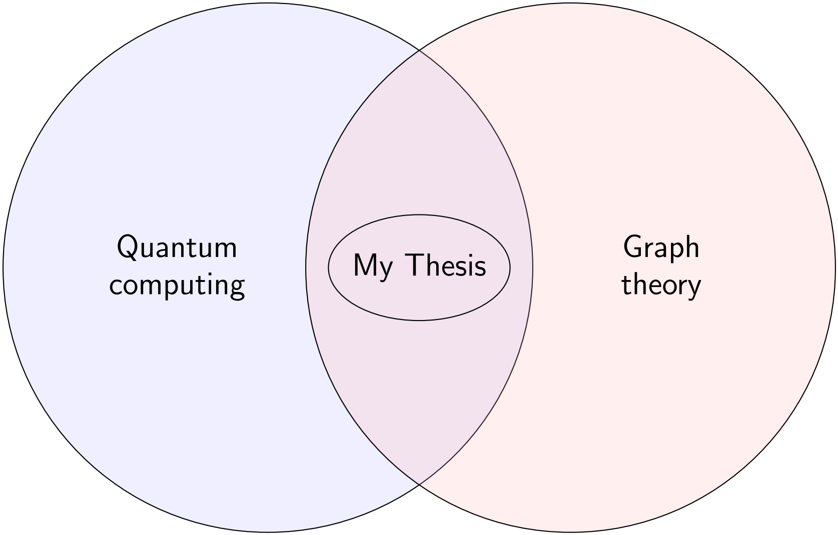
Local equivalences of graph states

Doctoral thesis defense

Nathan Claudet

17/11/25

Scope



Quantum computing

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Theorem (Shor's algorithm, 1994)

There exist an **efficient** *quantum algorithm that finds the prime factors of an integer.*

$$15 = 5 \times 3$$

$$221 = 13 \times 17$$

$$269535011 = 12923 \times 20857$$

How to build a quantum computer

- In classical computing, bits: 0 or 1.
- In quantum computing, quantum bits (qubits): $|0\rangle$, $|1\rangle$, but also $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

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$+$ ways to interact with the qubits to create **entanglement**.

Quantum entanglement

- Quantum states are composed of qubits.

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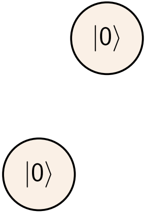

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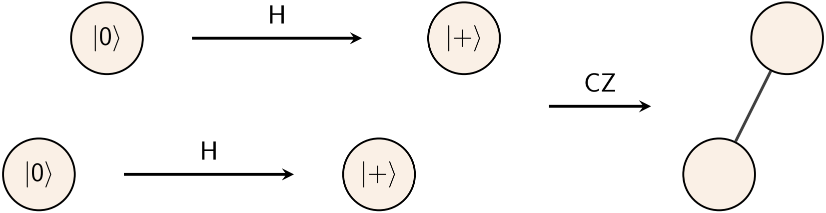

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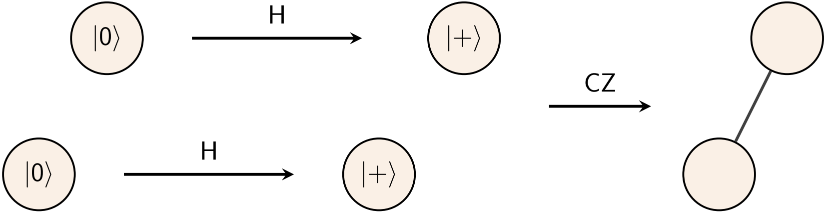

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Qubits are **entangled** when they cannot be described as separate entities.

Possible applications of quantum computing

- Optimisation (variational algorithms, HHL[1] algorithm...)

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Possible applications of quantum computing

- Optimisation (variational algorithms, HHL[1] algorithm...)
- Simulation of quantum systems (drug discovery, material science...)
- Communication & cryptography (safe encryption methods, key distribution...)

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Graphs

Definition

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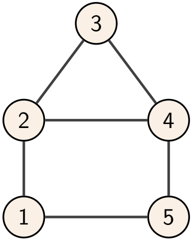
A graph is composed of two sets, a set $V \in \mathbb{N}$ of (labeled) vertices and a set $E \in V^2$ of edges linking vertices.



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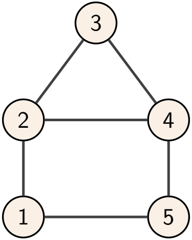

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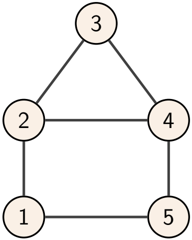

- Graphs in this thesis are simple (no multiples edges, no loops) and undirected (edges do not have a direction).
- Graphs are useful for representing complex structures (social networks, road networks...).

Structure of the presentation

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- Local complementation, application to graph state sharing.
- Graphically characterising the entanglement of graph states.
- Applications: algorithm & graph states up to 19 qubits.

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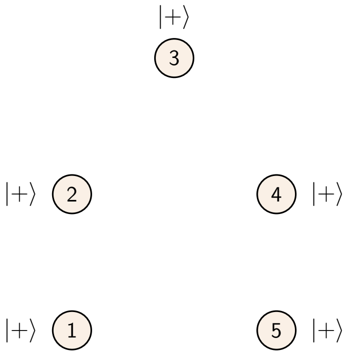

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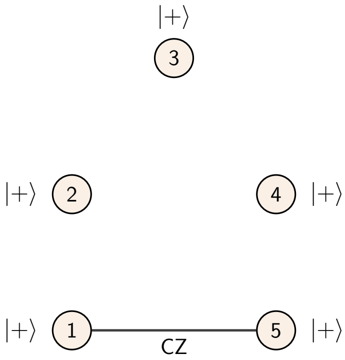

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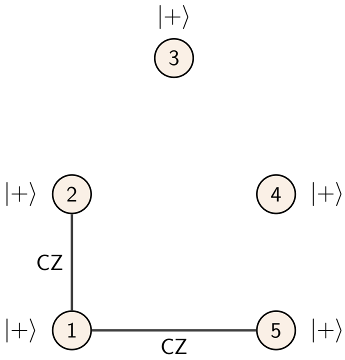

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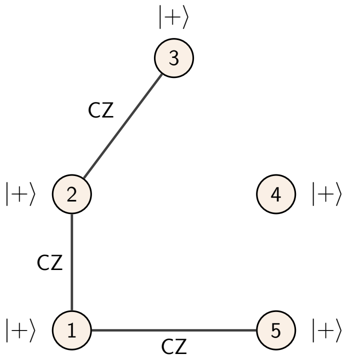

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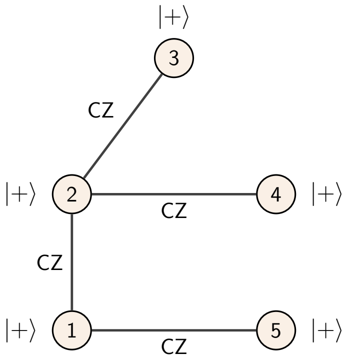

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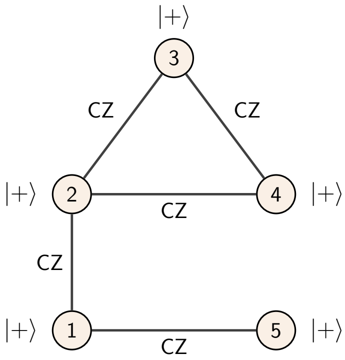

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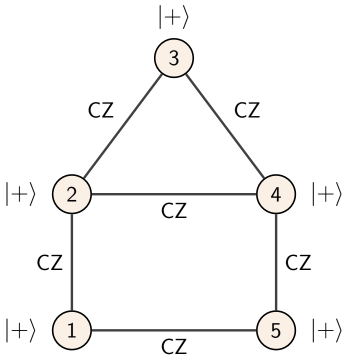
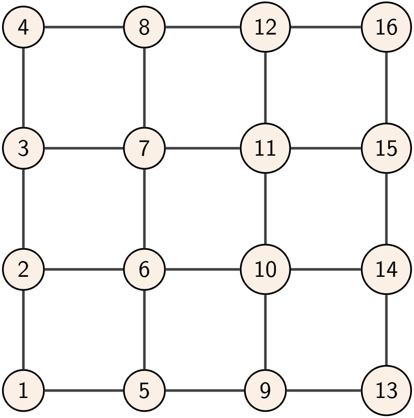


Measurement-based quantum computing (MBQC)

- Introduced in the early 2000's by Hans Briegel and Robert Raussendorf[1].
- Graph states are the resources for MBQC.



