Numerically Solving PDEs While Mapping Between Manifolds

Nathan King

Department of Mathematics
Simon Fraser University

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Overview of Research

- Research involves numerical methods for solving PDEs (and variational problems) defined on surfaces.
- Work with are embedding methods, which embed the surface PDE in a higher dimensional space.
- For example, a PDE defined on a sphere is embedded in $\mathbb{R}^3$.
- Embedding methods allow for standard, well studied numerical methods on cartesian grids to be used.

Outline

Level Set Method

- Level Set method illustrated with heat equation on a surface $\mathcal{S}$.
- Heat equation defined on $\mathcal{S}$ is

$$\frac{\partial u}{\partial t} = \Delta_{\mathcal{S}} u.$$

- Embed $\mathcal{S}$ using a level set function $\phi : \mathbb{R}^3 \to \mathbb{R}$.
- Zero level set represents $\mathcal{S}$, that is $\mathcal{S} = \{\phi = 0\}$.
- Take ϕ to be the signed distance function of $\mathcal{S}$.
- Signed distance function has property that $\|\nabla \phi\| = 1$.

Level Set Method

- Intrinsic gradient of u on $\mathcal{S}$ can be written in terms of gradients in $\mathbb{R}^3$,

$$\nabla_{\mathcal{S}} u = \mathcal{P}_{\nabla \phi} \nabla u.$$

- Operator

$$\mathcal{P}_v = I - \frac{v\, v^T}{\|v\|^2}$$

projects any given vector into a plane orthogonal to v.

- Heat equation on $\mathcal{S}$ becomes

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathcal{P}_{\nabla \phi} \nabla u),$$

which is now defined in $\mathbb{R}^3$.

Closest Point Method

- Want to create embedding PDE which replaces intrinsic surface gradients with standard gradients.
- Obviously the embedding PDE and surface PDE will not agree for long times, however this is accurate initially and is sufficient to update the solution in time.
- A closest point representation of the surface is used.
- For any point $\mathbf{x} \in \mathbb{R}^3$, let $CP(\mathbf{x})$ denote the closest point to $\mathbf{x}$ in $\mathcal{S}$.
- CP is a function from $\mathbb{R}^3$ to $\mathbb{R}^3$ that returns values lying in $\mathcal{S}$.

Closest Point Method

- Embed the surface PDE by replacing surface gradients, $\nabla_{\mathcal{S}}$, with standard gradients, ∇, in $\mathbb{R}^3$.
- Alternate between two steps:
  1. Extend surface data into $\mathbb{R}^3$ using the closest point function, i.e. replace $u(\mathbf{x})$ with $u(CP(\mathbf{x}))$.
  2. Compute solution to embedding PDE using standard finite differences on Cartesian grid for one time step.

Mapping Between Manifolds

- Let $\mathcal{M}$ denote the source manifold and $\mathcal{N}$ the target manifold.
- Signed distance functions of $\mathcal{M}$ and $\mathcal{N}$ are ϕ and ψ, respectively.
- Goal is to compute a vector function $u : \mathcal{M} \to \mathcal{N}$ that minimizes

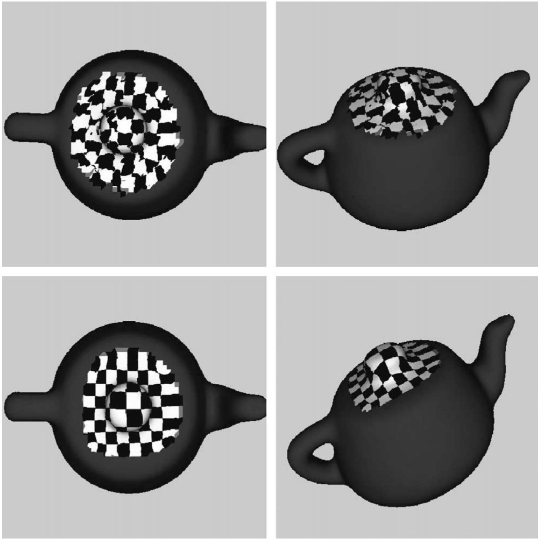
$$E = \frac{1}{2} \int \left\| \mathcal{P}_{\nabla \phi} \nabla u \right\|^2 \delta(\phi) d\mathcal{M}.$$

- Gradient descent flow is

$$\frac{\partial u}{\partial t} = \mathcal{P}_{\nabla \psi}(\nabla \cdot (\mathcal{P}_{\nabla \phi} \nabla u)).$$

- $\mathcal{P}_{\nabla \psi}$ is the projection operator onto the tangent space of $\mathcal{N}$.

Mapping Between Manifolds



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