Local equivalence of stabilizer states

a graphical characterisation

Nathan Claudet and Simon Perdrix

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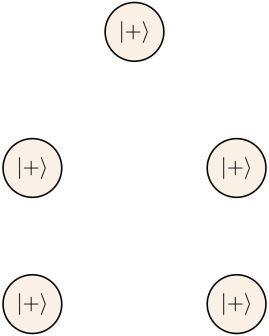
arXiv:2409.20183

Graph states, local unitary equivalence, local Clifford equivalence & local complementation

Graph states

A graph state is a quantum state represented by an undirected[1] and simple[2] graph. The vertices represent the qubits and the edges represent entanglement.



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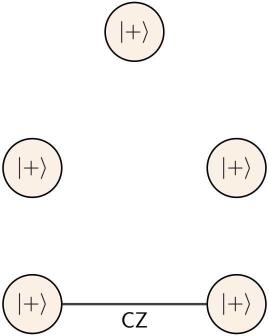

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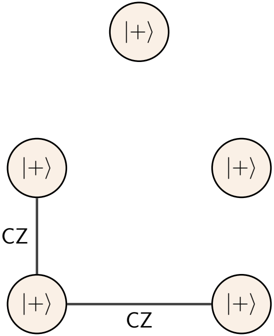

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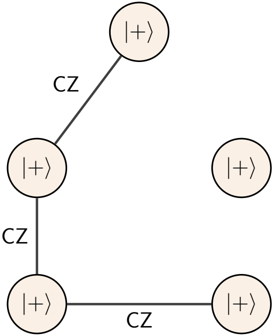

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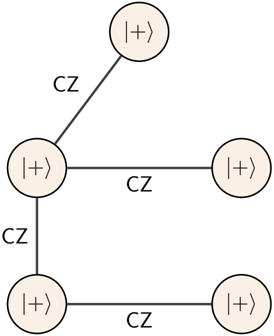

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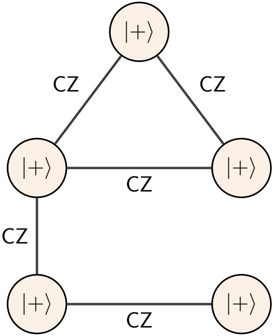

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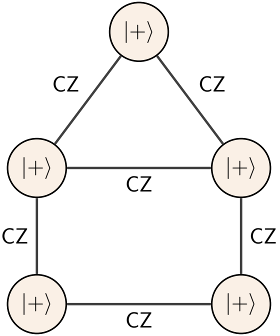

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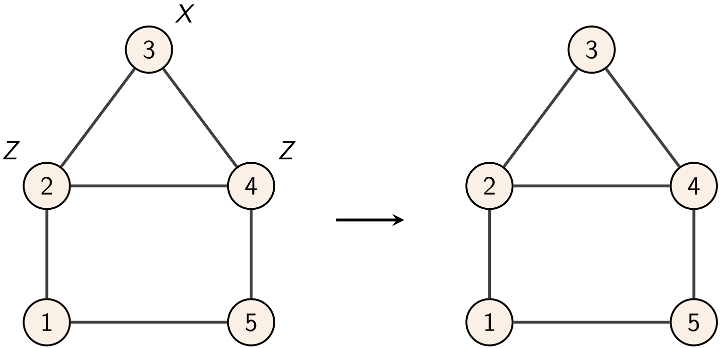
Stabilizer states

Graph states are a subfamilly of stabilizer states because for each vertex u, applying X on u and applying Z on the neighbours of u leaves the graph state invariant.



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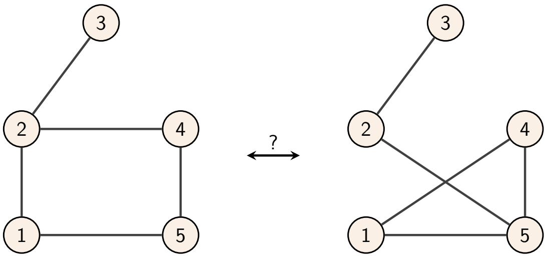

