Nonlinear Periodic Waves and Their Modulations— An Introductory Course

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Preface

Theories come and go, but examples remain forever.

(ascribed to I.M. Gelfand)

The main aim of this book is to give a detailed introduction into the theory of nonlinear waves and their modulations. At present this theory has reached a quite mature form, though many important questions are not answered yet. At the same time, its most developed parts are mathematically involved and for this reason they are rather difficult for study by young researchers, especially physicists. As a result, a wonderful theory having its origin in outstanding investigations of Whitham, Lighthill, Benjamin, Feir and many others on nonlinear water waves has lost to much extent its links with applied science and is used very rarely for solving concrete physical problems. Since this theory might have a number of important applications, it seems topical to give its presentation in an accessible for beginners form.

In the author's opinion, the best way of studying such a subject is to consider thoroughly a few relevant examples illustrating the most essential features of the theory. Therefore, in this book we do not even try to present a general mathematical approach. Rather, we consider in much detail several problems having clear physical formulation. These examples illustrate typical behaviour and may serve as models for solving other analogous problems.

Since the theory has its roots in dynamics of compressible fluids and water waves, a considerable part of Chapter 1 is devoted to their discussion with the aim to introduce necessary basic concepts used throughout the book. This Chapter makes the exposition to much extent self-contained. Influence of dispersion, nonlinearity and viscosity effects on evolution of a wave pulse is discussed. The fundamental notion of the Riemann invariants is introduced. The Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations are derived and their simplest solutions are studied.

In Chapter 2 we consider the universality of the KdV and NLS equations and indicate the physical conditions when one could expect that nonlinear waves are described by these equations. In this Chapter the reader will find several additional examples of the use of a singular perturbation theory for derivation of nonlinear wave equations in various physical situations. We consider also some other wave equations interesting from a physical point of view which will be discussed in further development of the theory.

The Whitham theory of modulations in its original form is presented in Chapter 3. The results obtained here form a starting point for development of the modern theory of nonlinear integrable equations and of the so-called "finite-gap integration method" which are developed in Chapter 4. We confine ourselves to the simplest but the most important for applications case of one-phase solutions. The method is formulated in such a form that yields directly the physically relevant solutions without necessity to impose any additional constraint on the parameters defining the solution.

The finite-gap integration method is applied to finding the periodic solutions of many wave equations in Chapter 5.In addition, it leads to simple and direct methods of derivation of modulational Whitham equations. One such method is presented in Chapter 5 with its applications to the previously discussed periodic solutions of the nonlinear wave equations.

The techniques developed in Chapters 4 and 5 are applied in the next Chapters to two typical problems. In Chapter 6 we expose the theory of collisionless shock waves in dispersive systems described by the KdV equation. We show how the dispersion and nonlinear effects lead to formation of a region of fast oscillations after the wave-breaking point. In Chapter 7 we consider the modulationally unstable systems described by the focusing NLS equation and some other equations. The nonlinear stage of evolution of the modulational instability is studied. It is shown that the Whitham theory permits one to describe the formation of solitons from an initially localized disturbance.

At the end of each Chapter exercises are provided to amplify the discussion of important topics such an singular perturbation theory, Riemann invariants, finite-gap integration method, Whitham equations and their solutions. A collection of useful formulas from the elliptic functions theory, the algebraic resolvents theory, and solutions to exercises are given in three Appendices.

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Chapter 1

Introduction and basic concepts

1.1 Examples of wave motion

Everybody who studied elementary physics knows that wave motion is ubiquitous. It is enough to recall sound waves, water waves, electromagnetic waves to make it clear that a considerable part of physics consists of investigation of behaviour of waves. At the same time, wave phenomena of different kind are united by common mathematical methods applied to their description. To introduce the main concepts, which will be used throughout this book, let us consider a few typical examples of wave motion.

1.1.1 *Sound*

The stationary state of air is described by constant air density $\rho = \rho_0$ and constant pressure $p = p_0$ which do not depend on space coordinates and time. If density and/or pressure are disturbed somewhere, this disturbance propagates in the form of a sound wave, i.e., density and pressure become functions of space and time coordinates: $\rho = \rho(\mathbf{r}, t)$, $p = p(\mathbf{r}, t)$. If the velocity of air at the point $\mathbf{r}$ and at the moment t is denoted by $\mathbf{v}(\mathbf{r}, t)$, then its motion is governed by the well known continuity equation and the Euler equation,

$$\rho_t + \nabla(\rho\mathbf{v}) = 0, \qquad \mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = -(1/\rho)\,\nabla p, \tag{1.1}$$

where subscripts denote differentiation with respect to the corresponding variables. The main physical effect leading to the propagation of a sound wave is the elasticity of air, that is, the increase of pressure with increase of the density. Therefore, Eqs (1.1) must be complemented by the equation of state giving the dependence of the pressure on the density,

$$p = p(\rho). \tag{1.2}$$

In a weak sound wave, the changes of ρ and p are small compared to their equilibrium values,

$$p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad p_0 = p(\rho_0), \tag{1.3}$$

where $|p'| \ll p_0, |\rho'| \ll \rho_0$, and the velocity $\mathbf{v}$ may also be considered as a small variable. Let us consider the simplest case of a plane wave propagating in x direction, so that p', ρ', and $\mathbf{v} = (v, 0, 0)$ do not depend on y and z coordinates. Then substitution of Eqs. (1.3) into Eqs. (1.1) and linearization with respect to the small variables p', ρ', v give

$$\rho'_t + \rho_0 v_x = 0, \qquad v_t + (1/\rho_0)\, p'_x = 0. \tag{1.4}$$

Now we express p' in terms of ρ' with the use of the relation (1.2):

$$p' = (dp/d\rho)\, \rho'. \tag{1.5}$$

(Note in passing that, generally speaking, the pressure p depends on two thermodynamical variables, say, the density ρ and the entropy density s. During fast sound vibrations there is no heat exchange between points at distances of the order of the wavelength magnitude. Hence, we have $s = \text{const}$ in the sound wave, so that it is implied that the derivative in Eq. (1.5) is taken at constant s: $dp/d\rho \equiv (dp/d\rho)_s$.) On substitution of Eq. (1.5) into the first equation (1.4) and exclusion of v from the system (1.4), we obtain the well-known wave equation

$$p'_{xx} - (1/c^2) p'_{tt} = 0, \tag{1.6}$$

where

$$c = \sqrt{dp/d\rho} \tag{1.7}$$

is the sound velocity. It is clear that for a small amplitude sound wave the derivative has to be taken at $\rho = \rho_0$ and, hence, it is constant. It is important to note that the dependence (1.2) must be such that the thermodynamic inequality

$$dp/d\rho > 0 \tag{1.8}$$

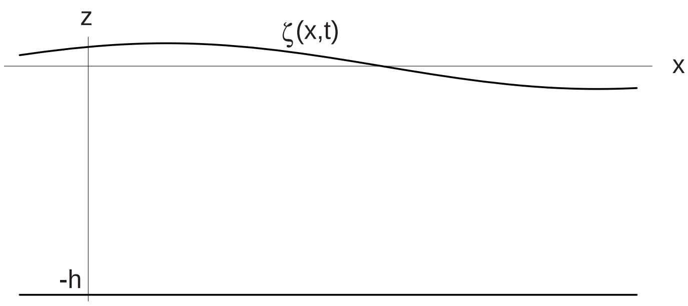

Fig. 1.1 Schematic plot of the gravity wave propagating along a water surface in the basin with depth h. The displacement of the surface from an undisturbed horizontal position is denoted by $\zeta(x,t)$ where x is the coordinate along the direction of propagation and t is the time variable.

is fulfilled.

The general solution of Eq. (1.6) has the form

$$p' = f(x - ct) + g(x + ct), \tag{1.9}$$

where f and g are arbitrary functions to be determined from the initial conditions. Any initial disturbance creates two pulses propagating in opposite directions. After long enough time, when these two pulses do not overlap, they propagate without change of their form. This is closely related to the fact that all harmonics $p' \propto \exp[i(kx - \omega t)]$ propagate with the same phase velocity $c = \omega/k$. It means that weak sound waves do not disperse.

1.1.2 *Gravity water waves (linear theory)*

If the surface of water contained in a horizontal basin with depth h (see Fig. 1.1) is disturbed, then gravity force will try to restore the equilibrium. As a result, the surface water waves appear. The motion of water under the surface is governed again by the Euler equation with the gravity force added:

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p/\rho + \mathbf{g}, \tag{1.10}$$

where $\mathbf{g}$ is the gravity acceleration constant directed vertically down, that is, $\rho(\mathbf{r}, t)\mathbf{g}$ is the gravity force acting on a unit volume of fluid at the point $\mathbf{r}$ and at the moment t. At usual conditions water may be considered as an incompressible fluid, that is, its density does not change under action of pressure of the order of magnitude $\sim \rho g h$. Therefore, we suppose that the density of water is constant, so that the equation of state (1.2) becomes simply

$$\rho = \text{const.} \tag{1.11}$$

On substitution of Eq. (1.11) into the continuity equation (1.1), it also simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1.12}$$

The Euler equation for the incompressible fluid can be written in the form

$$\mathbf{v}_t - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla(p/\rho + v^2/2),$$

which gives at once that

$$\partial(\nabla \times \mathbf{v})/\partial t = \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})].$$

This equation shows that if $\nabla \times \mathbf{v}$ vanishes everywhere, then it remains equal to zero during all motion of the fluid. We shall consider here motion with

$$\nabla \times \mathbf{v} = 0.$$

This equation means that the velocity field $\mathbf{v}$ is potential, that is, it can be expressed as

$$\mathbf{v} = \nabla\varphi = (\varphi_x, \varphi_y, \varphi_z), \tag{1.13}$$

where the potential $\varphi = \varphi(\mathbf{r}, t)$ by virtue of Eq. (1.12) must satisfy the Laplace equation

$$\Delta\varphi = 0. \tag{1.14}$$

Let us consider here waves with so small velocity of the fluid motion that the second term in the left hand side of Eq. (1.10) is negligibly small compared with the first one. If the period of the wave motion is denoted by T, and during the period a typical liquid particle moves on a distance a (amplitude of wave), then we estimate the velocity of fluid as $v \sim a/T$. Since the characteristic space dimension is the wavelength L, we may rewrite the condition $(\mathbf{v}\nabla)\mathbf{v} \ll \mathbf{v}_t$ as

$$\frac{1}{L}\left(\frac{a}{T}\right)^2 \ll \frac{1}{T}\left(\frac{a}{T}\right)$$

or

$$a \ll L, \tag{1.15}$$

that is, the wave amplitude must be much smaller than the wavelength. In this approximation, Eq. (1.10) becomes

$$\mathbf{v}_t = -(1/\rho)\nabla p + \mathbf{g},$$

and after substitution of Eq. (1.13) it can easily be integrated to give

$$p = -\rho g z - \rho\varphi_t, \tag{1.16}$$

where the integration constant is chosen equal to zero in accordance with the condition that there is no external pressure on the surface (we neglect here the atmospheric pressure). Equation (1.16) holds in all fluid's volume including its surface. If we denote the equation of surface as

$$z = \zeta(x, y, t), \tag{1.17}$$

then we get from Eq. (1.16) the following 'dynamical' boundary condition:

$$g\zeta + \varphi_t|_{z=\zeta} = 0. \tag{1.18}$$

Let us consider now a fluid particle 'attached' to the water surface. It moves with velocity

$$\mathbf{v}|_{z=\zeta(x,y)} = \nabla\varphi|_{z=\zeta(x,y)} = (\varphi_x, \varphi_y, \varphi_z)|_{z=\zeta(x,y)}\,.$$

Since a vector normal to the surface (1.17) has the components

$$\mathbf{n} = (-\zeta_x, -\zeta_y, 1)\Big/\sqrt{1+\zeta_x^2+\zeta_y^2},$$

the normal to the surface component of velocity is equal to

$$v_n = \frac{-\zeta_x\varphi_x - \zeta_y\varphi_y + \varphi_z}{\sqrt{1+\zeta_x^2+\zeta_y^2}}. \tag{1.19}$$

On the other hand, if we associate fluid particles with some arbitrary surface $F(x, y, z) = \text{const}$, then during the motion these particles will lay on the same moving surface, and hence, its total derivative with respect to time must be equal to zero,

$$F_t + (\mathbf{v}\nabla)F = 0.$$

Applying this condition to the water surface $z - \zeta(x, y, t) = 0$, we obtain the relation

$$\zeta_t - v_x\zeta_x - v_y\zeta_y + v_z = 0.$$

Hence, the normal component of the surface velocity,

$$v_n = \mathbf{v}\cdot\mathbf{n} = \frac{-v_x\zeta_x - v_y\zeta_y + v_z}{\sqrt{1+\zeta_x^2+\zeta_y^2}},$$

can be written as

$$v_n = \zeta_t \Big/ \sqrt{1+\zeta_x^2+\zeta_y^2}\,.$$

Since it must coincide with the velocity (1.19) of liquid particles, we obtain another boundary condition

$$\zeta_t = \varphi_z - \zeta_x\varphi_x - \zeta_y\varphi_y \tag{1.20}$$

of kinematical origin.

For small amplitude waves satisfying the condition (1.15), we have $\zeta \sim a$ and $|\zeta_x|, |\zeta_y| \sim a/L \ll 1$, hence, the last two terms in the right hand side of Eq. (1.20) are small compared to the first one, and in this linear limit the kinematical boundary condition reduces to

$$\zeta_t = \varphi_z|_{z=0}\,, \tag{1.21}$$

where with the same accuracy we have replaced $z = \zeta$ by $z = 0$. Analogous replacement can be done in the linear limit of the dynamical boundary condition Eq. (1.18):

$$g\zeta + \varphi_t|_{z=0} = 0.$$

After differentiation of this relation with respect to time t, we can eliminate the variable ζ_t by means of Eq. (1.21) to obtain

$$(g\varphi_z + \varphi_{tt})|_{z=0} = 0. \tag{1.22}$$

Thus, we have found the boundary conditions for the potential φ at the free surface. Besides that, it is clear that at the bottom of the basin $z = -h$

the normal component of velocity must be equal to zero, and this gives the boundary condition at the second surface confining the fluid:

$$\varphi_z|_{z=-h} = 0. \tag{1.23}$$

As a result, the problem of wave motion on the water surface is reduced to the Laplace equation (1.14) for the potential function φ with two boundary conditions (1.22) and (1.23).

Let us consider a plane wave propagating along x axis. We suppose that the potential φ and, hence, the velocity and the amplitude do not depend on y coordinate. We shall look for the harmonic wave solution in the form

$$\varphi = \cos(kx - \omega t) f(z), \tag{1.24}$$

where ω is the angular frequency of vibrations and k the wavenumber which are connected with the period T and the wavelength L by the well-known relations

$$\omega = 2\pi/T, \quad k = 2\pi/L. \tag{1.25}$$

On substitution of Eq. (1.24) into Eq. (1.14), we obtain equation for the function $f(z)$,

$$d^2 f/dz^2 - k^2 f = 0,$$

with obvious general solution

$$f(z) = C_1 e^{kz} + C_2 e^{-kz},$$

where C_1 and C_2 are the integration constants. Then the condition (1.23) gives $C_2 = C_1 e^{-2kh}$ and the solution (1.24) takes the form

$$\varphi = C \cos(kx - \omega t) \cosh[k(z + h)], \tag{1.26}$$

where C is still arbitrary constant. The second boundary condition (1.22) yields the relation between the frequency ω and the wavenumber k:

$$\omega^2 = gk \ \tanh(kh). \tag{1.27}$$

If the wavelength is much greater then the basin's depth, $L \gg h$, that is, $kh \ll 1$, then the dispersion relation (1.27) reduces to

$$\omega = \sqrt{gh}\, k. \tag{1.28}$$

In this limit of very long waves all quantities depend only on the phase

$$\theta = k(x - \sqrt{gh}\, t) \tag{1.29}$$

so that the equation of the water surface has the form

$$\zeta(x,t) = A\cos[k(x - \sqrt{gh}\, t)]. \tag{1.30}$$

As we see, all such waves, independently of their wave number k, propagate with the same phase velocity

$$V = \sqrt{gh}. \tag{1.31}$$

This means that if any initial disturbance

$$\zeta(x, t = 0) = \zeta_0(x) \tag{1.32}$$

is presented as a sum of its Fourier harmonics,

$$\zeta_0(x) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,\frac{dk}{2\pi}$$

(we use complex notations for the Fourier integral, so that $\zeta(-k) = \zeta^*(k)$), then after time t all these harmonics move on the same distance $\sqrt{gh}\, t$ and their sum gives the same form of the pulse moving with the velocity V:

$$\zeta(x,t) = \int_{-\infty}^{\infty} A(k)e^{ik(x-\sqrt{gh}\, t)}\,\frac{dk}{2\pi} = \zeta_0(x - \sqrt{gh}\, t). \tag{1.33}$$

If instead of Eq. (1.24) we had looked for the solution in the form $\varphi = \cos(kx + \omega t)f(z)$, we would obtain the wave moving in the opposite direction: $\zeta(x,t) = \zeta_0(x + \sqrt{gh}\, t)$. One might derive for water waves the wave equation analogous to Eq. (1.6), which describes waves propagating in both directions (see below).

Thus, the theory of long infinitesimal water waves is very simple—disturbances propagate without change of the form with the velocity equal to the phase velocity of harmonic waves.

1.1.3 *Plane electromagnetic wave in vacuum*

It is well known that the electromagnetic field in vacuum, when there is no any electric charges and currents, is described by the Maxwell equations

$$\begin{aligned} \nabla \times \mathbf{E} &= -(1/c)\,(\partial \mathbf{H}/\partial t), & \nabla \cdot \mathbf{H} &= 0, \\ \nabla \times \mathbf{H} &= (1/c)\,(\partial \mathbf{E}/\partial t), & \nabla \cdot \mathbf{E} &= 0, \end{aligned} \tag{1.34}$$

where $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ are the electric and magnetic fields of the wave, correspondingly, and c is the light velocity.

We suppose that a plane electromagnetic wave propagates along x axis, and the field vectors $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to each other and to the direction of propagation. Then the coordinate system can be chosen so that $\mathbf{E}$ is directed along y axis, $\mathbf{E} = (0, E, 0)$, $\mathbf{H}$ is directed along z axis, $\mathbf{H} = (0, 0, H)$, and both E and H depend only on the space coordinate x and the time t. As a result, the system (1.34) reduces to two equations

$$E_x = -(1/c)\,H_t, \quad -H_x = (1/c)\,E_t.$$

Elimination of H yields the wave equation for the electric field E:

$$E_{xx} - (1/c^2)E_{tt} = 0. \tag{1.35}$$

Again we find the dispersion relation

$$\omega = ck \tag{1.36}$$

for harmonic waves

$$E = Ae^{i(kx-\omega t)},$$

and the general solution

$$E(x, t) = f(x - ct) + g(x + ct)$$

of the wave equation (1.35).

The examples considered in this section have one common characteristic feature—wave pulses depending on only one space coordinate and propagating in a definite direction do not change their form during the propagation. It is clear from the above discussion that this simple behaviour is related directly with the special form of the dispersion relation which states that the frequency of vibrations ω in the harmonic wave is proportional to its wavevector k,

$$\omega = Vk,$$

where the proportionality coefficient V is called the phase velocity of the wave. It is equal to the sound velocity $V = \sqrt{dp/d\rho}$ for sound waves, to $V = \sqrt{gh}$ for linear water waves in a shallow basin ($kh \ll 1$), and to the light velocity $V = c$ for electromagnetic waves in vacuum. In all these cases the phase velocity V does not depend on the wavevector k, but it is clear that this is rather an exception than a general rule. Indeed, the exact dispersion law for water waves is given by Eq. (1.27) which shows that in general there is dependence of ω on k, and even in the long wavelength limit small corrections to the linear dispersion law (1.28) must have cumulative effect on the long time behaviour of the pulse. If the phase velocities ω/k of harmonics corresponding to different values of k differ from each other, then after long enough propagation time these harmonics will be shifted with respect to each other, so that their sum will give the form of the pulse different from the initial form. We infer from this consideration that the dependence of $V = \omega/k$ on k leads to gradual deformation of the pulse form.

In a similar way, when we consider an electromagnetic wave propagating through a bulk medium, its interaction with charged particles of the matter results in the dependence of ω/k on k. This phenomenon is very well known from elementary physics where it is usually formulated as the dependence of the speed of light on its wavelength and is called dispersion. Thus, we have come to the problem, how dispersion influences on the propagation of a wave pulse.

1.2 Dispersion

1.2.1 *Evolution of a wave pulse propagating on a shallow water surface (linear theory)*

Let a wave pulse propagate along the positive x direction so that it can be presented as a sum of the Fourier harmonics of the form

$$\zeta = A(k) \exp[i(kx - \omega t)], \tag{1.37}$$

where the dispersion dependence of the frequency ω on the wavevector k is given by Eq. (1.27). Now we take into account two terms of a Taylor series expansion of $\omega(k)$ in powers of a small parameter $hk \ll 1$,

$$\omega(k) \cong \sqrt{gh}\, k \left(1 - \tfrac{1}{6}(hk)^2\right). \qquad (1.38)$$

To simplify the following treatment, we use a simple trick to obtain the evolution equation for the pulse variable $\zeta(x,t)$. Each Fourier harmonic (1.37) after differentiation with respect to time t is multiplied by $-i\omega$ and after differentiation with respect to space coordinate x by ik. Hence, to obey the dispersion relation (1.38), each harmonic must satisfy the differential equation

$$\zeta_t + \sqrt{gh}\left(\zeta_x + \tfrac{1}{6}h^2\zeta_{xxx}\right) = 0. \qquad (1.39)$$

By virtue of linearity of this equation, the sum of the Fourier harmonics representing $\zeta(x,t)$ must also satisfy it. Therefore, Eq. (1.39) governs the evolution of pulses propagating in the positive x direction in the linear (negligibly small amplitude) approximation.

For further convenience, let us make transformation to the frame of reference moving with the velocity $\sqrt{gh}$ and to dimensionless coordinates

$$x' = (6^{1/3}/h)(x - \sqrt{gh}\; t), \quad t' = \sqrt{g/h}\; t$$

(a unit of length is proportional to the depth h, and a unit of time equals to this length divided by the phase velocity $\sqrt{gh}$). Then, replacing ζ by a more universal notation u for the amplitude and omitting the primes, we arrive at the evolution equation

$$u_t + u_{xxx} = 0. \qquad (1.40)$$

Due to the above transformation, the dispersion relation also simplifies and becomes

$$\omega(k) = -k^3. \qquad (1.41)$$

A solution of Eq. (1.40) may be expressed as a Fourier integral

$$u(x,t) = \int_{-\infty}^{\infty} A(k)\exp[i(kx + k^3 t)]\,\frac{dk}{2\pi}. \qquad (1.42)$$

Let at the initial moment $t = 0$ the wave have the form

$$u(x, 0) = u_0(x).$$

Putting $t = 0$ in Eq. (1.42), we obtain the equation

$$u_0(x) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,\frac{dk}{2\pi},$$

and, hence, the Fourier amplitudes $A(k)$ are given by

$$A(k) = \int_{-\infty}^{\infty} u_0(x')e^{-ikx'}\,dx'.$$

Substitution of this expression into Eq. (1.42) yields the solution of the initial value problem in the form

$$u(x, t) = \int_{-\infty}^{\infty} u_0(x')G(x - x', t)\,dx', \tag{1.43}$$

where $G(x, t)$ denotes the Green function of Eq. (1.40),

$$G(x, t) = \int_{-\infty}^{\infty} e^{i(kx+k^3t)}\,\frac{dk}{2\pi} = \frac{1}{\pi}\int_0^{\infty} \cos(kx + k^3 t)\,dk. \tag{1.44}$$

Taking into account the definition of the Airy function,

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{1}{3}k^3 + xk\right)dk,$$

we can represent the Green function as

$$G(x, t) = \frac{1}{(3t)^{1/3}}\,\mathrm{Ai}\left(\frac{x}{(3t)^{1/3}}\right). \tag{1.45}$$

Formulas (1.43–1.45) give the solution of the problem about evolution of the pulse with a given initial form in our hydrodynamical system. This solution is applicable to initial profiles whose Fourier harmonics have only small enough wave vectors $k \ll 1$ (i.e., which in the old dimensional variables satisfy the inequality $k \ll 1/h$ or $h/L \ll 1$, where L is a pulse's width). Therefore, the initial pulse width must be much greater then unity.

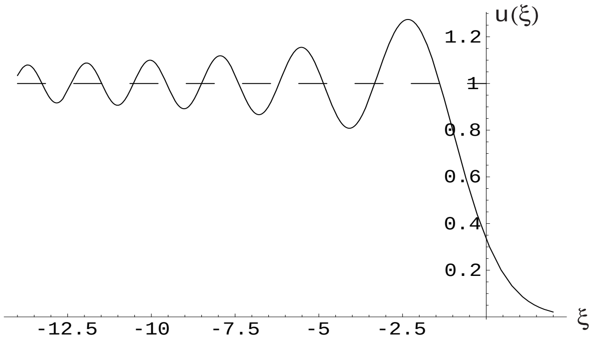

Fig. 1.2 The plot of the pulse profile $u(x,t)$ (see Eq. (1.47)) as a function of the self-similar variable $\xi = x/(3t)^{1/3}$.

As a typical example, one might consider a so long pulse, that at small enough time the existence of its trailing edge does not influence on evolution of the leading edge. Such an idealization is called a step-like pulse, and we shall often consider it in this book.

So, let the initial pulse be

$$u_0(x) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{1.46}$$

Then at time t the pulse profile is given by

$$u(x,t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{0} \mathrm{Ai}\left(\frac{x-x'}{(3t)^{1/3}}\right) dx' = \int_{x/(3t)^{1/3}}^{\infty} \mathrm{Ai}(z)\, dz. \tag{1.47}$$

We see that this profile depends only on the self-similar variable

$$\xi = x/(3t)^{1/3}. \tag{1.48}$$

The plot of the function (1.47) is shown in Fig. 1.2. At large x, when $\xi \gg 1$, we can use the asymptotic formula for the Airy function

$$\mathrm{Ai}(z) \cong \frac{1}{2\sqrt{\pi} z^{1/4}} \exp\left(-\tfrac{2}{3} z^{3/2}\right), \quad z \gg 1,$$

and find by integration of Eq. (1.47) by parts the asymptotic formula

$$u(x,t) \cong \frac{1}{2\sqrt{\pi}}\,\xi^{-3/4}\exp\left(-\tfrac{2}{3}\xi^{3/2}\right), \quad \xi \gg 1. \tag{1.49}$$

In the opposite limit of large negative x, when $-\xi \gg 1$, analogous calculation with the use of the formula

$$\mathrm{Ai}(-z) \cong \frac{1}{\sqrt{\pi}z^{1/4}}\cos\left(\tfrac{2}{3}z^{3/2} - \tfrac{\pi}{4}\right), \quad z \gg 1$$

gives

$$u(x,t) \cong 1 + \frac{1}{\sqrt{\pi}}\,\xi^{-3/4}\sin\left(\tfrac{2}{3}\xi^{3/2} - \tfrac{\pi}{4}\right), \quad \xi < 0,\ |\xi| \gg 1. \tag{1.50}$$

These asymptotic formulas and Fig. 1.2 show that far ahead of the leading edge $u(x,t)$ decays very fast and behind the front the oscillations around the initial constant value $u_0 = 1$ arise due to the dispersion effects. As a whole, the pulse does not move in our reference frame system, that is, dispersion does not influence on the velocity of the pulse. However, one should note that the dispersion law (1.41) has a special property that not only the phase velocity $V = \omega/k = -k^2$ vanishes at $k = 0$, but the known from elementary physics group velocity $v_g = d\omega/dk = -3k^2$ vanishes also. Even more, we have $dv_g/dk = d^2\omega/dk^2 = -6k|_{k=0} = 0$, that is, the group velocity is maximal at $k = 0$. Therefore, interference of comprising the pulse Fourier harmonics with $k \ll 1$ does not change its velocity and leads only to formation of oscillations behind the front and to exponential decay before the front. Such kind of interference is called caustic because it was revealed for the first time in investigations of a light intensity distribution near caustics.

Thus, if the pulse consists of harmonics with wavevectors near the extremum value of the group velocity, the interference of these harmonics leads during the evolution to creation of oscillations of the amplitude which are described by the Airy function.

Now let us consider another example when the group velocity does not vanish and has no extremum at wavevectors characteristic for the pulse under consideration.

1.2.2 Evolution of electromagnetic pulse in dispersive linear medium

In Sec. 1.1.3 we saw that an electromagnetic wave propagates in vacuum with the constant light velocity independently of the frequency of the wave. However, when such an electromagnetic wave enters into a material medium, it interacts with charged particles (say, with electrons) of the matter and influences on their motion. Consequently, the electrons begin to radiate electromagnetic waves and the resulting electromagnetic field will differ from the incident one. From a macroscopic point of view, one says that the external electromagnetic field causes the polarization of the medium and this polarization is described by the electric dipole moment $\mathbf{P}(\mathbf{r}, t)$ of a unit volume. For high frequency electromagnetic fields the magnetic susceptibility is equal to unity, so that the Maxwell equations for the electromagnetic field inside the matter read

$$\begin{array}{ll} \nabla\cdot\mathbf{D} = 0, & \nabla\cdot\mathbf{H} = 0, \\ \nabla\times\mathbf{E} = -(1/c)(\partial\mathbf{H}/\partial t), & \nabla\times\mathbf{H} = (1/c)(\partial\mathbf{D}/\partial t), \end{array} \tag{1.51}$$

where the electric displacement $\mathbf{D}$ is defined by the relation

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}.$$

In isotropic medium the vectors $\mathbf{P}$ and, hence, $\mathbf{D}$ are parallel to the vector $\mathbf{E}$. A simple calculation analogous to that made in Sec. 1.1.3, gives the equation,

$$E_{xx} - (1/c^2)E_{tt} = (4\pi/c^2)\,P_{tt}, \tag{1.52}$$

governing the evolution of the electric field $\mathbf{E}$ ($\mathbf{E} = (0, E, 0)$, $\mathbf{P} = (0, P, 0)$) during its propagation along x axis. If the field strength is small enough compared to typical fields at the molecular scale, it is natural to suppose that the polarization P is proportional to the field E,

$$P = \chi E, \tag{1.53}$$

where a susceptibility constant χ characterizes the properties of the matter. We imply here tacitly that χ does not depend on the frequency ω. Then Eq. (1.52) reduces immediately to

$$E_{xx} - (\epsilon/c^2)E_{tt} = 0, \tag{1.54}$$

where $\epsilon = 1 + 4\pi\chi$, and we return to the case discussed previously with the constant speed of the wave propagation

$$V = c/\sqrt{\epsilon}.$$

However, simple physical arguments show that the dielectric susceptibility χ cannot be independent of the wave frequency ω. Indeed, at very low frequencies electrons are able to follow the changes of the electromagnetic field, so that in the limit $\omega \to 0$ the parameters χ and ϵ take the values measured in electrostatic experiments with a capacitor. At extremely high frequencies electrons are not able to respond to the electromagnetic field because of their inertia, so that in the limit $\omega \to \infty$ one may expect that $\chi \to 0$ and $\epsilon \to 1$, and, hence, such an electromagnetic wave propagates through the matter with the vacuum light velocity. It becomes clear that at intermediate frequencies the response of the medium to an electromagnetic field is not instantaneous and is determined by the inertia of electrons. Hence, the polarization depends on the field values at the earlier moments of time,

$$P(x,t) = \int_0^\infty \tilde{\chi}(t')E(x, t-t')\,dt'. \tag{1.55}$$

If the field E is a monochromatic wave

$$E = E_\omega e^{i(kx-\omega t)},$$

then the polarization is also a monochromatic wave proportional to E,

$$P_\omega = \chi(\omega)E_\omega, \tag{1.56}$$

where

$$\chi(\omega) = \int_0^\infty \tilde{\chi}(t')e^{i\omega t'}\,dt'. \tag{1.57}$$

Thus, for the monochromatic wave we have the relation similar to Eq. (1.53) but with the susceptibility $\chi(\omega)$ depending on the frequency ω. Correspondingly, the substitution of $E = E_\omega \exp[i(kx-\omega t)]$ and $P = \chi(\omega)E_\omega \exp[i(kx-\omega t)]$ into Eq. (1.52) yields

$$k^2 = (\omega^2/c^2)\epsilon(\omega), \tag{1.58}$$

where $\epsilon(\omega) = 1 + 4\pi\chi(\omega)$ is the dielectric constant dependent of the frequency ω. Equation (1.58) determines the dependence of the wavevector on the frequency ω, so it has a meaning of the dispersion relation for electromagnetic waves propagating through the medium with the dielectric constant $\epsilon(\omega)$.

Let now the pulse of the electromagnetic field $E(x,t)$ propagating in the positive direction of the x axis have at the initial moment the Fourier integral representation

$$E(x,0) = E_0(x) = \int_{-\infty}^{\infty} E(k)e^{ikx}\,\frac{dk}{2\pi}. \tag{1.59}$$

Then after time t this pulse transforms to the Fourier integral form

$$E(x,t) = \int_{-\infty}^{\infty} E(k)e^{i(kx-\omega(k)t)}\,\frac{dk}{2\pi} = \int_{-\infty}^{\infty} E(k)e^{it[kx/t-\omega(k)]}\,\frac{dk}{2\pi}, \tag{1.60}$$

where $\omega = \omega(k)$ is the function defined implicitly by Eq. (1.58). We suppose that for Fourier harmonics giving the main contribution into the integrals (1.59,1.60) the derivatives $\omega' = d\omega/dk$ and $\omega'' = d^2\omega/dk^2$ do not vanish. Then we can estimate the integral (1.60) at large values of time t by the method of a stationary phase which is based on the statement that at a given ratio x/t and in the limit $t \to \infty$ the essential values of k correspond to some vicinity of such k_s where the function $f(k) = kx/t - \omega(k)$ has extremum values. Indeed, let the derivative

$$df/dk|_{k=k_s} = x/t - \omega'(k)|_{k=k_s} = 0 \tag{1.61}$$

vanish at $k = k_s$. Then the integral (1.60) is equal approximately to

$$E(x,t) \cong \int_{-\infty}^{\infty} E(k)e^{it[k_s x/t-\omega(k_s)]} \cdot e^{-it\omega''(k_s)(k-k_s)^2/2}\,\frac{dk}{2\pi}, \tag{1.62}$$

and because of fast oscillations of the function $\exp[-it\omega''(k_s)(k-k_s)^2/2]$ it converges at

$$|k - k_s| \lesssim 1/\sqrt{t|\omega''(k_s)|}.$$

Hence, the essential interval of k becomes very narrow at large enough values of t. Now we neglect variation of a slow function $E(k)$ within this interval, $E(k) \cong E(k_s)$, and use the formula

$$\int_{-\infty}^{\infty} e^{\pm ix^2}\, dx = \sqrt{\pi} e^{\pm \frac{i\pi}{4}} \tag{1.63}$$

to obtain the asymptotic formula

$$E(x,t) \cong \frac{\sqrt{2\pi}\, E(k_s)}{\sqrt{t|\omega''(k_s)|}} \exp\left[i\left(k_s x - \omega_s t + \frac{\pi}{4}\operatorname{sgn}\omega''(k_s)\right)\right] \tag{1.64}$$

with $\omega_s = \omega(k_s)$.

Let us consider this formula in some detail. First of all, the factor

$$\exp(i\theta), \quad \theta = k_s x - \omega_s t + \theta_s$$

shows that in vicinity of the point x at the moment t the field $E(x,t)$ is a harmonic wave with the wavevector k_s and the frequency ω_s. However, one should remember that the wavevector k_s, and, consequently, the frequency ω_s are the functions of x/t defined by Eq. (1.61). These values of k_s and ω_s agree with definitions

$$k = \theta_x, \quad \omega = -\theta_t. \tag{1.65}$$

Indeed, a simple calculation gives

$$\frac{\partial\theta}{\partial x} = k_s + x\frac{\partial k_s}{\partial x} - t\left.\frac{d\omega}{dk}\right|_{k_s}\frac{\partial k_s}{\partial x} = k_s + t\left(\frac{x}{t} - \left.\frac{d\omega}{dk}\right|_{k_s}\right)\frac{\partial k_s}{\partial x} = k_s,$$

$$-\frac{\partial\theta}{\partial t} = -x\frac{\partial k_s}{\partial t} + t\left.\frac{d\omega}{dk}\right|_{k_s}\frac{\partial k_s}{\partial t} + \omega_s = \omega_s - t\left(\frac{x}{t} - \left.\frac{d\omega}{dk}\right|_{k_s}\right)\frac{\partial k_s}{\partial t} = \omega_s,$$

where Eq. (1.61) was used. So, we see that the harmonic k_s arrives at the point x at the moment t moving, according to Eq. (1.61), with the group velocity

$$v_g(k_s) = \left. d\omega/dk\right|_{k_s}. \tag{1.66}$$

One may say that the wavevector k_s and the frequency ω_s propagate with the group velocity $v_g(k_s)$.

Since the spectrum of harmonics is determined by the initial width L of the pulse, that is, the essential harmonics in the Fourier integral (1.59) correspond to the wavevectors contained in the interval $(k_s - \Delta k/2, k_s +$

$\Delta k/2$), with $\Delta k \cong 2\pi/L \ll k_s$, then the difference between the propagation speeds of wavevectors $(k_s - \Delta k/2)$ and $(k_s + \Delta k/2)$ means widening of the wave packet: after large time t these two harmonics come to the different points with coordinates

$$x_+ = d\omega/dk|_{k_s+\Delta k/2} \cdot t, \qquad x_- = d\omega/dk|_{k_s-\Delta k/2} \cdot t,$$

correspondingly. Consequently, the width of the wave packet becomes equal approximately to

$$|x_+ - x_-| \cong |\omega_s''| \cdot \Delta k \cdot t \sim |\omega_s''| \cdot t/L. \tag{1.67}$$

Thus, at asymptotically large values of time the pulse width is proportional to time t.

According to Eq. (1.64), the amplitude of the harmonic with the wavevector k_s is equal to

$$\frac{\sqrt{2\pi}E(k_s)}{\sqrt{t|\omega''(k_s)|}}, \tag{1.68}$$

that is it decreases with time t. This decrease corresponds exactly to the noted above widening of the wave packet and agrees with the conservation of energy. Indeed, up to certain constant factors we have an estimate

$$\int\limits_{x_-}^{x_+} |E|^2\,dx \sim \frac{2\pi|E(k_s)|^2}{t|\omega_s''|} \cdot \frac{|\omega_s''| \cdot t}{L} \sim |E(k_s)|^2 \Delta k = \text{const},$$

which has a meaning of the energy conservation law.

The formulas obtained imply that $\omega_s'' \neq 0$. Otherwise, we have to take into account the next term of a Taylor series expansion in the integral (1.62) to obtain

$$E(x,t) \cong E(k_s)e^{i(k_s x - \omega_s t)} \int\limits_{-\infty}^{\infty} e^{i\left[(k-k_s)(x-\omega_s' t) - \frac{1}{6}\omega_s'''(k-k_s)^3 t\right]}\,\frac{dk}{2\pi}.$$

Then we return to the case considered in the preceding section about caustic behaviour of the pulse.

The theory based on the asymptotic formula (1.64) may be formulated in terms of modulation of the harmonic wave. Let a harmonic with the wavevector k_0 and the frequency $\omega_0 = \omega(k_0)$ is modulated by the envelope function $\mathcal{E}(x,t)$,

$$E(x,t) = \mathcal{E}(x,t)\exp[i(k_0x - \omega_0 t)]. \tag{1.69}$$

If a width L of the envelope function $\mathcal{E}(x,t)$ is much greater than the wavelength $2\pi/k_0$, i.e., $L \gg 1/k_0$, then the above asymptotic theory shows that after sufficiently long time t it has the order of magnitude $\sim |\omega_0''|t/L$, and the amplitude of the envelope function becomes equal to $\sim \mathcal{E}/\sqrt{|\omega_0''|t}$. One may ask, whether it is possible to formulate the theory of evolution of a wave packet as an initial value problem for the envelope $\mathcal{E}(x,t)$ itself. Such a theory has to be based on the supposition that the initial spectral width of the wave packet is very narrow, $\Delta k \ll k_0$, and, hence, the evolution of the envelope corresponds to the longest time scale of the problem. Therefore, one may distinguish slow evolution of the envelope from fast vibrations of the 'carrier' wave $\exp[i(k_0x - \omega_0 t)]$. Let us turn to derivation of the evolution equation for the envelope $\mathcal{E}(x,t)$.

Our starting point is naturally Eq. (1.52). At first, let us calculate the polarization $P(x,t)$ which, in our approximation, can be represented in the form analogous to Eq. (1.69),

$$P(x,t) = \mathcal{P}(x,t)\exp[i(k_0x - \omega_0 t)], \tag{1.70}$$

with a slow envelope $\mathcal{P}(x,t)$. Substitution of Eqs. (1.69,1.70) into Eq. (1.55) yields the relationship between the two envelope functions,

$$\mathcal{P}(x,t) = \int_0^\infty \tilde{\chi}(t')\mathcal{E}(x,t-t')e^{i\omega_0 t'}\,dt',$$

where it is implied that the frequency ω_0 is connected with the wavevector k_0 by the dispersion relation (1.58). Now we suppose that the pulse $\mathcal{E}(x,t)$ is so long that the characteristic time of its evolution is much longer than the response time characterizing the susceptibility function $\tilde{\chi}(t')$. This means that the envelope $\mathcal{E}(x,t-t')$ varies little within the characteristic time of integration over t'. Therefore, we can expand $\mathcal{E}(x,t-t')$ in the integrand into the Taylor series,

$$\mathcal{P}(x,t) \cong \int_0^\infty \tilde{\chi}(t')\left[\mathcal{E}(x,t) - (\partial\mathcal{E}/\partial t)\cdot t' + \tfrac{1}{2}(\partial\mathcal{E}/\partial t^2)\cdot t'^2\right] e^{i\omega_0 t'}\,dt'$$

$$= \mathcal{E}(x,t)\int_0^\infty \tilde{\chi}(t')e^{i\omega_0 t'}\,dt' + \partial\mathcal{E}(x,t)/\partial t\cdot i(d/d\omega_0)\int_0^\infty \tilde{\chi}(t')e^{i\omega_0 t'}\,dt'$$

$$- \tfrac{1}{2}(\partial^2\mathcal{E}(x,t)/\partial t^2)\cdot(d^2/d\omega_0^2)\int_0^\infty \tilde{\chi}(t')e^{i\omega_0 t'}\,dt',$$

and, taking into account Eq. (1.57), we obtain

$$\mathcal{P}(x,t) \cong \chi(\omega_0)\mathcal{E}(x,t) + i\frac{\partial\mathcal{E}(x,t)}{\partial t}\cdot\chi'(\omega_0) - \frac{1}{2}\frac{\partial^2\mathcal{E}(x,t)}{\partial t^2}\cdot\chi''(\omega_0), \quad (1.71)$$

where $\chi'(\omega_0) = d\chi/d\omega|_{\omega_0}$, $\chi''(\omega_0) = d^2\chi/d\omega^2|_{\omega_0}$. According to the asymptotic theory, the dispersion of a wave packet is determined by the terms with the second derivative $\omega''(k)$ which can be expressed by Eq. (1.58) in terms of the first and second derivatives of $\epsilon(\omega) = 1 + 4\pi\chi(\omega)$. Just for this reason we keep in the Taylor expansion the terms up to the second order derivatives of $\chi(\omega)$. On substitution of Eqs. (1.69), (1.70), and (1.71) into Eq. (1.52), we can neglect with the same accuracy the derivatives of $\mathcal{E}(x,t)$ of orders higher than the second one,

$$\begin{aligned}&\mathcal{E}_{xx} + 2ik_0\mathcal{E}_x - k_0^2\mathcal{E} - (1/c^2)\mathcal{E}_{tt} + (2i\omega_0/c^2)\mathcal{E}_t + (\omega_0^2/c^2)\mathcal{E}\\ &= (4\pi/c^2)\chi(\omega_0)\left(\mathcal{E}_{tt} - 2i\omega_0\mathcal{E}_t - \omega_0^2\mathcal{E}\right) + (4\pi/c^2)i\chi'(\omega_0)\left(-2i\omega_0\mathcal{E}_{tt} - \omega_0^2\mathcal{E}_t\right)\\ &\qquad + (4\pi/c^2)\chi''(\omega_0)(\omega_0^2/2)\mathcal{E}_{tt}. \end{aligned} \quad (1.72)$$

The coefficient before the envelope $\mathcal{E}$ in this expression is collected to

$$-k_0^2 + \omega_0^2/c^2 + (4\pi/c^2)\omega_0^2\chi(\omega_0) \equiv -k_0^2 + (\omega_0^2/c^2)(1 + 4\pi\chi(\omega_0)),$$

and, hence, it vanishes by virtue of the dispersion relation (1.58) for the carrier harmonic wave. Differentiation of Eq. (1.58) gives

$$dk/d\omega = (\omega/kc^2)\epsilon(\omega) + (\omega^2/2kc^2)\epsilon'(\omega),$$

and, hence, the terms with the first derivatives of $\mathcal{E}(x,t)$,

$$2ik_0\left\{\mathcal{E}_x + \left[(\omega_0/k_0c^2)(1 + 4\pi\chi(\omega_0)) + (4\pi\omega_0^2/2k_0c^2)\chi'(\omega_0)\right]\mathcal{E}_t\right\}$$

transform to

$$2ik_0 \left(\mathcal{E}_x + dk/d\omega|_{\omega_0} \mathcal{E}_t\right).$$

If we confined ourselves to this approximation, we would obtain the evolution equation

$$\mathcal{E}_x + (1/v_g)\mathcal{E}_t = 0, \tag{1.73}$$

where v_g is the group velocity defined by Eq. (1.66) with k_s replaced by k_0. Obviously, Eq. (1.73) is solved by

$$\mathcal{E} = \mathcal{E}_0(x - v_g t),$$

where $\mathcal{E}_0(x)$ is the initial profile of the envelope. Thus, in this approximation the envelope propagates without change of its form with the group velocity corresponding to the carrier wave frequency. To find the change of the form of $\mathcal{E}(x,t)$, we have to take into account the terms with the second derivatives of $\mathcal{E}(x,t)$, and we can transform these terms with the accepted accuracy by the use of Eq. (1.73), i.e., making the replacement

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} \cong -k' \frac{\partial}{\partial t}\frac{\partial \mathcal{E}}{\partial x} \cong (k')^2 \frac{\partial^2 \mathcal{E}}{\partial t^2}, \quad k' \equiv \left.\frac{dk}{d\omega}\right|_{\omega_0}. \tag{1.74}$$

Then the coefficient before $\partial^2 \mathcal{E}/\partial t^2$ becomes equal to

$$(k')^2 - (1/c^2)(1 + 4\pi\chi) - (4\pi/c^2)\left(2\omega_0\chi' + (\omega_0^2/2)\chi''\right) \equiv -kk'',$$

where the formula given by the second derivative of Eq. (1.58) with respect to ω was used. Thus, we have arrived at the evolution equation for the envelope function,

$$i\left(\mathcal{E}_x + k'\mathcal{E}_t\right) - \tfrac{1}{2}k''\mathcal{E}_{tt} = 0. \tag{1.75}$$

This form of the equation is most convenient for the problems when the pulse with the input envelope dependence $\mathcal{E}_0(t)$ on time t enters into the medium at some input coordinate $x = x_i$ and we are interested in its output time dependence at some output coordinate $x = x_f$. Such kind of problems is often considered in fiber optics where Eq. (1.75) is commonly used. But if we are interested in time evolution of a pulse with the initial profile $\mathcal{E}_0(x)$, then the second time derivative in Eq. (1.75) should be replaced by the second space derivative with the use of Eq. (1.74), so we get the equation

$$i\left(\mathcal{E}_t + v_g\mathcal{E}_x\right) + \tfrac{1}{2}\omega''\,\mathcal{E}_{xx} = 0, \tag{1.76}$$

where the relation $\omega'' = d^2\omega/dk^2 = -k''/(k')^3$ was taken into account. Throughout this book we shall use this form of the evolution equation for the envelope function. It is convenient to make a Galileo transformation to the frame of reference moving with the group velocity v_g,

$$x' = x - v_g t, \quad t' = t,$$

and to introduce the dimensionless variables as follows. Let the initial envelope have the width L. Then we take L as a unit of length and the 'dispersion' time as a unit of time,

$$x'' = x'/L, \quad t'' = \omega'' t'/2L^2.$$

As a result, Eq. (1.76) simplifies to

$$i\mathcal{E}_t + \mathcal{E}_{xx} = 0, \tag{1.77}$$

where we have dropped primes in new space and time variables x'' and t''.

The Green function of this equation can easily be found by the Fourier method. Let the solution of Eq. (1.77) be represented as a sum of the Fourier harmonics, each satisfying this equation,

$$\mathcal{E}(x,t) = \int\limits_{-\infty}^{\infty} \mathcal{E}(k) e^{i(kx-k^2t)} \frac{dk}{2\pi}. \tag{1.78}$$

Then the Fourier amplitudes are determined by the initial envelope

$$\mathcal{E}(x,0) = \mathcal{E}_0(x) = \int\limits_{-\infty}^{\infty} \mathcal{E}(k) e^{ikx} \frac{dk}{2\pi},$$

namely,

$$\mathcal{E}(k) = \int\limits_{-\infty}^{\infty} \mathcal{E}_0(x') e^{-ikx'}\, dx'.$$

We substitute this expression into Eq. (1.78), evaluate the integral by means of Eq. (1.63) and obtain the desired formula

$$\mathcal{E}(x,t) = \int\limits_{-\infty}^{\infty} \mathcal{E}_0(x') G(x-x',t)\, dx', \tag{1.79}$$

where the Green function is given by

$$G(x-x',t) = \frac{1}{2\sqrt{\pi i t}} \exp\left[\frac{i(x-x')^2}{4t}\right]. \tag{1.80}$$

Formulas (1.79,1.80) solve in principle the problem of the envelope evolution.

As an example let us consider the initial envelope in the form of a Gaussian function

$$\mathcal{E}_0(x) = \mathcal{E}_0 e^{-x^2}.$$

Simple calculation of the Gaussian integral in Eq. (1.79) yields the envelope profile as a function of time t:

$$\mathcal{E}(x,t) = \frac{\mathcal{E}_0}{\sqrt{1+4it}} \exp\left(-\frac{x^2}{1+4it}\right).$$

In dimensional units of x and t the field strength is equal to

$$\mathcal{E}(x,t) = \frac{\mathcal{E}_0}{\sqrt{1+2i\omega'' t/L^2}} \exp\left[-\frac{x^2}{L^2(1+2i\omega'' t/L^2)}\right],$$

and the intensity is given by

$$|\mathcal{E}(x,t)|^2 = \frac{\mathcal{E}_0^2}{\sqrt{1+(2\omega'' t/L^2)^2}} \exp\left\{-\frac{2x^2}{L^2[1+(2\omega'' t/L^2)^2]}\right\}. \tag{1.81}$$

As we see, dispersion effects lead to widening of the pulse so that its width grows with time according to

$$a(t) = L\sqrt{1+(2\omega'' t/L^2)^2}. \tag{1.82}$$

At $t \gg L^2/2|\omega''|$ (i.e., $t \gg 1$ in dimensionless units) this dependence becomes linear,

$$a(t) \sim |\omega''| t/L,$$

in agreement with the asymptotic formula Eq. (1.67). In this asymptotic limit the amplitude (1.81) has the order of magnitude

$$\mathcal{E} \sim \mathcal{E}_0 L/\sqrt{|\omega''| t},$$

and this estimate agrees with Eq. (1.68), if one takes into account that the Fourier harmonics corresponding to the initial pulse $\mathcal{E}_0 \exp(-x^2/L^2)$ are equal approximately to $\mathcal{E}(k)|_{k\sim 1/L} \sim \mathcal{E}_0 L$.

Thus, the examples considered above show that dispersion leads to widening of wave packets and to appearing of oscillations at sharp fronts. However, in classical physics there are situations where dispersion is negligibly small and evolution of a pulse is determined mainly by nonlinear terms in the corresponding evolution equations. A typical example is the propagation of an intense sound pulse with formation of a shock wave. Let us turn to consideration of these nonlinear effects.

1.3 Nonlinearity

1.3.1 *Rarefaction wave and a dam problem*

We shall start with a simple problem about a compressible gas flow. Let a large container be filled with gas with density equal to ρ_0, and let one plane wall of the container be situated at $x = 0$. We suppose that at the moment $t = 0$ this wall is taken away (a 'dam' is broken), and the gas starts its expansion into an empty half-space. Then, the initial distribution of the density is given by

$$\rho(x, t = 0) = \begin{cases} \rho_0, & x \le 0, \\ 0, & x > 0, \end{cases} \tag{1.83}$$

and the initial velocity of the gas is equal to zero everywhere,

$$v(x, t = 0) = 0. \tag{1.84}$$

The problem is to find distributions of the density $\rho(x, t)$ and the flow velocity $v(x, t)$ at all values of time t.

As we know from Sec. 1.1.1, the gas flow is governed by the Euler equation

$$v_t + vv_x = -(1/\rho)p_x \tag{1.85}$$

and the continuity equation

$$\rho_t + (\rho v)_x = 0, \tag{1.86}$$

where we have taken into account that the velocity has only one component in x direction and all variables depend only on x and t. The pressure p

depends on the density ρ according to the equation of state

$$p = p(\rho), \tag{1.87}$$

which completes the system of equations.

To solve the posed problem, we notice that the initial conditions do not contain any parameters with a dimension of length or time (the opposite wall is situated so far that it does not influence on the flow at small enough values of time). However, there is a variable with a dimension of velocity—a local sound velocity

$$c = \sqrt{dp/d\rho}. \tag{1.88}$$

Therefore, the variables $\rho(x,t)$ and $v(x,t)$ can depend on x and t only through combination x/t which also has a dimension of velocity. Hence, we write the system in the form

$$\begin{aligned} &\rho_t + \rho v_x + v\rho_x = 0, \\ &v_t + vv_x + (c^2/\rho)\rho_x = 0, \end{aligned} \tag{1.89}$$

where $c^2 = dp/d\rho$ is a known function of ρ, and introduce the variable $\xi = x/t$. The solutions of Eqs. (1.89) with ρ and v depending only on $\xi = x/t$ are called self-similar. Since

$$\partial/\partial x = (1/t)d/d\xi, \quad \partial/\partial t = -(\xi/t)d/d\xi,$$

the system (1.89) transforms to

$$\begin{aligned} &(v-\xi)\rho' + \rho v' = 0, \\ &(c^2/\rho)\rho' + (v-\xi)v' = 0, \end{aligned} \tag{1.90}$$

where prime denotes differentiation with respect to ξ. This system has nontrivial solutions only if $(v-\xi)^2 = c^2$, that is,

$$\xi = x/t = v \pm c. \tag{1.91}$$

On substitution of this relation into any equation (1.90), we get $\pm c\rho' = \rho v'$, hence

$$v = \pm \int^{\rho} c(\rho)d\rho/\rho. \tag{1.92}$$

Formulas (1.91) and (1.92) solve in principle our problem. It remains to satisfy the initial conditions (1.83) and (1.84). We shall do it for the case of polytropic dependence (1.87), namely,

$$p = p_0(\rho/\rho_0)^\gamma. \tag{1.93}$$

For adiabatic flow of an ideal gas the constant γ is equal to the ratio $\gamma = c_p/c_V$ of the specific heat capacities at constant pressure (c_p) and at constant volume (c_V). This constant is always greater than unity, e.g., for a monatomic gas it equals to $\gamma = 5/3$, and for a diatomic gas to $\gamma = 7/5$. From Eq. (1.93) we find the dependence of the sound velocity c on the density ρ,

$$c = c_0(\rho/\rho_0)^{\frac{\gamma-1}{2}}, \tag{1.94}$$

where $c_0 = \sqrt{\gamma p_0/\rho_0}$ is the sound velocity in a gas with the density ρ_0. The solution (1.91), (1.92) describes a non-uniform region between an undisturbed gas at rest with the density ρ_0 and vacuum with the zero density of the gas, hence the name 'rarefaction wave' of this solution. At the left edge x^- of this non-uniform region we have $v = 0$, and it moves to the left with the sound velocity:

$$x^- = -c_0 t. \tag{1.95}$$

This means that we have to take sign 'minus' in Eq. (1.91). Consequently, Eqs. (1.92) and (1.94) give

$$v = \frac{2c_0}{\gamma - 1}\left(1 - \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}\right), \tag{1.96}$$

where the integration constant is chosen according to the condition that $v = 0$ at $\rho = \rho_0$. Substitution of Eqs. (1.94) and (1.96) into Eq. (1.91) permits one to obtain the distribution of the density

$$\rho = \rho_0\left[\frac{2c_0 - (\gamma-1)x/t}{c_0(\gamma+1)}\right]^{\frac{2}{\gamma-1}}, \tag{1.97}$$

and its substitution into Eq. (1.96) gives the analogous distribution of the flow velocity

$$v = \frac{2}{\gamma+1}\left(c_0 + \frac{x}{t}\right). \tag{1.98}$$

This velocity vanishes at $x = -c_0 t$ where the density is equal to ρ_0. The right edge of the non-uniform region at the boundary with the vacuum

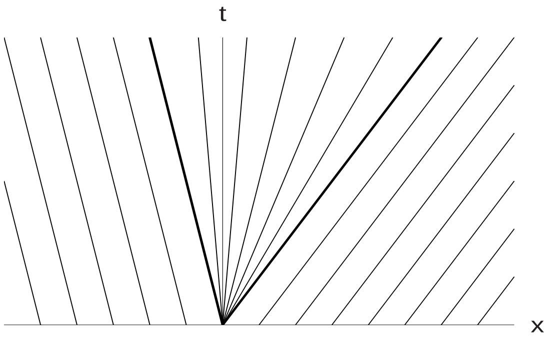

Fig. 1.3 Characteristics of the system (1.89) corresponding to the self-similar solution (1.100).

moves with the velocity

$$v^{+} = x^{+}/t = 2c_0/(\gamma - 1), \tag{1.99}$$

which corresponds to vanishing of the density (1.97).

Formula (1.98) can be written in the form

$$x = \left(-c_0 + \frac{\gamma + 1}{2} v\right) t, \tag{1.100}$$

where v varies in the interval $0 \leq v \leq 2c_0/(\gamma - 1)$. The self-similar solution found here corresponds to the set of straight lines in the plane (x, t), outgoing from the origin $(0, 0)$ (see Fig. 1.3) with $v = \text{const}$ along each line. These straight lines are called characteristics of the system (1.89), corresponding to the self-similar solution (1.100). As was noticed above, this solution is valid until the left edge x^{-} reaches the opposite wall of the container.

1.3.2 *Hopf equation and wave breaking*

In the above solution the most important role was played by the pressure—some flow velocity distribution is acquired due to action of the pressure on the gas initially compressed in a container. Therefore, the flow velocity in the rarefaction wave has the order of magnitude of the sound velocity, $v \sim$

c_0. However, there are many problems where the gas has some distribution of the flow velocity even at the initial moment $t = 0$. In these cases the evolution of the flow differs considerably from the rarefaction wave and we shall turn here to some peculiarities of such a flow.

We shall start here with the simplest limiting case when one can neglect a role of pressure what is possible if the initial velocity is much greater then the sound velocity,

$$v \gg c_0.$$

Then the last term in the second equation (1.89) can be omitted, and we arrive at the equation

$$v_t + vv_x = 0, \tag{1.101}$$

which is often called the Hopf equation. The density variable disappears from this equation which determines the evolution of the velocity distribution independently of the density. When $v(x,t)$ is found, then one can substitute this velocity distribution into the first equation (1.89) and find the density distribution. One may say that Eq. (1.101) describes a flow of 'dust' because the pressure in an ideal gas of particles with mass m is given by

$$p = NkT = (\rho/m)kT$$

(k is the Boltzmann constant and T the absolute temperature), and this pressure vanishes as $m \to \infty$, i.e., for a gas of 'heavy' dust particles.

As we know from Sec. 1.1.2, if the coefficient before v_x in Eq. (1.101) were constant, $v_t + cv_x = 0$, we would have the solution in the form

$$v = v_0(x - ct),$$

where $v_0(x)$ is the initial distribution of the flow velocity at $t = 0$. Therefore, it seems natural to check whether the solution of Eq. (1.101) is given implicitly by the expression

$$v = v_0(x - vt). \tag{1.102}$$

Indeed, its differentiation gives $v_t = -v_0'v - v_0'v_tt$, $v_x = v_0' - v_0'v_xt$, where $v_0'(x) = dv_0/dx$, and, hence,

$$v_t = -\frac{vv_0'}{1 + v_0' \cdot t}, \quad v_x = \frac{v_0'}{1 + v_0' \cdot t}. \tag{1.103}$$

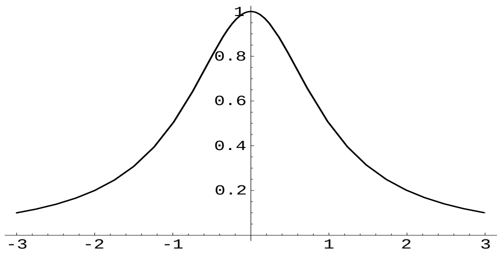

Fig. 1.4 The initial profile of the pulse: $v_0(\bar{x}) = 1/(1 + \bar{x}^2)$.

Substitution of Eqs. (1.103) into Eq. (1.101) shows at once that Eq. (1.102) is the solution satisfying the initial condition $v = v_0(x)$ at $t = 0$.

From a more formal point of view, it is worth to notice that we have found the general solution of Eq. (1.101) in the form

$$x - vt = w(v), \tag{1.104}$$

where $w(v)$ is an arbitrary function. If we put here $t = 0$, we find that $w(v)$ is an inverse function of the initial distribution $v_0(x)$, and we denote this function as $\bar{x}(v)$. Then the solution (1.102) of the Hopf equation can be written in the form

$$x = \bar{x}(v) + vt,$$

which means that the velocity v is constant along the straight lines

$$x = \bar{x} + v_0(\bar{x})t, \qquad v = v_0(\bar{x}), \tag{1.105}$$

in the plane (x, t), where $\bar{x}$ is the value of x at which the initial distribution has the value v of the initial velocity, i.e., it is the solution of the equation $v = v_0(\bar{x})$. The straight lines (1.105) are called the characteristics of Eq. (1.101), and v is the Riemann invariant constant along these characteristics. In what follows we shall generalize essentially these notions, so

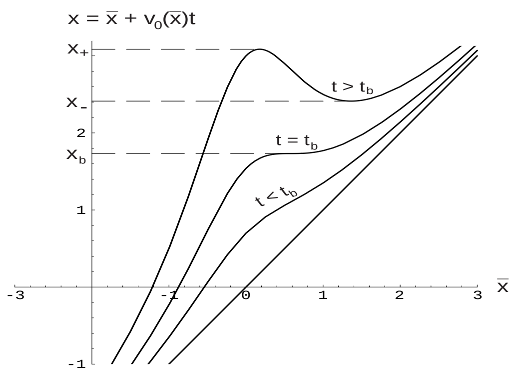

Fig. 1.5 Curves (1.105) for different values of time t.

that they can be used for solutions of equations much more complex than Eq. (1.101).

The solution (1.105) permits one to understand from geometrical considerations how a pulse evolves. Let the initial distribution be given by the function $v_0(\bar{x}) = 1/(1+\bar{x}^2)$ (see Fig. 1.4). Formula (1.105) gives the map of the initial points $\bar{x}$ into the corresponding points at the moment of time t. At $t = 0$ this is the identical map $x = \bar{x}$, but with growth of t its deviation from $x = \bar{x}$ increases proportionally to the values of $v_0(\bar{x})$. A set of curves $x = x(\bar{x})$ defined by Eq. (1.105) with $v_0(\bar{x}) = 1/(1+\bar{x}^2)$ is shown in Fig. 1.5 for different values of time t. We see that with growth of t the region of the plot $x(\bar{x})$ corresponding to points $\bar{x}$ with $v_0' < 0$ flattens, and at some moment $t = t_b$ the plot acquires a point with a horizontal tangent line. At greater times $t > t_b$ there arises a region $x^- < x < x^+$, where each value of x corresponds to three values of $\bar{x}$. This means that after $t = t_b$ the faster particles begin to overtake the slower ones. Three values $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ corresponding to one value of x give the three values of the initial velocities $v_0(\bar{x}_i)$, $i = 1, 2, 3$, of particles which come to the point x at the moment t. Hence, the plot of $v(x)$ is three-valued in the region $x^- < x < x^+$, as it is shown in Fig. 1.6. Such a deformation of the velocity distribution can also be understood from Eq. (1.102)—after time t the point of the initial profile is shifted along x axis to the distance proportional to the value of v, that

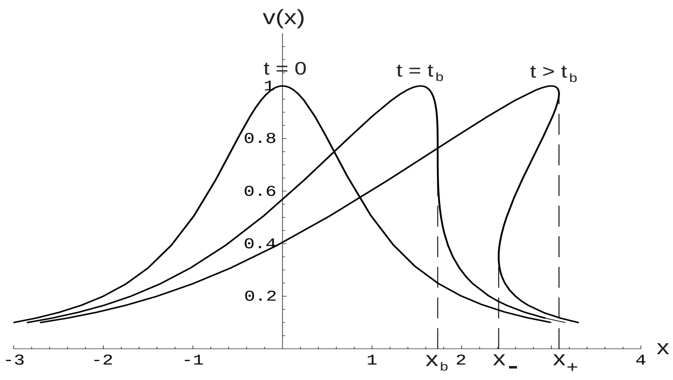

Fig. 1.6 Deformation of the profile with time t. After the moment t_b the profile becomes a multi-valued function.

is, the points at the top of the profile with greater values of v 'move' faster than the points at the bottom of the profile with smaller values of v.

The boundaries of the multi-valued region can easily be found by noticing that at these points the plot of $x(\bar{x})$ has a horizontal tangent line (see Fig. 1.5), that is,

$$\partial x/\partial\bar{x} = 1 + v_0'(\bar{x})t = 0. \tag{1.106}$$

Hence, Eqs. (1.105) and (1.106) define the functions $x^{\pm}(t)$ in a parametric form,

$$x^{\pm} = \bar{x} - v_0(\bar{x})/v_0'(\bar{x}), \quad t = -1/v_0'(\bar{x}), \tag{1.107}$$

where $\bar{x}$ plays the role of the parameter. The wave-breaking point x_b corresponds to the inflexion point of the curve $x(\bar{x})$, where $\partial^2 x/\partial\bar{x}^2 = 0$, so the corresponding value of $\bar{x}$ is determined by the equation

$$v_0''(\bar{x}_b) = 0. \tag{1.108}$$

If we find this value of $\bar{x}_b$ and substitute it into Eqs. (1.107), then we obtain the point (x_b, t_b) in the plane (x, t), where the boundary curves $x^+(t)$ and $x^-(t)$ merge together. These curves corresponding to the initial distribution of Fig. 1.4 are shown in Fig. 1.7.

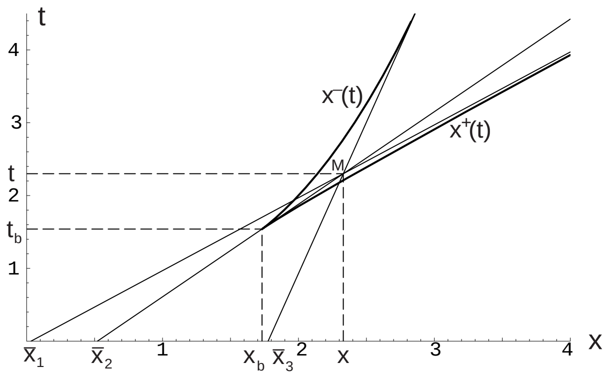

Fig. 1.7 Caustic curves around the multi-valued region in the plane (x, t).

The point M is situated inside the multi-valued region, and the particles with the initial coordinates $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ come into this point at the moment t. It is useful to look at this figure from another point of view. At each value of $\bar{x}$ Eq. (1.105) determines the straight line in the plane (x, t)—the characteristic of Eq. (1.101). Variation of $\bar{x}$ gives the family of straight lines whose envelope is determined by the known from elementary differential geometry equations

$$x(\bar{x}) = 0, \quad \partial x/\partial \bar{x} = 0,$$

that is, just by Eqs. (1.107). In geometric optics the envelope of a system of rays is called caustic. Therefore, the envelopes of characteristics are also called caustics. Figure 1.8 illustrates the envelope property of caustics for the example of characteristics (1.105) with $v_0(\bar{x}) = 1/(1 + \bar{x}^2)$.

Let us clarify a form of the caustic curve in vicinity of its cusp point. According to Eq. (1.108), the derivatives of the functions (1.107) with respect to $\bar{x}$ are equal at this point to zero:

$$\left.\frac{dt}{d\bar{x}}\right|_{\bar{x}=\bar{x}_b} = \left.\frac{v_0''}{{v_0'}^2}\right|_{\bar{x}=\bar{x}_b} = 0, \quad \left.\frac{dx}{d\bar{x}}\right|_{\bar{x}=\bar{x}_b} = \left.\frac{v_0 v_0''}{{v_0'}^2}\right|_{\bar{x}=\bar{x}_b} = 0.$$

The next derivatives do not vanish at this point, so that near the cusp point

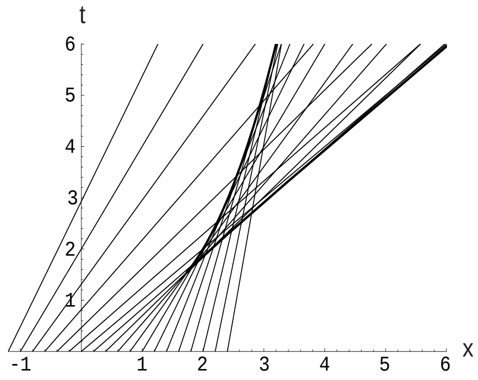

Fig. 1.8 Caustic curves as envelopes of characteristics.

we have

$$t - t_b = a_2(\bar{x} - \bar{x}_b)^2 + a_3(\bar{x} - \bar{x}_b)^3, \quad x - x_b = b_2(\bar{x} - \bar{x}_b)^2 + b_3(\bar{x} - \bar{x}_b)^3,$$

where a_2, a_3, b_2, b_3 are the coefficients of the Taylor series, and we have omitted the higher terms of the expansions. This linear system has a solution

$$(\bar{x} - \bar{x}_b)^2 = \alpha(t - t_b) + \beta(x - x_b), \quad (\bar{x} - \bar{x}_b)^3 = \gamma(t - t_b) + \delta(x - x_b),$$

which gives immediately the equation of the caustic curve,

$$[\alpha(t - t_b) + \beta(x - x_b)]^3 = [\gamma(t - t_b) + \delta(x - x_b)]^2.$$

After linear transformation

$$\tau = \alpha(t - t_b) + \beta(x - x_b), \quad \xi = \gamma(t - t_b) + \delta(x - x_b)$$

we obtain the equation

$$\tau^3 = \xi^2 \quad \text{or} \quad \tau = \xi^{2/3},$$

whose plot is shown in Fig. 1.9.

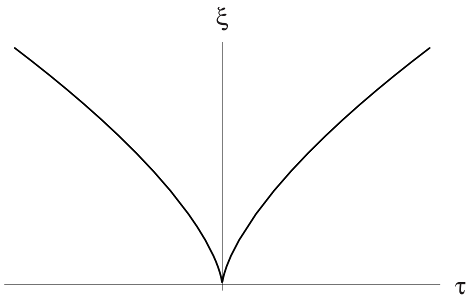

Fig. 1.9 The form of the caustic curve in vicinity of its cusp point.

As we see, the caustic curve has a cusp point at which its both branches have vertical tangent lines, and their curvatures

$$\left| \tau''/\left(1+\tau'^2\right)^{3/2} \right| \sim |\xi|^{-1/3}$$

go to infinity at $|\xi| \to 0$. In catastrophe theory language, the caustic is an example of a pleat catastrophe. The origin of this name can be understood from Fig. 1.10, where the set of characteristic lines form a surface in three dimensional space (x, t, v). This surface has a pleat in the region where three values of v correspond to one point (x, t). Projection of the boundary of this multi-valued region into the (x, t)-plane is the caustic curve.

Now, when the dependence of the velocity field on x at any moment t is known, we can return to finding the density distribution $\rho(x, t)$. It satisfies the continuity equation

$$\rho_t + v\rho_x + \rho v_x = 0, \tag{1.109}$$

which means that the number of particles in the interval $\Delta\bar{x}$ is equal to the number of particles in the interval Δx, $\Delta\bar{x}$ and Δx being related by Eq. (1.105), that is, $\Delta x = (\partial x/\partial\bar{x}) \cdot \Delta\bar{x}$. Hence, the solution of Eq. (1.109) is given by

$$\bar{x} = x - v(x,t)\,t, \quad \rho(x,t) = \rho_0(\bar{x})\,|\partial\bar{x}/\partial x| = \rho_0(\bar{x})|1 - v_x \cdot t|, \tag{1.110}$$

(we write an absolute value sign to make the density positive; since Eq. (1.109)

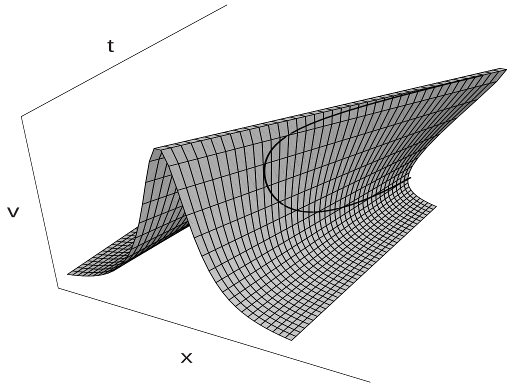

Fig. 1.10 The characteristics surface with pleat catastrophe. The curve lying on this surface shows the boundary of the multi-valued region, and its projection into (x, t)-plane is the caustic curve.

is homogeneous and linear in ρ, its solution does not depend on the sign of ρ), and this can be verified by a direct calculation:

$$\rho_t + v\rho_x + \rho v_x = \rho_0' \cdot (-t)(1 - v_x)(v_t + vv_x) + \rho_0 \cdot (-t)(v_t + vv_x)_x,$$

and this expression is equal to zero by Eq. (1.101).

However, as we know from Eq. (1.106), the derivative $\partial\bar{x}/\partial x$ is singular at the caustic curve, so that the density ρ becomes here infinite. Let us find the law of this singular behaviour of the density. Near the caustic point with coordinates (x_c, t_c) we have according to Eq. (1.105)

$$\begin{aligned} x - x_c = \bar{x} - \bar{x}_c &+ [v_0(\bar{x}) - v_0(\bar{x}_c)]t_c \\ &\cong [1 + v_0'(\bar{x}_c)t_c](\bar{x} - \bar{x}_c) + \tfrac{1}{2}v_0''(\bar{x}_c)t_c(\bar{x} - \bar{x}_c)^2 \\ &\qquad + \tfrac{1}{6}v_0'''(\bar{x}_c)t_c(\bar{x} - \bar{x}_c)^3. \end{aligned}$$

The first term in the last expression vanishes by Eq. (1.106). If t_c is equal to the wave-breaking moment t_b (and correspondingly $\bar{x}_c = \bar{x}_b$), then the second term vanishes by Eq. (1.108). Hence, in vicinity of the wave-breaking point x_b we have

$$x - x_b \cong \tfrac{1}{6}v_0'''(\bar{x}_b)t_c(\bar{x} - \bar{x}_b)^3,$$

so that $\partial x/\partial\bar{x} \propto (\bar{x}-\bar{x}_b)^2 \propto (x-x_b)^{2/3}$, and Eq. (1.110) yields here

$$\rho(x,t) \propto |x-x_b|^{-2/3}. \tag{1.111}$$

If $t = t_c > t_b$, then for small enough $|x-x_c|$ we have

$$x - x_c \cong \tfrac{1}{2}v_0''(\bar{x}_c)t_c(\bar{x}-\bar{x}_c)^2,$$

so that $\partial x/\partial\bar{x} \propto (\bar{x}-\bar{x}_c) \propto (x-x_c)^{1/2}$ and in vicinity of each branch of the caustic curve (but not near the cusp point) we obtain

$$\rho(x,t) \propto |x-x_c|^{-1/2}. \tag{1.112}$$

It is natural that in both cases (1.111) and (1.112) the integrals of density over x are finite in agreement with the finite value of the initial mass $\int \rho_0(x)\,dx$. Thus, during the evolution the region with a high density arises which becomes singular at the wave-breaking point with the order of singularity 2/3; just after the wave-breaking moment this singularity splits into the two singularities of the order 1/3 located at the two branches of the caustic curve.

The analysis of Eq. (1.101) leads to two general conclusions.

First, the phenomenon of wave breaking must be quite general and one may anticipate that taking into account of pressure does not exclude such behaviour. However, in this case we have no so simple interpretation of the region with many values of density as in the discussed above theory of non-interacting particles ('dust') whose macroscopic flow is described by the Hopf equation. It is known that in real flows of gas or plasma, for examples, viscosity or dispersion effects cannot be neglected and they lead to essentially new effects. If viscosity dominates over dispersion, then the gas flow is governed by the Navier-Stokes equation which for a one-dimensional flow has the form

$$v_t + vv_x = -p_x/\rho + \nu v_{xx},$$

This equation differs from the Euler equation by the additional term in the right hand side, where ν is a constant coefficient characterizing the gas viscosity. In the regions where the field $v(x,t)$ has a slow dependence on the coordinate x the last term can be dropped. However, at the wave-breaking point where $\partial v/\partial x \to \infty$ the viscosity term becomes the most essential one and cannot be neglected. In flows of plasma and many other problems the dispersion effects are more important than viscosity, and corresponding evolution equations also have terms with higher space derivatives (see, e.g., Eq. (1.39)) which are most essential at the wave-breaking point. So, the problem arises to determine how viscosity or dispersion effects influence on the wave evolution after the wave-breaking point. A considerable part of this book will be devoted to this problem.

Second, the method of solution of the Hopf equation (1.101) is very clear and one may expect that it can be generalized to some other equations. In fact, we 'marked' each particle of a fluid by its initial coordinate $\bar{x}$ and traced its movement with initial velocity $v_0(\bar{x})$ along straight characteristics in the plane (x, t). One may ask how to apply this idea to other partial differential equations of the first order. This method provides the solution correct up to the wave-breaking point and, as we shall find, its proper generalization works even after the wave-breaking point. Therefore, the next few sections will be devoted to several examples of integration of nonlinear hydrodynamical equations what will permit us to introduce important notions necessary for the subject of the book.

1.3.3 *Simple wave and a piston problem*

One may notice that both considered above problems about the self-similar rarefaction wave and the breaking of wave in a dust-like medium (with $p = 0$) have formally similar solution. Indeed, let us compare the self-similar solution (1.91,1.92) for the rarefaction wave,

$$x = [v \pm c(v)]t, \qquad v = \pm \int^{\rho} c(\rho) d\rho/\rho, \tag{1.113}$$

with the solution (1.105) for the evolution of the wave in media with $p = 0$,

$$x = \bar{x} + v(\bar{x})t, \tag{1.114}$$

which can be presented in the form

$$x = vt + \bar{x}(v), \tag{1.115}$$

where the density ρ is connected with the flow velocity v by Eq. (1.109). Equation (1.113) describes a non-uniform region arising from the decay of a step-like pulse initially located at $x = 0$. Therefore, this region shrinks to the point $x = 0$ in the limit $t \to 0$, and the characteristics (1.114) form a 'fan' made of straight lines meeting at the point $(0, 0)$ in the plane (x, t), see Fig. 1.3. The slope of the line, equal to a sum or a difference of a local velocity v of the flow and a local sound velocity $c(v)$, is constant. In the case of a 'dust matter' flow, Eq. (1.115), there is no sound velocity in accordance with the absence of pressure p, and the characteristic lines stem from different points of the x axis with the slope determined by the initial value of the velocity at these points.

