Paper Summary: Solving Variational Problems and PDEs Mapping into General Target Manifolds (Mémoli, Sapiro, Osher)

Nathan King

Department of Mathematics
Simon Fraser University

May 20, 2014

Introduction

- Solution to variational problems or PDEs which map data from a source manifold $\mathcal{M}$ to a target manifold $\mathcal{N}$.
- In [1] it was shown how to address this problem with general $\mathcal{M}$, while restricting $\mathcal{N}$ to be a hyperplane or hypersphere.
- Framework for a flat, open source manifold $\mathcal{M}$ with general target manifold $\mathcal{N}$ is derived here.
- Framework for general $\mathcal{M}$ and $\mathcal{N}$ is also stated.

Mapping Between Manifolds

- Let $\mathcal{N}$ be a $d-1$–dimensional manifold, represented by the zero level set of a higher dimensional function $\psi : \mathbb{R}^d \to \mathbb{R}$.
- For simplicity take ψ to be a signed distance function.
- Goal is to compute a map $\vec{u} : \mathcal{M} \to \mathcal{N}$ that minimizes

$$E[\vec{u}] = \int_{\mathcal{M}} e[\vec{u}]\, d\mathcal{M}.$$

- To illustrate the framework, take $e[\vec{u}] = \frac{1}{2}\|\mathbf{J}_{\vec{u}}\|^2_{\mathcal{F}}$.
- The Frobenius matrix norm is used, $\|\cdot\|^2_{\mathcal{F}} = \sum_{ij}(\cdot)^2_{ij}$.

Euler–Lagrange Equation

- The Euler–Lagrange equation of $E[\vec{u}]$ is

$$\Delta \vec{u} + \left(\sum_{k=1}^{d} \mathbf{H}_{\psi} \left[\frac{\partial \vec{u}}{\partial x_k}, \frac{\partial \vec{u}}{\partial x_k} \right] \right) \nabla \psi(\vec{u}) = 0,$$

- Notation of $\mathbf{A}[\vec{x}, \vec{y}] = \vec{y}^T \mathbf{A} \vec{x}$ is used.
- Solution to the Euler–Lagrange equation is a map $\vec{u}$ onto $\mathcal{N}$.
- Derivation involves classical variational techniques, but have to add a projection step to ensure $\vec{u} : \mathcal{M} \to \{\psi = 0\}$.

Derivation of Euler–Lagrange Equation

- Assume $\vec{u}$ is a map that minimizes $E[\vec{u}]$ and for $t > 0$ construct the variation
$$\vec{w}_t = \mathcal{P}_{\{\psi=0\}}(\vec{u} + t\vec{r}),$$
where $\vec{r}$ is compact C^∞ and $\mathcal{P}_{\{\psi=0\}} : \mathbb{R}^d \to \{\psi = 0\}$ is defined as
$$\mathcal{P}_{\{\psi=0\}}(\vec{v}) = \vec{v} - \psi(\vec{v})\nabla\psi(\vec{v}).$$
- Since the energy achieves a minimum at $t = 0$

$$\left.\frac{dE(\vec{u} + t\vec{r})}{dt}\right|_{t=0} = 0 \quad \iff \quad \sum_{ij} \int_{\mathcal{M}} \left.\left(\frac{\partial w_t^i}{\partial x_j} \frac{d\left(\frac{\partial w_t^i}{\partial x_j}\right)}{dt} \right)\right|_{t=0} d\mathcal{M} = 0.$$