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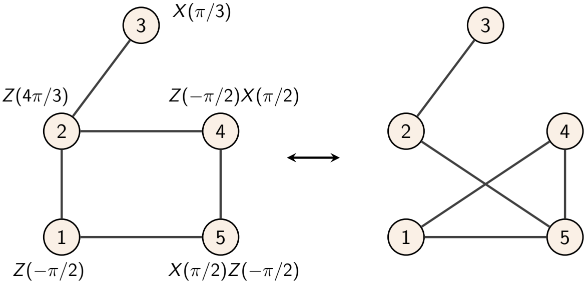
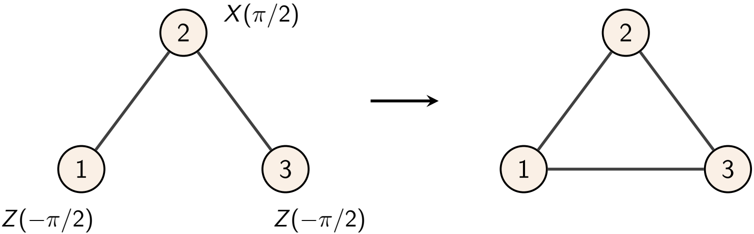


An easier subproblem: local Clifford equivalence

Two graph states are said **local Clifford equivalent** (or LC-equivalent) if there are related by unitaries in the local Clifford group.



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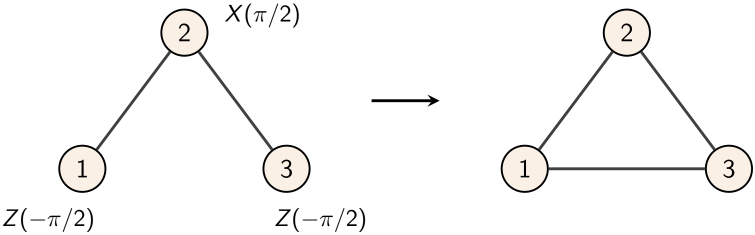


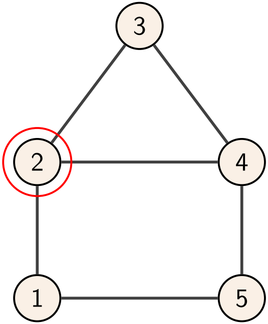
Theorem (Van den Nest, Dehaene, De Moor, 2004)

Two graph states are local Clifford equivalent iff the two corresponding graphs are related by **local complementations***.*

Local complementation

Definition (Kotzig, 1966)

A local complementation on a vertex u consists in complementing the (open) neighbourhood of u.



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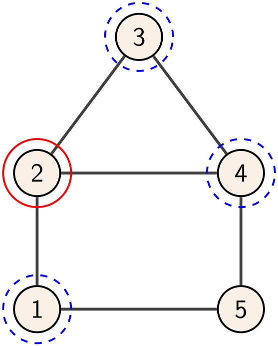

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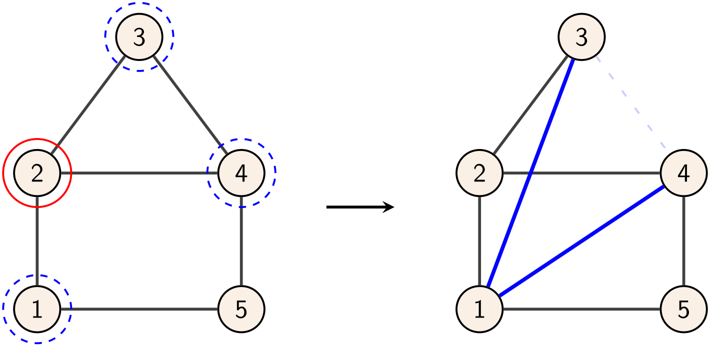

Algorithmic aspect of local Clifford equivalence

There exists an efficient algorithm (Bouchet, 1991) to recognise whether two graphs are related by local complementations, implying an efficient algorithm to recognise whether two graph states are local Clifford equivalent.

LU $\neq$ LC

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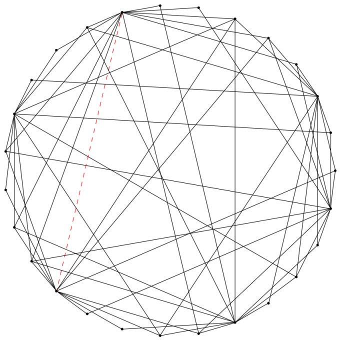

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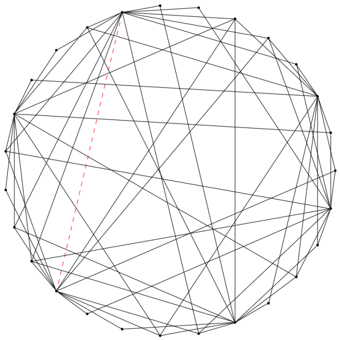

Consequence: local complementation does **not** capture the local unitary equivalence of graph states.

