Paper Summary: Direct Cortical Mapping via Solving Partial Differential Equations on Implicit Surfaces (Shi, Thompson, Dinov, Osher, Toga)

Nathan King

Department of Mathematics
Simon Fraser University

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Example: Heat Equation on a Surface

- Let $\mathcal{S}$ denote the surface and $\phi : \mathbb{R}^3 \to \mathbb{R}$ a level set function (for simplicity take ϕ as signed distance function).
- The zero level set function, i.e. $\phi = 0$, represents $\mathcal{S}$.
- The heat equation defined on $\mathcal{S}$ is

$$\frac{\partial u}{\partial t} = \Delta_{\mathcal{S}} u$$

- We extend the data u off $\mathcal{S}$ in normal direction, that is

$$\nabla u \cdot \nabla \phi = 0.$$

Example: Heat Equation on a Surface

- The intrinsic gradient of u on $\mathcal{S}$ can be represented in terms of gradients in $\mathbb{R}^3$ as

$$\nabla_{\mathcal{S}} u = \mathcal{P}_{\nabla\phi} \nabla u.$$

- The operator

$$\mathcal{P}_v = I - \frac{v \otimes v}{\|v\|^2} = I - \frac{v\, v^T}{\|v\|^2}$$

projects any given vector into a plane orthogonal to v.

- The heat equation defined on $\mathcal{S}$ becomes

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathcal{P}_{\nabla\phi} \nabla u),$$

which is now defined on $\mathbb{R}^3$.

Mapping Between Manifolds

- Let $\mathcal{M}$ denote the source manifold and $\mathcal{N}$ the target manifold.
- Signed distance functions of $\mathcal{M}$ and $\mathcal{N}$ are ϕ and ψ, respectively.
- Goal is to compute a vector function $u : \mathcal{M} \to \mathcal{N}$ that minimizes

$$E = \frac{1}{2} \int \|J_u^\phi\|^2 \delta(\phi) dx,$$

  where

$$J_u^\phi = \mathcal{P}_{\nabla \phi} \, J_u^T$$

  and J_u is the regular Jacobian in $\mathbb{R}^3$.
- The Frobenius matrix norm is used for $\|J_u^\phi\|^2 = \sum_{ij} (J_u^\phi)_{ij}^2$.

Mapping Between Manifolds

- From the first variation of E the gradient descent flow is

$$\frac{\partial u}{\partial t} = \mathcal{P}_{\nabla \psi(u(x,t))}(\nabla \cdot (\mathcal{P}_{\nabla \phi} J_u^T)).$$

- $\mathcal{P}_{\nabla \psi(u(x,t))}$ is the projection operator onto the tangent space of $\mathcal{N}$ at the point $u(x,t)$.

Brain Mapping Challenges

- There are two major challenges when extending this method to brain mapping.
- First, numerical schemes must be developed to incorporate landmark constraints such as sulcal.
- Also, the optimization of E is non-convex thus sufficient initialization is needed.
- The purpose of the paper by Shi et al. is to overcome these challenges when using the level set method.

Incorporating Sulcal Constraints

- Let $\{\mathcal{C}^k_{\mathcal{M}}\}$ and $\{\mathcal{C}^k_{\mathcal{N}}\}$, $k = 1, \ldots, K$, be the sets of sulcal curves for $\mathcal{M}$ and $\mathcal{N}$, respectively.
- Assume mapping between K pairs of curves are known.
- Now the variational problem becomes

$$u = \arg\min_u \; E(u),$$

  with boundary conditions $u(\mathcal{C}^k_{\mathcal{M}}) = \mathcal{C}^k_{\mathcal{N}}$ for all k.
- Boundary conditions are treated as Dirichlet type, therefore can view this as diffusion with heat flow block across sulcal curves.

Mesh Representation of Sulcal Constraints

- Approximate $\mathcal{C}^k_{\mathcal{M}}$ and $u(\mathcal{C}^k_{\mathcal{M}})$ by sampling L points, $p_1, \ldots, p_L$, uniformly along $\mathcal{C}^k_{\mathcal{M}}$.
- Extend boundary condition $u(p_i)$, $i = 1, \ldots, L$, normal to the surface at $2Q+1$ points $\hat{p}_{i,j}$, $j = -Q, \ldots, Q$.
- Start extending $u(p_i)$ at $\hat{p}_{i,0} = p_i$ and use

$$
\begin{array}{lcr}
\hat{p}_{i,j} = \hat{p}_{i,j+1} + h\nabla\phi(\hat{p}_{i,j-1}) & 1 \leq j \leq Q & \text{(outward)} \\
\hat{p}_{i,j} = \hat{p}_{i,j+1} - h\nabla\phi(\hat{p}_{i,j+1}) & -Q \leq j \leq -1 & \text{(inward).}
\end{array}
$$

- Triangulated mesh is created from the $L(2Q+1)$ points $\hat{p}_{i,j}$ and linear interpolation is used to compute $u(\mathcal{C}^k_{\mathcal{M}})$ within any triangle.

Numerical Schemes for ∇u and Δu

- Grid points x_1 and x_2 are **connected** if a line joining them does not cross the extended surface of $\mathcal{C}^k_{\mathcal{M}}$ for all k.
- For $u = \langle u^1, u^2, u^3 \rangle$ we approximate ∇u in the x direction as

$$D^x_+ u^d_{i,j,k} = \begin{cases} \frac{u^d_{i+1,j,k} - u^d_{i,j,k}}{h} & (i,j,k) \text{ and } (i+1,j,k) \text{ are connected,} \\ \frac{u^{\sim d}_{i+1,j,k} - u^d_{i,j,k}}{h} & \text{otherwise,} \end{cases}$$

  where $u^{\sim d}_{i+1,j,k}$ is the interpolated value of u^d at the intersection of the line connecting (i,j,k) and $(i+1,j,k)$.
- The Laplacian in the x direction is approximated by $D^x_- D^x_+$ and similarly in the y and z directions.

Initializing Optimization

- Sulcal curves give boundary conditions and we know u should be interpolated smoothly in areas between these sulcal curves.
- Shi et al. proposed a front propogating type approach to find the initial map based on the latter information.
- The sulcal curves act as the source of the front propagation initially, then move outward to find the map at their neighbouring points by searching locally for the best correlation of a feature called **landmark context**

$$\mathcal{L}\,\mathcal{C}_{\mathcal{M}}(p) = \langle d(p, \mathcal{C}^1_{\mathcal{M}}), d(p, \mathcal{C}^2_{\mathcal{M}}), \ldots, d(p, \mathcal{C}^K_{\mathcal{M}}) \rangle.$$