Derivation of Euler–Lagrange Equation

- We compute the terms $\frac{\partial w_t^i}{\partial x_j}$ and $\frac{d\left(\frac{\partial w_t^i}{\partial x_j}\right)}{dt}$ separately.
- $$\frac{\partial \vec{w}_t}{\partial x_j} = \left(\frac{\partial \vec{u}}{\partial x_j} + t\frac{\partial \vec{r}}{\partial x_j}\right) - \left[\nabla\psi(\vec{w}_t)\cdot\left(\frac{\partial \vec{u}}{\partial x_j} + t\frac{\partial \vec{r}}{\partial x_j}\right)\right]\nabla\psi(\vec{w}_t) \\ - \psi(\vec{w}_t)\mathbf{H}_\psi(\vec{w}_t)\left(\frac{\partial \vec{u}}{\partial x_j} + t\frac{\partial \vec{r}}{\partial x_j}\right).$$
- $$\left.\frac{\partial \vec{w}_t}{\partial x_j}\right|_{t=0} = \frac{\partial \vec{u}}{\partial x_j} - \left[\nabla\psi(\vec{u})\cdot\frac{\partial \vec{u}}{\partial x_j}\right]\nabla\psi(\vec{u}),$$
  since $\psi(\vec{u}) = 0$.
- Notice also that
  $$0 = \frac{\partial \psi(\vec{u})}{\partial x_j} = \nabla\psi(\vec{u})\frac{\partial \vec{u}}{\partial x_j} \quad\Rightarrow\quad \left.\frac{\partial \vec{w}_t}{\partial x_j}\right|_{t=0} = \frac{\partial \vec{u}}{\partial x_j}.$$

Derivation of Euler–Lagrange Equation

- Now calculate
$$\left.\frac{d\left(\frac{\partial w_t^i}{\partial x_j}\right)}{dt}\right|_{t=0} = \frac{\partial\left(\left.\frac{dw_t^i}{dt}\right|_{t=0}\right)}{\partial x_j}.$$

- $$\frac{dw_t^i}{dt} = \vec{r} - (\nabla\psi(\vec{w}_t)\cdot\vec{r})\nabla\psi(\vec{w}_t) - \psi(\vec{w}_t)\mathbf{H}_\psi(\vec{w}_t)\vec{r},$$
$$\Rightarrow \left.\frac{dw_t^i}{dt}\right|_{t=0} = \vec{r} - (\nabla\psi(\vec{u})\cdot\vec{r})\nabla\psi(\vec{u}).$$

- Then
$$\frac{\partial\left(\left.\frac{dw_t^i}{dt}\right|_{t=0}\right)}{\partial x_j} = \frac{\partial\vec{r}}{\partial x_j} - \nabla\psi(\vec{u})\left[\frac{\partial\vec{r}}{\partial x_j}\cdot\nabla\psi(\vec{u}) + \mathbf{H}_\psi\left(\vec{r}\cdot\frac{\partial\vec{u}}{\partial x_j}\right)\right]$$
$$-(\vec{r}\cdot\nabla\psi(\vec{u}))\left(\mathbf{H}_\psi\frac{\partial\vec{u}}{\partial x_j}\right).$$

Derivation of Euler–Lagrange Equation

- Now we put all the pieces together.

$$\left.\frac{dE}{dt}\right|_{t=0} = \sum_j \int_{\mathcal{M}} \left.\left(\frac{\partial \vec{w}_t}{\partial x_j} \frac{d\left(\frac{\partial \vec{w}_t}{\partial x_j}\right)}{dt} \right)\right|_{t=0} d\mathcal{M}$$
$$= \sum_j \int_{\mathcal{M}} \left(\frac{\partial \vec{r}}{\partial x_j} \frac{\partial \vec{u}}{\partial x_j} - (\vec{r} \cdot \nabla \psi(\vec{u})) \mathbf{H}_\psi \left[\frac{\partial \vec{u}}{\partial x_j}, \frac{\partial \vec{u}}{\partial x_j} \right] \right) d\mathcal{M}$$

$$\sum_{ij} \int_{\mathcal{M}} \frac{\partial \vec{r}}{\partial x_j} \frac{\partial \vec{u}}{\partial x_j} d\mathcal{M} = \sum_i \int_{\mathcal{M}} \nabla r^i \cdot \nabla u^i \, d\mathcal{M}$$
$$= \sum_i \int_{\partial \mathcal{M}} r^i \frac{\partial u^i}{\partial \mathbf{n}} d\mathcal{S} - \int_{\mathcal{M}} r^i \Delta u^i \, d\mathcal{M},$$

where $\nabla r^i \cdot \nabla u^i = \nabla \cdot (r^i \nabla u^i) - r^i \Delta u^i$ and then the divergence theorem is applied to get the bottom line.