LU=LC for some classes of graphs

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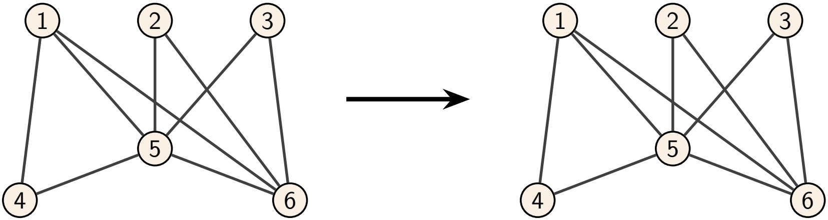
But what about local unitary equivalence for **any** graph ? Can we construct a graphical characterisation ?

Generalising local complementation to capture local unitary equivalence

A refinement of idempotent local complementations

A sequence of local complementations may leave the graph invariant.



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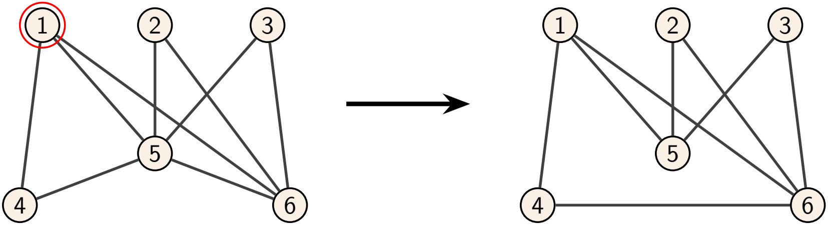

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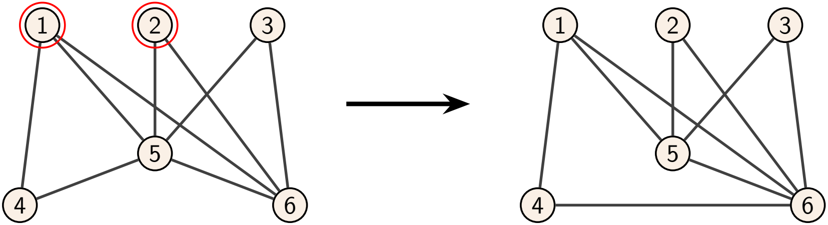

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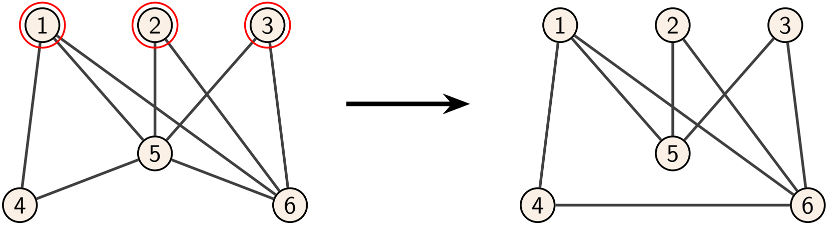

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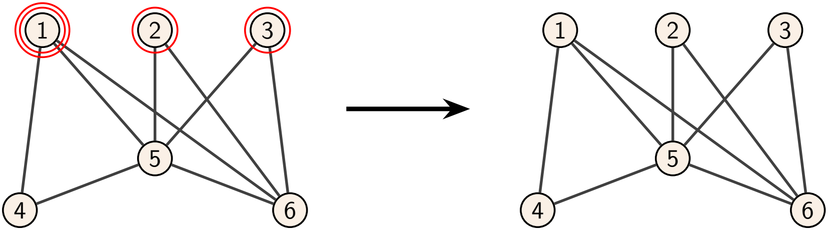

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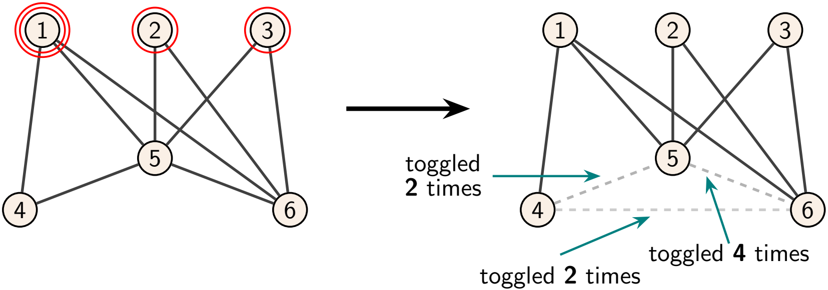