[1] Briegel, Raussendorf, A one-way quantum computer, Physical Review A, 2001

Local complementation

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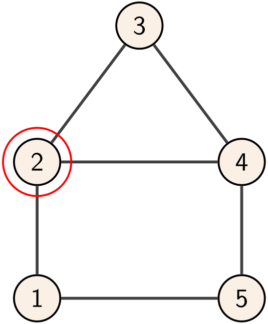
Definition (Kotzig, 1966)

A local complementation on a vertex u consists in complementing the (open) neighbourhood of u.



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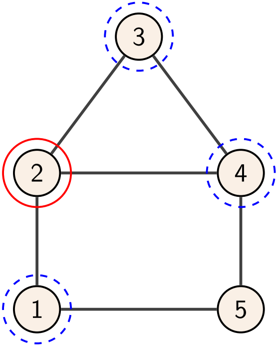

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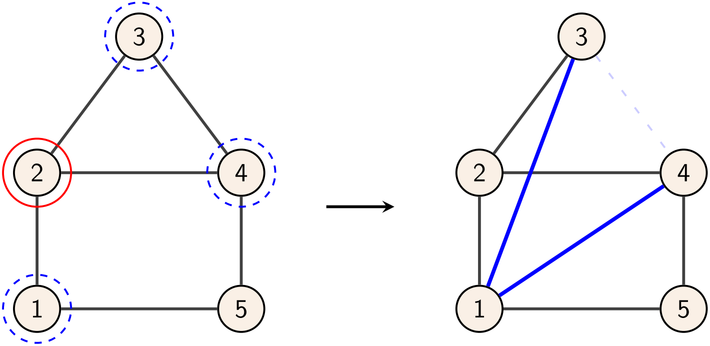

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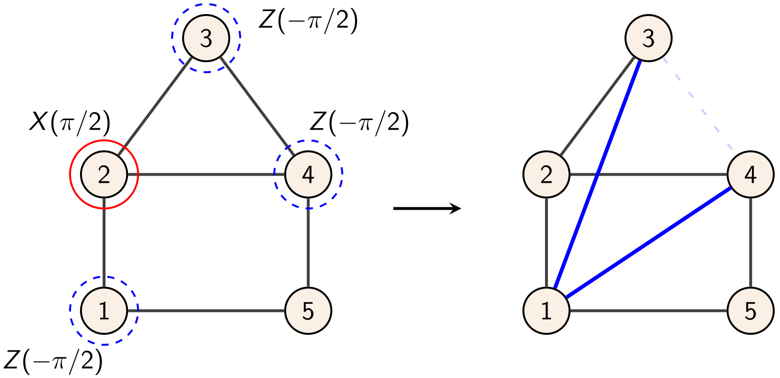

Algorithmic aspect of local Clifford equivalence

Theorem (Bouchet, 1991)

There exists an efficient algorithm to decide if two graphs are related by local complementations.

Application: quantum communication networks

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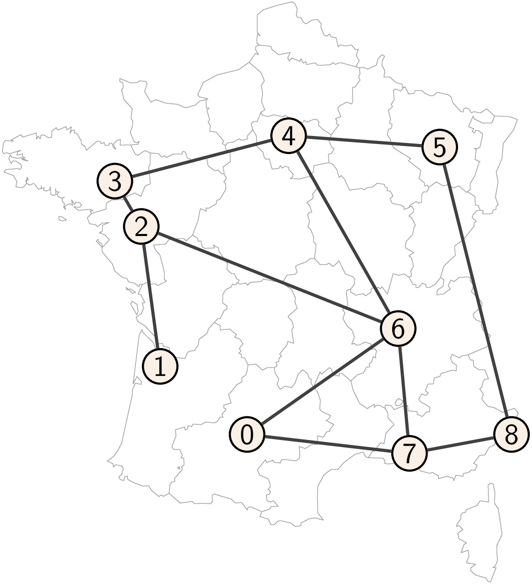

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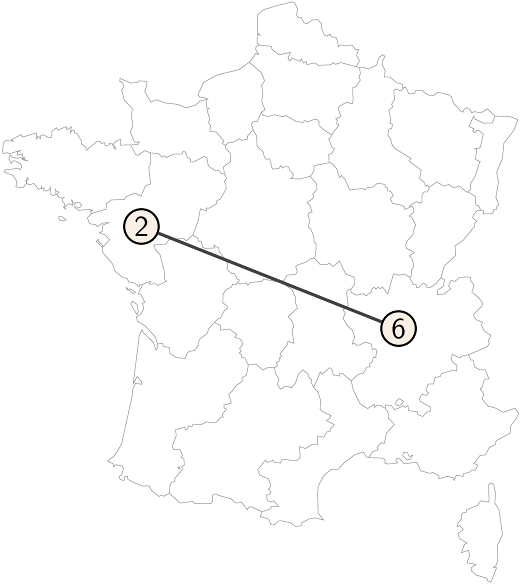

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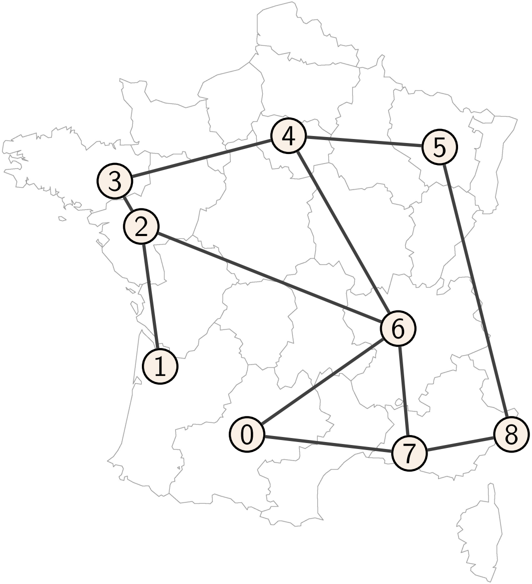

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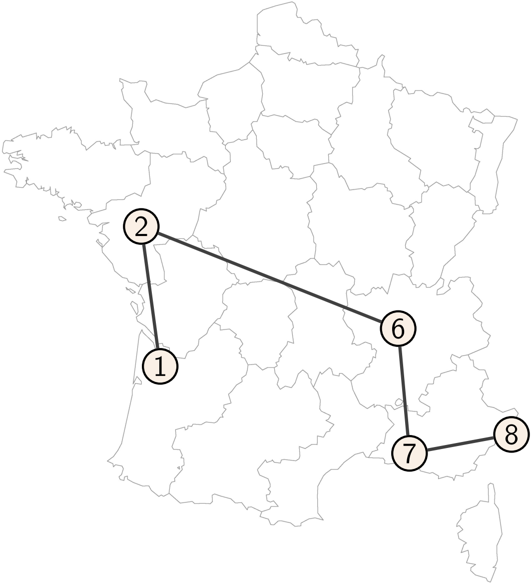

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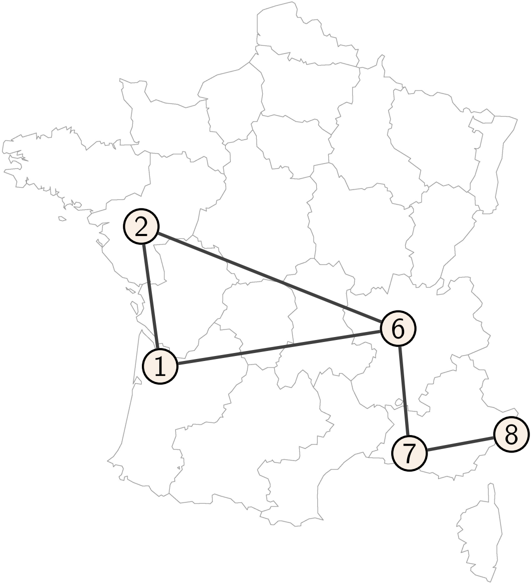