Let us try to find the solution of the system (1.89) combining the properties of the two particular cases considered above. They have a common property that the system of two equations can be reduced to one equation, so that one variable can be expressed in terms of the other. In the rarefaction wave, both variables v and ρ depend on the same self-similar variable $\xi = x/t$, hence, we can write $v = v(\rho)$ or $\rho = \rho(v)$. Equation (1.101) describing the 'dust matter' flow does not depend on the density ρ at all, but the functional dependence of ρ on v is obtained by the solution of Eq. (1.109). Therefore, it is natural to consider a class of solutions with one-to-one relationship between the flow velocity v and the gas density ρ,

$$v = v(\rho), \quad \rho = \rho(v).$$

Under this supposition, we may rewrite Eqs. (1.89) as

$$\rho_t + [d(\rho v)/d\rho]\, \rho_x = 0, \tag{1.116}$$

$$v_t + \left[v + (c^2/\rho)(d\rho/dv)\right]\, v_x = 0. \tag{1.117}$$

Noticing that

$$\rho_t/\rho_x = -\left(\partial x/\partial t\right)_\rho,$$

we obtain from Eq. (1.116) that

$$\left(\partial x/\partial t\right)_\rho = d(\rho v)/d\rho = v + \rho\, dv/d\rho,$$

and in the same way from Eq. (1.117) that

$$\left(\partial x/\partial t\right)_v = v + (c^2/\rho)(d\rho/dv). \tag{1.118}$$

Since ρ is a one-valued function of v, we have

$$\left(\partial x/\partial t\right)_\rho = \left(\partial x/\partial t\right)_v,$$

and, hence,

$$\rho(dv/d\rho) = (c^2/\rho)(d\rho/dv).$$

Thus, we have

$$dv/d\rho = \pm c/\rho, \tag{1.119}$$

and the function $v(\rho)$ (or $v(p)$) is determined by the relation

$$v = \pm \int^{\rho} c d\rho/\rho = \pm \int^{p} dp/\rho c. \tag{1.120}$$

Note that this result coincides with Eq. (1.113) corresponding to a self-similar solution. After substitution of $d\rho/dv = \pm\rho/c$ into Eq. (1.117) we obtain the equation

$$v_t + (v \pm c)v_x = 0 \tag{1.121}$$

similar to the Hopf equation (1.101). The only difference between Eq. (1.121) and Eq. (1.101) is that now a point of the wave profile with the flow velocity v moves with the velocity equal to the sum or difference of the flow velocity and the local sound velocity. Indeed, substituting Eq. (1.119) into Eq. (1.118), we obtain the equation

$$(\partial x/\partial t)_v = v \pm c(v) \tag{1.122}$$

whose integration gives

$$x = [v \pm c(v)]t + \bar{x}(v), \tag{1.123}$$

where $\bar{x}(v)$ is an arbitrary function of the velocity v, and $c(v)$ can be found with the use of Eq. (1.120) and the formula $c(\rho) = \sqrt{dp/d\rho}$. The straight lines (1.123) in the plane (x, t) are called characteristics of the system (1.116,1.117). It is clear that Eq. (1.123) reduces to Eq. (1.113), if $\bar{x}(v) = 0$, and to Eq. (1.115), if $c(v) = 0$, that is, we have got the generalization of the above solutions. The solution (1.123) may be rewritten in the form

$$v = v_0[x - (v \pm c(v))t] \tag{1.124}$$

similar to Eq. (1.102), $v_0(\bar{x})$ being the function inverse to $\bar{x}(v)$. Obviously, $v_0(x)$ defines the initial profile of the wave at $t = 0$. Points of the profile propagate with velocities

$$V = v \pm c.$$

Since V varies from one point to another, the profile changes its form during the propagation, and this can lead to the wave-breaking phenomenon. Thus, the analysis of the preceding section can be applied after appropriate modifications to various problems where the pressure plays an essential role. After the wave-breaking point the density ρ is also a multi-valued function and the solution becomes meaningless. (In contrast to the flow of 'dust' matter, now we cannot add densities of flows with different velocities.) Therefore, the solution (1.123,1.124), which is called usually a simple wave solution, describes only the evolution up to the wave-breaking point. The time t_b of wave breaking is determined by the condition that the profile has an inflexion point with a vertical tangent line,

$$(\partial x/\partial v)_{t_b} = 0, \quad \left(\partial^2 x/\partial v^2\right)_{t_b} = 0. \tag{1.125}$$

However, if a simple wave has contact with gas at rest and the wave breaking occurs at the boundary between these two regions, then a vertical tangent line of the profile $v = v(x)$ intersects the point corresponding to $v = 0$ and, hence, the wave-breaking condition becomes

$$(\partial x/\partial v)_{t_b}\big|_{v=0} = 0. \tag{1.126}$$

Let us illustrate the formulas obtained by an example of the polytropic gas with the dependence of the sound velocity on the density defined by Eq. (1.94). Then Eqs. (1.94) and (1.96) yield

$$c = c_0 \pm \tfrac{1}{2}(\gamma - 1)v, \tag{1.127}$$

and Eq. (1.121) simplifies to

$$v_t + \left(\pm c_0 + \tfrac{1}{2}(\gamma + 1)v\right) v_x = 0. \tag{1.128}$$

This equation coincides exactly with the Hopf equation (1.101) in the frame of reference moving with the velocity $\pm c_0$. Its solution (1.123) becomes

$$x = \left(\pm c_0 + \tfrac{1}{2}(\gamma + 1)v\right) t + \bar{x}(v), \tag{1.129}$$

and according to Eq. (1.96) the density ρ depends on the velocity v as

$$\rho = \rho_0 \left(1 \pm \tfrac{1}{2}(\gamma - 1)(v/c_0)\right)^{2/(\gamma-1)}. \tag{1.130}$$

Substitution of Eq. (1.129) into Eq. (1.125) gives the conditions for the wave-breaking moment for a polytropic gas:

$$t = -\frac{2}{\gamma+1}\,\bar{x}'(v), \qquad \bar{x}''(v) = 0. \tag{1.131}$$

If the breaking of wave occurs at the boundary with gas at rest, this moment is determined by

$$t = -\frac{2}{\gamma+1}\,\bar{x}'(0). \tag{1.132}$$

Let us apply the solution found to the 'piston' problem. Let a polytropic gas be located inside a pipe indefinitely long at one side ($x \to \infty$) and closed by a movable piston at the other side ($x = 0$). At some moment $t = 0$ the piston starts its motion according to the law $x = X(t)$, $X(0) = 0$. We have to calculate the corresponding flow of gas inside the pipe.

We have to satisfy the boundary condition at the piston,

$$v = \dot{X}(t) \quad \text{at} \quad X(t) = \left(c_0 + \tfrac{1}{2}(\gamma+1)v\right)t + \bar{x}(v),$$

which means that the flow velocity of the gas contacting with the piston is equal to the piston's velocity at any moment of time. These two formulas define the function $\bar{x}(v)$ in a parametric form:

$$\bar{x}(\tau) = X(\tau) - \left(c_0 + \tfrac{1}{2}(\gamma+1)\dot{X}(\tau)\right)\tau, \quad v(\tau) = \dot{X}(\tau).$$

The substitution of the first formula into Eq. (1.129) yields the solution of the piston problem

$$x = X(\tau) + \left[c_0 + \tfrac{1}{2}(\gamma+1)\dot{X}(\tau)\right](t-\tau), \quad v = \dot{X}(\tau). \tag{1.133}$$

These equations define parametrically the dependence of the flow velocity v on the coordinate x and the time t for a given law $X(t)$ of the piston's motion.

Let us specify these formulas for a simple case of the motion of the piston with constant acceleration,

$$X(t) = \tfrac{1}{2}at^2, \quad \dot{X}(t) = at, \quad a > 0. \tag{1.134}$$

Then Eqs. (1.133) transform to

$$x = \tfrac{1}{2}a\tau^2 + \left(c_0 + \tfrac{1}{2}(\gamma+1)a\tau\right)(t-\tau), \quad v = a\tau,$$

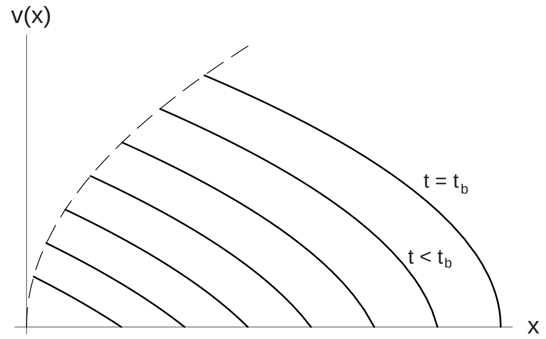

Fig. 1.11 Dependence of the flow velocity v on the space coordinate x at different values of time t up to the wave-breaking moment at $t = t_b$. The dashed line shows the velocity of the piston, moving with constant acceleration as a function of its position.

and these two formulas give

$$x = (1/2a)v^2 + \left(c_0 + \tfrac{1}{2}(\gamma+1)v\right)(t - v/a)\,. \tag{1.135}$$

This is a quadratic equation with respect to v, and its solution yields the explicit formula

$$v = -(1/\gamma)\left(c_0 - \tfrac{1}{2}(\gamma+1)at\right) + (1/\gamma)\sqrt{\left(c_0 - \tfrac{1}{2}(\gamma+1)at\right)^2 - 2\gamma a(x - c_0 t)}. \tag{1.136}$$

The sign before the square root is chosen so that $v = 0$ at the boundary $x = c_0 t$ with the gas at rest. The plot of $v(x)$ is a part of a parabola (see Fig. 1.11), which acquires a vertical tangent line at $x = c_0 t_b$ at the moment

$$t_b = 2c_0/[(\gamma+1)a]$$

in agreement with Eq. (1.132), since in our case $\bar{x}(v) = v^2/2a - (c_0 + (\gamma + 1)v/2)(v/a)$ (see Eq. (1.135)), and $\partial\bar{x}/\partial v|_{v=0} = -c_0/a$. The characteristics of the flow are straight lines (1.135) parametrized by the values of v from the interval $at \le v \le c_0 t$, that is, they stem from the line $x = at^2/2$ corresponding to the piston's motion (see Fig. 1.12). In this case one branch of the caustic curve is given by the straight line $x = c_0 t$ corresponding to the boundary with the gas at rest. Inside the caustic the solution becomes meaningless because the supposition about the one-valued dependence of ρ

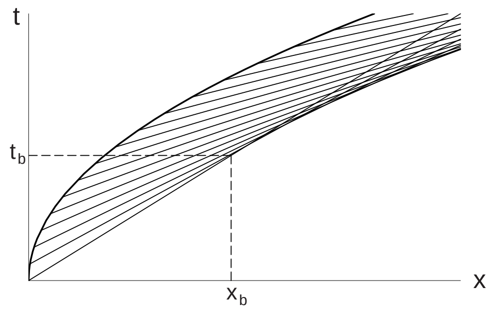

Fig. 1.12 Characteristics of the flow created by a uniformly accelerated piston. A parabolic boundary corresponds to the piston's position as a function of time t. Formation of the caustic curve around the multi-valued region is clearly seen.

on v fails in this region. To find the solution for $t > t_b$, one has to take into account the viscosity or dispersion effects which dominate at large values of the profile's gradient.

1.3.4 *Characteristics and Riemann invariants*

Let us look at the solutions found in the preceding sections from a more general point of view. The Hopf equation

$$u_t + uu_x = 0 \tag{1.137}$$

may be considered in the following way: In the plane (t, u) there is a vector field

$$\mathbf{a} = (1, u) \tag{1.138}$$

and Eq. (1.137) means that u is constant along integral curves of this field,

$$(\mathbf{a}\nabla)\, u \equiv (\partial/\partial t + u\partial/\partial x)\, u = 0.$$

Integral curves are defined as solutions of the ordinary differential equation

$$dt/1 = dx/u, \tag{1.139}$$

hence, they are straight lines

$$x = x_0 + ut, \quad u = u_0 \tag{1.140}$$

called characteristics of Eq. (1.137). Thus, we have found that the value u_0 corresponding to the initial coordinate x_0 is carried after time t to the point x determined by Eq. (1.140). Since characteristics depend on u, it is convenient to represent them as straight lines in three dimensional Cartesian coordinate system (t, x, u). These straight lines stem from the curve $(0, x, u_0(x))$ defined by the initial distribution of u at $t = 0$, and the entire set of characteristics forms the integral surface of the solution of Eq. (1.137). (An example of such a surface was shown in Fig. 1.10).

Obviously, this method of treatment can be generalized at once on any quasilinear partial differential equation (PDE) having the form

$$a(t, x, u)u_t + b(t, x, u)u_x = c(t, x, u). \tag{1.141}$$

This equation means that if we go out of the point (t_0, x_0) with the 'velocity' $\mathbf{a}_0 = (a(t_0, x_0, u_0),\ b(t_0, x_0, u_0))$, where $u_0 = u(t_0, x_0)$, then the value of $u(t, x)$ changes with the velocity $c_0 = c(t_0, x_0, u_0)$. In other words, we connect with Eq. (1.141) the vector field

$$\mathbf{A}(t, x, u) = (a(t, x, u),\ b(t, x, u),\ c(t, x, u)), \tag{1.142}$$

whose integral curves in the space (t, x, u) are defined as solutions of the system of ordinary differential equations (ODE)

$$\frac{dt}{a(t, x, u)} = \frac{dx}{b(t, x, u)} = \frac{du}{c(t, x, u)}. \tag{1.143}$$

These integral curves are also called characteristics of Eq. (1.141). They are straight lines lying in planes parallel to (t, x) plane only if $c \equiv 0$. The importance of characteristics follows from the statement that integral surfaces (solutions) of Eq. (1.141) are built from the characteristics defined as solutions of the system (1.143), so that solving quasilinear PDE is reduced to solving the system of ODE.

To prove the above statement, let us consider a surface made of characteristics and given in a parametric form,

$$t = t(s, \alpha, \beta), \quad x = x(s, \alpha, \beta), \quad u = u(s, \alpha, \beta), \tag{1.144}$$

where s is a parameter along characteristics and (α, β) are parameters which label different characteristics (e.g., (α, β) may be the coordinates of initial points (x_0, u_0) of characteristics in the plane $t = 0$). Then we have from Eqs. (1.143)

$$dt/ds = a(t, x, u), \quad dx/ds = b(t, x, u), \quad du/ds = c(t, x, u). \tag{1.145}$$

We are going to show that if some function $f(t, x, u)$ is constant along characteristics, then the equation

$$f(t, x, u) = \text{const} \tag{1.146}$$

defines implicitly the function $u = u(t, x)$ satisfying Eq. (1.141). Indeed, constancy of $f(t, x, u)$ along characteristics means that

$$\begin{aligned} 0 =& \frac{df}{ds} = \frac{\partial f}{\partial t}\frac{dt}{ds} + \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial u}\frac{du}{ds} = \frac{\partial f}{\partial t}a + \frac{\partial f}{\partial x}b + \frac{\partial f}{\partial u}c \\ =& \frac{\partial f}{\partial u}\left(a\frac{\partial f/\partial t}{\partial f/\partial u} + b\frac{\partial f/\partial x}{\partial f/\partial u} + c\right) = -\frac{\partial f}{\partial u}\left(a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} - c\right), \end{aligned} \tag{1.147}$$

where we have used the formulas $u_t = -f_t/f_u$, $u_x = -f_x/f_u$ for derivatives of an implicit function. Equation (1.147) shows that the function $u = u(t, x)$ defined by Eq. (1.146) represents a solution of Eq. (1.141).

Note, that Eq. (1.146) gives the first integral of the system (1.143). If we know other two first integrals,

$$g(t, x, u) = \text{const}, \quad h(t, x, u) = \text{const},$$

of Eqs. (1.143), then there is a functional dependence between the three first integrals,

$$f(t, x, u) = \Phi[g(t, x, u), h(t, x, u)]. \tag{1.148}$$

Indeed, let three integral surfaces $f = C_1$, $g = C_2$, $h = C_3$ have one common point (t, x, u). Normals to these surfaces at this point, proportional to the gradients ∇f, ∇g, ∇h, must be perpendicular to the common tangent vector (1.142). Hence, these three normals lay in one plane, and this means vanishing of the Jacobian $\partial(f, g, h)/\partial(t, x, u) = 0$, that is, existence of the functional dependence (1.148). In other words, the vector product $\nabla f \times \nabla g$ is directed along the vector (1.142) and therefore the intersection curve of the two integral surfaces $f = C_1$, $g = C_2$ must be a characteristic curve. Thus, the integral surfaces of Eq. (1.141) can intersect each other only along characteristics. Since a general solution of Eq. (1.141) must contain one arbitrary function of two variables, it follows from the above arguments that a general solution can be presented in the form (1.148), where $g(t, x, u) =$ const and $h(t, x, u) =$ const are two first integrals of Eqs. (1.143) and $\Phi(g, h)$ is an arbitrary function. This remark often provides a practical method of finding a general solution of Eq. (1.141).

Thus, the solving the quasilinear PDE (1.141) can be reduced to finding the solutions of the system (1.143) of ODE. We suppose that the initial data are given on some non-characteristic curve in (t, x, u) space, and if this curve intersects some characteristic, then there must be only one such a point of intersection. Then, if characteristics are found, the integral surface is formed by the characteristics stemming from the points of the initial data curve. This gives in principle the solution of the Cauchy problem.

Let us generalize this approach to a quasilinear system with two variables:

$$\begin{aligned} a_1\rho_t + a_2\rho_x + a_3u_t + a_4u_x = 0, \\ b_1\rho_t + b_2\rho_x + b_3u_t + b_4u_x = 0. \end{aligned} \tag{1.149}$$

Equations (1.89) of a one-dimensional isentropic gas flow have just this form. (For simplicity, we consider only homogeneous equations, though the theory can easily be generalized to the non-homogeneous case.)

We shall start from the observation that the Cauchy problem for the Eq. (1.141) does not have a solution, if the initial data are given on a characteristic curve of this equation—in this case the integral surface shrinks to this characteristic curve. In other words, if the function $u(x, t)$ is given on the characteristic curve, then we cannot calculate from Eq. (1.141) the derivatives of $u(x, t)$ along any direction intersecting the characteristic curve. Hence, we cannot use a Taylor series expansion for calculation of $u(x, t)$ in vicinity of the characteristic curve.

With this idea in mind, let us consider in the (t, x) plane a curve γ defined parametrically by expressions $t = T(s)$, $x = X(s)$, s being a parameter along the curve. We denote by

$$dx/ds = X' = \cos\phi, \quad dt/ds = T' = \sin\phi \tag{1.150}$$

the components of the vector tangent to γ. We suppose that the initial data for $\rho(x, t)$, $u(x, t)$ are given on the curve γ, and, hence, we can calculate the derivatives along γ:

$$\begin{aligned} \rho' \equiv d\rho/ds = \rho_x \cos\phi + \rho_t \sin\phi, \\ u' \equiv du/ds = u_x \cos\phi + u_t \sin\phi. \end{aligned} \tag{1.151}$$

Since Eqs. (1.149) must also be satisfied, we have a system of four equations for calculation of four derivatives ρ_t, ρ_x, u_t, u_x. This system does not have a solution (that is, γ is a characteristic curve), if its determinant is equal to zero. Hence, as a simple calculation shows, for a given solution $\rho(x,t)$, $u(x,t)$ the angle ϕ between the tangent line of the characteristic curve and the x axis is defined by the equation

$$\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} = \begin{vmatrix} a_1\cos\phi - a_2\sin\phi & a_3\cos\phi - a_4\sin\phi \\ b_1\cos\phi - b_2\sin\phi & b_3\cos\phi - b_4\sin\phi \end{vmatrix} = 0, \tag{1.152}$$

or

$$\Delta_{13}\cos^2\phi - (\Delta_{14} + \Delta_{23})\sin\phi\cos\phi + \Delta_{24}\sin^2\phi = 0, \tag{1.153}$$

where we have introduced the notation

$$\Delta_{ij} = a_i b_j - a_j b_i, \quad i,j = 1,2,3,4. \tag{1.154}$$

If Eq. (1.153) has two real roots for $\cot\phi$, then in vicinity of a given point (x,t) we can draw two families of characteristics corresponding to the solution $\rho(x,t)$, $u(x,t)$. In this case the system (1.149) is called hyperbolic for this given solution. If Eq. (1.153) has only one root, the system (1.149) is called parabolic. At last, if both roots are complex, then the system (1.149) is called elliptic. If the system (1.149) is linear, i.e., the coefficients a_i, b_i, $i = 1,2,3,4$, do not depend on ρ and u, then these definitions do not depend on the solution and characterize some region in the (x,t)-plane.

Let us suppose that both roots of Eq. (1.153) are real, so that there are two families of characteristics which we denote as C_1 and C_2. We want to find how the functions ρ and u vary along characteristics. At first we find from definitions (1.152) and (1.154) with the use of Eq. (1.153) the following identities:

$$\begin{aligned}
&(a_1B_1 - b_1A_1)/\sin\phi = (a_2B_1 - b_2A_1)/\cos\phi = -\Delta_{12},\\
&(a_3B_2 - b_3A_2)/\sin\phi = (a_4B_2 - b_4A_2)/\cos\phi = -\Delta_{34},\\
&(a_3B_1 - b_3A_1)/\sin\phi = (a_4B_1 - b_4A_1)/\cos\phi\\
&\qquad = -\Delta_{14} - \Delta_{42}\tan\phi = -\Delta_{32} - \Delta_{13}\cot\phi = -K,\\
&(a_1B_2 - b_1A_2)/\sin\phi = (a_2B_2 - b_2A_2)/\cos\phi\\
&\qquad = -\Delta_{14} - \Delta_{31}\tan\phi = -\Delta_{32} - \Delta_{24}\cot\phi = -L,
\end{aligned} \tag{1.155}$$

where the last two expressions serve also as definitions of K and L. Now we are able to construct linear combinations of Eqs. (1.149) which include the derivatives only along characteristics. If we multiply the first equation (1.149) by B_1, the second by A_1, and subtract one from another, then we obtain

$$\Delta_{12}(d\rho/ds) + K(du/ds) = 0. \tag{1.156}$$

In the same way, if we multiply the first equation (1.149) by B_2, the second by A_2, and subtract one from another, then we obtain

$$L(d\rho/ds) + \Delta_{34}(du/ds) = 0. \tag{1.157}$$

If neither of equations (1.156) and (1.157) is a trivial identity, they are equivalent to each other. Since the coefficients K and L depend on $\cot\phi$, the two roots of Eq. (1.153) lead to derivatives along the families C_1 and C_2 of characteristics. Thus, depending on the choice of the root $\cot\phi$, either Eq. (1.156) or Eq. (1.157) presents two equations for derivatives along each family of characteristics.

Let, for example, the derivatives of ρ and u along characteristics be defined by Eq. (1.156), that is,

$$\Delta_{12}\frac{d\rho}{ds_1} + K(\cot\phi_1)\frac{du}{ds_1} = 0, \quad \Delta_{12}\frac{d\rho}{ds_2} + K(\cot\phi_2)\frac{du}{ds_2} = 0, \tag{1.158}$$

where s_1 and s_2 are the length parameters along characteristics from the families C_1 and C_2, correspondingly. We suppose that all the coefficients a_i, b_i do not depend explicitly on the space coordinate x and time t, that is, they depend only on ρ and u: $a_i = a_i(\rho, u)$, $b_i = b_i(\rho, u)$. If we are able to find such integration factors $\alpha(\rho, u)$ and $\beta(\rho, u)$, that as relations

$$\begin{aligned} \alpha\Delta_{12} &= \partial\lambda_1/\partial\rho, \qquad \alpha K(\cot\phi_1) = \partial\lambda_1/\partial u, \\ \beta\Delta_{12} &= \partial\lambda_2/\partial\rho, \qquad \beta K(\cot\phi_2) = \partial\lambda_2/\partial u, \end{aligned} \tag{1.159}$$

so the compatibility conditions

$$\frac{\partial(\alpha\Delta_{12})}{\partial u} = \frac{\partial(\alpha K(\cot\phi_1))}{\partial\rho}, \quad \frac{\partial(\beta\Delta_{12})}{\partial u} = \frac{\partial(\beta K(\cot\phi_2))}{\partial\rho} \tag{1.160}$$

are fulfilled, then Eqs. (1.158) take the form

$$d\lambda_1/ds_1 = 0, \quad d\lambda_2/ds_2 = 0, \tag{1.161}$$

that is, the variables λ_1 and λ_2 are constant along characteristics C_1 and C_2, correspondingly. These variables λ_1 and λ_2 are called the Riemann invariants. As the characteristics define according to Eq. (1.150) the two vector fields $(\cos\phi_1, \sin\phi_1)$ and $(\cos\phi_2, \sin\phi_2)$ in the (x, t) plane, and the derivatives (1.161) have the meaning of derivatives along the integral curves of these vector fields, they can be rewritten in the form

$$\frac{\partial\lambda_1}{\partial x}\cos\phi_1 + \frac{\partial\lambda_1}{\partial t}\sin\phi_1 = 0, \quad \frac{\partial\lambda_2}{\partial x}\cos\phi_2 + \frac{\partial\lambda_2}{\partial t}\sin\phi_2 = 0,$$

or

$$\partial\lambda_1/\partial t + v_1\partial\lambda_1/\partial x = 0, \quad \partial\lambda_2/\partial t + v_2\partial\lambda_2/\partial x = 0, \tag{1.162}$$

where

$$v_1 = (dx/dt)_1 = \cot\phi_1, \quad v_2 = (dx/dt)_2 = \cot\phi_2 \tag{1.163}$$

are the so called characteristic speeds defined by Eq. (1.153). In the considered here case of two unknown functions ρ, u of two independent variables x, t it is always possible, in principle, to find the integration factors α and β, so that integration of Eqs. (1.159) determines the Riemann invariants as functions of ρ and u. Inverse expressions determine the variables ρ and u as functions of λ_1 and λ_2 and, consequently, we can obtain from Eq. (1.163) the velocities v_1 and v_2 as functions of the Riemann invariants:

$$v_1 = v_1(\lambda_1, \lambda_2), \quad v_2 = v_2(\lambda_1, \lambda_2). \tag{1.164}$$

As a result of all these transformations, the system of equations (1.149) is reduced to extremely simple and convenient form (1.162, 1.164).

Let us apply this approach to the system describing an isentropic flow of gas (see Eqs. (1.89))

$$\rho_t + \rho u_x + u\rho_x = 0, \quad u_t + uu_x + (c^2/\rho)\rho_x = 0, \tag{1.165}$$

that is, the coefficients of the system (1.149) are given by

$$\begin{array}{llll} a_1 = 1, & a_2 = u, & a_3 = 0, & a_4 = \rho, \\ b_1 = 0, & b_2 = c^2/\rho, & b_3 = 1, & b_4 = u, \end{array}$$

so that

$$\Delta_{13} = 1, \quad \Delta_{14} = \Delta_{23} = u, \quad \Delta_{24} = u^2 - c^2,$$

and Eq. (1.153) takes the form

$$\cos^2\phi - 2u\sin\phi\cos\phi + (u^2 - c^2)\sin^2\phi \equiv (\cos\phi - u\sin\phi)^2 - c^2\sin^2\phi = 0.$$

Thus, we obtain at once the characteristic speeds

$$v_{1,2} = \cot\phi_{1,2} = u \pm c. \tag{1.166}$$

The coefficient K in Eqs. (1.155) is equal to $K(\cot\phi_{1,2}) = \pm c$, and $\Delta_{12} = c^2/\rho$, whence the compatibility equations (1.160) become

$$\frac{\partial}{\partial u}\left(\frac{\alpha c^2}{\rho}\right) = \frac{\partial}{\partial \rho}(\alpha c), \quad \frac{\partial}{\partial u}\left(\frac{\beta c^2}{\rho}\right) = -\frac{\partial}{\partial \rho}(\beta c).$$

Since the sound velocity c depends on ρ only, it is clear that these equations are satisfied, if one puts $\alpha = \beta = 1/c$. Then we obtain from Eqs. (1.159)

$$\partial\lambda_1/\partial\rho = c/\rho, \quad \partial\lambda_1/\partial u = 1, \quad \partial\lambda_2/\partial\rho = c/\rho, \quad \partial\lambda_2/\partial u = -1,$$

and, hence, the Riemann invariants are given by

$$\lambda_1(\rho, u) = u + \int^{\rho} c d\rho/\rho, \quad \lambda_2(\rho, u) = u - \int^{\rho} c d\rho/\rho. \tag{1.167}$$

They evolve according to Eqs. (1.162,1.166), or

$$\frac{\partial\lambda_1}{\partial t} + (u + c)\frac{\partial\lambda_1}{\partial x} = 0, \quad \frac{\partial\lambda_2}{\partial t} + (u - c)\frac{\partial\lambda_2}{\partial x} = 0. \tag{1.168}$$

In the case of a polytropic gas flow, when the sound velocity is given by Eq. (1.94), the integral in Eqs. (1.167) is readily calculated, and up to inessential integration constant we find the following expressions for the Riemann invariants

$$\lambda_1 = u + 2c/(\gamma - 1), \quad \lambda_2 = u - 2c/(\gamma - 1), \tag{1.169}$$

Now we can express in terms of the Riemann invariants the flow velocity u and the sound velocity c,

$$u = \tfrac{1}{2}(\lambda_1 + \lambda_2), \quad c = \tfrac{1}{4}(\gamma - 1)(\lambda_1 - \lambda_2), \tag{1.170}$$

as well as the characteristic speeds,

$$\begin{aligned} v_1(\lambda_1, \lambda_2) &= dx_1/dt = u + c = \tfrac{1}{4}(1+\gamma)\lambda_1 + \tfrac{1}{4}(3-\gamma)\lambda_2, \\ v_2(\lambda_1, \lambda_2) &= dx_2/dt = u - c = \tfrac{1}{4}(3-\gamma)\lambda_1 + \tfrac{1}{4}(1+\gamma)\lambda_2. \end{aligned} \tag{1.171}$$

Thus, the system (1.165) is reduced to a simple form

$$\begin{aligned} &\partial\lambda_1/\partial t + \tfrac{1}{4}\left[(1+\gamma)\lambda_1 + (3-\gamma)\lambda_2\right]\partial\lambda_1/\partial x = 0, \\ &\partial\lambda_2/\partial t + \tfrac{1}{4}\left[(3-\gamma)\lambda_1 + (1+\gamma)\lambda_2\right]\partial\lambda_2/\partial x = 0. \end{aligned} \tag{1.172}$$

The results obtained permit us to look at the simple wave solutions found in Sec. 1.3.3 from a new point of view. Comparison of equations (1.120) and (1.167) shows that in the simple wave solution one of the two Riemann invariants is constant (even equal to zero in our notations). Hence, one of Eqs. (1.172) is satisfied identically, and the other coincides actually with the Hopf equation. It becomes obvious that in this case the characteristics of either C_1 or C_2 family must be straight lines. Indeed, along these characteristics one Riemann invariant is constant by definition, and the other is constant because it is constant in the entire simple wave solution. But if both λ_1 and λ_2 are constant, then the characteristic speeds v_1 and v_2 are constant too, and, hence, the solutions of Eqs. (1.171) are straight lines. Note, that the characteristics of the second family are not straight lines even in the case of a simple wave solution. For example, if one considers the self-similar solution of Eqs. (1.172) in the form of a rarefaction wave (see Sec. 1.3.1), in which

$$x = \tfrac{1}{4}\left((1+\gamma)\lambda_1 + (3-\gamma)\lambda_2\right)t, \quad \lambda_2 = \text{const}, \tag{1.173}$$

that is, the one family of characteristics consists of straight lines satisfying the first equation (1.171), then the second equation (1.171) after exclusion of λ_1 with help of Eq. (1.173) takes the form

$$\frac{dx}{dt} = \frac{3-\gamma}{1+\gamma} \cdot \frac{x}{t} + \frac{2(\gamma-1)}{1+\gamma}\lambda_2$$

and has the solution

$$x = Ct^{(3-\gamma)/(1+\gamma)} + \lambda_2 t, \tag{1.174}$$

where C is the integration constant. The constant Riemann invariant λ_2 can be determined from the matching condition on the boundary with gas at rest. The constant C is determined by the initial point which lies on this characteristic and corresponds to the moment $t = t_0$. According to Eqs. (1.171), the characteristic curves are the trajectories along which small disturbances propagate with the local sound velocity c on the 'background' flow with velocity u, so that depending on the direction of propagation the full velocity is equal to either sum or difference of the flow velocity and the sound velocity. Therefore, if one creates a small disturbance somewhere in the region of a simple wave flow, it splits into two disturbances one propagating in the direction of the flow and the other in the opposite direction, and their trajectories are given by Eqs. (1.173) and (1.174), respectively.

1.3.5 *Hodograph transform and a general case of a polytropic gas flow*

To complete the consideration of the gas flow, we have to consider the case when both Riemann invariants are variable, that is, to find a general solution of the system

$$\frac{\partial \lambda_1}{\partial t} + v_1(\lambda_1, \lambda_2)\frac{\partial \lambda_1}{\partial x} = 0, \quad \frac{\partial \lambda_2}{\partial t} + v_2(\lambda_1, \lambda_2)\frac{\partial \lambda_2}{\partial x} = 0 \tag{1.175}$$

(see Eqs. (1.162)). But in the region where both λ_1 and λ_2 are not constant, we may treat them as coordinates in the plane (x, t), that is, consider x and t as functions of λ_1 and λ_2:

$$x = x(\lambda_1, \lambda_2), \quad t = t(\lambda_1, \lambda_2). \tag{1.176}$$

Let us transform the system (1.175) to a new coordinate system. To this end, we have to express the derivatives with respect to x and t in terms of the derivatives with respect to λ_1 and λ_2. If we have a function $\Phi(x, t) =$

$\Phi[x(\lambda_1,\lambda_2),t(\lambda_1,\lambda_2)]$, then its differentiation with respect to λ_1 and λ_2 gives the system

$$\Phi_{\lambda_1}=\Phi_x\cdot x_{\lambda_1}+\Phi_t\cdot t_{\lambda_1},\quad \Phi_{\lambda_2}=\Phi_x\cdot x_{\lambda_2}+\Phi_t\cdot t_{\lambda_2},$$

from which we obtain the relations

$$\begin{aligned}\Phi_x&=J^{-1}\left(\Phi_{\lambda_1}\cdot t_{\lambda_2}-\Phi_{\lambda_2}\cdot t_{\lambda_1}\right),\\ \Phi_t&=J^{-1}\left(\Phi_{\lambda_2}\cdot x_{\lambda_1}-\Phi_{\lambda_1}\cdot x_{\lambda_2}\right),\end{aligned}\tag{1.177}$$

where $J=x_{\lambda_1}t_{\lambda_2}-x_{\lambda_2}t_{\lambda_1}$ is the Jacobian of the transformation of coordinates. If we put here Φ equal to λ_1 or λ_2, we get

$$\begin{aligned}\lambda_{1,x}&=J^{-1}t_{\lambda_2},\quad \lambda_{1,t}=-J^{-1}x_{\lambda_2},\\ \lambda_{2,x}&=-J^{-1}t_{\lambda_1},\quad \lambda_{2,t}=J^{-1}x_{\lambda_1}.\end{aligned}\tag{1.178}$$

Substitution of these expressions into Eqs. (1.175) yields the equations

$$x_{\lambda_2}-v_1(\lambda_1,\lambda_2)t_{\lambda_2}=0,\quad x_{\lambda_1}-v_2(\lambda_1,\lambda_2)t_{\lambda_1}=0\tag{1.179}$$

for x and t as functions of λ_1 and λ_2. The transition from Eqs. (1.175) to Eqs. (1.179) is called a hodograph transform. Its remarkable feature is that in the case of homogeneous equations (as Eqs. (1.175)) with coefficients independent explicitly on x and t the resulting system is linear with respect to the functions $x(\lambda_1,\lambda_2)$ and $t(\lambda_1,\lambda_2)$.

Now let us recall that in a simple wave case with one Riemann invariant constant, say, $\lambda_2=\lambda_{20}$, the solution of the only remaining equation

$$\lambda_{1,t}+v_1(\lambda_1,\lambda_{20})\lambda_{1,x}=0$$

is given by

$$x-v_1(\lambda_1,\lambda_{20})t=w(\lambda_1,\lambda_{20})$$

(see, e.g., Eq. (1.123)), where w is an arbitrary function. Therefore, let us try to look for a solution of Eqs. (1.179) in the form

$$x-v_1(\lambda_1,\lambda_2)t=w_1(\lambda_1,\lambda_2),\quad x-v_2(\lambda_1,\lambda_2)t=w_2(\lambda_1,\lambda_2).\tag{1.180}$$

Equations (1.179) lead to certain conditions for functions w_1 and w_2. Indeed, differentiation of the first equation (1.180) with respect to λ_2 and of the second equation with respect to λ_1 gives

$$-(\partial v_1/\partial\lambda_2)t=\partial w_1/\partial\lambda_2,\quad -(\partial v_2/\partial\lambda_1)t=\partial w_2/\partial\lambda_1.$$

Elimination of t with the help of the difference of Eqs. (1.180) yields the equations for w_1 and w_2,

$$\frac{1}{w_1 - w_2}\frac{\partial w_1}{\partial \lambda_2} = \frac{1}{v_1 - v_2}\frac{\partial v_1}{\partial \lambda_2}, \qquad \frac{1}{w_1 - w_2}\frac{\partial w_2}{\partial \lambda_1} = \frac{1}{v_1 - v_2}\frac{\partial v_2}{\partial \lambda_1}, \tag{1.181}$$

where the right hand sides are considered as known quantities. The symmetry of these equations with respect to exchange $v_i \leftrightarrow w_i$ prompts to consider the evolution of the Riemann invariants with 'characteristic velocities' w_i, that is, according to the system

$$\frac{\partial \lambda_1}{\partial \tau} + w_1(\lambda_1, \lambda_2)\frac{\partial \lambda_1}{\partial x} = 0, \qquad \frac{\partial \lambda_2}{\partial \tau} + w_2(\lambda_1, \lambda_2)\frac{\partial \lambda_2}{\partial x} = 0. \tag{1.182}$$

It may easily be verified that the conditions (1.181) mean that the two flows (1.175) and (1.182) commute with each other, i.e., $\lambda_{1,\tau t} = \lambda_{1,t\tau}$, $\lambda_{2,\tau t} = \lambda_{2,t\tau}$. They say that Eqs. (1.182) represent a symmetry of Eqs. (1.175), because a solution of Eq. (1.175) transforms into another solution under the action of the flow (1.182). Thus, if one knows the symmetry of the system (1.175), its particular solution is given by Eqs. (1.180).

In the case of a polytropic gas flow, when v_1 and v_2 are given by Eq. (1.171), the system (1.181) takes the form

$$\frac{1}{w_1 - w_2}\frac{\partial w_1}{\partial \lambda_2} = \frac{n}{\lambda_1 - \lambda_2}, \qquad \frac{1}{w_1 - w_2}\frac{\partial w_2}{\partial \lambda_1} = \frac{n}{\lambda_1 - \lambda_2}, \tag{1.183}$$

where we have introduced the parameter

$$n = \frac{3-\gamma}{2(\gamma - 1)}. \tag{1.184}$$

As follows from Eqs. (1.183), we have $\partial w_1/\partial \lambda_2 = \partial w_2/\partial \lambda_1$, so that the functions w_1 and w_2 may be presented in the form

$$w_1 = \partial \chi/\partial \lambda_1, \qquad w_2 = \partial \chi/\partial \lambda_2. \tag{1.185}$$

Substitution of these expressions into any equation (1.183) yields the equation for the potential χ:

$$\frac{\partial^2 \chi}{\partial \lambda_1 \partial \lambda_2} - \frac{n}{\lambda_1 - \lambda_2}\left(\frac{\partial \chi}{\partial \lambda_1} - \frac{\partial \chi}{\partial \lambda_2}\right) = 0. \tag{1.186}$$

This equation is called the Euler-Poisson equation and its theory is developed quite well. Here we shall only give a few simple examples of its solutions. It is remarkable that for any integer value of n the solution of

Eq. (1.186) can be expressed in a closed form in terms of two arbitrary functions $F(\lambda_1)$ and $G(\lambda_2)$ and their derivatives.

If $n = 0$ (that is, $\gamma = 3$), then the general solution of Eq. (1.186) is evident,

$$\chi(\lambda_1, \lambda_2) = F(\lambda_1) + G(\lambda_2), \quad (n = 0), \tag{1.187}$$

where $F(\lambda_1)$ and $G(\lambda_2)$ are arbitrary functions. The case $n = 1$ corresponds to a monatomic gas with $\gamma = 5/3$ and is described by the following solution of the Euler-Poisson equation:

$$\chi(\lambda_1, \lambda_2) = \frac{F(\lambda_1) + G(\lambda_2)}{\lambda_1 - \lambda_2}, \quad (n = 1); \tag{1.188}$$

and if $n = 2$ (which corresponds to a diatomic gas with $\gamma = 7/5$), then the solution has the form

$$\chi(\lambda_1, \lambda_2) = \frac{F(\lambda_1) + G(\lambda_2)}{(\lambda_1 - \lambda_2)^3} + \frac{F'(\lambda_1) - G'(\lambda_2)}{2(\lambda_1 - \lambda_2)^2}, \quad (n = 2). \tag{1.189}$$

When the potential $\chi(\lambda_1, \lambda_2)$ is found, the solution of Eqs. (1.172) is given by the expressions

$$\begin{aligned} x - \tfrac{1}{4}\left((1+\gamma)\lambda_1 + (3-\gamma)\lambda_2\right) t &= \partial\chi/\partial\lambda_1, \\ x - \tfrac{1}{4}\left((3-\gamma)\lambda_1 + (1+\gamma)\lambda_2\right) t &= \partial\chi/\partial\lambda_2. \end{aligned} \tag{1.190}$$

The functions F and G are to be determined from the initial and boundary conditions of the problem.

Note, that the matching of different regions of a solution (gas at rest or vacuum, simple waves, a general solution) can only occur along the characteristic curves. For example, in the dam problem (Sec. 1.3.1) the solution consists of three regions—gas at rest, the rarefaction wave described by the simple wave solution, and the region with the zero gas density (vacuum). Correspondingly, the boundaries between these regions (see Eqs. (1.95) and (1.99)) are the characteristics of the hydrodynamical equations. It is clear then that between the region described by the general solution with varying both Riemann invariants λ_1, λ_2, and the region of a steady flow, where $\rho = \text{const}$, $u = \text{const}$ and, hence, both Riemann invariants are also constant, $\lambda_1 = \text{const}$, $\lambda_2 = \text{const}$, there must exist the region described by a simple wave solution with only one constant Riemann invariant.

Let us illustrate the method of Riemann invariants by the example of gas evolution after removing the plane walls of the container in the form of a slab. So, we suppose that a monatomic gas ($\gamma = 5/3$) is confined by plane walls located at $x = \pm a$, so that the initial density is equal to

$$\rho(x,0) = \begin{cases} \rho_0, & |x| \le a, \\ 0, & x > a, \end{cases} \tag{1.191}$$

and the initial flow velocity is everywhere equal to zero,

$$u(x,0) = 0. \tag{1.192}$$

At the moment $t = 0$ the container's walls are removed and the gas starts its expansion into an empty space. As we know from Sec. 1.3.1, at first two rarefaction waves appear with centres at the points $x = \pm a$. The outside edges of the non-stationary regions expand into an empty space with velocities $u_\pm = \pm 3c_0$, and the inside edges move to the centre of the container with the sound velocity $\mp c_0$, where $c_0 = \sqrt{\gamma p_0/\rho_0}$. These two regions are described by the formulas,

$$\begin{aligned} &\lambda_1 = \mathrm{const} = 3c_0, \quad x - (c_0 + \tfrac{2}{3}\lambda_2)t = a, \quad a - c_0 t \le x \le a + 3c_0 t, \\ &\lambda_2 = \mathrm{const} = -3c_0, \quad x - (\tfrac{2}{3}\lambda_1 - c_0)t = -a, \quad -a - 3c_0 t \le x \le -a + c_0 t, \end{aligned} \tag{1.193}$$

and this solution of Eqs. (1.172) is valid up to the moment $t_{col} = a/c_0$ of collision of the two rarefaction waves in the centre of the container. After the collision moment the region arises described by the general solution

$$x - \tfrac{1}{3}(2\lambda_1 + \lambda_2)t = \partial\chi/\partial\lambda_1, \quad x - \tfrac{1}{3}(\lambda_1 + 2\lambda_2)t = \partial\chi/\partial\lambda_2, \tag{1.194}$$

where χ is given by Eq. (1.188) and the functions $F(\lambda_1)$ and $G(\lambda_2)$ must satisfy the following boundary conditions. Due to the symmetry of the problem, the flow velocity u must vanish at $x = 0$ for all values of time $t > t_{col}$. By Eqs. (1.170) and (1.194) this condition means that

$$(\partial\chi/\partial\lambda_1 + \partial\chi/\partial\lambda_2)_{\lambda_2 = -\lambda_1} = 0. \tag{1.195}$$

Besides that, the general solution (1.194) matches with the simple wave regions along the characteristics (1.193). Due to the symmetry, it is sufficient to consider only one characteristic curve along the boundary, say, with the region where $\lambda_1 = 3c_0 = \mathrm{const}$. Then we have the condition

$$\partial\chi/\partial\lambda_2|_{\lambda_1 = 3c_0} = x - \left(c_0 + \tfrac{2}{3}\lambda_2\right)t = a. \tag{1.196}$$

After substitution of Eq. (1.188) into (1.195), we find that $G'(-\lambda) = -F'(\lambda)$ and, hence, $G(\lambda) = F(\lambda)$, where $F(\lambda)$ is an even function of λ and the integration constant is included into a still arbitrary function F. The second boundary condition (1.196) gives

$$(F(3c_0) + F(\lambda))/(3c_0 - \lambda) = a\lambda + C,$$

where C is the integration constant. Since F must be an even function, we have $C = 3c_0 a$ and, hence,

$$F(\lambda) = a[(3c_0)^2 - \lambda^2]. \tag{1.197}$$

Thus, the potential χ is given by

$$\chi(\lambda_1, \lambda_2) = a\frac{2(3c_0)^2 - \lambda_1^2 - \lambda_2^2}{\lambda_1 - \lambda_2}, \tag{1.198}$$

and Eqs. (1.194) yield the solution

$$t = -\frac{3}{\lambda_1 - \lambda_2}\left(\frac{\partial\chi}{\partial\lambda_1} - \frac{\partial\chi}{\partial\lambda_2}\right) = 12a \cdot \frac{9c_0^2 - \lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^3}, \tag{1.199}$$

$$x = \frac{\lambda_1 + \lambda_2}{2}t + \frac{\partial\chi}{\partial\lambda_1} + \frac{\partial\chi}{\partial\lambda_2} = \frac{a(\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2}\left[6 \cdot \frac{9c_0^2 - \lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2} - 1\right]. \tag{1.200}$$

These formulas determine in an implicit form the dependence of the Riemann invariants λ_1 and λ_2 on the coordinates x and t. With the use of Eqs. (1.170) we can express t and x in terms of the physical variables u and c:

$$t = \frac{a}{2c}\left(\frac{c_0^2 - u^2/9}{c^2} + 1\right), \quad x = \frac{au}{2c}\left(\frac{c_0^2 - u^2/9}{c^2} + \frac{1}{3}\right). \tag{1.201}$$

Thus, we have obtained the general solution describing the region between two rarefaction waves (see Fig. 1.13). At $\lambda_1 = 3c_0$ Eqs. (1.199,1.200) become

$$t = \frac{36c_0 a}{(3c_0 - \lambda_2)^2}, \quad x = \frac{a\,(3c_0 + \lambda_2)\,(15c_0 + \lambda_2)}{(3c_0 - \lambda_2)^2}$$

and they determine parametrically the characteristic curve along which the general solution matches with the simple wave. Elimination of λ_2 gives the equation for this characteristic in the explicit form:

$$x = 3c_0 t\left(1 - \sqrt{a/c_0 t}\right)\left(1 - \sqrt{a/9c_0 t}\right) \tag{1.202}$$

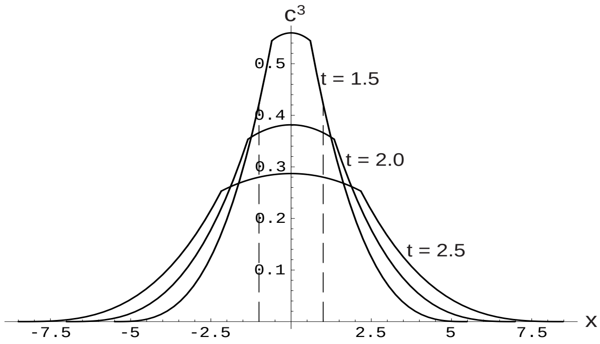

Fig. 1.13 Density profiles at different moments of time. Dashed lines show the initial position of the container's walls.

At asymptotically large values of time $t \gg a/c_0$, when $c \ll c_0$, the simple wave region between the characteristic (1.202) and $x = 3c_0 t$ becomes relatively small, so that almost all flow is described by the approximate formulas resulting from Eqs. (1.201)

$$u \simeq x/t, \quad \rho \propto c^3 \simeq (a/2t)\left(c_0^2 - u^2/9\right). \tag{1.203}$$

Since the density ρ in this solution vanishes at the characteristics $u = x/t = \pm 3c_0$, we may neglect the simple wave regions and suppose that the general solution has direct contact with vacuum. From Eqs. (1.170) and (1.203) we find the Riemann invariants at asymptotically large $t \gg a/c_0$:

$$\lambda_{1,2} = x/t \pm 3\left[(a/2t)\left(c_0^2 - x^2/9t^2\right)\right]^{1/3}, \quad t \gg a/c_0. \tag{1.204}$$

The equations for characteristics

$$dx/dt = v_{1,2} = x/t \pm \left[(a/2t)\left(c_0^2 - x^2/9t^2\right)\right]^{1/3} \tag{1.205}$$

can be integrated (see Exercise 1.3) and permit us to draw the characteristic curves in the (x, t) plane; see Fig. 1.14. In this limit the boundaries between the gas and vacuum become the straight lines $x = \pm 3c_0 t$ (see Eq. (1.202)), which are the envelopes of the characteristics of the general solution.

Thus, we see that the hodograph method enables us to give a detailed

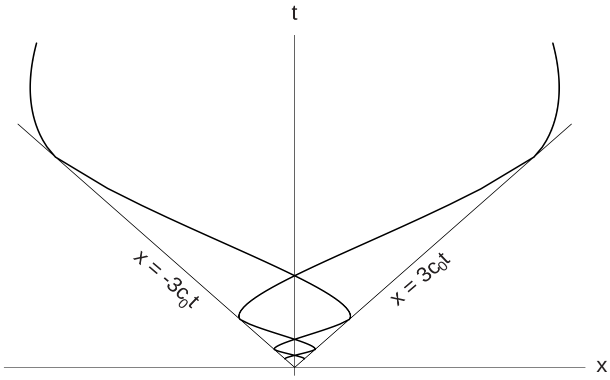

Fig. 1.14 Characteristics of the general solution at asymptotically large values of time.

treatment of solutions of the hydrodynamical equations even for the problems with varying both Riemann invariants.

1.4 Nonlinearity and viscosity: Burgers equation

1.4.1 *Derivation of the Burgers equation*

Examples of the preceding section show that the nonlinearity of hydrodynamical equations influences drastically on the evolution of wave pulses. The most essential feature is the breaking of wave followed by formation of the multi-valued region of the solution. Although in the case of 'dust' flow some meaning can be given to this solution, in the cases of real gases or compressible fluids the multi-valued regions become meaningless because of friction between the components with different flow velocities, so that at each point the fluid can have only one value of the flow velocity. In one-dimensional flow the effects of viscosity may be presented as an additional 'pressure' proportional to the gradient of velocity u and having the opposite sign. In other words, the Euler equation (1.85) is replaced by the so-called Navier-Stokes equation

$$u_t + uu_x = -(1/\rho)\,(p - \eta u_x)_x\,,$$

where η is a viscosity coefficient which may be considered as an empirical constant characterizing the properties of the gas. The continuity equation,

$$\rho_t + (u\rho)_x = 0,$$

expressing the conservation of mass, holds for a viscous fluid on the same footing as for an inviscid one. It is supposed that all corrections to the equation of state due to viscosity are taken into account by the additional term $-\eta\partial u/\partial x$ in the equation of motion, and, hence, in this approximation we may use the previous relation of the pressure with the density. We shall take it as for a polytropic gas $p = A\rho^\gamma$. As a result, we arrive at the system

$$u_t + uu_x + (c^2/\rho)\rho_x - (\eta/\rho)u_{xx} = 0, \tag{1.206}$$

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{1.207}$$

where $c^2 = dp/d\rho = A\gamma\rho^{\gamma-1}$ is a squared local sound velocity.

At first, we shall consider how viscosity influences on the propagation of a linear sound, that is, of a small amplitude wave

$$\rho = \rho_0 + \rho' \exp[i(kx - \omega t)], \quad u = u' \exp[i(kx - \omega t)],$$

where $\rho' \ll \rho_0$. Substitution of these expressions into Eqs. (1.206,1.207) followed by their linearization with respect to small amplitudes ρ' and u' leads to a homogeneous algebraic system whose solvability condition provides the dispersion relation for the sound waves. Assuming that viscosity is small ($\eta^2k^2 \ll \rho_0 c^2$), what is true for long enough wavelengths, we find

$$\omega = c_0 k - i\eta k^2/2\rho_0, \quad \eta^2 k^2/\rho_0 c^2 \ll 1. \tag{1.208}$$

The real part of this relation coincides with the dispersion relation for the sound waves (see Sec. 1.1.1), whereas the imaginary part leads to an exponential decay of the wave amplitude according to the law

$$\exp(-i\omega t) = \exp\left(-(\eta k^2/2\rho)t\right)\exp(-ic_0 kt).$$

In the long wave limit the damping of the wave may be very slow.

Now let us consider the influence of viscosity on the evolution of a wave pulse with small but finite amplitude. As we know, the propagation of a nonlinear wave pulse in the positive x direction through a medium without viscosity is described by the Hopf-type equation (1.128) which leads to the wave-breaking phenomenon. We are interested in the situation when the influence of viscosity during the evolution by the moment of the wave breaking is of the same order of magnitude as the influence of nonlinearity, that is, the time scale of the evolution is adjusted to the scales of the amplitude and its spatial gradient. To estimate these scales, let us introduce a small parameter ε characterizing the wave pulse amplitude, and suppose that the slow evolution of the pulse occurs at the time scale $t \sim \varepsilon^{-a}$, if the gradient is characterized by the space scale $x \sim \varepsilon^{-b}$, where both scaling indices a and b are positive, $a, b > 0$. The condition that the evolutional, nonlinear, and viscous terms in Eqs. (1.206) have the same order of magnitude, i.e.,

$$u_t \sim uu_x \sim u_{xx},$$

permits us to determine the scaling indices:

$$\epsilon \cdot \epsilon^a \sim \epsilon^2 \cdot \epsilon^b \sim \epsilon \cdot \epsilon^{2b},$$

whence, $a = 2$, $\quad b = 1$. Thus, the natural 'slow' variables characterizing the pulse evolution because of the nonlinearity and viscosity effects are

$$\tau = \epsilon^2 t, \quad \xi = \epsilon(x - c_0 t), \tag{1.209}$$

where we have taken into account that in the first approximation the pulse moves with the sound velocity c_0 (see Eq. (1.127)). In these new variables Eqs. (1.206,1.207) take the form

$$\begin{aligned} &\epsilon u_\tau + (u - c_0)u_\xi + (c^2/\rho)\rho_\xi - \epsilon(\eta/\rho)u_{\xi\xi} = 0, \\ &\epsilon\rho_\tau + (u - c_0)\rho_\xi + \rho u_\xi = 0. \end{aligned} \tag{1.210}$$

Now let us expand the variables u and ρ into the Taylor series in powers of a small parameter ε,

$$u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots, \quad \rho = \rho_0 + \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)} + \dots, \tag{1.211}$$

so that

$$c^2/\rho = (c_0^2/\rho_0)\left[1 + \varepsilon(\gamma - 2)(\rho^{(1)}/\rho_0) + \dots\right].$$

Substitution of these series expansions into Eqs. (1.210) gives in the first order in ε the relation $-c_0\rho^{(1)}_\xi + \rho_0 u^{(1)}_\xi = 0$, that is

$$\rho^{(1)} = (\rho_0/c_0)\, u^{(1)}, \tag{1.212}$$

where we took into account that the disturbances of the density and the flow velocity go to zero at $\xi \to \pm\infty$. The evolution of the pulse form $u^{(1)}$ is governed by the terms of the second order in ε in Eqs. (1.210):

$$\begin{aligned} u_\tau^{(1)} + u^{(1)}u_\xi^{(1)} &+ (c_0^2/\rho_0^2)(\gamma-2)\rho^{(1)}\rho_\xi^{(1)} \\ &- (\eta/\rho_0)u_{\xi\xi}^{(1)} - c_0u_\xi^{(2)} + (c_0^2/\rho_0)\rho_\xi^{(2)} = 0, \\ \rho_\tau^{(1)} + u^{(1)}\rho_\xi^{(1)} &+ \rho^{(1)}u_\xi^{(1)} - c_0\rho_\xi^{(2)} + \rho_0u_\xi^{(2)} = 0. \end{aligned}$$

Elimination of $\rho^{(1)}$ with the help of Eq. (1.212) gives

$$\begin{aligned} &u_\tau^{(1)} + (\gamma-1)u^{(1)}u_\xi^{(1)} - (\eta/\rho_0)u_{\xi\xi}^{(1)} - c_0u_\xi^{(2)} + (c_0^2/\rho_0)\rho_\xi^{(2)} = 0, \\ &u_\tau^{(1)} + 2u^{(1)}u_\xi^{(1)} - (c_0^2/\rho_0)\rho_\xi^{(2)} + c_0u_\xi^{(2)} = 0. \end{aligned}$$

Addition of these two equations enables us to eliminate $u^{(2)}$ and $\rho^{(2)}$ and to obtain the equation describing the evolution of $u^{(1)}(\xi,\tau)$:

$$u_\tau^{(1)} + \tfrac{1}{2}(\gamma+1)u^{(1)}u_\xi^{(1)} - (\eta/2\rho_0)u_{\xi\xi}^{(1)} = 0. \tag{1.213}$$

Now we may return to the usual space (x) and time (t) coordinates and identify $\varepsilon u^{(1)}$ with u. As a result we arrive at the equation

$$u_t + \left(c_0 + \tfrac{1}{2}(\gamma+1)u\right)u_x = (\eta/2\rho_0)u_{xx}. \tag{1.214}$$

If we neglect viscosity ($\eta = 0$), we return to Eq. (1.128) describing the evolution of a simple wave with steepening and wave-breaking phenomena. If we neglect nonlinearity, then the linear equation

$$u_t + c_0u_x = (\eta/2\rho_0)\,u_{xx}$$

reproduces the long wavelength limit (1.208) of the dispersion relation. Equation (1.214) is called the Burgers equation. For further investigation it is convenient to transform it to a canonical form by transition to the frame of reference moving with the sound velocity c_0 and introduction of a new variable $\tilde{u}$ according to $u = (2/(\gamma+1))\tilde{u}$, that is, $\tilde{u}_t + \tilde{u}\tilde{u}_{x'} = \mu\tilde{u}_{x'x'}$, where

$$\mu = \eta/2\rho_0 \tag{1.215}$$

After dropping tilde in $\tilde{u}$ and prime in x', we arrive at the final form of the Burgers equation

$$u_t + uu_x = \mu u_{xx}. \tag{1.216}$$

As it was mentioned above, at $\mu = 0$ the Burgers equation becomes the Hopf equation. But this does not mean that solutions of the Burgers equation tend in the limit $\mu \to 0$ to solutions of the Hopf equation, because the small parameter μ enters into the Burgers equation as a coefficient before the highest derivative of the unknown function u. Therefore, investigation of the behaviour of solutions at $\mu \to 0$ is an important problem whose solution will clarify how viscosity influences on the evolution of the pulse.

1.4.2 *Formation of a shock wave*

Let a profile of the pulse have at the initial moment the form

$$u(x, 0) = u_0(x). \tag{1.217}$$

We want to calculate its evolution governed by the Burgers equation (1.216). The most remarkable fact about the Burgers equation is that by means of the so-called Cole-Hopf substitution

$$u = -2\mu\psi_x/\psi \tag{1.218}$$

it is reduced to the linear heat conduction equation

$$\psi_t = \mu\psi_{xx}. \tag{1.219}$$

This relationship of the two equations is expressed most evidently by an easily verified identity

$$u_t + uu_x - \mu u_{xx} = -2\mu\partial_x \left[\left(\psi_t - \mu\psi_{xx} \right) / \psi \right].$$

It is clear that if the function $\psi(x, t)$ satisfies Eq. (1.219), then the function $u(x, t)$ calculated according to Eq. (1.218) satisfies the Burgers equation (1.216). Thus, the posed problem can be solved in the following way—by means of Eq. (1.218) we find the initial function $\psi(x, 0)$ corresponding to $u_0(x)$ (see Eq. (1.217)), then we calculate the evolution of $\psi(x, t)$ according to Eq. (1.219), and finally we return to $u(x, t)$ by means of Eq. (1.218).

A general solution of the Cauchy problem for the heat equation (1.219) is well known and can be found by the classical Fourier method. Let us represent a solution of Eq. (1.219) in the form

$$\psi(x, t) = \int A(k) \exp[i(kx + \mu k^2 t)] \frac{dk}{2\pi}, \tag{1.220}$$

where each Fourier harmonic satisfies Eq. (1.219). The function $A(k)$ is determined by the initial distribution $\psi(x,0)$:

$$\psi(x,0) = \int A(k)e^{ikx}\frac{dk}{2\pi}, \quad A(k) = \int \psi(x,0)e^{-ikx}dx.$$

Substitution of the last expression into Eq. (1.220) and integration over k yield the solution of the Cauchy problem

$$\psi(x,t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{\infty}\psi(x',0)\exp\left[-\frac{(x-x')^2}{4\mu t}\right]dx'. \tag{1.221}$$

From Eqs. (1.217,1.218) we have

$$\psi(x,0) = \exp\left(-\frac{1}{2\mu}\int_0^x u_0(\xi)d\xi\right),$$

(where the value of the integration constant is inessential); hence,

$$\psi(x,t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\mu}\left(\int_0^{x'}u_0(\xi)d\xi + \frac{(x-x')^2}{2t}\right)\right]dx',$$

and by Eq. (1.218) we obtain

$$u(x,t) = \frac{\int_{-\infty}^{\infty}\frac{x-x'}{t}\exp\left[-\frac{1}{2\mu}G(x';x,t)\right]dx'}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\mu}G(x';x,t)\right]dx'}, \tag{1.222}$$

where

$$G(x';x,t) = \int_0^{x'}u_0(\xi)d\xi + \frac{(x-x')^2}{2t}. \tag{1.223}$$

Thus, we have obtained the solution of the Cauchy problem for the Burgers equation in a closed analytical form.

Let us consider the behaviour of this solution in the limit $\mu \to 0$. Since μ appears in a denominator of the exponent in Eq. (1.222), it is clear that main contributions into integrals are made by vicinities of the points where the function $G(x')$ has minimum values. These points are the roots of the equation

$$\partial G/\partial x' = u_0(x') - (x-x')/t = 0. \tag{1.224}$$

Suppose at first, that this equation has only one solution $x' = \bar{x}$. Then the integrals can be calculated for small μ by means of an asymptotic formula

$$\int F(x') \exp\left(-\frac{G(x')}{2\mu}\right) dx' \cong F(\bar{x})\sqrt{\frac{4\pi\mu}{|G''(\bar{x})|}} \exp\left(-\frac{G(\bar{x})}{2\mu}\right). \tag{1.225}$$

The less we take value of μ, the more accurate result this formula gives. Thus, at $\mu \to 0$ and under condition that there is only one critical point $\bar{x}$, the approximate solution of the Burgers equation has the form

$$u(x,t) \simeq (x - \bar{x})/t = u_0(\bar{x}). \tag{1.226}$$

It coincides exactly with the solution (1.105) of the Hopf equation, that is, the critical point (1.224) corresponds to the characteristic curve of the Hopf equation.

But we know that after the wave-breaking moment the characteristics of the Hopf equation begin to intersect, so our supposition about existence of only one critical point fails. As was shown in Sec. 1.3.2, for a typical initial pulse there are three characteristics which intersect at each point of the multi-valued region, that is, Eq. (1.224) must have three solutions $x' = \bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ for x and t lying inside the caustic curve; see Fig. 1.7. Therefore, we have to determine which critical point from the set $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ gives the most essential contribution into the integrals of Eq. (1.222). According to Eq. (1.225), the value of each integral is determined by the corresponding value of $G_i \equiv G(\bar{x}_i)$ at $x' = \bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$:

$$G_i(x,t) = \int_0^{\bar{x}_i} u_0(\xi) d\xi + \frac{(x - \bar{x}_i)^2}{2t}, \quad i = 1,2,3, \tag{1.227}$$

$\bar{x}_i$ being the roots of the equation

$$u_0(\bar{x}_i) = (x - \bar{x}_i)/t, \quad i = 1,2,3. \tag{1.228}$$

Since $\bar{x}_i$ are functions of x and t defined implicitly by Eq. (1.228), each function G_i depends also on x and t. In Fig. 1.15 the plots of the functions G_1, G_2 G_3 against x are shown at $t = 2$ for the initial pulse form $u_0(x) = 1/(1 + x^2)$. As we see, for $x > x_s$ the function G_1 is less then G_2 and G_3. Hence, in this case just the point $\bar{x}_1$ makes the main contribution into the solution (1.222) in the limit of small μ, and one may neglect contributions of the other characteristics. Thus, at $x > x_s$ the solution of the Burgers

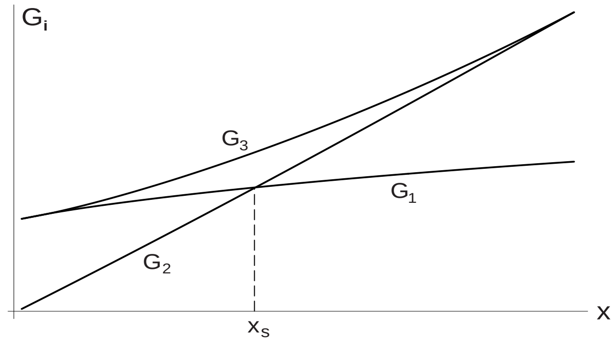

Fig. 1.15 The dependence of the functions G_1, G_2, G_3 defined by Eq.(1.227) on x at $t = 2$ for the initial distribution $u_0(x) = 1/(1 + x^2)$.

equation is presented with a good accuracy by the formula

$$u(x, t) \simeq (x - \bar{x}_1)/t = u_0(\bar{x}_1), \tag{1.229}$$

that is, we choose the lower branch between three branches of the multi-valued solution. Similarly, for $x < x_s$ we have $G_2 < G_1$, G_3, and the solution of the Burgers equation is approximated by the upper branch

$$u(x, t) \simeq (x - \bar{x}_2)/t = u_0(\bar{x}_2) \tag{1.230}$$

of the three-valued solution of the Hopf equation. Thus, we see that the solution (1.222) of the Burgers equation almost everywhere, except the vicinity of the point $x = x_s$, coincides actually with either upper or lower branch of the three-valued solution of the Hopf equation with zero viscosity. In vicinity of the point $x = x_s$ the functions G_1 and G_2 are close to each other and it is impossible to distinguish their contributions into the integrals in Eq. (1.222). Here the solution has a fast transition from Eq. (1.229) to Eq. (1.230) which is called the shock wave. The width of the transition region tends to zero as $\mu \to 0$, so that in this limit the solution of the Burgers equation is presented by the two branches of the solution of the Hopf equation joined by a jump-like transition at $x = x_s$. The condition

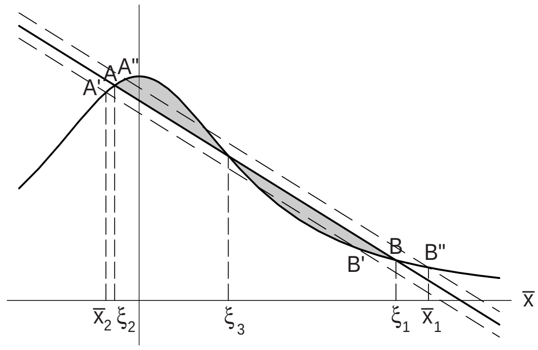

Fig. 1.16 Determination of the shock wave position at the moment t from the initial velocity distribution: The straight line AB with the slope $-1/t$ is chosen so that the filled areas are equal to each other. Then after time t the segment AB transforms into a shock wave discontinuity connecting the single-valued parts of the solution of the Hopf equation.

$G_1 = G_2$, that is,

$$\int_0^{\xi_1} u_0(\xi)d\xi + \frac{(x-\xi_1)^2}{2t} = \int_0^{\xi_2} u_0(\xi)d\xi + \frac{(x-\xi_2)^2}{2t}, \tag{1.231}$$

where ξ_1 is the minimum value of the root $\bar{x}_1$ corresponding to the lower branch (1.229) and ξ_2 is the maximum value of root $\bar{x}_2$ corresponding to the upper branch (1.230), defines the position of the shock wave. Indeed, taking into account Eqs. (1.229,1.230), we rewrite Eq. (1.231) in the form

$$\tfrac{1}{2}\left[u_0(\xi_1) + u_0(\xi_2)\right](\xi_1 - \xi_2) = \int_{\xi_2}^{\xi_1} u_0(\xi)d\xi. \tag{1.232}$$

Formula (1.232) has a vivid geometric interpretation. The values ξ_i correspond to the points of intersection of the initial profile $u_0(\bar{x})$ with straight lines $(x-\bar{x})/t$ having the slope $-1/t$ (see Fig. 1.16). The condition (1.232) means that the straight line must be drawn so that areas of two trapezia $\xi_1\xi_2AB$, one with the straight line AB and another with the curved line AB, must be equal to each other, or, in other word, the filled areas must be equal to each other. After time t the straight segment AB transforms into a vertical segment which has a meaning of a shock wave discontinuity in the approximate solution of the Burgers equation (see Fig. 1.17). In the

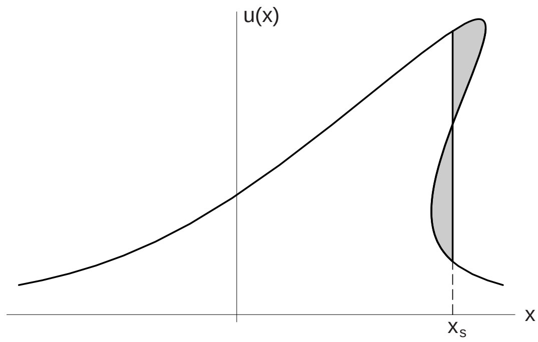

Fig. 1.17 The shock wave solution of the Burgers equation with the initial velocity profile $u_0(x) = 1/(1 + x^2)$.

corresponding to the same moment t (i.e., with the same slope $-1/t$) segment $A'B'$ only the point $\bar{x}_2 < \xi_2$ gives contribution into the solution of the Burgers equation, and in the segment $A''B''$ only the point $\bar{x}_1 > \xi_1$ does the same. Therefore, the values $u_0(\bar{x}_1)$ and $u_0(\bar{x}_2)$ are carried along the characteristics (1.229) and (1.230), correspondingly. Recall that the graph in Fig. 1.17 is obtained from the graph in Fig. 1.16 by translation of each its point to the right on the distance $u_0(\bar{x})t$ proportional to the ordinate of the initial graph. This transformation conserves as straight lines so areas between the curves. Thus, we come to the following geometrical method of finding the approximate solution of the Burgers equation. At first we draw a multi-valued solution of the Hopf equation and then draw a vertical segment so that it makes equal areas with the parts of the solution in this multi-valued region.

Let us calculate the velocity of propagation of the shock wave. Taking the limit $\bar{x}_1 \to \xi_1 + 0$ of Eq. (1.229) and the limit $\bar{x}_2 \to \xi_2 - 0$ of Eq. (1.230), we find the equations

$$x_s = \xi_1 + u_0(\xi_1)t, \quad x_s = \xi_2 + u_0(\xi_2)t$$

for the coordinate x_s of the shock wave. They give

$$t = -\frac{\xi_1 - \xi_2}{u_0(\xi_1) - u_0(\xi_2)} \tag{1.233}$$

and

$$\dot{x}_s = [1 + tu_0'(\xi_1)]\,\dot{\xi}_1 + u_0(\xi_1), \quad \dot{x}_s = [1 + tu_0'(\xi_2)]\,\dot{\xi}_2 + u_0(\xi_2),$$

where dot stands for the time derivative. One half of the sum of the last expressions leads after simple transformation with account of Eq. (1.233) to

$$\begin{aligned}&\dot{x}_s = V_s = \tfrac{1}{2}\left(u_0(\xi_1) + u_0(\xi_2)\right) + \tfrac{1}{u_0(\xi_1)-u_0(\xi_2)}\Big\{u_0(\xi_1)\dot{\xi}_1 - u_0(\xi_2)\dot{\xi}_2 \\ &-\tfrac{1}{2}\left[\left(u_0(\xi_1) + u_0(\xi_2)\right)\left(\dot{\xi}_1 - \dot{\xi}_2\right) + \left(u_0'(\xi_1)\dot{\xi}_1 + u_0'(\xi_2)\dot{\xi}_2\right)(\xi_1 - \xi_2)\right]\Big\}.\end{aligned}$$

The expression in curly braces is the time derivative of Eq. (1.232) and, hence, equals to zero. Thus, the velocity of the shock wave propagation is equal to

$$V_s = \tfrac{1}{2}\left(u_0(\xi_1) + u_0(\xi_2)\right). \tag{1.234}$$

This formula has very important physical interpretation. To make it clear, notice that the Hopf equation $u_t + uu_x = 0$ may be written as a conservation law

$$u_t + (u^2/2)_x = 0. \tag{1.235}$$

Let us write it in an integral form supposing that it holds even for solutions with a shock wave discontinuity,

$$\begin{aligned}\tfrac{1}{2}u^2(x_2,t) - \tfrac{1}{2}u^2(x_1,t) &= \frac{d}{dt}\int_{x_2}^{x_s} u(x,t)dx + \frac{d}{dt}\int_{x_s}^{x_1} u(x,t)dx \\ &= u_0(\xi_2)V_s - u_0(\xi_1)V_s + \int_{x_2}^{x_s} u_t(x,t)dx + \int_{x_s}^{x_1} u_t(x,t)dx\end{aligned}$$

where we have taken into account that $u(x_s + 0) = u_0(\xi_1)$, $u(x_s - 0) = u_0(\xi_2)$. Taking the limits $x_2 \to x_s - 0$, $x_1 \to x_s + 0$, we make integrals equal to zero and obtain

$$\left(u_0(\xi_2) - u_0(\xi_1)\right)V_s = \tfrac{1}{2}\left(u_0^2(\xi_2) - u_0^2(\xi_1)\right), \tag{1.236}$$

which gives at once the relation (1.234). Equation (1.236) may be presented as a condition that the velocity V_s must be chosen so that the integral form of the conservation law (1.235) fulfills, that is,

$$V_s[u(x,t)] = [u^2(x,t)/2], \tag{1.237}$$

where square brackets denote jumps of the corresponding quantities at the shock wave discontinuity. This remark permits one to generalize the theory on intensive shock waves which are not described by the Burgers equation.

Our consideration of the shock wave phenomenon in the framework of the Burgers equation theory was based on the supposition that the transition region approximated by a jump-like discontinuity is sufficiently narrow and its width goes to zero in the limit of vanishing viscosity $\mu \to 0$. To clarify this point, let us solve the following simple problem. Suppose that the moving pulse is so long that one may neglect its evolution at the trailing edge and take

$$u \to \begin{cases} u_1 = \text{const}, & x \to +\infty, \\ u_2 = \text{const}, & x \to -\infty, \end{cases} \tag{1.238}$$

where $u_2 > u_1$. We shall look for the stationary profile moving with the velocity V,

$$u = u(x - Vt)$$

that is, $u(x,t)$ depends only on $\xi = x - Vt$. Then the Burgers equation becomes

$$-Vu_\xi + uu_\xi = \mu u_{\xi\xi}$$

with an evident first integral

$$\tfrac{1}{2}u^2 - Vu + C = \mu u_\xi. \tag{1.239}$$

The boundary conditions (1.238) permit us to express the velocity of propagation V and the integration constant C in terms of u_1 and u_2,

$$V = \tfrac{1}{2}(u_1 + u_2), \quad C = \tfrac{1}{2}u_1 u_2,$$

so that Eq. (1.239) becomes

$$(u - u_1)(u_2 - u) = -2\mu u_\xi.$$

Integration of this equation gives

$$u(x,t) = u_1 + \frac{u_2 - u_1}{1 + \exp\left[\frac{u_2 - u_1}{2\mu}(x - Vt)\right]}, \quad V = \tfrac{1}{2}(u_1 + u_2). \tag{1.240}$$

Note that the velocity V coincides with the shock wave velocity (1.234). As we see, the width of the transition region (the shock wave's width) is equal to

$$\kappa = 2\mu/(u_2 - u_1), \tag{1.241}$$

and indeed it tends to zero as $\mu \to 0$ at given values of u_1 and u_2.

Thus, the main result of this subsection is that the effects of nonlinearity and viscosity lead to formation of shock waves.

1.5 Nonlinearity and dispersion: Korteweg-de Vries equation

1.5.1 *Derivation of the Korteweg-de Vries equation*

In Secs. 1.1.2 and 1.2.1 we have considered linear waves on the surface of the shallow water and found that if the wavelength is much greater than the basin's depth h and the amplitude is much less than h, then such linear waves obey the dispersion relation (see Eq. (1.38))

$$\omega(k) = \sqrt{gh}\, k \left(1 - \tfrac{1}{6}(hk)^2\right). \tag{1.242}$$

The corresponding evolution equation for the wave amplitude $\zeta(x,t)$ of displacement of the surface from the horizontal plane $z = 0$ has the form

$$\zeta_t + \sqrt{gh}\left(\zeta_x + \tfrac{1}{6}h^2\zeta_{xxx}\right) = 0. \tag{1.243}$$

Now we want to consider such waves whose amplitude is still small ($|\zeta| \ll h$), but not so small that one can neglect the nonlinear effects. (This situation is usually expressed as 'amplitude is small but finite'.) One should distinguish here two cases: (1) the nonlinear effects dominate over dispersion, and (2) the nonlinear effects have the same order of magnitude as the dispersion effects. Equation (1.243) corresponds to the third possible case when the dispersion effects dominate over the nonlinear ones.

For derivation of the evolution equations, let us return to the basic equations for the surface waves formulated in Sec. 1.1.2. The flow of water is potential and its velocity may be represented as a gradient of the potential

$$\mathbf{v} = (v_{(x)}, 0, v_{(z)}) = (\varphi_x, 0, \varphi_z), \tag{1.244}$$

where subscripts in parenthesis denote vector's components and we assume that all the variables depend on only one horizontal coordinate x and on a vertical coordinate z. For incompressible fluid, the continuity equation simplifies to $\operatorname{div}\mathbf{v} = 0$, and, hence, $\varphi(x,z)$ satisfies the Laplace equation

$$\varphi_{xx} + \varphi_{zz} = 0, \qquad -h < z < \zeta(x,t), \tag{1.245}$$

in the volume of fluid between the bottom $z = -h$ and the surface $z = \zeta(x,t)$. It is evident that the vertical component of velocity must vanish at the bottom what gives one boundary condition,

$$\varphi_z = 0 \quad \text{at} \quad z = -h. \tag{1.246}$$

The 'fluid particles' on the surface move with the surface not intersecting it what gives another boundary condition (see Eq. (1.20))

$$\zeta_t = \phi_z - \zeta_x\phi_x \quad \text{at} \quad z = \zeta(x,t). \tag{1.247}$$

Since we have two unknown functions $\varphi(x,z,t)$ and $\zeta(x,t)$, we have to have one more boundary condition for their calculation. In the linear approximation it was Eq. (1.18) which must be modified here to the case of waves with finite amplitudes. With the use of the vector analysis formula

$$\nabla(\mathbf{v}\cdot\mathbf{v}) = 2(\mathbf{v}\nabla)\mathbf{v} + 2\mathbf{v}\times(\nabla\times\mathbf{v})$$

and taking into account that for the potential flow $\nabla\times\mathbf{v}\equiv 0$, we rewrite the Euler equation (1.10) in the form

$$\mathbf{v}_t + \tfrac{1}{2}\nabla(v^2) = -\nabla p/\rho + \mathbf{g}.$$

As $\mathbf{v} = \nabla\varphi$, this equation can be integrated once to give

$$\varphi_t + \tfrac{1}{2}(\nabla\varphi)^2 = -p/\rho - gz.$$

Neglecting the change of the atmospheric pressure p at the water surface during the wave propagation, we obtain the necessary boundary condition

$$\varphi_t + \tfrac{1}{2}\left(\varphi_x^2 + \varphi_z^2\right) + gz = 0 \quad \text{at} \quad z = \zeta(x,t). \tag{1.248}$$

If we drop here the nonlinear terms, we return to the linear boundary condition (1.18).