Derivation of Euler–Lagrange Equation

- Now we want to determine when $\frac{dE}{dt}\big|_{t=0} = 0$,

$$\frac{dE}{dt}\bigg|_{t=0} = \int_{\partial\mathcal{M}} \vec{r} \cdot \mathbf{J}_{\vec{u}}\, \mathbf{n}\, d\mathcal{S} - \int_{\mathcal{M}} \vec{r} \cdot \left\{ \Delta\vec{u} + \left(\sum_k \mathbf{H}_\psi \left[\frac{\partial\vec{u}}{\partial x_k}, \frac{\partial\vec{u}}{\partial x_k} \right] \right) \nabla\psi(\vec{u}) \right\} d\mathcal{M},$$

- Boundary condition is eliminated since the support of $\vec{r}$ is compactly included in $\mathcal{M}$.
- Thus for other integral term we need

$$\Delta\vec{u} + \left(\sum_{k=1}^{d} \mathbf{H}_\psi \left[\frac{\partial\vec{u}}{\partial x_k}, \frac{\partial\vec{u}}{\partial x_k} \right] \right) \nabla\psi(\vec{u}) = 0.$$

Gradient Descent Flow

- The gradient descent flow for the above Euler–Lagrange equations is

$$
\begin{aligned}
&\frac{\partial \vec{u}}{\partial t} = \Delta \vec{u} + \left(\sum_{k=1}^{d} \mathbf{H}_{\psi} \left[\frac{\partial \vec{u}}{\partial x_k}, \frac{\partial \vec{u}}{\partial x_k} \right] \right) \nabla \psi(\vec{u}), \\
&\vec{u}(x,0) = \vec{u}_0(x), \qquad x \in \mathcal{M}, \\
&\mathbf{J}_{\vec{u}} \mathbf{n}|_{\partial \mathcal{M}} = 0,
\end{aligned}
\tag{1}
$$

where the vector field $\vec{u}_0(x)$ is the initial data we want to process.

Simple Verification of Gradient Descent

- It is necessary that, for $\vec{u}_0 \in \{\psi = 0\}$, the solution $\vec{u}$ to equation (1) also belongs to $\{\psi = 0\}$.
- *Proposition 1. A regular solution to equation (1) holds* $\psi(\vec{u}(x,t)) = 0\ \forall x \in \mathcal{M}, \forall t \geq 0$ *of regularity.*
- If the initial data is on $\{\psi = 0\}$ then this property is true for $t = 0$.
- Define $v(x,t) = \psi(\vec{u}(x,t))$ and consider

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \nabla\psi(\vec{u}) \cdot \frac{\partial \vec{u}}{\partial t}, \\
&= \Delta\vec{u} \cdot \nabla\psi(\vec{u}) + \sum_{k=1}^{d} \mathbf{H}_\psi(\vec{u}) \left[\frac{\partial \vec{u}}{\partial x_k}, \frac{\partial \vec{u}}{\partial x_k}\right] \nabla\psi(\vec{u}) \cdot \nabla\psi(\vec{u}) \\
&= \Delta\vec{u} \cdot \nabla\psi(\vec{u}) + \sum_{k=1}^{d} \mathbf{H}_\psi(\vec{u}) \left[\frac{\partial \vec{u}}{\partial x_k}, \frac{\partial \vec{u}}{\partial x_k}\right].
\end{aligned}$$

Simple Verification of Gradient Descent

- Now $\frac{\partial v}{\partial x_i} = \nabla\psi(\vec{u})\frac{\partial \vec{u}}{\partial x_i}$

$$\Rightarrow \frac{\partial^2 v}{\partial x_i^2} = \left(\mathbf{H}_\psi(\vec{u})\frac{\partial \vec{u}}{\partial x_i}\right)\cdot\frac{\partial \vec{u}}{\partial x_i} + \nabla\psi(\vec{u})\cdot\frac{\partial^2 \vec{u}}{\partial x_i^2}$$
$$= \frac{\partial^2 \vec{u}}{\partial x_i^2}\cdot\nabla\psi(\vec{u}) + \mathbf{H}_\psi(\vec{u})\left[\frac{\partial \vec{u}}{\partial x_i}, \frac{\partial \vec{u}}{\partial x_i}\right].$$

- Summing up components for $i = 1, \ldots, d$ we have

$$\Delta v = \Delta\vec{u}\cdot\nabla\psi(\vec{u}) + \sum_{k=1}^{d}\mathbf{H}_\psi(\vec{u})\left[\frac{\partial \vec{u}}{\partial x_k}, \frac{\partial \vec{u}}{\partial x_k}\right] = \frac{\partial v}{\partial t},$$

which means v evolves by heat flow.