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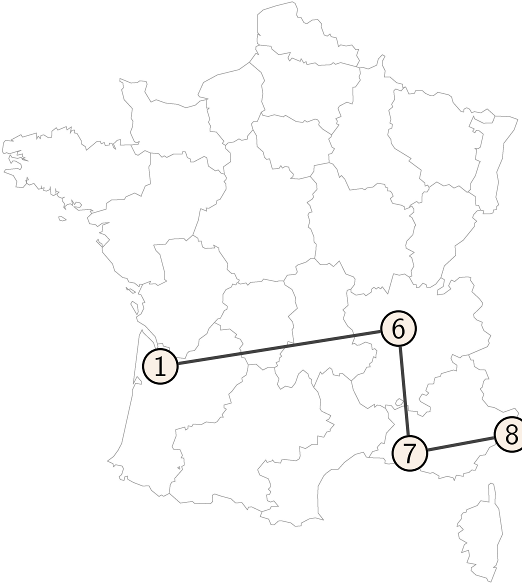

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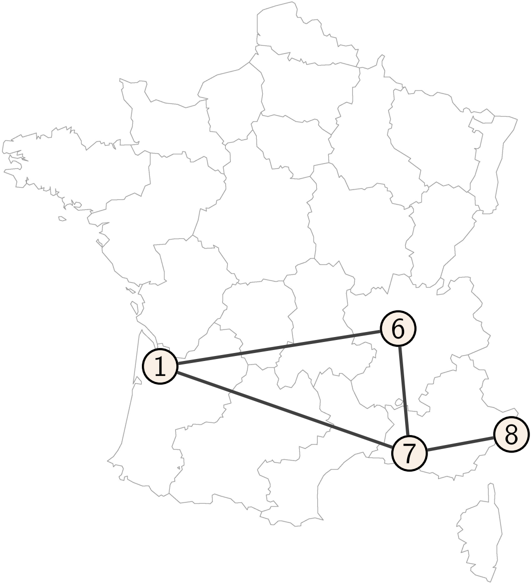

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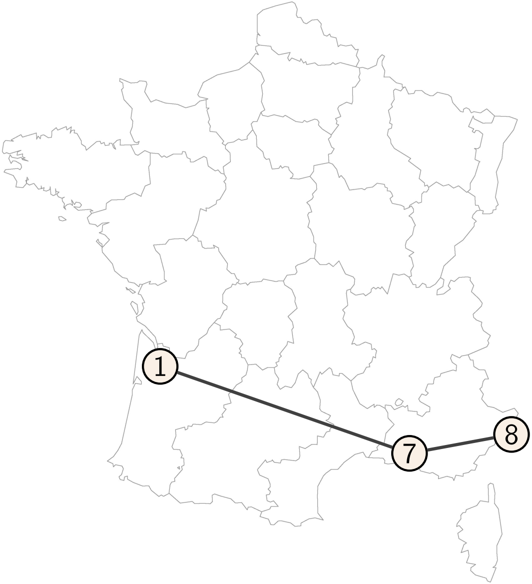

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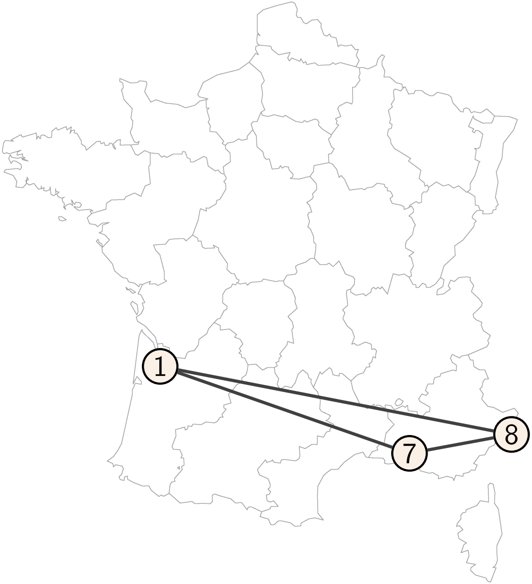

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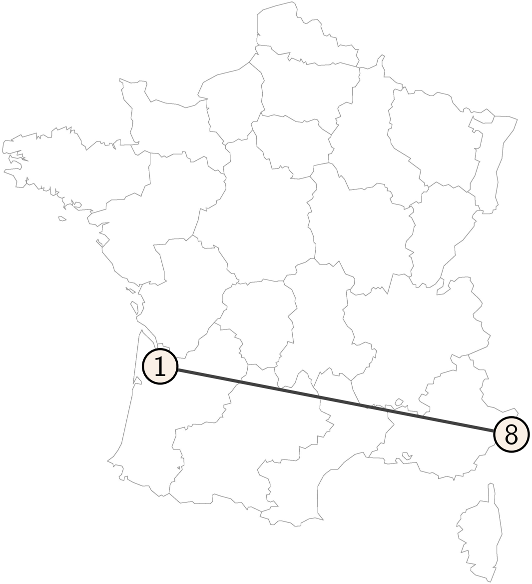

Creating arbitrary graph states with local complementation

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For any integer n, there exist states with n qubits such that $\log(n)$ EPR-pairs over any qubits can be created using local quantum operations.

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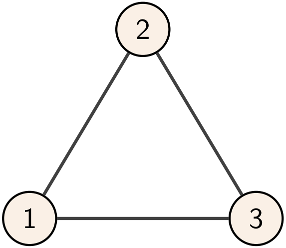
Proposition

If G is k-vertex-minor universal, any graph state on any k qubits of $|G\rangle$ can be induced by local quantum operations.

K_3 is 2-vertex-minor universal.



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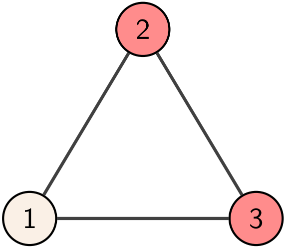

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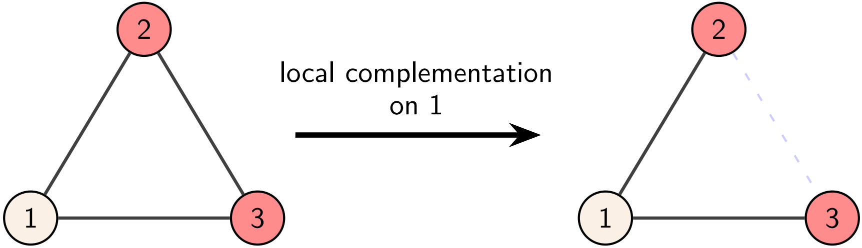

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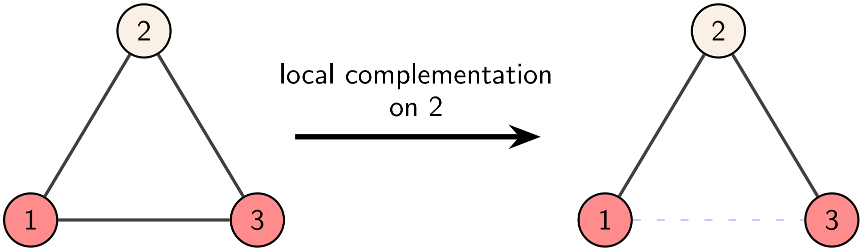

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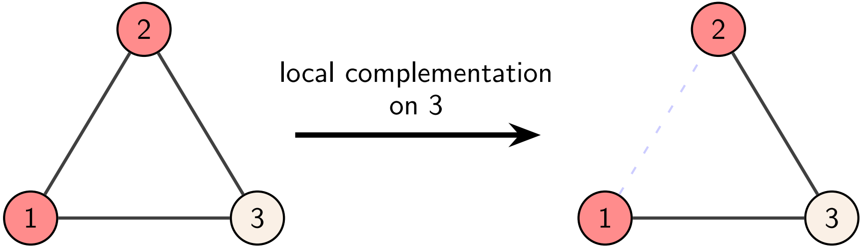
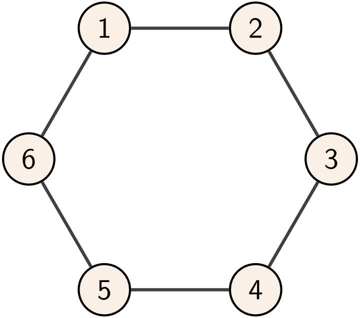


k-vertex-minor universal graphs : example 2

C_6 is 3-vertex-minor universal.