The system (1.245–1.248) comprise the full set of equations describing the surface waves without any restrictions on the wavelength or the basin's depth. It is convenient to introduce small parameters which control the approximations related to the shallow water and the long wavelength limits.

First of all, we introduce instead of $\zeta(x,t)$ the variable η according to

$$\zeta = \alpha h \eta \tag{1.249}$$

Due to the factor h the variable η is dimensionless and the parameter α controls the magnitude of ζ, i.e., the condition $|\zeta| \ll h$ takes the form

$$\alpha \ll 1. \tag{1.250}$$

Since the nonlinear effects are determined by the magnitude of the amplitude ζ, α may be considered as a parameter characterizing the nonlinear effects. Instead of the vertical coordinate z we introduce the variable Z according to

$$h + z = hZ. \tag{1.251}$$

Then the equation of the water surface $z = \zeta$ becomes

$$Z = 1 + \alpha h. \tag{1.252}$$

Now we have to choose an appropriate parameter along the horizontal x axis. As it was mentioned above, we are going to consider waves with the wavelength much longer than the depth h. Therefore, we define a new horizontal coordinate X according to

$$x = hX/\sqrt{\beta}. \tag{1.253}$$

It is again dimensionless and the long wavelength limit corresponds to

$$\beta \ll 1 \tag{1.254}$$

(a square root of β is introduced for convenience of writing the following formulas). Since linear waves propagate with the velocity $\sqrt{gh}$, we can find an appropriate time scale by dividing the horizontal scale $h/\sqrt{\beta}$ by $\sqrt{gh}$ to obtain $\sqrt{h/\beta g}$, that is,

$$t = \sqrt{h/\beta g}\, T, \tag{1.255}$$

T being a new dimensionless time variable. From the expression for the phase, $kx - \omega t = (kh/\sqrt{\beta})\, X - \omega\sqrt{h/\beta g}\, T = KX - \Omega T$, we find that the wavevector K and the frequency Ω in the new variables are related with those in the old ones by the relations

$$K = \left(h/\sqrt{\beta}\right) k, \quad \Omega = \sqrt{h/\beta g}\,\omega. \tag{1.256}$$

The dispersion relation (1.242) takes in the new variables the form

$$\Omega = K\left(1 - \tfrac{1}{6}\beta K^2\right). \tag{1.257}$$

Thus, the dispersion effects are controlled by the parameter β and they can be neglected in the limit $\beta \to 0$.

At last, if we demand that the linear boundary condition (1.18), $\varphi_t + g\zeta = 0$, transforms in new variables into $\Phi_T + \eta = 0$, then we get a new potential Φ defined by the formula

$$\varphi = \alpha h\sqrt{gh/\beta}\,\Phi. \tag{1.258}$$

Now let us rewrite all the equations and boundary conditions in the new dimensionless variables. The Laplace equation (1.245) transforms to

$$\beta\Phi_{XX} + \Phi_{ZZ} = 0; \tag{1.259}$$

the boundary condition (1.246) at the bottom to

$$\Phi_Z = 0 \quad \text{at} \quad Z = 0; \tag{1.260}$$

the kinematical boundary condition (1.247) at the surface to

$$(1/\beta)\Phi_Z = \eta_T + \alpha\eta_X\Phi_X \quad \text{at} \quad Z = 1 + \alpha\eta, \tag{1.261}$$

and the dynamic boundary condition (1.248) at the surface to

$$\Phi_T + \eta + \tfrac{1}{2}\alpha\left(\Phi_X^2 + (1/\beta)\Phi_Z^2\right) = 0 \quad \text{at} \quad Z = 1 + \alpha\eta. \tag{1.262}$$

Thus, we have arrived at a quite complicated system of equations for the dimensionless variables where the nonlinear effects are controlled by the parameter α and the dispersion effects by the parameter β.

Now let us turn to simplification of the above system using the smallness of parameters the α and β. At first, due to smallness of the coefficient β in Eq. (1.259), it is natural to represent Φ as a series expansion in the form

$$\Phi = \sum_{n=0}^{\infty} \frac{1}{n!}\Phi_n(X,T)Z^n. \tag{1.263}$$

Then Eq. (1.259) gives a recurrence relation for the functions Φ_n,

$$\Phi_n = -\beta\Phi_{n-2,XX}.$$

From the boundary condition (1.260) we obtain at once that $\Phi_1 = 0$, hence, all Φ_n with odd n are equal to zero. Denoting

$$\Phi_0 = F(X,T), \tag{1.264}$$

we obtain the expansion for Φ in powers of a small parameter β,

$$\Phi = F - \tfrac{1}{2}\beta F_{XX}Z^2 + \tfrac{1}{24}\beta^2 F_{XXXX}Z^4 - \dots, \tag{1.265}$$

where all the coefficients are expressed in terms of the only function $F(X,T)$ and its X derivatives.

We have to substitute the series (1.265) into the boundary conditions (1.261) and (1.262). As we are interested in the lowest order effects of nonlinearity and dispersion, we can keep after substitution only the terms up to the first order in α and β. As a result, we obtain

$$\eta_T + [F_X(1+\alpha\eta)]_X - \tfrac{1}{6}\beta F_{XXXX} = 0, \tag{1.266}$$

$$\eta + F_T + \tfrac{1}{2}\alpha F_X^2 - \tfrac{1}{2}\beta F_{XXT} = 0. \tag{1.267}$$

It is convenient to introduce a mean value of the horizontal velocity,

$$U = \int_0^1 \Phi_X dZ \cong F_X - \tfrac{1}{6}\beta F_{XXX}. \tag{1.268}$$

Then Eq. (1.266) can be rewritten as

$$\eta_T + [U(1+\alpha\eta)]_X = 0,$$

and Eq. (1.267) differentiated with respect to X transforms with the accepted accuracy to

$$U_T + \alpha U U_X + \eta_X - \tfrac{1}{3}\beta U_{XXT} = 0.$$

At last, if we define

$$\rho = 1 + \alpha\eta, \quad u = \alpha U, \tag{1.269}$$

then the above equations take the form

$$\begin{aligned} &\rho_T + (u\rho)_X = 0,\\ &u_T + u u_X + \rho_X - (\beta/3)u_{XXT} = 0, \end{aligned} \tag{1.270}$$

called the Boussinesq equations.

The system (1.266,1.267) may also be transformed in a different way. We can express η in terms of F by Eq. (1.267) and substitute the result into Eq. (1.266) to obtain

$$-F_{TT} + F_{XX} - 2\alpha F_X F_{XT} - \alpha F_{XX} F_T + \tfrac{1}{2}\beta F_{XXTT} - \tfrac{1}{6}\beta F_{XXXX} = 0.$$

In the linear approximation we obtain the wave equation $F_{TT} = F_{XX}$ which may be used for transformations of the correction terms of the next approximation, so that the dispersion correction becomes $(\beta/3)F_{XXXX}$. To transform the nonlinear terms, let us suppose that the wave propagates in the positive direction of the X axis, i.e., in the linear approximation we have $F_T = -F_X$, and this relation can be substituted into the nonlinear terms. Thus, we get

$$F_{TT} = F_{XX} + 3\alpha F_X F_{XX} + \tfrac{1}{3}\beta F_{XXXX}.$$

Differentiation of this equation with respect to X and replacement of F_X by $F_X \cong U + \frac{1}{6}\beta U_{XX} \cong U + \frac{1}{6}\beta U_{TT}$ yields

$$U_{TT} = U_{XX} + 3\alpha\left(U_X^2 + UU_{XX}\right) + \tfrac{1}{3}\beta U_{XXXX}$$

or

$$U_{TT} = \left(U + \tfrac{3}{2}\alpha U^2 + \tfrac{1}{3}\beta U_{XX}\right)_{XX}. \tag{1.271}$$

This equation is also called the Boussinesq equation.

Linearization of the system (1.270) near the stationary solution $u = 0$, $\rho = 1$ gives the system

$$\rho_T + u_X = 0, \quad u_T + \rho_X - \tfrac{1}{3}\beta u_{XXT} = 0,$$

describing the linear waves $\rho = \rho' \exp[i(KX - \Omega T)]$, $u = u' \exp[i(KX - \Omega T)]$ with the dispersion law

$$\Omega \cong \pm K\left(1 - \tfrac{1}{6}\beta K^2\right),$$

which coincides, as one should expect, with Eq. (1.257).

The opposite limiting case of a very long wavelength, when dispersion is negligibly small, is described by the system (1.270) with $\beta = 0$, that is,

$$\rho_T + (u\rho)_X = 0, \quad u_T + uu_X + \rho_X = 0, \tag{1.272}$$

which is called the shallow water equations. It has a familiar form of the compressible fluid flow equations. Comparison with Eqs. (1.89) shows that now the 'sound velocity' c is related with the 'density' ρ by the equation

$$c = \sqrt{\rho},$$

that is the shallow water hydrodynamics corresponds to the exponent

$$\gamma = 2 \tag{1.273}$$

in Eq. (1.94). The Riemann invariants (1.169) are equal to

$$\lambda_1 = u + 2\sqrt{\rho}, \quad \lambda_2 = u - 2\sqrt{\rho}; \tag{1.274}$$

the characteristic velocities (1.171) to

$$v_1(\lambda_1, \lambda_2) = \tfrac{3}{4}\lambda_1 + \tfrac{1}{4}\lambda_2, \quad v_2(\lambda_1, \lambda_2) = \tfrac{1}{4}\lambda_1 + \tfrac{3}{4}\lambda_2, \tag{1.275}$$

and, hence, the system (1.272) is reduced to the diagonal form

$$\frac{\partial \lambda_1}{\partial T} + v_1(\lambda_1, \lambda_2)\frac{\partial \lambda_1}{\partial X} = 0, \quad \frac{\partial \lambda_2}{\partial T} + v_2(\lambda_1, \lambda_2)\frac{\partial \lambda_2}{\partial X} = 0. \tag{1.276}$$

It can be reduced to the Euler-Poisson equation (1.186) with $n = 1/2$. In this case the general solution of the Euler-Poisson equation cannot be expressed as simply as it can be done for integer n (see, e.g., Eqs. (1.187–1.189)). However, simple wave solutions with one Riemann invariant constant can readily be investigated. As was shown in Sec. 1.3.3, the equation

$$u_T + \left(\pm 1 + \tfrac{3}{2}u\right) u_X = 0 \tag{1.277}$$

has the solution

$$X = \left(\pm 1 + \tfrac{3}{2}u\right) T + \bar{X}(u),$$

where $\bar{X}(u)$ is the function inverse to the initial distribution $u_0(X)$. Therefore, all results concerning the wave-breaking phenomenon hold for the shallow water waves. However, in vicinity of the wave-breaking point the gradient of the profile $u(X)$ becomes very large and we cannot drop in Eqs. (1.270) the term $-(\beta/3)u_{XXT}$ describing the dispersion effects. Thus, we arrive at the problem how dispersion effects influence on the wave behaviour near the wave-breaking point.

In the preceding section we solved an analogous problem for the case when viscosity was the main effect determining the wave behaviour near the wave-breaking point. The analysis was drastically simplified by reduction of the Navier-Stokes equations (1.206,1.207) to the Burgers equation (1.214). Let us make a similar simplification for the case of the Boussinesq system (1.270). At first, we assume that we consider only waves propagating in the positive direction of X axis. Then, in the frame of reference moving with the group velocity of linear waves, the wave evolves slowly due to small effects of nonlinearity and dispersion. We are interested in evolution of waves with the amplitude $u \sim \varepsilon$ when these two effects have the same order of magnitude.

To determine the scaling indices, we introduce the slow variables

$$\tau = \varepsilon^a, \qquad \xi = \varepsilon^b (X - T),$$

$$\frac{\partial}{\partial T} = \varepsilon^a \frac{\partial}{\partial \tau} - \varepsilon^b \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial X} = \varepsilon^b \frac{\partial}{\partial \xi},$$

so that the system (1.270) transforms to

$$\begin{aligned} \varepsilon^a \rho_\tau - \varepsilon^b \rho_\xi + \varepsilon^b u_\xi \rho + \varepsilon^b u \rho_\xi = 0, \\ \varepsilon^a u_\tau - \varepsilon^b u_\xi + \varepsilon^b u u_\xi + \varepsilon^b \rho_\xi - \tfrac{1}{3}\beta \left(\varepsilon^{2b+a} u_{\xi\xi\tau} - \varepsilon^{3b} u_{\xi\xi\xi}\right) = 0. \end{aligned}$$

As usual, we take expansions $\rho = 1 + \varepsilon \rho^{(1)} + \varepsilon^2 \rho^{(2)} + \ldots, \quad u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \ldots.$ with account of only the first approximation. Then the first equation takes the form

$$\varepsilon^{a+1} \rho_\tau^{(1)} - \varepsilon^{b+1} \rho_\xi^{(1)} + \varepsilon^{b+1} u_\xi^{(1)} + \varepsilon^{b+2} u^{(1)} \rho_\xi^{(1)} = 0$$

and assuming that $a > b$ we get the relation in the first approximation $\rho_\xi^{(1)} = u_\xi^{(1)}$. Then the second equation transforms in the main approximation to

$$\varepsilon^{a+1} u_\tau^{(1)} + \varepsilon^{b+2} u^{(1)} u_\xi^{(1)} - \tfrac{1}{3} \left(\varepsilon^{2b+a+1} u_{\xi\xi\tau}^{(1)} - \varepsilon^{3b+1} u_{\xi\xi\xi}^{(1)}\right) = 0,$$

hence $a + 1 = b + 2 = 3b + 1$, that is

$$a = \tfrac{3}{2}, \qquad b = \tfrac{1}{2}.$$

Therefore, the slow variables characterizing the wave evolution due to the nonlinearity and dispersion effects are

$$\tau = \varepsilon^{3/2} T, \quad \xi = \varepsilon^{1/2} (X - T). \tag{1.278}$$

In these variables the Boussinesq equations (1.270) take the form

$$\begin{aligned}&\varepsilon\rho_\tau + (u-1)\rho_\xi + \rho u_\xi = 0,\\ &\varepsilon u_\tau + (u-1)u_\xi + \rho_\xi + \tfrac{1}{3}\beta\varepsilon(u_{\xi\xi\xi} - \varepsilon u_{\xi\xi\tau}) = 0.\end{aligned} \tag{1.279}$$

Now we expand u and ρ in powers of the small parameter ε,

$$\rho = 1 + \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)} + \dots, \quad u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots.$$

Substitution of these expansions into Eqs. (1.279) gives at the first order in ε the relationship

$$\rho_\xi^{(1)} = u_\xi^{(1)}, \quad \text{so that} \quad \rho^{(1)} = u^{(1)}, \tag{1.280}$$

(the integration constant is equal to zero because at $\xi \to \pm\infty$ the flow velocity vanishes together with the surface displacement). The evolution of the wave profile is governed by the second order terms in ε:

$$\begin{aligned}&\rho_\tau^{(1)} + u^{(1)}\rho_\xi^{(1)} + \rho^{(1)}u_\xi^{(1)} - \rho_\xi^{(2)} + u_\xi^{(2)} = 0,\\ &u_\tau^{(1)} + u^{(1)}u_\xi^{(1)} + \tfrac{1}{3}\beta u_{\xi\xi\xi}^{(1)} + \rho_\xi^{(2)} - u_\xi^{(2)} = 0.\end{aligned}$$

Adding these two equations and taking into account Eq. (1.280), we obtain the evolution equation for $u^{(1)}$:

$$u_\tau^{(1)} + \tfrac{3}{2}u^{(1)}u_\xi^{(1)} + \tfrac{1}{6}\beta u_{\xi\xi\xi}^{(1)} = 0.$$

Returning to the variables X and T and denoting $\varepsilon u^{(1)} = u$, we arrive at the equation

$$u_T + \left(1 + \tfrac{3}{2}u\right)u_X + \tfrac{1}{6}\beta u_{XXX} = 0. \tag{1.281}$$

If we neglect dispersion, that is put $\beta = 0$, we obtain, as one should expect, the equation (1.277) for a simple wave propagating in the positive X direction.

Let us return to the physical coordinates x and t and the variable $\zeta(x,t)$ by means of the relations $u = \varepsilon u^{(1)} = \varepsilon\rho^{(1)} = \rho - 1 = \alpha\eta = \zeta/h$, $\partial/\partial T = (\sqrt{h/\beta g})\partial/\partial t$, $\partial/\partial X = (h/\sqrt{\beta})\partial/\partial x$, so that we obtain

$$\zeta_t + \sqrt{gh}\left(1 + \tfrac{3}{2}(\zeta/h)\right)\zeta_x + \tfrac{1}{6}h^2\sqrt{gh}\,\zeta_{xxx} = 0. \tag{1.282}$$

This is the famous Korteweg-de Vries (KdV) equation derived in 1895 in theoretical investigation of nonlinear waves propagating on the surface of a shallow water. In the linear limit of infinitesimal amplitude ζ it reproduces

Eq. (1.243). At small but finite values of the amplitude it describes in addition the nonlinear effects.

The KdV equation plays the fundamental role in the theory of nonlinear dispersive waves. Although it cannot be reduced to linear equation by a simple substitution, as it was possible in the case of the Burgers equation, the KdV equation has many wonderful mathematical properties which enable one to investigate its solutions quite effectively. To perform such investigations, it is convenient to transform the KdV equation to the canonical form. To this end, let us change the frame of reference and transform to the dimensionless variables in a usual way,

$$x' = (x - \sqrt{gh}\, t)/h, \quad t' = \sqrt{g/h}\, t, \quad \eta = \zeta/h.$$

Then we get $\eta_{t'} + \frac{3}{2}\eta\eta_{x'} + \frac{1}{6}\eta_{x'x'x'} = 0$. If one makes now the transformation $t'' = at'$, $x'' = bx'$, $\eta = c\tilde{u}$, then the constant parameters may be chosen so that the coefficients of the equation

$$a\tilde{u}_{t''} + \tfrac{3}{2}bc\tilde{u}\tilde{u}_{x''} + \tfrac{1}{6}b^3\tilde{u}_{x''x''x''} = 0$$

acquire the desired values. One of the standard choices is $a = \left(\frac{3}{4}\right)^2$, $b = \frac{3}{2}$, $c = \frac{3}{2}$, which gives the final form of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.283}$$

where primes and tilde are omitted.

In the next subsection we shall find some elementary solutions of this equation.

1.5.2 *Cnoidal wave and soliton*

Harmonic sinusoidal waves play the fundamental role in the linear theory. Although in the case of the nonlinear KdV equation (1.283) the general solution cannot be represented as a sum of harmonic waves, nevertheless particular solutions describing the progressive nonlinear periodic waves occur extremely useful, and we shall turn here to their study.

We look for the solution of Eq. (1.283) in the form

$$u = u(x - Vt), \tag{1.284}$$

where V is the propagation velocity. Introducing the variable

$$\xi = x - Vt, \tag{1.285}$$

we find that $u(\xi)$ satisfies the ordinary differential equation

$$u_{\xi\xi\xi} = Vu_\xi - 6uu_\xi$$

with an obvious first integral

$$u_{\xi\xi} = B + Vu - 3u^2,$$

B being the integration constant. After multiplication of this equation by u_ξ and integration we get

$$\tfrac{1}{2}u_\xi^2 = -A + Bu + \tfrac{1}{2}Vu^2 - u^3 = f(u), \tag{1.286}$$

where A is the second integration constant. This equation has real solutions, if the polynomial $f(u)$ in the right hand side has three real zeros which we shall denote as α, β, γ and assume that they are ordered according to $\alpha > \beta > \gamma$. It is clear that the real oscillating solution corresponds to motion of u between two zeros where $f(u) \geq 0$, that is, in the interval

$$\beta \leq u \leq \alpha. \tag{1.287}$$

The constants A, B, V can be expressed in terms of α, β, γ as follows:

$$A = -\alpha\beta\gamma, \quad -B = \alpha\beta + \beta\gamma + \gamma\alpha, \quad V = 2(\alpha + \beta + \gamma). \tag{1.288}$$

Then we obtain from Eq. (1.286) that

$$du/d\xi = \pm\sqrt{2} \cdot \sqrt{(\alpha - u)(u - \beta)(u - \gamma)}$$

and the periodic solution of the KdV equation is given implicitly by

$$\sqrt{2}\,\xi = \int_u^\alpha \frac{du'}{\sqrt{(\alpha - u')(u' - \beta)(u' - \gamma)}}, \tag{1.289}$$

where the third (additive with respect to ξ) integration constant is chosen so that $u(\xi)$ has the maximum value α at $\xi = 0$. The standard change of the integration variable,

$$u' = \alpha - (\alpha - \beta)\sin^2\varphi', \quad \sin\varphi' = \sqrt{(\alpha - u')/(\alpha - \beta)},$$

transforms the integral (1.289) to

$$\sqrt{2}\,\xi = \frac{2}{\sqrt{\alpha - \gamma}} \int_0^\varphi \frac{d\varphi'}{\sqrt{1 - m\sin^2\varphi'}} = \frac{2}{\sqrt{\alpha - \gamma}} F(\varphi, m), \tag{1.290}$$

where the parameter m is defined by

$$m = (\alpha - \beta)/(\alpha - \gamma), \tag{1.291}$$

and $F(\varphi, m)$ denotes the elliptic integral of the first kind (see Appendix A). The inverse to $F(\varphi, m)$ function is denoted as

$$\varphi = \mathrm{am}\left(\sqrt{(\alpha-\gamma)/2}\;\xi, m\right),$$

and its sine is called the Jacobi elliptic sine,

$$\begin{aligned}\sin \mathrm{am}\left(\sqrt{\alpha - \gamma/2}\,\xi, m\right) = \mathrm{sn}\left(\sqrt{(\alpha-\gamma)/2}\,\xi, m\right)\\ = \sin\varphi = \sqrt{(\alpha - u)/(\alpha-\beta)}.\end{aligned}$$

As a result of these transformations, our solution $u(\xi)$ of the KdV equation is represented in the form

$$u = \alpha - (\alpha - \beta)\,\mathrm{sn}^2\left(\sqrt{(\alpha-\gamma)/2}\,(x - Vt), m\right). \tag{1.292}$$

By virtue of the identity $\mathrm{sn}^2 z + \mathrm{cn}^2 z = 1$ this solution could be expressed in terms of the elliptic cosine function cn. Just this was done by Korteweg and de Vries in their celebrated paper, and by analogy with 'cosinusoidal' solution of linear equations they named the solution (1.292) as 'cnoidal wave'. The identity $m\,\mathrm{sn}^2(z, m) + \mathrm{dn}^2(z, m) = 1$ gives another representation of the solution

$$u(x, t) = (\alpha - \gamma)\,\mathrm{dn}^2\left(\sqrt{(\alpha-\gamma)/2}\,(x - Vt), m\right) + \gamma. \tag{1.293}$$

The properties of the cnoidal wave are determined by the three real parameters α, β, and γ. In particular, the propagation velocity V and the parameter m are given by Eqs. (1.288) and (1.291), correspondingly. The function $\mathrm{dn}(z, m)$ varies within the interval $\sqrt{1-m} \le \mathrm{dn}(z, m) \le 1$, so that the amplitude of the cnoidal wave u may be defined by

$$2a = u_{max} - u_{min} = \alpha - \beta. \tag{1.294}$$

Then Eq. (1.293) may be written in the form

$$u(x, t) = (2a/m)\,\mathrm{dn}^2\left(\sqrt{a/m}\,(x - Vt), m\right) + \gamma, \tag{1.295}$$

where the velocity V is also expressed in terms of the constants a, m, γ with the use of Eqs. (1.288), (1.291), (1.294):

$$\begin{aligned} &\alpha = 2a/m, \quad \beta = 2a\,(1/m - 1) + \gamma, \\ &V = 2(\alpha + \beta + \gamma) = 4a\,(2/m - 1) + 6\gamma. \end{aligned} \tag{1.296}$$

The appearance of the additive constant γ in the solution (1.295) reflects the invariance of the KdV equation with respect to transformation

$$x' = x + 6\gamma t, \quad t' = t, \quad u' = u + \gamma.$$

In fact, this is the Galileo invariance of the KdV equation—the transition to the moving reference frame is compensated by the change of the basin's depth.

By analogy with the linear case, let us define the phase

$$\theta = 2\sqrt{a/m}\,(x - Vt), \tag{1.297}$$

and the corresponding wavevector and frequency

$$k = 2\sqrt{a/m}, \quad \omega = 2\sqrt{a/m}\,V. \tag{1.298}$$

The frequency is related to the wavevector by the formula

$$\omega = kV = k(2k^2 - 4a + 6\gamma). \tag{1.299}$$

The most important and typical for nonlinear waves feature of this dispersion relation is that it depends on the amplitude a of the wave. The wavelength L defined as the distance between two neighbouring wave crests can be found from the statement that the real period of the elliptic function $\mathrm{dn}(z, m)$ is equal to $2\mathrm{K}(m)$, where $\mathrm{K}(m)$ is the complete elliptic integral of the first kind, that is,

$$L = 2\sqrt{m/a}\,\mathrm{K}(m). \tag{1.300}$$

Note also the formula for a mean value of u,

$$\bar{u} = \frac{1}{L}\int_0^L u(x,t)dx = \frac{2a}{m}\frac{\mathrm{E}(m)}{\mathrm{K}(m)} + \gamma, \tag{1.301}$$

found with the help of Eq. (A.19).

Now let us turn to the linear limit $a \ll 1$. In this limit the parameter m is also much less than unity ($m \ll 1$), but the wavevector (1.298) remains finite. Since in the limit $m \to 0$ we have (see Eqs. (A.16))

$\mathrm{dn}^2(z,m) = 1 - m\,\mathrm{sn}^2(z,m) \to 1 - m\sin^2 z$, Eq. (1.295) transforms in the linear approximation to

$$u(x,t) \cong a\cos\left(2\sqrt{a/m}\,(x - Vt)\right) + 2a/m + \gamma - a. \tag{1.302}$$

The undisturbed surface corresponds to $u = 0$, hence, the constant γ in the main approximation must be chosen so that the additive constant in Eq. (1.302) equals to zero, i.e., $\gamma = -2a/m + a \cong -2a/m = -k^2/2$. Then in the same approximation the dispersion relation (1.299) becomes, as one could expect, $\omega = -k^3$, and the solution (1.302) coincides with the harmonic wave solution of the linear equation $u_t + u_{xxx} = 0$.

In the opposite limit $m \to 1$ we obtain according to Eq. (A.17) the solution

$$u(x,t) = \frac{2a}{\cosh^2(\sqrt{a}(x - Vt))} + \gamma.$$

This solution describes a solitary wave or soliton. In this case it is natural to assume that the surface displacement vanishes at $x \to \pm\infty$, i.e., $\gamma = 0$. Then Eq. (1.296) gives the soliton velocity

$$V_s = 4a, \tag{1.303}$$

and the soliton solution becomes

$$u(x,t) = \frac{2a}{\cosh^2(\sqrt{a}(x - 4at))}. \tag{1.304}$$

The wavelength (1.300) goes to infinity in the limit $m \to 1$, i.e., at m close to unity the periodic wave represents a sequence of widely separated solitons. Thus, we have found that the KdV equation describes a solitary hump on the water surface which moves with the velocity proportional to the hump's height and has a width proportional to a square root of the hump's height. Just such a solitary wave was observed for the first time by Scott Russell in 1844, and its description started investigations of nonlinear waves and solitons. Existence of solitons seems especially wonderful in view of the spreading of wave pulses due to the dispersion effects discussed in Sec. 1.2. We infer from this fact that the dispersion effects can be compensated entirely by the nonlinear effects, and this conclusion changes drastically a qualitative picture of a pulse evolution compared to the linear theory, since an arbitrary pulse can eventually transform not only to slowly modulated harmonic waves with small amplitudes but also to solitons with finite amplitudes. For example, in the problem about the step-like pulse decay the dispersion effects dominate at small time $t \ll 1$ (see Sec. 1.2.1). As follows from Eq. (1.47), the region of oscillations spreads out with time. Therefore, the role of dispersion effects decreases gradually and the role of nonlinear effects becomes more important. One may suppose that at asymptotically large time $t \gg 1$ the region of wave oscillations can be described as a modulated cnoidal wave with slowly varying parameters a, m, γ (or α, β, γ, or any other equivalent set of parameters). These parameters change little in one wavelength $\sim 1/k$ and one period $\sim 1/\omega$, and we arrive at the problem of derivation and solution of equations describing such slow evolution of the parameters of the cnoidal wave.

In a similar way, if in the problem of the wave breaking we take into account the dispersion effects, we shall find that they lead to formation of oscillations in the multi-valued region of the solution of the Hopf equation $u_t + 6uu_x = 0$. Sometimes this region of oscillations arising after wave breaking due to dispersion is called a 'collisionless shock wave'. Here the adjective 'collisionless' means that this region is not caused by viscosity as it takes place for a usual shock wave studied in Sec. 1.4. Again we may assume that the region of oscillations can be described as a slowly evolving modulated cnoidal wave.

The idea of averaging of equations for the parameters a, m, γ (or any other set) over fast oscillations with the wavelength $\sim 1/k$ and the frequency $\sim 1/\omega$ was introduced into the nonlinear wave theory by G. B. Whitham in his famous theory of wave modulations, and it occurred very fruitful. We shall consider it in details in the following Chapters of the book. But at first we should mention that the soliton-type solution is not the only result of common action of the dispersive and nonlinear effects. In the next section we shall consider one more consequence of the interplay of these two effects.

1.6 Nonlinearity and dispersion: nonlinear Schrödinger equation

1.6.1 *Derivation of the nonlinear Schrödinger equation in nonlinear optics*

In Sec. 1.2.2 we have considered the spreading of the electromagnetic (light) pulses in a dispersive medium. Due to interaction of the electromagnetic field of the wave with charged particles of the medium the latter becomes polarized, and in the case of very weak fields $E(x,t)$ the dipole moment $P(x,t)$ of a unit volume depends linearly on $E(x,t)$ (see Eq. (1.55)),

$$P^L(x,t) = \int_0^\infty \tilde{\chi}(t')E(x,t-t')dt', \tag{1.305}$$

where a superscript L denotes that a linear part of the polarization is considered. Formula (1.305) expresses the inertial properties of the medium which lead to delay of the response to the field action. For harmonic waves with $E = E_\omega \exp(ikx - i\omega t)$ and $P^L = P^L_\omega \exp(ikx - i\omega t)$ we have

$$P^L_\omega = \chi(\omega)E_\omega, \tag{1.306}$$

where $\chi(\omega) = \int_0^\infty \tilde{\chi}(t)\exp(i\omega t)\,dt$ (see Eq. (1.57)). Hence, the equation

$$E_{xx} - (1/c^2)E_{tt} = (4\pi/c^2)P_{tt} \tag{1.307}$$

for the electromagnetic field propagating through the medium yields the dispersion relation

$$k^2 = (\omega^2/c^2)\epsilon(\omega), \quad \epsilon(\omega) = 1 + 4\pi\chi(\omega). \tag{1.308}$$

If the electromagnetic field is presented as a real part of the expression

$$E(x,t) = \mathcal{E}(x,t)\exp[i(k_0x - \omega_0t)], \tag{1.309}$$

where $\mathcal{E}(x,t)$ is the envelope function changing little in one wavelength $2\pi/k_0$ and one period $2\pi/\omega_0$, then a slow evolution of $\mathcal{E}(x,t)$ is governed by the equation

$$i\left(\mathcal{E}_t + \omega'(k)\mathcal{E}_x\right) + \tfrac{1}{2}\omega''(k)\mathcal{E}_{xx} = 0 \tag{1.310}$$

(see Eq. (1.76)), where the derivatives $\omega'(k)$ and $\omega''(k)$ are calculated at the wavevector k_0 of the carrier wave. As it follows from this equation, the wave packet moves as a whole with the group velocity $v_g = \omega'(k)$ and its form changes due to the dispersion term proportional to $\omega''(k)$.

From physical reasoning it is clear that the linear approximation (1.305) is valid as long as one can neglect the nonlinear effects which have an order of magnitude of a ratio of the electromagnetic field of the wave E to a typical value of the 'atomic' fields E_{atomic} in the medium. For example, in a hydrogen atom $E_{atomic} \sim e^2/a_B \sim 6\cdot 10^9 V\cdot cm^{-1}$ ($a_B = \hbar^2/me^2$ is the

Bohr radius), and in semiconductors E_{atomic} is about $\sim 10^7 V \cdot cm^{-1}$. At the same time, in the laser beams with intensity $10^9 W \cdot cm^{-2}$ the electric field is equal to $E \sim 6 \cdot 10^5 V \cdot cm^{-1}$, which may be large enough for the long distance pulse evolution. Thus, we add to the formula (1.305) the nonlinear term,

$$P = P^L + P^{NL}, \tag{1.311}$$

where the nonlinear polarization is given by

$$\begin{aligned} P^{NL} &= \iint_0^\infty \tilde{\chi}^{(2)}(t', t'') E(x, t - t') E(x, t - t'')\, dt' dt'' \\ &+ \iiint_0^\infty \tilde{\chi}^{(3)}(t', t'', t''') E(x, t - t') E(x, t - t'') E(x, t - t''')\, dt' dt'' dt''', \end{aligned} \tag{1.312}$$

and we have assumed for simplicity that the medium is isotropic so that the susceptibility tensor $\tilde{\chi}$ reduces to a scalar quantity. It is clear that functions $\tilde{\chi}^{(2)}(t', t'')$ and $\tilde{\chi}^{(3)}(t', t'', t''')$ multiplied by symmetric in E expressions are symmetric in their arguments. In media with a centre of inversion the vectors $\mathbf{E}$ and $\mathbf{P}$ change their signs under the inversion transformation $\mathbf{x} \to -\mathbf{x}$, i.e. $\mathbf{E} \to -\mathbf{E}$, $\mathbf{P} \to -\mathbf{P}$. Therefore, in such media the susceptibility $\tilde{\chi}^{(2)}$ is equal to zero identically. We shall consider this case in what follows. In calculation of the nonlinear polarization at the frequency ω_0, i.e., $\mathcal{P}^{NL}(x, t) \exp(ik_0 x - i\omega_0 t)$, we have to substitute into Eq. (1.312) the real values of the electric field of the wave,

$$E(x, t) = \mathcal{E}(x, t) e^{i(k_0 x - \omega_0 t)} + \mathcal{E}^*(x, t) e^{-i(k_0 x - \omega_0 t)}, \tag{1.313}$$

where the asterisk denotes complex conjugation. We write here only three terms

$$\begin{aligned} &\mathcal{P}^{NL}(x, t) \exp[i(k_0 x - \omega_0 t)] = \textstyle\iiint_0^\infty \tilde{\chi}^{(3)}(t', t'', t''')\, dt' dt'' dt''' \\ &\times \{\mathcal{E}(x, t - t') \mathcal{E}^*(x, t - t'') \mathcal{E}(x, t - t''') e^{i(k_0 x - \omega_0 t) + i\omega_0 (t' - t'' + t''')} \\ &+ \mathcal{E}(x, t - t') \mathcal{E}(x, t - t'') \mathcal{E}^*(x, t - t''') e^{i(k_0 x - \omega_0 t) + i\omega_0 (t' + t'' - t''')} \\ &+ \mathcal{E}^*(x, t - t') \mathcal{E}(x, t - t'') \mathcal{E}(x, t - t''') e^{i(k_0 x - \omega_0 t) + i\omega_0 (-t' + t'' + t''')}\}, \end{aligned} \tag{1.314}$$

which have the factor $\exp[i(k_0 x - \omega_0 t)]$ corresponding to the carrier wave. The other terms are either complex conjugate of Eq. (1.314), or describe the field conversion into the harmonic with the frequency $3\omega_0$. Since we are interested in the evolution of the envelope $\mathcal{E}(x, t)$, the conversion effects can be neglected. Further, supposing a slow evolution of the envelope $\mathcal{E}(x,t)$, we can expand the fields in powers of t', t'', t''', keeping in the main approximation only the first terms of the expansion,

$$\mathcal{E}(x,t-t') \cong \mathcal{E}(x,t-t'') \cong \mathcal{E}(x,t-t''') \cong \mathcal{E}(x,t).$$

As a result we obtain for this Kerr-type nonlinearity the expression

$$\mathcal{P}^{NL}(\omega_0) \cong 3\chi^{(3)}(\omega_0)|\mathcal{E}(x,t)|^2\mathcal{E}(x,t), \tag{1.315}$$

where

$$\chi^{(3)}(\omega_0) = \iiint_0^\infty \tilde{\chi}^{(3)}(t',t'',t''')e^{i\omega_0(-t'+t''+t''')}\,dt'dt''dt''',$$

and we have used the symmetry of the function $\tilde{\chi}^{(3)}(t',t'',t''')$ with respect to interchange of its arguments. In practice, $\chi^{(3)}(\omega_0)$ may be considered as a phenomenological constant characterizing the properties of the medium.

Thus, the total polarization is equal to the sum of the expressions (1.306) and (1.315) multiplied by the factor $\exp[i(k_0x-\omega_0t)]$. Its substitution into Eq. (1.307) leads to Eq. (1.72) with the additional term

$$-(4\pi\omega_0^2/c^2)\cdot 3\chi^{(3)}(\omega_0)|\mathcal{E}(x,t)|^2\mathcal{E}(x,t)$$

in the right hand side. Transformations similar to those of Sec. 1.2.2 yield the evolution equation for the envelope,

$$i\left(\frac{\partial\mathcal{E}}{\partial t}+\omega'(k)\frac{\partial\mathcal{E}}{\partial x}\right)+\tfrac{1}{2}\omega''(k)\frac{\partial^2\mathcal{E}}{\partial x^2}+\frac{6\pi\chi^{(3)}}{\epsilon(\omega)}|\mathcal{E}(x,t)|^2\mathcal{E}(x,t)=0, \tag{1.316}$$

where we have dropped the subscript '0' in k_0 and ω_0. This equation differs from Eq. (1.310) by the last term in the left hand side and is called the nonlinear Schrödinger (NLS) equation. It describes the evolution of the envelope of the wave packet under influence of weak dispersion and nonlinear effects.

To simplify the following formulas, it is convenient to transform the NLS equation to its canonical form. At first, we make the Galileo transformation to the frame of reference moving with the group velocity $v_g = \omega'(k)$,

$$x' = x - v_g t, \quad t' = t,$$

and after that introduce as a unit of length the pulse width l and as a unit of time the 'dispersion time' (see Sec. 1.2.2),

$$x'' = x'/l, \quad t'' = (\omega''/2l^2)t'$$

and obtain

$$i\mathcal{E}_{t''} + \mathcal{E}_{x''x''} + (12\pi\chi^{(3)}l^2/\omega''\epsilon(\omega))|\mathcal{E}|^2\mathcal{E} = 0.$$

At last, we introduce the dimensionless amplitude of the envelope

$$\mathcal{E}(x'', t'') = \sqrt{|\omega''\epsilon(\omega)/6\pi\chi^{(3)}l^2|}\, u(x'', t'').$$

and obtain finally the equation

$$iu_t + u_{xx} \pm 2|u|^2u = 0, \tag{1.317}$$

where we have dropped the primes in x'' and t''. The sign of the nonlinear term is determined by the sign of the product $\omega''\chi^{(3)}$. For $\omega''\chi^{(3)} < 0$ the NLS equation

$$iu_t + u_{xx} - 2|u|^2u = 0 \tag{1.318}$$

is called 'defocusing' because for the positive dispersion $\omega'' > 0$ the above condition means that $\chi^{(3)} < 0$, i.e., the refraction constant $n = \sqrt{\epsilon}$ decreases due to the nonlinear correction:

$$n + \delta n = (\epsilon + 12\pi\chi^{(3)}|\mathcal{E}|^2)^{1/2}, \quad \delta n \cong 6\pi\chi^{(3)}|\mathcal{E}|^2/n.$$

For $\omega''\chi^{(3)} > 0$ the NLS equation

$$iu_t + u_{xx} + 2|u|^2u = 0 \tag{1.319}$$

is called 'focusing', because the refraction constant increases due to the nonlinear correction.

The NLS equation may be generalized as follows,

$$iu_t + u_{xx} + g(|u|^2)u = 0, \tag{1.320}$$

where the function $g(|u|^2)$ is chosen according to some physical reasoning with the aim to model the corresponding nonlinear phenomena.

The NLS equation may be written in the form similar to the hydrodynamical equations by means of the so-called Madelung substitution,

$$u(x, t) = \sqrt{\rho(x, t)}\exp(i\theta(x, t)), \tag{1.321}$$

where $\rho(x,t)$ and $\theta(x,t)$ are real functions. Then separation of the real and imaginary parts in Eq. (1.320) yields

$$\begin{aligned}
&\tfrac{1}{2}\rho_t + \rho_x\theta_x + \rho\theta_{xx} = 0,\\
&\theta_t + \theta_x^2 - g(\rho) + \rho_x^2/4\rho^2 - \rho_{xx}/2\rho = 0.
\end{aligned}$$

After differentiation of the second equation with respect to x and introduction of the variable $\tau = 2t$, we arrive at the system

$$\begin{aligned}
&\rho_\tau + (\rho v)_x = 0,\\
&v_\tau + v v_x - \left[g(\rho)/2 + \rho_{xx}/4\rho - \rho_x^2/8\rho^2\right]_x = 0,
\end{aligned} \tag{1.322}$$

where $v = \theta_x$. It looks like the Boussinesq system (1.270) with more complex dispersion terms and the dependence of the 'pressure' on the 'density' ρ defined by the equation

$$p(\rho) = -\tfrac{1}{2}\int^{\rho} \rho' \frac{dg(\rho')}{d\rho'}\, d\rho'. \tag{1.323}$$

In particular, for the defocusing NLS equation we have

$$p(\rho) = \int^{\rho} \rho'\, d\rho' = \tfrac{1}{2}\rho^2 \quad \text{(defocusing NLS equation)}, \tag{1.324}$$

and for the focusing NLS equation

$$p(\rho) = -\int^{\rho} \rho'\, d\rho' = -\tfrac{1}{2}\rho^2 \quad \text{(focusing NLS equation)}. \tag{1.325}$$

These formulas show that properties of the focusing and defocusing NLS equations are very different. If the initial pulse envelope $\rho_0(x)$ is smooth enough, we can neglect the dispersive terms in Eqs. (1.322) and find that the initial stage of the evolution is described by the shallow water equations (1.272)

$$\rho_\tau + (\rho v)_x = 0, \qquad v_\tau + v v_x \pm \rho_x = 0, \tag{1.326}$$

where 'plus' sign corresponds to the defocusing NLS equation, and 'minus' sign to the focusing one. In the case of the defocusing NLS equation these equations coincide literally with the shallow water equations, but in the focusing case the 'pressure' term ρ_x has the opposite sign. However, it is known that a homogeneous state of the medium is stable if only $dp/d\rho > 0$, when the pressure tends to restore mechanical equilibrium what leads to sound waves propagating through the medium. But if $dp/d\rho < 0$, as we have in the focusing NLS equation case, the pressure tends to increase a density disturbance which in turn increase a local pressure. This means that the homogeneous state is unstable with respect to density disturbances. In other words, the wave with a constant amplitude is unstable with respect to amplitude modulations, if it is described by the focusing NLS equation, that is, if

$$\omega''(k)\chi^{(3)} > 0. \tag{1.327}$$

Here we have met a new effect caused by the nonlinearity—the modulational instability of solutions of some nonlinear evolution equations. Of course, the used above mechanical analogy is valid as long as one can neglect the dispersion terms in Eqs. (1.322). When the disturbance grows, the gradients of ρ increase also, and for description of the evolution far from the initial homogeneous state we have to use the full system (1.322) or Eq. (1.319).

Because of the pointed out difference between the two types of the NLS equation, we shall consider their solutions separately. To simplify analysis, note that the generalized NLS equation (1.320) as well as its particular cases (1.317) are invariant with respect to the Galileo transformation

$$x' = x + vt, \quad t' = t, \quad u'(x', t') = u(x, t) \exp\left[\tfrac{i}{2}v\left(x + \tfrac{1}{2}vt\right)\right]. \tag{1.328}$$

1.6.2 *Dark soliton solution of the defocusing NLS equation*

Let us study some particular solutions of the defocusing NLS equation

$$iu_t + u_{xx} - 2|u|^2 u = 0. \tag{1.329}$$

First of all, note that it has a solution with the constant amplitude

$$u(x, t) = u_0 e^{-2iu_0^2 t}, \tag{1.330}$$

where the amplitude $u_0 = \text{const}$ can be made real and positive by means of an appropriate choice of a constant phase factor. By the Galileo transformation (1.328) it is transformed into the solution

$$u(x, t) = u_0 \exp[ikx - i(k^2 + 2u_0^2)t], \tag{1.331}$$

where we have denoted $k = -v/2$. This is a general solution with the constant amplitude of the NLS equation. By virtue of the Galileo invariance, it is sufficient to investigate disturbances of the particular solution

(1.330), and then the behaviour of disturbances of the solution (1.331) can be found by means of the transformation (1.328). (To avoid a possible misunderstanding, note that the wavenumber k and the frequency $\omega = k^2+2\omega_0^2$ of the envelope $u(x,t)$ in Eq. (1.331) correspond to small deviations from the wavenumber and the frequency of the carrier wave.)

We shall look for the solution of the NLS equation in the form

$$u(x,t) = [u_0 + a(x,t)]\exp[-2iu_0^2 t + i\varphi(x,t)], \tag{1.332}$$

where it is supposed that the modulation of the amplitude is small, i.e., $|a| \ll u_0$. On substitution of Eq. (1.332) into (1.329) and separation of the real and imaginary parts, we obtain the equations

$$\begin{aligned} &a_t + u_0\varphi_{xx} + 2a_x\varphi_x + a\varphi_{xx} = 0, \\ -&u_0(\varphi_t + 4u_0 a) + a_{xx} - a\varphi_t + (u_0+a)\varphi_x^2 - 6u_0a^2 - 2a^3 = 0. \end{aligned} \tag{1.333}$$

In the linear approximation it reduces to the system

$$a_t + u_0\varphi_{xx} = 0, \quad \varphi_t + 4u_0 a - (1/u_0)\, a_{xx} = 0, \tag{1.334}$$

which has a solution

$$a = a_0\exp[i(Kx - \Omega t)], \quad \varphi = \varphi_0\exp[i(Kx - \Omega t)]$$

provided K and Ω are related by the dispersion relation

$$\Omega = \pm K\sqrt{K^2 + 4u_0^2} \tag{1.335}$$

for the waves of modulations. We see that in the long wave limit,

$$K \ll 2u_0, \tag{1.336}$$

the dispersion is small. Two signs in Eq. (1.335) correspond to the two directions in which the modulational wave can propagate.

On the other hand, we know that in the dispersionless limit the NLS equation reduces to the shallow water equations and, hence, the evolution of the envelope can lead to its steepening followed by the wave breaking. Our experience gained in the investigation of the Boussinesq equations suggests that again the common action of the dispersion and nonlinear effects may lead to existence of solitons. Let us confirm this supposition by studying a small amplitude modulational wave propagating in only one direction, as it was done in the Boussinesq equations case when they were reduced to the KdV equation.

At first, we introduce again 'natural' units of length and time. Since we suppose that the dispersion effects are small, the typical length must be much greater than $(2u_0)^{-1}$ (see Eq. (1.336)). Hence, we introduce the space coordinate

$$X = 2u_0\beta x, \tag{1.337}$$

where the parameter β controls the dispersion effects, and the condition $\beta \ll 1$ means that $x \gg (2u_0)^{-1}$ at $X \sim 1$. We shall consider waves propagating in the positive direction of the x axis which correspond to the upper sign in Eq. (1.335), i.e., $\Omega \cong 2u_0 K$. Rewriting the phase as

$$Kx - \Omega t = (K/2u_0\beta)(X - 4u_0^2\beta t) = (K/2u_0\beta)(X - T),$$

we find the proper time variable

$$T = 4u_0^2\beta t. \tag{1.338}$$

To control the nonlinear effects, we introduce the parameter α according to the formula

$$a = u_0\alpha A, \tag{1.339}$$

i.e., the condition $\alpha \ll 1$ corresponds to $|a| \ll u_0$ at $A \sim 1$. At last, to make the first equation (1.334) dimensionless ($A_T + \Phi_{XX} = 0$), a new phase variable Φ must be defined by the equation

$$\varphi = (\alpha/\beta)\Phi. \tag{1.340}$$

In new variables, the system (1.333) takes the form

$$\begin{aligned} &A_T + \Phi_{XX} + 2\alpha A_X\Phi_X + \alpha A\Phi_{XX} = 0, \\ &\Phi_T + A + \alpha A\Phi_T - \beta^2 A_{XX} + \alpha\Phi_X^2 + \tfrac{3}{2}\alpha A^2 = 0, \end{aligned} \tag{1.341}$$

where we have neglected the terms of the order of magnitude $\sim \alpha^2$ and kept only the terms giving the main contributions into the dispersion and nonlinear effects.

As in the cases of the Burgers equation (Sec. 1.4.1) and the KdV equation (Sec. 1.5.1), we are interested in such a value of the amplitude A that leads to the same order of magnitude of the dispersion and nonlinear effects. As usual, we assume that the slow evolution of the pulse with $A \sim \varepsilon \ll 1$ takes the time scale $T \sim \varepsilon^{-a}$, if the pulse changes considerably at distances of the space scale $X \sim \varepsilon^{-b}$. At the same time, the phase scales as

$\Phi \sim \varepsilon^c$. The scaling indices are found by equating the effects of the kinematical (Hopf-like) nonlinearity Φ_X^2, the dynamic (forced by the 'pressure') nonlinearity A^2, and the dispersion A_{XX} on the evolution of Φ:

$$\Phi_T \sim \Phi_X^2 \sim A^2 \sim A_{XX}, \quad \text{or} \quad \varepsilon^{c+a} \sim \varepsilon^{2c+2b} \sim \varepsilon^2 \sim^{1+2b},$$

and these estimates give

$$a = \tfrac{3}{2}, \quad b = c = \tfrac{1}{2}. \tag{1.342}$$

Further, we introduce the slow variables

$$\tau = \varepsilon^{3/2} T, \quad \xi = \varepsilon^{1/2}(X - T) \tag{1.343}$$

and expand A and Φ into the series expansions

$$A = \varepsilon A^{(1)} + \varepsilon^2 A^{(2)} + \ldots, \quad \Phi = \varepsilon^{1/2}(\Phi^{(1)} + \varepsilon\Phi^{(2)} + \ldots), \tag{1.344}$$

where we have taken into account that according to our definitions the leading terms are $A \sim \varepsilon$, $\Phi \sim \varepsilon^{1/2}$. Substitution of these expansions and variables into Eqs. (1.341) yields up to the terms of order $\sim \varepsilon$:

$$-A_\xi^{(1)} + \Phi_{\xi\xi}^{(1)} + \varepsilon[A_\tau^{(1)} + 2\alpha A_\xi^{(1)}\Phi_\xi^{(1)} + \alpha A^{(1)}\Phi_{\xi\xi}^{(1)} - A_\xi^{(2)} + \Phi_{\xi\xi}^{(2)}] + \ldots = 0,$$

$$\begin{aligned} -\Phi_\xi^{(1)} + A^{(1)} + \varepsilon[\Phi_\tau^{(1)} - \alpha A^{(1)}\Phi_\xi^{(1)} + \alpha(\Phi_\xi^{(1)})^2 \\ + \tfrac{3}{2}\alpha(A^{(1)})^2 - \beta^2 A_{\xi\xi}^{(1)} - \Phi_\xi^{(2)} + A^{(2)}] + \ldots = 0. \end{aligned}$$

In the main order we have

$$A^{(1)} = \Phi_\xi^{(1)}. \tag{1.345}$$

Then the next order terms can be transformed to

$$\begin{aligned} A_\tau^{(1)} + 3\alpha A^{(1)} A_\xi^{(1)} - A_\xi^{(2)} + \Phi_{\xi\xi}^{(2)} = 0, \\ A_\tau^{(1)} + 3\alpha A^{(1)} A_\xi^{(1)} - \beta^2 A_{\xi\xi\xi}^{(1)} - \Phi_{\xi\xi}^{(2)} + A_\xi^{(2)} = 0, \end{aligned}$$

and their addition gives the KdV equation for $A^{(1)}$:

$$A_\tau^{(1)} + 3\alpha A^{(1)} A_\xi^{(1)} - \tfrac{1}{2}\beta^2 A_{\xi\xi\xi}^{(1)} = 0. \tag{1.346}$$

When $A^{(1)}(\xi, \tau)$ is found, the phase can be calculated by integration of the relation (1.345). In the initial variables

$$A^{(1)} = a/u_0\alpha\varepsilon, \quad t = \tau/4u_0^2\beta\varepsilon^{3/2}, \quad x = (\varepsilon\xi + \tau)/2u_0\beta\varepsilon^{3/2},$$

Eq. (1.346) takes the form

$$a_t + 2u_0 a_x + 6aa_x - (1/4u_0)a_{xxx} = 0 \tag{1.347}$$

of the evolution equation for the amplitude of modulation.

Let us write here the soliton solution (see Sec. 1.5.2) of Eq. (1.347),

$$a(x,t) = -\frac{\tilde{V}/2}{\cosh^2[\sqrt{u_0\tilde{V}}(x-(2u_0-\tilde{V})t)]}, \tag{1.348}$$

where $\tilde{V} > 0$ is the velocity of the soliton with respect to the frame of reference moving with the velocity $2u_0$ of propagation of linear waves. The intensity $|u(x,t)|^2 = |u_0 + a(x,t)|^2$ in the main approximation is equal to

$$|u(x,t)|^2 \cong u_0^2\left(1 - \frac{\tilde{V}/u_0}{\cosh^2[\sqrt{u_0\tilde{V}}(x-(2u_0-\tilde{V})t)]}\right). \tag{1.349}$$

Note that change of the sign of the dispersion term in the KdV equation leads to change of the sign of the soliton solution. The solution with diminished local amplitude on the constant background is called a 'dark soliton'. Thus, we have found that the defocusing NLS equation has a dark soliton solution at least in the small amplitude limit.

Naturally, one may ask what happens when the modulation amplitude a increases. Unfortunately, the obtaining of the general periodic solution of the NLS equation is not a simple task and we shall discuss it later, when the appropriate methods are developed. Here we shall only obtain the dark soliton solution with arbitrary value of the amplitude.

Let us look for the solution of Eq. (1.329) in the form

$$u(x,t) = u_0[F(x-Vt) + iG(x-Vt)]e^{-2iu_0^2 t}, \tag{1.350}$$

where $F(\xi)$ and $G(\xi)$ are real functions depending only on the variable $\xi = x - Vt$. On substitution of Eq. (1.350) into Eq. (1.329) and separation of the real and imaginary parts, we get the system

$$\begin{aligned} F_{\xi\xi} + VG_\xi + 2u_0^2(1-F^2-G^2)F = 0,\\ G_{\xi\xi} - VF_\xi + 2u_0^2(1-F^2-G^2)G = 0. \end{aligned} \tag{1.351}$$

Multiplying the first equation by G, the second by F, and subtracting one equation from the other, we find the first integral

$$F_\xi G - G_\xi F - (V/2)(1-F^2-G^2) = 0, \tag{1.352}$$

where the integration constant is chosen according to the condition that $|u(x,t)|^2 \to u_0^2$ or $F^2 + G^2 \to 1$ at $|\xi| \to \infty$, where F_ξ, $G_\xi \to 0$. Then, multiplying the first equation (1.351) by F_ξ, the second by G_ξ, and adding these two equations, we find another first integral,

$$F_\xi^2 + G_\xi^2 - u_0^2(1 - F^2 - G^2)^2 = 0. \tag{1.353}$$

The form of the two first integrals shows, that it is convenient to introduce polar coordinates

$$F = r\cos\theta, \quad G = r\sin\theta, \tag{1.354}$$

in which the first integrals become

$$r^2\theta_\xi = -\frac{V}{2}(1 - r^2), \quad r_\xi^2 + r^2\theta_\xi^2 = u_0^2(1 - r^2)^2. \tag{1.355}$$

Hence, we have

$$r_\xi^2 = u_0^2(1 - r^2)^2\left[1 - (V/2u_0)^2(1/r^2)\right]$$

and

$$(dr/d\theta)^2 = r_\xi^2/\theta_\xi^2 = r^2\left[(2u_0/V)^2 r^2 - 1\right].$$

Simple integration of this equation yields

$$r\cos(\theta - \theta_0) = V/2u_0, \tag{1.356}$$

where θ_0 is the integration constant. This is the equation of a straight line in the polar coordinates, and θ_0 determines the slope of this line in the (F, G) plane. Depending on θ_0, we obtain different representations of the solution of the NLS equation. Let us choose $\theta_0 = \frac{\pi}{2}$, so that the line (1.356) is parallel to the F axis,

$$G = r\sin\theta = V/2u_0 = \text{const}. \tag{1.357}$$

As follows from this relation, we must have $r \geq |V/2u_0|$. On the other hand, at $|\xi| \to \infty$ the line (1.356) has to go to the points on the circle $r^2 = 1$. Consequently, the solution exists only if $|V/2u_0| \leq 1$, when there are real points of intersection of the line (1.357) with the circle $r^2 = 1$. Let us suppose for definiteness that $V > 0$ and introduce the parameter ϕ according to

$$G = V/2u_0 = \sin\phi. \tag{1.358}$$

Now, taking into account that $G_\xi = 0$, we get from Eq. (1.353) the equation

$$F_\xi^2 = u_0^2(\cos^2\phi - F^2), \quad (|F| \leq \cos\phi),$$

whose integration yields

$$F = \cos\phi \tanh(u_0 \cos\phi \cdot \xi),$$

where the integration constant is chosen so that F has the maximum value at $\xi = 0$. As a result, the solution (1.350) becomes

$$u(x,t) = u_0[\cos\phi \cdot \tanh(u_0\cos\phi\cdot(x - 2u_0\sin\phi\cdot t)) + i\sin\phi]e^{-2iu_0^2 t}, \quad (1.359)$$

and its squared modulus is equal to

$$|u(x,t)|^2 = u_0^2\left(1 - \frac{\cos^2\phi}{\cosh^2[u_0\cos\phi(x - 2u_0\sin\phi\cdot t)]}\right). \quad (1.360)$$

At $x \to \pm\infty$ the solution (1.359) has the asymptotic behaviour

$$u(x,t) \to \begin{cases} u_0\exp[i\phi - 2iu_0^2 t], & x \to +\infty, \\ u_0\exp[i(\pi - \phi) - 2iu_0^2 t], & x \to -\infty, \end{cases}$$

That is, after transition through the dark soliton, the phase of the background wave acquires a finite shift equal to $\pi - 2\phi$. The angle ϕ characterizes also the intensity in the centre of the soliton, and if $\cos\phi < 1$, the soliton is called 'grey'. With decrease of $\cos^2\phi$ the depth of the soliton decreases and for $\cos^2\phi \ll 1$ we have to return to the small amplitude soliton solution (1.348, 1.349). This limit corresponds to

$$2u_0 - V = \tilde{V} \ll 2u_0,$$

when in the first approximation we have $\sin\phi = 1 - \tilde{V}/2u_0$, $\cos\phi \cong (\tilde{V}/2u_0)^{1/2}$, and indeed Eq. (1.360) transforms to Eq. (1.349).

As we see, it is easy to find some particular solutions of nonlinear evolution equations, and success depends on the choice of the ansatz in which the solution is looked for. But if we want to solve such problems as decay of a step-like discontinuity or formation of oscillations after the wave-breaking point, we have to know the periodic solutions in the form convenient for applications to these concrete problems.

1.6.3 *Modulational instability and the soliton solution of the focusing nonlinear Schrödinger equation*

The focusing NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{1.361}$$

has also the solution with the constant amplitude

$$u(x,t) = u_0 \exp(2iu_0^2 t). \tag{1.362}$$

In the end of Sec. 1.6.1 we presented some arguments in favour of instability of this solution with respect to small modulations of the amplitude. Let us confirm this statement by a direct calculation. We look for the solution in the form

$$u(x,t) = [u_0 + a(x,t)] \exp[2iu_0^2 t + i\phi(x,t)]. \tag{1.363}$$

We substitute it into Eq. (1.361) and linearize with respect to small variables $a(x,t)$ and $\varphi(x,t)$ to obtain the system similar to Eqs. (1.334),

$$a_t + u_0\phi_{xx} = 0, \quad \phi_t - 4u_0 a - (1/u_0)a_{xx} = 0. \tag{1.364}$$

It has the solution

$$a = a_0 \exp[i(Kx - \Omega t)], \quad \phi = \phi_0 \exp[i(Kx - \Omega t)],$$

with the dispersion relation

$$\Omega = \pm K\sqrt{K^2 - 4u_0^2} \tag{1.365}$$

of the linear waves of modulations, which differs from the relation (1.335) for the defocusing NLS equation case only by the sign before the term $4u_0^2$, but this difference changes drastically the properties of the modulational waves. In the defocusing case the frequency Ω is real at any value of the wavevector K, whereas in the focusing case at small enough values of the wavevector,

$$K < 2u_0, \tag{1.366}$$

the frequency Ω becomes complex. This means that a long wavelength disturbance grows with time exponentially,

$$a \propto \exp\left(\sqrt{4u_0^2 - K^2}\cdot t\right), \quad K < 2u_0. \tag{1.367}$$

Such exponential growth is called the modulational instability of the solution (1.362).

It is clear that the obtained here exponential growth takes place as long as the amplitude of modulations is small enough: $|a| \ll u_0$. When a becomes of the order of magnitude $a \sim u_0$, the nonlinear effects, which are not taken into account in the linearized system (1.364), become most essential. As we shall see later, in the NLS equation case the growth of a gradually slows down and after reaching the maximum value the evolution of the disturbance changes its direction—it flattens going to the initial state.

As we know from the considered above examples of modulationally stable systems (the KdV and the defocusing NLS equations), the common action of the dispersion and nonlinearity effects can lead to existence of the soliton solutions. The same is also true for the focusing NLS equation, and this soliton solution on the zero background field can be found by the same method which was used in the preceding section with the result

$$u(x,t) = u_0 \exp[iVx/2 - i(V^2/4 - u_0^2)t]/\cosh[u_0(x - Vt)], \qquad (1.368)$$

where u_0 is the amplitude of the soliton and V its velocity. The soliton's width is connected with its amplitude, and V is an independent parameter. It can be shown that the solution (1.368) is stable with respect to small disturbances, and therefore it plays a fundamental role in physical processes described by the focusing NLS equation.

Existence of stable soliton solutions of the focusing NLS equation suggests another formulation of the problem about modulational instability of the solution (1.362). Suppose that the disturbance is localized in space. Hence, its Fourier expansion contains as stable harmonics with $K > 2u_0$, so unstable ones with $K < 2u_0$. Evolution of the stable harmonics is described by the linear theory with the use of the Fourier method. As we know, the dispersion of linear waves with $K > 2u_0$ will lead to the spreading out of this part of the disturbance, and, hence, its amplitude at asymptotically large time t decreases according to the law $t^{-1/2}$. However, the harmonics with $K < 2u_0$ will grow with time, and the fastest growth has the harmonic $K = \sqrt{2}u_0$ which corresponds to the increment $\Im\Omega = 2u_0^2$. Thus, one may expect that in vicinity of the initially localized disturbance arises the region of oscillations with a wavelength of the order of magnitude $\sim 1/u_0$. Formation of this oscillation region will saturate the instability and lead to the train of solitons described by the modulated periodic wave. As a result, we come again to the problem of description of the evolution of the modulated periodic solution now for the case of the focusing NLS equation.

Now we are in position to formulate the main subjects considered in the next Chapters of this book. We want:

- to develop the method of finding the periodic solutions for a wide enough class of nonlinear evolution equations interesting from a physical point of view,
- to develop the method of description of the modulated periodic waves,
- to apply these methods to some typical problems as, e.g., the decay of a step-like discontinuity, formation of oscillations in dispersive systems after the wave-breaking point, or the nonlinear stage of the modulational instability.

Before proceeding to this topics, we shall consider briefly in the following Chapter some nonlinear evolution equations which will be used further as examples illustrating the methods under consideration.

Bibliographic remarks

The material of this Chapter is quite standard and more details can be found in many excellent books (see, e.g., Lighthill (1978), Landau and Lifshits (1988), von Mises (1958), Whitham (1974)). An early review article by Kadomtsev and Karpman (1971) may be recommended as an introduction into the subject of this book. Some examples and derivations were taken from current literature. Parabolic equation (1.75) (as well as nonlinear Schrödinger equation (1.316)) was derived by the method of Menyuk (1989). The solution (1.199,1.200) of the problem about the gas evolution after removing the container walls was given in a different form by Nosov and Kamchatnov (1976) with application to the theory of inelastic interactions between atomic nuclei at high energies. The Burgers equation and the KdV equation are derived following to the classical paper by Su and Gardner (1969). The small-amplitude approximation for the dark soliton solution of the defocusing NLS equation was considered by Kivshar (1990) and Kivshar and Luther-Davies (1998), and the solution (1.359) was obtained by the method of Kosevich and Kovalev (1989).

Exercises on Chapter 1

Exercise 1.1
Solve the Hopf equation $u_t + uu_x = 0$ for the initial condition $u_0(x) = 1/(1+x^2)$. Find the caustic curve and the space and time coordinates of the wave-breaking point.

Exercise 1.2
Show that the formulas (1.171) for the characteristic velocities can be written in the form

$$v_i = \left(1 - \frac{L}{\partial L/\partial\lambda_i}\frac{\partial}{\partial\lambda_i}\right)V,$$

where $L = (\lambda_1\lambda_2)^{-n}$, $n = \frac{3-\gamma}{2(\gamma-1)}$, $V = \frac{1}{4}(3-\gamma)(\lambda_1+\lambda_2)$.

Exercise 1.3
Integrate Eqs. (1.205) for the characteristic curves.

Exercise 1.4
Derive the KdV equation from the Boussinesq equation (1.271).

Exercise 1.5
Obtain the soliton solution (1.368) by the method used in Sec. 1.6.2 for derivation of the dark soliton solution of the defocusing NLS equation.

Chapter 2

Nonlinear wave equations in physics

In the preceding Chapter we have considered the KdV equation and two types of the NLS equation related with two physical systems—gravity water waves and packets of electromagnetic waves propagating through a nonlinear media. In this Chapter we shall discuss various situations described by nonlinear wave equations. They will serve us as examples to which the methods under consideration can be applied.

2.1 Korteweg-de Vries equation and modified Korteweg-de Vries equation

The presented in the preceding Chapter derivations of the KdV equation for shallow water waves and small amplitude modulations of the plane wave solution of the defocusing NLS equation were based on the following suppositions:

(1) Only waves propagating in definite direction of the x axis were considered. In other words, it was supposed that characteristic velocities of the linearized initial system had essentially different values what led to separation of the initial pulse into two parts propagating independently of each other.
(2) Only main (quadratic) nonlinearity was taken into account in study of influence of nonlinear effects on the pulse form. This means that nonlinearity is weak and its higher order contributions (cubic, etc.) may be neglected.
(3) Dispersion effects were also considered as small, i.e., in a Taylor

series expansion of the phase velocity in powers of the wavevector only the first nontrivial term was taken into account.

Then the resulting KdV equation can be easily interpreted. For example, in the equation

$$\zeta_t + \sqrt{gh}\,\zeta_x + \tfrac{3}{2}\sqrt{g/h}\,\zeta\zeta_x + \tfrac{1}{6}h^2\sqrt{gh}\,\zeta_{xxx} = 0 \tag{2.1}$$

the first two terms correspond to the propagation of linear waves in the positive x direction without taking into account the dispersion effects. The third term describes the nonlinear effects, so that three first terms correspond to the Hopf equation for a shallow water wave. Finally, the last term corresponds to the first nontrivial term in the series expansion of the dispersion relation $\omega(k)$ in powers of k for linear surface waves. Since all corrections are taken into account in the first order of the perturbation theory, all of them enter additively into Eq. (2.1). Note that in the case of the surface wave the nonlinearity is quadratic, hence, it is taken into account exactly in the KdV equation, whereas in the case of small amplitude dark solitons of the defocusing NLS equation we neglected the cubic terms and, hence, Eq. (1.347) is approximate as with regard to the nonlinear effects so to the dispersion effects.

The above consideration shows that the KdV equation must be ubiquitous—anytime, when the expansion of nonlinear terms in powers of the amplitude and the expansion of the phase velocity $\omega(k)/k$ in powers of the wavevector k begin with quadratic terms, we have to obtain in the main approximation the KdV equation describing the evolution of wave pulses. Let us consider a few typical examples.

2.1.1 *Ion-acoustic nonlinear waves in plasma*

We shall start with the example of ion-acoustic plasma waves which can propagate without damping, if the temperature of electrons T_e is much greater than the temperature of ions (see, e.g., Lifshitz and Pitaevskii, 1979) which will be supposed here equal to zero. Since the ion's mass M is much greater than the electron's one, the latter can be neglected, if the frequency of wave vibrations is not too high. Let us consider a disturbance of the density of ions n (n is the number of ions in the unit volume, n_0 is the equilibrium value of the density) depending only on the x coordinate. Then conservation of the number of particles is expressed by the continuity equation

$$n_t + (nu)_x = 0, \tag{2.2}$$

u being the ion velocity along the x axis.

Since plasma consists of charged particles (for simplicity we suppose that ions have the electric charge $+e$, so that $-e$ is the electron charge), the disturbance can lead to their local separation and, as a consequence, to the electric field described by the potential φ. Therefore, the ions are driven by the force $-e\varphi_x$ directed along the x axis, and their equation of motion reads

$$u_t + uu_x = -(e/M)\varphi_x. \tag{2.3}$$

At last, the potential φ satisfies the Poisson equation

$$\varphi_{xx} = -4\pi e(n - n_e), \tag{2.4}$$

where n_e is the density of electrons. For low frequency vibrations we may suppose that electrons can reach the thermal equilibrium instantaneously and, consequently, their density is given by the Boltzmann distribution

$$n_e = n_0 \exp\left(e\varphi/T_e\right), \tag{2.5}$$

where the temperature is measured in the energy units. On substitution of Eq. (2.5) into Eq. (2.4), we obtain the equation

$$\varphi_{xx} = -4\pi e\left(n - n_0 \exp\left(e\varphi/T_e\right)\right), \tag{2.6}$$

which together with Eqs. (2.2) and (2.3) comprise the closed system of equations describing one-dimensional ion-acoustic waves.

Linearization of this system with respect to small deviations $n' = n - n_0$, φ and u gives the system for linear waves

$$n'_t + n_0 u_x = 0, \quad u_t = -(e/M)\varphi_x, \quad \varphi_{xx} = -4\pi e n' + (1/r_D^2)\varphi, \tag{2.7}$$

where we have introduced the parameter with a dimension of length,

$$r_D = \sqrt{T_e/4\pi e^2 n_0}, \tag{2.8}$$

called the Debye radius and characterizing the plasma properties. Looking for the harmonic wave solution n', u, $\varphi \propto \exp[i(kx - \omega t)]$, we readily find the dispersion relation for linear waves

$$\omega = \sqrt{\frac{T_e}{M}} \frac{k}{\sqrt{1+r_D^2 k^2}}. \tag{2.9}$$

Hence, the dispersion effects are small, if the wavelength $2\pi/k$ is much greater than the Debye radius r_D, that is,

$$\omega/k = c_0 \left(1 - \tfrac{1}{2} r_D^2 k^2\right), \quad k r_D \ll 1, \tag{2.10}$$

where

$$c_0 = \sqrt{T_e/M} \tag{2.11}$$

is the ion sound speed in the long wavelength limit.

Now let us consider situation when the nonlinear effects dominate over the dispersive ones. This means that we consider the long wavelength limit and finite amplitude of the wave. It is clear that for very long waves φ_{xx} goes to zero and Eq. (2.6) reduces to the algebraic relation between the densities of ions and electrons and the potential φ,

$$n = n_e = n_0 \exp\left(e\varphi/T_e\right). \tag{2.12}$$

This is called the approximation of a quasi-neutral plasma, and in this approximation we can eliminate the potential φ from Eq. (2.3) with the use of Eq. (2.12) to obtain

$$u_t + u u_x + \left(T_e/M\right)\left(n_x/n\right) = 0. \tag{2.13}$$

This equation together with the continuity equation (2.2) comprise the hydrodynamical type system for the polytropic gas with $\gamma = 1$ when the pressure p depends on the density n according to the law

$$p = \left(T_e/M\right) n. \tag{2.14}$$

The linear sound velocity defined by the relation $c_0^2 = dp/dn = T_e/M$ coincides, naturally, with the velocity (2.11) of linear waves. The simple wave propagating in the positive x direction obeys Eq. (1.128), that is, for $\gamma = 1$, to

$$u_t + (c_0 + u) u_x = 0. \tag{2.15}$$

This is a usual Hopf equation with well-known properties.

According to the above consideration, the dispersion effects can be taken into account by adding to Eq. (2.15) the term correctly reproducing the dispersion correction in Eq. (2.10). As a result, we arrive at the KdV equation

$$u_t + (c_0 + u)\, u_x + \tfrac{1}{2} c_0 r_D^2\, u_{xxx} = 0, \tag{2.16}$$

which describes the nonlinear ion-acoustic waves in the approximation of small dispersion. It has the well-known solutions in the form of the cnoidal wave and its limiting case—soliton. More strict derivation of Eq. (2.16) by the singular perturbation method used in Chapter 1 is suggested to the reader as an exercise (see Ex. 2.1).

2.1.2 *Shallow water waves with account of surface tension*

The discussed above approach to the KdV equation indicates also the conditions when it may arise as the evolution equation for weakly nonlinear and dispersive waves. For instance, the dispersive term u_{xxx} corresponds to the cubic term in the series expansion of the dispersion relation in powers of the wavevector k,

$$\omega(k) \cong c_0(k + \beta_1 k^3 + \beta_2 k^5 + \ldots). \tag{2.17}$$

If for some reasons the coefficient β_1 vanishes, then the dispersion effects are described by the next term $c_0\beta_2 k^5$ of the expansion, and, hence, the evolution equation will have the term with the fifth derivative of u with respect to x.

As a simple example of such a situation, let us consider waves on the surface of a shallow water with taking into account the surface tension. Now the dynamic boundary condition (1.248) has to include the term describing the Laplace pressure arising due to a curvature of the surface. Since the motion of water occurs with velocities much less than the sound velocity (just for this reason water may be considered as an incompressible fluid), the Laplace pressure may be calculated from the condition of static equilibrium which reads that variation $\delta\zeta$ of the surface leads to zero variation of the full work consisting of the work necessary for a change of the volume,

$$-\int (p_1 - p_2)\delta\zeta\, dS,$$

(p_1 and p_2 are the values of pressure above and below the surface, correspondingly, and dS is the surface element), and for change of the surface area,

$$T_c \int dS,$$

where T_c is the surface tension (capillarity) coefficient. Thus, we have the condition

$$-\int (p_1 - p_2)\delta\zeta \, dS + T_c \delta \int dS = 0. \tag{2.18}$$

If $\zeta = \zeta(x, y)$ is the surface equation, then in the limit of long waves, $|\zeta_x|$, $|\zeta_y| \ll 1$, we obtain

$$\int dS = \int \sqrt{1 + \zeta_x^2 + \zeta_y^2} \, dxdy \cong \int \left(1 + \tfrac{1}{2}\zeta_x^2 + \tfrac{1}{2}\zeta_y^2\right) \, dxdy,$$

so that

$$\delta \int dS \cong \int (\zeta_x \delta\zeta_x + \zeta_y \delta\zeta_y) \, dxdy = -\int (\zeta_{xx} + \zeta_{yy})\delta\zeta \, dxdy.$$

On substitution of this equation into the second term of Eq. (2.18) and of $dS \cong dxdy$ into the first one, we find the equilibrium condition

$$p_1 - p_2 = T_c(\zeta_{xx} + \zeta_{yy}).$$

This pressure difference divided by the density ρ must be added to the right hand side of Eq. (1.248). Hence, in a one-dimensional case of the wave propagating along the x axis, we obtain the boundary condition

$$\varphi_t + \tfrac{1}{2}\left(\varphi_x^2 + \varphi_z^2\right) + g\zeta - (T_c/\rho)\zeta_{xx} = 0. \tag{2.19}$$

Other calculations coincide actually with those of Sec. 1.5.1. In fact, since the nonlinear terms remain the same, it suffices to find the dispersion relation for linear waves. In the linear limit we have the condition

$$\varphi_t + g\zeta - (T_c/\rho)\zeta_{xx} = 0,$$

which after differentiation with respect to t and substitution of Eq. (1.21) gives

$$\varphi_{tt} + g\varphi_z - (T_c/\rho)\varphi_{zxx} = 0 \quad \text{at} \quad z = 0. \tag{2.20}$$

Solution of Eq. (1.14), $\varphi_{xx} + \varphi_{zz} = 0$, with boundary conditions (2.20) and (1.23) yields the dispersion relation

$$\omega^2 = \left(gk + (T_c/\rho)\,k^3\right)\tanh(kh). \tag{2.21}$$

Its series expansion in powers of k has the form

$$\omega^2 \cong ghk^2\left[1 - \left(\frac{h^2}{3} - \frac{T_c}{g\rho}\right)k^2 + \left(\frac{2h^4}{15} - \frac{T_c h^2}{gh}\right)k^4 + \dots\right]. \tag{2.22}$$

If the coefficient before k^2 does not vanish, then this term makes the main contribution into the dispersion effects in the long wave limit and the next term may be neglected. Consequently, we arrive at the following modification of the KdV equation (2.1),

$$\zeta_t + \sqrt{gh}\,\zeta_x + \tfrac{3}{2}\sqrt{g/h}\,\zeta\zeta_x + \sqrt{gh}\left(h^2/6 - T_c/2g\rho\right)\zeta_{xxx} = 0. \tag{2.23}$$

The coefficient in the last term as a function of h changes its sign at

$$h_c = \sqrt{3T_c/g\rho}. \tag{2.24}$$

If $h > h_c$, this coefficient is positive and Eq. (2.23) describes the soliton of elevation; if $h < h_c$, it describes the soliton of lowering of the water surface. If $h = h_c$, this coefficient vanishes, and we have to take into account the next term of the expansion (2.22) which in this particular case gives

$$\omega = \sqrt{gh}\;k\left(1 - \tfrac{1}{90}(h_c k)^4\right). \tag{2.25}$$

Consequently, instead of the KdV equation we arrive at the evolution equation

$$\zeta_t + \sqrt{gh_c}\,\zeta_x + \tfrac{3}{2}\sqrt{g/h_c}\,\zeta\zeta_x + \sqrt{gh_c}\,(h_c^4/90)\,\zeta_{xxxxx} = 0. \tag{2.26}$$

It can be derived more strictly by the used above method of the singular perturbation theory, but we shall not do it here.