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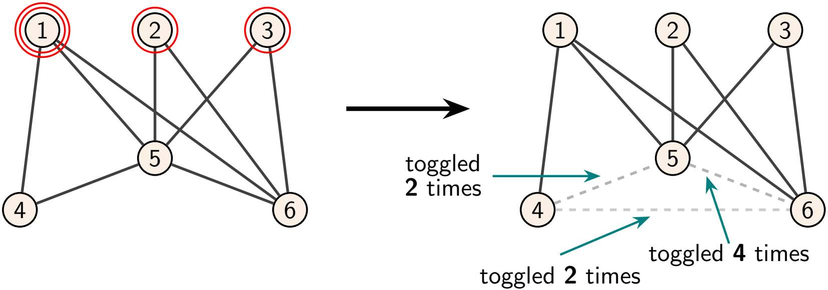

A **2-local complementation** consists in toggling every edge that was toggled 2 mod 4 times by the idempotent local complementations.
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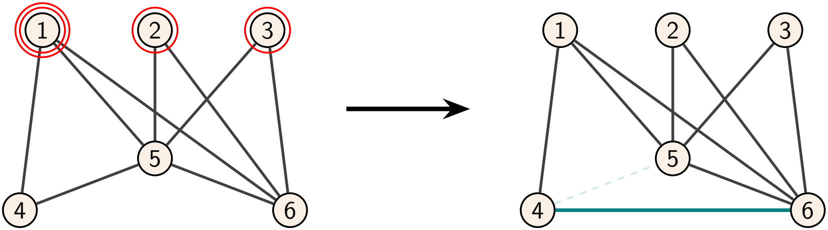

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r-local complementation

3-local complementation is a refinement of idempotent 2-local complementation, and so on...
$\to$ Infinite family of graphical operations parametrised by an integer r:

r-**local complementations**

1-local complementation = local complementation.

Main results

Theorem (this work)

Two graphs are related by r-local complementations iff the two corresponding graph states are related by local unitaries in the level r+1 of the Clifford hierarchy.

For $r = 1$, we recover local Clifford $\Leftrightarrow$ local complementation.

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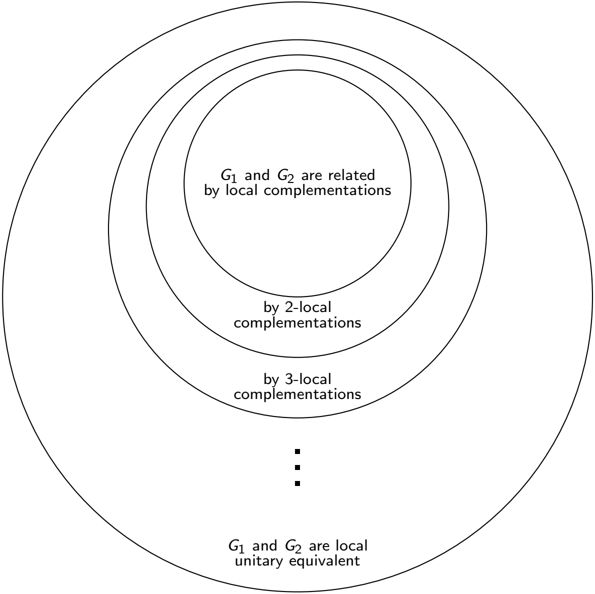
Corollary

If two graph states are local unitary equivalent, the local unitaries can be chosen to be in the Clifford hierarchy.