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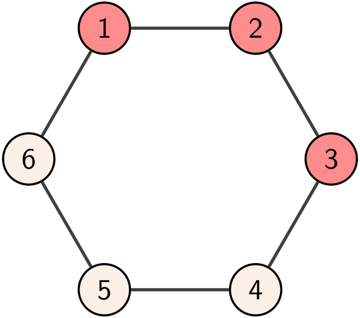

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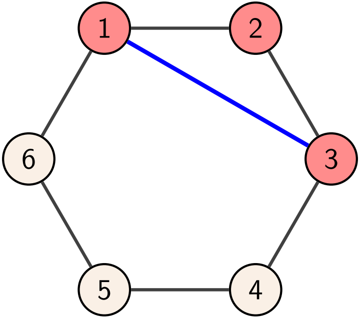

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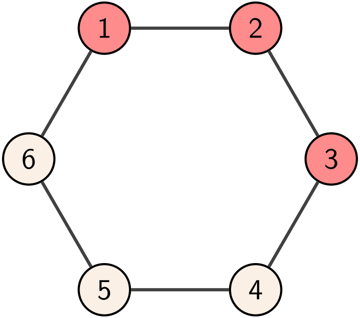

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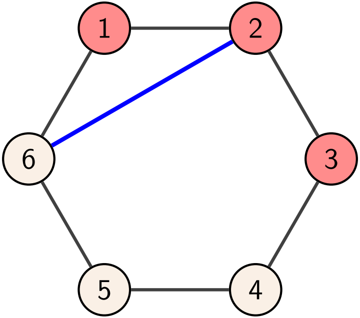

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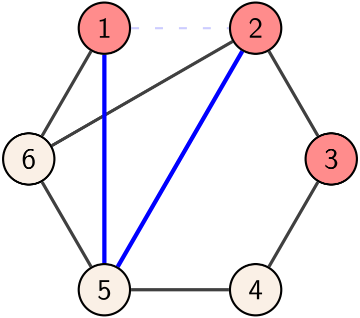

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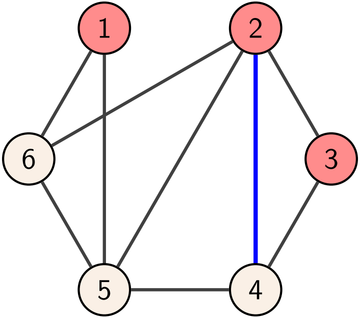

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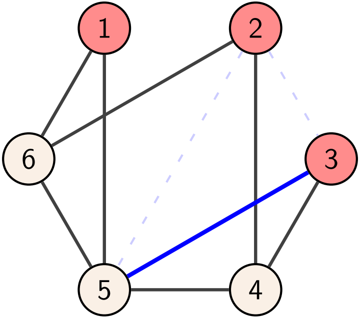

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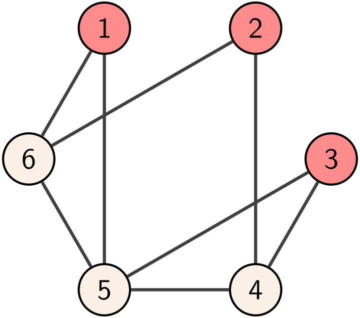

To induce the complete graph on $\{1, 2, 3\}$: Local complementation on 2.
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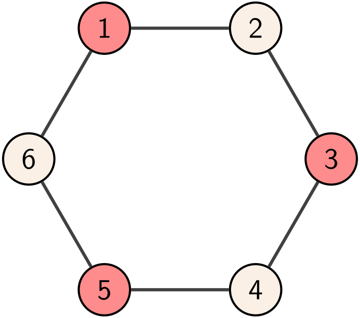

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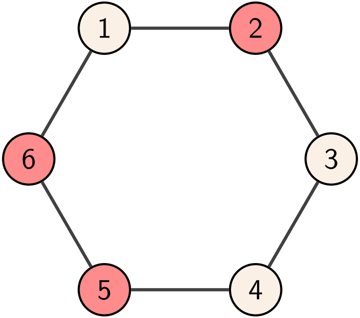

Existence of k-vertex-minor universal graphs for large k

For an arbitrary k, existence of k-vertex-minor universal graphs?

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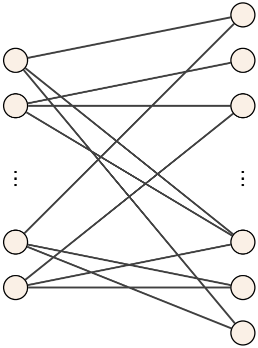
Theorem (Cautrès, C, Mhalla, Perdrix, Savin, Thomassé, 2024)

For any integer n, there exist graphs of order n that are $\sqrt{n}$-vertex-minor universal. This is optimal up to a multiplicative factor.



Local unitary equivalence and local Clifford equivalence

Entanglement of graph states

Graph states are useful entangled resources (MBQC, quantum networks...)
$\rightarrow$ It is a fundamental problem to know whether two graph states have the same entanglement, i.e. the graph states are **local unitary equivalent** (or LU-equivalent).

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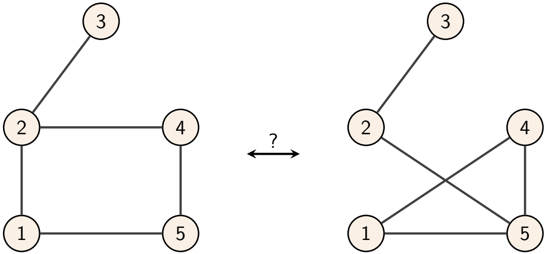

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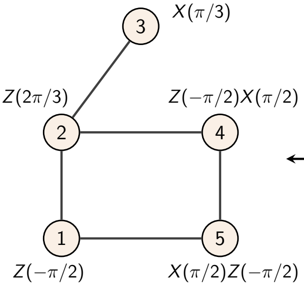
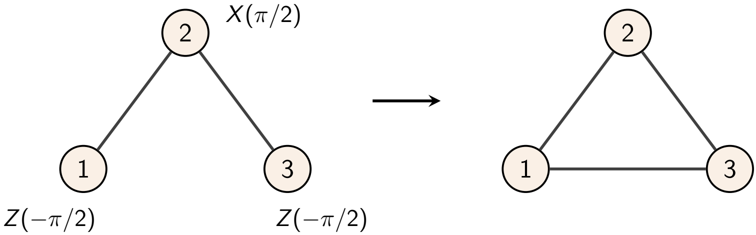


An easier subproblem: local Clifford equivalence

Two graph states are said **local Clifford equivalent** (or LC-equivalent) if they are related by unitaries in the local Clifford group.
Single-qubit Clifford group $= < H, Z(\pi/2) >$.



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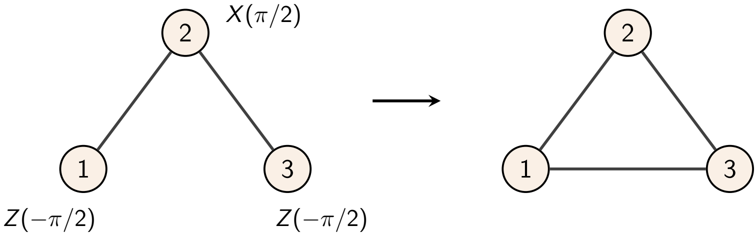

Theorem (Van den Nest, Dehaene, De Moor, 2004)

Two graph states are LC-equivalent iff the two corresponding graphs are related by **local complementations***.*

The LU=LC conjecture

Formulated in the early 2000's [1].

Conjecture

LU=LC i.e. if two graph states are LU-equivalent (local unitaries) then they are LC-equivalent (local Clifford).

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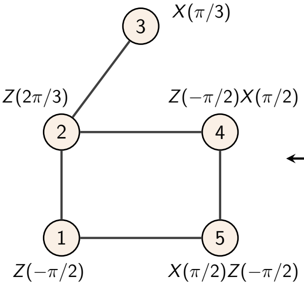

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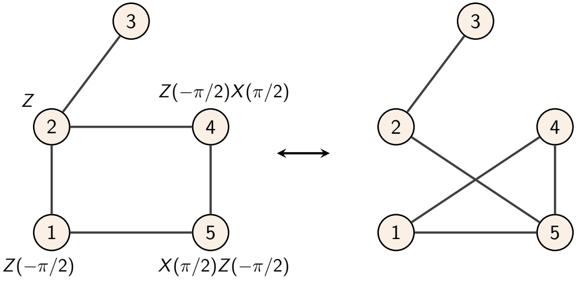

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LU $\neq$ LC

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There exist graph states that are LU-equivalent but not LC-equivalent.