To sum up, the form of the dispersion term in the evolution equation is determined by the first nontrivial term in the series expansion of the frequency ω in powers of the wavevector k.

2.1.3 *Waves in nonlinear lattices*

The form of the nonlinear term in evolution equations for weakly nonlinear waves is determined by the first nontrivial term of the series expansion in powers of the wave amplitude. For example, the equations for surface water waves include the boundary conditions with quadratic terms which yield the quadratic nonlinearity $\zeta\zeta_x$ in the KdV equation. In a similar way, the equations for ion-acoustic waves contain quadratic terms in the Euler equation and in the expansion of $\exp(\Phi)$ in powers of Φ, which also lead to the quadratic nonlinearity uu_x in the resulting equation for weakly nonlinear waves.

Consequently, one may expect that if the expansion in powers of an amplitude begins with a cubic term, that is, if coefficients in quadratic terms vanish for some reasons, then the resulting evolution equation for weakly nonlinear waves will contain a cubic nonlinearity rather than a quadratic one. This supposition can be confirmed by simple examples of waves in nonlinear lattices which are often used for modelling crystals and other discrete systems. It is worth to note that interest to nonlinear waves was stimulated to large extent by a famous work of Fermi, Pasta and Ulam (1955) on vibrations of nonlinear lattices.

Let us consider a set of atoms situated at distance d from their neighbours along a straight line which is taken as the x axis. Let the atoms be enumerated by an index n, $-\infty < n < +\infty$. When atoms are shifted from their equilibrium positions, the system tends to restore the equilibrium giving rise to waves propagating along the lattice. If the wave amplitude is small, we arrive at the well-known problem about elastic linear waves in a one-dimensional lattice which is considered in many books on the solid state theory (see, e.g., Kittel, 1966). The dispersion effects arise here due to discreteness of the lattice. In the limit of waves with a wavelength much longer than the lattice constant d, the dispersion effects disappear and waves become usual sound waves with the frequency proportional to the wavevector. With increase of the wave amplitude the interatomic interaction becomes nonlinear, and we are interested in situation when the nonlinear and dispersion effects have the same order of magnitude.

So, let $x_n = nd$ denote the coordinate of the n-th atom in the equilibrium position, when there is no wave, and let q_n be the displacement of the n-th atom from equilibrium. Then the Newton equation for this atom reads

$$\ddot{q}_n = f(q_{n+1} - q_n) - f(q_n - q_{n-1}), \tag{2.27}$$

where the atomic mass is taken equal to unity, and $f(q_{n+1} - q_n)$ denotes the force acting from the $(n+1)$-th atom on the n-th one. We take into account interaction between the nearest neighbours only. In a linear (harmonic) approximation of infinitesimal values of q_n, the restoring force is proportional to the difference $Q_n = q_{n+1} - q_n$ of the distance between the neighbouring atoms from its mean value d (Hook's law),

$$f(Q_n) \cong \omega_0^2 Q_n,$$

where the coefficient ω_0^2 characterizes linear elastic waves in the lattice. For finite but small values of Q_n it is necessary to take into account the next term of the series expansion of $f(Q_n)$ in powers of Q_n. Fermi, Pasta and Ulam considered two cases. The most natural supposition is that the next term is quadratic in Q_n,

$$f(Q_n) \cong \omega_0^2 Q_n + \gamma_2 Q_n^2. \tag{2.28}$$

But if for some reasons the coefficient γ_2 vanishes, then the main nonlinear contribution is due to the cubic term,

$$f(Q_n) \cong \omega_0^2 Q_n + \gamma_3 Q_n^3. \tag{2.29}$$

We shall consider in detail just this last case leaving the first one as an exercise for the reader (see Ex. 2.2).

Since we consider long waves with a wavelength much greater than the lattice constant d, the displacements q_n change little when we pass from the n-th atom to the $(n+1)$-th atom. Therefore, instead of a discrete variable n we introduce a continuous variable x which takes values $nd, n = 0, \pm1, \pm2, \ldots$, at the sites of the lattice, and expand $q_{n\pm1} - q_n = q(nd \pm d) - q(nd)$ into the series in powers of small d (actually, the small parameter is the ratio d/L, L being a wavelength),

$$q_{n\pm1} - q_n = \pm q_x \cdot d + \tfrac{1}{2} q_{xx} \cdot d^2 \pm \tfrac{1}{6} q_{xxx} \cdot d^3 + \tfrac{1}{24} q_{xxxx} \cdot d^4 \pm \ldots,$$

where all the derivatives are taken at $x = nd$. Substituting this expansion into Eq. (2.27) with f defined by Eq. (2.29), we obtain ($\gamma \equiv \gamma_3$)

$$q_{tt} = \omega_0^2 d^2 q_{xx} + \tfrac{1}{12} \omega_0^2 d^4 q_{xxxx} + 3\gamma d^4 q_x^2 q_{xx}. \tag{2.30}$$

Notice that this equation contains only the derivatives of q but not the displacement itself. This means that the amplitude of the wave is measured by the variable $u = q_x$ rather than by q. In fact, the solution with constant displacement $q_n = \text{const}$ does not correspond to any wave motion but reflects a translational invariance of Eq. (2.27). Therefore we differentiate Eq. (2.30) with respect to x to obtain the equation for $u = q_x$,

$$u_{tt} = \left(\omega_0^2 d^2 u + \tfrac{1}{12}\omega_0^2 d^4 u_{xx} + \gamma d^4 u^3\right)_{xx}. \tag{2.31}$$

Note that this equation differs from the Boussinesq equation (1.271) only by the nonlinear term $(u^3)_{xx}$ instead of $(u^2)_{xx}$.

The first two terms in the right hand side of Eq. (2.31) correspond to the dispersion relation,

$$\omega(k) = \omega_0 dk \left(1 - \tfrac{1}{24}(dk)^2\right), \tag{2.32}$$

of linear waves. The dispersion effects are small when, as it was supposed in derivation of Eq. (2.31), $d^2k^2 \ll 1$, so the natural space variable is

$$X = (\sqrt{12}/d)\,\beta x, \tag{2.33}$$

where $\beta \ll 1$, and, consequently, as the time variable we take

$$T = \sqrt{12}\,\omega_0 \beta t. \tag{2.34}$$

At last, we scale the amplitude variable as

$$u = (\omega_0/\sqrt{\gamma} d)\,\alpha U, \tag{2.35}$$

where the condition $\alpha \ll 1$ means that the nonlinear correction to the force is small (i.e., $\omega_0^2 d^4 \gg \gamma d^4 u^2$), and it is assumed that $\gamma > 0$. In these new variables Eq. (2.31) takes the form

$$U_{TT} = (U + \beta^2 U_{XX} + \alpha^2 U^3)_{XX} \tag{2.36}$$

Now we suppose that the amplitude $U \sim \varepsilon \ll 1$ is such that during the propagation of the wave pulse along the X axis it evolves slowly under the action of nonlinear and dispersion effects, and this slow evolution develops at time scale $T \sim \varepsilon^{-a}$ and X scale (i.e., gradients of the pulse profile in the moving with velocity of linear waves reference frame) $X - T \sim \varepsilon^{-b}$. Transition to the slow variables

$$\tau = \varepsilon^a T, \quad \xi = \varepsilon^b (X - T)$$

describing the evolution of the pulse form, yields the evolution equation

$$\varepsilon^{2a} U_{\tau\tau} + 2\varepsilon^{a+b} U_{\tau\xi} = \varepsilon^{4b}\beta^2 U_{\xi\xi\xi\xi} + \varepsilon^{2b}\alpha^2 (U^3)_{\xi\xi}.$$

The requirement that for a wave with the amplitude $U \sim \varepsilon$ the dispersion and nonlinear effects have the same order of magnitude gives $\varepsilon^{a+b+1} \sim \varepsilon^{4b+1} \sim \varepsilon^{2b+3}$, and, consequently,

$$a = 3, \quad b = 1.$$

The expansion $U = \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots$ yields in a usual way the equation for $U^{(1)}$,

$$U_\tau^{(1)} + \tfrac{3}{2}\alpha^2 (U^{(1)})^2 U_\xi^{(1)} + \tfrac{1}{2}\beta^2 U_{\xi\xi\xi}^{(1)} = 0.$$

Returning to the initial variables

$$U^{(1)} = \frac{\sqrt{\gamma} d}{\omega_0 \alpha \varepsilon}\, u, \quad t = \frac{1}{\sqrt{12}\omega_0 \beta} \cdot \frac{\tau}{\varepsilon^3}, \quad x = \frac{d}{\sqrt{12}\beta}\left(\frac{\tau}{\varepsilon^3} + \frac{\xi}{\varepsilon}\right),$$

we obtain the equation

$$u_t + \omega_0 d u_x + (3\gamma d^2/2\omega_0)\, u^2 u_x + (\omega_0 d^3/24)\, u_{xxx} = 0. \tag{2.37}$$

Naturally, its linear part reproduces the dispersion relation (2.32), and, as we expected, it has a cubic nonlinearity $u^2 u_x$.

Equation (2.37) is called a modified KdV (mKdV) equation, and it also describes the evolution of nonlinear waves in weakly nonlinear dispersive systems. By means of a change of variables

$$u \to \sqrt{3/2|\gamma|}(\omega_0/d) u, \quad t \to (\omega_0/3)t, \quad x \to (2/d)(x - \omega_0 d t)$$

it transforms to the canonical form

$$u_t \pm 6u^2 u_x + u_{xxx} = 0, \tag{2.38}$$

where the sign of the nonlinear term is determined by the sign of γ. In contrast to the KdV equation, we cannot change it by means of the transformation $u \to -u$. Therefore, the properties of the mKdV equation depend on the sign of the nonlinear term.

Similar derivation of the evolution equation for waves in the lattice with quadratic nonlinearity of the form Eq. (2.28) yields the KdV equation

$$u_t + \omega_0 d u_x + (\gamma d^2/\omega_0)\, u u_x + (\omega_0 d^3/24)\, u_{xxx} = 0, \tag{2.39}$$

which by the change of variables

$$u \to (\omega_0^2/\gamma d^2)\, u, \quad t \to (\omega_0/3)\, t, \quad x \to (2/d)(x - \omega_0 d t)$$

transforms to the canonical form

$$u_t + 6uu_x + u_{xxx} = 0.$$

The presented derivation of the KdV and mKdV equations does not provide any indication that the solutions of these two equations are related to each other. In fact, they are related, and this remarkable discovery of R. Miura (1968) has played an important role in the theory of solitons. Miura has found that any solution $v(x,t)$ of the mKdV equation

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

is mapped by the transformation

$$u(x,t) = v^2 \pm v_x \tag{2.40}$$

into the solution of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

where we have changed the sign of nonlinear term to make the coefficients of the Miura transformation real. Indeed, it is easy to verify the identity

$$u_t - 6uu_x + u_{xxx} = (2v \pm \partial/\partial x)\,(v_t - 6v^2 v_x + v_{xxx}),$$

which yields at once the above statements. Periodic and soliton solutions of the mKdV equation can be found in the same way as those of the KdV equation in Sec. 1.5.2.

The presented examples show that if dispersion is small in the long wave limit, then slow evolution of the wave due to dispersion and nonlinear effects is governed by the 'KdV-type' equation.

2.2 Nonlinear Schrödinger equation

2.2.1 *Waves in a chain of interacting pendula*

Let us consider a simple example of a wave propagating along a chain of interacting pendula. Oscillations of a single pendulum obey the equation

$$\ddot{q} = -\omega_0^2 \sin q, \tag{2.41}$$

where q is the angle between the pendulum and the vertical line, and $\omega_0 = \sqrt{g/l}$ is the frequency of oscillations in the linear limit $|q| \ll 1$.

Let the pendula be situated at the points $x_n = nd$ of the x axis and the corresponding angles be denoted as q_n. We suppose that the pendula hang on an elastic string, so that the motion of the n-th pendulum is governed by the equation

$$\ddot{q}_n = \omega_1^2(q_{n+1} - q_n) - \omega_1^2(q_n - q_{n-1}) - \omega_0^2 \sin q_n, \tag{2.42}$$

where $\omega_1^2(q_{n+1} - q_n)$ is the elastic force acting from the $(n+1)$-th pendulum on the n-th one due to the torsion of the string.

The continuous limit $d \to 0$ can be investigated in the same way as it was done in the preceding section. We suppose that d is so small compared to the wavelength, that one can neglect the dispersion correction due to the discrete structure of the chain. Denoting by $c = \omega_1 d$ the velocity of linear waves in the limit $\omega_0 \to 0$, we obtain the equation

$$q_{tt} - c^2 q_{xx} + \omega_0^2 \sin q = 0. \tag{2.43}$$

This is the so-called sine-Gordon equation which often appears in different physical problems and is of a great interest as itself. Here we are interested in the limit of small but finite amplitudes when it is enough to consider the main nonlinear correction,

$$q_{tt} - c^2 q_{xx} + \omega_0^2 q - \omega_0^2 \gamma q^3 = 0. \tag{2.44}$$

In the case of the sine-Gordon equation we have $\gamma = 1/6$, but we shall consider γ as an arbitrary parameter which may be useful for some generalizations.

In the linear approximation Eq. (2.44) gives the dispersion relation

$$\omega = \sqrt{\omega_0^2 + c^2 k^2} \tag{2.45}$$

for linear waves. In the long wave limit we have

$$\omega \cong \omega_0 + c^2 k^2 / 2\omega_0, \qquad k \ll \omega_0 / c, \tag{2.46}$$

that is this situation differs from the considered in the preceding section where the dispersion relation had the form $\omega \cong ck + ak^3$, that is, there was a small correction to the relation $\omega = ck$. Therefore, now we cannot obtain the KdV equation or its modification as some approximation to Eq. (2.44) for description on nonlinear waves. Nevertheless, we may consider a slow evolution of the envelopes of the wave packets due to small effects of weak dispersion and nonlinearity. Here we shall try to understand the evolution of the envelope of the wave packet of waves governed by Eq. (2.44).

Recall that in Sec. 1.6.1 we have already considered a typical physical problem of this kind about evolution of the electromagnetic wave pulse propagating through the nonlinear medium, and have found that this evolution is described by the nonlinear Schrödinger equation (1.316). All terms of this equation,

$$i\left(\frac{\partial\mathcal{E}}{\partial t}+\omega'(k)\frac{\partial\mathcal{E}}{\partial x}\right)+\frac{\omega''(k)}{2}\frac{\partial^2\mathcal{E}}{\partial x^2}+\frac{6\pi\chi^{(3)}}{\epsilon(\omega)}|\mathcal{E}(x,t)|^2\mathcal{E}(x,t)=0, \qquad (2.47)$$

can easily be interpreted: (a) the first two terms in parenthesis describe the motion of the packet as a whole with the group velocity $\omega'(k)$ (this is the dispersion effect of the first order); (b) the next term describes the dispersion spreading of the packet (the dispersion effect of the second order); (c) the last term describes the influence of a cubic nonlinearity (see Eq. (1.315)) on the packet evolution provided the amplitude $\mathcal{E}$ is such that this effect has the same order of magnitude as the dispersion spreading of the packet.

This example shows that in the problem of the envelope evolution there are several scales of time: (i) the fastest scale corresponding to oscillations with the frequency $\omega(k)$; (ii) slower scale corresponding to the packet's motion with the group velocity $\omega'(k)$; (iii) the slowest scale corresponding to deformation of the packet's profile due to weak dispersive and nonlinear effects. In derivation of the KdV equation in the preceding section we actually used the same idea of different scales of time, but there we had only two scales. Now we have to generalize this approach to more number of scales of time.

In nonlinear mechanics there is a well-developed perturbation theory of weakly nonlinear systems. One may ask, how one can combine this theory with the idea of many scales of time. To answer this question, let us consider a simple but instructive example of a nonlinear pendulum whose motion is governed by the equation

$$\ddot{q}+\omega_0^2 q-\omega_0^2\gamma q^3=0, \qquad (2.48)$$

that is, we have neglected the interactions between the pendula in Eq. (2.44). Suppose, that we are interested in the motion with the initial data

$$q(0)=\varepsilon A, \qquad \dot{q}(0)=0, \qquad (2.49)$$

where ε is a small parameter controlling the order of magnitude of the amplitude of vibrations. Let us look for a solution in the form of a series expansion

$$q = \varepsilon u = \varepsilon(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots). \tag{2.50}$$

On substitution of this expansion into Eq. (2.48), we obtain at the first order in ε the equation

$$\ddot{u}_0 + \omega_0^2 u_0 = 0 \tag{2.51}$$

with evident solution

$$u_0 = A\cos\omega_0 t. \tag{2.52}$$

At the second order in ε we obtain for u_1 the same equation as Eq. (2.51), because the nonlinear correction is of the order $\sim \varepsilon^3$. Hence, we may consider u_1 as being included in u_0 and put $u_1 = 0$ what agrees with the initial conditions (2.49). At the third order in ε we obtain

$$\ddot{u}_2 + \omega_0^2 u_2 = \omega_0^2 \gamma u_0^3,$$

or, after substitution of Eq. (2.52) into the right hand side,

$$\ddot{u}_2 + \omega_0^2 u_2 = \omega_0^2 \gamma A^3 \cos^3\omega_0 t = \tfrac{1}{4}\omega_0^2\gamma A^3(\cos 3\omega_0 t + 3\cos\omega_0 t).$$

We see that the right hand side has a resonant driving force at the eigenfrequency ω_0 of the operator on the left hand side. This means a resonant growth of the term u_2,

$$u_2 \propto t\sin\omega_0 t,$$

that is, our perturbation theory fails at the time $t \sim 1/\varepsilon^2$, when the correction u_2 becomes of the same order of magnitude as u_0. It is clear, however, that in reality u oscillates with some frequency slightly different from ω_0, and u_2 remains small at arbitrary large values of time. How have we to modify the perturbation theory to make it valid at long times? An answer to this question follows from the exact solution of Eq. (2.41) in which $q(t)$ is the elliptic function of t with the period of vibrations given by

$$\frac{4}{\omega_0}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - m\sin^2\phi}} = \frac{4}{\omega_0}\mathrm{K}(m), \qquad m = \sin^2(\varepsilon A/2),$$

where $\mathrm{K}(m)$ is the complete elliptic integral of the first kind. For small ε we have $(4/\omega_0)\mathrm{K}(m) \cong (2\pi/\omega_0)(1+\varepsilon^2 A^2/16)$, and, hence, the frequency is equal to

$$\omega = \omega_0 \left(1 - \tfrac{1}{16}\varepsilon^2 A^2\right). \tag{2.53}$$

Thus, we have found that nonlinearity leads to dependence of the frequency on the amplitude, and we have overlooked this point in our perturbation theory. In fact, we expanded

$$u = A\cos\omega t = A\cos\left[\omega_0\left(1 - \tfrac{1}{16}\varepsilon^2 A^2\right)\right] + \dots$$

in powers of ε and obtained

$$u = A\cos\omega_0 t + \tfrac{1}{16}\omega_0 A^3\varepsilon^2 \cdot t\sin\omega_0 t,$$

where the correction term is of order $\sim \varepsilon^2$ and grows linearly with time. We infer from this example that we have to expand in powers of ε not only the amplitude (see Eq. (2.50)), but also the frequency,

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{2.54}$$

and choose the coefficients $\omega_1, \omega_2, \dots$ in such a way that resonant forces disappear. For the first time this problem was met in calculations of a long time motion of planets in celestial mechanics, hence the name 'a secular perturbation theory'.

So, let us turn to a modified secular perturbation theory. For convenience of calculations, let us rewrite Eq. (2.48) as follows,

$$(\omega_0^2/\omega^2)\ddot{u} + \omega_0^2 u = \omega_0^2\gamma u^3 - (1 - \omega_0^2/\omega^2)\ddot{u}, \tag{2.55}$$

so that the exact solution

$$u = A\cos\omega t \tag{2.56}$$

makes the left hand side of Eq. (2.55) equal to zero. The right hand side equals to zero in the first approximation, if $\omega = \omega_0$. This is a trivial result taken into account in writing Eq. (2.54). In the next approximation $\sim \varepsilon$ the nonlinear term does not make any contribution yet. The condition

$$\varepsilon(\omega_0^2/\omega^2)\ddot{u}_1 + \varepsilon\omega_0^2 u_1 = -(1-\omega_0^2/\omega^2)(-\omega^2 A\cos\omega t + \varepsilon\ddot{u}_1) \cong \varepsilon\cdot 2\omega_1\omega_0 A\cos\omega t$$

gives $\omega_1 = 0$ and u_1 may be taken equal to zero as well. In the approximation $\sim \varepsilon^2$ we obtain the corrections due to the nonlinear term:

$$\begin{aligned}
&\varepsilon^2(\omega_0^2/\omega^2)\ddot{u}_2 + \varepsilon^2\omega_0^2 u_2 \\
&\qquad = \omega_0^2\gamma\varepsilon^2 A^3\cos^3\omega t - (1-\omega_0^2/\omega^2)(-\omega^2 A\cos\omega t + \varepsilon^2\ddot{u}_2) \\
&\qquad \cong \varepsilon^2\cdot\tfrac{1}{4}\omega_0^2\gamma A^3(\cos 3\omega t + 3\cos\omega t) + \varepsilon^2\cdot 2\omega_0\omega_2 A\cos\omega t.
\end{aligned} \tag{2.57}$$

Here in the right hand side there are resonant (secular) terms $\propto \cos\omega t$ and the condition that they cancel each other yields the correction term to the frequency,

$$\omega_2 = -\tfrac{3}{8}\omega_0\gamma A^2,$$

so the frequency of vibrations in this approximation is equal to

$$\omega = \omega_0 + \varepsilon^2\omega_2 = \omega_0\left(1 - \tfrac{3}{8}\gamma\varepsilon^2 A^2\right). \tag{2.58}$$

For $\gamma = \frac{1}{6}$ this expression coincides with Eq.(2.53). The other terms of Eq. (2.57) allow one to calculate the amplitude of the third harmonic $\propto \cos 3\omega t$.

Thus, up to the term of the order of magnitude $\sim \varepsilon^2$ in the amplitude of vibrations, we have found that oscillations of the nonlinear pendulum are described by Eq. (2.56) with the frequency given by Eq. (2.58). But where are the different time scales in this calculation? It is clear that they are hidden in different scales of the frequency ω_0 and the nonlinear correction to it. Indeed, let us rewrite the solution (2.56) in the form

$$u = a(t)\cdot\exp(-i\omega_0 t) + a^*(t)\cdot\exp(i\omega_0 t), \tag{2.59}$$

where

$$a(t) = (A/2)\exp\left(\tfrac{3}{8}i\omega_0\gamma A^2\varepsilon^2 t\right). \tag{2.60}$$

Evidently, the last expression describes a slowly varying amplitude obeying the equation

$$da/dT_2 = \tfrac{3}{2}i\omega_0\gamma a^* a^2, \qquad |a|^2 = \tfrac{1}{4}|A|^2 = \mathrm{const}, \tag{2.61}$$

where $T_2 = \varepsilon^2 t$ is a slow time variable. Thus, the condition of elimination of secular terms may be formulated as a differential equation for slowly varying amplitudes. We have obtained this equation in a roundabout way by means of a calculation of the nonlinear correction to the frequency of vibrations. However, this calculation may be organized in such a way that leads directly to Eq. (2.61). Just this form of the secular perturbation theory will be used later for derivation of nonlinear evolution equations, so let us show how it can be done for this simple example.

We introduce a variable $u(t)$ by definition $q = \varepsilon u$ and rewrite Eq. (2.48) and the initial conditions (2.49) in the form

$$\ddot{u} + \omega_0^2 u = \gamma\omega_0^2\varepsilon^2 u^3, \qquad u(0) = A, \quad \dot{u}(0) = 0, \tag{2.62}$$

so that the small parameter ε measures a relative magnitude of the nonlinear term. We look for the solution in the form of a series expansion

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \dots, \tag{2.63}$$

where

$$u_0(t) = a(t)\exp(-i\omega_0 t) + a^*(t)\exp(i\omega_0 t), \tag{2.64}$$

that is, we imply from the very beginning that the nonlinear correction to the frequency is included into the slow dependence of the amplitude $a(t)$ on time t. This slowness is expressed by the following expansion of the derivative of $a(t)$ with respect to time t:

$$da/dt = \varepsilon f_1(a, a^*) + \varepsilon^2 f_2(a, a^*) + \dots. \tag{2.65}$$

We are going to obtain equations for f_1, $f_2, \dots$ from the condition that the secular terms have to be eliminated.

It is convenient to represent the expansion (2.65) in the form

$$da/dt = \varepsilon\,\partial a/\partial T_1 + \varepsilon^2 \partial a/\partial T_2 + \dots, \tag{2.66}$$

that is, we assume that the slow amplitude a is formally a function of many slow time variables of different scales,

$$a = a(T_1, T_2, \dots),$$

where $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t, \dots$. One should remember, however, that T_1, $T_2, \dots$ are not independent variables and correspond to different scales of a single time variable t. Introduction of the multi-scale variables T_1, $T_2, \dots$ enables one to order the series expansions in powers of ε, but when all necessary derivatives $\partial a/\partial T_n$ are found, the amplitude $a(t)$ has to be calculated according to Eq. (2.66) rather than by integration of the derivatives $\partial a/\partial T_n$ over 'many variables T_1, $T_2, \dots$'.

Substitution of Eq. (2.63) into Eq. (2.62) gives at the order $\sim \varepsilon^0$

$$\ddot{u}_0 + \omega_0^2 u_0 = 0$$

with the solution (2.64), where $a(t)$ is a slow function with the derivative given by Eq. (2.65). At the next order $\sim \varepsilon^1$ we obtain the equation for u_1,

$$\ddot{u}_1 + \omega_0^2 u_1 = 2i\omega_0 f_1 e^{-i\omega_0 t} + \text{c.c.}$$

The condition that the secular term has to be eliminated gives $f_1 = 0$ (recall that there is no frequency shift in this approximation), and, hence, we may put $u_1 = 0$. At the next order $\sim \varepsilon^2$ we obtain

$$\ddot{u}_2 + \omega_0^2 u_2 = 2i\omega_0 f_2 e^{-i\omega_0 t} - \gamma\omega_0^2 \left(a^3 e^{-3i\omega_0 t} + 3a^2 a^* e^{-i\omega_0 t}\right) + \text{c.c.},$$

and for elimination of the secular terms we choose

$$f_2(a, a^*) \equiv \partial a/\partial T_2 = \tfrac{3}{2} i\omega_0 \gamma a^2 a^*, \qquad (2.67)$$

which is just Eq. (2.61). Then u_2 is governed by the equation

$$\ddot{u}_2 + \omega_0^2 u_2 = -\gamma\omega_0^2 a^3 e^{-3i\omega_0 t} + \text{c.c.}$$

with the obvious solution

$$u_2 = (\gamma a^3/8) e^{-3i\omega_0 t} + \text{c.c.}$$

As was expected, there is no growth of the correction term u_2 with time t.

After we have gained this experience in consideration of a simple example of one nonlinear oscillator, let us return to the system of coupled pendula, where the propagation of a weak long wave with small but finite amplitude is governed by Eq. (2.44). We want to describe the evolution of a wave packet presented by a superposition of linear waves with the dispersion relation (2.45). We suppose that the packet is concentrated in a narrow band of wavenumbers with a small width $\sim \mu k_0$, $\mu \ll 1$, around the wavevector k_0 of the carrier wave, that is, the space width of the packet, $\sim (\mu k_0)^{-1}$, is much longer than the wavelength $2\pi/k_0$. We introduce a small parameter ε which measures the magnitude of the amplitude q. As we know from a linear analysis (see, e.g., Sec. 1.2), the dispersion effects are described by the second derivative of the envelope with respect to the space coordinate, that is in Eq. (2.44) they have the order of magnitude $\sim c^2(\mu k_0)^2\varepsilon$. The nonlinear effects are described by the last term in Eq. (2.44) and have the order of magnitude $\sim \omega_0^2\gamma\varepsilon^3$. Since we want them to be balanced, we get the relation between the parameters,

$$\mu^2\varepsilon \sim \varepsilon^3, \quad \text{or} \quad \mu \sim \varepsilon.$$

Now we can introduce the slow space coordinate,

$$X = \varepsilon x. \tag{2.68}$$

We shall not use here dimensionless coordinates to make the physical meaning of the results more transparent. To distinguish the processes at different time scales (oscillations with the frequency ω_0, motion with the group velocity, dispersive spreading of the packet), we introduce a sequence of time variables,

$$t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \ldots, \tag{2.69}$$

hence, the time derivative is to be calculated according to the rule

$$\partial/\partial t \to \partial/\partial t + \varepsilon\partial/\partial T_1 + \varepsilon^2\partial/\partial T_2 + \ldots. \tag{2.70}$$

We expand the amplitude $q(x,t)$ in powers of ε,

$$\begin{aligned} q(x,t) = \varepsilon & \left[a(X, T_1, T_2, \ldots)e^{i(k_0x-\omega(k_0)t)} + \text{c.c.}\right] \\ & + \varepsilon^2 u_1(x, t, X, T_1, T_2, \ldots) + \varepsilon^3 u_2(x, t, X, T_1, T_2, \ldots) + \ldots, \end{aligned} \tag{2.71}$$

so that $u = \varepsilon a(X, T_1, T_2, \ldots)$ is the complex envelope which evolution we are interested in. Emphasize again that T_1, $T_2, \ldots$ cannot be considered as independent variables. They enable us only to expand the time derivatives in powers of ε in a convenient way (see Eq. (2.70), and after calculation of contributions at different orders in ε we have to return to the derivative with respect to the actual time variable t. According to the rule (2.70) the second time derivative in Eq. (2.44) is expanded in powers of ε as follows,

$$\begin{aligned} q_{tt} =& \varepsilon\left\{\left[\varepsilon^2(\partial^2 a/\partial T_1^2) - 2i\omega(k_0)\left(\varepsilon(\partial a/\partial T_1) + \varepsilon^2(\partial a/\partial T_2) + \ldots\right)\right.\right. \\ & \left.\left. - \ \omega_0^2(k_0)a\right] e^{i(k_0x-\omega(k_0)t)} + \text{c.c.}\right\} \\ & + \varepsilon^2(\partial^2 u_1/\partial t^2) + 2\varepsilon^3(\partial^2 u_1/\partial t\partial T_1) + \varepsilon^3(\partial^2 u_2/\partial t^2) + \ldots, \end{aligned} \tag{2.72}$$

where only the necessary terms up to the order $\sim \varepsilon^3$ are written down (recall that the nonlinear term has the order of magnitude $\sim \varepsilon^3$). In a similar way, the second space coordinate derivative is equal to

$$q_{xx} = \varepsilon \left\{ \left[\varepsilon^2(\partial^2 a/\partial X^2) + 2ik_0(\partial a/\partial X) - k_0^2 a\right] e^{i(k_0 x - \omega(k_0)t)} + \text{c.c.}\right\} \\ + \varepsilon^2(\partial^2 u_1/\partial X^2) + 2\varepsilon^3(\partial^2 u_1/\partial x \partial X) + \varepsilon^3(\partial^2 u_2/\partial x^2)\dots, \tag{2.73}$$

and with the same accuracy

$$q^3 = \varepsilon^3 \left[a^3 e^{3i(k_0 x - \omega(k_0)t)} + 3a^2 a^* e^{i(k_0 x - \omega(k_0)t)} + \text{c.c.}\right] + \dots. \tag{2.74}$$

On substitution of these expressions into Eq. (2.44), we obtain the sequence of equations for the coefficients of powers of ε.

At the leading order we get

$$\omega^2(k_0) - c^2 k_0^2 - \omega_0^2 = 0, \tag{2.75}$$

which is the dispersion relation (2.45) for linear waves.

At the next order $\sim \varepsilon^2$ we obtain the equation

$$\partial^2 u_1/\partial t^2 - c^2(\partial^2 u_1/\partial x^2) + \omega_0^2 u_1 \\ = 2i\left[\omega(k_0)(\partial a/\partial T_1) + c^2 k_0(\partial a/\partial X)\right] e^{i(k_0 x - \omega(k_0)t)} + \text{c.c.}.$$

On the left hand side there is the operator of the linear approximation $L_1 = \partial^2/\partial t^2 - c^2\partial^2/\partial x^2 + \omega_0^2$ acting on u_1, and on the right hand side we have the driving force at the wavevector and frequency satisfying the dispersion relation of linear waves. Hence, this 'force' is in resonance with the eigenfunctions of the operator L_1 which leads to growth of u_1 as some power of time t. According to the formulated above rule, the envelope a has to evolve in such a way that the secular term is eliminated. (This procedure is similar to derivation of Eq. (2.67) which governs a slow evolution of the amplitude of the nonlinear pendulum.) Thus, at the order $\sim \varepsilon^2$ the envelope amplitude A satisfies the equation

$$\omega(k_0)\partial a/\partial T_1 + c^2 k_0 \partial a/\partial X = 0. \tag{2.76}$$

Since the dispersion relation (2.75) gives $c^2 k_0/\omega(k_0) = \omega'(k_0)$ and in this order $\partial a/\partial t \cong \varepsilon\partial a/\partial T_1$, $\partial a/\partial x = \varepsilon \partial a/\partial X$, the equation for the envelope $u = \varepsilon a$ has the form

$$u_t + \omega'(k_0)u_x = 0. \tag{2.77}$$

Thus, at the time scale $t \sim 1/\varepsilon$ the dispersion effects lead to the well-known motion of the wave packet with the group velocity $\omega'(k_0)$.

At last, at the order $\sim \varepsilon^3$ we obtain the equation

$$\begin{aligned} &\partial^2 u_2/\partial t^2 - c^2(\partial^2 u_2/\partial x^2) + \omega_0^2 u_2 \\ &= \left[2i\omega(k_0)(\partial a/\partial T_2) - \partial^2 a/\partial T_1^2 + c^2(\partial^2 a/\partial X^2) + 3\omega_0^2\gamma a^2 a^*\right] e^{i(k_0 x - \omega(k_0)t)} \\ &\quad + \omega_0^2 \gamma a^3 e^{3i(k_0 x - \omega(k_0)t)} + \text{c.c.} - 2(\partial^2 u_1/\partial t\partial T_1) + 2c^2(\partial^2 u_1/\partial x \partial X). \end{aligned}$$

The first term in the right hand side is again in resonance with the eigenfunctions of the operator L_1 in the left hand side, and for elimination of this secular term we require that the amplitude satisfies the equation

$$2i\omega(k_0)(\partial a/\partial T_2) - \partial^2 a/\partial T_1^2 + c^2(\partial^2 a/\partial X^2) + 3\omega_0^2\gamma a^2 a^* = 0. \tag{2.78}$$

The second derivative with respect to T_1 can be transformed with the help of Eq. (2.76) with the same accuracy,

$$\partial^2 a/\partial T_1^2 = (c^4 k_0^2/\omega^2(k_0))(\partial^2 a/\partial X^2),$$

and after that with the use of the identity

$$c^2 - c^4 k_0^2/\omega^2(k_0) = \omega(k_0)\omega''(k_0),$$

which follows from the dispersion relation (2.75), Eq. (2.78) transforms to

$$i a_{T_2} + \tfrac{1}{2}\omega''(k_0)\, a_{XX} + \tfrac{3}{2}(\omega_0^2\gamma/\omega(k_0))\, a^2 a^* = 0. \tag{2.79}$$

Note that in the limit of an infinitely long wavelength $k_0 \to 0$, when all pendula oscillate in phase, the wave frequency $\omega(k_0)$ goes to ω_0 and the nonlinear term in Eq. (2.79) coincides with that of Eq. (2.61).

Now we return to the physical time t and space x coordinates according to

$$\varepsilon a_X = a_x, \quad \varepsilon^2 a_{T_2} = a_t - \varepsilon a_{T_1} = a_t + \varepsilon(k_0/\omega(k_0))\, a_X = a_t + \omega'(k_0)\, a_x,$$

and arrive at the final equation for the envelope amplitude $u = \varepsilon a$:

$$i(u_t + \omega'(k_0)\, u_x) + \tfrac{1}{2}\omega''(k_0)\, u_{xx} + \tfrac{3}{2}(\gamma/\omega(k_0))|u|^2 u = 0. \tag{2.80}$$

This is the NLS equation which has exactly the same form as Eq. (2.47) derived for the envelope of the electromagnetic pulse propagating through a weakly nonlinear and dispersive medium. It is interesting that even in this simple mechanical model the wave with constant amplitude is modulationally unstable if $\gamma > 0$ (see Sec. 1.6.3).

After consideration of the two examples, an electromagnetic wave in a medium with cubic nonlinearity and a wave in a lattice of coupled nonlinear pendula, one might think that the cubic nonlinearity in the NLS equation for the envelope amplitude can arise only due to the cubic nonlinearity in the 'exact' wave equation (as Eq. (1.312) and (2.44)). However, if one takes into account that the dispersion effects are balanced by the nonlinear effects at the second order in the amplitude ($\sim \varepsilon^2$), then it is easy to see that the quadratic nonlinearity in the initial equations can lead at the first order in ε to the correction term in the form $u_1 \sim a^* a$ (apart of $u_1 \sim a^2 e^{2i\theta}$, $a^* e^{-2i\theta}$), and, hence, at the next order $\sim \varepsilon^2$ we can obtain the cubic term $a^2 a^*$ in the equation for the envelope. In the next section we shall consider an example when such a situation takes place.

2.2.2 *Modulation of a wave on the shallow water surface*

As we know, long waves on a shallow water surface are described by the KdV equation

$$\zeta_t + \sqrt{gh}\zeta_x + \tfrac{3}{2}\sqrt{g/h}\zeta\zeta_x + (h^2\sqrt{gh}/6)\zeta_{xxx} = 0.$$

This equation is derived under the supposition that the amplitude $\zeta \ll h$ is large enough for balance of the nonlinear $\sim \zeta\zeta_x$ and dispersion $\sim \zeta_{xxx}$ terms. However, this equation holds even for smaller values of the amplitude ζ as long as the nonlinear term is greater than the next correction term $\sim h^4\sqrt{gh}\,\zeta_{xxxxx}$ in the expansion of the dispersion relation. Hence, there is a region of the amplitude values,

$$(kh)^4 \ll \zeta/h \ll (kh)^2,$$

for the wave with the wavevector k, where the nonlinear term in the KdV equation may be considered as a small correction. In the treatment of such a wave we can adopt the point of view of the preceding section, that is, suppose that the carrier wave satisfies the linear equation, and the wave packet (superposition of linear waves with wavevectors from a narrow band around the wavevector k_0 of the carrier wave) evolves under influence of weak dispersion and nonlinearity effects. So we have to derive the evolution equation for the envelope of the wave packet.

We take the KdV equation in its canonical form,

$$q_t + 6qq_x + q_{xxx} = 0, \tag{2.81}$$

and introduce a small parameter ε which measures the magnitude of the amplitude $q = \varepsilon u$, so that Eq. (2.81) may be rewritten in the form

$$u_t + u_{xxx} = -6\,\varepsilon u u_x \tag{2.82}$$

with a small parameter ε as a coefficient in the nonlinear term. In the linear approximation we have the solution in the form of a travelling wave,

$$u = ae^{i\theta} + a^* e^{-i\theta}, \quad \theta = kx - \omega t, \quad \omega = -k^3, \tag{2.83}$$

where $a = \text{const}$. If we have a wave packet made of these linear waves, then the coefficient a may be considered as an envelope function slowly evolving under influence of the dispersion and nonlinearity effects. Hence, we write the series expansion in powers of ε:

$$u(x,t) = a(X, T_1, T_2, \ldots)e^{i\theta} + a^*(X, T_1, T_2, \ldots)e^{-i\theta} + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots, \tag{2.84}$$

where $X = \varepsilon x$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t, \ldots$, are the slow variables distinguishing different scales of space x and time t coordinates. From the above examples we know that at the first order in ε we get the motion of the envelope with the group velocity (and in the case under consideration also the generation of the second harmonic $\propto e^{\pm 2i\theta}$), and the envelope evolution arises only at the order $\sim \varepsilon^2$. Consequently, we have to keep the terms up to the order $\sim \varepsilon^2$. Then, the linear terms in the left hand side of Eq. (2.82) are equal to

$$\begin{aligned} u_t =& (\varepsilon a_{T_1} + \varepsilon^2 a_{T_2} - i\omega a)e^{i\theta} + \text{c.c.} \\ &+ \varepsilon u_{1,t} + \varepsilon^2 u_{1,T_1} + \varepsilon^2 u_{2,t} + \ldots, \end{aligned} \tag{2.85}$$

$$\begin{aligned} u_{xxx} = & (3ik\varepsilon^2 a_{XX} - 3k^2\varepsilon a_X - ik^3 a)e^{i\theta} + \text{c.c.} \\ &+ \varepsilon u_{1,xxx} + 3\varepsilon^2 u_{1,xxX} + \varepsilon^2 u_{2,xxx} + \ldots. \end{aligned} \tag{2.86}$$

Due to a small factor ε in the right hand side of Eq. (2.82), the nonlinear term uu_x may be calculated with the accuracy $\sim \varepsilon$,

$$\begin{aligned} uu_x = & ika^2 e^{2i\theta} - ika^{*2} e^{-2i\theta} + \varepsilon (aa^*)_X + ik\varepsilon u_1 (ae^{i\theta} - a^* e^{-i\theta}) \\ &+ \varepsilon u_{1,x}(ae^{i\theta} + a^* e^{-i\theta}) + \varepsilon a a_X e^{2i\theta} + \varepsilon a^* a^*_X e^{-2i\theta}. \end{aligned}$$

On substitution of these expressions into Eq. (2.82), we obtain a sequence of equations for the coefficients of the series expansion.

In the leading approximation $\sim \varepsilon^0$ we get the dispersion relation $\omega = -k^3$ for linear waves.

In the next approximation $\sim \varepsilon$ we obtain the equation for u_1:

$$u_{1,t} + u_{1,xxx} = [-(a_{T_1} - 3k^2 a_X)e^{i\theta} + \text{c.c.}] - 6ika^2 e^{2i\theta} + 6ika^{*2}e^{-2i\theta}. \quad (2.87)$$

The resonant terms in square brackets have to be eliminated, and this condition gives the equation

$$a_{T_1} - 3k^2 a_X = 0, \quad (2.88)$$

which means that the envelope amplitude moves with the group velocity. Then Eq. (2.87) is readily solved to give

$$u_1 = b + (a^2/k^2)\, e^{2i\theta} + (a^{*2}/k^2)\, e^{-2i\theta}, \quad (2.89)$$

where we have included the solution $u_1 = b$ of the homogeneous equation $u_{1,t} + u_{1,xxx} = 0$, because it cannot be considered as already taken into account in the initial approximation (2.83).

At the order $\sim \varepsilon^2$ Eq. (2.82) yields

$$\begin{aligned} u_{2,t} + u_{2,xxx} = -(u_{1,T_1} + 6(aa^*)_X) - [(a_{T_2} + 3ika_{XX})e^{i\theta} + \text{c.c.}] \\ -6iku_1(ae^{i\theta} - a^*e^{-i\theta}) - 6u_{1,x}(ae^{i\theta} + a^*e^{-i\theta}) \\ -[(6ika^2 + 6aa_X)e^{2i\theta} + \text{c.c.}] - 3u_{1,xxX}. \end{aligned} \quad (2.90)$$

We have to substitute u_1 from Eq. (2.89) into this equation and to eliminate the secular terms. Now we have the secular terms of two types—at the 'zero frequency' harmonic ($\propto e^0$) and at the main harmonic ($\propto e^{i\theta}$). Elimination of the first one gives with account of Eq. (2.88)

$$b_{T_1} = -6(aa^*)_X = -(2/k^2)(aa^*)_{T_1},$$

and, hence,

$$b = -(2/k^2)aa^*. \quad (2.91)$$

Since b has a meaning of a mean value in the Fourier expansion of u, we see that the wave packet changes a mean value of the water level at the order $\sim \varepsilon$. In this calculation we have taken into account that downstream the packet, where $a \to 0$, the shift of a mean value vanishes, $b \to 0$, and, hence, the integration constant in Eq. (2.91) must be equal to zero.

Elimination of the secular terms at the harmonic $e^{i\theta}$ gives

$$-(a_{T_2} + 3ika_{XX}) - 6ikba + 6ik(a^2/k^2)a^* - 6 \cdot 2ik \cdot (a^2/k^2)a^* = 0,$$

or, after substitution of Eq. (2.91),

$$a_{T_2} + 3ika_{XX} - (6i/k)a^2a^* = 0. \tag{2.92}$$

Returning to the physical variables x and t according to $\varepsilon a_X = a_x$, $\varepsilon^2 a_{T_2} = a_t - 3k^2 a_x$ yields the NLS equation for the envelope $u = \varepsilon a$:

$$i(u_t - 3k^2 u_x) - 3ku_{xx} + (6/k)|u|^2 u = 0. \tag{2.93}$$

Since the linear dispersion relation gives $\omega' = -3k^2$, $\omega'' = -6k$, this equation reproduces a usual structure of the NLS equation. Note that signs before the dispersive and nonlinear terms are opposite, that is, Eq. (2.93) is a defocusing NLS equation with the canonical form (1.329). Hence, the shallow water wave with constant amplitude is modulationally stable.

We would like to emphasize one subtle point in the presented derivation. Since the linear KdV operator $L_1 = \partial_t + \partial_{xxx}$ has the eigenfunction with the zero frequency, we have found a change of a mean value at the place of the wave packet (see Eq. (2.91)). Hence, the solution of the KdV equation (2.81) with constant amplitude $a = \text{const}$ has the form

$$q = \varepsilon u = 2\varepsilon a \cos\theta - 2\varepsilon^2 a^2/k^2 + (2\varepsilon^2 a^2/k^2)\cos 2\theta + \dots, \tag{2.94}$$

where

$$\theta = kx - \omega t, \qquad \omega = -k^3 - 6\varepsilon^2 a^2/k + \dots, \tag{2.95}$$

and the nonlinear correction to the frequency is written in accordance with the nonlinear term in Eq. (2.93). But the dispersion relation for the shallow water wave depends on the basin's depth! Therefore, the frequency shift in Eq. (2.95) calculated for the localized wave packet (recall that we take the integration constant equal to zero in Eq. (2.91)) differs from the frequency shift calculated for the Stokes wave with the constant mean depth. To clarify this difference, let us turn to the exact periodic solution of the KdV equation (see Sec. 1.5.2) and expand it into the Taylor series in powers of the amplitude. As it follows from comparison of linear periodic terms in Eqs. (2.94) and (1.302), to pass to the notations of the present section we have to replace a by $2\varepsilon a$ and u by q in formulas of Sec. 1.5.2. Hence, a mean amplitude given by Eq. (1.301) is equal to

$$\bar{q} = (4\varepsilon a/m)\mathrm{E}(m)/\mathrm{K}(m) + \gamma \cong (4\varepsilon a/m)(1 - m/2 - m^2/16) + \gamma$$

(see Eqs. (A.11)). By equating it to the constant term in Eq. (2.94), we find

$$\gamma = -4\varepsilon a/m + 2\varepsilon a + \varepsilon am/4 - 2\varepsilon^2 a^2/k^2,$$

and the velocity (1.296) is equal to

$$V = -8\varepsilon a/m + 4\varepsilon a + (3/2)\varepsilon am - 12\varepsilon^2 a^2/k^2. \tag{2.96}$$

The relationship between the wavenumber k and the parameter m follows from Eq. (1.300) and the series expansion

$$k^2 = 4\pi^2/L^2 = 2\pi^2\varepsilon a/m\mathrm{K}^2(m) \cong 8\varepsilon a/m - 4\varepsilon a - (3/4)\varepsilon am. \tag{2.97}$$

If we express m in terms of k^2 by means of Eq. (2.97) and substitute this m into Eq. (2.96), we shall find the phase velocity and the frequency of the wave,

$$V = -k^2 - 6\varepsilon^2 a^2/k^2, \quad \omega = kV = -k^3 - 6\varepsilon^2 a^2/k, \tag{2.98}$$

in agreement with Eq. (2.95). But if we assumed that the wave motion does not change the mean depth, that is $\bar{q} = 0$, and this assumption is natural for the treatment of the periodic waves with constant amplitude, we would obtain instead of Eq. (2.98)

$$V_{\mathrm{Stokes}} = -k^2 + 6\varepsilon^2 a^2/k^2, \quad \omega_{\mathrm{Stokes}} = -k^3 + 6\varepsilon^2 a^2/k, \tag{2.99}$$

with the opposite sign of the nonlinear corrections compared with Eq. (2.98). These expressions can be obtained by a direct calculation with the use of the perturbation theory (see Ex. 2.3). This remark shows that, generally speaking, we cannot reconstruct the NLS equation from known expressions for the dispersion and nonlinear corrections. At first we have to make sure that the nonlinear correction is calculated in a self-consistent way, that is, there is no contribution due to a local change of the mean intensity $|u|^2$. These 'dangerous' situations are possible when the linear operator of the equation under consideration (like $L_1 = \partial_x + \partial_{xxx}$ in the KdV equation) has constant eigenfunction.

For the first time expansions in powers of the amplitude were introduced by Stokes in 1847 for deep water waves. It was one of the first works on the theory of nonlinear waves, and we shall consider this problem in some detail in the next section.

2.2.3 *Deep water surface waves*

The dispersion relation

$$\omega^2 = gk \tanh(kh)$$

for waves propagating on the water surface in the basin with depth h was obtained in Sec. 1.1.2. If the basin's depth h is much greater then the wavelength, $kh \gg 1$, then the dispersion relation simplifies to

$$\omega = \sqrt{gk}. \tag{2.100}$$

Let us consider the evolution of a wave packet made of these linear waves under influence of the dispersion and nonlinear effects. Now in our disposal there is no such a simplified description as we had in the case of shallow water waves (namely, the KdV equation), and therefore we have to return to the exact equations and boundary conditions. For the first time this problem was considered by Stokes who used the perturbation theory with respect to a small parameter equal to the ratio of the wave amplitude to the wavelength with account of the dependence of the frequency on the amplitude. (Similar calculation is considered in Ex. 2.3 about shallow water waves described by the KdV equation.) These calculations are quite lengthy and to simplify them, we shall use a nice method of Rayleigh based on some special mathematical properties of a two-dimensional flow of incompressible fluid.

We suppose that a wave with a small but finite amplitude propagates with the velocity V on the deep water surface. We shall consider the flow in the reference frame moving downstream with the velocity V, where this flow is stationary, that is, the velocity field $\mathbf{v}(\mathbf{r}, t)$ does not depend on time. The flow remains potential in the moving reference frame, so $\mathbf{v}(\mathbf{r})$ can be represented as a gradient of some potential φ (we suppose that all quantities do not depend on y coordinate):

$$\mathbf{v} = (v_{(x)}, 0, v_{(z)}) = (\varphi_x, 0, \varphi_z), \tag{2.101}$$

where vector indices are shown in parenthesis. From the continuity equation for incompressible fluid, $\nabla \cdot \mathbf{v} = 0$, we obtain the Laplace equation for the potential,

$$\varphi_{xx} + \varphi_{zz} = 0, \quad -\infty < z < \zeta(x), \tag{2.102}$$

where $\zeta(x)$ is a stationary profile of the surface as a function of the space coordinate x.

At infinitely large values of z the potential φ has to satisfy the boundary condition

$$\varphi \to -Vx, \qquad z \to -\infty, \tag{2.103}$$

which means that far below the disturbed surface the liquid moves with the velocity opposite to the velocity of our reference frame, that is, in the laboratory frame it is at rest.

The Bernoulli law

$$v^2/2 + gz = \mathrm{const}$$

holds for the stationary flow of liquid including its surface, so we obtain the 'dynamic' boundary condition at the surface in the form

$$(\varphi_x^2 + \varphi_z^2)/2 + gz\big|_{z=\zeta} = \mathrm{const}. \tag{2.104}$$

To formulate the 'kinematical' boundary condition, let us introduce a so called stream function $\psi(x, z)$ according to

$$\mathbf{v} = (\psi_z, 0, -\psi_x), \tag{2.105}$$

so that the continuity equation is satisfied by definition. At large depth we have

$$\psi \to -Vz, \qquad z \to -\infty. \tag{2.106}$$

As follows from a simple calculation,

$$\psi_{xx} + \psi_{zz} = -\partial v_{(z)}/\partial x + \partial v_{(x)}/\partial z = -\varphi_{xz} + \varphi_{zx} = 0, \tag{2.107}$$

the function ψ also satisfies the Laplace equation. From Eqs. (2.101) and (2.105) we obtain the Cauchy-Riemann conditions, $\varphi_x = \psi_z$, $\varphi_z = -\psi_x$, for the function $w(s) = \varphi(x, z) + i\psi(x, z)$, $s = x + iz$, so that $w(z)$ is an analytical function of the complex variable s. The stream function $\psi(x, s)$ has very important physical meaning which explains its name. The streamlines of the current in a two dimensional flow are determined as solutions of the equation

$$dx/v_x = dz/v_z, \quad \text{or} \quad -v_z dx + v_x dz = 0,$$

which means that the tangential direction of the streamline coincides with the direction of the flow velocity. Taking into account Eq. (2.105), we obtain $\psi_x dx + \psi_z dz = 0$, and, hence, $\psi = \text{const}$ along the streamlines. But the surface of the fluid is formed by the streamlines, so ψ must be constant on the surface. Since ψ is defined up to an additive constant, we may choose

$$\psi(\zeta) = 0 \tag{2.108}$$

as the kinematical boundary condition.

Now, if the wavevector is equal to k, we can calculate the Fourier series expansions as for the profile $\zeta(x)$, so for the potential φ and the stream function ψ. The solutions of the Laplace equations (2.102) and (2.107), satisfying the boundary conditions (2.103) and (2.106), have the Fourier series expansions

$$\begin{aligned} \varphi &= -Vx + \alpha V e^{kz} \sin kx + \dots, \\ \psi &= -Vz + \alpha V e^{kz} \cos kx + \dots, \end{aligned} \tag{2.109}$$

where α is some constant related to the amplitude of the wave. It is easy to see that Eqs. (2.109) correspond to the complex potential $w = -Vs + i\alpha V e^{-iks}$. We shall confine ourselves to the approximation (2.109). The boundary condition (2.108) written as

$$\zeta = \alpha e^{k\zeta} \cos kx, \tag{2.110}$$

defines implicitly the profile $\zeta(x)$. Supposing that $k\zeta \ll 1$, we look for the solution of Eq. (2.110) in the form

$$\zeta = \alpha(1 + k\zeta + \tfrac{1}{2}k^2\zeta^2 + \dots) \cos kx, \quad \zeta = \zeta_0 + \zeta_1 + \zeta_2 + \dots,$$

hence, $\zeta_0 = \alpha \cos kx$, $\zeta_1 = \frac{1}{2}k\alpha^2(1 + \cos 2kx)$, $\zeta_2 = \frac{3}{8}k^2\alpha^3(3\cos kx + \cos 3kx), \dots$, that is,

$$\begin{aligned} \zeta = \tfrac{1}{2}k\alpha^2 + \alpha(1 + \tfrac{9}{8}k^2\alpha^2)\cos kx + \tfrac{1}{2}k\alpha^2 \cos 2kx \\ + \tfrac{3}{8}k^2\alpha^3 \cos 3kx + \dots. \end{aligned} \tag{2.111}$$

As we shall find soon, all written here terms of the Fourier expansion are correct.

Let us define the amplitude a of the wave by the equation

$$2a = \alpha(1 + \tfrac{9}{8}k^2\alpha^2),$$

which gives with the accuracy $\sim a(ka)^2$ the expression for α,

$$\alpha \cong 2a(1 - \tfrac{9}{2}(ka)^2), \tag{2.112}$$

and with the same accuracy the expansion (2.111) takes the form

$$\zeta = 2ka^2 + 2a\cos kx + 2ka^2\cos 2kx + 3k^2a^3\cos 3kx + \dots. \tag{2.113}$$

This is the famous Stokes expansion.

The remaining unused boundary condition (2.104) determines the relationship between the phase velocity V and the amplitude a, that is, the dispersion relation. On substitution of Eqs. (2.109) into Eq. (2.104) we obtain

$$V^2(1 - 2k\zeta + \alpha^2k^2e^{2k\zeta}) + g\zeta = \text{const},$$

or

$$\tfrac{1}{2}V^2(1 + \alpha^2k^2) + (g - kV^2 + \alpha^2k^3V^2)\zeta = \text{const}.$$

Since the left hand side cannot depend on x, we find

$$V^2k(1 - (\alpha k)^2) = g,$$

or, with the accepted accuracy,

$$V = \sqrt{g/k}\,(1 + 2(ak)^2). \tag{2.114}$$

As we see, the boundary conditions are satisfied up to the terms of order $\sim (ak)^2$, and this is the accuracy of the expansions (2.111).

The wave frequency is given by the dispersion relation

$$\omega = kV = \sqrt{gk}\,(1 + 2(ak)^2), \tag{2.115}$$

whose linear part coincides with Eq. (2.100). Thus, the leading term of the Fourier expansion for the deep water wave is equal to

$$\zeta \cong 2a\cos(kx - \omega t) = ae^{-2i\sqrt{gk}\,(ka)^2t} \cdot e^{i(kx - \sqrt{gk}\,t)} + \text{c.c.}, \tag{2.116}$$

that is, the constant amplitude envelope

$$u = a\exp(-2i\sqrt{gk}\,(ka)^2t)$$

satisfies the equation

$$idu/dt = 2\sqrt{gk}\,k^2|u|^2u. \tag{2.117}$$

In contrast to the shallow water wave, the constant term in the expansion (2.111) does not influence on the nonlinear shift of the frequency (a small change of infinite depth does not matter). Therefore, the nonlinear term in the NLS equation must coincide with the right hand side of Eq. (2.117). Collection of terms corresponding to the motion of the packet with the group velocity $\omega' = (g/k)^{1/2}/2$ and to its dispersive spreading with the coefficient $\omega''/2 = -g^{1/2}k^{-3/2}/8$ yields the NLS equation for the envelope amplitude of a deep water wave,

$$i(u_t + (\omega_0/2k)\,u_x) - (\omega_0/8k^2)\,u_{xx} - 2\omega_0 k^2|u|^2u = 0, \tag{2.118}$$

where $\omega_0 = \sqrt{gk}$ is the frequency of the carrier wave. This heuristical derivation of the NLS equation can be confirmed by a strict derivation with the use of the secular perturbation theory.

In contrast to the NLS equation for a shallow water wave, in the deep water case the dispersive and nonlinear terms have the same sign, hence the Stokes wave with constant amplitude is modulationally unstable. The discovery of the modulational instability of deep water waves and its experimental confirmation (Benjamin and Feir, 1967) made a sensation and stimulated considerably further development of the theory of nonlinear waves.

We have shown that the KdV and NLS equations have universal nature and under certain conditions appear as evolution equations for nonlinear waves. There are many other equations which are not so universal, though they appear in some important branches of physics and deserve very thorough study. In the rest part of this Chapter we shall discuss briefly some of these equations.

2.3 Derivative nonlinear Schrödinger equation

2.3.1 *Nonlinear Alfvén wave*

When a fluid consists of electrically charged particles (plasma) and it is imposed into an external magnetic field, its motion is described by the combined system of the hydrodynamical and Maxwell equations. Suppose that the ambient field $\mathbf{B}_0$ is directed along the x axis and it is disturbed by the wave propagating along this axis, that is,

$$\mathbf{B} = (B_0, B_{(y)}, B_{(z)}), \quad |B_{(y)}|, |B_{(z)}| \ll B_0. \tag{2.119}$$

In the long wave limit the fluid's flow is governed by the well-known magnetohydrodynamical equations which lead to several types of linear waves—a slow sound, a fast sound, and Alfvén's wave (see, e.g., Lifshitz and Pitaevskii, 1978). The first two are caused by the combined effect of the magnetic (Lorentz) force and plasma pressure and they become a usual sound wave in a plasma without external magnetic field. The origin of the Alfvén wave is a curvature of the magnetic field lines, so it exists even in incompressible fluid and disappears in the absence of the external magnetic field. The dispersion relation for a linear Alfvén wave has the form

$$\omega = c_A k, \tag{2.120}$$

where

$$c_A = B_0/\sqrt{4\pi m n}, \tag{2.121}$$

m being the ion's mass and n the density of particles. For shorter wavelength we have to take into account weak dispersion effects due to a finite value of the plasma frequency,

$$\omega_{pi} = \sqrt{4\pi n e^2/m}. \tag{2.122}$$

If the magnetic field disturbance is small but finite, we have to take into account also its influence on the density n of the plasma. Since the Alfvén velocity (2.121) depends on the density, we have the nonlinear self-action of the Alfvén wave. The propagation of such a nonlinear wave is governed by the equation

$$i\frac{\partial b_\pm}{\partial \tau} \mp \frac{c}{2\omega_{pi}}\frac{\partial^2 b_\pm}{\partial \xi^2} + i\frac{c_A^2}{4(c_A^2 - c_s^2)}\frac{\partial}{\partial \xi}\left(|b_\pm|^2 b_\pm\right) = 0. \tag{2.123}$$

where $b_\pm = (B_{(y)} \pm iB_{(z)})/B_0$, and $c_s = (dp/d\rho)^{1/2}$ is the sound velocity, $\rho = mn$ the mass density, $\tau = c_A t$, $\xi = x - c_A t$ are the time and space coordinates in the frame of reference moving with the Alfvén velocity. For any choice of signs of the coefficients in Eq. (2.123) this equation can be transformed to the canonical form. To be definite, let us consider the b_--wave propagating in a high-pressure plasma $c_s > c_A$. Then in new variables

(recall that $\tau = c_A t$ has a dimension of length)

$$t' = (\omega_{pi}/2c)\,\tau, \quad x' = (\omega_{pi}/c)\,\xi, \quad b_- = \left(2(c_s^2 - c_A^2)^{1/2}/c_A\right) u$$

we obtain (omitting primes for simplicity)

$$iu_t + u_{xx} - 2i\left(|u|^2 u\right)_x = 0. \tag{2.124}$$

Introducing the amplitude a and the phase ϕ of the wave according to

$$u = a\exp(i\phi), \tag{2.125}$$

we can transform Eq. (2.124) to the form

$$\begin{aligned} &a_t - 6a^2 a_x + (1/a)(a^2\phi_x)_x = 0,\\ &\phi_t - 2a^2\phi_x + \phi_x^2 - a_{xx}/a = 0. \end{aligned} \tag{2.126}$$

Another convenient form can be obtained, if one introduces the local frequency and the wavevector

$$\omega = -\phi_t, \qquad k = \phi_x, \tag{2.127}$$

so that Eqs. (2.126) transform to

$$\left(a^2\right)_t + \left(2a^2 k - 3a^4\right)_x = 0, \quad k_t + \omega_x = 0, \tag{2.128}$$

where

$$\omega = k^2 - 2a^2 k - a_{xx}/a \tag{2.129}$$

is the dispersion relation.

2.3.2 *Dispersionless limit, characteristic speeds and Riemann invariants*

The last term in the expression (2.129) describes the dispersion effects for the envelope a. When this term is dropped, we obtain the dispersionless equations, which may be written in a hydrodynamical form, if one introduces the 'density' ρ and the 'velocity' v according to

$$\rho = a^2, \qquad v = \phi_x = k. \tag{2.130}$$

Then we obtain the hydrodynamical type equations

$$\rho_t + (2v - 6\rho)\rho_x + 2\rho v_x = 0, \quad v_t + (2v - 2\rho)v_x - 2v\rho_x = 0. \tag{2.131}$$

Calculation of the characteristic speeds by the method of Sec. 1.3.4 gives

$$v_{\pm} = 2(v - 2\rho) \pm 2\sqrt{\rho(\rho - v)}. \tag{2.132}$$

We see that long waves with small amplitudes

$$\rho < v \tag{2.133}$$

are modulationally unstable, because the characteristic speeds become complex in this case. In the limit $v \to 0$ of infinitely long waves we obtain

$$v_{-} = -6\rho = -6a^2, \qquad v_{+} = -2\rho = -2a^2. \tag{2.134}$$

The system (2.131) can be written in terms of the Riemann invariants

$$\lambda_{\pm} = v - 2\rho \pm 2\sqrt{\rho(\rho - v)}, \tag{2.135}$$

so that

$$\frac{\partial \lambda_{\pm}}{\partial t} + v_{\pm}\frac{\partial \lambda_{\pm}}{\partial x} = 0, \tag{2.136}$$

where $v_{\pm} = v_{\pm}(\lambda_{+}, \lambda_{-})$ are expressed as functions of $\lambda_{\pm}$,

$$v_{+}(\lambda_{+}, \lambda_{-}) = \tfrac{1}{2}(3\lambda_{+} + \lambda_{-}), \quad v_{-}(\lambda_{+}, \lambda_{-}) = \tfrac{1}{2}(\lambda_{+} + 3\lambda_{-}). \tag{2.137}$$

Equations (2.136) with the characteristic speeds (2.137) coincide with the Euler hydrodynamical equations in the Riemann invariant form (1.172) for the value of the adiabatic constant $\gamma = 2$, but with quite nontrivial relationship of the Riemann invariants (2.135) with the physical variables ρ and v.

Substitutions

$$N = 4\rho(\rho - v), \qquad q = 2(2\rho - v)$$

transform the system (2.131) to the shallow water equations

$$N_t + qN_x + Nq_x = 0, \quad q_t + qq_x + N_x = 0.$$

2.3.3 *Modulational instability of a plane wave*

Equations (2.126) have a plane wave solution with the constant amplitude:

$$a = a_0 = \text{const}, \quad \phi = k_0 x - \omega_0 t, \quad \omega_0 = k_0^2 - 2a_0^2 k_0. \tag{2.138}$$

Let us linearize these equations with respect to small deviations from this solution:

$$a = a_0 + \tilde{a}(x,t), \quad \phi = k_0 x - \omega_0 t + \tilde{\phi}(x,t). \tag{2.139}$$

After simple calculation we obtain

$$\tilde{a}_t + 2k_0\tilde{a}_x + a_0\tilde{\phi}_{xx} - 6a_0^2\tilde{a}_x = 0, \quad \tilde{\phi}_t + 2(k_0 - a_0^2)\tilde{\phi}_x - 4a_0k_0\tilde{a} - \tilde{a}_{xx}/a_0 = 0.$$

If we look for the solution $\tilde{a}$, $\tilde{\phi} \propto \exp[i(Kx - \Omega t)]$, then we obtain the dispersion relation for the waves of small modulations,

$$\Omega/K = 2(k_0 - 2a_0^2) \pm \sqrt{4a_0^2(a_0^2 - k_0) + K^2}. \tag{2.140}$$

As we see, if $a_0^2 < k_0$, then there is the instability region of the modulational waves with small enough wavenumbers

$$K^2 < 4a_0^2(k_0 - a_0^2). \tag{2.141}$$

This formula specifies the result of the preceding subsection based on the hydrodynamical arguments. Note that the phase velocities (2.140) in the limit $K \to 0$ coincide with the characteristic speeds (2.132) what one could expect according to definition of the characteristic speeds as velocities of propagation of small disturbances.

2.3.4 *Small amplitude nonlinear waves on the constant background*

If amplitudes of the modulational waves are small but finite, then their evolution is governed by some nonlinear equations. Here we shall derive these equations for both types of linear waves corresponding to the two values of the characteristic speeds

$$\Omega/K = -4a_0^2 \pm \sqrt{4a_0^4 + K^2}, \tag{2.142}$$

that is, we take $a = a_0$, $k_0 = 0$, $\omega_0 = 0$. We consider small deviations from the solution $a = a_0$, $\phi = 0$, so that equations (2.126) for the variables $\tilde{a}$ and $\tilde{\phi}$, where $a = a_0 + \tilde{a}$, $\phi = \tilde{\phi}$, read

$$\begin{aligned} &\tilde{a}_t - 6(a_0 + \tilde{a})^2\tilde{a}_x + 2\tilde{a}_x\tilde{\phi}_x + (a_0 + \tilde{a})\tilde{\phi}_{xx} = 0, \\ &(a_0 + \tilde{a})\tilde{\phi}_t - 2(a_0 + \tilde{a})^3\tilde{\phi}_x + (a_0 + \tilde{a})\tilde{\phi}_x^2 - \tilde{a}_{xx} = 0. \end{aligned} \tag{2.143}$$

KdV limit

At first we consider the wave packet made of linear waves with the dispersion law

$$\Omega/K \simeq -6a_0^2 - K^2/4a_0^2 \tag{2.144}$$

(lower sign in Eq. (2.142)). The dispersion effects are small for long waves with wavevectors $K \ll 2a_0^2$. Therefore, we introduce the dimensionless space and time coordinates

$$X = 2a_0^2\beta x, \quad T = 12a_0^4\beta t, \tag{2.145}$$

so that $Kx - \Omega t = (K/2a_0^2\beta)(X + T)$. A small parameter $\beta \ll 1$ controls the dispersion effects. To control the nonlinear effects, we introduce the parameter $\alpha \ll 1$ according to

$$\tilde{a} = a_0\alpha A, \quad \tilde{\phi} = (\alpha/\beta)\Phi. \tag{2.146}$$

In new variables Eqs. (2.143) take the form

$$\begin{aligned} &3A_T - 3(1+\alpha A)^2 A_X + 2\alpha A_X\Phi_X + (1+\alpha A)\Phi_{XX} = 0,\\ &3(1+\alpha A)\Phi_T - (1+\alpha A)^3\Phi_X + \alpha(1+\alpha A)\Phi_X^2 - \beta^2 A_{XX} = 0. \end{aligned} \tag{2.147}$$

We suppose that in the reference frame moving with the characteristic velocity $-6a_0^2$ (equal to -1 in the units (2.145)) there is a slow evolution of A—if $A \sim \epsilon \ll 1$, then this evolution is essential for the time scale $T \sim \epsilon^{-a}$ and the space scale $X \sim \epsilon^{-b}$, and it is accompanied by the evolution of the phase with the order of magnitude $\Phi \sim \epsilon^c$. The scaling indices may be found from the estimate of the nonlinear and dispersion terms in Eqs. (2.147) under supposition that the lowest relevant nonlinearity is quadratic and of the same order of magnitude as the dispersion term,

$$\Phi_T \sim A\Phi_X \sim \Phi_X^2 \sim A_{XX},$$

which gives $\epsilon^{c+a} \sim \epsilon^{1+c+b} \sim \epsilon^{2c+2b} \sim \epsilon^{1+2b}$, and, consequently,

$$a = \tfrac{3}{2}, \quad b = \tfrac{1}{2}, \quad c = \tfrac{1}{2}. \tag{2.148}$$

So we introduce the slow variables

$$\tau = \epsilon^{3/2}T, \qquad \xi = \epsilon^{1/2}(X + T) \tag{2.149}$$

and expand the amplitude and the phase according to

$$A = \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \dots, \quad \Phi = \epsilon^{1/2}\left(\Phi^{(1)} + \epsilon\Phi^{(2)} + \dots\right). \tag{2.150}$$

Substitution into Eqs. (2.147) yields at the two lowest orders in ϵ:

$$\begin{aligned}
&\Phi^{(1)}_{\xi\xi} + \epsilon\left[3A^{(1)}_\tau - 6\alpha A^{(1)}A^{(1)}_\xi + 2\alpha A^{(1)}\Phi^{(1)}_\xi + \alpha A^{(1)}\Phi^{(1)}_{\xi\xi} + \Phi^{(2)}_{\xi\xi}\right] + \dots = 0,\\
&2\Phi^{(1)}_\xi + \epsilon\left[3\Phi^{(1)}_\tau - \beta^2 A^{(1)}_{\xi\xi} + 2\Phi^{(2)}_\xi\right] + \dots = 0.
\end{aligned}$$

At the lowest order we have

$$\Phi^{(1)}_\xi = \Phi^{(1)}_{\xi\xi} = 0,$$

and, hence, at the next order

$$3A^{(1)}_\tau - 6\alpha A^{(1)}A^{(1)}_\xi + \Phi^{(2)}_{\xi\xi} = 0, \quad 3\Phi^{(1)}_\tau - \beta^2 A^{(1)}_{\xi\xi} + 2\Phi^{(2)}_\xi = 0.$$

Differentiation of the second equation with respect to ξ and substitution of $\Phi^{(2)}_{\xi\xi}$ into the first equation leads to the KdV equation

$$3A^{(1)}_\tau - 6\alpha A^{(1)}A^{(1)}_\xi + \tfrac{1}{2}\beta^2 A^{(1)}_{\xi\xi\xi} = 0. \tag{2.151}$$

Returning to the ‘physical’ variables

$$t = \frac{1}{12a_0^4\beta}\cdot\frac{\tau}{\epsilon^{3/2}}, \quad x = \frac{1}{2a_0^2\beta}\cdot\left(\frac{\xi}{\epsilon^{1/2}} - \frac{\tau}{\epsilon^{3/2}}\right), \quad A^{(1)} = \frac{\tilde a}{a_0\alpha\epsilon},$$

we find the evolution equation for the modulational wave

$$\tilde a_t - 6a_0^2\tilde a_x - 6a_0\tilde a\tilde a_x + (1/4a_0^2)\,\tilde a_{xxx} = 0. \tag{2.152}$$

The linear part of this equation reproduces the dispersion relation (2.144).

mKdV limit

Now let us consider the wave packet made of linear waves propagating with another phase velocity

$$\Omega/K = -2a_0^2 + K^2/4a_0^2. \tag{2.153}$$

In this case we introduce the dimensionless variables in the following way:

$$X = 2a_0^2\beta x, \quad T = 4a_0^4\beta t, \quad \tilde a = a_0\alpha A, \quad \tilde\phi = (\alpha/\beta)\Phi,$$

where α and β control the nonlinear and dispersion effects, correspondingly. Instead of Eqs. (2.147) we obtain the system

$$\begin{aligned}&A_T - 3(1+\alpha A)^2 A_X + 2\alpha A_X \Phi_X + (1+\alpha A)\Phi_{XX} = 0,\\&(1+\alpha A)\Phi_T - (1+\alpha A)^3\Phi_X + \alpha(1+\alpha A)\Phi_X^2 - \beta^2 A_{XX} = 0.\end{aligned} \tag{2.154}$$

At first sight one might think that the leading nonlinearity is quadratic one, and again the estimates of the dispersion and nonlinear terms lead to the same indices (2.148) and, consequently, to the KdV equation. However, more careful inspection shows that quadratic nonlinear terms vanish, if one introduces the slow variables (2.149). Indeed, in this case at the leading order of the first Eq. (2.154) one would obtain

$$\Phi_\xi^{(1)} = 2A^{(1)}, \tag{2.155}$$

and quadratic nonlinear terms in the second Eq. (2.154) at the next order in ϵ are equal to

$$\alpha A^{(1)}\Phi_\xi^{(1)} - 3\alpha A^{(1)}\Phi_\xi^{(1)} + \alpha\big(\Phi_\xi^{(1)}\big)^2 = \alpha(2-6+4)\big(A^{(1)}\big)^2 \equiv 0.$$

Therefore, in this case the cubic nonlinearity becomes responsible for the slow evolution and it must have the same order of magnitude as the dispersion terms. Hence, Eqs. (2.154) lead to the estimates $A \sim \Phi_X$, $\Phi_T \sim A^2\Phi_X \sim A_{XX}$ or $\epsilon \sim \epsilon^{c+b}$, $\epsilon^{c+a} \sim \epsilon^{2+c+b} \sim \epsilon^{1+2b}$. Then the scaling indices are equal to $a = 3$, $b = 1$, $c = 0$, so that we have the slow variables

$$\tau = \epsilon^3 T, \qquad \xi = \epsilon(X+T), \tag{2.156}$$

and expansions

$$A = \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \dots, \quad \Phi = \Phi^{(1)} + \epsilon\Phi^{(2)} + \dots, \tag{2.157}$$

(recall that Eqs. (2.154) include only the derivatives of Φ which are small by virtue of (2.156)). Now Eqs. (2.154) must be expanded up to the cubic degree of ϵ, which gives

$$\begin{aligned}&-2A_\xi^{(1)} + \Phi_{\xi\xi}^{(1)} + \epsilon\big[-6\alpha A^{(1)}A_\xi^{(1)} + 2\alpha A_\xi^{(1)}\Phi_\xi^{(1)}\\&+\alpha\Phi_{\xi\xi}^{(1)} + \Phi_{\xi\xi}^{(2)} - 2A_\xi^{(2)}\big] + \dots = 0,\end{aligned} \tag{2.158}$$

$$\begin{aligned}&\alpha\big(-2A^{(1)} + \Phi_\xi^{(1)}\big)\Phi_\xi^{(1)} + \epsilon\big[\Phi_\tau^{(1)} - 2\alpha A^{(1)}\big(\Phi_\xi^{(2)} - 2A^{(2)}\big)\\&-3\alpha^2\big(A^{(1)}\big)^2\Phi_\xi^{(1)} + \alpha^2 A^{(1)}\big(\Phi_\xi^{(1)}\big)^2 - \beta^2 A_{\xi\xi}^{(1)}\big] + \dots = 0.\end{aligned} \tag{2.159}$$

At the leading order both equations yield the relation (2.155) by virtue of which Eq. (2.158) gives at the order $\sim \epsilon$

$$\Phi^{(2)}_{\xi\xi} - 2A^{(2)}_{\xi} = 0, \quad \text{so that} \quad \Phi^{(2)}_{\xi} - 2A^{(2)} = 0,$$

and, hence, Eq. (2.159) at the order $\sim \epsilon$ becomes

$$\Phi^{(1)}_{\tau} - 3\alpha^2 \big(A^{(1)}\big)^2 \Phi^{(1)}_{\xi} + \alpha^2 A^{(1)} \big(\Phi^{(1)}_{\xi}\big)^2 - \beta^2 A^{(1)}_{\xi\xi} = 0,$$

which upon substitution of Eq. (2.155) and differentiation with respect to ξ yields

$$2A^{(1)}_{\tau} - 6\alpha^2 \left(A^{(1)}\right)^2 A^{(1)}_{\xi} - \beta A^{(1)}_{\xi\xi\xi} = 0.$$

Returning to the 'physical' variables

$$t = (1/4a_0^4\beta) \cdot (\tau/\epsilon^3), \quad x = (1/2a_0^2\beta) \cdot (\xi/\epsilon - \tau/\epsilon^3), \quad A^{(1)} = \tilde{a}/(a_0\alpha\epsilon),$$

we obtain the mKdV equation

$$\tilde{a}_t - 2a_0^2 \tilde{a}_x - 6\tilde{a}^2 \tilde{a}_x - (1/4a_0^2)\tilde{a}_{xxx} = 0, \tag{2.160}$$

which describes a slow evolution of the wave packet propagating along characteristics with the speed $-2a_0^2$. The linear part of this equation reproduces the dispersion relation (2.153).

Thus, the derivative NLS equation has a quite rich structure and in the appropriate limits reduces to the KdV or mKdV equations. These limiting cases suggest that the DNLS equation itself has soliton solutions in a closed form. They will be studied in Chapter 5.

2.4 Spin waves in magnetic materials

As it is known, ferromagnetic materials are characterized by existence of a nonzero mean magnetic moment $\mathbf{M}$. In the ground state the magnetic moment does not depend on time and in an infinite crystal it is a constant vector $\mathbf{M}_0$. Disturbance of such a ground state propagates in the form of the so-called spin wave. To explain the origin of the corresponding evolution equation, let us consider a chain of atoms situated at the points $x_n = nd$ of the x axis. Each atom has the magnetic moment $\mathbf{M}_n$ proportional to its angular moment (spin) $\mathbf{S}_n$,

$$\mathbf{M}_n = -\gamma \, \mathbf{S}_n, \tag{2.161}$$

where $\gamma = |e|g/2mc$, e and m being the electron's charge and mass, correspondingly, and g is the so-called gyromagnetic ratio. Each magnetic moment is affected by the other atoms via the so-called exchange interaction of the quantum mechanical origin. For simplicity, we suppose that there exists only interaction between the nearest neighbours, so that the exchange energy U of the chain is given by

$$U = -2A \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1}, \tag{2.162}$$

where A is the coupling constant. Expression (2.162) corresponds to the Heisenberg model of ferromagnetic materials.