Simple Verification of Gradient Descent

- Notice also on $\partial\mathcal{M}$

$$\frac{\partial v}{\partial \mathbf{n}} = \nabla_x(\psi(\vec{u}))\cdot\mathbf{n} = \mathbf{J}_{\vec{u}}^T\nabla\psi(\vec{u})\cdot\mathbf{n} = (\nabla\psi(\vec{u}))^T\mathbf{J}_{\vec{u}}\mathbf{n} = (\nabla\psi(\vec{u}))^T\mathbf{0} = 0.$$

- Hence v follows heat flow with zero Neumann BCs and zero initial data.
- From uniqueness of the solution it follows that $v(x,t) = \psi(\vec{u}(x,t)) = 0$ for all $x \in \mathcal{M}$ and $t \geq 0$.

Generic Source and Target Manifolds

- Let $\mathcal{M} = \{x \in \mathbb{R}^m | \phi(x) = 0\}$ be a general manifold with signed distance function ϕ.
- Another projection step must be added for a generic $\mathcal{M}$.
- Energy density is now given by

$$e_\phi[\vec{u}] = \frac{1}{2} \|\mathbf{J}_{\vec{u}}^{\phi}\|_{\mathcal{F}}^2,$$

  where the Jacobian of $\vec{u}$ intrinsic to $\mathcal{M}$ is $\mathbf{J}_{\vec{u}}^{\phi} = \mathbf{J}_{\vec{u}} \mathcal{P}_{\nabla \phi}$.
- Energy is redefined as

$$E[\vec{u}] = \int_{\mathbb{R}^m} e_\phi[\vec{u}] \delta(\phi(x)) dx.$$

Generic Source and Target Manifolds

- The gradient descent flow becomes

$$\frac{\partial \vec{u}}{\partial t} = \nabla \cdot (\mathcal{P}_{\nabla\phi}\mathbf{J}_{\vec{u}}^T) + \left(\sum_{k=1}^{d}\sum_{r=1}^{m}\mathbf{H}_\psi\left[\frac{\partial \vec{u}}{\partial x_r}, \frac{\partial \vec{u}}{\partial x_k}\right](\mathcal{P}_{\nabla\phi})_{kr}\right)\nabla\psi.$$

- Columnwise divergence is applied, i.e. for a matrix A, $\nabla \cdot A = (\nabla \cdot \vec{A}_{v_1}|\cdots|\nabla \cdot \vec{A}_{v_r})$, where $\vec{A}_{v_i}$ is the ith column of A.
- p–harmonic maps can be implemented by changing the energy density as

$$e_{\phi,p}[\vec{u}] = \frac{1}{p}\|\mathbf{J}_{\vec{u}}^{\phi}\|_{\mathcal{F}}^{p}.$$

Numerical Examples

- Texture maps are constructed, noise added to them, and then diffusion applied using the above framework.
- $\mathcal{J}$ is a surface onto which an image $I \in D \subset \mathbb{R}^2$ is mapped to.
- The *texture map* is a map $\vec{T} : \mathcal{J} \to D$.
- To find $\vec{T}$ a multidimensional scaling approach is applied [2].
- Once $\vec{T}$ is known, it is inverted to obtain $\vec{u}_0 : D \to \mathcal{J}$.

Numerical Examples

- A noisy map $\vec{u} : D \to \mathcal{J}$ is built as

$$\vec{u}(x) = \mathcal{P}_{\mathcal{J}}\Big(\vec{u}_0(x) + \vec{\mathbf{n}}(x)\Big),$$

  where $\vec{\mathbf{n}} : D \to \mathcal{J}$ is a random map.
- Gradient descent flow (1) is applied with $\vec{u}$ as initial condition.
- Resulting map is inverted and used to paint the surface with a texture.
- Note, the map $\vec{u}$ is what is being processed, not the image I itself.

References

M. Bertalmío, L.T. Cheng, S. Osher, G. Sapiro,
Variational problems and partial differential equations on implicit surfaces,
J. Comput. Phys. 174 (2) (2001) 759–780.

G. Zigelman, R. Kimmel, N. Kiryati,
Texture mapping using surface flattening via multi–dimensional scaling,
Technion–CIS, Technical Report 2000–01, 2000.