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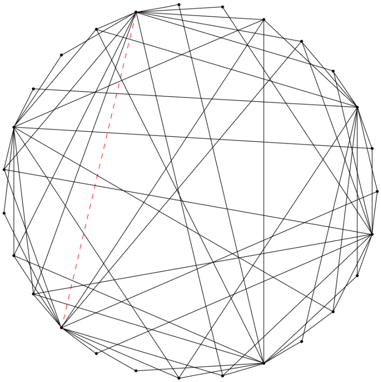
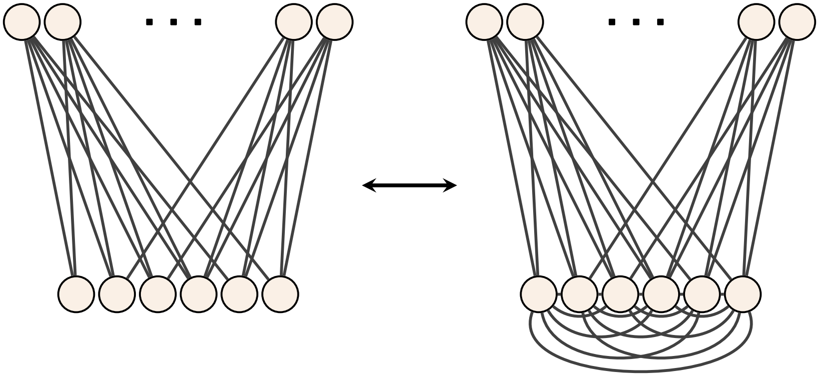
$\rightarrow$ A 27-qubit counterexample to the LU=LC conjecture.



Another look at the 27-vertex counterexample

The 27-vertex counterexample is LC-equivalent to a more structured pair of graphs (Tsimakuridze, Gühne, 2017).



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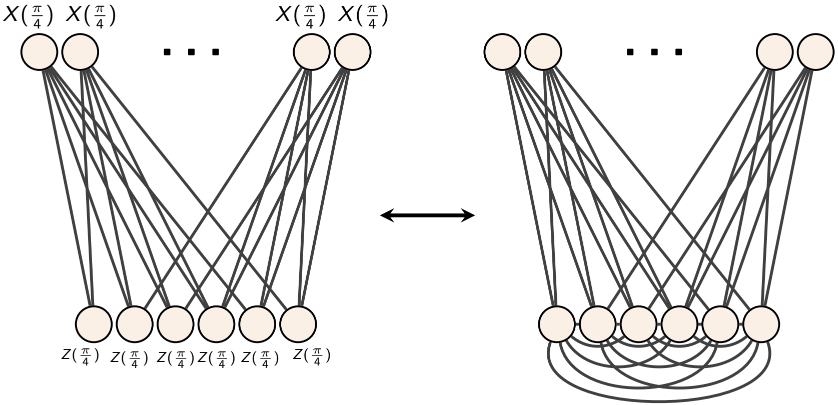

Other results after LU $\neq$ LC

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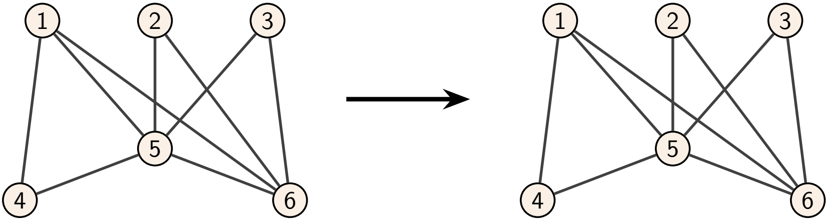
But what about LU-equivalence for **any** graph? Can we construct a graphical characterisation?

Generalizing local complementation to capture local unitary equivalence

A refinement of idempotent local complementations

A sequence of local complementations may leave the graph invariant.



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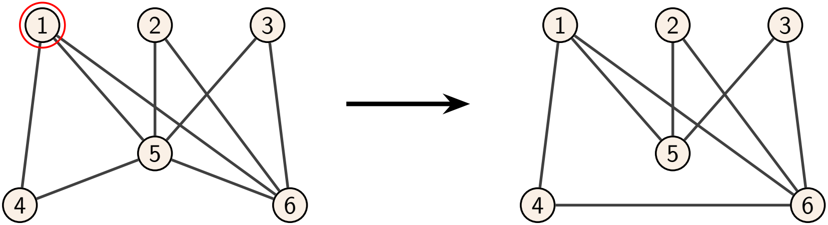

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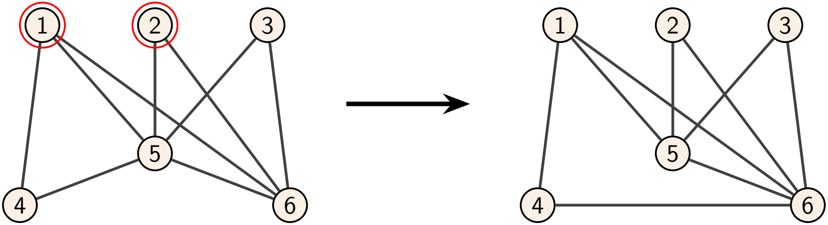

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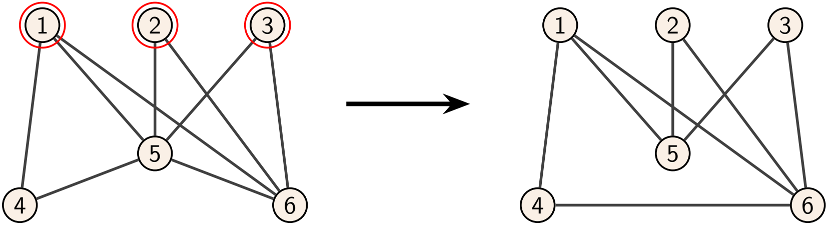

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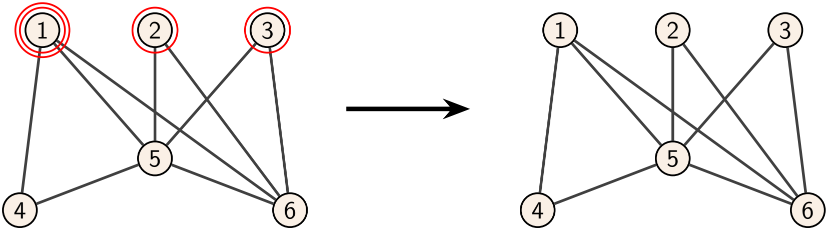

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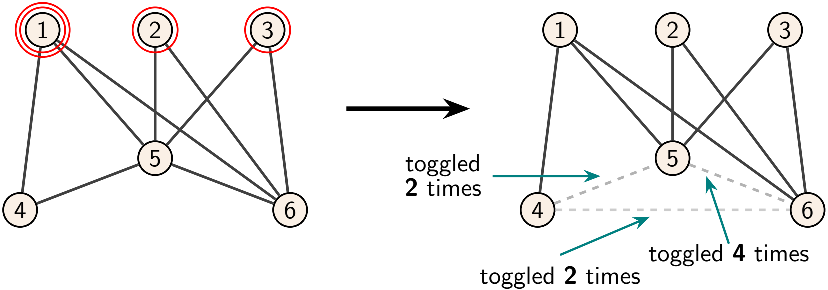

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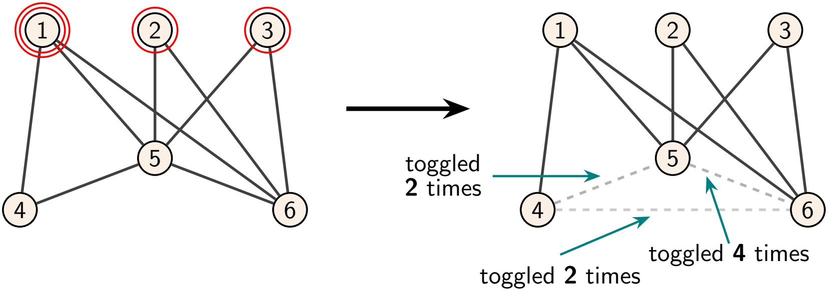

A **2-local complementation** consists in toggling every edge that was toggled 2 mod 4 times by the idempotent local complementations.
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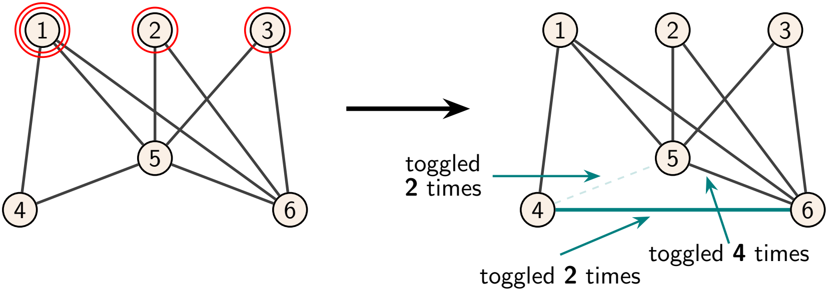