We can treat the energy U as a sum of energies of the magnetic moments $\mathbf{M}_n$ in some effective magnetic field $\mathbf{B}_n$,

$$U = -\sum_n \mathbf{M}_n \cdot \mathbf{B}_n, \quad \mathbf{B}_n = (2A/\gamma^2)(\mathbf{M}_{n-1} + \mathbf{M}_{n+1}). \tag{2.163}$$

From elementary mechanics it is known that the time derivative of the angular moment $\mathbf{S}_n$ is equal to the rotating moment $\mathbf{M}_n \times \mathbf{B}_n$ acting on the magnetic moment $\mathbf{M}_n$ situated in the magnetic field $\mathbf{B}_n$, that is, $\partial \mathbf{S}_n/\partial t = \mathbf{M}_n \times \mathbf{B}_n$, or

$$\partial \mathbf{S}_n/\partial t = 2A\, \mathbf{S}_n \times (\mathbf{S}_{n-1} + \mathbf{S}_{n+1}). \tag{2.164}$$

In the long wave limit, when the wavelength is much longer than the distance d between atoms, we may consider $\mathbf{S}_n$ as a continuous function of x, $\mathbf{S}_n = \mathbf{S}(x)|_{x=nd}$, and expand $\mathbf{S}_{n\pm1}$ into a Taylor series,

$$\mathbf{S}_{n\pm1} = \mathbf{S}(x) \pm d\, \partial \mathbf{S}/\partial x + \tfrac{1}{2} d^2\, \partial^2 \mathbf{S}/\partial x^2 + \ldots,$$

where all the derivatives in the right hand side are calculated at $x = nd$. As a result, Eq. (2.164) transforms to

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \tag{2.165}$$

where we got rid of the constant A by means of a simple substitution $\mathbf{S} \to \mathbf{S}/A$. This is the evolution equation for the so called classical Heisenberg model.

Besides the exchange energy (2.162) the so-called energy of magnetic anisotropy can also exist which depends on the orientation of the vector $\mathbf{S}$

with respect to the axes of the crystal lattice. In general case this leads to the Landau-Lifshitz equation

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S}, \tag{2.166}$$

where $J = \mathrm{diag}(J_1, J_2, J_3)$ is a diagonal matrix of the magnetic anisotropy constants. For instance, in the 'easy-axis' crystal the magnetic moments tend to align in the direction of some axis x which we take as the propagation axis. Then $J = \mathrm{diag}(-J, 0, 0)$, that is, $\mathbf{S} \times J\mathbf{S} = (0, JS_1S_3, -JS_1S_2)$, and Eq. (2.166) can be written as

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + J(\mathbf{S} \cdot \mathbf{n})(\mathbf{S} \times \mathbf{n}), \tag{2.167}$$

where $\mathbf{n}$ is a unit vector along x axis.

Thus, the spin waves in magnetic materials are described by the nonlinear equations, which, as one can show, have the soliton solutions. We shall consider them in Chapter 5 of this book.

2.5 Self-induced transparency

Many problems about the nonlinear wave propagation have appeared in optics due to invention of laser. Among first such problems was one about the propagation of an intensive light pulse through the material medium when the light frequency ω is very close to the frequency $\omega_0 = (E_b - E_a)/\hbar$ of the transition between the atomic levels a and b. Under this resonance condition one may neglect the existence of the other levels. If, in addition, we suppose that the levels a and b are not degenerate, then we arrive at the often used model of two-level atoms. In this model, the wave function of the atom may be represented as a time dependent linear combination of the normalized functions of the two levels,

$$\psi(\mathbf{r}, t) = a(t)\psi_a(\mathbf{r}) + b(t)\psi_b(\mathbf{r}), \tag{2.168}$$

where $\int \psi_a^*(\mathbf{r})\psi_b(\mathbf{r})d\mathbf{r} = \delta_{ab}$, and the equality

$$|a(t)|^2 + |b(t)|^2 = 1 \tag{2.169}$$

means that the total probability that an atom can be found in the upper or lower level is equal to unity. The functions $\psi_a(\mathbf{r})$ and $\psi_b(\mathbf{r})$ are the eigenfunctions of the unperturbed Hamiltonian H_0, $\hat{H}_0\psi_{a,b} = E_{a,b}\psi_{a,b}$, and interaction with the electromagnetic field $\mathbf{E}$ of a light wave is described by the term $V = -\mathbf{p} \cdot \mathbf{E}$, where $\mathbf{p}$ is an operator of the electric dipole moment of an atom. This interaction leads to transitions between the levels a and b, so that the wave function (2.168) satisfies the non-stationary Schrödinger equation

$$i\hbar\psi_t = H\psi = (H_0 + V)\psi. \tag{2.170}$$

Taking the interaction energy in the form $V = -\mathbf{p} \cdot \mathbf{E}$, we suppose that the light wavelength is much longer than the atom's size. This supposition fulfills very well in practice. Let us also suppose that the light wave is linearly polarized and the vectors $\mathbf{p}$ and $\mathbf{E}$ are parallel. Then Eq. (2.170), that is,

$$i\hbar(a_t\psi_a + b_t\psi_b) = E_a a\psi_a + E_b b\psi_b - pE(a\psi_a + b\psi_b),$$

after multiplication by ψ_a^* and integration over atomic volume, and similar operation with ψ_b^*, reduces to the system of equations for the amplitudes $a(t)$ and $b(t)$,

$$i\hbar a_t = E_a a - p_0 E b, \quad i\hbar b_t = E_b b - p_0 E a, \tag{2.171}$$

where $p_0 = \int \psi_a^* \hat{p} \psi_b d\mathbf{r}$ is the transition dipole matrix element. In this calculation we also supposed that atoms do not have a static dipole moment, $\int \psi_a^* \hat{p} \psi_a d\mathbf{r} = \int \psi_b^* \hat{p} \psi_b d\mathbf{r} = 0$, and took into account that the light field does not vary practically within the atomic volume.

If one neglects the interaction with the light field, then the amplitudes $a(t)$ and $b(t)$ oscillate with time according to

$$a = \tilde{a}\exp(-iE_a t/\hbar), \quad b = \tilde{b}\exp(-iE_b t/\hbar). \tag{2.172}$$

Let us represent the light field in the form of the modulated wave propagating along the x axis,

$$E = \tfrac{i}{2}\left[A(x,t)e^{i(kx-\omega t)} - A^*(x,t)e^{-i(kx-\omega t)}\right], \tag{2.173}$$

where the light frequency ω is close to the transition frequency ω_0,

$$\omega_0 = \omega + 2\Delta\omega, \quad |\Delta\omega| \ll \omega_0 = (E_b - E_a)/\hbar. \tag{2.174}$$

If we keep in Eqs. (2.171) the resonant terms only, we shall obtain equations for the amplitudes $\tilde{a}$ and $\tilde{b}$,

$$\tilde{a}_t = (p_0/2\hbar)A^*\tilde{b}\exp[-i(kx+2\Delta\omega t)], \quad \tilde{b}_t = -(p_0/2\hbar)A\tilde{a}\exp[i(kx+2\Delta\omega t)].$$

At last, introducing the variables

$$\alpha(t) = \tilde{a}\exp(ikx/2 + i\Delta\omega t), \quad \beta(t) = \tilde{b}\exp(-ikx/2 - i\Delta\omega t),$$

we arrive at the equations

$$\alpha_t - i\Delta\omega\alpha = (p_0/2\hbar)A^*\beta, \quad \beta_t + i\Delta\omega\beta = -(p_0/2\hbar)A\alpha. \tag{2.175}$$

In this derivation we suppose that all atoms have the same transition frequency ω_0, that is, we neglect an inhomogeneous line broadening caused by the Doppler shift due to the thermal motion of atoms.

The mean value of the transition dipole moment at the light frequency ω can be expressed in terms of the amplitudes α and β as

$$Np_{ab} = Np_0 a^* b = Np_0 \alpha^* \beta \exp[i(kx - \omega t)],$$

where N is a number of atoms in a unit volume. Therefore, it is convenient to introduce the variable

$$d = 2\alpha^*\beta \tag{2.176}$$

which describes the amplitude of transitions between the atomic levels at the point x at the moment t, i.e., the polarization of the medium. The dependence of d on time t is determined by the equation

$$d_t + 2i\Delta\omega d = (p_0/\hbar)An, \tag{2.177}$$

where the variable

$$n = |\beta|^2 - |\alpha|^2 \tag{2.178}$$

characterizes inversion of the two-level medium. According to Eq. (2.169), the polarization d and inversion n are related by the condition

$$|d|^2 + n^2 = 1. \tag{2.179}$$

The inversion variable satisfies the equation

$$n_t = -(p_0/2\hbar)(Ad^* + A^*d), \tag{2.180}$$

which also follows from Eqs. (2.175).

Equations (2.177) and (2.179) determine variation of the radiative properties of the medium under influence of the electromagnetic field of the light pulse. Now we have to take into account feedback action of the varying polarization d on the electromagnetic field. As follows from the equation (see Eq. (1.307))

$$E_{xx} - (1/c^2)E_{tt} = (4\pi/c^2)P_{tt},$$

the field component $A(x,t)$ in Eq. (2.173) with the frequency $\omega = ck$ satisfies the equation

$$A_x + (1/c)A_t = (2\pi N p_0 \omega/c)d,$$

because at this frequency the polarization is equal to $P = Np_{ab}$. Introducing the variable $\widetilde{\mathcal{E}} = (p_0/\hbar)A$, we obtain the equation

$$\widetilde{\mathcal{E}}_t + c\widetilde{\mathcal{E}}_x = \Omega^2 d, \tag{2.181}$$

where the parameter

$$\Omega = (2\pi N p_0^2 \omega/\hbar)^{1/2} \tag{2.182}$$

has a dimension of the frequency. At last, after transition to dimensionless coordinates

$$\zeta = \Omega x/c, \quad \tau = \Omega(t - x/c) \tag{2.183}$$

and introduction of the variable $\mathcal{E} = \widetilde{\mathcal{E}}/\Omega$ and the off-set parameter $\Delta = \Delta\omega/\Omega$, we arrive at the final form of the evolution equations,

$$\mathcal{E}_\zeta = d, \quad d_\tau + 2i\Delta d = n\mathcal{E}, \quad n_\tau = -\tfrac{1}{2}(\mathcal{E}d^* + \mathcal{E}^* d), \tag{2.184}$$

which describe the propagation of an intensive light pulse through a two-level medium.

It is worth to note that if we define a unit length vector $\mathbf{R} = (R_1, R_2, R_3)$ by the equations

$$d = R_1 - iR_2, \quad n = R_3, \tag{2.185}$$

and take the electromagnetic field in the form $\mathcal{E} = -iE$ with real $E(x,t)$, then the last two equations (2.184) give the evolution system for $\mathbf{R}$,

$$R_{1,\tau} = -2\Delta R_2, \quad R_{2,\tau} = 2\Delta R_1 + ER_3, \quad R_{3,\tau} = -ER_2. \tag{2.186}$$

These equations are often used and called the optical Bloch equations.

The nonlinear system (2.184) has a soliton solution which will be considered in Chapter 5. Here we only note that at the zero off-set parameter

$\Delta = 0$ and with real functions $\mathcal{E}$ and d this system reduces at once to the sine-Gordon equation by means of the substitution $d = \sin\phi$, $n = \cos\phi$, so that the last two equations (2.184) give

$$\mathcal{E} = \phi_\tau \tag{2.187}$$

and the first one transforms to

$$\phi_{\zeta\tau} = \sin\phi. \tag{2.188}$$

This is the sine-Gordon equation in the 'light-cone variables'. It is easy to show that this equation has a solution

$$\phi = 4\arctan[\exp(\tau/\tau_p + \zeta\tau_p)],$$

that is the pulse envelope is given by

$$\mathcal{E} = \frac{2/\tau_p}{\cosh(\tau/\tau_p + \zeta\tau_p)} = \frac{2/\tau_p}{\cosh[(1/\tau_p)(t + (1+\Omega^2\tau_p^2)(x/c))]}. \tag{2.189}$$

This soliton solution describes the process of propagation of a coherent light pulse through a two-level medium without damping, that is, at certain conditions the medium becomes transparent for propagation of intensive short pulses. This phenomenon is called the self-induced transparency (SIT). The velocity of the pulse

$$V = c/(1+\Omega^2\tau_p^2)$$

is less then the speed of light, and the area under the envelope profile is equal to 2π, $\int_{-\infty}^{\infty}\mathcal{E}d\tau = \phi(\tau = +\infty) - \phi(\tau = -\infty) = 2\pi$. Therefore, the SIT soliton is often called a 2π-pulse.

2.6 Stimulated Raman scattering

As the last example in this Chapter, let us consider equations of the propagation of a stimulated Raman scattering (SRS) pulse in a resonant medium. Now we have two electromagnetic waves propagating along the x axis,

$$E_i(x,t) = \mathcal{E}_i\exp[i(k_ix - \omega_it)] + \text{c.c.}, \quad i = 1, 2, \tag{2.190}$$

through the medium with a resonant transition at the frequency $\omega_1 - \omega_2$ ($\omega_1 > \omega_2$). The main principles of derivation of the corresponding evolution equations coincide with those used in the preceding section, though now the effect is quadratic with respect to the electromagnetic field, and one should use the second order approximation of the quantum-mechanical perturbation theory. Since calculations are rather tedious, we shall not give here the details and write down the results.

The radiative properties of the medium are described by the Bloch vector

$$\widetilde{\mathbf{R}} = (\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3), \quad \widetilde{R}_\pm = \widetilde{R}_1 \pm i\widetilde{R}_2, \tag{2.191}$$

which satisfies the equations

$$\begin{aligned}\partial\widetilde{R}_+/\partial t \quad &= 2i(b_1|\mathcal{E}_1|^2 + b_2|\mathcal{E}_2|^2)\widetilde{R}_+ + 2i\kappa\mathcal{E}_2\mathcal{E}_1^*\widetilde{R}_3, && (2.192)\\ \partial\widetilde{R}_3/\partial t \qquad\qquad &= i\kappa(\mathcal{E}_1\mathcal{E}_2^*\tilde{R}_+ + \mathcal{E}_1^*\mathcal{E}_2\widetilde{R}_-). && (2.193)\end{aligned}$$

Equation (2.192) is similar to the second equation (2.186), but now the frequency shift is caused by the dynamical Stark effect and the change of the refraction index (both effects arise at the second order in the fields $\mathcal{E}_i$), and in the last term on the right hand side we have got the product $\mathcal{E}_2\mathcal{E}_1^*$, because the transition between atomic levels is caused by absorption of a quantum $\hbar\omega_1$ and radiation of a quantum $\hbar\omega_2$, whereas in the SIT case we considered a one-photon process. For the same reason we have products of the field amplitudes in Eq. (2.193) instead of one amplitude in the last equation (2.184).

Equations for the envelope amplitudes read

$$\begin{aligned}(\partial/\partial x + (1/c)\partial/\partial t)\mathcal{E}_1 = -ib_1\widetilde{R}_3\mathcal{E}_1 + i\kappa\tilde{R}_-\mathcal{E}_2,\\ (\partial/\partial x + (1/c)\partial/\partial t)\mathcal{E}_2 = -ib_2\widetilde{R}_3\mathcal{E}_2 + i\kappa\widetilde{R}_+\mathcal{E}_1,\end{aligned} \tag{2.194}$$

where the right hand sides include not only the components of $\widetilde{\mathbf{R}}$, but also the field components leading to the two-photon process. It is remarkable that these equations can be transformed to a very symmetrical form, if one introduces a pseudo-vector $\widetilde{\mathbf{S}}$ with the components

$$\widetilde{S}_+ = \widetilde{S}_1 + i\widetilde{S}_2 = 2\mathcal{E}_2\mathcal{E}_1^*, \quad \widetilde{S}_- = \widetilde{S}_+^*, \quad \widetilde{S}_3 = \mathcal{E}_1\mathcal{E}_2^* - \mathcal{E}_2\mathcal{E}_1^*. \tag{2.195}$$

After introduction of the retarded time variable $t' = t - x/c$, it is easy to show that the total intensity (a length of the vector $\widetilde{\mathbf{S}}$) depends only on t',

$$\widetilde{S}_1^2 + \widetilde{S}_2^2 + \widetilde{S}_3^2 = \mathcal{E}_1\mathcal{E}_1^* + \mathcal{E}_2\mathcal{E}_2^* = I(t').$$

Then the Maxwell-Bloch equations can be written in the form

$$\begin{aligned}\partial\widetilde{R}_+/\partial t' &= i(b_+I + b_-\widetilde{S}_3)\widetilde{R}_+ + i\kappa\widetilde{S}_+\widetilde{R}_3,\\ \partial\widetilde{R}_3/\partial t' &= \kappa(\widetilde{R}_1\widetilde{S}_2 - \widetilde{R}_2\widetilde{S}_1);\end{aligned} \tag{2.196}$$

and

$$\begin{aligned}\partial\widetilde{S}_+/\partial x &= ib_-\widetilde{R}_3\widetilde{S}_+ + i\kappa\widetilde{R}_+\widetilde{S}_3,\\ \partial\widetilde{S}_3/\partial x &= \kappa(\widetilde{R}_2\widetilde{S}_1 - \widetilde{R}_1\widetilde{S}_2),\end{aligned} \tag{2.197}$$

where $b_\pm = b_1 \pm b_2$. The term $ib_+I\tilde{R}_+$ can be eliminated by means of a transformation to new variables

$$\tau = \kappa\int_{t_0}^{t'} I(t'')dt'', \quad \zeta = x;$$
$$\mathbf{R} = \widetilde{\mathbf{R}}\exp(ib_+\tau/\kappa), \quad \mathbf{S} = I^{-1/2}\widetilde{\mathbf{S}}\exp(ib_+\tau/\kappa), \quad \Delta = b_-/\kappa.$$

As a result, the systems (2.196,2.197) take the final form

$$\begin{aligned}\partial R_+/\partial\tau = i(\Delta R_+S_3 + R_3S_+), \quad \partial R_3/\partial\tau = \tfrac{i}{2}(R_+S_- - R_-S_+),\\ \partial S_+/\partial\zeta = i(\Delta S_+R_3 + S_3R_+), \quad \partial S_3/\partial\zeta = \tfrac{i}{2}(S_+R_- - S_-R_+).\end{aligned} \tag{2.198}$$

These equations will be investigated in some details in Chapter 5. Here, let us only show that if $\Delta = 0$, then some particular solutions can be obtained from the solutions of the sine-Gordon equation. From equations

$$\begin{aligned}\partial R_1/\partial\tau = -R_3S_2, \quad \partial R_2/\partial\tau = R_3S_1, \quad \partial R_3/\partial\tau = R_1S_2 - R_2S_1,\\ \partial S_1/\partial\zeta = -S_3R_2, \quad \partial S_2/\partial\zeta = S_3R_1, \quad \partial S_3/\partial\zeta = R_2S_1 - R_1S_2,\end{aligned} \tag{2.199}$$

we find that if the initial and boundary conditions

$$R_1(\zeta,\tau_0) = 0, \quad S_2(\zeta_0,\tau) = 0$$

are fulfilled, then in the region $\zeta \ge \zeta_0$, $\tau \ge \tau_0$ we have

$$R_1(\zeta,\tau) \equiv 0, \quad S_2(\zeta,\tau) \equiv 0,$$

and the system (2.199) reduces to

$$\partial R_2/\partial\tau = R_3S_1, \quad \partial R_3/\partial\tau = -R_2S_1, \quad \partial S_3/\partial\zeta = R_2S_1.$$

Taking into account that the vectors **R** and **S** have unit length, we represent them as

$$R_2 = \sin\phi, \quad R_3 = \cos\phi, \quad S_1 = \sin\psi, \quad S_3 = \cos\psi, \tag{2.200}$$

so that $\phi_\tau = \sin\psi, \quad \psi_\zeta = -\sin\phi$, and, hence,

$$(\phi+\psi)_{\tau\zeta} = -\sin(\phi+\psi), \quad (\phi-\psi)_{\tau\zeta} = -\sin(\phi-\psi). \tag{2.201}$$

These equations coincide with the sine-Gordon equation (2.188) after change of sign of one of the variables ζ or τ. If we know two different solutions of the sine-Gordon equation, we obtain at once the solution of the SRS equations (2.199). This simple example suggests that more complete theory of Eqs. (2.198) may be developed, and this will be done in Chapter 5.

Bibliographic remarks

Universality of the KdV and NLS equations was recognized long ago after their appearence in various physical contexts (see, e.g., papers by Korteweg and de Vries (1895), Berezin and Karpman (1964), Taniuti and Wei (1968), Su and Gardner (1969), about the KdV equation, and by Benney and Newell (1967), Zakharov (1968), Taniuti and Yadjima (1969), Asano, Taniuti and Yadjima (1969), Benney (1972) about the NLS equation). The NLS equation describes not only propagation of wave packets but also that of narrow stationary light beams in Kerr type medium, see Chiao, Garmire and Townes (1964), Talanov (1965), Zakharov (1967). Early review of the singular perturbation method can be found in Taniuti (1974) (see also other papers in this special volume). Good introduction into the singular perturbation method is given by Newell (1985) and Newell and Moloney (1992). The Rayleigh method for derivation of the Stokes expansion is presented in a classical book by Lamb (1932). The DNLS equation for nonlinear Alfvén waves was derived for the first time by Rogister (1971); generalization and more detailed treatment can be found in Kennel et al. (1988). Its reduction to the KdV and mKdV equations was discussed by Mjølhus (1989). This equation applies also to description of the self-steepening process of short pulses in optical wavegides, see Anderson and Lisak (1983). About Landau–Lifshitz equation for waves in magnetic materials see, e.g., Kittel (1966), Lifshitz and Pitaevskii (1978). Self-induced transparency phenomenon was predicted by McCall and Hahn (1969). Equations (2.184) were derived in

Lamb (1971). The stimulated Raman scattering equations were obtained in the form (2.198) by Steudel (1983); he indicated also their reduction under certain restrictions to two sine-Gordon equations (Steudel, 1994).

Exercises on Chapter 2

Exercise 2.1
Derive the KdV equation (2.16) for ion-acoustic waves by the method of the singular perturbation theory.

Exercise 2.2
Derive the KdV equation (2.39) describing the evolution of long weakly nonlinear waves in a nonlinear one-dimensional lattice with quadratic nonlinearity of the elastic force.

Exercise 2.3
Find the Stokes wave solution for the shallow water case by means of the perturbation theory applied to the KdV equation (2.82).

Exercise 2.4
Show that the propagation of a narrow light beam in medium with Kerr type nonlinearity (the dielectric constant $\epsilon = \epsilon_0 + \epsilon_1 |E|^2$) is described by the NLS equation

$$i\frac{\partial A}{\partial z} + \frac{1}{2k}\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right) + \frac{\epsilon_1}{2k\epsilon_0}|A|^2 A = 0$$

for the envelope $A(x, y, z)$ of the optical field $E = A\exp[i(kz - \omega t)]$, $k^2 = (\omega^2/c^2)\epsilon_0(\omega)$.

Exercise 2.5
Find fields intensities for the solution of the SRS equations corresponding to the solution $\phi = \psi = 2\arctan[\exp(\tau - \zeta)]$ of two sine–Gordon equations (2.201).

Chapter 3

Whitham theory of modulations

3.1 The general idea

For the first time we mentioned the idea of modulations in Sec. 1.2 of this book, where we studied the influence of dispersion effects on the evolution of a linear wave packet. Later weak nonlinear effects were taken into account with the use of the NLS equation. In the theory of the NLS equation it is assumed that linear waves provide good approximation to the system's motion, and we only have to include weak nonlinear effects. In particular, very long water waves with a very small amplitude can be described by the NLS equation (Sec. 2.2.2), although with increase of the amplitude and decrease of the wavelength we have to return to more exact description with the use of the 'strongly nonlinear' periodic solution of the KdV equation. So, we arrive at the problem of description of modulations of such strongly nonlinear periodic waves which cannot be approximated by linear wave packets. For instance, let the initial state be described by the modulated periodic wave (1.295), where the parameters a, m, γ defining the solution are some functions of x. How, then, do these parameters evolve? It is clear that if in the initial state the parameters a, m, γ change much in one wavelength $L \sim \sqrt{m/a}$, then there is no reason to expect that the evolving wave can be described as a modulated periodic wave. On the contrary, if the initial modulation is slow, that is the parameters a, m, γ change little in one wavelength, then one may expect that modulation remains slow for some time, and the evolution can be described as a slow change of the parameters a, m, γ. Thus, these parameters become slow functions of the space and time coordinates and we want to calculate how they change.

Thus, we have arrived at the problem of derivation of equation describing a slow evolution of the parameters a, m, γ. The existence of two scales of distance and time, namely, 'fast' oscillations with the wavelength L and period T and a slow evolution of a, m, γ suggests the method of approach to this problem—we may average the equations of motion for the parameters a, m, γ over 'fast' oscillations, and these averaged equations may be more appropriate for the analytical treatment of slow modulations. This approach was suggested by Whitham (1965) and the averaged modulation equations are called the Whitham equations.

Naturally, the idea of averaging applies not only to the KdV equation but to any evolution equation having periodic solutions. Therefore, let us formulate here this approach for rather common case of the evolution equation in the form

$$\Phi(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \dots) = 0. \tag{3.1}$$

We suppose that the wave variable $u(x,t)$ depends on x and t only in combination

$$u = u(\xi), \quad \xi = x - Vt, \tag{3.2}$$

and, hence, Eq. (3.1) can be reduced to the form

$$u_\xi^2 = F(u, V, A_i), \tag{3.3}$$

where A_i denotes a set of some parameters (they arise as integration constants in derivation of Eq. (3.3) from Eq. (3.1) and determine, for example, values of the wavelength and the amplitude). It is clear that periodic solutions correspond to oscillations of u between two zeros of $F(u)$. We shall denote these zeros as $u_1(V, A_i)$ and $u_2(V, A_i)$, $u_1 < u_2$, and according to Eq. (3.3) the function F must be positive in the interval $u_1 < u < u_2$. Let ξ_1 and ξ_2 be the values of ξ at which u has the values u_1 and u_2,

$$u(\xi_1; V, A_i) = u_1(V, A_i), \quad u(\xi_2; V, A_i) = u_2(V, A_i).$$

Then, as it follows from Eqs. (3.2) and (3.3), the wavelength is determined by the equation

$$L = L(V, A_i) = 2\int_{\xi_1}^{\xi_2} d\xi = 2\int_{u_1}^{u_2} \frac{du}{u_\xi} = 2\int_{u_1}^{u_2} \frac{du}{\sqrt{F(u; V, A_i)}}, \tag{3.4}$$

and the wavevector k and the frequency ω are equal to

$$k = k(V, A_i) = 2\pi/L(V, A_i), \quad \omega = \omega(V, A_i) = Vk(V, A_i). \tag{3.5}$$

In a weakly modulated wave $u(\xi; V, A_i)$ the parameters V and A_i are slow functions of x and t changing little in one wavelength and one period $2\pi/\omega$. We are interested in the physical variables averaged over fast oscillations of the wave according to the rule

$$\widetilde{\mathcal{F}}(x,t) = \frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta} \mathcal{F}(x',t)dx'. \tag{3.6}$$

Here it is supposed that the interval 2Δ is much longer than the wavelength L and much shorter than any characteristic size l in the problem (e.g., l may be a wave packet's width),

$$L \ll \Delta \ll l. \tag{3.7}$$

According to Whitham, in practice it is more convenient to average conservation laws rather than evolution equations. Indeed, let we know a conservation law

$$\mathcal{P}_t + \mathcal{Q}_x = 0. \tag{3.8}$$

Then from the identities

$$\begin{aligned}
\widetilde{(\mathcal{P}_t)} &= \frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta} \frac{\partial}{\partial t}\mathcal{P}(x',t)dx' = \left(\widetilde{\mathcal{P}}\right)_t, \\
\widetilde{(\mathcal{Q}_x)} &= \frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta} \frac{\partial}{\partial x}\mathcal{Q}(x',t)dx' = \frac{1}{2\Delta}[\mathcal{Q}(x'+\Delta,t) - \mathcal{Q}(x'-\Delta,t)] \\
&= \frac{\partial}{\partial x}\frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta} \mathcal{Q}(x',t)dx' = \left(\widetilde{\mathcal{Q}}\right)_x,
\end{aligned}$$

we find that the averaged conservation law can be written in the form

$$\frac{\partial}{\partial t}\widetilde{\mathcal{P}}(x,t,A_i) + \frac{\partial}{\partial x}\widetilde{\mathcal{Q}}(x,t,A_i) = 0, \tag{3.9}$$

that is, we can interchange operations of averaging the conservation laws and of the first order differentiation with respect to x and t. The averaged quantities depend on the space x and time t coordinates and on the averaging interval 2Δ. However, we have not used yet the difference of the scales (3.7). As follows from this condition, within the interval 2Δ the wavelength L is constant up to small corrections of the order of magnitude

$\varepsilon \sim \Delta/l$. Therefore, with the same accuracy we can replace averaging over the interval 2Δ by averaging over the wavelength and substitute

$$\overline{\mathcal{F}} = \frac{1}{L}\int_x^{x+L} \mathcal{F}(x', t)dx' \tag{3.10}$$

into Eq. (3.9) instead of $\widetilde{\mathcal{F}}$. In this approximation $\mathcal{F}(x', t)$ is the periodic function and explicit dependence of the averaged quantity $\overline{\mathcal{F}}$ on x and t disappears, so we can put $x = 0$ in Eq. (3.10). As a result, we arrive at the averaged conservation law

$$\frac{\partial}{\partial t}\overline{\mathcal{P}}(V, A_i) + \frac{\partial}{\partial x}\overline{\mathcal{Q}}(V, A_i) = 0, \tag{3.11}$$

where the dependence on x and t appears due to the slowly varying parameters only. In fact, Eq. (3.11) is a differential equation for these parameters, so we have to involve into consideration as many conservation laws as there are parameters $\{V,\, A_i\}$ in the wave-train solution of the evolution equation.

Before passing to the modulation equations for nonlinear waves, it is instructive to consider by this method the known example of linear wave modulations (a wave packet of linear waves).

3.2 Modulation of a linear wave

Let us consider a simple example of a dispersive linear wave whose evolution is governed by the Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0. \tag{3.12}$$

If we look for the solution in the form of the travelling wave $u = u(\xi) = u(x - Vt)$, we shall obtain

$$(V^2 - 1)u_{\xi\xi} + u = 0.$$

On multiplying this equation by u_ξ and integration, we obtain

$$u_\xi^2 = (a^2 - u^2)/(V^2 - 1),$$

where a is the integration constant. Further integration yields the periodic solution

$$u(\xi; V, a) = a\cos\left[(\xi - \xi_0)/\sqrt{V^2 - 1}\right], \tag{3.13}$$

that is a has a meaning of the wave amplitude. Another integration constant ξ_0 has a meaning of an initial phase and in principle it could be a slow function of x and t. But it disappears from quantities $\overline{\mathcal{F}}$ averaged according to the defined above procedure, so we shall take it equal to zero, $\xi_0 = 0$. As it follows from Eq. (3.13), the wavevector k and the frequency ω are equal to

$$k = 1/\sqrt{V^2 - 1}, \quad \omega = V/\sqrt{V^2 - 1}, \tag{3.14}$$

and elimination of V from these equations yields the dispersion relation

$$\omega = \sqrt{k^2 + 1}. \tag{3.15}$$

In a modulated wave the parameters V and a are the slow functions, so for derivation of the Whitham equations we need two conservation laws. It is easy to find them. Multiplication of Eq. (3.12) by either u_t or u_x leads to

$$\begin{aligned} &\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u^2\right)_t + (-u_t u_x)_x = 0, \\ &(-u_t u_x)_t + \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 - \tfrac{1}{2}u^2\right)_x = 0. \end{aligned} \tag{3.16}$$

Substitution of the solution (3.13) and obvious averaging yield the modulation equations

$$\begin{aligned} &[\,a^2V^2/(V^2 - 1)]_t + [\,a^2V/(V^2 - 1)]_x = 0, \\ &[\,a^2V/(V^2 - 1)]_t + [\,a^2/(V^2 - 1)]_x = 0. \end{aligned} \tag{3.17}$$

To show that these equations agree with the earlier obtained results, let us express the phase velocity V in terms of the wavevector k. On substitution of $V = \sqrt{k^2 + 1}/k$ into Eqs. (3.17), we obtain

$$\begin{aligned} &a^2\left(2kk_t + \frac{(2k^2+1)k_x}{\sqrt{k^2+1}}\right) + (k^2+1)\left((a^2)_t + \frac{k(a^2)_x}{\sqrt{k^2+1}}\right) = 0, \\ &(a^2)_t + \frac{k(a^2)_x}{\sqrt{k^2+1}} + a^2\left(\frac{(2k^2+1)k_t}{k(k^2+1)} + \frac{2k_x}{\sqrt{k^2+1}}\right) = 0. \end{aligned} \tag{3.18}$$

Multiplication of the second equation by $(k^2 + 1)$ and subtraction of the result from the first one gives the equation containing the wavevector k only. With the use of the relation $\omega'(k) = k(k^2 + 1)^{-1/2}$ this equation can be transformed to

$$k_t + \omega'(k)k_x = 0. \tag{3.19}$$

Substitution of this relation into the second equation (3.18) gives with account of $\omega''(k) = (k^2+1)^{-3/2}$ another modulation equation

$$(a^2)_t + (\omega'(k)\, a^2)_x = 0, \tag{3.20}$$

which has a meaning of the energy conservation law. Equations (3.19,3.20) are the modulation equations for the wave-train solution of Eq. (3.12).

As it follows from Eq. (3.19), the wavevector k is constant along characteristics determined by the equation

$$dx/dt = \omega'(k), \tag{3.21}$$

which has the general solution $x - \omega'(k)t = f(k)$, and, hence, at large x and t the wavevector propagates with the group velocity

$$x \cong \omega'(k)t. \tag{3.22}$$

This result has been already obtained by means of the asymptotic analysis in Sec. 1.2.2. If we rewrite Eq. (3.20) in the form

$$da/dt = a_t + \omega'(k)\, a_x = -\tfrac{1}{2} a\, \omega''(k)\, k_x$$

and take into account the relation $\omega''(k)k_x \cong 1/t$ which follows from Eq. (3.22) by differentiation with respect to x, we shall find that along characteristic the amplitude obeys the equation

$$da/dt \cong -\tfrac{1}{2} a/t,$$

and, hence, it varies with time as

$$a \propto t^{-1/2},$$

what agrees with Eq. (1.68) known from the asymptotic analysis.

Thus, we have reproduced the main results of the linear theory about influence of the dispersion effects on the evolution of the wave packet (i.e., of the modulated periodic wave) by means of the Whitham averaging method. Now let us proceed to a simple example of a nonlinear modulated wave train.

3.3 Modulation of a wave-train solution of the nonlinear Klein-Gordon equation

Here we shall generalize the results of the preceding section on the periodic solution of the nonlinear Klein-Gordon equation,

$$u_{tt} - u_{xx} + \Phi'(u) = 0, \tag{3.23}$$

which reduces to the linear equation (3.12) for $\Phi(u) = u^2/2$. The propagating with the phase velocity V wave,

$$u = u(\xi), \quad \xi = x - Vt, \tag{3.24}$$

satisfies the equation

$$u_\xi = \sqrt{2/(V^2-1)} \cdot \sqrt{A - \Phi(u)}, \tag{3.25}$$

where A is the integration constant, so that the variable u oscillates between two roots u_1 and u_2 of the equation

$$A - \Phi(u) = 0, \tag{3.26}$$

provided $\Phi(u) < A$ in the interval $u_1 \le u \le u_2$. Then integration of Eq. (3.25) gives the dependence of ξ on u:

$$\xi(u) = \sqrt{\frac{V^2-1}{2}} \int_{u_1}^{u} \frac{du}{\sqrt{A - \Phi(u)}}, \tag{3.27}$$

and the inverse function $u(\xi)$ is the periodic solution of Eq. (3.23). The wavelength is equal to

$$L = L(V, A) = \sqrt{2(V^2-1)} \int_{u_1}^{u_2} du/\sqrt{A - \Phi(u)}. \tag{3.28}$$

As in the linear case, the wave-train solution depends on two parameters V and A (apart of an inessential initial phase ξ_0). The two conservation laws are the obvious generalizations of Eqs. (3.16),

$$\begin{aligned} &\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \Phi(u)\right)_t + (-u_t u_x)_x = 0, \\ &(-u_t u_x)_t + \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 - \Phi(u)\right)_x = 0. \end{aligned} \tag{3.29}$$

Following Whitham, let us introduce the function

$$\begin{aligned} W(V,A) = (V^2-1)\oint u_\xi du &= \sqrt{2(V^2-1)}\oint\sqrt{A-\Phi(u)}du \\ &= \sqrt{V^2-1}\cdot G(A), \end{aligned} \tag{3.30}$$

where the contour integration is taken over the cycle around the interval $u_1 \le u \le u_2$, and the function

$$G(A) = \sqrt{2}\oint\sqrt{A-\Phi(u)}du \tag{3.31}$$

depends only on A. Its differentiation with respect to A gives the relation

$$G'(A) = 2^{-1/2}\oint du/\sqrt{A-\Phi(u)} = (V^2-1)^{-1/2}\oint d\xi = L/\sqrt{V^2-1},$$

where L is the wavelength (3.28). Hence, the partial derivative of W with respect to A is equal to

$$W_A = \sqrt{V^2-1}\cdot G'(A) = L. \tag{3.32}$$

Let us define the wavevector k by the equation

$$k = 1/L = 1/W_A, \tag{3.33}$$

where we have dropped for convenience the factor 2π. Since the frequency is equal to $\omega = kV$, Eqs. (3.32) and (3.33) yield the dispersion relation

$$\omega^2 = k^2 + (G'(A))^{-2} \tag{3.34}$$

which depends on the amplitude of the wave.

Calculation of the necessary quantities averaged according to the rule (3.10), that is,

$$\overline{\mathcal{F}(u)} = L^{-1}\int_0^L \mathcal{F}(u)d\xi = k\oint \mathcal{F}(u)du/u_\xi,$$

gives

$$\begin{aligned} &\overline{\tfrac{1}{2}u_t^2+\tfrac{1}{2}u_x^2} = \tfrac{1}{2}(V^2+1)\overline{u_\xi^2} = \frac{V^2+1}{V^2-1}\cdot\overline{(A-\Phi(u))} = \frac{k}{2}\frac{V^2+1}{V^2-1}W, \\ &\overline{-u_tu_x} = V\overline{u_\xi^2} = \frac{kV}{V^2-1}W, \\ &\overline{\Phi(u)} = A - \tfrac{1}{2}(V^2-1)\overline{u_\xi^2} = A - \frac{k}{2}W. \end{aligned}$$

Thus, all these quantities are expressed in terms of the function W, and after their substitution into Eqs. (3.29) we obtain the averaged conservation laws

$$\begin{aligned}&\big(kW/(V^2-1)+A\big)_t+\big(kVW/(V^2-1)\big)_x=0,\\&\big(kVW/(V^2-1)\big)_t+\big(kV^2W/(V^2-1)-A\big)_x=0.\end{aligned}\tag{3.35}$$

These modulation equations, where k is a function of V and A defined by Eq. (3.33), describe the slow evolution of the parameters V and A in the modulated nonlinear wave train. If we take into account that the derivative of W with respect to V is equal to

$$W_V=VG(A)/\sqrt{V^2-1}=VW/(V^2-1),$$

we can rewrite these equations in the form more convenient for further transformations,

$$\begin{aligned}&[k(VW_V+AW_A-W)]_t+[kV(VW_V+AW_A-W)-VA]_x=0,\\&(kW_V)_t+(kVW_V-A)_x=0.\end{aligned}\tag{3.36}$$

Partial differentiation in the first equation gives

$$\begin{aligned}&V[(kW_V)_t+(kVW_V-A)_x]+A[(kW_A)_t+(kVW_A-V)_x]\\&+k(W_AA_t+W_VV_t-W_t)+kV(W_AA_x+W_VV_x-W_x)\\&-W[k_t+(kV)_x]=0.\end{aligned}$$

Here, the first term vanishes by the second equation (3.36), the second term by Eq. (3.33), the third and forth terms because W does not depend explicitly on t and x. Thus, we arrive again at the equation

$$k_t+(kV)_x=0,\tag{3.37}$$

or, by the relationship $\omega=kV$, at the equation

$$k_t+\omega_x=0.\tag{3.38}$$

This equation always appears as a consequence of the Whitham equations and is called the equation of the 'conservation of waves', k being the density of waves and ω the flux of waves. The phase of a uniform periodic wave train is given by $\theta=kx-\omega t$, and though in a non-uniform modulated wave it becomes a more complicated function of x and t, this linear dependence remains a good approximation within relatively small intervals (3.7).

Therefore, we can define the local values of the wavevector and frequency by the equations

$$k = \theta_x, \quad \omega = -\theta_t, \tag{3.39}$$

and then Eq. (3.38) is a direct consequence of these definitions.

Substitution of $k = 1/W_A$ into Eq. (3.37) and the second equation (3.36) yields after simple transformations the Whitham equations in the form

$$W_{A,t} + VW_{A,x} - W_A V_x = 0, \quad W_{V,t} + VW_{V,x} - W_A A_x = 0. \tag{3.40}$$

Introducing a 'long' derivative

$$D/Dt = \partial/\partial t + V\partial/\partial x,$$

we rewrite Eqs. (3.40) in a more compact form

$$DW_A/Dt - W_A V_x = 0, \quad DW_V/Dt - W_A A_x = 0. \tag{3.41}$$

To calculate the characteristic velocities, let us substitute Eq. (3.30) into Eqs. (3.40). We obtain

$$\begin{aligned} &G''A_t + VG''A_x + (VG'/(V^2-1))V_t + (G'/(V^2-1))V_x = 0,\\ &VG'A_t + G'A_x - (G/(V^2-1))V_t - (VG/(V^2-1))V_x = 0, \end{aligned} \tag{3.42}$$

where the function $G(A)$ is defined by Eq. (3.31) and depends only on A. Equations (3.42) have a familiar form of hydrodynamical equations (1.149). The characteristic velocities are given by the roots of the equation (see Eq. (1.153))

$$(GG'' + V^2G'^2)\cot^2\phi - 2V(GG'' + G'^2)\cot\phi + (V^2GG'' + G'^2) = 0,$$

and are equal to

$$v_\pm = \cot\phi_\pm = \frac{V(GG'' + G'^2) \pm (V^2-1)\sqrt{-GG''G'^2}}{GG'' + V^2G'^2}.$$

It is convenient to define a 'velocity'

$$\tilde{V} = \sqrt{-G'^2/GG''} \tag{3.43}$$

which is real if $G'' < 0$, so that

$$v_\pm = \frac{V \mp \tilde{V}}{1 \mp V\tilde{V}}. \tag{3.44}$$

This formula looks like a relativistic law of the velocities addition, which is quite natural because of relativistic invariance of the Klein-Gordon equation (3.23). The characteristic curves in the (x,t) plane are defined as solutions of the equations

$$\frac{dx}{dt} = \frac{V \mp \tilde{V}}{1 \mp V\tilde{V}}. \tag{3.45}$$

The formulas (1.155) give $K = \pm G'^2/\tilde{V}$, so that Eqs. (1.156) transform to

$$\frac{dV}{V^2-1} \mp \sqrt{-\frac{G''}{G}}\,dA = 0,$$

and, hence, the Riemann invariants are equal to

$$\lambda_\pm = \int^V \frac{dV}{V^2-1} \mp \int^A \sqrt{-\frac{G''}{G}}\,dA. \tag{3.46}$$

Thus, the modulation equations (3.42) are reduced to the diagonal form

$$\partial\lambda_+/\partial t + v_+\partial\lambda_+/\partial x = 0, \quad \partial\lambda_-/\partial t + v_-\partial\lambda_-/\partial x = 0, \tag{3.47}$$

where the velocities $v_\pm$ defined by Eqs. (3.44) have to be expressed in terms of the Riemann invariants (3.46).

If $G'' < 0$, the characteristic velocities and the Riemann invariants are real. In contrast to the linear theory of the preceding section, where there was only one characteristic speed (3.21) equal to the group velocity, now we have two different characteristic speeds, and, hence, small disturbances of the wave train propagate along different families of the characteristic curves. In the linear limit $A \to 0$ we have $G'' \to 0$ and both characteristic speeds have the same limiting value $v_\pm \to 1/V$ equal to the linear group velocity $\omega'(k) = k/\sqrt{k^2+1} = k/\omega = 1/V$. So, the nonlinear characteristic speeds may be considered as generalizations of the notion of the group velocity. This nonlinear splitting of the group velocity means that an initially localized disturbance of a periodic wave train separates after some time into the two simple waves each propagating with its own value of the characteristic velocity.

It is instructive to consider this nonlinear splitting phenomenon for a simple example of weak nonlinearity. In Sec. 2.2.1 we have considered waves in a chain of coupled nonlinear pendulums whose motion at small enough amplitudes and in a long wave limit is governed by the Klein-Gordon equation

$$q_{tt} - q_{xx} + q - \gamma q^3 = 0 \tag{3.48}$$

(see Eq. (2.44)). Evolution of the envelope amplitude $u(x,t)$ of a wave packet, defined according to the relation $q(x,t) = u(x,t)\exp[i(kx - \omega(k)t)]$, is governed by the NLS equation (see Eq. (2.80)),

$$i(u_t + \omega_0'(k)\, u_x) + \tfrac{1}{2}\omega_0''(k)\, u_{xx} + \tfrac{3}{2}(\gamma/\omega_0(k))|u|^2 u = 0, \tag{3.49}$$

where

$$\omega_0(k) = \sqrt{k^2 + 1} \tag{3.50}$$

is the linear dispersion law and k is the wavevector of the carrier wave. For modulation stability of linear waves we suppose that $\gamma < 0$.

As we know from the preceding section, modulations of linear waves are governed by the equations (see Eqs. (3.19,3.20))

$$k_t + \omega_0'(k)\, k_x = 0, \quad (a^2)_t + (\omega_0'(k)\, a^2)_x = 0. \tag{3.51}$$

The main nonlinear effect is a nonlinear correction to the frequency due to the last term in Eq. (3.49). For a uniform wave we have

$$u = a\exp[i(3\gamma a^2/2\omega_0(k))\, t],$$

that is

$$\omega(k) = \omega_0(k) + \omega_2(k)\, a^2 + \dots, \tag{3.52}$$

where $\omega_0(k)$ is given by Eq. (3.50) and

$$\omega_2(k) = -3\gamma/2\omega_0(k). \tag{3.53}$$

Taking this correction into account, we replace the first equation (3.51) by the equation

$$k_t + [\omega_0'(k) + \omega_2'(k)a^2]k_x + \omega_2(k)(a^2)_x = 0,$$

which follows from the general law $k_t + \omega_x = 0$ with ω given by Eq. (3.52). Here the last term is the most important, because it causes changes of k due to changes of a. The term $\omega_2'(k)a^2 k_x$ only slightly corrects the already existing term $\omega_0'(k)k_x$ and we can drop it. In a similar way, the corrections to the second equation (3.51) are relatively small and can also be dropped out. As a result, we arrive at the following system of modulation equations for a weakly nonlinear wave train:

$$\begin{aligned} &k_t + \omega_0'(k)k_x + \omega_2(k)(a^2)_x = 0, \\ &(a^2)_t + \omega_0'(k)(a^2)_x + \omega_0''(k)a^2 k_x = 0. \end{aligned} \tag{3.54}$$

By the method of Sec. 1.3.4 we find the characteristic speeds

$$v_\pm = \omega_0'(k) \pm \sqrt{\omega_0''(k)\,\omega_2(k)\,a^2} \tag{3.55}$$

and the corresponding Riemann invariants

$$\lambda_\pm = \tfrac{1}{2}\int \sqrt{\omega_0''(k)/\omega_2(k)}\,dk \mp a. \tag{3.56}$$

Equation (3.55) clearly shows splitting of the linear group velocity $\omega_0'(k)$ into the two characteristic velocities.

Let us compare formulas (3.55) with the dispersion relation of a modulation wave of the uniform solution to the NLS equation (3.49). It has the form either (1.335) or (1.365) depending on the sign of $\omega_2(k)$, and in our present notation is given by

$$\Omega/K = \omega_0'(k) \pm \sqrt{\left(\tfrac{1}{2}\omega_0''(k)\right)^2 K^2 + \omega_0''(k)\,\omega_2(k)\,a^2}. \tag{3.57}$$

As we see, in the long wave limit $K \to 0$ the phase velocities of the modulation wave coincide with the characteristic velocities (3.55). This should be expected, because the system (3.54) has a meaning of the Whitham equations for weakly nonlinear waves, and the Whitham approach is based on the assumption that modulation has a space scale much greater than the wavelength of the nonlinear wave train under consideration. On the other hand, in the NLS equation approach it is also assumed that the envelope amplitude of a linear wave packet is a slow enough function, although the dispersion effects are taken into account with more accuracy than it is done in the Whitham theory. Formally, this difference manifests itself in the additional differential term in the dispersion relation for the NLS equation compared with the purely algebraic dispersion relation (3.34) in the Whitham approximation. To clarify this point, let us represent the envelope amplitude in the form

$$u = a(x,t)\exp(i\phi(x,t)) \tag{3.58}$$

and define the wavevector K and the frequency Ω of the modulation wave by the usual equations

$$K = \phi_x, \quad \Omega = -\phi_t. \tag{3.59}$$

Then substitution of Eq. (3.58) into the NLS equation leads to the system

$$\begin{aligned} &a_t + \omega_0'(k)a_x + \omega_0''(k)a_x\phi_x + \tfrac{1}{2}\omega_0''(k)a\phi_{xx} = 0, \\ &\phi_t + \omega_0'(k)\phi_x + \tfrac{1}{2}\omega_0''(k)\phi_x^2 + \omega_2(k)a^2 - \tfrac{1}{2}\omega_0''(k)a_{xx}/a = 0, \end{aligned} \tag{3.60}$$

and the last equation can be written in the form

$$\Omega = \omega_0'(k)\,K + \tfrac{1}{2}\omega_0''(k)\,K^2 + \omega_2(k)a^2 - \tfrac{1}{2}\omega_0''(k)\,a_{xx}/a \tag{3.61}$$

of the dispersion relation for the modulation wave with the wavevector K small compared with the wavevector k of the carrier wave. The first two terms in the right hand side of Eq. (3.61) correspond to the two terms of the Taylor expansion of the dispersion relation $\omega_0(k+K)$ in powers of $K \ll k$. The third term describes the nonlinear frequency shift (see Eq. (3.52)), and the last term is a specific NLS correction term which is absent in the Whitham theory in the developed here form. Thus, the Whitham theory does not include the NLS equation, but it has an advantage that it applies to strongly nonlinear waves with a large amplitude rather than to weakly nonlinear wave packets only as it takes place in the case of the NLS equation. In the next section we shall show that the Whitham approach can be made more precise to include the dispersion effects corresponding to the last term of Eq. (3.61).

Finally, let us note that if $G'' > 0$, then the characteristic velocities (3.44) and Riemann invariants (3.46) become complex. This means that a small disturbance will grow with time, that is, the wave train is modulationally unstable. This result agrees with the weak nonlinearity limit, when the condition $\omega_2 < 0$ (i.e., $\gamma > 0$) means modulation instability of a plane wave.

3.4 A variational approach to the modulation theory

If the evolution wave equation can be represented as the Euler-Lagrange equation corresponding to some action functional, that is, it is a consequence of the Hamilton principle,

$$\delta \int \Lambda(u, u_t, u_x) dx dt = 0, \tag{3.62}$$

with some Lagrangian Λ depending on the wave variable u and its space and time derivatives, then the Whitham averaging approach can be simplified. The variational principle (3.62) with fixed limits of integration gives

$$\int \left(\frac{\partial \Lambda}{\partial u} \delta u + \frac{\partial \Lambda}{\partial u_t} \delta u_t + \frac{\partial \Lambda}{\partial u_x} \delta u_x \right) dx dt = 0,$$

and after integration of the two last terms by parts (δu vanishes at the integration limits) we obtain the equation

$$\int \left(\frac{\partial \Lambda}{\partial u} - \frac{\partial}{\partial t} \frac{\partial \Lambda}{\partial u_t} - \frac{\partial}{\partial x} \frac{\partial \Lambda}{\partial u_x} \right) \delta u dx dt = 0,$$

which due to arbitrariness of the variation δu yields the Euler-Lagrange equation

$$\frac{\partial}{\partial t} \frac{\partial \Lambda}{\partial u_t} + \frac{\partial}{\partial x} \frac{\partial \Lambda}{\partial u_x} - \frac{\partial \Lambda}{\partial u} = 0. \tag{3.63}$$

(If a Lagrangian Λ depends on higher derivatives of u, then analogous calculations lead to additional terms in Eq. (3.63).) This form of the evolution equation permits one to write the conservation laws of energy and momentum in a general form. The conservation of energy is connected with invariance of the system with respect to time shifts $t \to t + t_0$ or, in other words, with the fact that Λ does not depend explicitly on time t. Therefore, its time derivative is equal to

$$\frac{\partial \Lambda}{\partial t} = \frac{\partial \Lambda}{\partial u} u_t + \frac{\partial \Lambda}{\partial u_t} u_{tt} + \frac{\partial \Lambda}{\partial u_x} u_{xt}.$$

Elimination of $\partial \Lambda / \partial u$ with the help of Eq. (3.63) gives the energy conservation law

$$\frac{\partial}{\partial t} \left(\frac{\partial \Lambda}{\partial u_t} u_t - \Lambda \right) + \frac{\partial}{\partial x} \left(\frac{\partial \Lambda}{\partial u_x} u_t \right) = 0. \tag{3.64}$$

If the system is invariant with respect to space shifts $x \to x + x_0$, that is Λ does not depend explicitly on x, then analogous calculation gives the momentum conservation law

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial u_t}u_x\right)+\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_x}u_x-\Lambda\right)=0. \tag{3.65}$$

As a simple example, note that the Lagrangian

$$\Lambda = \tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - \Phi(u) \tag{3.66}$$

leads to the Klein-Gordon equation (3.23) and the conservation laws (3.64) and (3.65) coincide with the known formulas (3.29).

Let us suppose now that we know the periodic solution

$$u = u(\theta, A) \tag{3.67}$$

of the evolution equation. Here θ denotes the phase which for a uniform wave train is equal to

$$\theta = kx - \omega t \tag{3.68}$$

with constant k and ω. In a modulated non-uniform wave the wavevector k and the frequency ω are defined by the equations

$$k = \theta_x, \quad \omega = -\theta_t, \tag{3.69}$$

and they themselves are slow functions of x and t. A parameter A in Eq. (3.67) characterizes the amplitude of the wave and also depends on x and t. Within small enough intervals of the x axis we may consider the wave train as uniform and average the Lagrangian Λ over the wavelength $L = 2\pi/k$ to obtain

$$\mathcal{L}(k,\omega,A) = \frac{1}{L}\int_0^L \Lambda(kx-\omega t, A)dx, \quad L = 2\pi/k, \tag{3.70}$$

where k, ω, and A are considered as constant. However, in larger scale, the parameters k, ω, A depend slowly on x and t, so that we have to replace k and ω by their definitions (3.69). After such a replacement the averaged Lagrangian

$$\mathcal{L} = \mathcal{L}(\theta_x, -\theta_t, A) \tag{3.71}$$

becomes a slow function of x and t in spite of the integration over x in Eq. (3.70). Whitham supposed that the slow evolution is governed again by the variational principle

$$\delta \int \mathcal{L}(\theta_x, -\theta_t, A) dx dt = 0. \tag{3.72}$$

Now we have two unknown functions $\theta(x,t)$ and $A(x,t)$ whose variation leads to two Euler-Lagrange equations similar to Eq. (3.63). Since in $\mathcal{L}$ there are no derivatives of A, variation with respect to A gives

$$\partial \mathcal{L} / \partial A = 0, \tag{3.73}$$

and since $\mathcal{L}$ does not depend on the phase θ itself but only on its derivatives, variation with respect to θ gives

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \theta_t} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial \theta_x} = 0. \tag{3.74}$$

It is convenient to return in this modulation equation to $k = \theta_x$ and $\omega = -\theta_t$ and write it in the form

$$\frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial x} \mathcal{L}_k = 0. \tag{3.75}$$

Then equations (3.73) and (3.75) should be complemented by the compatibility equation

$$k_t + \omega_x = 0 \tag{3.76}$$

which follows from the definitions (3.69). The averaged conservation laws analogous to Eqs. (3.64,3.65) are given by the equations

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \omega} \omega - \mathcal{L} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial k} \omega \right) &= 0, \\ \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial k} k \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial k} k - \mathcal{L} \right) &= 0, \end{aligned} \tag{3.77}$$

which relate the variational approach with the theory developed in the preceding section.

As a simple example, let us consider a linear Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0, \tag{3.78}$$

which corresponds to the Lagrangian

$$\Lambda = \tfrac{1}{2} u_t^2 - \tfrac{1}{2} u_x^2 - \tfrac{1}{2} u^2. \tag{3.79}$$

We are interested in a weakly non-uniform wave train

$$u = A\cos\theta, \tag{3.80}$$

where A, $k = \theta_x$, $\omega = -\theta_t$ are slow functions of x and t. Then we have locally

$$u = A\cos(kx - \omega t + \phi_0) \tag{3.81}$$

and averaging of the Lagrangian (3.79) gives

$$\mathcal{L} = \tfrac{1}{4}(\omega^2 - k^2 - 1)A^2. \tag{3.82}$$

This is a typical for linear problems result—the initial Lagrangian is quadratic with respect to the wave variable u, and, hence, the averaged Lagrangian is quadratic with respect to the amplitude A,

$$\mathcal{L} = D(\omega, k)A^2. \tag{3.83}$$

The variational equation (3.73) gives

$$D(\omega, k) = 0 \tag{3.84}$$

which is an implicit form of the dispersion relation. From $D(\omega(k), k) = 0$ we find the expression for the group velocity

$$v_g = d\omega/dk = -D_k/D_\omega. \tag{3.85}$$

The compatibility equation (3.76) transforms to a familiar form

$$k_t + \omega'(k)k_x = 0, \tag{3.86}$$

and Eq. (3.75), that is

$$(D_\omega a^2)_t + (D_k a^2)_x = 0,$$

can be transformed with the use of Eqs. (3.85,3.86) to

$$(a^2)_t + (\omega'(k)\, a^2)_x = 0. \tag{3.87}$$

Thus, we have reproduced the modulation equations (3.19,3.20) for the linear Klein-Gordon equation.

For better understanding the variational approach, let us return to the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + \Phi'(u) = 0, \tag{3.88}$$

whose uniform wave-train solution

$$u = u(\theta), \quad \theta = kx - \omega t \tag{3.89}$$

satisfies the equation

$$(\omega^2 - k^2)u_{\theta\theta} + \Phi'(u) = 0 \tag{3.90}$$

with obvious first integral

$$\tfrac{1}{2}(\omega^2 - k^2)u_\theta^2 + \Phi(u) = A. \tag{3.91}$$

We can average the Lagrangian

$$\Lambda = \tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - \Phi(u) \tag{3.92}$$

without knowing the explicit form of the periodic solution (3.89). Indeed, substitution of Eq. (3.89) into Λ and taking into account Eq. (3.91) give

$$\Lambda = \tfrac{1}{2}(\omega^2 - k^2)\,u_\theta^2 - \Phi = (\omega^2 - k^2)\,u_\theta^2 - A. \tag{3.93}$$

Averaging over the wavelength is equivalent to averaging over the period of the phase θ. We can normalize θ so that this period is equal to 2π, that is, ω, k, and A are related with each other by the dispersion relation following from Eq. (3.91),

$$2\pi = \sqrt{\frac{2}{\omega^2 - k^2}} \oint \frac{du}{\sqrt{A - \Phi(u)}}, \tag{3.94}$$

(cf. Eq. (3.28)). Then, the averaging of the Lagrangian (3.93) gives

$$\begin{aligned} \mathcal{L}(k, \omega, A) &= (1/2\pi)(\omega^2 - k^2) \int_0^{2\pi} u_\theta^2 d\theta - A \\ &= (1/2\pi)\sqrt{2(\omega^2 - k^2)} \oint \sqrt{A - \Phi(u)}\, du - A, \end{aligned} \tag{3.95}$$

and, hence, the modulation equations

$$\mathcal{L}_A = 0, \quad (\mathcal{L}_\omega)_t - (\mathcal{L}_k)_x = 0, \quad k_t + \omega_x = 0 \tag{3.96}$$

are found. Note that the first equation coincides with Eq. (3.94), and the averaged Lagrangian is related with the function W of the preceding section (see Eq. (3.30)) by a simple relation

$$\mathcal{L} = (k/2\pi)W(\omega/k, A) - A.$$

Now let us try to justify the averaged variational principle for modulation equations by means of the perturbation theory with respect to small parameter ε equal to a ratio of the wavelength to a characteristic scale of modulation. If the coordinates x and t are measured in units of the wavelength and the period, then slowly modulated functions change considerably at scales of order $\sim 1/\varepsilon$. We introduce the slow space and time variables

$$X = \varepsilon x, \quad T = \varepsilon t. \tag{3.97}$$

Then, the phase $\theta(x,t)$ can be represented in the form

$$\theta = \Theta(X,T)/\varepsilon, \tag{3.98}$$

so that the slowly varying wavevector and frequency are expressed in terms of the slow function $\Theta(X,T)$ by the formulas

$$k = \theta_x = \Theta_X, \quad \omega = -\theta_t = -\Theta_T. \tag{3.99}$$

The non-uniform modulated wave train depends on X and T as via θ so due to a slow dependence of the other parameters on X and T, that is,

$$u = u(\theta, X, T; \varepsilon), \tag{3.100}$$

where θ is defined by Eq. (3.98). Hence, the derivatives of u with respect to t and x are equal to

$$u_t = -\omega u_\theta + \varepsilon u_T, \quad u_x = k u_\theta + \varepsilon u_X, \tag{3.101}$$

where the first terms in the right hand sides correspond to the fast oscillations in the wave train, and the second terms to a slow modulation of the amplitude and the other parameters.

Let us rewrite the evolution equation (3.63) in the form

$$\Lambda_{1,t} + \Lambda_{2,x} - \Lambda_3 = 0, \tag{3.102}$$

where

$$\Lambda_1 = \partial\Lambda/\partial u_t, \quad \Lambda_2 = \partial\Lambda/\partial u_x, \quad \Lambda_3 = \partial\Lambda/\partial u. \tag{3.103}$$

On substitution of Eq. (3.100) into Λ_i we get these quantities as functions of θ, X, T, so that their derivatives with respect to x and t are given by the formulas similar to Eqs. (3.101) and then Eq. (3.102) takes the form

$$-\omega\Lambda_{1,\theta} + \varepsilon\Lambda_{1,T} + k\Lambda_{2,\theta} + \varepsilon\Lambda_{2,X} - \Lambda_3 = 0. \tag{3.104}$$

This equation with Λ_i given by the formulas

$$\Lambda_i = \Lambda_i(-\omega u_\theta + \varepsilon u_T, ku_\theta + \varepsilon u_X, u), \quad i = 1, 2, 3, \tag{3.105}$$

may be considered as the equation for the function $u(\theta, X, T)$, and the function $\theta(X, T)$ can be chosen so that this choice provides the desired properties of $u(\theta, X, T)$. This method is similar to the above used secular perturbation theory where the NLS equation for the envelope amplitude was obtained from the condition of elimination of resonant secular terms. Now, instead of the envelope amplitude, we have the slowly modulated parameters varying with x and t, and we shall eliminate the secular terms by the condition that $u(\theta, X, T)$ be a strictly periodic function of θ. Indeed, in this case the solution cannot have terms growing with time (or θ) what means the absence of secular terms. Moreover, we can choose θ so that the period of $u(\theta)$ is equal to 2π. Of course, the wavevector k and the frequency ω remain some functions of X and T.

To write down the periodicity condition for u, let us multiply Eq. (3.104) by u_θ and try to distinguish the total derivative with respect to θ, keeping in mind that k and ω do not depend on θ,

$$\begin{aligned}
-\omega\Lambda_{1,\theta}u_\theta &+ k\Lambda_{2,\theta}u_\theta + \varepsilon\Lambda_{1,T}u_\theta + \varepsilon\Lambda_{2,X}u_\theta - \Lambda_3 u_\theta \\
&= \partial/\partial\theta\big[(-\omega\Lambda_1 + k\Lambda_2)u_\theta\big] - \Lambda_1(-\omega u_{\theta\theta} + \varepsilon u_{\theta,T}) \\
&- \Lambda_2(ku_{\theta\theta} + \varepsilon u_{\theta,X}) - \Lambda_3 u_\theta \\
&+ \varepsilon(\Lambda_1 u_{\theta,T} + \Lambda_{1,T}u_\theta) + \varepsilon(\Lambda_2 u_{\theta,X} + \Lambda_{2,X}u_\theta) = 0,
\end{aligned}$$

or

$$\partial/\partial\theta\,[(-\omega\Lambda_1 + k\Lambda_2)u_\theta - \Lambda] + \varepsilon\partial/\partial T(u_\theta\Lambda_1) + \varepsilon\partial/\partial X(u_\theta\Lambda_2) = 0.$$

On integration of this equation over θ from 0 to 2π the first term vanishes due to the supposed periodicity of $u(\theta)$, so we arrive at the equation

$$\frac{\partial}{\partial T}\left(\frac{1}{2\pi}\int_0^{2\pi} u_\theta\Lambda_1 d\theta\right) + \frac{\partial}{\partial X}\left(\frac{1}{2\pi}\int_0^{2\pi} u_\theta\Lambda_2 d\theta\right) = 0. \tag{3.106}$$

Thus, for exclusion of secular terms, Eq. (3.104) has to be solved with taking into account the condition (3.106).

Let us show that for the case of the Klein-Gordon equation this procedure leads in the limit $\varepsilon \to 0$ to the modulation equations (3.96). Equation (3.104) transforms to the equation for u_θ,

$$(-\omega\Lambda_1^{(0)} + k\Lambda_2^{(0)})_\theta - \Lambda_3^{(0)} = 0,$$

where superscript (0) means that in the arguments of the functions (3.105) we have to put $\varepsilon \to 0$. Multiplication of this equation by $u_\theta^{(0)}$ and integration over θ gives the integral

$$(-\omega\Lambda_1^{(0)} + k\Lambda_2^{(0)})u_\theta^{(0)} - \Lambda^{(0)} = A(X,T) \qquad (3.107)$$

coinciding with the result of integration of Eq. (3.104) at $\varepsilon = 0$. It is easy to find from the Lagrangian (3.92) for the Klein-Gordon equation that $\Lambda_1 = u_t$, $\Lambda_2 = -u_x$, $\Lambda_3 = -\Phi'$, and hence,

$$\Lambda_1^{(0)} = -\omega u_\theta^{(0)}, \quad \Lambda_2^{(0)} = -k u_\theta^{(0)},$$

so that Eq. (3.107) transforms to Eq. (3.93). The condition that the period of u is equal to 2π (i.e., Eq. (3.94)) can be written as $\mathcal{L}_A = 0$ where $\mathcal{L}$ is defined by Eq. (3.95). At last, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} u_\theta^{(0)}\Lambda_1^{(0)} d\theta &= -\frac{\omega}{2\pi}\int_0^{2\pi} (u_\theta^{(0)})^2 d\theta \\
&= -\frac{\omega}{2\pi}\sqrt{\frac{2}{\omega^2 - k^2}}\oint\sqrt{A - \Phi(u)}\,du = -\mathcal{L}_\omega, \\
\frac{1}{2\pi}\int_0^{2\pi} u_\theta^{(0)}\Lambda_2^{(0)} d\theta &= -\frac{k}{2\pi}\int_0^{2\pi} (u_\theta^{(0)})^2 d\theta \\
&= -\frac{k}{2\pi}\sqrt{\frac{2}{\omega^2 - k^2}}\oint\sqrt{A - \Phi(u)}\,du = \mathcal{L}_k,
\end{aligned}$$

and, hence, Eq. (3.106) coincides with the second equation (3.96). Thus, the variational principle (3.72) is the first approximation of the perturbation theory with respect to a small parameter ε. Now we are able to make the next step and to find the correction of the next order in ε. This will also clarify the connection of the Whitham theory with the NLS theory for the envelope of weakly nonlinear waves.

We shall start from the variational principle

$$\delta\iint\left\{\frac{1}{2\pi}\int_0^{2\pi}\Lambda(\Theta_T u_\theta + \varepsilon u_T, \Theta_X u_\theta + \varepsilon u_X, u)d\theta\right\}dXdT = 0 \qquad (3.108)$$

with the corrected Lagrangian, where we have used Eqs. (3.99) and (3.101) to replace u_t and u_x by their derivatives with respect to θ, X, and T. It is remarkable that the main equations (3.104) and (3.106) can be obtained at once from Eq. (3.108). This functional depends on two functions, $u(\theta, X, T)$ and $\Theta(X, T)$. For variation of u the Lagrangian is given by Λ and the corresponding Euler-Lagrange equation reads

$$\Theta_T \Lambda_{1,\theta} + \Theta_X \Lambda_{2,\theta} + \varepsilon \Lambda_{1,T} + \varepsilon \Lambda_{2,X} - \Lambda_3 = 0,$$

which with account of Eqs. (3.99) coincides with Eq. (3.104). On the other hand, for variation of $\Theta(X, T)$ the effective Lagrangian is given by the integral in curly braces in Eq. (3.108), i.e., by $(1/2\pi)\int_0^{2\pi} \Lambda d\theta$, and this variation gives

$$\frac{\partial}{\partial T}\left[\frac{\partial}{\partial \Theta_T}\left\{\frac{1}{2\pi}\int_0^{2\pi}\Lambda\, d\theta\right\}\right] + \frac{\partial}{\partial X}\left[\frac{\partial}{\partial \Theta_X}\left\{\frac{1}{2\pi}\int_0^{2\pi}\Lambda\, d\theta\right\}\right] = 0, \quad (3.109)$$

so that differentiation of the integrals with respect to Θ_T and Θ_X reproduces Eq. (3.106). Thus, the variational principle (3.108) leads to the same equations as the periodicity condition, and it is more convenient in practice, because it is easier to calculate at first the averaged Lagrangian

$$\mathcal{L} = \overline{\Lambda} = \frac{1}{2\pi}\int_0^{2\pi} \Lambda(-\omega u_\theta + \varepsilon u_T, k u_\theta + \varepsilon u_X, u) d\theta, \quad (3.110)$$

and after that to obtain the modulation equations by differentiation.

Let us consider by this method the Klein-Gordon equation

$$u_{tt} - u_{xx} + u - \gamma u^3 = 0, \quad (3.111)$$

which corresponds to the Lagrangian

$$\Lambda = \tfrac{1}{2}(u_t^2 - u_x^2 - u^2) + \tfrac{1}{4}\gamma u^4. \quad (3.112)$$

The modulation equations are to be obtained from the variational principle

$$\delta \iint \mathcal{L}\, dX dT = 0, \quad (3.113)$$

where the averaged Lagrangian is given by

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} \left[\tfrac{1}{2}(-\omega u_\theta + \varepsilon u_T)^2 - \tfrac{1}{2}(k u_\theta + \varepsilon u_X)^2 - \tfrac{1}{2}u^2 + \tfrac{1}{4}\gamma u^4\right] d\theta. \quad (3.114)$$

We want to discuss here the connection of the Whitham theory with the NLS theory for the envelope amplitude. Therefore, we shall consider an almost periodic wave

$$u \cong A\cos\theta, \tag{3.115}$$

where A, $k = \theta_x$, and $\omega = -\theta_t$ are the slow functions of x and t, that is, up to higher harmonics which we are not interested in, we have

$$u_\theta \cong -A\sin\theta, \quad u_T \cong A_T\cos\theta, \quad u_X \cong A_X\cos\theta$$

(recall that the arguments of $u(\theta, X, T)$ are considered as independent of each other). On substitution of these expressions into Eq. (3.114), we obtain with the accuracy up to the order $\sim \varepsilon^2$ the averaged Lagrangian

$$\mathcal{L} = \tfrac{1}{4}(\omega^2 - k^2 - 1)A^2 + \tfrac{3}{2}\gamma A^4 + \tfrac{1}{4}\varepsilon^2(A_T^2 - A_X^2). \tag{3.116}$$

Variation of A gives the 'dispersion relation'

$$(\omega^2 - k^2 - 1)A + 12\gamma A^3 - \varepsilon^2(A_{TT} - A_{XX}) = 0. \tag{3.117}$$

Here the first term corresponds to the linear dispersion relation. The second term gives the nonlinear frequency shift,

$$\omega^2 \cong 1 + k^2 - 12\gamma A^2, \quad \omega \cong \sqrt{1+k^2} - 6\gamma A^2/\sqrt{1+k^2}, \tag{3.118}$$

which coincides with Eqs. (3.52,3.53), if one takes into account that $A = a/2$. The last term in Eq. (3.117) describes the dispersion effects neglected in the first approximation of the Whitham theory.

Variation of $\Theta(X,T)$ gives Eq. (3.109), or $\mathcal{L}_{\omega,T} - \mathcal{L}_{k,X} = 0$, that is,

$$(\omega A^2)_T + (kA^2)_X = 0, \tag{3.119}$$

and to complete the system (3.117,3.119), we have to add the compatibility condition

$$k_T + \omega_X = 0. \tag{3.120}$$

Equations (3.117,3.119,3.120) provide the complete modulational system of the Whitham equations for an almost uniform monochromatic wave (3.115) whose evolution is governed by the Klein-Gordon equation (3.111). Let us consider some properties of this system.

First of all, note that in a monochromatic wave the parameters $A = A_0 = \text{const}$, $k = k_0 = \text{const}$, $\omega = \omega_0 = \text{const}$ are connected with each other by the dispersion relation (3.118). Let us linearize Eqs. (3.117,3.119,3.120) with respect to small deviations from these constant values,

$$A = A_0 + A', \quad \omega = \omega_0 + \omega', \quad k = k_0 + k'.$$

Then A', ω', k', satisfy the system

$$\begin{aligned}
&(2\omega_0\omega - 2k_0k')A_0 + 36\gamma A_0^2 A' - \varepsilon^2(A'_{TT} - A'_{XX}) = 0,\\
&(2\omega_0 A_0 A' + A_0^2\omega')_T + (2k_0 A_0 A' + A_0^2 k')_X = 0,\\
&k'_T + \omega'_X = 0.
\end{aligned}$$

Looking for the solution in the form $A', k', \omega' \propto \exp[i(KX - \Omega T)]$, we find the dispersion relation for the modulation waves,

$$4(\omega_0\Omega - k_0K)^2 = (K^2 - \Omega^2)[\,\varepsilon^2(K^2 - \Omega^2) - 24\gamma A_0^2]. \tag{3.121}$$

In this problem there are two small parameters—the nonlinearity coupling constant γ and the parameter ε which measures slowness of the modulation. If $\gamma = 0$, the dispersion relation gives in the first approximation in ε the relation

$$\Omega/K = k_0/\omega_0 = \omega_0', \tag{3.122}$$

that is, if there is no nonlinear effects, the modulation wave propagates with the group velocity of linear wave packets. Of course, this is a natural result. Now let us take into account the leading nonlinear correction,

$$\Omega/K = k_0/\omega_0 + \Delta, \quad |\Delta| \ll k_0/\omega_0,$$

so that Eqs. (3.118,3.122) give $K^2 - \Omega^2 \cong K^2/\omega_0^2$, and, hence, Δ is determined by the equation

$$4\omega_0^2 K^2\Delta^2 \cong (K^2/\omega_0^2)[\varepsilon^2(K^2/\omega_0^2) - 24\gamma A_0^2].$$

Thus, in this approximation the dispersion relation takes the form

$$\frac{\Omega}{K} = \frac{k_0}{\omega_0} \pm \sqrt{\frac{\varepsilon^2 K^2}{4\omega_0^6} - \frac{6\gamma A_0^2}{\omega_0^4}}, \tag{3.123}$$

which with the account of $\omega' = k_0/\omega_0$, $\omega_0'' = \omega_0^{-3}$, $\omega_2 = -3\gamma/2\omega_0$, and $A_0 = a_0/2$, coincides exactly with the result (3.57) of the NLS equation theory. We see that in the order $\sim \varepsilon^2$ the Whitham theory reproduces the results of the theory of weakly nonlinear and almost harmonic wave packets.

It is easy to find the wave-train solution of Eqs. (3.117,3.119,3.120), which, if $\gamma > 0$, reduces to the NLS envelope soliton solution in the limit of the infinitely long wavelength. Thus, the NLS equation theory can be included in the Whitham theory, if one takes into account small corrections at the next order in the slowness parameter ε.