An infinite hierarchy of local equivalences



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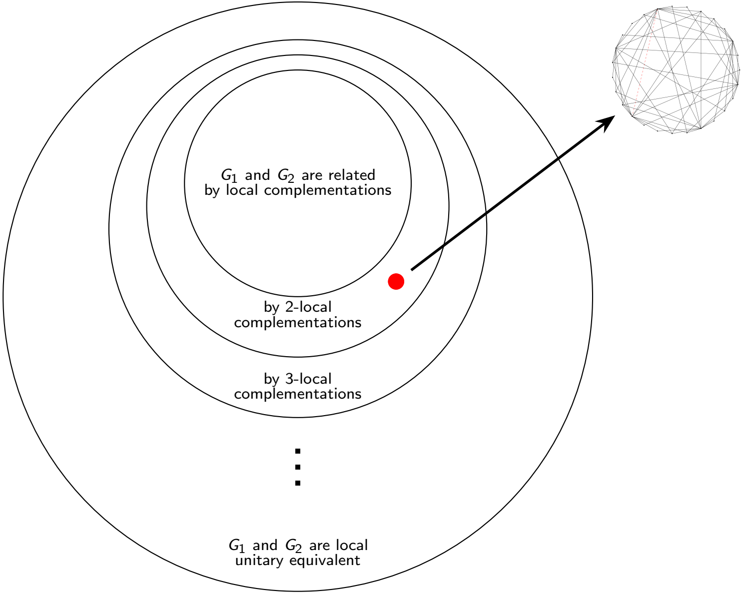

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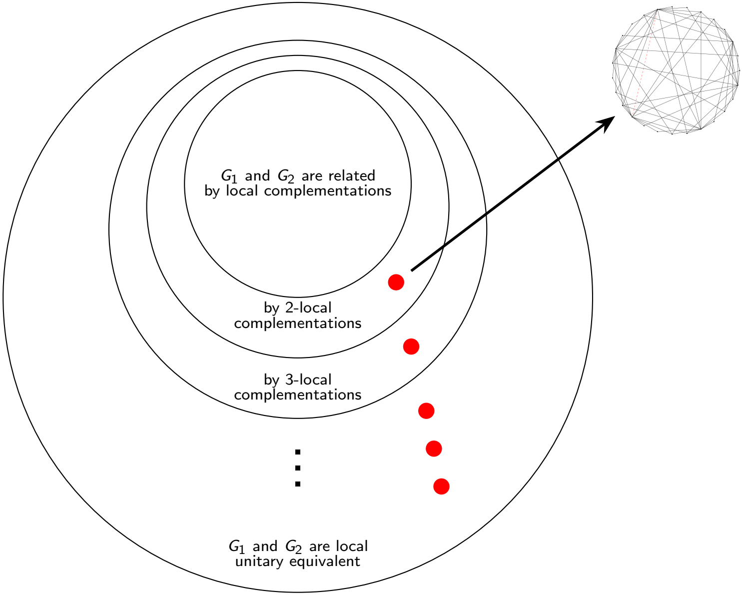
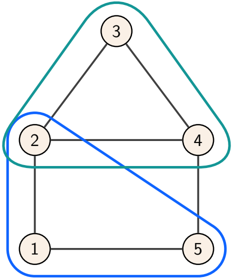


Proof sketch: Minimal local set

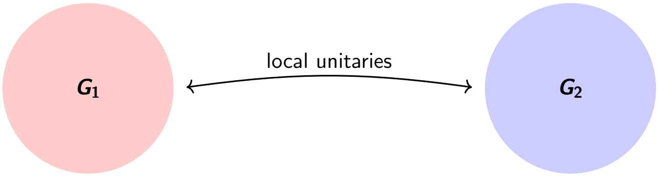


Minimal local sets are subsets of vertices that are invariant by local unitary equivalence and carry information on the possible local unitaries that maps graph states to other graph states.

Theorem (C, Perdrix, 2024)

Each vertex of a graph is covered by at least one minimal local set.

Proof sketch: Standard form for graph states



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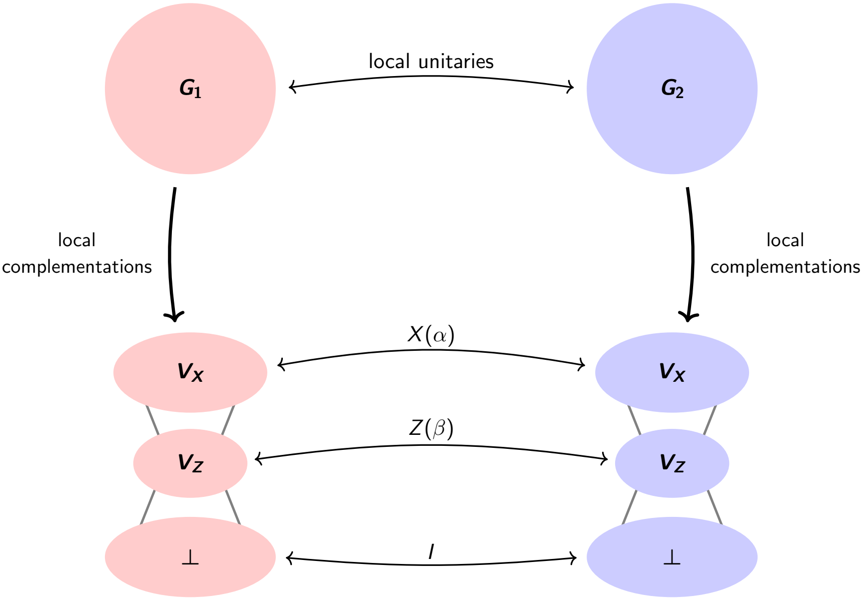