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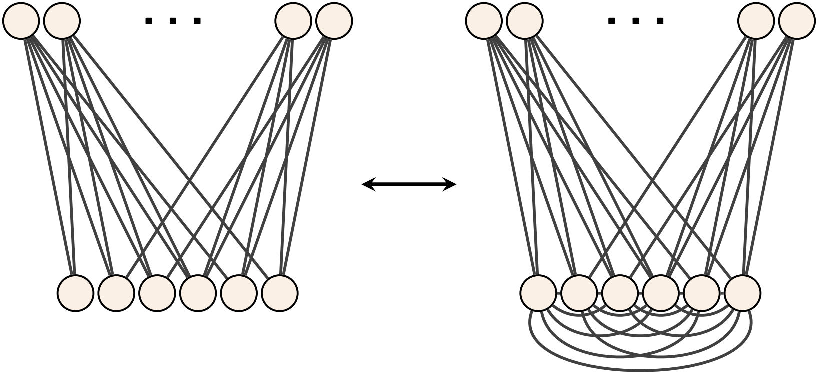

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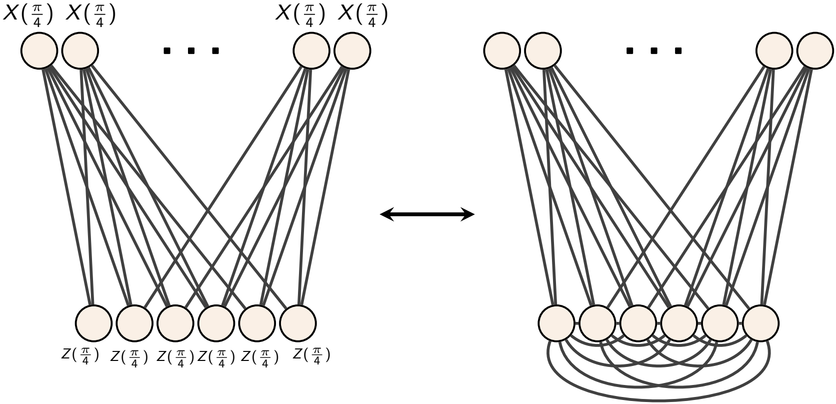

r-local complementation

3-local complementation is a refinement of idempotent 2-local complementation, and so on...
$\rightarrow$ Infinite family of graphical operations parametrised by an integer r:

r-**local complementations**

1-local complementation = local complementation.

Graphical characterization of entanglement

Recall: LC-equivalent $\Leftrightarrow$ related by local unitaries generated by finitely many H and $Z(\pi/2)$.
Define: **LC$_r$-equivalent** $\Leftrightarrow$ related by local unitaries generated by finitely many H and $Z(\pi/2^r)$.

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For $r = 1$, we recover local Clifford $\Leftrightarrow$ local complementation.

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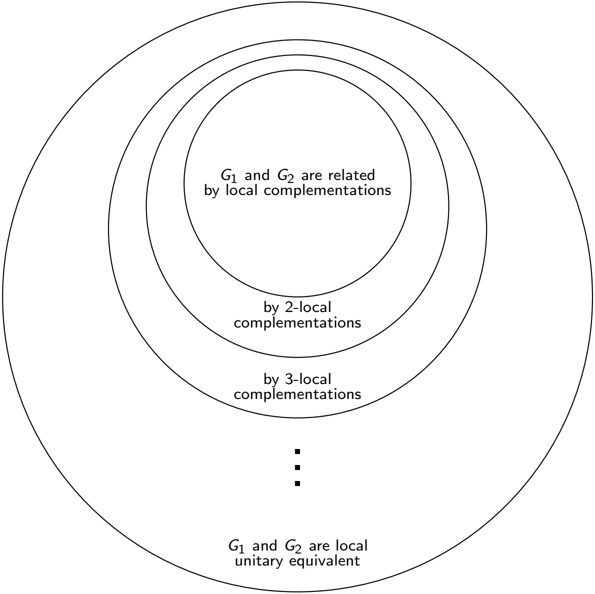
An infinite hierarchy of local equivalences



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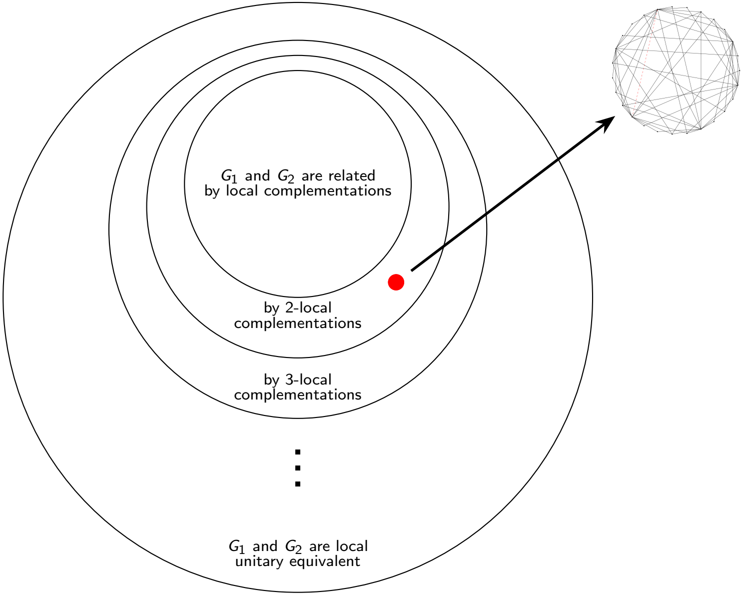

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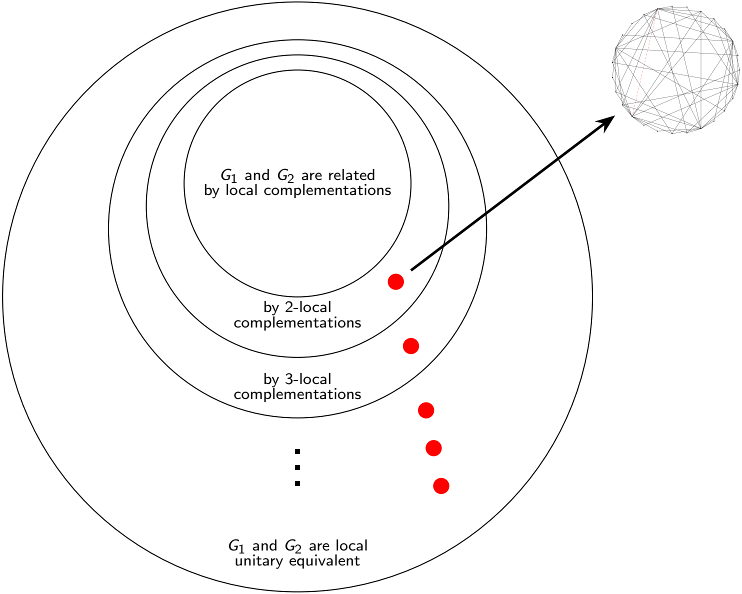
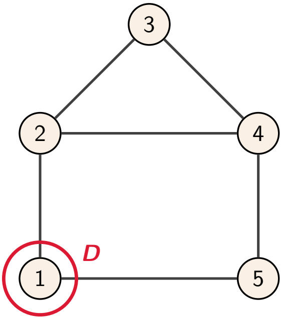


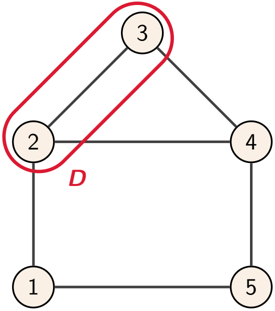
Proof that r-local complementation captures LU-equivalence

Minimal local sets

Definition (Odd neighbourhood)

Given a set of vertices D, the **odd neighbourhood** $Odd_G(D)$ of D is the set of vertices that are neighbours of an odd number of vertices in D.