3.5 Whitham modulation equations for a wave-train solution of the KdV equation

Let us turn to the key example of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3.124}$$

Its periodic wave-train solution ('cnoidal wave') was found in Sec. 1.5.2 by means of a simple substitution $u = u(\xi)$, $\xi = x - Vt$, so that two integrations led to the ordinary differential equation (see Eq. (1.286))

$$\tfrac{1}{2}u_\xi^2 = f(u), \tag{3.125}$$

where $f(u)$ is a third degree polynomial with the zeros α, β, γ ($\alpha \geq \beta \geq \gamma$):

$$f(u) = -A + Bu + \tfrac{1}{2}Vu^2 - u^3 = -(u-\alpha)(u-\beta)(u-\gamma). \tag{3.126}$$

Then the periodic solution is given by the formula (see Eq. (1.292))

$$u(x,t) = \alpha - (\alpha - \beta)\,\mathrm{sn}^2(\sqrt{(\alpha-\gamma)/2}\,(x - Vt), m), \tag{3.127}$$

where the phase velocity V and the parameter m are expressed in terms of α, β, γ (see Eqs. (1.291) and (1.296)):

$$V = 2(\alpha + \beta + \gamma), \quad m = (\alpha - \beta)/(\alpha - \gamma). \tag{3.128}$$

In the slowly modulated wave train the parameters A, B, V (or α, β, γ) become the slow functions of x and t, and we want to obtain equations which govern the evolution of these parameters. Since there are now three parameters, we have to average three conservation laws. It is easy to check by a direct calculation that the KdV equation has the following conservation laws,

$$\begin{aligned}
&u_t + (3u^2 + u_{xx})_x = 0,\\
&(\tfrac{1}{2}u^2)_t + (2u^3 + uu_{xx} - \tfrac{1}{2}u_x^2)_x = 0,\\
&(u^3 - \tfrac{1}{2}u_x^2)_t + (\tfrac{9}{2}u^4 + 3u^2u_{xx} + \tfrac{1}{2}u_{xx}^2 + u_xu_t)_x = 0.
\end{aligned} \tag{3.129}$$

At first, let us obtain the modulation equations for the parameters A, B, V. As in Sec. 3.3, it is convenient to express the averaged quantities in terms of the function

$$W(A,B,V) = -\sqrt{2}\oint\sqrt{-A+Bu+\tfrac{1}{2}Vu^2-u^3}\,du, \tag{3.130}$$

where the integration is taken over the period of u. The wavelength L and the wavevector k are equal to

$$L = 1/k = \int_0^L d\xi = \oint du/u_\xi = 2^{-1/2}\oint du/\sqrt{f(u)} = W_A. \tag{3.131}$$

It is easy to calculate the averaged quantities

$$\begin{aligned}
&\overline{u} = k\int_0^L u d\xi = k\oint u du/u_\xi = (k/\sqrt{2})\oint u du/\sqrt{f(u)} = -kW_B,\\
&\overline{\tfrac{1}{2}u^2} = k\int_0^L \tfrac{1}{2}u^2 du/u_\xi = -kW_V,\\
&\overline{u_\xi^2} = k\oint u_\xi^2 du/u_\xi = -kW.
\end{aligned} \tag{3.132}$$

The second derivatives $u_{\xi\xi}$ can be excluded from the conservation laws by means of the formula $u_{\xi\xi} = B+Vu-3u^2$, and after simple calculations we obtain the averaged conservation laws

$$\begin{aligned}
&(kW_B)_t + (kVW_B - B)_x = 0,\\
&(kW_V)_t + (kVW_V - A)_x = 0,\\
&[k(AW_A + BW_B + VW_V - W)]_t\\
&\qquad + [kV(AW_A + BW_B + VW_V - W) - \tfrac{1}{2}B^2 - AV]_x = 0.
\end{aligned} \tag{3.133}$$

The last equation can be rewritten in the form

$$\begin{aligned}
&A[(kW_A)_t + (kVW_A - V)_x] + B[(kW_B)_t + (kVW_B - B)_x]\\
&+ V[(kW_V)_t + (kVW_V - A)_x] - W[k_t + (kV)_x] = 0.
\end{aligned}$$

Here, the first term vanishes by $kW_A = 1$, and the second and third terms by the other Eqs. (3.133). As a result, we obtain a familiar equation

$$k_t + (kV)_x = 0. \tag{3.134}$$

On substitution of $k = 1/W_A$ into the first two equations (3.133) and introduction of the derivative $D/Dt = \partial/\partial t + V\partial/\partial x$, we arrive at the modulation equations

$$\frac{DW_A}{Dt} = W_A \frac{\partial V}{\partial x}, \quad \frac{DW_B}{Dt} = W_A \frac{\partial B}{\partial x}, \quad \frac{DW_V}{Dt} = W_A \frac{\partial A}{\partial x}. \tag{3.135}$$

In spite of the simple appearance of these equations, they are not very useful in practice. Therefore, let us transform them to the variables α, β, γ. By Eq. (1.288) we have the following relationships between the differentials,

$$\begin{aligned} dV &= 2(d\alpha + d\beta + d\gamma), \\ dB &= -[(\beta+\gamma)d\alpha + (\alpha+\gamma)d\beta + (\alpha+\beta)d\gamma], \\ dA &= -(\beta\gamma \cdot d\alpha + \alpha\gamma \cdot d\beta + \alpha\beta \cdot d\gamma). \end{aligned}$$

Hence, Eqs. (3.135) in terms of α, β, γ take the form

$$\begin{aligned} & W_{A,\alpha}\frac{D\alpha}{Dt} + W_{A,\beta}\frac{D\beta}{Dt} + W_{A,\gamma}\frac{D\gamma}{Dt} = 2W_A(\alpha_x + \beta_x + \gamma_x), \\ & W_{B,\alpha}\frac{D\alpha}{Dt} + W_{B,\beta}\frac{D\beta}{Dt} + W_{B,\gamma}\frac{D\gamma}{Dt} \\ & \qquad = -W_A[(\beta+\gamma)\alpha_x + (\alpha+\gamma)\beta_x + (\alpha+\beta)\gamma_x], \\ & W_{V,\alpha}\frac{D\alpha}{Dt} + W_{V,\beta}\frac{D\beta}{Dt} + W_{V,\gamma}\frac{D\gamma}{Dt} \\ & \qquad = -W_A[\beta\gamma \cdot \alpha_x + \alpha\gamma \cdot \beta_x + \alpha\beta \cdot \gamma_x], \end{aligned} \tag{3.136}$$

where

$$\begin{aligned} W_A &= \frac{1}{\sqrt{2}}\oint \frac{du}{\sqrt{f(u)}}, & W_{A,\alpha} &= \frac{1}{\sqrt{8}}\oint \frac{du}{(u-\alpha)\sqrt{f(u)}}, \\ W_B &= -\frac{1}{\sqrt{2}}\oint \frac{udu}{\sqrt{f(u)}}, & W_{B,\alpha} &= -\frac{1}{\sqrt{8}}\oint \frac{udu}{(u-\alpha)\sqrt{f(u)}}, \\ W_V &= -\frac{1}{\sqrt{2}}\oint \frac{(u^2/2)du}{\sqrt{f(u)}}, & W_{V,\alpha} &= -\frac{1}{\sqrt{8}}\oint \frac{(u^2/2)du}{(u-\alpha)\sqrt{f(u)}}, \end{aligned}$$

and similar expressions take place for the derivatives with respect to β and γ (everywhere $f(u) = -(u-\alpha)(u-\beta)(u-\gamma)$).

Of course, it is not evident how one can transform Eqs. (3.136) to the diagonal Riemann form. It was a great achievement of Whitham who found that the Riemann invariants are given by $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$. To show this, let us multiply the first equation (3.136) by p, the second by q, the third by r, add them, and choose the parameters p, q, r so that the coefficients before α_x vanish and before β_x and γ_x be equal to each other,

$$2p - q(\beta+\gamma) - r\beta\gamma = 0, \quad 2p - q(\alpha+\gamma) - r\alpha\gamma = 2p - q(\alpha+\beta) - r\alpha\beta.$$

These equations give at once $q = -r\alpha$, $p = -\frac{1}{2}r(\alpha\beta + \alpha\gamma - \beta\gamma)$, so that without loss of generality we can put $r = -2$ to obtain

$$p = \alpha\beta + \alpha\gamma - \beta\gamma, \quad q = 2\alpha, \quad r = -2.$$

Then the right hand side of the linear combination of Eqs. (3.136) takes the form

$$-2(\alpha-\beta)(\alpha-\gamma)W_A \frac{\partial(\beta+\gamma)}{\partial x}. \tag{3.137}$$

Now let us check that the coefficient of $D\alpha/Dt$ in the left hand side of the linear combination is equal to zero. Indeed, by means of a simple identity

$$\frac{d}{du}\left(2\sqrt{\frac{(u-\beta)(u-\gamma)}{-(u-\alpha)}}\right) = \frac{u^2 - 2\alpha u + \alpha\gamma + \alpha\beta - \beta\gamma}{(u-\alpha)\sqrt{f(u)}} \tag{3.138}$$

we obtain

$$pW_{A,\alpha} + qW_{B,\alpha} + rW_{V,\alpha} = \frac{1}{\sqrt{8}}\oint \frac{d}{du}\left(2\sqrt{\frac{(u-\beta)(u-\gamma)}{-(u-\alpha)}}\right)du = 0,$$

because the integrand is a total derivative of a periodic function.

The coefficients of $D\beta/Dt$ and $D\gamma/Dt$ are equal, correspondingly, to

$$K_1 = pW_{A,\beta} + qW_{B,\beta} + rW_{V,\beta} = -\frac{\alpha-\beta}{\sqrt{2}}\oint \frac{(u-\gamma)du}{(u-\beta)\sqrt{f(u)}},$$
$$K_2 = pW_{A,\gamma} + qW_{B,\gamma} + rW_{V,\gamma} = -\frac{\alpha-\gamma}{\sqrt{2}}\oint \frac{(u-\beta)du}{(u-\gamma)\sqrt{f(u)}},$$

where we have used in transformations the identity (3.138) with interchanged symbols α, β, γ. At first sight, the coefficients K_1 and K_2 seem to be different. However, their difference is equal to zero as an integral over the period of a periodic function,

$$K_1 - K_2 = \frac{\beta-\gamma}{\sqrt{2}}\oint \frac{u^2 - 2\alpha u + \alpha\beta + \alpha\gamma - \beta\gamma}{(u-\beta)(u-\gamma)\sqrt{f(u)}}du$$
$$= \frac{\beta-\gamma}{\sqrt{2}}\oint \frac{d}{du}\left(2\sqrt{\frac{-(u-\alpha)}{(u-\beta)(u-\gamma)}}\right)du = 0,$$

where we have used the identity similar to Eq. (3.138). Thus, $K_1 = K_2$ and $(\beta+\gamma)$ is the Riemann invariant. The coefficients K_1 and K_2 can be easily expressed in terms of the function W_A,

$$\begin{aligned} K_1 &= -(\alpha-\beta)W_A - 2(\alpha-\beta)(\beta-\gamma)W_{A,\beta}, \\ K_2 &= -(\alpha-\gamma)W_A - 2(\alpha-\gamma)(\beta-\gamma)W_{A,\gamma}, \end{aligned} \tag{3.139}$$

and the equality $K_1 = K_2$ leads to the identity

$$W_A = 2[(\alpha-\beta)W_{A,\beta} + (\alpha-\gamma)W_{A,\gamma}].$$

Its substitution into any equation (3.139) gives

$$K_1 = K_2 = -2(\alpha-\beta)(\alpha-\gamma)(W_{A,\beta} + W_{A,\gamma}) = 2(\alpha-\beta)(\alpha-\gamma)W_{A,\alpha},$$

where we have used the identity

$$W_{A,\alpha} + W_{A,\beta} + W_{A,\gamma} = \frac{1}{\sqrt{8}}\oint \frac{f'(u)du}{f^{3/2}(u)} = 0.$$

Now we equate the left hand side of the linear combination of equations,

$$2(\alpha-\beta)(\alpha-\gamma)W_{A,\alpha}\frac{D(\beta+\gamma)}{Dt},$$

to the right hand side (3.137) to obtain the equation

$$\frac{D(\beta+\gamma)}{Dt} + \frac{W_A}{W_{A,\alpha}}\frac{\partial(\beta+\gamma)}{\partial x} = 0, \tag{3.140}$$

and cyclic permutations of α, β, γ give the other two modulational equations

$$\frac{D(\gamma+\alpha)}{Dt} + \frac{W_A}{W_{A,\beta}}\frac{\partial(\gamma+\alpha)}{\partial x} = 0, \quad \frac{D(\alpha+\beta)}{Dt} + \frac{W_A}{W_{A,\gamma}}\frac{\partial(\alpha+\beta)}{\partial x} = 0. \tag{3.141}$$

Let us define new Riemann invariants $\lambda_1 \le \lambda_2 \le \lambda_3$,

$$\lambda_1 = -\tfrac{1}{2}(\alpha+\beta), \quad \lambda_2 = -\tfrac{1}{2}(\alpha+\gamma), \quad \lambda_3 = -\tfrac{1}{2}(\beta+\gamma), \tag{3.142}$$

that is,

$$\alpha = \lambda_3 - \lambda_1 - \lambda_2, \quad \beta = \lambda_2 - \lambda_1 - \lambda_3, \quad \gamma = \lambda_1 - \lambda_2 - \lambda_3, \tag{3.143}$$

and, consequently,

$$\begin{aligned} W_{A,\lambda_1} &= W_{A,\gamma} - W_{A,\alpha} - W_{A,\beta} = 2W_{A,\gamma}, \\ W_{A,\lambda_2} &= 2W_{A,\beta}, \quad W_{A,\lambda_3} = 2W_{A,\alpha}. \end{aligned}$$

Taking into account that $W_A = 1/k = L$, we obtain

$$\frac{W_A}{W_{A,\alpha}} = \frac{2W_A}{W_{A,\lambda_3}} = \frac{2}{\partial(\ln L)/\partial\lambda_3} = -\frac{2k}{\partial k/\partial\lambda_3},$$

and similar formulas for $W_A/W_{A,\beta}$ and $W_A/W_{A,\gamma}$. At last, since

$$V = 2(\alpha + \beta + \gamma) = -2(\lambda_1 + \lambda_2 + \lambda_3), \tag{3.144}$$

we arrive at the Whitham equations in the form

$$\frac{\partial\lambda_i}{\partial t} + v_i(\lambda_1, \lambda_2, \lambda_3)\frac{\partial\lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, \tag{3.145}$$

where the characteristic velocities v_i are given by

$$\begin{aligned} v_i &= -2(\lambda_1 + \lambda_2 + \lambda_3) + \frac{2}{\partial(\ln L)/\partial\lambda_i} \\ &= \left(1 - \frac{1}{\partial_i(\ln L)}\partial_i\right) V \\ &= \left(1 + \frac{k}{\partial_i k}\partial_i\right) V, \quad i = 1, 2, 3, \end{aligned} \tag{3.146}$$

where $\partial_i \equiv \partial/\partial\lambda_i$, $i = 1, 2, 3$. With the use of Eq. (1.300) we find the expression for the wavelength

$$L = \frac{1}{k} = \frac{2}{\sqrt{\lambda_3 - \lambda_1}}\,\mathrm{K}(m), \tag{3.147}$$

where

$$m = \frac{\alpha - \beta}{\alpha - \gamma} = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}. \tag{3.148}$$

Differentiation of $\ln L$ with respect to λ_i with the use of Eq. (A.12) gives

$$\begin{aligned} v_1 &= -2(\lambda_1+\lambda_2+\lambda_3) + \frac{4(\lambda_3-\lambda_1)(1-m)\mathrm{K}(m)}{\mathrm{E}(m)}, \\ v_2 &= -2(\lambda_1+\lambda_2+\lambda_3) - \frac{4(\lambda_3-\lambda_2)(1-m)\mathrm{K}(m)}{\mathrm{E}(m)-(1-m)\mathrm{K}(m)}, \\ v_3 &= -2(\lambda_1+\lambda_2+\lambda_3) + \frac{4(\lambda_3-\lambda_2)\mathrm{K}(m)}{\mathrm{E}(m)-\mathrm{K}(m)}, \end{aligned} \tag{3.149}$$

where $\mathrm{K}(m)$ and $\mathrm{E}(m)$ are the complete elliptic integrals of the first and second kind, respectively. Equations (3.145) and the characteristic velocities (3.149) were obtained by Whitham (1965). Formally, they are similar to the hydrodynamical equations written in the diagonal Riemann form (see Eqs. (1.172)), but now we have three Riemann invariants and the characteristic velocities are expressed in terms of the Riemann invariants by more complicated formulas.

Note that Eqs. (3.146) follow from the conservation of waves law (3.134), if we know expressions for the wavevector k (or the wavelength L) and the phase velocity V in terms of the Riemann invariants λ_i. Indeed, in this case Eq. (3.134) takes the form

$$\sum_{i=1}^{n}\left[\frac{\partial k}{\partial\lambda_i}\frac{\partial\lambda_i}{\partial t} + \left(\frac{\partial k}{\partial\lambda_i}V + k\frac{\partial V}{\partial\lambda_i}\right)\frac{\partial\lambda_i}{\partial x}\right] = 0,$$

n being the number of the Riemann invariants λ_i, $i=1,2,\ldots,n$, and since they can vary independently, each term in this sum must be equal to zero during the evolution, so we get at once the universal formula

$$v_i = \left(1 + \frac{k}{\partial_i k}\partial_i\right)V, \quad i=1,2,\ldots,n. \tag{3.150}$$

Hence, the main difficulty in diagonalization of the modulation equations consists in the proof that a certain set of parameters comprise the set of the Riemann invariants.

Suppose that we have found some solution $\lambda_i = \lambda_i(x,t)$, $i=1,2,3$, of the Whitham equations. Then the evolution of the modulated wave train is described by Eq. (3.127) written in terms of the Riemann invariants,

$$u(x,t) = (\lambda_3-\lambda_1)[h - 2m\,\mathrm{sn}^2(\sqrt{\lambda_3-\lambda_1}\,(x-Vt), m)], \tag{3.151}$$

where

$$h = \frac{\lambda_3 - \lambda_1 - \lambda_2}{\lambda_3 - \lambda_1}, \quad V = -2(\lambda_1 + \lambda_2 + \lambda_3), \quad m = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}. \tag{3.152}$$

Thus, we have developed the technique for solving problems connected with modulated periodic wave trains described by the Whitham equations. However, before proceeding to concrete applications of this technique, we shall discuss the results obtained above from a more general point of view. First of all, our success in diagonalization of the Whitham equations for the KdV wave train looks rather mysterious. From the general theory of the first order partial differential equations it is known that systems with three or more unknown variables can be diagonalized in very special cases only. Therefore, it is natural to ask, what is the reason of our success? To answer this question, we should examine more thoroughly the presentation of the periodic solution in the form (3.151,3.152), because the Riemann invariants λ_1, λ_2, λ_3 must be connected very deeply with the properties of the KdV equation. If we succeed in clarification of the specific properties of the KdV equation, then this may open the door to their generalization on the NLS equation and some other equations considered in Chapter 2. The next Chapter will be devoted to clarification and exact formulation of specific mathematical properties of the KdV equation and some other evolution equations.

Bibliographic remarks

We followed in our exposition to the classical papers by Whitham (1965a, 1965b, 1970) and to his book Whitham (1974). The universal form (3.150) for the Whitham characteristic velocities was indicated independently by Gurevich *et al* (1991) and Kudashev (1991) for the KdV equation, and by Kamchatnov (1990a, 1990b) for the DNLS equation and the Heisenberg model equation.

Exercises on Chapter 3

Exercise 3.1
Derive from the results of Sec. 3.2 for a linear Klein–Gordon equation the

averaged conservation law of 'wave action' (Whitham, 1965b),

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega}\right)+\frac{\partial}{\partial x}\left(\frac{v_g E}{\omega}\right)=0.$$

where E is the averaged density of energy.

Exercise 3.2
From the KdV equation $u_t + (3u^2 + u_{xx})_x = 0$ obtain the equation

$$(\overline{u})_t + (3\overline{u^2})_x = 0$$

averaged over the wavelength of the wave train. Consider the limit of well separated solitons $m \to 1$ and derive the corresponding Whitham equations.

Exercise 3.3
Find an asymptotic solution for $t \to \infty$ of the Whitham equations obtained in the preceding Exercise.

Exercise 3.4
(Lighthill, 1965) Let the averaged Lagrangian $\mathcal{L} = \mathcal{L}(k, \omega, A^2)$ and the dispersion relation $\omega = \omega(k, A^2)$ be known. Then elimination of A^2 from these relations yields $\mathcal{L} = \mathcal{L}(k, \omega)$ as a function only of two variables k and ω. Show that the Whitham equation (3.75) reduces then to the second order quasi-linear partial differential equation

$$\frac{\partial^2\mathcal{L}}{\partial\omega^2}\frac{\partial^2\theta}{\partial t^2}+2\frac{\partial^2\mathcal{L}}{\partial\omega\partial k}\frac{\partial^2\theta}{\partial t\partial x}+\frac{\partial^2\mathcal{L}}{\partial k^2}\frac{\partial^2\theta}{\partial x^2}=0.$$

It is known (see, e.g., Courant and Hilbert, 1962) that properties of this equation depend on the sign of the quantity

$$\mathcal{D}=\frac{\partial^2\mathcal{L}}{\partial\omega^2}\frac{\partial^2\mathcal{L}}{\partial k^2}-\left(\frac{\partial^2\mathcal{L}}{\partial\omega\partial k}\right)^2.$$

If $\mathcal{D} < 0$, then the equation for θ is hyperbolic and has real characteristics; but if $\mathcal{D} > 0$, then the equation for θ is elliptic, i.e., characteristics are complex and the system is unstable with respect to small perturbations. Formulate the stability criterion for the case of small but finite amplitudes A when $\omega = \omega(k, A^2)$ is approximated by the two terms of its series expansion in powers of A^2,

$$\omega = \omega_0(k) + \omega_2(k)\cdot A^2.$$

Chapter 4

Complete integrability of nonlinear wave equations

4.1 Complete integrability of the KdV equation

4.1.1 *Lamé equation*

After some inspection of the nonlinear wave-train solution (3.151) of the KdV equation, one may notice that the expression in square brackets coincides exactly with the expression appearing in the so called Lamé equation (see, e.g., Whittaker and Watson, 1927)

$$d^2\psi/dz^2 = [2m\,\mathrm{sn}^2(z,m) + A]\psi, \tag{4.1}$$

where $z = \sqrt{\lambda_3 - \lambda_1}\,x$, $m = (\lambda_3 - \lambda_2)/(\lambda_3 - \lambda_1)$, A is some constant, and we consider the dependence of $u(x,t)$ only on x. For better understanding the role which the parameters λ_1, λ_2, λ_3 play in the theory of the Lamé equation, let us represent it in terms of the Weierstrass $\wp$-function. To this end, let us return to Eq. (3.125), $u_\xi^2 = -2(u-\alpha)(u-\beta)(u-\gamma)$, and make a substitution $u = -2\,U + \frac{1}{3}(\alpha+\beta+\gamma)$, so that this equation transforms to

$$U_\xi^2 = 4(U - e_1)(U - e_2)(U - e_3), \tag{4.2}$$

where

$$\begin{aligned} e_1 &= \tfrac{1}{6}(\alpha+\beta-2\gamma) = \tfrac{1}{3}(\lambda_2+\lambda_3-2\lambda_1),\\ e_2 &= \tfrac{1}{6}(\alpha+\gamma-2\beta) = \tfrac{1}{3}(\lambda_1+\lambda_3-2\lambda_2),\\ e_3 &= \tfrac{1}{6}(\beta+\gamma-2\alpha) = \tfrac{1}{3}(\lambda_1+\lambda_2-2\lambda_3), \quad e_1 > e_2 > e_3, \end{aligned} \tag{4.3}$$

and λ_1, λ_2, λ_3 are defined by Eqs. (3.142). It is well known that the solution of Eq. (4.2) is given by the Weierstrass $\wp$-function (see Eq. (A.20))

$$U = \wp(\xi + c),$$

where c is the integration constant determined by the initial conditions. Hence, the periodic wave-train solution of the KdV equation takes the form

$$u(\xi) = -\tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) - 2\wp(\xi + c), \quad \xi = x - Vt, \tag{4.4}$$

where as the phase velocity V so the parameters e_1, e_2, e_3 of the $\wp$-function are expressed in terms of λ_1, λ_2, λ_3 (see Eqs. (3.144) and (4.3)). Generally speaking, the integration constant c has to be included into the solution to make it real and regular on the x axis. But for convenience of notations we shall drop it keeping in mind that it should be restored in the end of calculations by means of a simple replacement $x \to x + c$. Then, neglecting the dependence on t, we can rewrite the Lamé equation (4.1) in the form (see Whittaker and Watson, 1927)

$$d^2\psi/dx^2 = (2\wp(x) + l)\,\psi, \tag{4.5}$$

where again l is some constant.

Thus, the periodic solution $u(x)$ of the KdV equation is related with the eigenvalue problem

$$\psi_{xx} + u(x)\psi = -\lambda\psi, \tag{4.6}$$

where the eigenvalue λ is connected with the parameter l in Eq. (4.5) by the relation

$$\lambda = \tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) - l. \tag{4.7}$$

Using some formulas from the elliptic functions theory, it is not difficult to find the solution of Eq. (4.5). Let us look for it in the form

$$\psi_- = e^{-ipx}\frac{\sigma(x + a)}{\sigma(x)},$$

where $\sigma(x)$ is the Weierstrass σ-function, and p and a are the parameters to be determined. According to Eqs. (A.36), the logarithmic derivative of ψ_- is equal to

$$\psi_-'/\psi_- = -ip + \zeta(x + a) - \zeta(x),$$

so that (see Eq. (A.32))

$$\psi_-''/\psi_- = (\psi_-'/\psi_-)' + (\psi_-'/\psi_-)^2 = \wp(x) - \wp(x+a) + [\zeta(x+a) - \zeta(x) - ip]^2.$$

We have to choose p and a so that Eq. (4.5) is satisfied, that is, the identity

$$l + \wp(x) + \wp(x+a) = [\zeta(x+a) - \zeta(x) - ip]^2$$

fulfills. By the addition theorems for $\wp$-function (see Eq. (A.29)) and ζ-function (see Eq. (A.42)), this identity can be written in the form

$$l - \wp(a) + \tfrac{1}{4}\left[\frac{\wp'(x) - \wp'(a)}{\wp(x) - \wp(a)}\right]^2 = \left[\zeta(a) - ip + \tfrac{1}{2}\frac{\wp'(x) - \wp'(a)}{\wp(x) - \wp(a)}\right]^2.$$

This identity fulfills, if we choose

$$l = \wp(a), \quad ip = \zeta(a),$$

and from behaviour of the left and right hand sides near singularities we conclude that there is no other solutions. As a result, we have one solution of the Lamé equation (4.5) in the form

$$\psi_-(x) = e^{-x\zeta(a)}\frac{\sigma(x+a)}{\sigma(x)}, \quad \wp(a) = l. \tag{4.8}$$

The second order Lamé equation has to have another linearly independent solution. Since $\wp(x)$ is an even function, we see that if $\psi(x)$ is a solution, then $\psi(-x)$ is also a solution. Therefore, as a second solution we can take

$$\psi_+(x) = e^{x\zeta(a)}\frac{\sigma(x-a)}{\sigma(x)}, \quad \wp(a) = l, \tag{4.9}$$

where we have taken into account that $\sigma(x)$ is an odd function. These two solutions are linearly independent, if only a is not equal to one of the half-periods $\pm\omega_i$, $i = 1, 2, 3$, of the function $\wp(x)$. But if $a = -\omega_i$, and, hence, $l = \wp(\omega_i) = e_i$, then by Eqs. (A.37) we have

$$\sigma(x+\omega_i) = \sigma(x - \omega_i + 2\omega_i) = -e^{2\eta_i x}\sigma(x - \omega_i)$$

and

$$\psi_+(x) = -\psi_-(x).$$

As follows from Eqs. (4.3) and (4.7), the values $l = e_i$, $i = 1, 2, 3$, correspond to the eigenvalues λ equal to

$$\lambda = \lambda_1, \lambda_2, \lambda_3. \tag{4.10}$$

Thus, the Riemann invariants introduced in the preceding Chapter are equal to such eigenvalues of the problem (4.6) with $u(x)$ given by the periodic solution of the KdV equation, for which the two eigenfunctions (4.8) and (4.9) are linearly dependent.

Equation (4.6) coincides actually with the quantum mechanical Schrödinger equation for a particle moving in the periodic potential $u(x)$. As it is known from elementary solid state physics (Kittel, 1966), the eigenvalues of the Schrödinger equation with the periodic potential form the band structure. Let us consider the band structure of Eq. (4.5) in some more detail.

4.1.2 *Band structure of the Lamé equation*

Let us formulate at first some general statements about the spectrum of the equation

$$\psi_{xx} + u(x)\psi = -\lambda\psi \tag{4.11}$$

with the periodic potential $u(x)$,

$$u(x + L) = u(x). \tag{4.12}$$

Solutions of Eq. (4.11) can be presented as linear combinations of two basis solutions $\psi_+(x)$ and $\psi_-(x)$ examples of which we have seen above for the case of the Lamé equation. In general case it is convenient to choose these basis solutions by fixing the initial conditions at some $x = x_0$ as follows:

$$\psi_\pm(x_0) = 1, \quad \psi'_\pm(x_0) = \pm ik, \quad k = \sqrt{\lambda}. \tag{4.13}$$

In quantum mechanics this basis corresponds at $\lambda \to \infty$ to plane waves propagating in opposite directions,

$$\psi_\pm \sim e^{\pm ikx}, \quad \lambda = k^2 \to \infty. \tag{4.14}$$

The potential $u(x)$ is supposed to be real so that at real values of λ we have

$$\psi_+ = \psi_-^*. \tag{4.15}$$

Since Eq. (4.11) is invariant with respect to translations $x \to x + L$ (see Eq. (4.12)), the functions $\psi_\pm(x+L)$ can be expressed as linear combinations of $\psi_\pm(x)$. Thus, we can introduce the operator of translation $\mathbb{L}\psi = \psi(x+L)$ defined by its matrix elements in the basis (4.13):

$$\begin{aligned} \mathbb{L}\psi_+ &= a\psi_+ + b\psi_-, \\ \mathbb{L}\psi_- &= b^*\psi_+ + a^*\psi_-, \end{aligned} \qquad \mathbb{L} = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \tag{4.16}$$

where we have taken into account Eq. (4.15). Wronskian $\psi_+\psi_-' - \psi_-\psi_+'$ of two solutions of Eq. (4.11) is constant, so we have

$$\det \mathbb{L} = |a|^2 - |b|^2 = 1. \tag{4.17}$$

The trace of $\mathbb{L}$ is equal to

$$\operatorname{Sp} \mathbb{L} = a + a^* = 2a_R, \quad a = a_R + ia_I.$$

If we denote by $e^{\pm ipL}$ the eigenvalues of the operator $\mathbb{L}$, then they are equal to the roots of the quadratic equation $z^2 - 2a_R z + 1 = 0$, that is,

$$e^{\pm ipL} = a_R \pm \sqrt{a_R^2 - 1}, \tag{4.18}$$

and, hence,

$$a_R = \cos pL. \tag{4.19}$$

Common eigenfunctions of Eq. (4.11) and of the translation operator are called the Bloch functions. It is clear that the Bloch functions must have the form

$$\psi_\pm = e^{\pm ipx}\Psi(x),$$

where $\Psi(x)$ is a periodic function, $\Psi(x+L) = \Psi(x)$, so that

$$\mathbb{L}\psi_\pm = e^{\pm ipL}\psi_\pm, \tag{4.20}$$

and $p(\lambda) \to k = \sqrt{\lambda}$ as $\lambda \to \infty$, so that

$$a_R \cong \cos\sqrt{\lambda}L, \quad \lambda \to \infty. \tag{4.21}$$

The allowed bands of the Bloch spectrum correspond to the real values of the 'quasi-momentum' $p(\lambda)$, that is, they are determined by the inequality

$$|a_R| \leq 1. \tag{4.22}$$

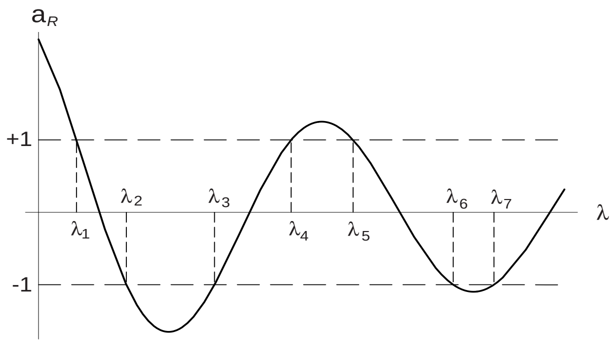

Fig. 4.1 Dependence of a_R on λ for the potential with several forbidden gaps of the spectrum of the Schrödinger equation.

Due to the asymptotic behaviour (4.20), gaps between allowed bands become infinitely narrow, or disappear at all, as $\lambda \to \infty$. A typical dependence of a_R on λ is shown in Fig. 4.1, where the allowed bands correspond to the intervals

$$\lambda_1 \le \lambda \le \lambda_2, \quad \lambda_3 \le \lambda \le \lambda_4, \dots .$$

At the end points of the allowed bands, where $|\cos pL| = 1$, both eigenvalues (4.18) coalesce into one value. This means that the eigenfunctions $\psi_\pm$ become linearly dependent.

This general analysis shows that in the case of the Lamé equation (4.5) we have three end points λ_1, λ_2, λ_3 of the allowed bands. Therefore, the Bloch spectrum consists of the two allowed bands

$$\lambda_1 \le \lambda \le \lambda_2, \quad \lambda_3 \le \lambda < \infty,$$

separated by the forbidden gap

$$\lambda_2 \le \lambda \le \lambda_3. \tag{4.23}$$

Thus, the periodic wave-train solution of the KdV equation corresponds to the potential of the problem (4.11) with only one forbidden gap.

It is remarkable that the eigenvalues λ_1, λ_2, λ_3 are the zeros of the polynomial arising naturally in the theory of the Lamé equation. Indeed, let a product of two basis functions be denoted by $g(x)$,

$$g(x) = \psi_+(x)\psi_-(x). \tag{4.24}$$

It is easy to prove by a direct calculation that if $\psi_\pm$ satisfy Eq. (4.11), then $g(x)$ satisfies the third order differential equation (Appel's theorem, see Whittaker and Watson (1927), Sec. 19.52)

$$g_{xxx} + 2u_x g + 4(\lambda + u)g_x = 0, \tag{4.25}$$

which after multiplication by g can be integrated once to give

$$P(\lambda) = -\tfrac{1}{4}g_x^2 + \tfrac{1}{2}gg_{xx} + (\lambda + u)g^2. \tag{4.26}$$

In the particular case of the Lamé equation we have (see Eq. (A.43) with $\wp(a) = l$)

$$g = \psi_+\psi_- = \sigma(x-a)\sigma(x+a)/\sigma^2(x) = -\sigma^2(a)(\wp(x) - l).$$

Substitution of this expression into Eq. (4.26) gives (up to some constant factor) with the use of Eqs. (A.20,A.21) and the formula $-(\lambda + u) = 2\wp + l$ the expression for $P(\lambda)$,

$$P(\lambda) = \mathrm{const}\cdot(l - e_1)(l - e_2)(l - e_3),$$

or, by Eqs. (4.3) and (4.7),

$$P(\lambda) = \mathrm{const}\cdot(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

If we define the function

$$\mu = \tfrac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + u(x)) = \tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) - U(x), \tag{4.27}$$

then Eq. (4.2) transforms to

$$\mu_x^2 = -4(\mu - \lambda_1)(\mu - \lambda_2)(\mu - \lambda_3). \tag{4.28}$$

Thus, if we accept that the most natural parameters, in terms of which the periodic solution of the KdV equation should be expressed, are the eigenvalues of the problem (4.11), then this periodic solution has to be connected intimately with the polynomial

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3). \tag{4.29}$$

This polynomial defines the differential equation

$$d\mu/dx = 2\sqrt{-P(\mu)}, \tag{4.30}$$

(μ varies inside the forbidden gap $\lambda_2 \leq \lambda \leq \lambda_3$, where $P(\lambda) \leq 0$), whose solution $\mu(x)$ is related with the periodic solution of the KdV equation by the formula

$$u(x) = 2\mu(x) - (\lambda_1 + \lambda_2 + \lambda_3). \tag{4.31}$$

It is worth also noting that Eq. (4.25) can be written in the form

$$\mathbb{R}g = 4\lambda g_x, \tag{4.32}$$

where the operator

$$\mathbb{R} = -\partial_x^3 - 4u\partial_x - 2u_x = -\partial_x^3 - 2(u\,\partial_x + \partial_x\, u) \tag{4.33}$$

is called the Lenard operator. Then the KdV equation can be written as

$$u_t = \mathbb{R}u, \tag{4.34}$$

and this equation also suggests that there exists a deep connection between the nonlinear KdV equation and linear Schrödinger equation (4.11). Now our aim is to clarify this connection and to formulate the theory of the KdV equation is such a form that all above formulas arise naturally and reproduce in addition the dependence of the solution $u(x,t)$ on time t.

4.1.3 *KdV equation as a compatibility condition of two linear equations*

For a particular example of the cnoidal wave we have found that solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{4.35}$$

are parameterized most naturally by the eigenvalues of a linear equation

$$\psi_{xx} + u\psi = -\lambda\psi. \tag{4.36}$$

We suppose that this is not an occasional coincidence and want to clarify this inter-connection of the two equations. We notice at once that there is no time dependence in Eq. (4.36), though the potential $u(x,t)$ evolves according to the KdV equation. One might think that such evolution must change the spectrum of Eq. (4.36). But in the case of the cnoidal wave the dependence of $u(x - Vt)$ on time is trivial and leads only to translation of the potential without change of its form, and, hence, the spectrum remains the same. This particular example suggests the question: Which evolution of the potential $u(x, t)$ leaves the spectrum of Eq. (4.36) unchanged? So, we introduce into Eq. (4.36) the dependence on time t,

$$\psi_{xx}(x, t) = -(\lambda + u(x, t))\psi(x, t). \tag{4.37}$$

Dependence of the potential $u(x, t)$ on time t causes the dependence of the eigenfunctions $\psi(x, t)$ on t in spite of the conservation of the eigenvalues λ. We suppose that the time dependence is determined by a linear differential equation which can be written in the form

$$\psi_t = \mathcal{A}\psi + \mathcal{B}\psi_x, \tag{4.38}$$

because all higher derivatives ψ_{xx}, $\psi_{xxx}, \ldots$ can be eliminated with the use of Eq. (4.37). The coefficients $\mathcal{A}$ and $\mathcal{B}$ can depend on as the spectral parameter λ so the potential $u(x, t)$ and its derivatives with respect to x. They should be chosen so that the compatibility condition

$$(\psi_{xx})_t = (\psi_t)_{xx}$$

fulfills, that is,

$$-u_t \cdot \psi = (\mathcal{A}_{xx} - 2\mathcal{B}_x(u + \lambda) - \mathcal{B}u_x)\psi + (2\mathcal{A}_x + \mathcal{B}_{xx})\psi_x. \tag{4.39}$$

First of all, we notice that this condition gives

$$\mathcal{A}_x = -\tfrac{1}{2}\mathcal{B}_{xx},$$

and, hence, we obtain the equation

$$u_t = \tfrac{1}{2}\mathcal{B}_{xxx} + 2\mathcal{B}_x(u + \lambda) + \mathcal{B}u_x. \tag{4.40}$$

If we choose $\mathcal{B}$ so that the parameter λ disappears from this equation, we shall arrive at the evolution equation which conserves the spectrum of Eq. (4.37). Since Eq. (4.40) has the term $2\mathcal{B}_x\lambda$, it is clear that supposing the polynomial dependence of $\mathcal{B}$ on λ,

$$\mathcal{B} = b_n + b_{n-1}\lambda + b_{n-2}\lambda^2 + \ldots + b_0\lambda^n, \tag{4.41}$$

we shall obtain a recursion formula which permits one to calculate all the coefficients b_k starting from a given constant value of $b_0 = \text{const}$. Let us see how it works.

In the simplest case $n = 0$, when $\mathcal{B} = b_0 = \text{const}$, we obtain an equation

$$u_t = b_0 u_x. \tag{4.42}$$

This is a trivial result which means that the spectrum does not change after the translational 'evolution'

$$u(x,t) = u(x + b_0 t).$$

For $n = 1$, when $\mathcal{B} = b_1 + b_0\lambda$, we choose $b_0 = 4$, so that Eq. (4.40) takes the form

$$u_t = \tfrac{1}{2} b_{1,xxx} + 2b_{1,x} + b_1 u_x + (2b_{1,x} + 4u_x)\lambda.$$

To eliminate the dependence on λ, we must choose $b_1 = -2u$, and then this equation reduces to

$$u_t = -u_{xxx} - 6uu_x.$$

But this is the KdV equation! We have obtained a remarkable result: The spectrum of the Schrödinger equation (4.36) does not change, if the potential $u(x,t)$ evolves according to the KdV equation (4.35). Just the discovery of this property of the KdV equation made by Gardner, Greene, Kruskal and Miura (1967) was the starting point of the modern theory of solitons. It can be expressed as a statement that the KdV equation is a compatibility condition of two linear equations

$$\psi_{xx} = -(u + \lambda)\psi, \qquad \psi_t = u_x\psi + (4\lambda - 2u)\psi_x. \tag{4.43}$$

4.1.4 *The KdV hierarchy and the conservation laws*

But our game is not over yet, and we can take $n = 2, 3, \ldots$. Let us do it in a general form and substitute Eq. (4.41) into Eq. (4.40). We obtain a chain of recursion relations,

$$\begin{aligned} 2b_{2,x} &= -\tfrac{1}{2} b_{1,xxx} - 2b_{1,x}u - b_1 u_x, \\ &\cdots \\ 2b_{n,x} &= -\tfrac{1}{2} b_{n-1,xxx} - 2b_{n-1,x}u - b_{n-1}u_x, \end{aligned} \tag{4.44}$$

where the last expression defines b_n for all integer n. The n-th evolution equation is given by

$$u_t = \tfrac{1}{2}b_{n,xxx} + 2b_{n,x}u + b_n u_x = -2b_{n+1,x}. \tag{4.45}$$

Here the Lenard operator (4.33) appears again, so that Eqs. (4.44) can be written as

$$\partial b_{n+1}/\partial x = \tfrac{1}{4}\mathbb{R}b_n, \quad \mathbb{R} = -\partial_x^3 - 4u\partial_x - 2u_x. \tag{4.46}$$

$\mathbb{R}$ is an example of the so-called recursion operator. Several first coefficients b_n are equal to

$$\begin{gathered} b_0 = 4, \quad b_1 = -2u, \quad b_2 = \tfrac{3}{2}u^2 + \tfrac{1}{2}u_{xx}, \\ b_3 = -\tfrac{5}{4}u^3 - \tfrac{5}{4}uu_{xx} - \tfrac{5}{8}u_x^2 - \tfrac{1}{8}u_{xxxx}, \dots . \end{gathered} \tag{4.47}$$

Thus, we have obtained an infinite sequence of equations (4.45), where the first one ($n = 1$) is the KdV equation, and equations corresponding to $n = 2, 3, \dots$ are called the higher KdV equations. This sequence as a whole is called the KdV hierarchy. The higher KdV equations do not have clear physical interpretation, but they are very important for understanding the properties of the KdV equation.

Comparison of Eq. (4.46) with the equation (see Eq. (4.32))

$$g_{xxx} = -2u_x g - 4(u + \lambda)g_x, \tag{4.48}$$

or

$$\lambda g_x = \tfrac{1}{4}\mathbb{R}g, \tag{4.49}$$

shows that the product of the basis functions after appropriate normalization can serve as a generating function of the coefficients b_n. Therefore, let us study the properties of $g(x,t)$ in some more detail. As follows from Eq. (4.38) with $\mathcal{A} = -\mathcal{B}_x/2$, the function $g(x,t)$ satisfies the 'temporal' counterpart of Eq. (4.48),

$$g_t = \mathcal{B}g_x - \mathcal{B}_x g. \tag{4.50}$$

This equation yields at once the conservation law

$$\frac{\partial}{\partial t}\left(\frac{1}{g}\right) - \frac{\partial}{\partial x}\left(\frac{\mathcal{B}}{g}\right) = 0. \tag{4.51}$$

In particular, for the KdV equation case we have

$$\frac{\partial}{\partial t}\left(\frac{1}{g}\right)+\frac{\partial}{\partial x}\left(\frac{-4\lambda+2u}{g}\right)=0. \tag{4.52}$$

If we expand Eq. (4.51) or (4.52) into series in powers of λ^{-1}, we arrive at an infinite sequence of the conservation laws. The densities of the corresponding integrals of motion are given by the coefficients of the expansion

$$\frac{1}{g}=\sum_{n=0}^{\infty}\frac{g_{-n}}{\lambda^n}=1+\frac{g_{-1}}{\lambda}+\frac{g_{-2}}{\lambda^2}+\dots. \tag{4.53}$$

Thus, we have found that the KdV equation (as well as the other equations of the KdV hierarchy) has an infinite number of integrals of motion.

Combining Eq. (4.48) with Eq. (4.50) written for the KdV case, that is, with

$$g_t=(4\lambda-2u)g_x+2u_xg, \tag{4.54}$$

we find that $g(x,t)$ satisfies the linearized KdV equation

$$g_t+6ug_x+g_{xxx}=0. \tag{4.55}$$

Moreover, this equation can also be written as a generating function of the conservation laws. Indeed, let us represent Eq. (4.48) in the form

$$2ug_x=-2(ug)_x-4\lambda g_x-g_{xxx},$$

and then Eq. (4.55) transforms at once to

$$g_t+(-6ug-2g_{xx}-12\lambda g)_x=0. \tag{4.56}$$

Hence, the coefficients of the series expansion

$$g=\sum_{n=0}^{\infty}\frac{g_n}{\lambda^n}=1+\frac{g_1}{\lambda}+\frac{g_2}{\lambda^2}+\dots \tag{4.57}$$

are also the densities of integrals of motion of the KdV equation. Substitution of this expansion into Eq. (4.49) gives the recursion relation for the coefficients

$$g_{n+1,x}=\tfrac{1}{4}\mathbb{R}g_n. \tag{4.58}$$

Several first coefficients are equal to

$$\begin{aligned} &g_0 = 1, \quad g_1 = -\tfrac{1}{2}u, \quad g_2 = \tfrac{3}{8}u^2 + \tfrac{1}{8}u_{xx}, \\ &g_3 = -\tfrac{5}{16}u^3 + \tfrac{5}{32}u_x^2 - \left(\tfrac{5}{16}uu_x + \tfrac{1}{32}u_{xxx}\right)_x, \dots . \end{aligned} \tag{4.59}$$

Note that up to constant factors and additional terms equal to total derivatives with respect to x, which do not change values of integrals of motion, these densities coincide with those of Eqs. (3.129) used in derivation of the Whitham equations. Coefficients g_{-n} of the expansion (4.53) define actually the same densities which differs from g_n only by total derivatives. Indeed, from $g \cdot (1/g) = 1$ we can calculate with the use of Eqs. (4.59) several first coefficients g_{-n},

$$\begin{aligned} &g_0 = 1, \quad g_{-1} = \tfrac{1}{2}u, \quad g_{-2} = -\tfrac{1}{8}u^2 - \tfrac{1}{8}u_{xx}, \\ &g_{-3} = \tfrac{1}{16}u^3 - \tfrac{1}{32}u_x^2 + \left(\tfrac{3}{16}uu_x + \tfrac{1}{32}u_{xxx}\right)_x, \dots , \end{aligned} \tag{4.60}$$

and comparison with Eqs. (4.59) confirms our assertion. More careful inspection shows that for these first coefficients the exact relationship

$$g_n = 2\frac{\widehat{\delta}}{\delta u}g_{-n-1} \tag{4.61}$$

takes place, where $\widehat{\delta}/\delta u$ denotes the operator of variational derivative

$$\frac{\widehat{\delta}}{\delta u} = \frac{\partial}{\partial u} - \frac{\partial}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial u_{xx}} - \dots + (-1)^n\frac{\partial^n}{\partial x^n}\frac{\partial}{\partial u^{(n)}}, \tag{4.62}$$

with the number of terms determined by the highest order derivative $u^{(n)}$ in the expression for g_{-n-1}. (Check of particular cases of Eq. (4.61) becomes easier, if one takes into account that the variational derivative of total derivatives is equal identically to zero by definition, because the value of any functional of a total derivative does not depend on variations of functions inside the interval of integration.) Supposing that Eq. (4.61) holds for all integer n, we arrive at the relation

$$g = 2\lambda\frac{\widehat{\delta}}{\delta u}\frac{1}{g}, \tag{4.63}$$

which we are going to prove.

Let us introduce a convenient representation of the Bloch functions

$$\psi_\pm = \exp\left\{i\int_0^x \chi_\pm(x)dx\right\}, \tag{4.64}$$

and separate the real and imaginary parts of $\chi_\pm$,

$$\chi_\pm = \pm\sqrt{\lambda}\,\chi_R - i\chi_I. \tag{4.65}$$

(Introduction of the factor $\sqrt{\lambda}$ will be clear from what follows.) Substitution of Eq. (4.64) into equation $\psi_{xx} = -(u+\lambda)\psi$ (see Eqs. (4.43)) and separation of the real and imaginary parts give

$$\chi_{R,x} + 2\chi_R\chi_I = 0, \quad \chi_{I,x} - \lambda(\chi_R^2 - 1) + \chi_I^2 + u = 0. \tag{4.66}$$

The first equation gives

$$\chi_I = -\tfrac{1}{2}\,(\ln\chi_R)_x\,,$$

so that Eq. (4.64) takes the form

$$\psi_\pm = \frac{1}{\sqrt{\chi_R}}\exp\left(\pm i\sqrt{\lambda}\int^x \chi_R dx\right). \tag{4.67}$$

Since

$$g = \psi_+\psi_- = 1/\chi_R, \tag{4.68}$$

Eq. (4.67) can be written as

$$\psi_\pm = \sqrt{g}\exp\left(\pm i\sqrt{\lambda}\int^x \frac{dx}{g}\right). \tag{4.69}$$

This formula makes clear an introduction of the factor $\sqrt{\lambda}$ into Eq. (4.65)—in the limit $\lambda\to\infty$ we have $1/g = \chi_R \to 1$ by the second equation (4.66) in agreement with our normalization of the function $g(x,t)$. Since the function g is expressed in terms of u and its derivatives with respect to x (see Eqs(4.57–4.59)), it is periodic with the same wavelength L.

Taking into account that Bloch functions are multiplied by $\exp(\pm ipL)$ when translated on one wavelength, we find that the quasi-momentum p is equal to

$$p = \frac{\sqrt{\lambda}}{L}\int_0^L \chi_R dx = \frac{\sqrt{\lambda}}{L}\int_0^L \frac{dx}{g}. \tag{4.70}$$

Hence, the variational derivative of p with respect to u is equal to

$$\frac{\delta p}{\delta u} = \frac{\sqrt{\lambda}}{L}\,\widehat{\frac{\delta}{\delta u}}\,\frac{1}{g}. \tag{4.71}$$

On the other hand, the variational derivative of p can be calculated directly with the use of the Schrödinger equation $\psi_{xx} = -(u+\lambda)\psi$. Let functions $\psi^{(1)}$ and $\psi^{(2)}$ satisfy the Schrödinger equations with different potentials $u^{(1)}$ and $u^{(2)}$, correspondingly, but with the same eigenvalue λ,

$$\psi^{(1)}_{xx} = -(u^{(1)}+\lambda)\psi^{(1)}, \quad \psi^{(2)}_{xx} = -(u^{(2)}+\lambda)\psi^{(2)}.$$

Multiplying the first equation by $\psi^{(2)}$, the second by $\psi^{(1)}$, and subtracting the second from the first, we obtain after integration over the wavelength the following relation

$$\left[\psi^{(1)}_x\psi^{(2)} - \psi^{(1)}\psi^{(2)}_x\right]_0^L = \int_0^L (u^{(2)} - u^{(1)})\psi^{(1)}\psi^{(2)}dx. \tag{4.72}$$

As follows from Eqs. (4.69), the derivatives

$$(\psi_\pm)_x = \frac{1}{2g}(g_x \pm 2i\sqrt{\lambda})\psi_\pm \tag{4.73}$$

are the eigenfunctions of the translation operator $\mathbb{L}$ with the same eigenvalues as $\psi_\pm$ themselves. Therefore, putting $\psi^{(1)} = \psi^{(1)}_-$, $\psi^{(2)} = \psi^{(2)}_+$ in Eq. (4.72) gives

$$\begin{aligned}\left(e^{i(p_2-p_1)L} - 1\right)\left(\psi^{(1)}_{-,x}(0)\psi^{(2)}_+(0) - \psi^{(1)}_-(0)\psi^{(2)}_{+,x}(0)\right)& \\ = \int_0^L (u^{(2)} - u^{(1)})\psi^{(1)}_-\psi^{(2)}_+ dx.&\end{aligned}$$

Taking the difference of the potentials $u^{(2)} - u^{(1)} = \delta u$ small, we have

$$\begin{gathered}(\psi_-)_x\psi_+ - \psi_-(\psi_+)_x \cong -2i\sqrt{\lambda}, \quad \psi_-\psi_+ \cong g, \\ e^{i(p_2-p_1)L} - 1 \cong i(p_2-p_1)L = i\delta pL,\end{gathered}$$

consequently, the above relation gives

$$2\sqrt{\lambda}\,L\delta p = \int_0^L \delta u \cdot g dx,$$

and, hence,

$$\frac{\delta p}{\delta u} = \frac{1}{2\sqrt{\lambda}\,L}g. \tag{4.74}$$

Comparison of Eqs. (4.71) and (4.74) proves Eq. (4.63). This formula and corresponding recursion relation (4.61) permit us to represent the recursion relation (4.58) in the form

$$\frac{\partial}{\partial x}\widehat{\frac{\delta}{\delta u}}\, g_{-n-1} = \tfrac{1}{4}\mathbb{R}\,\widehat{\frac{\delta}{\delta u}}\, g_{-n}. \tag{4.75}$$

Now let us define the integrals of motion I_n by the relation

$$I_n = \int_0^L \mathcal{P}_n dx = -16\int_0^L g_{-n-2}dx. \tag{4.76}$$

The first several densities are equal to

$$\mathcal{P}_{-1} = -8u, \quad \mathcal{P}_0 = 2u^2, \quad \mathcal{P}_1 = -u^3 + \tfrac{1}{2}u_x^2, \ldots, \tag{4.77}$$

and they are related by the recursion relation

$$\frac{\partial}{\partial x}\widehat{\frac{\delta}{\delta u}}\,\mathcal{P}_n = \tfrac{1}{4}\mathbb{R}\,\widehat{\frac{\delta}{\delta u}}\,\mathcal{P}_{n-1}. \tag{4.78}$$

Comparison of Eqs. (4.47) and (4.59) and coincidence of the corresponding recursion relations (4.46) and (4.58) show that

$$b_n = 4g_n. \tag{4.79}$$

Hence, the right hand side of the evolution equation (4.45) can be written in the form

$$-2b_{n+1,x} = -8g_{n+1,x} = -16\frac{\partial}{\partial x}\widehat{\frac{\delta}{\delta u}}\, g_{-n-2} = \frac{\partial}{\partial x}\frac{\delta I_n}{\delta u},$$

that is, the KdV hierarchy can be represented in terms of the integrals of motion of the KdV equation,

$$u_t = \frac{\partial}{\partial x}\frac{\delta I_n}{\delta u}. \tag{4.80}$$

By virtue of Eqs. (4.76) and (4.78), the integrals of motion I_n are related by the recursion relation

$$\frac{\partial}{\partial x}\frac{\delta I_{n+1}}{\delta u} = \tfrac{1}{4}\mathbb{R}\frac{\delta I_n}{\delta u}. \tag{4.81}$$

4.1.5 *KdV equation as a Hamiltonian system*

As we have found in the preceding section, the KdV equation can be represented as

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}, \quad H = I_1 = \int_0^L \mathcal{P}_1 dx = \int_0^L \left(-u^3 + \tfrac{1}{2}u_x^2\right) dx. \tag{4.82}$$

The derivative of any functional

$$F[u] = \int_0^L \mathcal{F}(u, u_x, u_{xx}, \ldots, u^{(n)})dx$$

with respect to time t, if u evolves according to the KdV equation (4.82), is given by

$$\frac{\partial F}{\partial t} = \int_0^L \left(\frac{\partial \mathcal{F}}{\partial u} u_t + \frac{\partial \mathcal{F}}{\partial u_x}\frac{\partial u_t}{\partial x} + \frac{\partial \mathcal{F}}{\partial u_{xx}}\frac{\partial^2 u_t}{\partial x^2} + \ldots + \frac{\partial \mathcal{F}}{\partial u^{(n)}}\frac{\partial^{(n)} u_t}{\partial x^{(n)}}\right) dx,$$

or, after integration by parts with taking into account the periodicity of $u(x)$, we have

$$\begin{aligned}\frac{\partial F}{\partial t} &= \int_0^L \left(\frac{\partial \mathcal{F}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{F}}{\partial u_x} + \ldots + (-1)^n \frac{\partial^n}{\partial x^n}\frac{\partial \mathcal{F}}{\partial u^{(n)}}\right) u_t dx \\ &= \int_0^L \left(\widehat{\frac{\delta}{\delta u}}\mathcal{F}\right) u_t dx = \int_0^L \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta H}{\delta u} dx = \{F, H\},\end{aligned} \tag{4.83}$$

where we have defined the Poisson bracket

$$\{F, H\} = \int_0^L \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta H}{\delta u} dx \tag{4.84}$$

for any two functionals $F[u]$ and $H[u]$. Then the KdV equation (4.82) can be written in a Hamiltonian form

$$u_t = \{u, H\}, \tag{4.85}$$

if one takes into account the formula

$$\frac{\delta u(x)}{\delta u(x')} = \delta(x - x'),$$

which follows from $u(x) = \int u(x')\delta(x-x')dx$, where $\delta(x-x')$ is the Dirac δ-function, so that

$$\{u, H\} = \int \frac{\delta u(x)}{\delta u(x')} \frac{\partial}{\partial x'} \frac{\delta H}{\delta u(x')} dx' = \int \delta(x-x') \frac{\partial}{\partial x'} \frac{\delta H}{\delta u(x')} dx' = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)}$$

in accordance with Eq. (4.82).

To justify the definition of the Poisson bracket by the expression (4.84), we have to check that it is skew symmetric,

$$\{F, H\} = -\{H, F\}, \tag{4.86}$$

and satisfies the Jacobi identity,

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0. \tag{4.87}$$

Equation (4.86) is proved at once by integration of Eq. (4.84) by parts. In proof of Eq. (4.87) we shall follow to Gardner (1971).

For convenience, we shall take the wavelength L equal to 2π, so that the Fourier expansion of $u(x,t)$ has the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{inx}. \tag{4.88}$$

Since the derivative of a functional $F[u]$ with respect to the Fourier coefficient u_k is equal to

$$\frac{\partial F}{\partial u_k} = \int_0^{2\pi} \frac{\delta F}{\delta u} \frac{\partial u}{\partial u_k} dx = \int_0^{2\pi} \frac{\delta F}{\delta u} e^{ikx} dx, \tag{4.89}$$

the Fourier expansion for the functional derivative $\delta F/\delta u$ is given by

$$\frac{\delta F}{\delta u(x)} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\partial F}{\partial u_{-k}} e^{ikx}. \tag{4.90}$$

If the Hamiltonian H is expressed in terms of u_k by means of substitution of Eq. (4.88) into Eq. (4.82), then the KdV equation can be written as

$$\sum_{k=-\infty}^{\infty} \frac{du_k}{dt} e^{ikx} = \sum_{k=-\infty}^{\infty} \frac{\partial H}{\partial u_{-k}} \frac{ik}{2\pi} e^{ikx},$$

and, hence,

$$\frac{d}{dt}\left(\frac{2\pi u_k}{ik}\right)=\frac{\partial H}{\partial u_{-k}}. \tag{4.91}$$

Introducing the notations

$$q_n=2\pi u_n/in, \quad p_n=u_{-n}, \quad n>0, \tag{4.92}$$

we find that Eqs. (4.91) with $k=n>0$ yield a half of the Hamilton equations

$$\frac{dq_n}{dt}=\frac{\partial H}{\partial p_n}, \quad n>0, \tag{4.93}$$

and equations corresponding to $k=-n<0$ yield another half,

$$\frac{dp_n}{dt}=\frac{du_{-n}}{dt}=\frac{i}{2\pi}(-n)\frac{\partial H}{\partial u_n}=-\frac{\partial H}{\partial(2\pi u_n/in)}=-\frac{\partial H}{\partial q_n}. \tag{4.94}$$

A canonical Poisson bracket of two functionals $F[u]$ and $G[u]$ given by

$$\{F,G\}=\sum_{n=1}^{\infty}\left(\frac{\partial F}{\partial q_n}\frac{\partial G}{\partial p_n}-\frac{\partial F}{\partial p_n}\frac{\partial G}{\partial q_n}\right) \tag{4.95}$$

transforms by virtue of Eqs. (4.89) to

$$\begin{aligned}
\{F,G\}&=\frac{i}{2\pi}\sum_{n=1}^{\infty}\left(n\frac{\partial F}{\partial u_n}\frac{\partial G}{\partial u_{-n}}-n\frac{\partial F}{\partial u_{-n}}\frac{\partial G}{\partial u_n}\right)=\frac{i}{2\pi}\sum_{n=-\infty}^{\infty}n\frac{\partial F}{\partial u_n}\frac{\partial G}{\partial u_{-n}}\\
&=\iint_0^{2\pi}dxdy\frac{\delta F}{\delta u(x)}\frac{\delta G}{\delta u(y)}\frac{d}{dx}\left(\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in(x-y)}\right)\\
&=-\iint_0^{2\pi}dxdy\left(\frac{d}{dx}\frac{\delta F}{\delta u(x)}\right)\frac{\delta G}{\delta u(y)}\delta(x-y)\\
&=\int_0^{2\pi}\frac{\delta F}{\delta u(x)}\frac{\partial}{\partial x}\frac{\delta G}{\delta u(x)}dx,
\end{aligned}$$

which coincides with the definition (4.84). Since the validity of the Jacobi identity for the bracket (4.95) is well known, we infer that this identity holds for the bracket (4.84), too. Thus, we have shown that the KdV equation is a Hamiltonian one.

In the preceding section we have shown that the KdV equation has an infinite number of the integrals of motion I_n. The statement that I_n does not depend on time can be expressed by the equation

$$\{I_n, H\} = \{I_n, I_1\} = 0. \tag{4.96}$$

It is natural to ask whether the integrals I_m conserve under evolution of $u(x,t)$ according to the higher KdV equations (4.80), that is, in other words, whether the integrals I_n and I_m commute with respect to the Poisson bracket (4.84),

$$\{I_n, I_m\} = 0. \tag{4.97}$$

This can be proved by the mathematical induction method by virtue of the property

$$\int \mathcal{F}(\mathbb{R}\mathcal{G})dx = -\int \mathcal{G}(\mathbb{R}\mathcal{F})dx$$

of the recursion operator, which can be checked by integration by parts, and with the use of the recursion formula (4.78):

$$\begin{aligned}
\{I_n, I_m\} &= \int \left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_n\right)\frac{\partial}{\partial x}\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_m\right)dx \\
&= \tfrac{1}{4}\int \left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_n\right)\mathbb{R}\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{m-1}\right)dx \\
&= -\tfrac{1}{4}\int \left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{m-1}\right)\mathbb{R}\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_n\right)dx \\
&= -\int \left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{m-1}\right)\frac{\partial}{\partial x}\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{n+1}\right)dx = \{I_{n+1}, I_{m-1}\}.
\end{aligned}$$

Subsequent use of this identity reduces $\{I_n, I_m\}$ to $\{I_{n+m+1}, I_1\}$ equal to zero according to Eq. (4.96).

Thus, we have proved that the KdV equation is an infinite-dimensional Hamiltonian system with an infinite number of the integral of motion in involution (i.e., commuting with each other). In the case of finite-dimensional Hamiltonian systems of $2n$ Hamilton equations with n integrals of motion in involution the Liouville theorem is valid which states that such a system can be integrated in quadratures (see Whittaker (1927); Arnold (1974)), or, in other words, it is completely integrable. The completely integrable systems can be transformed to the 'action-angle' variables, in which n action variables I_i are constant and n angle variables ϕ_i depend on time linearly,

$\phi_i = \omega_i t + \phi_{i0}$, that is, the motion is quasi-periodic with frequencies ω_i, $i = 1, \dots, n$. Analogous theorem can be proved for the KdV equation (Zakharov and Faddeev, 1971; Flaschka and McLaughlin, 1974), that is in this sense the KdV equation is completely integrable. We shall not consider here this proof and note only that this theory is based on the possibility to represent the KdV equation as a compatibility condition of two linear systems with spectral parameter λ.

4.1.6 *Periodic solution of the KdV equation*

In the preceding subsection we have supposed that $u(x, t)$ is a periodic function of x with the wavelength L. Now let us recall that the cnoidal wave has a special property—for this potential $u(x)$ the spectrum of the Schrödinger equation consists of two bands $\lambda_1 \le \lambda \le \lambda_2$, $\lambda_3 \le \lambda < \infty$ separated by one forbidden gap $\lambda_2 \le \lambda \le \lambda_3$. At the same time, the eigenvalues λ_1, λ_2, λ_3 are the zeros of the polynomial $P(\lambda)$ connected with the function g by the relation (4.26). This observation suggests that the periodic solutions of the KdV equation are distinguished by the condition that the integral of motion

$$\tfrac{1}{2} g g_{xx} - \tfrac{1}{4} g_x^2 + (\lambda + u) g^2 = P(\lambda) \tag{4.98}$$

be a polynomial in λ, and in the simplest case of the cnoidal wave (the one-phase periodic solution) this is a polynomial of the third degree. Correspondingly, the product g of basis functions must be polynomial in λ, too.

To develop this idea, let as pass from a scalar Schrödinger equation, $\psi_{xx} = -(u + \lambda)\psi$, to the system of the first order equations,

$$\psi_x = \begin{pmatrix} -i\zeta & -1 \\ u & i\zeta \end{pmatrix} \psi, \quad \text{where} \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4.99}$$

so that

$$\psi_{1,xx} = -i\zeta \psi_{1,x} - \psi_{2,x} = -(u + \zeta^2)\psi_1,$$

that is ψ_1 can be identified with the solution of the scalar Schrödinger equation corresponding to the eigenvalue $\lambda = \zeta^2$. Correspondingly, from the other Eq. (4.43) we find

$$\begin{aligned}\psi_{1,t} &= (-4i\zeta^3 + 2iu\zeta + u_x)\psi_1 - (4\zeta^2 - 2u)\psi_2,\\ \psi_{2,t} &= -i\zeta\psi_{1,t} - \psi_{1,tx}\\ &= (4u\zeta^2 - 2iu_x\zeta - 2u^2 - u_{xx})\psi_1 + (4i\zeta^3 - 2iu\zeta - u_x)\psi_2,\end{aligned}$$

that is the time evolution of ψ is determined by the equation

$$\psi_t = \begin{pmatrix} -4i\zeta^3 + 2iu\zeta + u_x & -4\zeta^2 + 2u \\ 4u\zeta^2 - 2iu_x\zeta - 2u^2 - u_{xx} & 4i\zeta^3 - 2iu\zeta - u_x \end{pmatrix}\psi. \tag{4.100}$$

Thus, the KdV equation can be represented as a compatibility condition of two linear systems of the first order equations (4.99) and (4.100). Let these linear systems have the basis solutions

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{4.101}$$

similar to the Bloch functions for the Schrödinger equation. Considering them as spinors, we can build a vector with spherical components (see Landau and Lifshitz, 1989)

$$f = -\tfrac{i}{2}(\psi_1\phi_2 + \psi_2\phi_1), \quad g = \psi_1\phi_1, \quad h = -\psi_2\phi_2, \tag{4.102}$$

which generalizes the used above product $g = \psi_+\psi_-$ of the Schrödinger equation. It is easy to find from Eqs. (4.99) and (4.100), that f, g and h satisfy the linear systems

$$\begin{aligned} f_x &= -iug - ih,\\ g_x &= -2if - 2i\zeta g,\\ h_x &= -2iuf + 2i\zeta h,\end{aligned} \tag{4.103}$$

and

$$\begin{aligned} f_t &= -i(4u\zeta^2 - 2iu_x\zeta - 2u^2 - u_{xx})g + i(-4\zeta^2 + 2u)h,\\ g_t &= 2i(-4\zeta^2 + 2u)f + 2(-4i\zeta^3 + 2iu\zeta + u_x)g,\\ h_t &= -2i(4u\zeta^2 - 2iu_x\zeta - 2u^2 - u_{xx})f - 2(-4i\zeta^3 + 2iu\zeta + u_x)h.\end{aligned} \tag{4.104}$$

These systems correspond to Eqs. (4.48) and (4.54) in the scalar representation. Now, let us show that Eq. (4.98) follows directly from these systems. Indeed, it is easy to check that they have the integral of motion

$$f^2 - gh = P(\zeta^2), \tag{4.105}$$

that is, the space and time derivatives of the left hand side of Eq. (4.105) are equal identically to zero. This means that the 'length' of the vector with spherical components Eq. (4.102) is constant during the evolution of $u(x,t)$ according to the KdV equation. From Eq. (4.103) we find

$$f = \tfrac{i}{2}g_x - \zeta g, \quad h = if_x - ug = -\tfrac{1}{2}g_{xx} - i\zeta g_x - ug, \tag{4.106}$$

and substitution of these equations into Eq. (4.105) gives

$$\tfrac{1}{2}gg_{xx} - \tfrac{1}{4}g_x^2 + (\zeta^2 + u)g^2 = P(\zeta^2), \tag{4.107}$$

which coincides with Eq. (4.98) with account of $\zeta^2 = \lambda$.

The left hand side of Eq. (4.107) contains only ζ^2, so we have to find polynomial in ζ^2 solutions corresponding to a given polynomial $P(\zeta^2)$. The simplest nontrivial solution corresponds to the first degree polynomial

$$g = \zeta^2 - \mu, \tag{4.108}$$

where $\mu(x,t)$ as a new unknown variable. Then Eqs. (4.106) give

$$f = -\zeta^3 + \mu\zeta - \tfrac{i}{2}\mu_x. \quad h = -u\zeta^2 + i\zeta\mu_x + \tfrac{1}{2}\mu_{xx} + u\mu,$$

and substitution of Eq. (4.108) into Eq. (4.107) yields the identity

$$\begin{aligned}\zeta^6 - (2\mu - u)\zeta^4 + &(\mu^2 - 2\mu u - \tfrac{1}{2}\mu_{xx})\zeta^2 + u\mu^2 + \tfrac{1}{2}\mu\mu_{xx} - \tfrac{1}{4}\mu_x^2 \\ &= \zeta^6 - s_1\zeta^4 + s_2\zeta^2 - s_3,\end{aligned} \tag{4.109}$$

where s_i, $i = 1,2,3$, are the coefficients of the polynomial $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$,

$$s_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad s_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad s_3 = \lambda_1\lambda_2\lambda_3. \tag{4.110}$$

From Eq. (4.109) we find that u and μ are connected by the relation

$$u = 2\mu - s_1 = 2\mu - (\lambda_1 + \lambda_2 + \lambda_3), \tag{4.111}$$

coinciding with Eq. (4.31). Notice now, that if we put the spectral parameter $\lambda = \zeta^2$ equal to μ in Eq. (4.105), then we obtain $f(\mu) = \sqrt{P(\mu)}$ because of $g|_{\zeta^2=\mu} = 0$. Hence, the second equation (4.103) gives the equation

$$\mu_x = 2if(\mu) = 2\sqrt{-P(\mu)}, \tag{4.112}$$

which coincides with Eq. (4.30) and determines the x-dependence of μ. The same substitution $\zeta^2 = \mu$ into the second equation (4.104) gives with account of Eq. (4.111) the equation

$$\mu_t = 2i(4\mu - 2u)f = 2s_1 \cdot 2\sqrt{-P(\mu)} = 2s_1\mu_x. \tag{4.113}$$

Thus, μ depends on x and t only in combination $\xi = x + 2s_1t$, that is, the phase velocity V is connected with the zeros of the polynomial $P(\lambda)$ by the relation

$$V = -2s_1 = -2(\lambda_1 + \lambda_2 + \lambda_3) \tag{4.114}$$

coinciding with Eq. (3.144).

Thus, the simplest solution of the systems (4.103,4.104) with the polynomial dependence of f, g, h on $\zeta^2 = \lambda$ reproduces immediately the main formulas related with the cnoidal wave solution of the KdV equation.

4.1.7 *Periodic solution of the KdV hierarchy*

As it follows from Sec. 4.1.6, the dependence of the cnoidal wave $u(\xi)$ on x or $\xi = x - Vt$ is determined by the linear system (4.103) which is the same for all evolution equations of the KdV hierarchy (4.45). The second linear system (4.104) allows us to find the dependence on t, that is, in a simple case of the cnoidal wave the value of the phase velocity V as a function of the eigenvalues λ_1, λ_2, λ_3. We know the generating function of equations (4.45), namely, all evolution equations can be written with taking into account Eq. (4.78) as

$$u_t = -8g_x, \tag{4.115}$$

where the series expansion of g in powers of $1/\lambda$ has the form (4.57). Then the KdV equation is determined by the coefficient g_2 and the higher KdV equations by the coefficients $g_3, g_4, \ldots$. On the other hand, $g(x,t)$ satisfies Eq. (4.48), that is,

$$g_{xxx} + 2u_xg + 4(u + \lambda)g_x = 0. \tag{4.116}$$

To determine the generating function of the phase velocities for the higher KdV equations, we have to find the cnoidal wave solution satisfying the two equations (4.115) and (4.116). Equation (4.115) shows that we can define a potential,

$$u = \phi_x, \quad g = -\tfrac{1}{8}\phi_t, \tag{4.117}$$

so that Eq. (4.116) takes the form

$$\phi_t\phi_{txx} - \tfrac{1}{2}\phi_{tx}^2 + 2(\phi_x + \lambda)\phi_t^2 = \text{const} = C^2 \tag{4.118}$$

analogous to Eq. (4.107). According to Eqs. (4.117), the cnoidal wave solution has the form

$$\phi = \Phi(\xi) + \sigma x - \rho t, \quad \xi = x - Vt, \tag{4.119}$$

where σ and ρ are some constants. Since $u = \Phi' + \sigma$, $\phi_t = -V\Phi' - \rho$, we have

$$\phi_x = u, \quad \phi_t = -Vu + V\sigma - \rho, \quad \phi_{tx} = -Vu', \quad \phi_{txx} = -Vu''. \tag{4.120}$$

We are interested in the expression for V in terms of λ_1, λ_2, λ_3; therefore, it is convenient to pass to the variable μ (see Eq. (4.31))

$$u(\xi) = 2\mu(\xi) - s_1, \tag{4.121}$$

where μ satisfies the equations

$$\begin{aligned}
&(\mu')^2 = -4P(\mu) = -4\prod_{i=1}^{3}(\mu - \lambda_i) = -4(\mu^3 - s_1\mu^2 + s_2\mu - s_3),\\
&\mu'' = -2(3\mu^2 - 2s_1\mu + s_2).
\end{aligned} \tag{4.122}$$

Hence, Eqs. (4.120) can be expressed in terms of μ,

$$\begin{aligned}
&\phi_x = 2\mu - s_1, \quad \phi_t = -V(2\mu - s_1) + V\sigma - \rho,\\
&\phi_{tx} = -2V\mu', \quad \phi_{txx} = -2V\mu''.
\end{aligned}$$

Substitution of these expressions and Eqs. (4.122) into Eq. (4.118) gives after simple transformations

$$\begin{aligned}
&8V[V(2\lambda - s_1) - (V\sigma - \rho)]\mu^2\\
&+ 8[(V\sigma - \rho)(V\sigma - \rho - 2V(\lambda - s_1)) - V^2 s_1(2\lambda - s_1)]\mu\\
&+ 2[2s_2V(V\sigma - \rho + Vs_1) + V^2 s_1^2(\lambda - s_1)\\
&- (\lambda - s_1)(V\sigma - \rho)(V\sigma - \rho + 2Vs_1) - 4V^2 s_3] = C^2.
\end{aligned} \tag{4.123}$$

Since the left hand side cannot depend on μ, we obtain the relation

$$V\sigma - \rho = V(2\lambda - s_1),$$

which makes the coefficients before μ^2 and μ equal to zero. Its substitution into the rest term of Eq. (4.123) gives

$$16V^2(\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3) = 16V^2 P(\lambda) = C^2.$$

Hence, we have

$$V = \frac{C}{4\sqrt{P(\lambda)}} = \frac{C}{4\lambda^{3/2}}\left(1 + \frac{s_1}{2\lambda} + \dots\right).$$

The remaining undetermined constant C can be chosen so that the coefficient before λ^{-1} in the right hand side is equal to the phase velocity for the KdV equation case, $V = V_1 = -2s_1$, that is $C = -16\lambda^{3/2}$. As a result, we arrive at the generating function of the phase velocities for the KdV hierarchy,

$$V = -\frac{4\lambda^{3/2}}{\sqrt{P(\lambda)}} = -4 + \frac{V_1}{\lambda} + \frac{V_2}{\lambda^2} + \frac{V_3}{\lambda^3} + \dots, \tag{4.124}$$

where

$$V_1 = -2s_1, \quad V_2 = 2s_2 - \tfrac{3}{2}s_1^2, \quad V_3 = 3s_1 s_2 - \tfrac{5}{4}s_1^3 - 2s_3, \dots. \tag{4.125}$$

If we introduce separate time variables for every equation (4.80) of the KdV hierarchy,

$$u_{t_n} = \frac{\partial}{\partial x}\frac{\delta I_n}{\delta u}, \tag{4.126}$$

then their common cnoidal wave solution can be written in the form

$$u = u(\xi), \quad \xi = x - \sum_{n=1}^{\infty} V_n t_n, \tag{4.127}$$

where $u(\xi)$ is given by the equation

$$u(\xi) = \lambda_3 - \lambda_1 - \lambda_2 - 2(\lambda_3 - \lambda_2)\,\mathrm{sn}^2(\sqrt{\lambda_3 - \lambda_1}\,\xi, m), \tag{4.128}$$

with $m = (\lambda_3 - \lambda_2)/(\lambda_3 - \lambda_1)$.

4.1.8 *Another derivation of the Whitham equations*

Introduction of the linear systems permits one not only to find the periodic solution, what is not difficult in the KdV case without use of this new technique, but also to derive very easily the corresponding modulational equations. Now we have in our disposal a simple expression (4.52) for the generating function of the conservation laws and averaging of the generating function simplifies calculations drastically compared with those presented in Sec. 3.5. Moreover, since the parameters defining the periodic solution are the zeros of the polynomial $P(\lambda)$, we obtain the Whitham equations directly in the diagonal Riemann form. The only thing which should be taken into account is that normalization of the functions f, g, h must not depend on a slow evolution of λ_i. Therefore, we replace the identity Eq. (4.105) by

$$\left(f/\sqrt{P(\lambda)}\right)^2 - \left(g/\sqrt{P(\lambda)}\right)\cdot\left(h/\sqrt{P(\lambda)}\right) = 1 \tag{4.129}$$

and make similar changes $f \to f/\sqrt{P(\lambda)}$, $g \to g/\sqrt{P(\lambda)}$, $h \to h/\sqrt{P(\lambda)}$ in other formulas. Then the generating function (4.52) of the conservation laws takes the form

$$\frac{\partial}{\partial t}\left(\sqrt{P(\lambda)}\cdot\frac{1}{g}\right) + \frac{\partial}{\partial x}\left(\sqrt{P(\lambda)}\cdot\frac{2u-4\lambda}{g}\right) = 0. \tag{4.130}$$

Obviously, this replacement does not change any results for a strictly uniform unmodulated wave, because for them $P(\lambda) = \text{const}$, but the modulated wave satisfies the same condition (4.129) in all non-uniform region with the varying parameters λ_i.

We transform Eq. (4.130) with the use of Eqs. (4.108) and (4.111) to the form

$$\frac{\partial}{\partial t}\left(\sqrt{P(\lambda)}\cdot\frac{1}{\lambda-\mu}\right) + \frac{\partial}{\partial x}\left[\sqrt{P(\lambda)}\left(-4-\frac{2s_1}{\lambda-\mu}\right)\right] = 0.$$

According to Eq. (4.112), we have $dx = d\mu/(2\sqrt{-P(\mu)})$, hence, the averaged generating function of the conservation laws is given by

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\sqrt{P(\lambda)}\cdot\frac{1}{L}\oint\frac{d\mu}{2(\lambda-\mu)\sqrt{-P(\mu)}}\right)\\&+\frac{\partial}{\partial x}\left[\sqrt{P(\lambda)}\left(-4-2s_1\cdot\frac{1}{L}\oint\frac{d\mu}{2(\lambda-\mu)\sqrt{-P(\mu)}}\right)\right] = 0,\end{aligned} \tag{4.131}$$

where the wavelength is equal (up to an inessential here constant factor) to

$$L = \oint \frac{d\mu}{2\sqrt{-P(\mu)}} = \frac{1}{2}\oint \frac{d\mu}{\sqrt{(\lambda_1-\mu)(\lambda_2-\mu)(\lambda_3-\mu)}}, \tag{4.132}$$

provided integration is taken over the cycle around the gap $\lambda_2 \le \mu \le \lambda_3$ in the spectrum of the Lamé equation. Let us study the limits of Eq. (4.131) at $\lambda \to \lambda_i$, $i = 1, 2, 3$. Since in these formulas we have $\sqrt{P(\lambda)} \to 0$, the terms with derivatives of as $1/L$ so the integrals over μ are multiplied by the vanishing factor and do not contribute into the resulting expression for the left hand side of Eq. (4.131). However, differentiation of $\sqrt{P(\lambda)}$ gives the factors

$$\frac{1}{\sqrt{\lambda-\lambda_i}}\frac{\partial\lambda_i}{\partial t} \quad \text{and} \quad \frac{1}{\sqrt{\lambda-\lambda_i}}\frac{\partial\lambda_i}{\partial x} \quad i = 1, 2, 3,$$

singular at $\lambda \to \lambda_i$. Hence, to satisfy Eq. (4.131), the coefficients before $(\lambda-\lambda_i)^{-1/2}$ must vanish, that is, the slow evolution of λ_i must satisfy the equations

$$\begin{aligned} &\frac{1}{L}\oint \frac{d\mu}{2(\lambda_i-\mu)\sqrt{-P(\mu)}} \cdot \frac{\partial\lambda_i}{\partial t} \\ &+ \left(-4 - 2s_1\frac{1}{L}\oint \frac{d\mu}{2(\lambda_i-\mu)\sqrt{-P(\mu)}}\right) \cdot \frac{\partial\lambda_i}{\partial x} = 0, \quad i = 1, 2, 3. \end{aligned} \tag{4.133}$$

As follows from Eq. (4.132), the integral here is equal to

$$\oint \frac{d\mu}{(\lambda_i-\mu)\sqrt{-P(\mu)}} = -4\frac{\partial L}{\partial\lambda_i},$$

As a result, Eqs. (4.133) take the form

$$\frac{\partial\lambda_i}{\partial t} + \left(-2s_1 + \frac{2L}{\partial_i L}\right)\frac{\partial\lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, \tag{4.134}$$

coinciding exactly with Eqs. (3.145,3.146).