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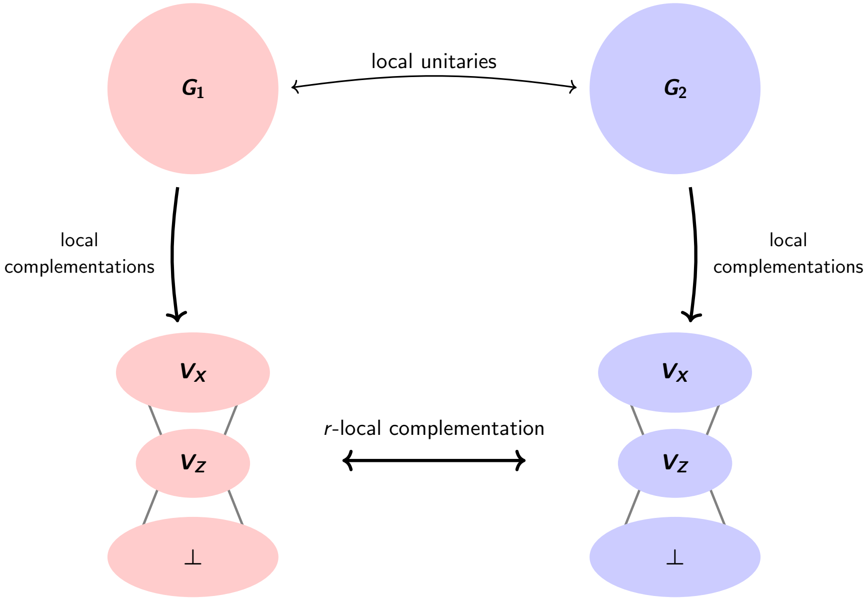
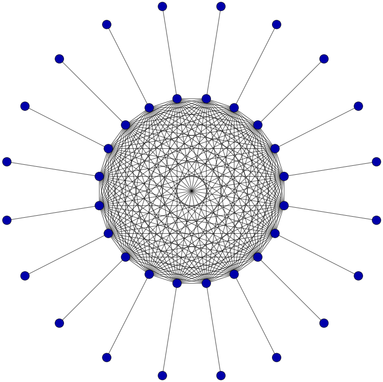
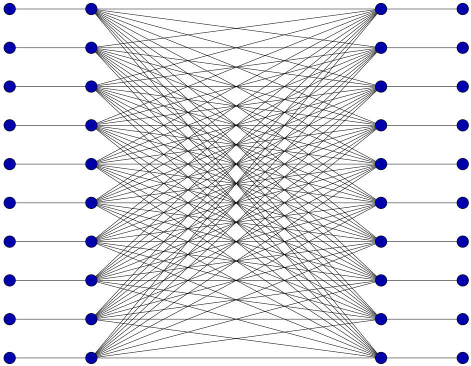


Application 1: local equivalence of repeater graph states

It was conjectured that LU=LC holds for some repeater graph states (Tzitrin, 2018). We showed that this is indeed the case.





Application 2: LU=LC for graph states up to 19 qubits

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Theorem (C, Perdrix, 2025)

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Application 3: A quasi-polynomial algorithm to decide local unitary equivalence

Previously: exponential algorithm for deciding local unitary equivalence of graph states (Burchardt, de Jong, Vandré, 2024).

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Theorem (C, Perdrix, 2025)

There exists an algorithm that decides whether two graph states are local unitary equivalent with runtime $n^{log_2(n)+O(1)}$.

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Open questions:

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- Does there exist a counter-example to LU=LC between 20 and 26 qubits ?
- Does there exist a polynomial-time algorithm for local unitary equivalence ?

Thanks

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