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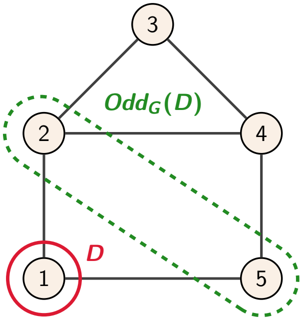

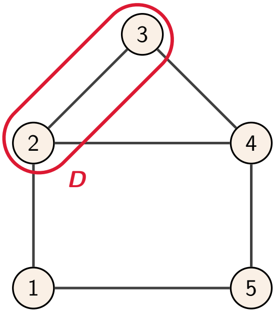

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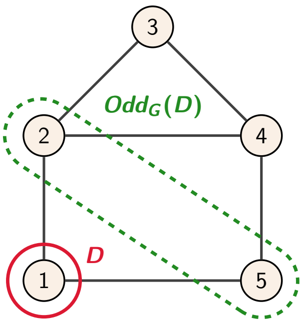
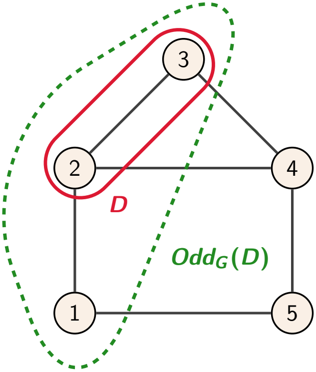
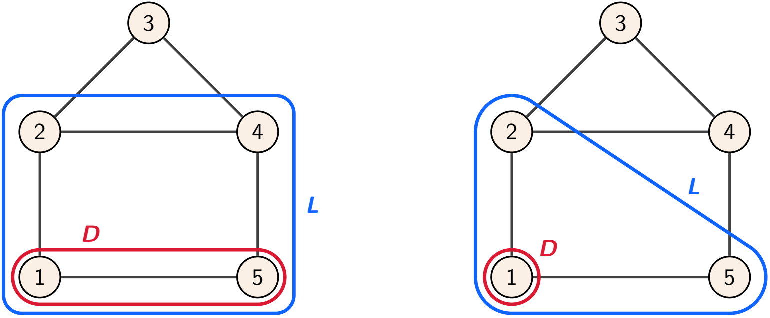




Minimal local sets

Definition (Høyer, Mhalla, Perdrix, 2006)

A **local set** is a non-empty vertex set of the form $L = D \cup Odd_G(D)$.
A **minimal local set** is a local set that is minimal by inclusion (i.e it doesn't strictly contain another local set).



a local set

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Proposition

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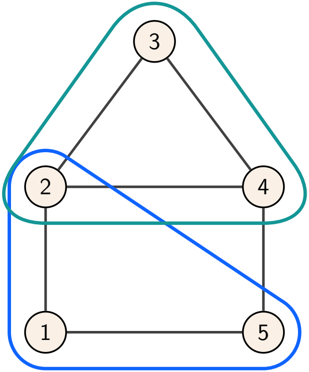
Minimal local sets carry information on the possible local unitaries that map graph states to other graph states.

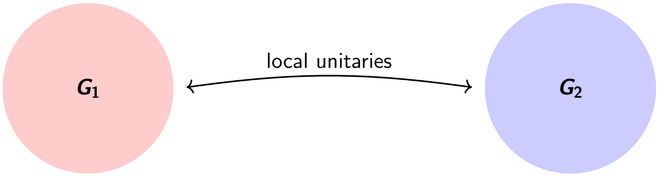
Minimal local sets cover any graph

Theorem (C, Perdrix, 2024)

Each vertex of a graph is covered by at least one minimal local set.



Proof sketch: Standard form for graph states



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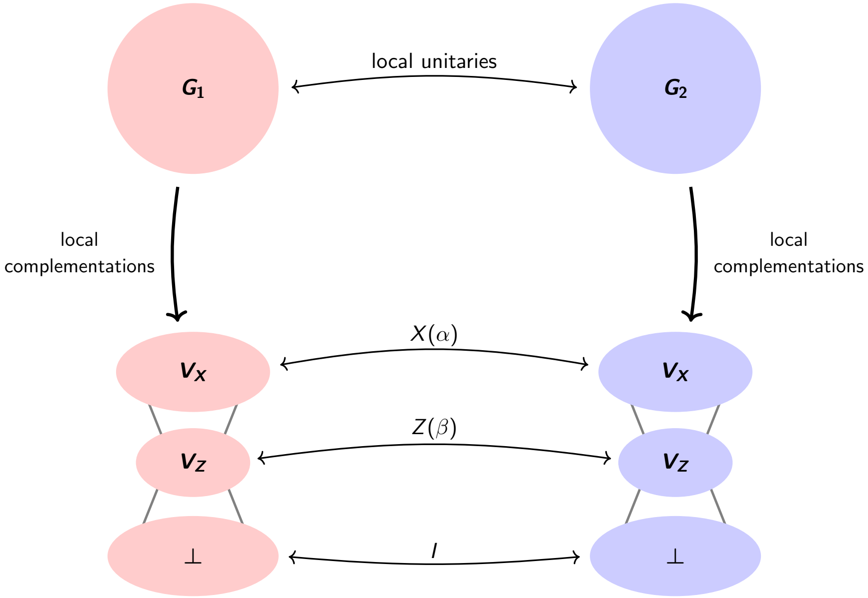

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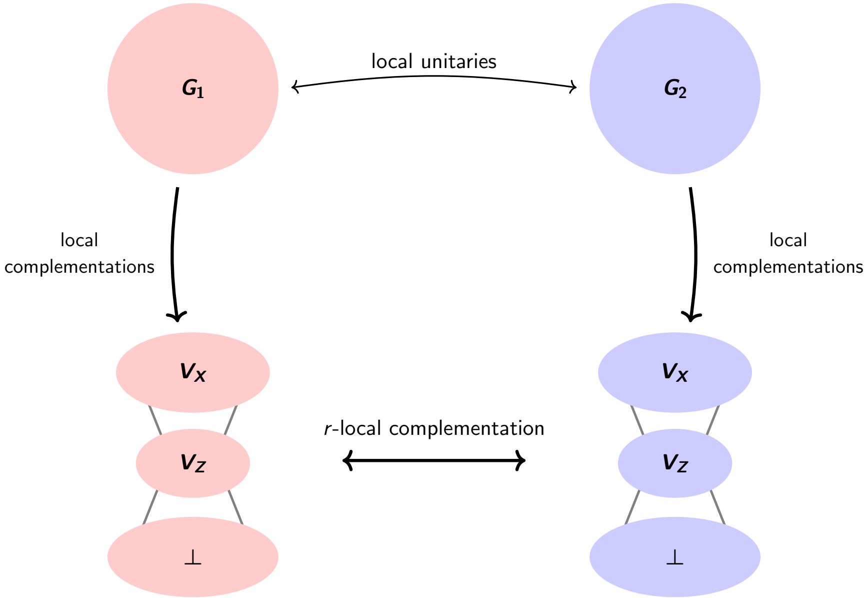

Application 1: a toolbox to prove LU=LC for classes of graphs

A bound on r

Proposition (C, Perdrix, 2025)

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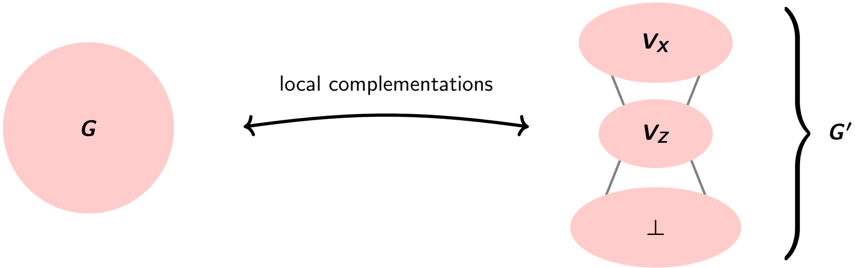
Proposition (C, Perdrix, 2025)

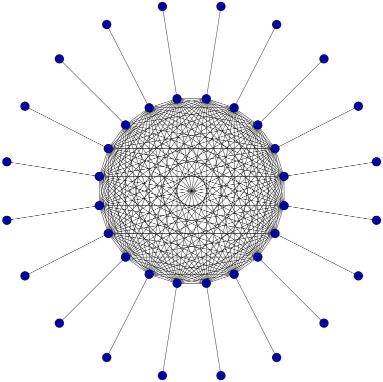
LU=LC for graph states up to 19 qubits.