As was remarked in the end of Sec. 3.5, the general formulas (3.146) for the Whitham velocities are correct when the phase velocity V is expressed in terms of the Riemann invariants. The above derivation shows that the eigenvalues λ_1, λ_2, λ_3 are the Riemann invariants for all equations of the KdV hierarchy. Hence, we can obtain the Whitham equations for the higher

KdV equation from the generating functions

$$v_i = \left(1 - \frac{L}{\partial_i L}\partial_i\right)\frac{-4\lambda^{3/2}}{\sqrt{P(\lambda)}}, \quad i = 1, 2, 3. \tag{4.135}$$

The coefficient before λ^{-1} in the right hand side of Eqs. (4.135) gives Eqs. (3.146), and the coefficients before λ^{-2}, $\lambda^{-3}, \dots$ give the Whitham velocities for modulations of the cnoidal waves evolving according to the higher KdV equations.

To sum up, we have shown that the presentation of the KdV equation as a compatibility condition of two linear systems leads to a convenient parameterization of the periodic solution (the cnoidal wave) in terms of the eigenvalues λ_i, $i = 1, 2, 3$, which play a role of the Riemann invariants for the corresponding modulation Whitham equations. We may try to generalize these results in two directions.

First, we may consider solutions of the systems (4.103,4.104) corresponding to polynomials $P(\lambda)$ (see Eq. (4.107)) of higher degrees in λ. In this way, we arrive at multi-phase quasi-periodic solutions depending on several phases $\theta_i = x - V_i t$, $i = 1, 2, \dots, m$, where the degree of the polynomial is equal to $2m + 1$, and its $2m + 1$ zeros λ_i are the parameters defining the solution. The corresponding Whitham equations consist of $2m + 1$ equations for $2m + 1$ Riemann invariants λ_i. In this book we shall not consider this quite complicated theory.

Second, we may try to find some other equations of physical interest which admit presentation as compatibility conditions of two linear systems having the spectral parameter λ. It turns out that many important equations can be presented in this form, and methods analogous to the developed in this section permit one to find as periodic solutions of these equations so the corresponding Whitham equations. We shall investigate this possibility in what follows.

4.2 The AKNS scheme

4.2.1 *A general formulation*

It seems that a natural generalization of Eqs. (4.99,4.100) may be

$$\psi_x = \begin{pmatrix} F & G \\ H & -F \end{pmatrix}\psi = \mathbb{U}\psi, \quad \psi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\psi = \mathbb{V}\psi, \tag{4.136}$$

where $\psi = (\psi_1, \psi_2)^T$, and the matrix elements of $\mathbb{U}$ and $\mathbb{V}$ depend on as variables evolving according to the evolution equation under consideration so the spectral parameter λ. The compatibility condition $\psi_{xt} = \psi_{tx}$ of the linear systems (4.136) leads to the equations

$$\begin{aligned} &\partial F/\partial t - \partial A/\partial x + CG - BH = 0, \\ &\partial G/\partial t - \partial B/\partial x + 2\,(BF - AG) = 0, \\ &\partial H/\partial t - \partial C/\partial x + 2\,(AH - CF) = 0, \end{aligned} \tag{4.137}$$

or in a matrix form,

$$\mathbb{U}_t - \mathbb{V}_x + [\mathbb{U}, \mathbb{V}] = 0, \tag{4.138}$$

where $[\mathbb{U}, \mathbb{V}] = \mathbb{U}\mathbb{V} - \mathbb{V}\mathbb{U}$ denotes a commutator of two matrices. The matrix elements are to be chosen so that Eqs. (4.137) give the evolution equations for the variables $u_k(x,t)$ where the index k enumerates the dependent functions. It is not evident whether such a choice is possible for a given evolution equations, but when we are able to find such a presentation of the evolution equations, we have in our disposal a powerful mathematical technique similar to that developed in the preceding sections for the KdV equation. With the use of the two sets of the basis solutions (ψ_1, ψ_2) and (ϕ_1, ϕ_2), we define again the 'squared basis functions'

$$f = -\tfrac{i}{2}\left(\psi_1\phi_2 + \psi_2\phi_1\right), \quad g = \psi_1\phi_1, \quad h = -\psi_2\phi_2, \tag{4.139}$$

which satisfy the linear systems

$$\begin{aligned} \partial f/\partial x &= -iHg + iGh, \\ \partial g/\partial x &= 2iGf + 2Fg, \\ \partial h/\partial x &= -2iHf - 2Fh, \end{aligned} \tag{4.140}$$

and

$$\begin{aligned} \partial f/\partial t &= -iCg + iBh, \\ \partial g/\partial t &= 2iBf + 2Ag, \\ \partial h/\partial t &= -2iCf - 2Ah. \end{aligned} \tag{4.141}$$

The matrices $\mathbb{U}$ and $\mathbb{V}$ have zero traces, hence, the Wronskian $\psi_1\phi_2 - \psi_2\phi_1$, as it is easy to verify, does not depend on x and t,

$$(\psi_1\phi_2 - \psi_2\phi_1)_x = 0, \quad (\psi_1\phi_2 - \psi_2\phi_1)_t = 0.$$

Therefore, the quantity $f^2 - gh = (-1/4)(\psi_1\phi_2 - \psi_2\phi_1)^2$ is an integral of motion depending only on the spectral parameter λ,

$$f^2 - gh = P(\lambda) \tag{4.142}$$

(compare with Eqs. (4.105–4.107) for the KdV case). By analogy with the KdV equation case one may suppose that periodic solutions are distinguished by the condition that $P(\lambda)$ be a polynomial in λ, and this supposition is confirmed by the results obtained as its consequences. As a rule, the simplest nontrivial solution corresponds to the polynomial $P(\lambda)$ of the third or fourth degree in λ (if only it is not a polynomial in even degrees in $\zeta = \sqrt{\lambda}$, as it takes place in the KdV case). Therefore, we suppose that

$$P(\lambda) = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4, \tag{4.143}$$

where λ_i are the polynomial's zeros which parameterize the solution. For equations which will be considered below, the systems (4.140,4.141) have a solution in a polynomial form

$$f = f_0\lambda^2 - f_1\lambda + f_2, \quad g(\lambda) = g_0(\lambda)(\lambda-\mu), \quad h(\lambda) = h_0(\lambda)(\lambda-\mu^*), \tag{4.144}$$

with the coefficients depending on $u_k(x,t)$. Then Eq. (4.142) yields four integrals of motion and Eqs. (4.140,4.141) yield formulas connecting u_k with the coefficients of the polynomial (4.143) (cf. Eq. (4.109)). It turns out that the dynamics is described most conveniently in terms of the variable $\mu(x,t)$ equal to the zero of the function g. Indeed, if we substitute $\lambda = \mu$ into Eq. (4.142), we obtain $f(\mu) = \sqrt{P(\mu)}$, and then the same substitution into the second equations of the systems (4.140,4.141) gives the equations for μ,

$$\frac{\partial\mu}{\partial x} = -\frac{2iG(\mu)}{g_0(\mu)}f(\mu), \quad \frac{\partial\mu}{\partial t} = -\frac{2iB(\mu)}{g_0(\mu)}f(\mu), \tag{4.145}$$

where $G(\mu) = G(\lambda)|_{\lambda=\mu}$, etc. As we shall see, the coefficients before $f(\mu)$ in Eqs. (4.145) are constant,

$$\left.\frac{G}{g_0}\right|_{\lambda=\mu} = \text{const}, \quad \left.\frac{B}{g_0}\right|_{\lambda=\mu} = \text{const}, \tag{4.146}$$

so that the solutions of Eqs. (4.145) can easily be expressed in terms of the elliptic functions. The phase velocity is equal to

$$V = \frac{B}{G}\bigg|_{\lambda=\mu}, \tag{4.147}$$

and can be expressed in terms of λ_i, $i = 1,2,3,4$. When $\mu(x - Vt)$ is found, we can calculate the variables $u_k(x,t)$ and, thus, obtain the periodic solution of the equations under consideration.

The zeros λ_i of the polynomial $P(\lambda)$ are the parameters defining the periodic solution. In a weakly modulated wave train they become slow functions of x and t, and, according to Whitham, their evolution is governed by the equations which follow from the averaged conservation laws. As we have seen above for the KdV equation case, it is convenient to use the generating function of the conservation laws which leads directly to the modulational equations in the diagonal Riemann form. Equation (4.51) suggests that in a general case of the systems (4.136–4.141) the generating function must have the form

$$\frac{\partial}{\partial t}\frac{G}{g} - \frac{\partial}{\partial x}\frac{B}{g} = 0, \qquad \frac{\partial}{\partial t}\frac{H}{h} - \frac{\partial}{\partial x}\frac{C}{h} = 0, \tag{4.148}$$

and these formulas can be verified by a direct calculation with the use of Eqs. (4.137,4.140,4.141). The series expansions of Eqs. (4.148) in powers of λ^{-1} give an infinite sequence of the conservation laws. To average them, we have to normalize the functions f, g, h so that this normalization condition does not depend on the slow evolution of λ_i. We choose the condition $f^2 - gh = 1$ (cf. Eq. (4.129)), and then the first equation (4.148) takes the form

$$\frac{\partial}{\partial t}\left(\sqrt{P(\lambda)}\frac{G(\lambda)}{g(\lambda)}\right) - \frac{\partial}{\partial x}\left(\sqrt{P(\lambda)}\frac{B(\lambda)}{g(\lambda)}\right) = 0.$$

Averaging of this equation over the wavelength L according to the rule

$$\overline{\mathcal{F}} = \frac{1}{L}\int_0^L \mathcal{F}dx$$

with taking into account $dx = \mathrm{const}\cdot d\mu/\sqrt{-P(\mu)}$ gives the generating function of the averaged conservation laws

$$\frac{\partial}{\partial t}\left[\frac{\sqrt{P(\lambda)}}{L}\oint\frac{G(\lambda)d\mu}{g_0(\lambda)(\lambda-\mu)\sqrt{-P(\mu)}}\right] - \frac{\partial}{\partial x}\left[\frac{\sqrt{P(\lambda)}}{L}\oint\frac{B(\lambda)d\mu}{g_0(\lambda)(\lambda-\mu)\sqrt{-P(\mu)}}\right]=0.$$

As in the KdV equation case, the condition of vanishing the singular at $\lambda\to\lambda_i$ coefficients before the terms with the factor $(\lambda-\lambda_i)^{-1/2}$, arising due to differentiation of $\sqrt{P(\lambda)}$ with respect to x and t, leads to equations

$$\oint\frac{G(\lambda_i)d\mu}{g_0(\lambda_i)(\lambda_i-\mu)\sqrt{-P(\mu)}}\cdot\frac{\partial\lambda_i}{\partial t} - \oint\frac{B(\lambda_i)d\mu}{g_0(\lambda_i)(\lambda_i-\mu)\sqrt{-P(\mu)}}\cdot\frac{\partial\lambda_i}{\partial x}=0,\quad i=1,2,3,4.$$

Thus, we arrive at the Whitham equations for the Riemann invariants λ_i,

$$\frac{\partial\lambda_i}{\partial t}+v_i(\lambda)\frac{\partial\lambda_i}{\partial x}=0,\quad i=1,2,3,4,\tag{4.149}$$

with the characteristic speeds

$$v_i=-\frac{I_2(\lambda_i)}{I_1(\lambda_i)},\quad i=1,2,3,4,\tag{4.150}$$

where

$$I_1(\lambda_i)=\oint\frac{G}{g_0(\lambda_i-\mu)}\frac{d\mu}{\sqrt{P(\mu)}},\quad I_2(\lambda_i)=\oint\frac{B}{g_0(\lambda_i-\mu)}\frac{d\mu}{\sqrt{P(\mu)}}.\tag{4.151}$$

As we shall see below, these integrals can be easily evaluated in every particular case.

To apply this sketch of the general theory to particular equations, we have to show that we can choose the matrices $\mathbb{U}$ and $\mathbb{V}$ in Eq. (4.136) so that Eqs. (4.137) lead really to physically interesting equations.

4.2.2 *Nonlinear equations and the AKNS scheme*

The most important example of the nonlinear equation which can be presented as a compatibility condition of linear systems (4.136) was found by

Zakharov and Shabat in 1971 for the matrix $\mathbb{U}$ having the form

$$\mathbb{U} = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}, \tag{4.152}$$

and such a matrix $\mathbb{V}$ that Eqs. (4.137) reduce to the NLS equation. Therefore, the problem $\psi_x = \mathbb{U}\psi$ with $\mathbb{U}$ given by Eq. (4.152) is called the Zakharov-Shabat problem. In the paper Ablowitz, Kaup, Newell and Segur (1974), the Zakharov-Shabat spectral problem was studied for a fairly general form of the $\mathbb{V}$–matrix. We shall call the AKNS scheme any compatibility condition of linear systems (4.136) with the 2×2 matrices $\mathbb{U}$ and $\mathbb{V}$.

Let us consider some examples corresponding to the Zakharov-Shabat $\mathbb{U}$-matrix given by Eq. (4.152), when Eqs. (4.137) take the form

$$\begin{aligned} A_x &= Cq - Br, \\ B_x + 2i\lambda B &= q_t - 2Aq, \\ C_x - 2i\lambda C &= r_t + 2Ar. \end{aligned} \tag{4.153}$$

As in the considered above case of the KdV hierarchy, it is natural to suppose that A, B, C are some polynomial functions in λ,

$$A = \sum_{i=0}^{n} A_{n-i}\lambda^i, \quad B = \sum_{i=0}^{n} B_{n-i}\lambda^i, \quad C = \sum_{i=0}^{n} C_{n-i}\lambda^i, \tag{4.154}$$

so that the coefficients A_i, B_i, C_i can be found with the use of the recursion relations following from Eqs. (4.153). We shall not develop here the general theory and confine ourselves to a particular case $n = 3$. Substitution of the polynomials (4.154) into Eqs. (4.153) gives first of all that $B_0 = C_0 = 0$, $A_{0,x} = 0$, that is $A_0 = a_0$ does not depend on x, and then the other coefficients are determined by the equations

$$\begin{aligned} &A_{1,x} = C_1 q - B_1 r, \quad A_{2,x} = C_2 q - B_2 r, \quad A_{3,x} = C_3 q - B_3 r; \\ &iB_1 = -A_0 q, \quad B_{1,x} + 2iB_2 = -2A_1 q, \quad B_{2,x} + 2iB_3 = -2A_2 q, \\ &B_{3,x} = q_t - 2A_3 q; \quad -iC_1 = A_0 r, \quad C_{1,x} - 2iC_2 = 2A_1 r, \\ &C_{2,x} - 2iC_3 = 2A_2 r, \quad C_{3,x} = r_t + 2A_3 r. \end{aligned}$$

Hence, we find step by step

$$\begin{aligned}
&B_1 = ia_0 q, \quad C_1 = ia_0 r, \quad A_{1,x} = 0, \quad \text{that is} \quad A_1 = a_1;\\
&B_2 = -\tfrac{1}{2}a_0 q_x + ia_1 q, \quad C_2 = \tfrac{1}{2}a_0 r_x + ia_1 r,\\
&A_{2,x} = \tfrac{1}{2}a_0(qr)_x, \quad \text{that is} \quad A_2 = \tfrac{1}{2}a_0 qr + a_2;\\
&B_3 = -\tfrac{i}{4}a_0(q_{xx} - 2q^2 r) - \tfrac{1}{2}a_1 q_x + ia_2 q,\\
&C_3 = -\tfrac{i}{4}a_0(r_{xx} - 2qr^2) + \tfrac{1}{2}a_1 r_x + ia_2 r,\\
&A_3 = \tfrac{i}{4}a_0(q_x r - r_x q) + \tfrac{1}{2}a_1 qr + a_3,
\end{aligned}$$

where a_1, a_2, a_3 do not depend on x. Thus, the evolution equations

$$\begin{aligned}
q_t + \tfrac{i}{4}a_0(q_{xxx} - 6qrq_x) + \tfrac{1}{2}a_1(q_{xx} - 2q^2 r) - ia_2 q_x - 2a_3 q = 0,\\
r_t + \tfrac{i}{4}a_0(r_{xxx} - 6qrr_x) - \tfrac{1}{2}a_1(r_{xx} - 2r^2 q) - ia_2 r_x + 2a_3 r = 0
\end{aligned} \tag{4.155}$$

are the compatibility conditions for the linear systems (4.136) provided the $\mathbb{U}$-matrix is given by Eq. (4.152) and the matrix elements of $\mathbb{V}$ are equal to

$$\begin{aligned}
A &= a_0\lambda^3 + a_1\lambda^2 + \left(\tfrac{1}{2}a_0 qr + a_2\right)\lambda + \tfrac{i}{4}a_0(q_x r - r_x q) + \tfrac{1}{2}a_1 qr + a_3,\\
B &= ia_0 q\lambda^2 + \left(-\tfrac{1}{2}a_0 q_x + ia_1 q\right)\lambda - \tfrac{i}{4}a_0(q_{xx} - 2q^2 r) - \tfrac{1}{2}a_1 q_x + ia_2 q,\\
C &= ia_0 r\lambda^2 + \left(\tfrac{1}{2}a_0 r_x + ia_1 r\right)\lambda - \tfrac{i}{4}a_0(r_{xx} - 2r^2 q) + \tfrac{1}{2}a_1 r_x + ia_2 r.
\end{aligned} \tag{4.156}$$

We see that if we take $a_0 = -4i$, $a_1 = a_2 = a_3 = 0$, and $q = -1$, $r = u$, then Eqs. (4.155) reduce to the KdV equation and $\mathbb{U}$ and $\mathbb{V}$ coincide with the found earlier matrices (4.99,4.100). But now there are some other interesting possibilities. If we take $a_0 = -4i$, $a_1 = a_2 = a_3 = 0$, and $r = \pm q = \pm iu$, then Eqs. (4.155) reduce to the mKdV equation

$$u_t \pm 6u^2 u_x + u_{xxx} = 0, \tag{4.157}$$

and the corresponding $\mathbb{V}$-matrix has the form

$$\mathbb{V} = \begin{pmatrix} -4i\lambda^3 \pm 2iu^2\lambda & 4iu\lambda^2 - 2u_x\lambda - iu_{xx} \mp 2iu^3 \\ \pm 4iu\lambda^2 \pm 2u_x\lambda \mp iu_{xx} - 2iu^3 & 4i\lambda^3 \mp 2iu^2\lambda \end{pmatrix}. \tag{4.158}$$

Another choice $a_0 = a_2 = a_3 = 0$, $a_1 = -2i$, and $q = iu$, $r = \pm iu^*$ leads to the NLS equation with any sign of the nonlinear term,

$$iu_t + u_{xx} \pm 2|u|^2 u = 0, \tag{4.159}$$

and the corresponding matrices $\mathbb{U}$ and $\mathbb{V}$ are equal to

$$\mathbb{U} = \begin{pmatrix} -i\lambda & iu \\ \pm iu^* & i\lambda \end{pmatrix}, \quad \mathbb{V} = \begin{pmatrix} -2i\lambda^2 \pm i|u|^2 & 2iu\lambda - u_x \\ \pm 2iu^*\lambda \pm u_x^* & 2i\lambda^2 \mp i|u|^2 \end{pmatrix}. \tag{4.160}$$

Hence, the mKdV and NLS equations are the completely integrable equations.

In a similar way, if we look for the matrix elements of $\mathbb{V}$ in the form

$$A = \frac{a(x,t)}{\lambda - \Delta}, \quad B = \frac{b(x,t)}{\lambda - \Delta}, \quad C = \frac{c(x,t)}{\lambda - \Delta},$$

then Eqs. (4.153) reduce to

$$\begin{aligned} &a_x = cq - br, \quad q_t = 2ib, \quad r_t = -2ic, \\ &b_x + \Delta q_t = -2aq, \quad c_x + \Delta r_t = 2ar. \end{aligned} \tag{4.161}$$

Comparison with the SIT equations (see Eqs. (2.184))

$$\mathcal{E}_\zeta = d, \quad d_\tau + 2i\Delta d = \mathcal{E}n, \quad n_\tau = -\tfrac{1}{2}(\mathcal{E}d^* + \mathcal{E}^*d) \tag{4.162}$$

shows that if we replace $x \to \tau$, $t \to \zeta$ and put $q = \mathcal{E}/2$, $r = -\mathcal{E}^*/2$, $a = (i/4)n$, $b = -(i/4)d$, $c = -(i/4)d^*$, then Eqs. (4.161) reproduce these equations, which are, hence, the compatibility conditions for the linear systems

$$\psi_\tau = \begin{pmatrix} -i\lambda & \mathcal{E}/2 \\ -\mathcal{E}^*/2 & i\lambda \end{pmatrix}\psi, \quad \psi_\zeta = \frac{i}{4(\lambda - \Delta)}\begin{pmatrix} n & -d \\ -d^* & -n \end{pmatrix}\psi. \tag{4.163}$$

Thus far, we have considered the linear systems with the Zakharov-Shabat matrix $\mathbb{U}$. Some other simple forms of the $\mathbb{U}$-matrix also yield evolution equations of physical interest. For example, if we suppose that all matrix elements of the $\mathbb{U}$-matrix are proportional to the spectral parameter λ, we arrive at the Heisenberg model equation

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} \tag{4.164}$$

as a compatibility condition of the linear systems (4.136) with the matrix elements

$$F = -\tfrac{1}{2}i\lambda S_3, \quad G = -\tfrac{1}{2}i\lambda S_-, \quad H = -\tfrac{1}{2}i\lambda S_+ \tag{4.165}$$

and

$$\begin{aligned} A &= \tfrac{1}{2}i\lambda^2 S_3 + \tfrac{1}{4}\lambda\left[(S_-)_x S_+ - S_-(S_+)_x\right], \\ B &= \tfrac{1}{2}i\lambda^2 S_- + \tfrac{1}{2}\lambda\left[(S_3)_x S_- - S_3(S_-)_x\right], \\ C &= \tfrac{1}{2}i\lambda^2 S_+ - \tfrac{1}{2}\lambda\left[(S_3)_x S_+ - S_3(S_+)_x\right], \end{aligned} \tag{4.166}$$

where $S_\pm = S_1 \pm iS_2$. A slight generalization of these expressions given by

$$F = -\tfrac{i}{2}\lambda S_3, \quad G = -\tfrac{i}{2}\sqrt{\lambda^2+J}\,S_-, \quad H = -\tfrac{i}{2}\sqrt{\lambda^2+J}\,S_+, \tag{4.167}$$

and

$$\begin{aligned} A &= \tfrac{i}{2}\left(\lambda^2+J\right)S_3 + \tfrac{1}{4}\lambda\left[(S_-)_x S_+ - S_-(S_+)_x\right], \\ B &= \tfrac{i}{2}\lambda\sqrt{\lambda^2+J}\,S_- + \tfrac{1}{2}\sqrt{\lambda^2+J}\,\left[(S_3)_x S_- - S_3(S_-)_x\right], \\ C &= \tfrac{i}{2}\lambda\sqrt{\lambda^2+J}\,S_+ - \tfrac{1}{2}\sqrt{\lambda^2+J}\,\left[(S_3)_x S_+ - S_3(S_+)_x\right], \end{aligned} \tag{4.168}$$

leads to the Landau-Lifshitz equation

$$\mathbf{S}_t = \mathbf{S}\times\mathbf{S}_{xx} + J\left(\mathbf{S}\cdot\mathbf{n}\right)\left(\mathbf{S}\times\mathbf{n}\right), \tag{4.169}$$

for spin waves in a one-axis ferromagnet.

If we suppose that the matrix elements of $\mathbb{U}$ are the polynomial functions quadratic in λ, then we obtain the DNLS equation

$$iu_t + u_{xx} - 2i(|u|^2u)_x = 0 \tag{4.170}$$

as a compatibility condition for the linear systems with the matrices $\mathbb{U}$ and $\mathbb{V}$ defined by the matrix elements

$$\begin{aligned} &F = -2i\lambda^2, \quad G = 2\lambda u, \quad H = 2\lambda u^*, \\ &A = -\left(8i\lambda^4 + 4i\,|u|^2\,\lambda^2\right), \quad B = 8\lambda^3 u + \left(2iu_x + 4\,|u|^2\,u\right)\lambda, \\ &C = 8\lambda^3 u^* + \left(-2iu_x^* + 4\,|u|^2\,u^*\right)\lambda. \end{aligned} \tag{4.171}$$

At last, the SRS equations (see Eqs. (2.198))

$$\begin{aligned} \partial R_+/\partial\tau &= i(\Delta R_+ S_3 + R_3 S_+), \quad & \partial R_3/\partial\tau &= \tfrac{i}{2}(R_+S_- - R_-S_+), \\ \partial S_+/\partial\zeta &= i(\Delta S_+ R_3 + S_3 R_+), \quad & \partial S_3/\partial\zeta &= \tfrac{i}{2}(S_+R_- - S_-R_+). \end{aligned} \tag{4.172}$$

correspond to the systems

$$\psi_\tau = \mathbb{U}\psi, \qquad \psi_\zeta = \mathbb{V}\psi, \tag{4.173}$$

with the matrix elements

$$\begin{aligned} &F = -i\lambda S_3, \quad G = (i\sigma + \lambda) S_+, \quad H = (i\sigma - \lambda) S_-, \\ &A = \frac{i}{2}\left(\Delta - \frac{1}{2\lambda + \Delta}\right) R_3, \quad B = \frac{i\sigma + \lambda}{2\lambda + \Delta} R_+, \quad C = \frac{i\sigma - \lambda}{2\lambda + \Delta} R_-, \end{aligned} \tag{4.174}$$

where the parameter σ is connected with Δ by the relation $\sigma^2 = \frac{1}{4}\left(1 - \Delta^2\right)$.

Of course, there are many other equations of physical interest which can be presented as compatibility conditions of linear systems having a spectral parameter λ, but the above examples are representative enough and we shall turn in the next Chapter to investigation of their periodic solutions.

Bibliographic remarks

The inverse scattering transform method was discovered in a famous paper by Gardner, Green, Kruskal, and Miura (1967). This discovery led to a fast progress in the soliton theory and many details can be found in the textbooks by Ablowitz and Segur (1981), Calogero and Degasperis (1982), Dodd, Eilbeck, Gibbon, and Morris (1982), Karpman (1973), Lamb (1980), Zakharov, Manakov, Novikov, and Pitaevskii (1980). The finite-gap integration method for the solution of the periodic problem for the KdV equation was developed by Novikov (1974), Lax (1974), Dubrovin (1975), Its and Matveev (1975), McKean and van Moerbeke (1975). In our exposition we used some ideas of a simple approach by Alber (1979). The periodic solution for the KdV hierarchy was obtained by Pavlov (1994). The Zakharov-Shabat problem was introduced into the soliton theory in the paper by Zakharov and Shabat (1971) with the corresponding $\mathbb{U}$–$\mathbb{V}$-pair for the NLS equation, and this paper opened the door to further discoveries of integrable evolution equations. This approach was generalized by Ablowitz, Kaup, Newell and Segur (1974). For the sine-Gordon equation the $\mathbb{U}$–$\mathbb{V}$-pair was found by Ablowitz, Kaup, and Newell (1973), for the SIT equations by Ablowitz, Kaup, Newell (1974), for the Heisenberg model by Takhtajan (1977), for the Landau-Lifshitz equation by Borovik and Robuk (1981), for the DNLS equation by Kaup and Newell (1978), for SRS equations by Kaup (1983) and Steudel (1983). The multi-phase Whitham equations for the KdV equation case were obtained by Flaschka, Forest and McLaughlin (1979). We used for derivation of the Whitham equations the method of Kamchatnov (1994).

Exercises on Chapter 4

Exercise 4.1

Show that the identity $\{I_n, I_1\} = 0$ follows from the recursion relation $\{I_n, I_m\} = \{I_{n+1}, I_{m-1}\}$ proven in Sec. 4.1.5. This makes the proof of the commutation relation $\{I_n, I_m\} = 0$ (see Eq. (4.97)) independent of Eq. (4.96)).

Exercise 4.2

(Lax, 1968) We can rewrite the first equation (4.43) as the eigenvalue problem

$$\mathbb{L}\psi = \lambda\psi$$

for the linear operator

$$\mathbb{L} = -\mathbb{D}^2 - u, \qquad \mathbb{D} = \partial/\partial x.$$

Then, if we eliminate the term $\lambda\psi_x$ from the second equation (4.43), we shall obtain

$$\psi_t = -\mathbb{A}\psi, \qquad \mathbb{A} = 4\mathbb{D}^3 + 3(u\mathbb{D} + \mathbb{D}u).$$

Show that the KdV equation can be presented in an operator form

$$\partial\mathbb{L}/\partial t = [\mathbb{L}, \mathbb{A}],$$

where brackets denote the commutator of operators.

Exercise 4.3

(Dubrovin, 1975; McKean and van Moerbeke, 1975) The result of the preceding exercise can be written in the form

$$[\mathbb{L}, \mathbb{A}_1] = -\frac{\partial}{\partial x}\frac{\delta I_1}{\delta u},$$

where $\mathbb{A}_1 = 4\mathbb{D}^3 + 3(u\mathbb{D} + \mathbb{D}u)$ and $I_1 = \int \mathcal{P}_1 dx$, $\mathcal{P}_1 = -u^3 + \frac{1}{2}u_x^2$. Show that the higher KdV equations can be presented in the Lax operator form

$$\partial\mathbb{L}/\partial t = [\mathbb{L}, \mathbb{A}_n],$$

where $\mathbb{A}_n$ are given by the expression

$$\mathbb{A}_n = -\tfrac{1}{4}\sum_{k=1}^{n+1}\left[\left(\mathbb{D}\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{k-2}\right) - 2\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{k-2}\right)\mathbb{D}\right]\mathbb{L}^{n-k+1}.$$

Exercise 4.4

Show that the soliton solution of the KdV equation can be obtained from Eqs. (4.112–4.115) when the allowed band between λ_1 and λ_2 shrinks into one eigenvalue $\lambda_1 = \lambda_2$.

Exercise 4.5

(Steudel, 1983) The two-photon propagation (TPP) equations describe the propagation of two electromagnetic waves with the frequencies ω_1 and ω_2 in a medium with a resonance transition at the frequency $\omega_1 + \omega_2$. These equations are similar to the SRS equations and acquire symmetric form, if we introduce the vector $\mathbf{S}$ with the components

$$S_1 = \mathcal{E}_1^*\mathcal{E}_2^* + \mathcal{E}_1\mathcal{E}_2, \quad S_2 = i(\mathcal{E}_1^*\mathcal{E}_2^* - \mathcal{E}_1\mathcal{E}_2), \quad S_3 = \mathcal{E}_1\mathcal{E}_1^* + \mathcal{E}_2\mathcal{E}_2^*,$$

so that in the appropriate coordinate system the TPP equations read

$$\begin{aligned} \partial R_+/\partial\tau &= i\left(\Delta R_+ S_3 + R_3 S_+\right), & \partial R_3/\partial\tau &= \tfrac{i}{2}\left(R_+ S_- - R_- S_+\right),\\ \partial S_+/\partial\zeta &= i\left(\Delta S_+ R_3 - S_3 R_+\right), & \partial S_3/\partial\zeta &= \tfrac{i}{2}\left(S_+ R_- - S_- R_+\right), \end{aligned}$$

where $S_\pm = S_1 \pm iS_2$, $\mathbf{R}$ is the Bloch vector, and the vectors $\mathbf{R}$ and $\mathbf{S}$ are normalized according to the conditions

$$R_1^2 + R_2^2 + R_3^2 = 1, \qquad -S_1^2 - S_2^2 + S_3^2 = 1.$$

Show that the TPP equations can be presented as a compatibility condition of the linear systems (4.173) with the matrix elements

$$\begin{aligned} &F = -i\lambda S_3, \quad G = (\lambda+\sigma)\,S_+, \quad H = (\lambda-\sigma)\,S_-,\\ &A = \frac{i}{2}\left(\Delta + \frac{1}{2\lambda+\Delta}\right)R_3, \quad B = -\frac{\lambda+\sigma}{2\lambda+\Delta}R_+, \quad C = -\frac{\lambda-\sigma}{2\lambda+\Delta}R_-, \end{aligned}$$

where the parameter σ is connected with Δ by the relation $\sigma^2 = \frac{1}{4}\left(1+\Delta^2\right)$.

Chapter 5

Periodic solutions

5.1 Focusing nonlinear Schrödinger equation

As we have seen in the preceding Chapters, the KdV and NLS equations arise in various physical situations and in this sense they are universal. The theory of the periodic solutions of the KdV equation has already been developed, and in connection with this problem we have introduced some new concepts concerning the complete integrability of the KdV equation. Now our aim is to show that similar methods work for some other equations and we shall begin with the focusing NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{5.1}$$

As we know, it is completely integrable, that is, it can be presented as a compatibility condition of two linear systems (4.136), where the matrix elements of $\mathbb{U}$ and $\mathbb{V}$ are given by (see Eqs. (4.160))

$$\begin{aligned} &F = -i\lambda, \quad G = iu, \quad H = iu^*; \\ &A = -2i\lambda^2 + i|u|^2, \quad B = 2iu\lambda - u_x, \quad C = 2iu^*\lambda + u_x^*. \end{aligned} \tag{5.2}$$

According to the rules formulated in the preceding chapter, a periodic solution corresponds to the polynomial in λ solution of systems (4.140) and (4.141), and this solution has the form

$$f = \lambda^2 - f_1\lambda + f_2, \quad g = iu(\lambda - \mu), \quad h = iu^*(\lambda - \mu^*), \tag{5.3}$$

provided the coefficients of these polynomials satisfy the equations

$$\begin{aligned} &f_{1,x} = f_{1,t} = 0, \quad f_{2,x} = -i\,|u|^2\,(\mu - \mu^*)\,, \\ &f_{2,t} = -u_x^* u\mu - u_x u^* \mu^*; \end{aligned} \tag{5.4}$$

$$u_x = 2iu\,(\mu - f_1)\,, \quad u_t = 2i\,|u|^2\,u + 2f_1 u_x + 4iuf_2; \tag{5.5}$$

$$u_x\mu + u\mu_x = -2iuf_2, \quad u_t\mu + u\mu_t = 2i\,|u|^2\,u\mu + 2f_2 u_x, \tag{5.6}$$

as well as their complex conjugate. Besides that, we must satisfy the condition (4.142) which on inserting Eqs. (5.3) takes the form

$$\begin{aligned} &\left(\lambda^2 - f_1\lambda + f_2\right)^2 + |u|^2\,(\lambda - \mu)\,(\lambda - \mu^*) \\ &\qquad\qquad = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4. \end{aligned} \tag{5.7}$$

Comparison of the coefficients before λ^k on both sides of Eq. (5.7) leads to the conservation laws

$$2f_1 = s_1, \qquad f_1^2 + 2f_2 + |u|^2 = s_2, \tag{5.8}$$

$$2f_1 f_2 + |u|^2\,(\mu + \mu^*) = s_3, \quad f_2^2 + |u|^2\,\mu\mu^* = s_4. \tag{5.9}$$

Equations (5.8) give

$$f_1 = \tfrac{1}{2}s_1, \quad f_2 = \tfrac{1}{2}\left(s_2 - \tfrac{1}{4}s_1^2 - |u|^2\right), \tag{5.10}$$

that is $f_1 = \text{const}$ in accordance with Eqs. (5.4) for f_1, and Eqs. (5.4) for f_2 reduce to

$$\frac{\partial\,|u|^2}{\partial x} = 2i\,|u|^2\,(\mu - \mu^*)\,, \quad \frac{\partial\,|u|^2}{\partial t} = 2is_1\,|u|^2\,(\mu - \mu^*) = s_1\frac{\partial\,|u|^2}{\partial x}. \tag{5.11}$$

Equations (5.5) yield the important relation between u and μ,

$$u_x = 2iu\left(\mu - \tfrac{1}{2}s_1\right), \quad u_t = 2i\left(s_2 - \tfrac{1}{4}s_1^2\right)u + s_1 u_x, \tag{5.12}$$

and Eqs. (5.6) give the evolution equations for μ,

$$\mu_x = -2if\,(\mu) = -2i\sqrt{P\,(\mu)}, \quad \mu_t = -2is_1 f\,(\mu) = s_1\mu_x. \tag{5.13}$$

Thus, as in the case of the KdV equation (see Eqs. (4.112) and (4.113)), the zero μ of the function g in the complex plane λ is the most convenient variable whose evolution is determined by the polynomial $P(\lambda)$, but this time the connection between u and μ is more complicated than it was for the KdV equations (cf. Eqs. (5.11) and (4.111)).

As follows from Eqs. (5.11) and (5.13), $|u|^2$ and μ depend only on the phase

$$\xi = x - Vt, \tag{5.14}$$

where the nonlinear phase velocity V of the periodic solution of the NLS equation is expressed in terms of the zeros $\lambda_i, i = 1, 2, 3, 4$, of the polynomial $P(\lambda)$,

$$V = -s_1 = -\sum_{i=1}^{4} \lambda_i. \tag{5.15}$$

The dependence of the field intensity $|u|^2$ and μ on the phase ξ is determined by equations

$$d\,|u|^2/d\xi = 2i\,|u|^2\,(\mu - \mu^*)\,, \quad d\mu/d\xi = -2i\sqrt{P\,(\mu)}. \tag{5.16}$$

From Eq. (5.12) we have also

$$u\,(x,t) = \exp\left[2i\left(s_2 - \tfrac{1}{4}s_1^2\right)t\right]\widetilde{u}\,(\xi)\,, \tag{5.17}$$

provided $\widetilde{u}(\xi)$ satisfies the equation

$$d\widetilde{u}/d\xi = 2i\left(\mu - \tfrac{1}{2}s_1\right)\widetilde{u}. \tag{5.18}$$

At first sight one might think that similarly to the KdV equation case the periodic solution of the NLS equation could be found by integration of Eq. (5.16) followed by integration of Eq. (5.18) which gives $\widetilde{u}(\xi)$ and, consequently, $u(x,t)$ according to Eq. (5.17). However, more careful inspection shows that there is essential difference between the KdV and NLS equations. In the KdV equation theory the reality of $u(x,t)$ leads immediately by Eq. (4.111) to reality of the variable μ, so that μ moves along the real axis in the λ plane around the segment $\lambda_2 \le \mu \le \lambda_3$, where $P(\mu) \le 0$, that is, around the forbidden gap in the spectrum of the corresponding linear equation $\psi_{xx} = -(u+\lambda)\psi$. On the contrary, in the NLS equation theory both variables u and μ are complex, the locus of the μ variable in the λ plane is not evident, and its determination is the problem which has to be solved before integration of Eq. (5.16). It is easy to see that this locus can be found by means of Eqs. (5.9) which has not been used yet. Indeed, these equations yield μ and μ^* as functions of $|u|^2$, so that $|u|^2$ can serve as a parameter along the 'trajectory' of μ in the λ plane. So we introduce the parameter

$$\nu = |u|^2 \tag{5.19}$$

and obtain from Eqs. (5.9)

$$\begin{aligned} &\mu + \mu^* = \tfrac{1}{2}s - q/\nu, \\ &\mu\mu^* = -(1/4\nu)\left[\nu^2 - 2\nu\left(p + \tfrac{1}{8}s^2\right) + sq + p^2 - 4r\right], \end{aligned} \tag{5.20}$$

where we have introduced the notations

$$\begin{aligned} &s = s_1, \quad p = s_2 - \tfrac{3}{8}s_1^2, \quad q = \tfrac{1}{2}s_1\left(s_2 - \tfrac{1}{4}s_1^2\right) - s_3, \\ &r = s_4 + \tfrac{1}{16}s_1^2\left(s_2 - \tfrac{3}{16}s_1^2\right) - \tfrac{1}{4}s_1 s_3. \end{aligned} \tag{5.21}$$

Thus, we see that μ and μ^* are the solutions of the quadratic equation with the coefficients given by expressions (5.20). The discriminant of this quadratic equation is equal to $\mathcal{R}(\nu)/\nu^2$, where

$$\mathcal{R}(\nu) = \nu^3 - 2p\nu^2 + \left(p^2 - 4r\right)\nu + q^2, \tag{5.22}$$

that is,

$$\mu, \mu^* = \tfrac{1}{4}s - [q \pm \sqrt{\mathcal{R}(\nu)}]/2\nu. \tag{5.23}$$

The polynomial $\mathcal{R}(\nu)$ is called the Ferrari cubic resolvent of the polynomial $P(\lambda)$ (see, e.g., van der Waerden, 1971). It appeared for the first time in solving in radicals the fourth degree algebraic equation $P(\lambda) = 0$. There is an important relationship between the zeros λ_i, $i = 1, 2, 3, 4$, of the polynomial $P(\lambda)$ and the zeros ν_i, $i = 1, 2, 3$, of the resolvent $\mathcal{R}(\nu)$, which can be found from the identity (5.7) (see Appendix B) and is expressed by the formulas

$$\begin{aligned} &\nu_1 = -\tfrac{1}{4}\left(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4\right)^2, \quad \nu_2 = -\tfrac{1}{4}\left(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4\right)^2, \\ &\nu_3 = -\tfrac{1}{4}\left(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4\right)^2. \end{aligned} \tag{5.24}$$

The quantity $\nu = |u|^2$ is real by definition and can oscillate between two positive zeros (5.24) of the resolvent $\mathcal{R}(\nu)$. Therefore, the zeros λ_i of the polynomial $P(\lambda)$ have to consist of two complex conjugate pairs

$$\lambda_1 = \alpha + i\gamma, \quad \lambda_2 = \beta + i\delta, \quad \lambda_3 = \alpha - i\gamma, \quad \lambda_4 = \beta - i\delta, \tag{5.25}$$

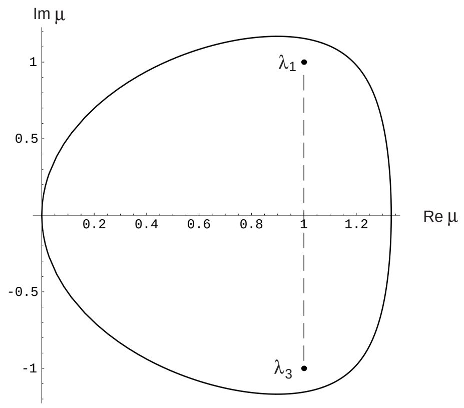

Fig. 5.1 The locus of μ in the complex plane λ for the periodic solution of the focusing NLS equation. The chosen values of the spectrum are equal to $\lambda_1 = 1 + i$, $\lambda_2 = 2 + 2i$, $\lambda_3 = 1 - i$, $\lambda_4 = 2 - 2i$. For different values of λ_i this curve deforms but its topological behaviour remains the same, i.e., it always goes around the two branching points λ_1 and λ_3.

and Eqs. (5.24) yield

$$\nu_1 = -\left(\alpha - \beta\right)^2, \quad \nu_2 = \left(\gamma - \delta\right)^2, \quad \nu_3 = \left(\gamma + \delta\right)^2. \tag{5.26}$$

This means that ν oscillates in the interval $\nu_2 \le \nu \le \nu_3$, where the resolvent $\mathcal{R}(\nu)$ is negative, and for the trajectory of μ we have

$$\begin{aligned} \mu\left(\nu\right) &= \tfrac{1}{4}s - [q + i\sqrt{-\mathcal{R}\left(\nu\right)}]/2\nu, \quad \nu_2 \le \nu \le \nu_3, \\ q &= \sqrt{-\nu_1\nu_2\nu_3}, \end{aligned} \tag{5.27}$$

and μ^* is the complex conjugate of this expression. The graph of this curve is shown in Fig. 5.1. It goes around the cut between two branching points λ_1 and λ_3 of the Riemann surface

$$l^2 = P\left(\lambda\right).$$

The evolution equation for $\nu = |u|^2$ can be obtained from the first equation (5.16) after substitution of the expression (5.27):

$$d\nu/d\xi = 2\sqrt{-\mathcal{R}(\nu)}. \tag{5.28}$$

Since $\mathcal{R}(\nu)$ is a cubic polynomial, the solution of the last equation is readily expressed in terms of the elliptic functions what gives us a simple expression for the intensity,

$$\begin{aligned}\nu = |u(x,t)|^2 &= \nu_3 + (\nu_2 - \nu_3)\,\mathrm{sn}^2\left(\sqrt{\nu_3-\nu_1}\,\xi, m\right) \\ &= (\gamma+\delta)^2 - 4\gamma\delta\,\mathrm{sn}^2\left(\sqrt{(\alpha-\beta)^2+(\gamma+\delta)^2}\,\xi, m\right),\end{aligned} \tag{5.29}$$

where the parameter m of the elliptic function equals to

$$m = (\nu_3-\nu_2)/(\nu_3-\nu_1) = 4\gamma\delta/[(\alpha-\beta)^2+(\gamma+\delta)^2], \tag{5.30}$$

and the initial phase ξ_0 is taken equal to zero so that the intensity is maximal at $x=0, t=0$.

For calculation of $u(x,t)$ it is convenient to pass to the Weierstrass elliptic functions. To this end, let us rewrite the resolvent in the form

$$\mathcal{R}(\nu) = -16 \times 4\,(\upsilon-e_1)(\upsilon-e_2)(\upsilon-e_3), \tag{5.31}$$

where

$$\begin{aligned}&\upsilon = \tfrac{1}{6}p - \tfrac{1}{4}\nu, \quad \nu = \tfrac{2}{3}p - 4\upsilon \quad p = \tfrac{1}{2}(\nu_1+\nu_2+\nu_3); \\ &e_1 = \tfrac{1}{12}(\nu_2+\nu_3-2\nu_1), \quad e_2 = \tfrac{1}{12}(\nu_1+\nu_3-2\nu_2), \\ &e_3 = \tfrac{1}{12}(\nu_1+\nu_2-2\nu_3).\end{aligned} \tag{5.32}$$

Substituting Eqs. (5.31) and (5.32) into Eq. (5.28), we find the equation

$$\frac{d\upsilon}{d(2\xi)} = -\sqrt{4(\upsilon-e_1)(\upsilon-e_2)(\upsilon-e_3)},$$

which gives

$$\upsilon = \wp(2\xi+c), \quad \nu = \tfrac{2}{3}p - 4\wp(2\xi+c). \tag{5.33}$$

The integration constant c is determined by the initial condition and we shall choose it as follows: $\nu=\nu_3$ (ν is maximal and υ is minimal) at $\xi=0$, i.e., $\wp(c)=e_3$ and, hence, $c=\omega'$ (ω and ω' are the half-periods of the

$\wp$-function; see Appendix A). On substitution of Eq. (5.33) into Eq. (5.27), we find μ as a function of ξ:

$$\mu(\xi) = \frac{s}{4} + \frac{q}{8}\frac{1}{\wp(2\xi+\omega') - p/6} - \frac{i}{2}\frac{1}{\nu}\frac{d\nu}{d(2\xi)}. \tag{5.34}$$

Then Eq. (5.18) gives

$$\widetilde{u}(\xi) = \sqrt{\nu}\exp\left[i\left(-\frac{s}{2}\xi + \frac{q}{8}\int_0^\xi \frac{d(2\xi)}{\wp(2\xi+\omega') - p/6}\right)\right]. \tag{5.35}$$

The integral can be easily calculated by means of the formula

$$\int_0^\xi \frac{1}{\wp(\xi) - \wp(\kappa)}d\xi = \frac{1}{\wp'(\kappa)}\left[\ln\frac{\sigma(\kappa-\xi)}{\sigma(\kappa+\xi)} + 2\zeta(\kappa)\xi\right], \tag{5.36}$$

(see Eq. (A.49)) where ζ and σ are the Weierstrass functions. Choosing κ according to

$$\wp(\kappa) = \tfrac{1}{6}p, \tag{5.37}$$

we find $\wp'(\kappa)$ from the differential equation for $\wp(\kappa)$:

$$\wp'(\kappa) = -\tfrac{1}{4}\sqrt{\nu_1\nu_2\nu_3} = -\tfrac{i}{4}q.$$

After simple calculation with the use of the identities (A.35) and (A.43), we obtain the periodic solution for the NLS equation in the form

$$\begin{aligned} u(x,t) =& 2\exp[-i(\alpha+\beta)x - 2i(\alpha^2+\beta^2-\gamma^2-\delta^2)t - 2\zeta(\kappa)\xi - \eta'\kappa] \\ &\times \frac{\sigma(2\xi+\kappa+\omega')}{\sigma(2\xi+\omega')\,\sigma(\kappa)}, \qquad \xi = x + 2(\alpha+\beta)t, \end{aligned} \tag{5.38}$$

where $\eta' = \zeta(\omega')$.

Let us investigate its soliton limit $m = 1$ corresponding to the following values of λ_i,

$$\lambda_1 = \lambda_2 = \alpha + i\gamma, \qquad \lambda_3 = \lambda_4 = \alpha - i\gamma. \tag{5.39}$$

It is convenient to find the limit of Eq. (5.38) in two steps. At first we take $\alpha = \beta$ so that

$$\nu_1 = 0, \quad \nu_2 = (\gamma-\delta)^2, \quad \nu_3 = (\gamma+\delta)^2,$$

and then $p = \frac{1}{2}(\nu_2 + \nu_3)$, $e_1 = \frac{1}{12}(\nu_2 + \nu_3)$. Hence, the condition (5.37) takes the form $\wp(\kappa) = e_1$ so that we can take

$$\kappa = -\omega, \quad \text{and, then,} \quad \zeta(\kappa) = -\eta.$$

Taking into account that (see Eq. (A.39))

$$-\frac{\sigma(2\xi + \omega' - \omega)}{\sigma(\omega)} = e^{-\eta(2\xi+\omega')}\sigma_1(2\xi + \omega'),$$

and also the identity (A.35), we transform Eq. (5.38) to

$$u(x,t) = 2\exp[-2i\alpha x - 2i(2\alpha^2 - \gamma^2 - \delta^2)t]\frac{\sigma_1(2\xi + \omega')}{\sigma(2\xi + \omega')}.$$

The ratio of σ-functions can be reduced to the Jacobi elliptic functions with the use of Eqs. (A.41) and (A.39),

$$\begin{aligned}\frac{\sigma_1}{\sigma} &= \frac{\sigma_1}{\sigma_3}\cdot\frac{\sigma_3}{\sigma} = \sqrt{e_1 - e_3}\cdot\frac{\mathrm{cn}(2\sqrt{e_1 - e_3}\,\xi + i\mathrm{K}', m)}{\mathrm{sn}(2\sqrt{e_1 - e_3}\,\xi + i\mathrm{K}', m)}\\ &= -i\sqrt{e_1 - e_3}\cdot\mathrm{dn}(2\sqrt{e_1 - e_3}\,\xi, m).\end{aligned}$$

Since in the case under consideration $\sqrt{e_1 - e_3} = \frac{1}{2}(\gamma + \delta)$, we obtain

$$\begin{aligned}u\,(x,t) = &(\gamma + \delta)\exp\left[-2i\alpha x - 2i\left(2\alpha^2 - \gamma^2 - \delta^2\right)t\right]\\ &\times \mathrm{dn}\left[(\gamma + \delta)\,\xi, 4\gamma\delta/(\gamma + \delta)^2\right],\end{aligned} \tag{5.40}$$

where $\xi = x + 4\alpha t$. Then in the limit $\delta = \gamma$ (see Eq. (A.17)) we arrive at the well-known soliton solution (1.368)

$$u\,(x,t) = 2\gamma\exp[-2i\alpha x - 4i\left(\alpha^2 - \gamma^2\right)t]\;\mathrm{sech}[2\gamma\,(x + 4\alpha t)]. \tag{5.41}$$

As another example, let us discuss the case of small wave modulations with $\delta \ll \gamma$. Now we have a small parameter

$$m \simeq 4\gamma\delta/[\gamma^2 + (\alpha - \beta)^2] \propto \delta/\gamma \ll 1.$$

Equation (5.29) for small δ and m gives

$$|u(x,t)|^2 \cong \gamma^2 + 2\gamma\delta\cos\left[2\sqrt{(\alpha - \beta)^2 + \gamma^2}\,(\xi + 2(\alpha + \beta)t)\right]. \tag{5.42}$$

Thus, the wavevector K and the frequency Ω of the small modulation wave are given by

$$K = 2\sqrt{(\alpha - \beta)^2 + \gamma^2}, \quad \Omega = -4(\alpha + \beta)\sqrt{(\alpha - \beta)^2 + \gamma^2}. \tag{5.43}$$

Elimination of β yields the well-known dispersion relation

$$\Omega = K\left(\sqrt{K^2 - 4\gamma^2} - 4\alpha\right). \tag{5.44}$$

Now let us consider derivation of the Whitham equations which govern the slow evolution of a modulated periodic solution of the NLS equation. As it was shown above, the parameters λ_i, $i = 1, 2, 3, 4$, are the slow varying Riemann invariants which obey the equations (see Eqs. (4.149))

$$\partial\lambda_i/\partial t + v_i\, \partial\lambda_i/\partial x = 0, \quad i = 1, 2, 3, 4, \tag{5.45}$$

where the characteristic speeds are given by Eqs. (4.150,4.151). In the NLS equation case we have

$$G = iu, \quad B = -u_x + 2i\lambda u, \quad g = iu\,(\lambda - \mu)\,.$$

With the use of the first equation (5.12) we find

$$G/g_0 = 1, \qquad B/g_0 = s_1 + 2\,(\lambda - \mu)\,, \tag{5.46}$$

and the integrals (4.151) are equal to

$$I_1\,(\lambda_i) = \oint \frac{d\mu}{(\lambda_i - \mu)\sqrt{P\,(\mu)}}, \quad I_2\,(\lambda_i) = s_1 I_1\,(\lambda_i) + 2L, \tag{5.47}$$

where

$$L = \oint \frac{d\mu}{\sqrt{P\,(\mu)}} \tag{5.48}$$

is the wavelength (multiplied by a constant factor inessential in this case) of the periodic solution. Using the obvious formula

$$I_1\,(\lambda_i) = -2\,\partial L/\partial\lambda_i,$$

we find that the characteristic speeds are equal to

$$v_i = -\left[s_1 - (\partial \ln L/\partial\lambda_i)^{-1}\right], \quad i = 1, 2, 3, 4. \tag{5.49}$$

If we express the wavelength (5.48) in terms of the complete elliptic integral of the first kind,

$$L = \frac{\mathrm{K}\,(m)}{\sqrt{(\lambda_1 - \lambda_4)\,(\lambda_2 - \lambda_3)}}, \tag{5.50}$$

and substitute it into Eq. (5.49), then we obtain the formulas

$$\begin{aligned}
v_1 &= -\sum \lambda_i - \frac{2\,(\lambda_1-\lambda_3)\,(\lambda_1-\lambda_2)\,\mathrm{K}\,(m)}{(\lambda_1-\lambda_2)\,\mathrm{K}\,(m)+(\lambda_2-\lambda_3)\,\mathrm{E}\,(m)},\\
v_2 &= -\sum \lambda_i - \frac{2\,(\lambda_1-\lambda_2)\,(\lambda_2-\lambda_4)\,\mathrm{K}\,(m)}{(\lambda_1-\lambda_2)\,\mathrm{K}\,(m)-(\lambda_1-\lambda_4)\,\mathrm{E}\,(m)},\\
v_3 &= -\sum \lambda_i + \frac{2\,(\lambda_1-\lambda_3)\,(\lambda_3-\lambda_4)\,\mathrm{K}\,(m)}{(\lambda_3-\lambda_4)\,\mathrm{K}\,(m)-(\lambda_1-\lambda_4)\,\mathrm{E}\,(m)},\\
v_4 &= -\sum \lambda_i + \frac{2\,(\lambda_2-\lambda_4)\,(\lambda_3-\lambda_4)\,\mathrm{K}\,(m)}{(\lambda_3-\lambda_4)\,\mathrm{K}\,(m)+(\lambda_2-\lambda_3)\,\mathrm{E}\,(m)}.
\end{aligned} \tag{5.51}$$

Note that Eq. (5.49) can be written in the universal form

$$v_i = (1 + (k/\partial_i k)\,\partial_i)\,V, \tag{5.52}$$

(see Eq. (3.150)), where $\partial_i \equiv \partial/\partial\lambda_i$, $V = -s_1 = -\sum \lambda_i$ is the phase velocity (see Eq. (5.15)), and $k = 2\pi/L$ is the wavenumber.

5.2 Defocusing NLS equation

For the defocusing NLS equation

$$iu_t + u_{xx} - 2|u|^2 u = 0 \tag{5.53}$$

all calculations are similar to those for the focusing case. Now the matrix elements of $\mathbb{U}$ and $\mathbb{V}$ are equal to (see Eqs. (4.160))

$$\begin{aligned}
&F = -i\lambda, \quad G = iu, \quad H = -iu^*;\\
&A = -2i\lambda^2 - i|u|^2, \quad B = 2iu\lambda - u_x, \quad C = -2iu^*\lambda - u_x^*,
\end{aligned} \tag{5.54}$$

and the solution of Eqs. (4.140) and (4.141) has the form

$$f = \lambda^2 - f_1\lambda + f_2, \quad g = iu(\lambda-\mu), \quad h = -iu^*(\lambda-\mu^*), \tag{5.55}$$

provided

$$\begin{aligned}
&f_{1,x} = f_{1,t} = 0, \quad f_{2,x} = -i|u|^2(\mu-\mu^*),\\
&f_{2,t} = u_x^* u\mu + u_x u^* \mu^*;
\end{aligned} \tag{5.56}$$

and

$$\begin{aligned}
u_x = 2iu(\mu - f_1), &\quad u_t = -2i|u|^2 u + 2f_1 u_x + 4iuf_2,\\
(u\mu)_x = -2iuf_2, &\quad (u\mu)_t = -2i|u|^2 u\mu + 2f_2 u_x.
\end{aligned} \tag{5.57}$$

Then the identity $f^2 - gh = P(\lambda)$ leads to the relations

$$f_1 = \tfrac{1}{2}s_1, \quad f_2 = \tfrac{1}{2}(s_2 - \tfrac{1}{4}s_1^4 - |u|^2) \tag{5.58}$$

and

$$u_x = 2iu(\mu - \tfrac{1}{2}s_1), \quad u_t = 2i(s_2 - \tfrac{1}{4}s_1^2)u + s_1 u_x. \tag{5.59}$$

Comparison with Eqs. (5.7,5.8,5.9) shows that now the locus of μ is given by the equation

$$\mu = \tfrac{1}{4}s - [q \pm \sqrt{\mathcal{R}\,(-\nu)}]/2\nu \tag{5.60}$$

and, correspondingly, the intensity $\nu = |u|^2$ obeys the equation

$$d\nu/d\xi = 2\sqrt{-\mathcal{R}\,(-\nu)}, \tag{5.61}$$

where $\xi = x + s_1 t$. Now the roots $-\nu_i$ of the polynomial $\mathcal{R}(-\nu)$ must be positive, hence, according to Eqs. (5.24), the parameters λ_i must be real. Let us enumerate λ_i so that $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$, introduce for convenience $\bar{\nu}_i = -\nu_i$, and change enumeration of $\bar{\nu}_i$ so that inequalities $\bar{\nu}_1 \le \bar{\nu}_2 \le \bar{\nu}_3$ hold, that is,

$$\begin{aligned} &\bar{\nu}_1 = \tfrac{1}{4}\left(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4\right)^2, \quad \bar{\nu}_2 = \tfrac{1}{4}\left(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4\right)^2, \\ &\bar{\nu}_3 = \tfrac{1}{4}\left(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4\right)^2. \end{aligned} \tag{5.62}$$

Equation (5.61) is solved in a usual way (see Sec. 1.5.2 and Appendix A)

$$\begin{aligned} \nu = |u\,(x,t)|^2 &= \bar{\nu}_1 + (\bar{\nu}_2 - \bar{\nu}_1)\,\mathrm{sn}^2\left(\sqrt{\bar{\nu}_3 - \bar{\nu}_1}\,\xi, m\right) \\ &= \tfrac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2 \\ &\quad + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)\,\mathrm{sn}^2\left(\sqrt{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)}\,\xi, m\right), \end{aligned} \tag{5.63}$$

where $\xi = x - Vt$, $V = -s_1 = -\sum \lambda_i$ is the phase velocity, and m equals to

$$m = \frac{\bar{\nu}_2 - \bar{\nu}_1}{\bar{\nu}_3 - \bar{\nu}_1} = \frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}. \tag{5.64}$$

The soliton limit corresponds to $m = 1$, that is $\bar{\nu}_2 = \bar{\nu}_3$ or $\lambda_2 = \lambda_3$. With the use of Eq. (A.17) we obtain after simple transformations

$$|u(x,t)|^2 = \tfrac{1}{4}(\lambda_4 - \lambda_1)^2 - \frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}{\cosh^2\left[\sqrt{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}\,\xi\right]}, \tag{5.65}$$

where $\xi = x + (\lambda_1 + 2\lambda_2 + \lambda_4)t$. If we define parameters u_0 and ϕ by the relations

$$u_0 = \tfrac{1}{2}(\lambda_4 - \lambda_1), \quad \cos\phi = 2\sqrt{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}/(\lambda_4 - \lambda_1) \le 1,$$

then the soliton solution takes the form

$$|u(x,t)|^2 = u_0^2 \left(1 - \frac{\cos^2\phi}{\cosh^2[u_0\cos\phi(x + (2u_0\sin\phi + 4\lambda_2)t)]}\right). \tag{5.66}$$

It reduces to Eq. (1.360) at $\lambda_2 = 0$. The difference between Eqs. (1.360) and (5.66) is caused by the fact that ansatz (1.350) corresponds to the modulation of the standing wave $\propto \exp(-2iu_0^2 t)$, whereas the solution (5.66) corresponds to a more general soliton propagating on the background of the travelling wave.

Formulas for $u(x,t)$ can be found in the same way as was used for the focusing NLS equation and we shall not consider here this calculation. The Whitham equations coincide exactly with those for the focusing case, but now the Riemann invariants λ_i are real variables.

5.3 Self-induced transparency

Equations of the self-induced transparency (see Sec. 2.5)

$$\mathcal{E}_\zeta = d, \quad d_\tau + 2i\Delta d = \mathcal{E}n, \quad n_\tau = -\tfrac{1}{2}(\mathcal{E}d^* + \mathcal{E}^* d), \tag{5.67}$$

describe the propagation of the electromagnetic pulse with the electric field envelope $\mathcal{E}(\zeta,\tau)$ through a resonant medium of two-level atoms in the direction of the $\zeta = x$ axis, Δ is the frequency offset parameter of the atomic transitions from the oscillation frequency of the electromagnetic field of the wave, and d and n are the dipole moment of the transitions (polarization) and the population of the atoms, respectively. They are related by the normalization condition

$$|d|^2 + n^2 = 1, \tag{5.68}$$

which reflects the conservation of probability—the total probability that an atom can be found in the upper or lower level is equal to unity. We neglect in Eqs. (5.67) the so called inhomogeneous line broadening and assume that all atoms have the same frequency shift Δ. The systems (5.67) is written in the variables

$$\zeta = x, \quad \tau = t - x, \tag{5.69}$$

where x and t are the dimensionless spatial coordinate and time.

According to the Sec. 4.2.2, Eqs. (5.67) can be written as a compatibility condition of the linear systems (4.163) with the matrix elements

$$\begin{aligned} &F = -i\lambda, \qquad G = \mathcal{E}/2, \qquad H = -\mathcal{E}^*/2; \\ &A = \tfrac{i}{4}n/(\lambda - \Delta), \quad B = -\tfrac{i}{4}d/(\lambda - \Delta), \quad C = -\tfrac{i}{4}d^*/(\lambda - \Delta). \end{aligned} \tag{5.70}$$

As usual, we introduce the 'squared basis functions' f, g, h whose evolution on ζ and τ is governed by the equations

$$\begin{aligned} &f_\tau = -iHg + iGh, \quad g_\tau = 2iGf + 2Fg, \quad h_\tau = -2iHf - 2Fh, \\ &f_\zeta = -iCg + iBh, \quad g_\zeta = 2iBf + 2Ag, \quad h_\zeta = -2iCf - 2Ah. \end{aligned} \tag{5.71}$$

As we know, the quantity

$$f^2 - gh = P(\lambda) \tag{5.72}$$

does not change during this evolution.

Periodic solution is distinguished by the condition that $P(\lambda)$ be a polynomial in λ. Therefore we look for the solution of Eqs. (5.71) in the polynomial form and find

$$\begin{aligned} &f = \lambda^2 - \Delta^2 - f_1(\lambda - \Delta) + an, \\ &g = \tfrac{1}{2}\mathcal{E}(\lambda - \mu), \quad h = -\tfrac{1}{2}\mathcal{E}^*(\lambda - \mu^*), \end{aligned} \tag{5.73}$$

where

$$\mu = 2ad/i\mathcal{E} + \Delta, \tag{5.74}$$

and f_1 and a are constants which can be expressed in terms of λ_i with the use of Eq. (5.72):

$$f_1 = \tfrac{1}{2}s_1 = \tfrac{1}{2}\sum_{i=1}^{4}\lambda_i, \quad a^2 = \prod_{i=1}^{4}(\Delta - \lambda_i) = P(\Delta), \quad a = \sqrt{P(\Delta)}. \tag{5.75}$$

The variable μ depends only on the phase

$$\xi = \tau - \zeta/V, \quad V = 4a = 4\sqrt{P(\Delta)}, \tag{5.76}$$

and is a solution of the equation

$$d\mu/d\xi = -2i\sqrt{P(\mu)}. \tag{5.77}$$

In variables (5.69) the phase has the form $\xi = t-(1+1/V)x$ and the 'real' velocity of the wave $V/(1+V)$ must be less than unity (the speed of light) what determines the sign before the root in Eqs. (5.75) and (5.76).

If μ is known, then the field $\mathcal{E}$ can be found with the help of equations which follow from

$$\begin{aligned}
&\partial\mathcal{E}/\partial\tau = -2if_1\mathcal{E} + 2i\mu\mathcal{E},\\
&\partial\mathcal{E}/\partial\zeta = d = (i(\mu-\Delta)/2a)\,\mathcal{E} = -(i\Delta/2a)\mathcal{E} + (1/4a)2i\mu\mathcal{E},
\end{aligned}$$

so that

$$\mathcal{E} = \exp(-is_1\tau - (i\Delta/2a)\zeta)\widetilde{\mathcal{E}}, \tag{5.78}$$

where $\widetilde{\mathcal{E}}$ satisfies the equation

$$d\widetilde{\mathcal{E}}/d\xi = 2i\mu\widetilde{\mathcal{E}}. \tag{5.79}$$

We take the quantity

$$\nu = |\mathcal{E}|^2/4 \tag{5.80}$$

as a parameter along the locus of μ. The constraint (5.72) after substitution of Eq. (5.73) takes the form

$$\begin{aligned}
&[(\lambda-\Delta)^2 - (f_1-2\Delta)(\lambda-\Delta) + an]^2 + \nu[(\lambda-\Delta)^2 - (\mu+\mu^*-2\Delta)(\lambda-\Delta)\\
&\quad + (\mu-\Delta)(\mu^*-\Delta)] = \prod_{i=1}^{4}[\lambda-\Delta-(\lambda_i-\Delta)]\\
&= (\lambda-\Delta)^4 - \tilde{s}_1(\lambda-\Delta)^3 + \tilde{s}_2(\lambda-\Delta)^2 - \tilde{s}_3(\lambda-\Delta) + \tilde{s}_4,
\end{aligned} \tag{5.81}$$

where $\tilde{s}_i$ can be expressed according to the Viète formulas in terms of $\lambda_i-\Delta$ in the same way as s_i are expressed in terms of λ_i; in particular, $\tilde{s}_4 = \prod(\lambda_i-\Delta) = a^2 = P(\Delta)$. Comparing coefficients of the $(\lambda_i-\Delta)^k$, we obtain the system

$$\begin{aligned}
&2(f_1-2\Delta) = \tilde{s}_1 = s_1 - 4\Delta,\\
&(f_1-2\Delta)^2 + 2an + \nu = \tilde{s}_2,\\
&2(f_1-2\Delta)an + \nu(\mu+\mu^*-2\Delta) = \tilde{s}_3,\\
&(an)^2 + \nu(\mu-\Delta)(\mu^*-\Delta) = \tilde{s}_4,
\end{aligned} \tag{5.82}$$

which coincides with the corresponding system in the preceding sections. Its solution has the form

$$\begin{aligned} f_1 = \tfrac{1}{2}s, \quad an = \left(\tfrac{1}{4}s - \Delta\right)^2 + \tfrac{1}{2}(p - \nu), \\ \mu = \tfrac{1}{4}s - [q + i\sqrt{-\mathcal{R}(\nu)}]/2\nu, \end{aligned} \tag{5.83}$$

where all notations coincide with those used earlier. In writing Eq. (5.83) we have made use of that p, q, and r do not change when the zeros λ_i are shifted by the constant term Δ. This also ensures that μ and $\nu = |\mathcal{E}|^2/4$ are independent of the frequency offset parameter Δ.

As in the preceding sections, we find that intensity ν satisfies the equation

$$d\nu/d\xi = 2\sqrt{-\mathcal{R}(\nu)}, \tag{5.84}$$

and its solution gives a simple expression for the intensity,

$$\begin{aligned} |\mathcal{E}|^2 = 4\nu &= 4\left[\nu_3 + (\nu_2 - \nu_3)\,\mathrm{sn}^2\left(\sqrt{\nu_3 - \nu_1}\,\xi, m\right)\right] \\ &= 4\,(\gamma + \delta)^2 - 16\gamma\delta\,\mathrm{sn}^2\left(\sqrt{(\alpha - \beta)^2 + (\gamma + \delta)^2}\;\xi, m\right), \end{aligned} \tag{5.85}$$

where the parameter m of elliptic functions equals to

$$m = (\nu_3 - \nu_2)/(\nu_3 - \nu_1) = 4\gamma\delta/[(\alpha - \beta)^2 + (\gamma + \delta)^2]. \tag{5.86}$$

The field itself can be expressed in terms of the elliptic functions, too. We shall write down here only the soliton solution for $\alpha = \beta$, $\gamma = \delta$, when we obtain

$$\mathcal{E}(\tau, \xi) = 4\gamma \exp\left(-2i\alpha\tau - \frac{i\zeta}{2}\frac{\alpha - \Delta}{(\alpha - \Delta)^2 + \gamma^2}\right)\mathrm{sech}[2\gamma(\tau - \zeta/V)], \tag{5.87}$$

where the soliton velocity equals to

$$V = 4[(\alpha - \Delta)^2 + \gamma^2]. \tag{5.88}$$

This is a famous 2π-pulse of the self-induced transparency discovered by McCall and Hahn (1969).

The Whitham equations can be found with the use of the generating function

$$(\partial/\partial\zeta)(G/g) - (\partial/\partial\tau)(B/g) = 0, \tag{5.89}$$

On substitution of Eqs. (5.70) and (5.73) into Eq. (5.89), we obtain

$$\frac{\partial}{\partial\zeta}\frac{\sqrt{P(\lambda)}}{\lambda-\mu}+\frac{\partial}{\partial\tau}\left(\sqrt{P(\lambda)}\cdot\frac{\mu-\Delta}{4\sqrt{P(\Delta)}\,(\lambda-\mu)\,(\lambda-\Delta)}\right)=0, \qquad (5.90)$$

where the normalization condition $f^2 - gh = 1$ was used. Now we average Eq. (5.90) over fast oscillations with the period defined by the expression

$$\frac{2\pi}{\omega}=\frac{1}{2}\oint\frac{d\mu}{\sqrt{-P(\mu)}}. \qquad (5.91)$$

Again the condition for vanishing of the singular terms which arise as a result of differentiation of $\sqrt{P(\lambda)}$ with respect to ζ and τ gives the equations

$$\overline{\frac{1}{\lambda_i-\mu}}\cdot\frac{\partial\lambda_i}{\partial\zeta}+\left[\frac{1}{4\sqrt{P(\Delta)}}\cdot\overline{\frac{1}{\lambda_i-\mu}}-\frac{1}{4\sqrt{P(\Delta)}\,(\lambda_i-\Delta)}\right]\frac{\partial\lambda_i}{\partial\tau}=0.$$

Thus, we obtain the Whitham equations in the diagonal Riemann form

$$\partial\lambda_i/\partial\zeta+(1/v_i)(\partial\lambda_i/\partial\tau)=0, \qquad (5.92)$$

where the characteristic velocities are given by

$$\frac{1}{v_i}=\left[\frac{1}{4\sqrt{P(\Delta)}}+\frac{1}{4\sqrt{P(\Delta)}\,(\lambda_i-\Delta)}\cdot\frac{1}{2}\left(\frac{\partial\ln\omega}{\partial\lambda_i}\right)^{-1}\right]. \qquad (5.93)$$

Taking into account the relations (see Eq. (5.76))

$$\frac{1}{4\sqrt{P(\Delta)}}=\frac{1}{V},\qquad \frac{1}{4\sqrt{P(\Delta)}\,(\lambda_i-\Delta)}=-2\frac{\partial}{\partial\lambda_i}\frac{1}{V},$$

we rewrite Eq. (5.93) in a simple form

$$1/v_i=(1+(\omega/\partial_i\omega)\,\partial_i)\,(1/V). \qquad (5.94)$$

On introducing the wavevector $k=\omega/V$ of the nonlinear wave, we return to the universal formula (3.150) with V given by Eq. (5.76).

5.4 The Heisenberg model

According to Sec. 2.4, a local magnetic moment in the Heisenberg model is described by the vector $\mathbf{S}=(S_1,S_2,S_3)$ whose evolution is governed by the isotropic Landau-Lifshitz equation

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \tag{5.95}$$

where the vector $\mathbf{S}$ is normalized by the condition $\mathbf{S}^2 = 1$. As usual, we consider a one-dimensional wave propagating along the x axis.

As was indicated in Sec. 4.2.2, Eq. (5.95) is a compatibility condition of the linear systems (4.136) with the matrix elements given by

$$F = -\tfrac{1}{2}i\lambda S_3, \quad G = -\tfrac{1}{2}i\lambda S_-, \quad H = -\tfrac{1}{2}i\lambda S_+ \tag{5.96}$$

and

$$\begin{aligned} A &= \tfrac{1}{2}i\lambda^2 S_3 + \tfrac{1}{4}\lambda\left[(S_-)_x S_+ - S_-(S_+)_x\right], \\ B &= \tfrac{1}{2}i\lambda^2 S_- + \tfrac{1}{2}\lambda\left[(S_3)_x S_- - S_3(S_-)_x\right], \\ C &= \tfrac{1}{2}i\lambda^2 S_+ - \tfrac{1}{2}\lambda\left[(S_3)_x S_+ - S_3(S_+)_x\right], \end{aligned} \tag{5.97}$$

where $S_\pm = S_1 \pm iS_2$, $(S_3)_x = \partial S_3/\partial x$, etc.

The 'squared basis functions' (4.139), whose evolution is governed by the Eqs. (4.140), (4.141), now have the form

$$\begin{aligned} &f = S_3\lambda^2 - f_1\lambda + f_2, \quad g = -iS_-\lambda(\lambda - \mu), \\ &h = -iS_+\lambda(\lambda - \mu^*), \end{aligned} \tag{5.98}$$

and they must satisfy the condition (4.142) where $P(\lambda)$ is again a fourth degree polynomial (4.143). Comparing the coefficients of λ^k on both sides of the identity $f^2 - gh = P(\lambda)$ leads to equations

$$\begin{aligned} &2S_3 f_1 + \left(1 - S_3^2\right)(\mu + \mu^*) = s_1, \\ &f_1^2 + 2S_3 f_2 + \left(1 - S_3^2\right)\mu\mu^* = s_2, \\ &2f_1 f_2 = s_3, \quad f_2^2 = s_4. \end{aligned} \tag{5.99}$$

From the last two equations we find at once that f_1 and f_2 are constant,

$$f_1 = s_3/(2\sqrt{s_4}), \quad f_2 = \sqrt{s_4}. \tag{5.100}$$

It is easy to show that μ depends only on the phase

$$\xi = x - Vt, \quad V = \tfrac{1}{2}s_1, \quad d\mu/d\xi = -i\sqrt{P(\mu)}. \tag{5.101}$$

From Eqs. (4.140) and (4.141) we find also

$$\partial S_3/\partial x = -\tfrac{i}{2}S_+S_-(\mu - \mu^*), \tag{5.102}$$

and

$$\partial S_-/\partial x = -i\left(f_1 - \mu S_3\right)S_-, \quad \partial S_-/\partial t = -if_2 S_- - \tfrac{1}{2}s_1\,\partial S_-/\partial x.$$

The last two equations lead to

$$S_- = \exp\left(-i\sqrt{s_4}\,t\right)\tilde{S}_-, \tag{5.103}$$

provided $\tilde{S}_-$ satisfies the equation

$$d\tilde{S}_-/d\xi = -i\left(f_1 - \mu S_3\right)\tilde{S}_-. \tag{5.104}$$

The first two equations (5.99) determine the locus of μ. We take

$$\nu = 2\sqrt{s_4}\,S_3 \tag{5.105}$$

as a parameter along it so that the solution of these equations yields

$$\mu\left(\nu\right) = \frac{2s_1 s_4 - s_3\nu + 2i\sqrt{s_4}\sqrt{\mathcal{R}\left(\nu\right)}}{4s_4 - \nu^2}, \tag{5.106}$$

where

$$\mathcal{R}\left(\nu\right) = \nu^3 - s_2\nu^2 + \left(s_1 s_3 - 4s_4\right)\nu + 4s_2 s_4 - s_1^2 s_4 - s_3^2 \tag{5.107}$$

is again the cubic resolvent of the polynomial $P(\lambda)$. The zeros ν_1, ν_2, ν_3 of the resolvent (5.107) are related to the zeros $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the polynomial $P(\lambda)$ by the formulas (see Appendix B)

$$\nu_1 = \lambda_1\lambda_3 + \lambda_2\lambda_4, \quad \nu_2 = \lambda_1\lambda_4 + \lambda_2\lambda_3, \quad \nu_3 = \lambda_1\lambda_2 + \lambda_3\lambda_4. \tag{5.108}$$

When λ_i are equal to

$$\lambda_1 = \alpha + i\gamma, \quad \lambda_2 = \beta + i\delta, \quad \lambda_3 = \alpha - i\gamma, \quad \lambda_4 = \beta - i\delta, \tag{5.109}$$

these formulas give

$$\nu_1 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2, \quad \nu_2 = 2\left(\alpha\beta + \gamma\delta\right), \quad \nu_3 = 2\left(\alpha\beta - \gamma\delta\right). \tag{5.110}$$

This means that ν oscillates in the interval $\nu_3 \le \nu \le \nu_2$, where $|S_3| \le 1$ and $\mathcal{R}(\nu) \ge 0$. The locus of μ is shown in Fig. 5.2.

From Eqs. (5.102), (5.105), and (5.106) we find the equation for ν:

$$d\nu/d\xi = \sqrt{\mathcal{R}\left(\nu\right)}. \tag{5.111}$$

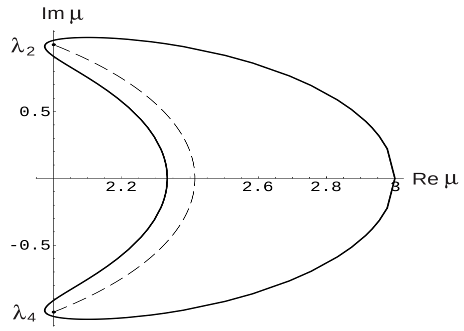

Fig. 5.2 The locus of μ in the complex plane λ for the periodic solution of the Heisenberg model equation. The chosen values of the spectrum are equal to $\lambda_1 = 1+i$, $\lambda_2 = 2+2i$, $\lambda_3 = 1-i$, $\lambda_4 = 2-2i$. Now the curve goes around the cut between two branching points λ_2 and λ_4, and it is topologically equivalent to the contour around the cut between the branching points λ_1 and λ_3 (cf. Fig. 5.1.).

