict local minimum if

$$f(x^*) < f(x) \quad \forall x \in (x^* - \epsilon, x^* + \epsilon) \backslash \{x^*\} \tag{12}$$

Definition 2.4 (Global Minimum; $(\mathbb{R}, d)$)**:**
$x^* \in \mathbb{R}$ is a global minimum of $f : \mathbb{R} \to \mathbb{R}$ if

$$f(x^*) \leq f(x) \quad \forall x \in \mathbb{R} \tag{13}$$

x^* is a strict global minimum if

$$f(x^*) < f(x) \quad \forall x \in \mathbb{R} \backslash \{x^*\} \tag{14}$$

Definition 2.5 (Extremum)**:**
If x^* is a local minimum or maximum, then x^* is an extremum.

2.2 Unconstrained Smooth Parameter Optimization

Summary. Deriving first and second order necessary and sufficient optimiality conditions in Euclidean space $\mathbb{R}^n$.
Definitions:

1. Continuously Bounded Differentiable Function
2. Compact Set
3. Stationary Point
4. Unit Sphere
5. Hessian Matrix

Theorems:

1. Heine-Borel Property
2. Weierstrass Extreme Value
3. First Order Necessary Condition; $(\mathbb{R}, d)$
4. Taylor's Formula; Using Lagrange Form of the Mean-Value of the Remainder
5. Second Order Necessary Condition; $(\mathbb{R}, d)$
6. Second Order Sufficient Condition; $(\mathbb{R}, d)$
7. First Order Necessary Condition; $(\mathbb{R}^n, d)$

Definition 2.6 (Continuously Bounded Differentiable Function)**:**
Defining

- $C^k(\Omega; \mathbb{R})$: real-valued k-times continuously differentiable functions on the set Ω. So $f \in C^k(\Omega; \mathbb{R})$ is k-times differentiable, and each derivative is continuous. e.g., $\Omega \subseteq \mathbb{R}^n$
- $C_b^k(\Omega; \mathbb{R})$: real-valued k-times continuously differentiable functions on the set Ω that are bounded

Definition 2.7 (Compact Set)**:**
Given

- $(M, \mathcal{T})$: a topological space
- $\Omega \subseteq M$: a set

Ω is compact if

- $(E_i)_{i \in I}$: open cover of Ω
- $(E_j)_{j \in J}$: finite subcover where $J \subseteq I$
- For any $(E_i)_{i \in I}$, there exists $(E_j)_{j \in J}$.

Theorem 2.1 (Heine-Borel Property)**:**
A metric space has the Heine-Borel property iff every compact set is closed and bounded. In particular, $\mathbb{R}^n$ with the sandard metric has this property.

Theorem 2.2 (Weierstrass Extreme Value)**:**
Let

- Ω: a compact set
- $f : \omega \to \mathbb{R}$ a continous function.

Then f has extremums on Ω.

Remark. If Ω is compact, then the class of functions is automatically $C_b^k(\Omega; \mathbb{R})$

Theorem 2.3 (First Order Necessary Condition; $(\mathbb{R}, D)$)**:**
Given

- $f \in C^1(\Omega; \mathbb{R})$
- $x^* \in \Omega^\circ \subseteq \mathbb{R}$

If x^* is a local minimum for f, then

$$\frac{\partial f}{\partial x}_{x^*} = 0$$

Definition 2.8 (Stationary Point)**:**
x^* is a stationary point for f if

$$\partial_x f|_{x^*} = 0 \tag{15}$$

Theorem 2.4 (Taylor's Formula; Using Language Form of the Mean-Value of the Remainder)**:**
Assume

- $f \in C_b^2(\mathbb{R};\mathbb{R})$
- x, x^*
- $\delta x \equiv x - x^*$

there exists a point y between x and x^* such that

$$f(x) = f(x^*) + \delta_x f|_{x^*}\delta x + \frac{1}{2}\delta_x^2 f|_y \delta x^2 \tag{16}$$

Theorem 2.5 (Second Order Necessary Condition; $(\mathbb{R}, d)$)**:**
If $x^* \in \Omega^\circ \subseteq \mathbb{R}$ is a local minimum for $f \in C_b^2(\Omega;\mathbb{R})$, then

$$\frac{\partial^2 f}{\partial x^2}\Big|_{x^*} \geq 0 \tag{17}$$

Theorem 2.6 (Second Order Sufficient Condition; $(\mathbb{R}, d)$)**:**
Prerequisites

- Let $x^* \in \Omega^\circ \subseteq \mathbb{R}$
- Assume $f \in C_b^2(\Omega;\mathbb{R})$
- If $\frac{\partial f}{\partial x}|_{x^*} = 0$ and $\frac{\partial^2 f}{\partial x^2}|_{x^*} > 0$ hold

Then x^* is a strict local minimum for f.

Definition 2.9 (Unit Sphere)**:**
Defining

- $S^n(x, d) \subset \mathbb{R}^{n+1}$ for $n \in \mathbb{N}$: unit sphere
- $x \in \mathbb{R}^{n+1}$: a point
- d: a metric on $\mathbb{R}^{n+1}$

Then the complete definition is $S^n(x, d) \equiv \{y \in \mathbb{R}^{n+1} : d(x, y) = 1\}$

Remark. If S^n is denoted without x and d, then it can be assumed that $x = 0$ and d is the standard l_2 metric.

Theorem 2.7 (First Order Necessary Condition; $(\mathbb{R}^n, d)$)**:**
Given

- $f \in C^1(\Omega; \mathbb{R})$
- $x^* \in \Omega^\circ \subseteq \mathbb{R}^n$

If x^* is a local minimum for f, then

$$\nabla f\big|_{x^*} \equiv \left(\frac{\partial f}{\partial x_1}\bigg|, \cdots, \frac{\partial f}{\partial x_n}\bigg|_{x^*} \right) = 0 \in \mathbb{R}^n$$

Definition 2.10 (Hessian Matrix)**:**
Given

- $f \in C^2(\mathbb{R}^n; \mathbb{R})$
- $\mathbb{S}(n; \mathbb{R}) = \mathbb{S}(n)$: an $n \times n$ symmetric matrix

The Hessian matrix of f evaluated at $x \in \mathbb{R}$ is

$$\nabla_x^{\otimes 2} f = \nabla_x^2 = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial f}{\partial x_1 x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \tag{18}$$

Remark. Every $A \in \mathbb{S}(n)$ can be expressed as $A = U \Lambda U^T$ where the columns of U are the eigenvectors of A and the diagonal of Λ is the eigenvalues of A.

3 Constrained Parameter Optimization

3.1 Second Order Conditions in Arbitrary Dimensions

Summary. Theorems: Figure out what constrained vs unconstrained is

1. Second Order Necessary Condition; $(\mathbb{R}^n, d)$
2. Second Order Sufficent Condition; $(\mathbb{R}^n, d)$

Summary. Definitions

1. Big O Notation

3.2 Equality Constrained Smooth Parameter Optimization

1. Regular Point of M
2. Tangent Space (Geometric)
3. Tangent Space (Curves)
4. First Order Necessary Condition (Geometric)
5. Lagrange Multipliers
6. First Order Necessary Condition (Analytic)