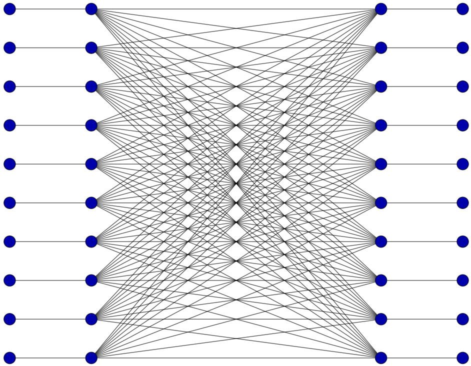
A graphical characterization for LU=LC



Example: LU=LC for repeater graph states

It was conjectured that LU=LC holds for some repeater graph states (Tzitrin, 2018). We showed that this is indeed the case.





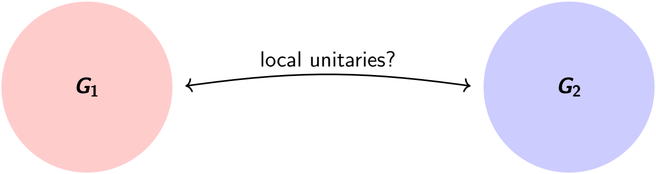
Application 2: a quasi-polynomial algorithm for LU-equivalence

Algorithm for LU-equivalence

Theorem (C, Perdrix, 2025)

There exists an algorithm that decides if two graph states are LU-equivalent with runtime $n^{log_2(n)+O(1)}$.

The algorithm



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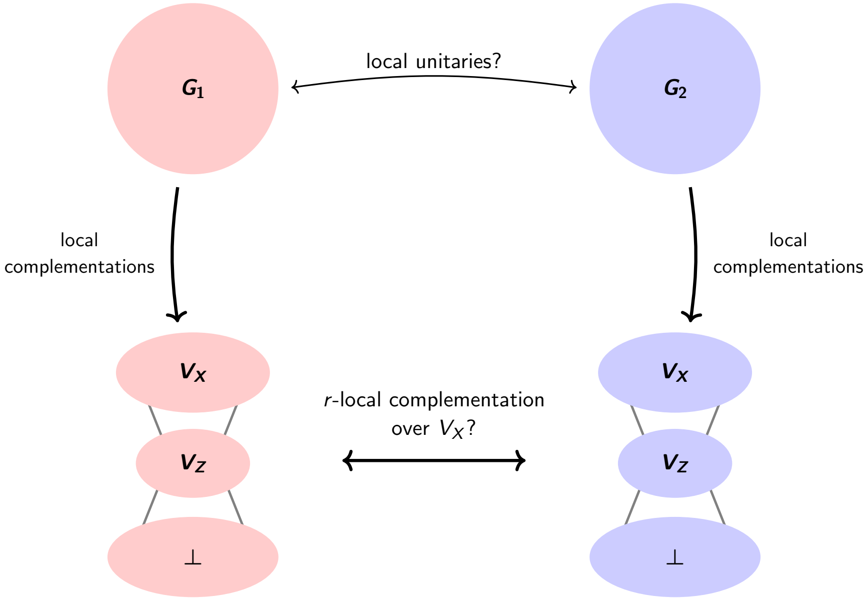
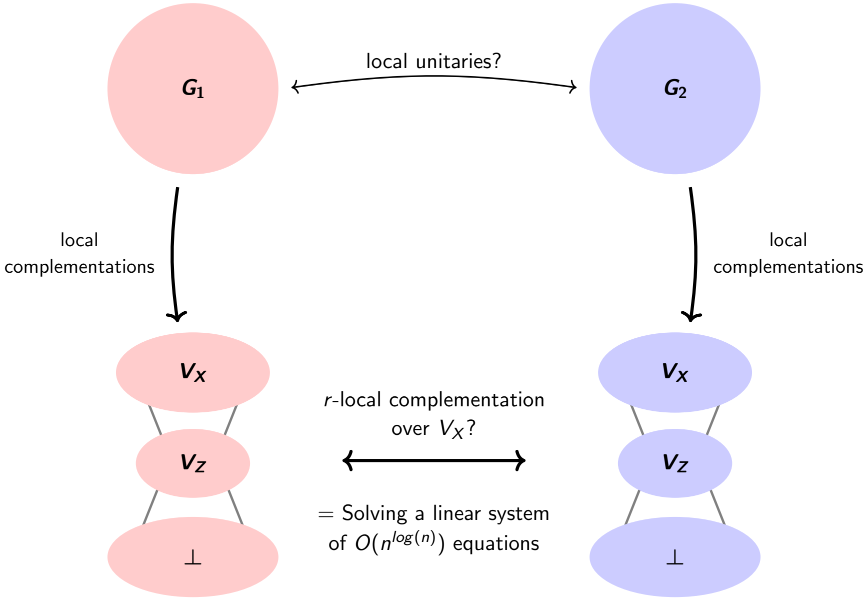


The algorithm



Summary

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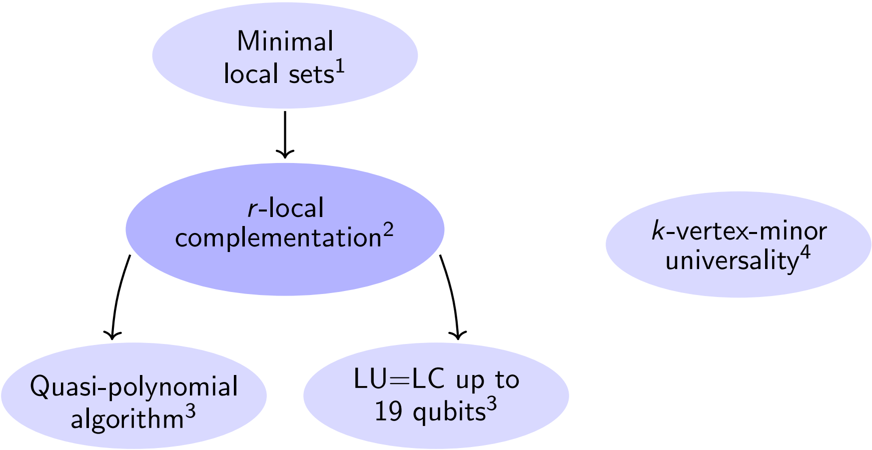

[1] C, Perdrix, Covering a Graph with Minimal Local Sets, WG 2024

[2] C, Perdrix, Local Equivalence of Stabilizer States: a Graphical Characterisation, STACS 2025, QIP 2025

[3] C, Perdrix, Deciding Local Unitary Equivalence of Graph States in Quasi-Polynomial Time, ICALP 2025

[4] Cautrès, C, Mhalla, Perdrix, Savin, Thomassé, Vertex-Minor Universal Graphs for Generating Entangled Quantum Subsystems, ICALP 2024

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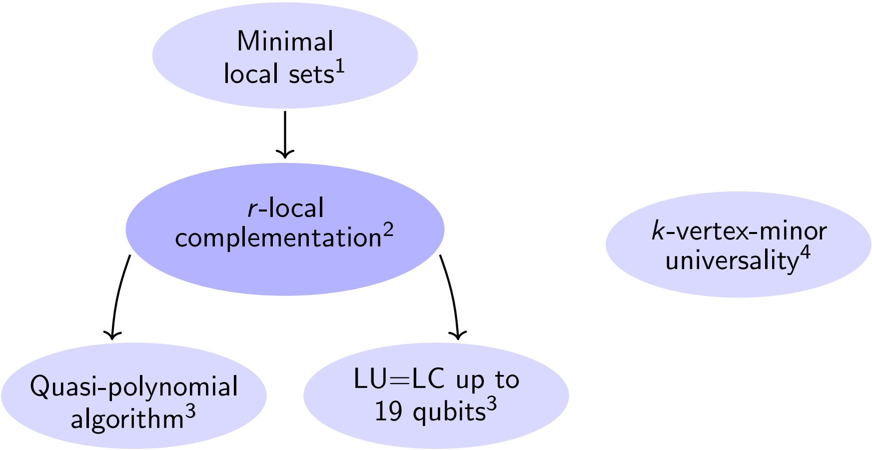
- Does there exist a polynomial-time algorithm for LU-equivalence?
- How does the orbit of a graph state by local unitary operations compare to the orbit by local Clifford operations/local complementation?
- Can the techniques developed in this thesis be applied to generalizations of graph states, like hypergraph states?

Thank you



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