1D gas dynamics

$P = \frac{mK}{1+\alpha} n^{\alpha+1}$ so that $\frac{1}{m} P_x = K n^{\alpha} n_x$

and Euler equation reads: $u_t + u u_x + K n^{\alpha-1} n_x = 0$

Let's define $r_i(x,t) \overset{\text{def}}{=} \frac{u}{2} \pm \frac{\sqrt{K}}{\alpha} n^{\alpha/2}$ where $(\pm) = \binom{1}{2}$

$$\begin{cases} (r_i)_x = \frac{1}{2} u_x \pm \frac{\sqrt{K}}{2} n^{\frac{\alpha}{2}-1} n_x \\ (r_i)_t = \frac{1}{2} u_t \pm \frac{\sqrt{K}}{2} n^{\frac{\alpha}{2}-1} n_t \end{cases}$$

n_t $= -n_x u - n u_x$: this is the equation of continuity

u_t can be expressed in terms of u_x and n_x thanks to Euler equation.

one thus has: $(r_i)_t = -\frac{1}{2} u u_x - \frac{K}{2} n^{\alpha-1} n_x \pm \frac{\sqrt{K}}{2} n^{\frac{\alpha}{2}-1} (-n_x u - n u_x)$

$$= -u \left[\frac{u_x}{2} \pm \frac{\sqrt{K}}{2} n^{\frac{\alpha}{2}-1} n_x \right] \pm \sqrt{K} n^{\frac{\alpha}{2}} \left[-\frac{u_x}{2} \mp \frac{\sqrt{K}}{2} n^{\frac{\alpha}{2}-1} n_x \right]$$

$$= -u (r_i)_x \pm \sqrt{K} n^{\alpha/2} (-r_i)_x$$

thus $$\boxed{(r_i)_t + (u \pm \sqrt{K} n^{\alpha/2}) (r_i)_x = 0}$$

NB: from (A2)
$[K] = [P][L^3]^{1+\alpha} M^{-1}$
$= M L^{-1} T^{-2} L^{3+3\alpha} M^{-1}$
$= L^{3\alpha+2} T^{-2}$
and $[\sqrt{K} n^{\alpha/2}] = L T^{-1} = [u]$

From the definition of the Riemann invariants it is clear that:

$$r_1 + r_2 = u \qquad r_1 - r_2 = \frac{2}{\alpha} \sqrt{K} n^{\alpha/2}$$

so the quantity $V_i = u \pm \sqrt{K} n^{\alpha/2}$ can be written as:

$$V_i = r_1 + r_2 \pm \frac{\alpha}{2} (r_1 - r_2) = \left(1 \pm \frac{\alpha}{2}\right) r_1 + \left(1 \mp \frac{\alpha}{2}\right) r_2$$

that is: $$\begin{cases} V_1 = \left(1+\frac{\alpha}{2}\right) r_1 + \left(1-\frac{\alpha}{2}\right) r_2 \\ V_2 = \left(1-\frac{\alpha}{2}\right) r_1 + \left(1+\frac{\alpha}{2}\right) r_2 \end{cases}$$

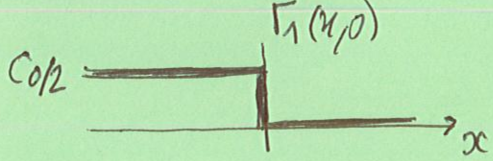
GD2

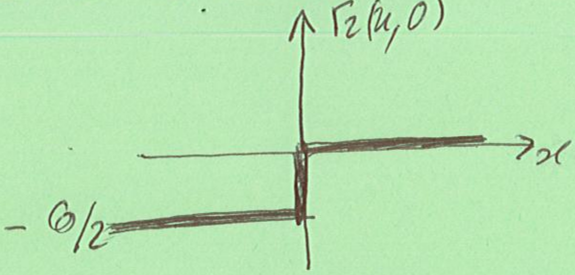
$\boxed{\text{Dam break}}$

$(\alpha = 2)$

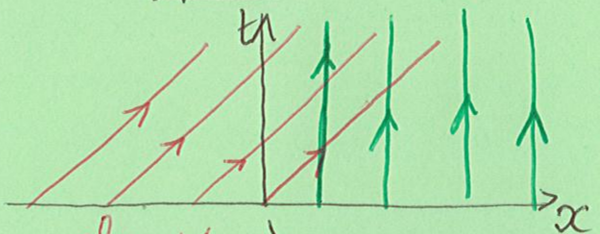
r_1 and r_2 both satisfy: $r_t + 2r\, r_x = 0$
with the initial condition:

$(c_0 = \sqrt{K n_0})$

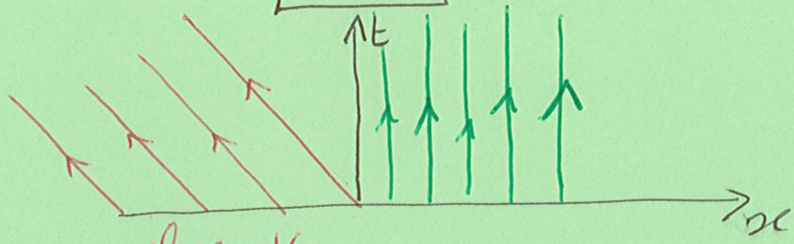




hence the characteristics are:

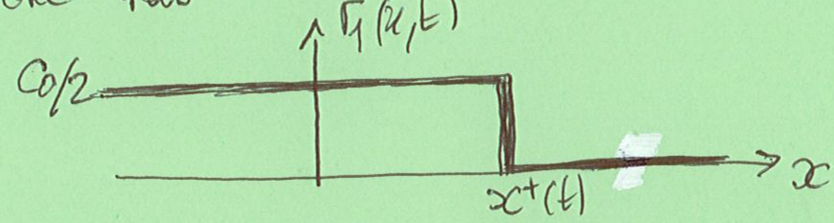


here there is clearly a shock

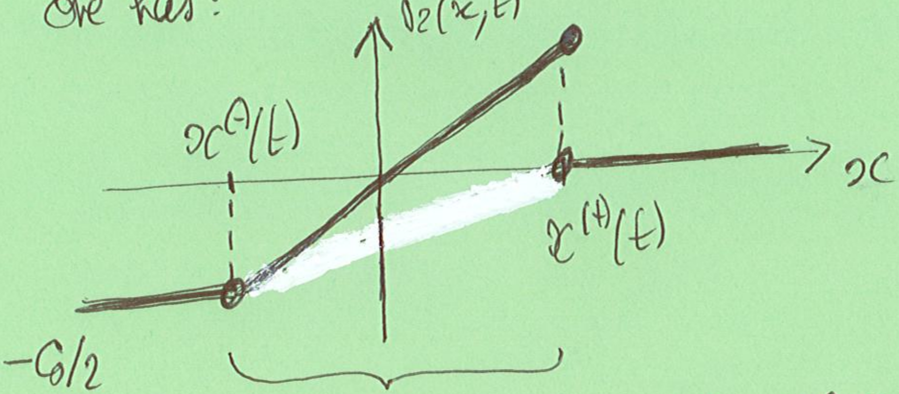


here there is clearly a rarefaction wave.

- if $r_1(x,t)$ is piece-wise constant in $[-\infty, x^{(+)}(t)]$ and $[x^{(+)}(t), +\infty]$ one has:



if $r_2(x,t)$ is piece-wise constant in $[-\infty, x^{(-)}(t)]$ and $[x^{(+)}(t), +\infty]$ one has:

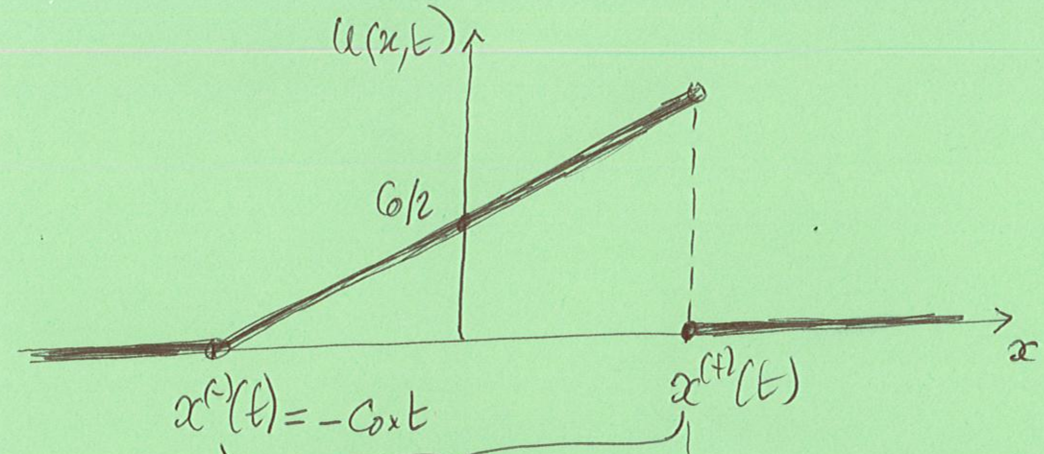


here r_2 depend on ξ and since $(\xi - 2r_2)\dfrac{dr_2}{d\xi} = 0$

one has $r_2 = \dfrac{\xi}{2} = \dfrac{x}{2t}$

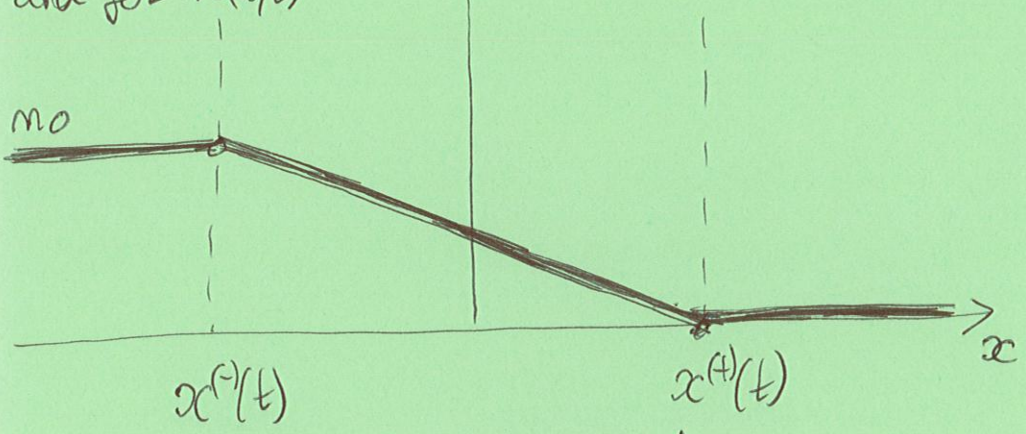
r_2 should be continuous at $x^{(-)}(t) \Longrightarrow \boxed{x^{(-)}(t) = -c_0 \times t}$

then one has $u = r_1 + r_2$ and $n = \frac{r_1 - r_2}{\sqrt{\kappa}}$ $\left(\Leftrightarrow \frac{n}{n_0} = \frac{r_1 - r_2}{c_0}\right)$ GD3



here $u = \frac{c_0}{2} + \frac{x}{2t} = \frac{c_0}{2}\left(1 + \frac{x}{c_0 t}\right)$

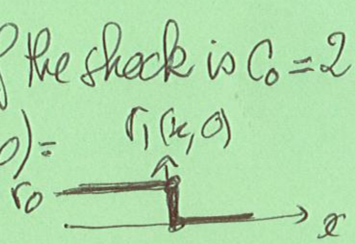
and for $n(x,t)$ =



here $n = \frac{1}{\sqrt{\kappa}}\left(\frac{n_0\sqrt{\kappa}}{2} - \frac{x}{2t}\right)$

$= \frac{n_0}{2}\left(1 - \frac{x}{c_0 t}\right)$ = n cancels at $x = c_0 t$, thus $x^{(+)}(t) = c_0 \times t$

remark: the above result shows that the velocity of the shock is $c_0 = 2r_0$ where r_0 $(= c_0/2)$ is the value of $r_1(x<0, t=0)$ =



- the eq. verified by r_1 is $r_t + (r^2)_x = 0$.
  Hence, from the conservation law seen in the course, one is tempted to compute the velocity of the shock as: $U = r_0^2/r_0 = r_0$ which is not correct.
- However, it is difficult to assess which is the correct conserved quantity: the eq. can also be written as: $(r^2)_t + \left(\frac{4r^3}{3}\right)_x = 0$ and in this case $U = 4/3\, r_0$ - see the discussion in Whitham ---