Its solution gives us the expression for the third component S_3,

$$S_3 = \frac{\alpha\beta - \gamma\delta + 2\gamma\delta\,\mathrm{sn}^2\left[\sqrt{(\alpha-\beta)^2+(\gamma+\delta)^2}\,\xi, m\right]}{\sqrt{(\alpha^2+\gamma^2)(\beta^2+\delta^2)}}, \tag{5.112}$$

where $m = (\nu_3-\nu_2)/(\nu_3-\nu_1) = 4\gamma\delta/[(\alpha-\beta)^2+(\gamma+\delta)^2]$ and $\nu=\nu_3$ at $\xi = 0$.

For calculation of S_- it is convenient to pass to the Weierstrass elliptic functions:

$$\nu = \nu_3 + \frac{(\nu_1-\nu_3)(\nu_2-\nu_3)}{4\left[\wp(\xi)-e_3\right]},$$

where

$$e_1 = \tfrac{1}{12}(2\nu_1-\nu_2-\nu_3), \quad e_2 = \tfrac{1}{12}(2\nu_2-\nu_1-\nu_3),$$
$$e_3 = \tfrac{1}{12}(2\nu_3-\nu_1-\nu_2).$$

From (5.104) we find

$$\tilde{S}_- = \sqrt{1-S_3^2}\,\exp\left(i\sqrt{s_4}\int_0^\xi \frac{s_1\nu-2s_3}{4s_4-\nu^2}d\xi\right).$$

Calculation analogous to the NLS case yields

$$S_-(x,t) = S_1 - iS_2 = \frac{\alpha\beta + \gamma\delta}{\sqrt{(\alpha^2+\gamma^2)(\beta^2+\delta^2)}}$$
$$\exp\left(-\frac{i\sqrt{(\alpha^2+\gamma^2)(\beta^2+\delta^2)}}{\alpha\delta+\beta\gamma}\left[(\gamma+\delta)x + (\alpha\gamma+\beta\delta)t\right] - \left[\zeta(\kappa)+\zeta(\tilde\kappa)\right]\xi \right)$$
$$\times \frac{\sigma(\xi+\kappa)\,\sigma(\xi+\tilde\kappa)}{\sigma(\kappa)\,\sigma(\tilde\kappa)\,\sigma_3^2(\xi)} \quad (5.113)$$

where $\xi = x - (\alpha+\beta)t$ and the parameters κ and $\tilde\kappa$ are determined by the equations

$$\begin{aligned} \wp(\kappa) &= e_3 + \frac{(\nu_1-\nu_3)(\nu_2-\nu_3)}{4\left(2\sqrt{s_4}-\nu_3\right)}, \\ \wp(\tilde\kappa) &= e_3 - \frac{(\nu_1-\nu_3)(\nu_2-\nu_3)}{4\left(2\sqrt{s_4}+\nu_3\right)}. \end{aligned} \quad (5.114)$$

Let us consider the soliton limit of the general periodic solution, when $m=1$, that is,

$$\begin{aligned} &\lambda_1 = \lambda_2 = \alpha + i\gamma, \quad \lambda_3 = \lambda_4 = \alpha - i\gamma, \\ &e_1 = e_2 = \tfrac{1}{3}\gamma^2, \quad e_3 = -\tfrac{2}{3}\gamma^2. \end{aligned} \quad (5.115)$$

Formula (5.112) gives at once

$$S_3(x,t) = 1 - [2\gamma^2/(\alpha^2+\gamma^2)]\,\mathrm{sech}^2\left[\gamma(x-2\alpha t)\right]. \quad (5.116)$$

It is convenient to find the corresponding limit of Eq. (5.113) in two steps. Let us, at first, put $\alpha^2+\gamma^2 = \beta^2+\delta^2$. Then from the first equation (5.114) we get $\wp(\kappa) = e_2$, hence, $\kappa = \omega_2$. After that the second equation (5.114) with λ_i given by Eqs. (5.115) gives the relations

$$\sinh(\gamma\tilde\kappa) = -i\alpha/\sqrt{\alpha^2+\gamma^2}, \quad \cosh(\gamma\tilde\kappa) = \gamma/\sqrt{\alpha^2+\gamma^2},$$

by means of which the expression (5.113) can be transformed to

$$S_-(x,t) = \frac{i\gamma\exp\left\{-i\left[\alpha x - (\alpha^2-\gamma^2)t\right]\right\}}{\alpha^2+\gamma^2}\cdot\frac{\gamma\sinh(\gamma\xi) - i\alpha\cosh(\gamma\xi)}{\cosh^2(\gamma\xi)}, \quad (5.117)$$

where $\xi = x - 2\alpha t$.

To find the Whitham equations, let us transform B from Eq. (5.97) with the help of Eq. (5.102) to the form

$$B = \tfrac{1}{2} i\lambda S_- \left(\tfrac{1}{2}s_1 + \lambda - \mu\right), \tag{5.118}$$

that is,

$$G/g_0 = \tfrac{1}{2}, \qquad B/g_0 = -\tfrac{1}{2}\left[\tfrac{1}{2}s_1 + (\lambda - \mu)\right], \tag{5.119}$$

and these expressions actually coincide with Eqs. (5.46) for the NLS equation case. The characteristic speeds are equal to

$$v_i = \tfrac{1}{2}\left[\sum \lambda_i - (\partial \ln L/\partial\lambda_i)^{-1}\right], \quad i = 1, 2, 3, 4, \tag{5.120}$$

and differ from Eq. (5.49) by the factor $-1/2$. They can be written in the universal form (3.150),

$$v_i = (1 + (k/\partial_i k)\,\partial_1)\,V, \quad i = 1, 2, 3, 4, \tag{5.121}$$

with the phase velocity $V = \frac{1}{2}s_1 = \frac{1}{2}\sum \lambda_i$.

5.5 Uniaxial ferromagnet (easy axis case)

A wave propagating along the x axis in a uniaxial ferromagnet is described by the Landau-Lifshitz equation (see Sec. 2.4)

$$\mathbf{S}_t = \mathbf{S}\times\mathbf{S}_{xx} + J\,(\mathbf{S}\cdot\mathbf{n})\,(\mathbf{S}\times\mathbf{n}), \tag{5.122}$$

where $\mathbf{S}\,(x,t)$, as in the preceding section, is the local magnetization normalized by the condition of unit length ($\mathbf{S}^2 = 1$), J is the anisotropy constant, and $\mathbf{n}$ the unit vector along the x axis (the easiest magnetization axis). Let us find the periodic solution of Eq. (5.122). This equation can be presented as a compatibility condition of systems (4.136) with the coefficients (4.167,4.168),

$$\begin{aligned}
&F = -\tfrac{i}{2}\lambda S_3, \quad G = -\tfrac{i}{2}\sqrt{\lambda^2 + J}\,S_-, \quad H = -\tfrac{i}{2}\sqrt{\lambda^2 + J}\,S_+;\\
&A = \tfrac{i}{2}\left(\lambda^2 + J\right)S_3 + \tfrac{1}{4}\lambda\left[(S_-)_x\,S_+ - S_-\,(S_+)_x\right],\\
&B = \tfrac{i}{2}\lambda\sqrt{\lambda^2 + J}\,S_- + \tfrac{1}{2}\sqrt{\lambda^2 + J}\left[(S_3)_x\,S_- - S_3\,(S_-)_x\right],\\
&C = \tfrac{i}{2}\lambda\sqrt{\lambda^2 + J}\,S_+ - \tfrac{1}{2}\sqrt{\lambda^2 + J}\left[(S_3)_x\,S_+ - S_3\,(S_+)_x\right].
\end{aligned}$$

The systems (4.140) and (4.141) are satisfied by

$$\begin{aligned} &f = S_3\lambda^2 - f_1\lambda + f_2, \quad g = -iS_-\sqrt{\lambda^2+J}\,(\lambda-\mu)\,, \\ &h = -iS_+\sqrt{\lambda^2+J}\,(\lambda-\mu^*)\,, \end{aligned} \tag{5.123}$$

and the condition $f^2 - gh = P(\lambda)$ leads to the equations

$$\begin{aligned} &2f_1S + \left(1-S^2\right)(\mu+\mu^*) = s_1, \\ &2f_1f_2 + \left(1-S^2\right)J\,(\mu+\mu^*) = s_3, \\ &f_1^2 + 2f_2S + \left(1-S^2\right)(J+\mu\mu^*) = s_2, \\ &f_2^2 + \left(1-S^2\right)J\mu\mu^* = s_4, \end{aligned} \tag{5.124}$$

where $S \equiv S_3$. The equation for μ coincides with that for the isotropic case,

$$d\mu/d\xi = \sqrt{-P\,(\mu)}, \tag{5.125}$$

where $\xi = x - \frac{1}{2}s_1t$, and the components of vector $\mathbf{S}$ satisfy the equations

$$\begin{aligned} \partial S/\partial x &= -\tfrac{i}{2}\left(1-S^2\right)(\mu-\mu^*)\,, \\ \partial S_-/\partial x &= -i\,(f_1-\mu S)\,S_-, \\ \partial S_-/\partial t &= -i\,(f_2-JS)\,S_- - \tfrac{1}{2}s_1\partial S_-/\partial x. \end{aligned} \tag{5.126}$$

From the first and the second pairs of Eqs. (5.124) we obtain

$$\begin{aligned} &2f_1\,(S - f_2/J) = s_1 - s_3/J, \\ &f_1^2/J - (S - f_2/J) = (s_2 - s_4/J)/J - 1, \end{aligned}$$

which give

$$f_2 = (s_3 - s_1J)/2f_1 + JS \tag{5.127}$$

and

$$f_1^2 = (1/2J)\left[\sqrt{P_2\,(J)} - J^2 + s_2J - s_4\right], \tag{5.128}$$

where

$$P_2\,(J) = \prod_{i=1}^{4}\left(J+\lambda_i^2\right). \tag{5.129}$$

Taking into account Eq. (5.127), we transform the last equation (5.126) to the form

$$\partial S_-/\partial t = -i[(s_3 - s_1 J)/2f_1]\, S_- - (s_1/2)\partial S_-/\partial x,$$

which together with the other Eqs. (5.126) yield

$$S_- = \exp\left[it(s_1 J - s_3)/2f_1\right] \tilde{S}_-\left(\xi\right), \tag{5.130}$$

where $\tilde{S}$ satisfies the equation

$$d\tilde{S}_-/d\xi = -i\left(f_1 - \mu S\right)\tilde{S}_-. \tag{5.131}$$

As a parameter along the locus of μ in the complex plane we take $S \equiv S_3$. Then it follows from Eqs. (5.124), (5.127), (5.128) that

$$\mu + \mu^* = (s_1 - 2f_1 S)/(1 - S^2),$$

$$\mu\mu^* = \frac{s_2 - f_1^2 - \left[\left(s_3 - s_1 J\right)/f_1\right]\, S - \left(1 + S^2\right) J}{1 - S^2},$$

which lead to the expression for μ,

$$\mu\left(S\right) = \frac{s_1 - 2f_1 S + 2i\sqrt{J\mathcal{R}\left(S\right)}}{2\left(1 - S^2\right)}, \tag{5.132}$$

where the resolvent equals to

$$\begin{aligned} \mathcal{R}\left(\nu\right) = \nu^4 + \frac{s_3 - s_1 J}{f_1 J}\nu^3 - \frac{s_2}{J}\nu^2 + \left(\frac{s_1 f_1}{J} - \frac{s_3 - s_1 J}{f_1 J}\right)\nu \\ + \frac{4s_2 - 4f_1^2 - s_1^2 - 4J}{4J}. \end{aligned} \tag{5.133}$$

The constant f_1 is determined by the formula (5.128) and it can be found that the positive sign of f_1 yields a stable ground state solution $\mathbf{S} = (0, 0, 1)$.

The zeros ν_i, $i = 1, 2, 3$, of the resolvent are connected with the zeros λ_i, $i = 1, 2, 3, 4$, by the relations (see Appendix B):

$$\begin{aligned} \nu_1 = (4f_1 J)^{-1}[(\lambda_1 - \lambda_3)(\lambda_2' - \lambda_4') + (\lambda_2 - \lambda_4)(\lambda_1' - \lambda_3')]^{-1} \\ \times\{(\lambda_1 - \lambda_3)[2(\lambda_1 + \lambda_3)(\lambda_2' - \lambda_4')J \\ + (\lambda_2\lambda_4' - \lambda_4\lambda_2')((\lambda_1 + \lambda_3)^2 - (\lambda_1' - \lambda_3')^2)] \\ +(\lambda_2 - \lambda_4)[2(\lambda_2 + \lambda_4)(\lambda_1' - \lambda_3')J \\ + (\lambda_1\lambda_3' - \lambda_3\lambda_1')((\lambda_2 + \lambda_4)^2 - (\lambda_2' - \lambda_4')^2)]\}, \end{aligned} \tag{5.134}$$

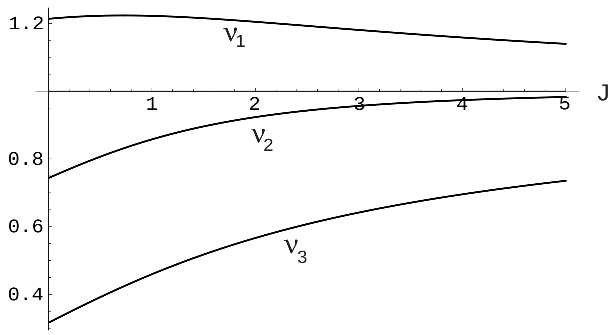

Fig. 5.3 The dependence of the resolvent's zeros ν_1, ν_2, ν_3 on the anisotropy constant J at the spectrum parameters values $\lambda_1 = 1+i,\ \lambda_2 = 2+2i,\ \lambda_3 = 1-i,\ \lambda_4 = 2-2i$. The plot is schematic since the curves for ν_1 and ν_2 are too close to the line $\nu = 1$. The curve for ν_4 is located below $\nu = -1$ and is not shown in the figure.

where

$$\lambda_i' = \sqrt{\lambda_i^2 + J}, \tag{5.135}$$

ν_2 and ν_3 are obtained from ν_1 by interchange of indices $3 \leftrightarrow 4$ and $3 \leftrightarrow 2$, respectively, and ν_4 can be found by means of the formula

$$\nu_4 = (s_1 J - s_4)/f_1 J - (\nu_1 + \nu_2 + \nu_3). \tag{5.136}$$

The equation for S can be obtained from Eqs. (5.126) and (5.132):

$$dS/d\xi = \sqrt{J\mathcal{R}\,(S)}. \tag{5.137}$$

Equations (5.125) and (5.137) permit us to calculate the wavelength L in two ways

$$L = \frac{1}{2}\oint \frac{d\mu}{\sqrt{-P\,(\mu)}} = \int_{\nu_2}^{\nu_3} \frac{d\nu}{\sqrt{J\mathcal{R}\,(\nu)}},$$

what leads to equality of the parameters of the elliptic integrals in terms of which L is expressed in both cases,

$$m = \frac{(\nu_2 - \nu_3)\,(\nu_1 - \nu_4)}{(\nu_1 - \nu_3)\,(\nu_2 - \nu_4)} = \frac{(\lambda_1 - \lambda_3)\,(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)\,(\lambda_2 - \lambda_3)}, \tag{5.138}$$

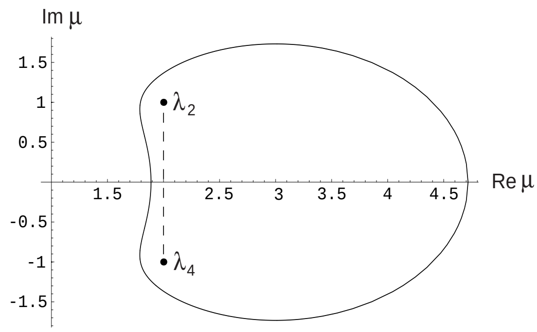

Fig. 5.4 The locus of μ in the complex plane λ for the periodic solution of the one-axis ferromagnet equation. The chosen values of the spectrum are equal to $\lambda_1 = 1 + i,\ \lambda_2 = 2 + i,\ \lambda_3 = 1 - i,\ \lambda_4 = 2 - i$, and the anisotropy constant is equal to $J = 5$. The curve is deformed compared with that in Fig. 5.2, which corresponds to $J = 0$, and it goes again around the cut between the branching points λ_2 and λ_4.

and useful relations

$$\begin{aligned} J\,(\nu_1 - \nu_3)\,(\nu_2 - \nu_4) &= (\lambda_1 - \lambda_4)\,(\lambda_3 - \lambda_2) \\ &= (\alpha - \beta)^2 + (\gamma + \delta)^2 \,, \\ J\,(\nu_1 - \nu_4)\,(\nu_2 - \nu_3) &= (\lambda_1 - \lambda_3)\,(\lambda_4 - \lambda_2) = 4\gamma\delta, \end{aligned} \tag{5.139}$$

where, as usual, $\lambda_1 = \alpha + i\gamma$, $\lambda_2 = \beta + i\delta$, $\lambda_3 = \alpha - i\gamma$, $\lambda_4 = \beta - i\delta$. In Fig. 5.3 the plots of ν_i as functions of J are shown for the case of the parameters values $\lambda_1 = 1 + i,\ \lambda_2 = 2 + 2i$. For other choice of the parameters values the curves are deformed but their order is the same. Hence, in our case the variable S oscillates in the interval

$$-1 \le \nu_3 \le S \le \nu_2 < 1. \tag{5.140}$$

The corresponding trajectory of μ (see Eq. (5.132)) is shown in Fig. 5.4. The solution of Eq. (5.137) can be easily expressed in terms of the elliptic functions:

$$S_3 = \frac{(\nu_2 - \nu_4)\nu_3 - (\nu_2 - \nu_3)\nu_4 \,\mathrm{sn}^2[\sqrt{J(\nu_1 - \nu_3)(\nu_2 - \nu_4)}\,\xi/2, m]}{\nu_2 - \nu_4 - (\nu_2 - \nu_3)\,\mathrm{sn}^2[\sqrt{J(\nu_1 - \nu_3)(\nu_2 - \nu_4)}\,\xi/2, m]}. \tag{5.141}$$

Now, let us calculate S_-. Insertion of Eqs. (5.132) and (5.137) into Eq. (5.131) yields

$$\widetilde{S}_- = \sqrt{1 - S_3^2} \exp\left[\frac{i}{2}\int_0^\xi \frac{s_1 S_3 - 2f_1}{1 - S_3^2} d\xi\right], \tag{5.142}$$

where $S_3(\xi)$ is given by Eq. (5.141). It is convenient to pass to the Weierstrass functions,

$$\mathrm{sn}^2\left(\sqrt{J(\nu_1 - \nu_3)(\nu_2 - \nu_4)}\,\xi/2, m\right) = (e_1 - e_3)/(\wp(\xi) - e_3),$$

where

$$\begin{aligned} e_1 &= -\tfrac{1}{12}\left[s_2 + 3J(\nu_1\nu_4 + \nu_2\nu_3)\right], \\ e_2 &= -\tfrac{1}{12}\left[s_2 + 3J(\nu_1\nu_3 + \nu_2\nu_4)\right], \\ e_3 &= -\tfrac{1}{12}\left[s_2 + 3J(\nu_1\nu_2 + \nu_3\nu_4)\right]. \end{aligned} \tag{5.143}$$

The integrand in Eq. (5.142) can be written as follows:

$$\frac{s_1 S_3 - 2f_1}{1 - S_3^2} = \frac{s_1 - 2f_1}{2(1-\nu_3)}\frac{\wp(\xi) - \wp(\rho)}{\wp(\xi) - \wp(\kappa)} - \frac{s_1 + 2f_1}{2(1+\nu_3)}\frac{\wp(\xi) - \wp(\rho)}{\wp(\xi) - \wp(\widetilde{\kappa})},$$

where $\rho, \kappa, \tilde{\kappa}$ are determined by the formulas

$$\begin{aligned} \wp(\rho) &= e_3 + (J/4)(\nu_1 - \nu_3)(\nu_2 - \nu_3), \\ \wp(\kappa) &= e_3 + \frac{J(\nu_1 - \nu_3)(\nu_2 - \nu_3)(1 - \nu_4)}{4(1 - \nu_3)}, \\ \wp(\widetilde{\kappa}) &= e_3 + \frac{J(\nu_1 - \nu_3)(\nu_2 - \nu_3)(1 + \nu_4)}{4(1 + \nu_3)}. \end{aligned} \tag{5.144}$$

Integration can be performed with the use of Eq. (A.49). As a final result, we obtain

$$\begin{aligned} S_- \equiv S_1 - iS_2 = \sqrt{1 - \nu_3^2}\exp\{&it(s_1 J - s_3)/2f_1 + i\xi(s_1\nu_3 - 2f_1)/2(1 - \nu_3^2) \\ &- (\zeta(\kappa) + \zeta(\widetilde{\kappa}))\xi\} \cdot \frac{\sigma^2(\rho)\sigma(\xi + \kappa)\sigma(\xi + \tilde{\kappa})}{\sigma(\kappa)\sigma(\tilde{\kappa})\sigma(\xi + \rho)\sigma(\xi - \rho)}. \end{aligned} \tag{5.145}$$

Formulas (5.141) and (5.145) give the general expression for the one-phase periodic solution of the Landau-Lifshitz equation for the case of a uniaxial ferromagnet.

In the soliton limit the two pairs of the complex conjugate zeros λ_i coalesce into one pair,

$$\lambda = \lambda_1 = \lambda_2 = \lambda_3^* = \lambda_4^* = \alpha + i\gamma, \qquad \lambda' = \sqrt{\lambda^2 + J}. \tag{5.146}$$

In this limit the zeros ν_i of the resolvent transform to

$$\begin{aligned} &\nu_1 = \nu_2 = 1, \quad \nu_3 = (\lambda\lambda' + \lambda^*\lambda'^*)/(\lambda'\lambda^* + \lambda\lambda'^*), \\ &\nu_4 = -(1/J)\left(\lambda\lambda^* + \lambda'\lambda'^*\right). \end{aligned} \tag{5.147}$$

Then Eq. (5.141) gives

$$S_3 = \frac{\nu_4(1-\nu_3) + (\nu_3 - \nu_4)\cosh^2(\gamma\xi)}{1 - \nu_3 + (\nu_3 - \nu_4)\cosh^2(\gamma\xi)}, \qquad \xi = x - 2\alpha t. \tag{5.148}$$

Taking into account the following formulas (see Eqs. (5.139)),

$$\begin{aligned} &(1-\nu_3)\,(1-\nu_4) = 2\gamma^2/J, \\ &\sum \nu_i = 2 + \nu_3 + \nu_4 = (s_1 J - s_3)/f_1 J = 2 - 2(\alpha^2 + \gamma^2)/J, \end{aligned}$$

we find that $(1-\nu_3)$ and $(1-\nu_4)$ are the roots of a simple quadratic equation which gives

$$\begin{aligned} \nu_3 &= -(1/J)\left(\alpha^2 + \gamma^2 - \sqrt{(\alpha^2 - \gamma^2 + J)^2 + 4\alpha^2\gamma^2}\right), \\ \nu_4 &= -(1/J)\left(\alpha^2 + \gamma^2 + \sqrt{(\alpha^2 - \gamma^2 + J)^2 + 4\alpha^2\gamma^2}\right). \end{aligned} \tag{5.149}$$

On introducing the angle ϑ between $\mathbf{S}$ and $\mathbf{n}$, so that

$$S_3 = \cos\vartheta,$$

the soliton solution can be written as

$$\tan\left(\frac{\vartheta}{2}\right) = \frac{\gamma^2/J}{\Omega\cosh^2(\gamma\xi) - (\Omega - \Omega_1)/2}, \tag{5.150}$$

where

$$\Omega = (1/J)\sqrt{(\alpha^2 - \gamma^2 + J)^2 + 4\alpha^2\gamma^2}, \qquad \Omega_1 = (1/J)\left(\alpha^2 - \gamma^2 + J\right). \tag{5.151}$$

The Whitham equations coincide with those for the isotropic Heisenberg model.

5.6 Stimulated Raman scattering

As it was shown in Sec. 2.6, the SRS equations have the form

$$\begin{aligned}\partial R_+/\partial\tau &= i\left(\Delta R_+S_3 + R_3S_+\right), \quad \partial R_3/\partial\tau = \tfrac{i}{2}\left(R_+S_- - R_-S_+\right),\\ \partial S_+/\partial\zeta &= i\left(\Delta S_+R_3 + S_3R_+\right), \quad \partial S_3/\partial\zeta = \tfrac{i}{2}\left(S_+R_- - S_-R_+\right),\end{aligned} \tag{5.152}$$

where $S_\pm = S_1 \pm iS_2$ and Δ is the relative dynamical Stark shift coefficient. The vectors $\mathbf{R}$ and $\mathbf{S}$ characterize the radiative properties of the medium and the radiation fields, correspondingly, and they are normalized according to the unit length conditions

$$R_1^2 + R_2^2 + R_3^2 = 1, \qquad S_1^2 + S_2^2 + S_3^2 = 1. \tag{5.153}$$

As we know, Eqs. (5.152) can be presented as a compatibility condition of Eqs. (4.173), if one takes the coefficients (4.174):

$$\begin{aligned}&F = -i\lambda S_3, \quad G = (i\sigma + \lambda)\,S_+, \quad H = (i\sigma - \lambda)\,S_-,\\ &A = \frac{i}{2}\left(\Delta - \frac{1}{2\lambda + \Delta}\right)R_3, \quad B = \frac{i\sigma + \lambda}{2\lambda + \Delta}R_+, \quad C = \frac{i\sigma - \lambda}{2\lambda + \Delta}R_-,\end{aligned}$$

where the parameter σ is connected with Δ by the relation $\sigma^2 = \frac{1}{4}\left(1 - \Delta^2\right)$. Functions (4.139) have now the form

$$\begin{aligned}&f = S_3\lambda^2 - f_1\lambda + f_2, \quad g = (i\sigma + \lambda)\,S_+\,(\lambda - \mu),\\ &h = (i\sigma - \lambda)\,S_-\,(\lambda - \mu^*),\end{aligned} \tag{5.154}$$

provided f_1, f_2, μ, μ^* satisfy the system

$$\begin{aligned}&2f_1S_3 + \left(1 - S_3^2\right)(\mu + \mu^*) = s_1,\\ &2f_1f_2 + \left(1 - S_3^2\right)\sigma^2\,(\mu + \mu^*) = s_3,\\ &f_1^2 + 2f_2S_3 + \left(1 - S_3^2\right)\left(\sigma^2 + \mu\mu^*\right) = s_2,\\ &f_2^2 + \left(1 - S_3^2\right)\sigma^2\mu\mu^* = s_4,\end{aligned} \tag{5.155}$$

and the following equations are also fulfilled

$$\partial S_3/\partial\tau = -i\left(1 - S_3^2\right)(\mu - \mu^*), \quad \partial S_+/\partial\tau = -2i\,(f_1 - \mu S_3)\,S_+, \tag{5.156}$$

$$\begin{aligned}&R_+S_-\,(\mu^* + \Delta/2) = R_-S_+\,(\mu + \Delta/2),\\ &f\,(-\Delta/2)\,R_+ + \tfrac{1}{2}\,(\mu + \Delta/2)\,R_3S_+ = 0.\end{aligned} \tag{5.157}$$

The evolution equations for μ have the form:

$$\begin{aligned}
\partial\mu/\partial\tau &= -2if\,(\mu) = -2i\sqrt{P\,(\mu)},\\
\partial\mu/\partial\zeta &= (R_+/(2\mu+\Delta)\,S_+)(\partial\mu/\partial\tau).
\end{aligned} \tag{5.158}$$

Let us rewrite the relations (5.157) in the form

$$\frac{R_+}{(\mu+\Delta/2)\,S_+} = \frac{R_-}{(\mu^*+\Delta/2)\,S_-} = -\frac{R_3}{2f\,(-\Delta/2)} = -\frac{2}{V}, \tag{5.159}$$

where, as we shall see, V is the phase velocity of the wave. From these equations we find

$$\frac{1-R_3^2}{(1-S_3^2)\,(\mu+\Delta/2)\,(\mu^*+\Delta/2)} = \frac{R_3^2}{4f^2\,(-\Delta/2)} = \frac{4}{V^2}.$$

If we put $\lambda = -\Delta/2$ into the relation $f^2 - gh = P(\lambda)$, then we obtain

$$\left(1-S_3^2\right)(\mu+\Delta/2)\,(\mu^*+\Delta/2) = 4\left[P\,(-\Delta/2) - f^2\,(-\Delta/2)\right],$$

and, hence, the preceding equation gives

$$V = 4\sqrt{P\,(-\Delta/2)}. \tag{5.160}$$

Thus, μ depends only on the phase

$$\xi = \tau - \zeta/V, \quad d\mu/d\xi = -2i\sqrt{P\,(\mu)}. \tag{5.161}$$

The last equation of the system (5.152) can also be transformed with the help of Eqs. (5.156) and (5.159) to the form

$$\partial S_3/\partial\zeta = -(1/V)\,\partial S_3/\partial\tau,$$

that is, S_3 depends only on the phase ξ, too.

The system (5.155) actually coincides with the analogous system (5.124) in the preceding section, so we can use the solution found there. For f_1 and f_2 we have

$$f_1^2 = (1/2\sigma^2)\left[\sqrt{P_2\,(\sigma^2)} - \sigma^4 + s_2\sigma^2 - s_4\right], \tag{5.162}$$

where

$$P_2(\sigma^2) = \prod_{i=1}^{4}\left(\sigma^2+\lambda_i^2\right),$$

and

$$f_2 = (s_3 - s_1\sigma^2)/2f_1 + \sigma^2 S_3. \tag{5.163}$$

The sign of f_1 is determined by the stability condition of the solution $S_3 = R_3 = -1$. The choice of the negative sign, $f_1 = -\sqrt{f_1^2}$, leads to this stable solution.

Equation for S_+ in Eq. (5.152) and Eqs. (5.156), (5.159), (5.163) yield

$$\frac{\partial S_+}{\partial \zeta} = -\frac{2i}{V}\left[4f_1\sigma^2 - \frac{\left(s_3 - s_1\sigma^2\right)\Delta}{f_1}\right] S_+ - \frac{1}{V}\frac{\partial S_+}{\partial \tau},$$

that is,

$$S_+ = \exp\left\{-(2i/V)\left[4f_1\sigma^2 - \left(s_3 - s_1\sigma^2\right)\Delta/f_1\right]\zeta\right\}\widetilde{S}_+, \tag{5.164}$$

where $\tilde{S}_+$ depends only on the phase ξ and is determined by the equation

$$d\widetilde{S}_+/d\xi = -2i\,(f_1 - \mu S_3)\,\widetilde{S}_+. \tag{5.165}$$

The parameter μ is expressed in terms of S_3 by the formula (5.132), where $\mathcal{R}$ differs from Eq. (5.133) by the sign before f_1 and J has to be replaced by σ^2. The equation for $S = S_3$ coincides with Eq. (5.137):

$$dS/d\xi = \sqrt{\sigma^2\mathcal{R}(S)}. \tag{5.166}$$

In Fig. 5.5 the plots of ν_i as functions of σ^2 are shown in the case of the parameters values $\lambda_1 = 1 + i, \lambda_2 = 2 + 2i$. As we see, for $\sigma^2 > 0$ the resolvent's zeros are ordered according to $\nu_1 < -1 < \nu_2 < \nu_3 < 1 < \nu_4$ and S_3 oscillates in the interval

$$-1 < \nu_2 \le S_3 \le \nu_3 < 1, \tag{5.167}$$

where $\mathcal{R}(S_3) \ge 0$. For $\sigma^2 < 0$ we have $\nu_4 < \nu_1 < -1 < \nu_2 < \nu_3 < 1$ and S_3 oscillates in the same interval where now $\mathcal{R}(S_3) \le 0$, but the expression under the radical sign in Eq. (5.166) remains positive.

Periodic solution of Eq. (5.166) gives the desired expression for S_3

$$S_3 = \frac{(\nu_2 - \nu_4)\,\nu_3 - (\nu_2 - \nu_3)\,\nu_4\,\mathrm{sn}^2\left(\sqrt{\sigma^2\,(\nu_1 - \nu_3)\,(\nu_2 - \nu_4)}\,\xi, m\right)}{\nu_2 - \nu_4 - (\nu_2 - \nu_3)\;\mathrm{sn}^2\left(\sqrt{\sigma^2\,(\nu_1 - \nu_3)\,(\nu_2 - \nu_4)}\,\xi, m\right)}, \tag{5.168}$$

where the initial phase equals to zero.

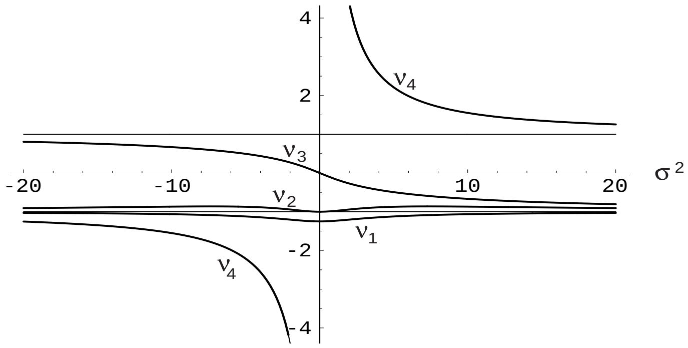

Fig. 5.5 The dependence of the resolvent's zeros ν_1, ν_2, ν_3 on the dynamic Stark shift parameter σ^2 for the periodic solution of the SRS equations. The spectral parameters are equal to $\lambda_1 = 1+i,\ \lambda_2 = 2+2i,\ \lambda_3 = 1-i,\ \lambda_4 = 2-2i$. For other values of λ_i these curves are deformed but their ordering remains the same.

The calculation of S_+ analogous to that of the preceding section yields

$$S_+ = -\sqrt{1-\nu_3^2}\exp\Bigg\{-\frac{2i}{V}\left[4f_1\sigma^2 - \frac{\left(s_3 - s_1\sigma^2\right)\Delta}{f_1}\right]\zeta + \frac{i\left(s_1\nu_3 - 2f_1\right)}{1-\nu_3^2}\xi \\ - \left(\zeta\left(\kappa\right) + \zeta\left(\widetilde{\kappa}\right)\right)\xi\Bigg\}\cdot\frac{\sigma\left(\xi+\kappa\right)\sigma\left(\xi+\widetilde{\kappa}\right)\sigma^2\left(\rho\right)}{\sigma\left(\kappa\right)\sigma\left(\widetilde{\kappa}\right)\sigma\left(\xi+\rho\right)\sigma\left(\xi-\rho\right)}, \tag{5.169}$$

where $\xi = \tau - \zeta/V$,

$$V = 4\sqrt{P\left(-\Delta/2\right)} = 4\sqrt{\left[\left(\alpha+\Delta/2\right)^2+\gamma^2\right]\left[\left(\beta+\Delta/2\right)^2+\delta^2\right]}, \tag{5.170}$$

and

$$\begin{aligned} e_1 &= -s_2/3 - \sigma^2\left(\nu_1\nu_4 + \nu_2\nu_3\right), \\ e_2 &= -s_2/3 - \sigma^2\left(\nu_1\nu_3 + \nu_2\nu_4\right), \\ e_3 &= -s_2/3 - \sigma^2\left(\nu_1\nu_2 + \nu_3\nu_4\right). \end{aligned} \tag{5.171}$$

The parameters ρ, κ, $\tilde{\kappa}$ are determined by the formulas

$$\begin{aligned}
\wp(\rho) &= e_3 + \sigma^2(\nu_1-\nu_3)(\nu_2-\nu_3), \\
\wp(\kappa) &= e_3 + [\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_3)(1-\nu_4)]/(1-\nu_3), \\
\wp(\widetilde{\kappa}) &= e_3 + [\sigma^2(\nu_1-\nu_3)(\nu_2-\nu_3)(1+\nu_4)]/(1+\nu_3).
\end{aligned} \tag{5.172}$$

The formulas (5.168) and (5.169) give the expressions for $\mathbf{S}$ in the general solution. The components of vector $\mathbf{R}$ can be found with the help of (5.159); in particular

$$R_3 = (4/V)f(-\Delta/2) = (1/V)\left(S_3 + 2f_1\Delta + 2/f_1)\left(s_3 - s_1\sigma^2\right)\right). \tag{5.173}$$

Let us consider the soliton limit of this solution when we have

$$\lambda_1 = \lambda_2 = \alpha + i\gamma, \quad \lambda_3 = \lambda_4 = \alpha - i\gamma.$$

Then $s_1 = 4\alpha$, $s_3 = 4\alpha(\alpha^2+\gamma^2)$, $f_1 = -2\alpha$, and Eq. (5.173) gives

$$1 + R_3 = (1/V)(1+S_3), \tag{5.174}$$

where the soliton velocity equals to

$$V = 4\left[(\alpha+\Delta/2)^2 + \gamma^2\right]. \tag{5.175}$$

The general formulas for resolvent's zeros reduce now to

$$\begin{aligned}
\nu_1 &= \nu_2 = -1, \\
\nu_3 &= (1/\sigma^2)\left(\alpha^2+\gamma^2 - \sqrt{(\alpha^2+\gamma^2+\sigma^2)^2 - 4\gamma^2\sigma^2}\right), \\
\nu_4 &= (1/\sigma^2)\left(\alpha^2+\gamma^2 + \sqrt{(\alpha^2+\gamma^2+\sigma^2)^2 - 4\gamma^2\sigma^2}\right).
\end{aligned} \tag{5.176}$$

The expression (5.168) gives

$$S_3 = \frac{\nu_4(1+\nu_3) + (\nu_3-\nu_4)\cosh^2(2\gamma\xi)}{1+\nu_3-(\nu_3-\nu_4)\cosh^2(2\gamma\xi)},$$

so that

$$1 + S_3 = 2\frac{(1+\nu_3)(1+\nu_4)/(\nu_4-\nu_3)}{\cosh(4\gamma\xi) + (\nu_3+\nu_4-2)/(\nu_4-\nu_3)}. \tag{5.177}$$

If $\sigma^2 > 0$, we can introduce the parameter χ according to

$$\tanh(2\chi) = 2\gamma\sigma/(\alpha^2+\gamma^2+\sigma^2), \tag{5.178}$$

and then the soliton solution reads

$$1 + S_3 = V\,(1 + R_3) = \frac{2\gamma}{\sigma} \frac{\sinh(2\chi)}{\cosh(4\gamma\xi) + \cosh(2\chi)}. \tag{5.179}$$

If $\sigma^2 = -\varphi^2 < 0$, then we introduce the parameter χ according to

$$\tan(2\chi) = \frac{2\varphi\gamma}{\alpha^2 + \gamma^2 - \varphi^2}, \tag{5.180}$$

so that

$$1 + S_3 = V\,(1 + R_3) = \frac{2\gamma}{\varphi} \frac{\sin(2\chi)}{\cosh(4\gamma\xi) + \cos(2\chi)}. \tag{5.181}$$

Since we have

$$G = (i\sigma + \lambda)\, S_+, \quad B = \frac{i\sigma + \lambda}{2\lambda + \Delta} R_+, \quad g = (i\sigma + \lambda)\, S_+\,(\lambda - \mu)\,, \tag{5.182}$$

ζ plays the role of t, τ plays the role of x, the generating function of conservation laws has the form (5.89) and after insertion of Eq. (5.182) can be transformed to

$$\overline{\frac{1}{\lambda_i - \mu}} \cdot \frac{\partial \lambda_i}{\partial \zeta} + \frac{1}{4\sqrt{P(-\Delta/2)}} \left[\overline{\frac{1}{\lambda_i - \mu}} \cdot - \frac{1}{\lambda_i + \Delta/2} \right] \frac{\partial \lambda_i}{\partial \tau} = 0. \tag{5.183}$$

Thus, we obtain the Whitham equations in the diagonal Riemann form

$$\partial\lambda_i/\partial\xi + (1/v_i)\partial\lambda_i/\partial\tau = 0, \quad i = 1, 2, 3, 4, \tag{5.184}$$

with the characteristic velocities

$$\frac{1}{v_i} = \left[1 + \frac{1}{2} \frac{1}{\lambda_i + \Delta/2} \left(\frac{\partial \ln \omega}{\partial \lambda_i} \right)^{-1} \right] \cdot \frac{1}{4\sqrt{P(-\Delta/2)}}, \tag{5.185}$$

where ω is defined as in Eq. (5.91), and these speeds can be written in a universal form (3.150) with V given by Eq. (5.170).

5.7 Derivative nonlinear Schrödinger equation

The derivative nonlinear Schrödinger (DNLS) equation

$$iu_t + u_{xx} \pm 2i(|u|^2 u)_x = 0 \tag{5.186}$$

plays important role in physics of ultrashort optical pulses, wave propagation in magnetized plasma, etc (see Sec. 2.3). Here we shall find its periodic solution. We shall discuss the DNLS equation (5.186) with a minus sign before the last term. This sign can be easily inverted by means of simple substitutions.

The DNLS equation can be expressed as a compatibility condition (see Eq. (4.137)), if we take the coefficients

$$\begin{aligned} &F=-2i\lambda^2, \quad G=2\lambda u, \quad H=2\lambda u^*, \\ &A=-\left(8i\lambda^4+4i\,|u|^2\,\lambda^2\right), \quad B=8\lambda^3 u+\left(2iu_x+4\,|u|^2\,u\right)\lambda, \\ &C=8\lambda^3 u^*+\left(-2iu_x^*+4\,|u|^2\,u^*\right)\lambda. \end{aligned}$$

Now the polynomial $P(\lambda)$ contains only even degrees of λ and a non-trivial periodic solutions correspond to the eighth degree polynomial. Therefore, we assume that $P(\lambda)$ is equal to

$$P(\lambda)=\prod_{i=1}^{4}\left(\lambda^2-\lambda_i^2\right)=\lambda^8-s_1\lambda^6+s_2\lambda^4-s_3\lambda^2+s_4, \tag{5.187}$$

where $\pm\lambda_i$ are the zeros of the polynomial. Then Eqs. (4.140) and (4.141) lead to the expressions

$$f=\lambda^4-s_1\lambda^2+f_2, \quad g=u\lambda\left(\lambda^2-\mu\right), \quad h=u^*\lambda\left(\lambda^2-\mu^*\right), \tag{5.188}$$

and u satisfies the equations

$$u_x=-4iu\left(f_1-\mu\right), \quad u_t=8iu\left[2f_2-\left(f_1-\mu\right)\left(2f_1+|u|^2\right)\right], \tag{5.189}$$

where the quantities f_1, f_2, $|u|^2$, μ, μ^* are connected by the following constraint

$$\left(\lambda^4-f_1\lambda^2+f_2\right)^2-|u|^2\lambda^2\left(\lambda^2-\mu\right)\left(\lambda^2-\mu^*\right)=P(\lambda). \tag{5.190}$$

The dependence of μ on x and t can be obtained from Eqs. (4.140) and (4.141), if one puts there $\lambda^2=\mu$ and takes into account that $f\left(\mu^{1/2}\right)=\sqrt{P\left(\mu^{1/2}\right)}$:

$$\mu_x=4i\sqrt{P\left(\mu^{1/2}\right)}, \quad \mu_t=8i\left(2f_1+|u|^2\right)\sqrt{P\left(\mu^{1/2}\right)}. \tag{5.191}$$

As in the preceding cases, we see that the constraint (5.190) can be considered as an equation for the locus of μ in the complex λ plane, and, as in the NLS equation case, the variable $\nu=|u|^2$ is a natural parameter along this locus. Comparing the coefficients of λ^k on both sides of Eq. (5.190) gives

$$\begin{aligned} &2f_1+\nu=s_1, \quad f_1^2+2f_2+\nu\left(\mu+\mu^*\right)=s_2, \\ &2f_1f_2+\nu\mu\mu^*=s_3, \quad f_2^2=s_4. \end{aligned} \tag{5.192}$$

We can obtain from this system the expression for μ,

$$\mu\left(\nu\right)=\left(1/8\nu\right)\left[4s_2\pm8\sqrt{s_4}-\left(\nu-s_1\right)^2\pm i\sqrt{-\mathcal{R}\left(\nu\right)}\right], \tag{5.193}$$

where $\mathcal{R}\left(\nu\right)$ is the fourth-degree polynomial in ν,

$$\begin{aligned} &\mathcal{R}\left(\nu\right)=\nu^4-4s_1\nu^3+\left(6s_1^2-8s_2\pm48\sqrt{s_4}\right)\nu^2 \\ &-\left(4s_1^3-16s_1s_2+64s_3\pm32s_1\sqrt{s_4}\right)\nu+\left(-s_1^2+4s_2\pm8\sqrt{s_4}\right)^2. \end{aligned} \tag{5.194}$$

Here it is supposed that $\sqrt{s_4}=\lambda_1\lambda_2\lambda_3\lambda_4$. The zeros of the resolvent (5.194) are related to the zeros of $P\left(\lambda\right)$ by simple symmetric formulas—the zeros

$$\begin{aligned} &\nu_1=\left(\lambda_1+\lambda_2+\lambda_3-\lambda_4\right)^2, \quad \nu_2=\left(\lambda_1+\lambda_2-\lambda_3+\lambda_4\right)^2, \\ &\nu_3=\left(\lambda_1-\lambda_2+\lambda_3+\lambda_4\right)^2, \quad \nu_4=\left(-\lambda_1+\lambda_2+\lambda_3+\lambda_4\right)^2 \end{aligned} \tag{5.195}$$

correspond to the upper sign in Eq. (5.194), and the zeros

$$\begin{aligned} &\nu_1=\left(\lambda_1+\lambda_2+\lambda_3+\lambda_4\right)^2, \quad \nu_2=\left(\lambda_1+\lambda_2-\lambda_3-\lambda_4\right)^2, \\ &\nu_3=\left(\lambda_1-\lambda_2+\lambda_3-\lambda_4\right)^2, \quad \nu_4=\left(-\lambda_1+\lambda_2+\lambda_3-\lambda_4\right)^2 \end{aligned} \tag{5.196}$$

correspond to the lower sign. This can be proved by a simple check of the Viète formulas.

As it follows from Eq. (5.191) and the first formula (5.192), the variable μ depends only on the phase

$$\xi=x+2s_1t, \quad d\mu/d\xi=\pm4i\sqrt{P\left(\mu^{1/2}\right)}. \tag{5.197}$$

From Eqs. (5.189) and (5.192) we find that

$$d\left|u\right|^2/d\xi=4i\left|u\right|^2\left(\mu-\mu^*\right),$$

and then Eq. (5.193) gives

$$d\nu/d\xi=\sqrt{-\mathcal{R}\left(\nu\right)}. \tag{5.198}$$

This equation can be easily solved by means of the elliptic functions. If ν is known, then $u(x,t)$ can be obtained from Eq. (5.189). With the help of Eq. (5.192) we obtain

$$\partial u/\partial t = 16i\sqrt{s_4}\, u + 2s_1\partial u/\partial x,$$

so that

$$u(x,t) = \exp\left(16i\sqrt{s_4}t\right)\tilde{u}(\xi), \tag{5.199}$$

where $\tilde{u}(\xi)$ satisfies the equation

$$d\tilde{u}/d\xi = 4i\left(-\tfrac{1}{2}s_1 + \tfrac{1}{2}\nu + \mu\right)\tilde{u}. \tag{5.200}$$

It is clear that the zeros λ_i should be such numbers that ν oscillates between two positive values. The polynomial $\mathcal{R}(\nu)$ has four zeros ν_i which are given by Eqs. (5.195) or (5.196) depending on the choice of the sign in Eq. (5.194). If only two ν_i are real and positive, then let us enumerate λ_i so that these ν_i are $\nu_1,\ \nu_2$, and $\nu_1 \geq \nu_2$. If all ν_i are real and positive, then let us enumerate λ_i so that $\nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4$. Thus, as it is clear from Eq. (5.198), the variable ν can oscillate in the intervals $\nu_1 \geq \nu \geq \nu_2$ or $\nu_3 \geq \nu \geq \nu_4$, where $\mathcal{R}(\nu) \leq 0$.

Let us list the $\lambda_i,\ i = 1,2,3,4$, corresponding to the periodic solutions.

(i) The zeros λ_i consist of two complex conjugate pairs:

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta, \quad \lambda_3 = \gamma - i\delta, \quad \lambda_4 = \gamma + i\delta. \tag{5.201}$$

Then Eq. (5.196) yields

$$\begin{aligned} &\nu_1 = 4(\alpha+\gamma)^2, \quad \nu_2 = 4(\alpha-\gamma)^2, \\ &\nu_3 = -4(\beta-\delta)^2, \quad \nu_4 = -4(\beta+\delta)^2, \end{aligned} \tag{5.202}$$

and Eq. (5.195) results in the complex values of ν_i.

(ii) If

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta, \quad \lambda_3 = \gamma - i\delta, \quad \lambda_4 = -\gamma - i\delta, \tag{5.203}$$

then Eqs. (5.195) yields Eqs. (5.202) and Eqs. (5.196) becomes inapplicable. In this case the variable ν oscillates in the interval $\nu_1 \geq \nu \geq \nu_2$.

(iii) All four λ_i are real and

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4. \tag{5.204}$$

Both (5.195) and (5.196) yield the real and positive ν_i corresponding to different periodic solutions for which the variable ν oscillates in the intervals $\nu_1 \geq \nu \geq \nu_2$ or $\nu_3 \geq \nu \geq \nu_4$.

(*iv*) If two λ_i are complex conjugate and two others are real

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta, \quad \lambda_3 = \gamma, \quad \lambda_4 = \delta, \tag{5.205}$$

then Eq. (5.195) yields

$$\begin{aligned} \nu_1 = (2\alpha + \gamma - \delta)^2, \quad \nu_2 = (2\alpha - \gamma + \delta)^2, \\ \nu_3 = (\gamma + \delta + 2i\beta)^2, \quad \nu_4 = (\gamma + \delta - 2i\beta)^2, \end{aligned} \tag{5.206}$$

and Eq. (5.196) leads to the same values of ν_i with different sign before δ.

Now we shall turn to finding the periodic solutions. Let us discuss at first the case when the variable ν oscillates in the interval $\nu_1 \geq \nu \geq \nu_2$ and ν_3, ν_4 are also real. We shall choose the initial conditions so that $\nu = \nu_1$ at $\xi = 0$. Then Eq. (5.198) leads to the solution

$$\nu = \frac{\nu_1(\nu_2 - \nu_4) + (\nu_1 - \nu_2)\nu_4 \,\mathrm{sn}^2\left(\sqrt{(\nu_1 - \nu_3)(\nu_2 - \nu_4)}\,\xi/2, m\right)}{\nu_2 - \nu_4 + (\nu_1 - \nu_2)\,\mathrm{sn}^2\left(\sqrt{(\nu_1 - \nu_3)(\nu_2 - \nu_4)}\,\xi/2, m\right)}, \tag{5.207}$$

where

$$m = \frac{(\nu_1 - \nu_2)(\nu_3 - \nu_4)}{(\nu_1 - \nu_3)(\nu_2 - \nu_4)}. \tag{5.208}$$

Now we introduce the zeros of the Weierstrass cubic by means of the expressions

$$\begin{aligned} e_1 &= \tfrac{1}{12}\left[2(\nu_1 - \nu_3)(\nu_2 - \nu_4) - (\nu_1 - \nu_2)(\nu_3 - \nu_4)\right], \\ e_2 &= \tfrac{1}{12}\left[2(\nu_1 - \nu_2)(\nu_3 - \nu_4) - (\nu_1 - \nu_3)(\nu_2 - \nu_4)\right], \\ e_3 &= -\tfrac{1}{12}\left[(\nu_1 - \nu_2)(\nu_3 - \nu_4) + (\nu_1 - \nu_3)(\nu_2 - \nu_4)\right]. \end{aligned} \tag{5.209}$$

Then Eq. (5.207) can be expressed in the form

$$\nu = \nu_1 \frac{\wp(\xi) - \wp(\rho)}{\wp(\xi) - \wp(\kappa)}, \tag{5.210}$$

where the parameters κ and ρ are defined by the equations

$$\begin{aligned} \wp(\kappa) &= e_3 - \tfrac{1}{4}(\nu_1 - \nu_2)(\nu_1 - \nu_3), \\ \wp(\rho) &= e_3 - \tfrac{1}{4}(\nu_4/\nu_1)(\nu_1 - \nu_2)(\nu_1 - \nu_3). \end{aligned} \tag{5.211}$$

After substitution of Eqs. (5.210) into Eq. (5.200), one can integrate the resulting equation by means of Eq. (A.49) and get the expression for the periodic solution of the DNLS equation

$$\begin{aligned} u\,(x,t) = &\exp[i(-s_1 + \tfrac{3}{2}\nu_1 + \sqrt{\nu_1\nu_2\nu_3\nu_4}/2\nu_1)\xi \\ &+ (3\zeta\,(\kappa) - \zeta\,(\rho))\,\xi + 16i\sqrt{s_4}t] \\ &\times \sqrt{\nu_1}\frac{\sigma\,(\kappa)\,\sigma\,(\xi+\rho)\,\sigma\,(\xi-\kappa)}{\sigma\,(\rho)\,\sigma^2\,(\xi+\kappa)}, \qquad \nu_1 \geq \nu \geq \nu_2. \end{aligned} \tag{5.212}$$

The case when ν oscillates in the interval $\nu_3 \geq \nu \geq \nu_4$ can be considered in the same way. The initial conditions are chosen so that $\nu = \nu_4$ at $\xi = 0$. For ν we get the expression

$$\nu = \nu_4 \frac{\wp\,(\xi) - \wp\,(\rho)}{\wp\,(\xi) - \wp\,(\kappa)}, \tag{5.213}$$

where κ and ρ are now defined by

$$\begin{aligned} \wp\,(\kappa) &= e_3 - \tfrac{1}{4}\,(\nu_2 - \nu_4)\,(\nu_3 - \nu_4)\,, \\ \wp\,(\rho) &= e_3 - \tfrac{1}{4}(\nu_1/\nu_4)\,(\nu_2 - \nu_4)\,(\nu_3 - \nu_4)\,. \end{aligned} \tag{5.214}$$

The corresponding periodic solution of the DNLS equation takes the form

$$\begin{aligned} u\,(x,t) = &\exp[i(-s_1 + \tfrac{3}{2}\nu_4 + \sqrt{\nu_1\nu_2\nu_3\nu_4}/2\nu_4)\xi \\ &+ (3\zeta\,(\kappa) - \zeta\,(\rho))\,\xi + 16i\sqrt{s_4}t] \\ &\times \sqrt{\nu_4}\frac{\sigma\,(\kappa)\,\sigma\,(\xi+\rho)\,\sigma\,(\xi-\kappa)}{\sigma\,(\rho)\,\sigma^2\,(\xi+\kappa)}, \qquad \nu_3 \geq \nu \geq \nu_4. \end{aligned} \tag{5.215}$$

Let us consider the soliton limit of Eq. (5.212) when $\nu_2 = \nu_3$, that is, $m = 1$ and

$$\begin{aligned} e_1 = e_2 = a &= \tfrac{1}{12}\,(\nu_1 - \nu_2)\,(\nu_2 - \nu_4)\,, \\ e_3 = -2a &= -\tfrac{1}{6}\,(\nu_1 - \nu_2)\,(\nu_2 - \nu_4)\,. \end{aligned} \tag{5.216}$$

By means of the well-known limiting expressions for the Weierstrass functions (A.47) we obtain from Eq. (5.212)

$$\begin{aligned} u(x,t) = &-\exp\Big\{i\left[-s_1+\tfrac{3}{2}\nu_1+\tfrac{1}{2}\nu_2\sqrt{\nu_4/\nu_1}\right]\xi+ \\ &+\left[3\sqrt{3a}\coth\left(\sqrt{3a}\kappa\right)-\sqrt{3a}\coth\left(\sqrt{3a}\rho\right)\right]\xi+16i\sqrt{s_4}t\Big\} \\ &\times\frac{\sqrt{\nu_1}\sinh\left[\sqrt{3a}(\xi+\rho)\right]\sinh\left[\sqrt{3a}(\xi-\kappa)\right]}{\sinh\left(\sqrt{3a}\rho\right)\sinh^2\left[\sqrt{3a}(\xi+\kappa)\right]}. \end{aligned} \tag{5.217}$$

Equations (5.211) give in this limit

$$\begin{aligned} \sqrt{3a}\coth\left(\sqrt{3a}\,\kappa\right) &= -\tfrac{1}{2}i(\nu_1-\nu_2), \\ \sqrt{3a}\coth\left(\sqrt{3a}\,\rho\right) &= -\tfrac{1}{2}\sqrt{\nu_4/\nu_1}\,(\nu_1-\nu_2). \end{aligned} \tag{5.218}$$

Let us denote

$$2\theta=\sqrt{3a}\,\xi, \qquad \cos^2\frac{\Gamma}{2}=\frac{\nu_2-\nu_4}{\nu_1-\nu_4}, \tag{5.219}$$

so that

$$\begin{aligned} &\sinh\left(\sqrt{3a}\kappa\right)=i\cos(\Gamma/2), \qquad \cosh\left(\sqrt{3a}\kappa\right)=\sin(\Gamma/2), \\ &\sinh\left(\sqrt{3a}\rho\right)=\frac{i\sqrt{\nu_1}\cos(\Gamma/2)}{\sqrt{\nu_1\cos^2(\Gamma/2)+\nu_4\sin^2(\Gamma/2)}}, \\ &\cosh\left(\sqrt{3a}\rho\right)=\frac{\sqrt{\nu_4}\sin(\Gamma/2)}{\sqrt{\nu_1\cos^2(\Gamma/2)+\nu_4\sin^2(\Gamma/2)}}. \end{aligned} \tag{5.220}$$

Simple transformation of Eq. (5.217) with the use of Eqs. (5.218)–(5.220) results in the soliton-like solution

$$\begin{aligned} u(x,t) = &\tfrac{1}{2}\exp\left[i\left(-s_1+\frac{3}{2}\nu_2\right)\xi+16i\sqrt{s_4}t-\tfrac{1}{2}\sqrt{-\nu_1\nu_4}\xi\right] \\ &\times\frac{\cosh(2\theta+i\Gamma/2)}{\cosh(2\theta-i\Gamma/2)}\left(\sqrt{\nu_1}+\sqrt{\nu_4}+(\sqrt{\nu_1}-\sqrt{\nu_4})\frac{\cosh(2\theta+i\Gamma/2)}{\cosh(2\theta-i\Gamma/2)}\right). \end{aligned} \tag{5.221}$$

This solution describes a soliton propagating on the constant background.

Let us discuss two particular cases of this solution. Consider the case

$$\lambda_1 = \lambda_4 = \alpha + i\beta, \qquad \lambda_2 = \lambda_3 = \alpha - i\beta, \tag{5.222}$$

so that

$$\nu_1 = 16\alpha^2, \quad \nu_2 = \nu_3 = 0, \quad \nu_4 = -16\beta^2, \quad \cos^2(\Gamma/2) = \beta^2/(\alpha^2+\beta^2).$$

The last formula prompts the parametrization

$$\alpha = \Delta \sin(\Gamma/2), \quad \beta = \Delta \cos(\Gamma/2). \tag{5.223}$$

Substitution of these expressions into Eq. (5.221) leads to the soliton solution

$$u(x,t) = 4\Delta \sin\Gamma \frac{\exp(2i\Phi)}{\exp(2\theta) + \exp(-2\theta + i\Gamma)} \frac{\exp(4\theta) + \exp(-i\Gamma)}{\exp(4\theta) + \exp(i\Gamma)}, \tag{5.224}$$

where

$$\begin{aligned} \Phi &= 2\Delta^2(\cos\Gamma)\,x - 8\Delta^4(\cos 2\Gamma)\,t, \\ \theta &= 2\Delta^2(\sin\Gamma)\,x - 8\Delta^4(\sin 2\Gamma)\,t. \end{aligned} \tag{5.225}$$

Let now all λ_i be real and equal to

$$\lambda_1 = \tfrac{1}{2}(\alpha+\beta), \quad \lambda_2 = \lambda_3 = \tfrac{1}{2}\beta, \quad \lambda_4 = -\tfrac{1}{2}(\alpha-\beta), \tag{5.226}$$

so that

$$\nu_1 = 4\beta^2, \quad \nu_2 = \nu_3 = \alpha^2, \quad \nu_4 = 0, \quad \cos^2(\Gamma/2) = \alpha^2/\left(4\beta^2\right),$$

and substitution into Eq. (5.221) gives

$$u(x,t) = \alpha \exp(i\Phi) \frac{\cosh(2\theta)\cosh(2\theta + i\Gamma/2)}{\cosh^2(2\theta - i\Gamma/2)}, \tag{5.227}$$

where

$$\begin{aligned} \Phi &= \left(\alpha^2+\beta^2\right)x + \left[\left(\alpha^2+\beta^2\right)^2 - 4\beta^4\right]t, \\ \theta &= \tfrac{1}{4}\alpha\sqrt{4\beta^2-\alpha^2}\left[x + \left(\alpha^2+2\beta^2\right)t\right]. \end{aligned} \tag{5.228}$$

This is the 'bright' soliton on the constant background, as one can see from the expression for the intensity,

$$\nu = |u(x,t)|^2 = \frac{4\alpha^2\beta^2}{\alpha^2 + (4\beta^2-\alpha^2)\tanh^2(2\theta)}. \tag{5.229}$$

In the same way one can consider the soliton limit of the periodic solution (5.205). The final result for the case (5.226) has the form

$$u(x,t) = i\alpha \exp(i\Phi) \frac{\sinh(2\theta)\cosh(2\vartheta - i\Gamma/2)}{\cosh^2(2\theta + i\Gamma/2)}, \tag{5.230}$$

where $\sin^2(\Gamma/2) = \alpha^2/\left(4\beta^2\right)$ and Φ, θ are given by Eqs. (5.228). The intensity is given by

$$\nu = |u(x,t)|^2 = \frac{4\alpha^2\beta^2}{\alpha^2 + (4\beta^2 - \alpha^2)\coth^2(2\theta)}. \tag{5.231}$$

This is the 'dark' soliton on the constant background.

The Whitham equations in the DNLS equation case can be derived in a usual way. Now we have $G = 2\lambda u,\ g = u\lambda\left(\lambda^2 - \mu\right)$, and the expression for B can be written in the form $B = 4\lambda u\left[s_1 + 2\left(\lambda^2 - \mu\right)\right]$, that is,

$$G/g_0 = 2,\ B/g_0 = 2\left[s_1 + 2\left(\lambda^2 - \mu\right)\right].$$

This coincides with Eq. (5.46) after replacement $\lambda \to \lambda^2$. The averaging with the use of Eq. (5.197) leads to actually the same Whitham equations for the Riemann invariants as for the NLS equation case,

$$\partial\lambda_i^2/\partial t + v_i \partial\lambda_i^2/\partial x = 0, \quad i = 1,2,3,4,$$

with the velocities given by Eqs. (5.51,5.52) where λ_i are replaced by $2\lambda_i^2$ and $V = -2s_1 = -2\sum\lambda_i^2$.

Bibliographic remarks

The multi-phase quasi-periodic solution of the NLS equation was obtained for the first time by Kotlyarov (1976) and Its and Kotlyarov (1976). These authors indicated also the importance of the condition $f^2 - gh = P(\lambda)$ for distinguishing the 'real' solutions (see also Tracy, Chen and Lee, 1984). For more detailed treatment see papers by Ma and Ablowitz (1981), Previato (1985), Forest and Lee (1986), Mertsching (1987), Tracy and Chen (1988). A general theory of the multi-phase quasi-periodic solutions of integrable equations is presented in the book by Belokolos, Bobenko, Enolskii, Its and Matveev (1994). Concrete calculations with the use of algebraic resolvents were given by Kamchatnov (1990c) and this approach was generalized on many other equations (see Kamchatnov, 1997), in particular on equations considered in this Chapter. Whitham equations for modulated NLS wave train were derived by Forest and Lee (1986) and Pavlov (1987). The method of derivation used in this Chapter was suggested by Kamchatnov (1990a, 1994).

Exercises on Chapter 5

Exercise 5.1
(Kamchatnov and Ginovart, 1996) Find the periodic solution of the two-photon propagation equations formulated in Exercise 4.5.

Exercise 5.2
(Kamchatnov and Steudel, 1997) The degenerate two-photon propagation (DTPP) equations describe the propagation of an electromagnetic wave with the frequency ω and the envelope electric field $\mathcal{E}$ in a medium with a resonance transition at the frequency 2ω. They differ from the TPP equations (see Exercise 4.5) only by the normalization condition for the vector $\mathbf{S}$,

$$-S_1^2 - S_2^2 + S_3^2 = 0.$$

Find the periodic solution of the DTPP equations and investigate its soliton limit.

Exercise 5.3
(Kamchatnov, Steudel and Zabolotskii, 1997) Show that the TPP equations (Exercise 4.5) can be transformed in the limit of weak excitation of the upper level of medium molecules $|R_\pm| \ll 1$ and relatively small intensity of one of the two participating waves $|\mathcal{E}_2| \ll |\mathcal{E}_1|$ to the equations

$$\begin{aligned}
-i\frac{\partial R_+}{\partial \tau} &= \Delta R_+ - S_+ + \tfrac{1}{2}\Delta S_+ S_- R_+ + \tfrac{1}{2} R_+ R_- S_+,\\
i\frac{\partial S_+}{\partial \zeta} &= \Delta S_+ + R_+ - \tfrac{1}{2}\Delta R_+ R_- S_+ + \tfrac{1}{2} S_+ S_- R_+.
\end{aligned}$$

which after replacements

$$S_+ \to S_+ \exp[i\Delta(\zeta - \tau)], \quad R_+ \to R_+ \exp[-i\Delta(\zeta - \tau)],$$

and

$$S_\pm \to -S_\pm, \qquad \zeta \to -\zeta,$$

take the form

$$\begin{aligned} i\partial R_+/\partial\tau + S_+ + \tfrac{1}{2}\Delta S_+ S_- R_+ + \tfrac{1}{2} R_+ R_- S_+ &= 0, \\ -i\partial S_+/\partial\zeta + R_+ + \tfrac{1}{2}\Delta R_+ R_- S_+ + \tfrac{1}{2} S_+ S_- R_+ &= 0. \end{aligned}$$

The last terms in these equations can be neglected in the limit of strong dynamic Stark shift $|\Delta| \gg 1$, and we arrive at the Thirring model equations,

$$\begin{aligned} i\partial R_+/\partial\tau + S_+ + \tfrac{1}{2}\Delta S_+ S_- R_+ &= 0, \\ -i\partial S_+/\partial\zeta + R_+ + \tfrac{1}{2}\Delta R_+ R_- S_+ &= 0. \end{aligned}$$

Find the corresponding limit of the $\mathbb{U}$–$\mathbb{V}$-pair matrix elements and derive the periodic and soliton solutions.

Exercise 5.4
(Kamchatnov, 1996) One may notice that the phase velocity V in the considered above particular cases is proportional to $1/\sqrt{P(\Delta)}$ (for properly chosen time and space variables), where Δ is the point of location of a pole singularity of the coefficients A, B, C in the complex λ plane. Give a general proof of this statement supposing that these coefficients have the form

$$B(\lambda) = \frac{b(\lambda)}{\lambda - \Delta}, \quad C(\lambda) = \frac{c(\lambda)}{\lambda - \Delta}, \quad A(\lambda)|_{pole} = \frac{a(\Delta)}{\lambda - \Delta},$$

where only a necessary for us singular part of $A(\lambda)$ is written.

Chapter 6

Dissipationless shock wave

6.1 Main equations and boundary conditions

In the preceding Chapters we have developed analytical techniques permitting one to find periodic solutions of a wide class of nonlinear wave equations and to describe slow evolution of modulations of these wave trains. As a result, we are now in a position to give a solution of the problem posed in the Introduction about formation and development of the oscillation region arising after wave breaking of a simple wave. Let us recall what this problem is about.

As we know, in many cases the evolution of a weakly nonlinear dispersive wave is described by the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{6.1}$$

If the initial distribution $u_0(x)$ is smooth enough, that is, the second term in this equation is much greater than the last one, then at first stage of the wave evolution we can neglect the dispersive term and reduce the KdV equation to the Hopf equation

$$u_t + 6uu_x = 0. \tag{6.2}$$

We can see at once from this nonlinear equation, that the parts of the distribution $u(x,t)$ with higher values of u propagate faster than the parts with lower values of u. This leads to steepening of the wave and, if the initial distribution $u_0(x,0)$ has an inflexion point, at some moment t_b the point x_b appears where the distribution $u(x,t)$ has a vertical tangent line ($u_x \to \infty$ as $x \to x_b$, $t \to t_b$). This point (x_b, t_b) is called a wave-breaking point of the wave $u(x,t)$. After the moment t_b the distribution $u(x,t)$, $t > t_b$, becomes multi-valued—there is a region of x where to each x correspond three values of $u(x)$ (see Fig. 1.6). Analytical solution of Eq. (6.2) is given in an implicit form by the formula (see Eq. (1.105))

$$x - 6ut = \bar{x}(u), \tag{6.3}$$

where $\bar{x}(u)$ is a function inverse to the initial distribution $u_0(x)$. It is clear that such a multi-valued solution does not have any physical sense and its appearance is connected with the fact that we have neglected the dispersive effects in vicinity of the wave-breaking point where the last term in Eq. (6.2) cannot be considered as small compared with the second nonlinear term. On the other hand, as it was shown in Sec. 1.2.1, a linear part of the KdV equation gives rise to oscillations in the distribution $u(x,t)$ (see Fig. 1.2). Therefore, it is natural to suppose that the dispersive effects will modify the wave-breaking phenomenon in such a way that the steepening front evolves into the region $x^- < x < x^+$ filled by relatively fast regular oscillations of the distribution $u(x,t)$. This is the basis supposition of the developed below theory of a 'dissipationless shock wave'.

Gurevich and Pitaevskii (1973) were the first who indicated that evolution of the region of regular oscillations ('dissipationless shock wave') can be described at large enough values of time by the Whitham modulation theory. This means that in this region the wave $u(x,t)$ is represented as a modulated cnoidal wave (see Eq. (3.151)). Here it is convenient to pass from the parameters λ_1, λ_2, λ_3 to another set of the parameters,

$$r_1 = -\lambda_3, \quad r_2 = -\lambda_2, \quad r_3 = -\lambda_1, \qquad r_1 \le r_2 \le r_3, \tag{6.4}$$

which are used traditionally in the theory of modulations of the KdV wave trains. Then the periodic solution (3.151,3.152) takes the form

$$u(x,t) = r_2 + r_3 - r_1 - 2(r_2 - r_1)\,\mathrm{sn}^2(\sqrt{r_3 - r_1}\,\xi, m) \tag{6.5}$$

where

$$\xi = x - Vt, \quad V = 2(r_1 + r_2 + r_3), \quad m = (r_2 - r_1)/(r_3 - r_1). \tag{6.6}$$

The wavelength is equal to

$$L = 2\mathrm{K}(m)/\sqrt{r_3 - r_1}, \tag{6.7}$$

and the maximum and minimum values of u are given by

$$u_{max} = r_2 + r_3 - r_1, \quad u_{min} = r_1 + r_3 - r_2, \tag{6.8}$$

that is, the amplitude of oscillations is characterized by the parameter

$$a = \tfrac{1}{2}\left(u_{max} - u_{min}\right) = r_2 - r_1. \tag{6.9}$$

Following to Gurevich and Pitaevskii (1973), we suppose that the region of oscillation $u(x,t)$ is presented by the modulated wave (6.5), that is, the parameters r_1, r_2, r_3 are functions of x and t and their evolution is governed by the Whitham equations

$$\partial r_i/\partial t + v_i(r)\partial r_i/\partial x = 0, \quad i = 1,2,3, \tag{6.10}$$

where the characteristic speeds $v_i(r)$ are given by (see Eq. (3.146))

$$v_i(r) = \left(1 - (\partial_i \ln L)^{-1}\partial_i\right)V, \quad \partial_i = \partial/\partial r_i, \quad i = 1,2,3, \tag{6.11}$$

or in explicit form by

$$\begin{aligned} v_1(r_1,r_2,r_3) &= 2(r_1+r_2+r_3) + \frac{4(r_2-r_1)\mathrm{K}(m)}{\mathrm{E}(m)-\mathrm{K}(m)}, \\ v_2(r_1,r_2,r_3) &= 2(r_1+r_2+r_3) - \frac{4(r_2-r_1)(1-m)\mathrm{K}(m)}{\mathrm{E}(m)-(1-m)\mathrm{K}(m)}, \\ v_3(r_1,r_2,r_3) &= 2(r_1+r_2+r_3) + \frac{4(r_3-r_1)(1-m)\mathrm{K}(m)}{\mathrm{E}(m)}. \end{aligned} \tag{6.12}$$

At the trailing edge $x^-(t)$ of the oscillation region, where $r_1 = r_2$, $m = 0$, the amplitude a of oscillations vanishes and the solution (6.5) goes to the limit

$$u(x,t)|_{r_1=r_2} = r_3. \tag{6.13}$$

At the leading edge $x^+(t)$ of the oscillation region, where $r_2 = r_3$, $m = 1$, the solution (6.5) goes to the soliton solution

$$u(x,t)|_{r_2=r_3} = \frac{2(r_3-r_1)}{\cosh^2[\sqrt{r_3-r_1}(x-V_s t)]} + r_1, \quad V_s = 2(r_1+2r_3). \tag{6.14}$$

The averaged over the wavelength value of $u(x,t)$,

$$\bar{u} = 2(r_3-r_1)\mathrm{E}(m)/\mathrm{K}(m) + r_1 + r_2 - r_3, \tag{6.15}$$

in these limiting points is equal, correspondingly, to

$$\bar{u}|_{r_1=r_2} = r_3, \quad \bar{u}|_{r_2=r_3} = r_1. \tag{6.16}$$

The coalescence of the Riemann invariants leads to the coalescence of the Whitham velocities (6.12). We have

$$v_1|_{r_1=r_2} = v_2|_{r_1=r_2} = 12r_1 - 6r_3, \quad v_3|_{r_1=r_2} = 6r_3, \tag{6.17}$$

and

$$v_1|_{r_2=r_3} = 6r_1, \quad v_2|_{r_2=r_3} = v_3|_{r_2=r_3} = 2r_1 + 4r_3. \tag{6.18}$$

As we see, at the trailing edge point $x = x^-(t)$, where $u(x,t)$ and its averaged value coincide with the Riemann invariant r_3 (see Eqs. (6.13) and (6.16)), its evolution is governed by the limiting form of the Whitham equation

$$\partial r_3/\partial t + 6r_3\partial r_3/\partial x = 0, \quad r_2 = r_1, \quad x = x^-(t), \tag{6.19}$$

which coincides with the Hopf equation (6.2) for the variable $u(x,t)$ in the dispersionless limit. In a similar way, at the leading edge point $x = x^+(t)$, where the averaged value of $\bar{u}(x,t)$ coincides with the Riemann invariant r_1 (see Eq. (6.16)), its evolution is governed again by the same Hopf equation

$$\partial r_1/\partial t + 6r_1\partial r_1/\partial x = 0, \quad r_2 = r_3, \quad x = x^+(t). \tag{6.20}$$

Thus, we arrive at the conclusion that at the trailing edge of the oscillation region the Whitham equations satisfy the conditions

$$v_1|_{r_1=r_2} = v_2|_{r_1=r_2}, \quad v_3|_{r_1=r_2} = r^-, \tag{6.21}$$

and at the leading edge to the conditions

$$v_1|_{r_2=r_3} = r^+, \quad v_2|_{r_2=r_3} = v_3|_{r_2=r_3}, \tag{6.22}$$

where $r^\pm$ are the corresponding edge values of the multi-valued solution

$$x - 6rt = \bar{x}(r) \tag{6.23}$$

of the Hopf equation

$$r_t + 6rr_x = 0, \tag{6.24}$$

which corresponds to the initial distribution $r = u_0(x)$. These boundary conditions determine as values of r^+ and r^- at the limiting points so, according to Eq. (6.23), the laws of their propagation. Then, if we are able to find the global solution of the Whitham equations (6.10), then we can obtain r_1, r_2, r_3 within the entire region $x^-(t) < x < x^+(t)$ and, hence, the description of the oscillation region. Hence, we have to find the method of solving the Whitham equations.

6.2 The generalized hodograph method

As it was shown in Sec. 1.3.5, the gas flow equations can be reduced to linear equations by means of the hodograph transform in which the space x and time t coordinates are considered as functions of the dependent variables (ρ, v) or of the Riemann invariants (λ^+, λ^-). Then the solution of these hydrodynamical equations can be obtained in an implicit form, $x = x(\lambda^+, \lambda^-)$, $t = t(\lambda^+, \lambda^-)$, which permits one to investigate analytically various problems about the gas flow.

Similar approach (the generalized hodograph method) was introduced by S. Tsarev (1984) as a method of solving the hydrodynamical-type equations with number of unknown functions greater than two. Here we explain some basic notions of this method for the case of the Whitham equations (6.10). Let us look for the solution of these equations in the form similar to Eq. (6.3),

$$x - v_i(r)t = w_i(r), \quad i = 1, 2, 3. \tag{6.25}$$

Differentiation of these equations with respect to r_j gives the relations

$$-(\partial v_i/\partial r_j)t = \partial w_i/\partial r_j, \quad i \neq j,$$

from which the variable t can be excluded by means of Eqs. (6.25),

$$t = -(w_i - w_j)/(v_i - v_j),$$

so that the functions w_i have to satisfy the Tsarev equations

$$\frac{1}{w_i - w_j}\frac{\partial w_i}{\partial r_j} = \frac{1}{v_i - v_j}\frac{\partial v_i}{\partial r_j}, \quad i \neq j. \tag{6.26}$$

Thus, if we find the solution $w_i(r)$ of these equations for given $v_i(r)$, then we shall obtain the solution (6.25) of the Whitham equations (6.10).

To find the solution of Eqs. (6.26), it is important to notice that these equations can be presented as a commutation condition of the Whitham equations (6.10) and equations

$$\partial r_i/\partial\tau + w_i(r)\partial r_i/\partial x = 0, \quad i = 1, 2, 3. \tag{6.27}$$

Indeed, the compatibility condition

$$\partial^2 r_i/\partial\tau\partial t = \partial^2 r_i/\partial t\partial\tau$$

yields after simple calculations the equation

$$w_j\partial v_i/\partial r_j + v_i\partial w_i/\partial r_j = v_j\partial w_i/\partial r_j + w_i\partial v_i/\partial r_j$$

equivalent to Eq. (6.26). This observation can be formulated as a statement that evolution of r_i according to Eqs. (6.27) is a symmetry of the Whitham equations (6.10).

We know from the theory of the complete integrability of the KdV equation that it has an infinite number of symmetries connected with the higher KdV equations—if we introduce a parameter τ into $u(x, t, \tau)$, then the evolution of the solution $u(x, t, \tau)$ of the KdV equation according to

$$u_\tau = \frac{\partial}{\partial x}\frac{\delta I_n}{\delta u}, \quad n = 2, 3, \dots,$$

transforms $u(x, t, \tau)$ again into the solution of the KdV equation because of commutation of integrals of motion I_n with respect to the Poisson bracket (4.84). One may suggest that these symmetries are 'inherited' after averaging, so that we can take the Whitham velocities for the higher KdV equations as the functions w_i in Eqs. (6.27). Since the generating function of the Whitham velocities is known (see Eq. (4.135)),

$$w_i = \left(1 - \frac{1}{\partial_i \ln L}\partial_i\right)\frac{-4\lambda^{3/2}}{\sqrt{P(\lambda)}}, \quad \partial_i = \partial/\partial r_i, \tag{6.28}$$

where $P(\lambda) = (\lambda + r_1)(\lambda + r_2)(\lambda + r_3)$, we can obtain a variety of important particular solutions of the Whitham equation.

To justify the above suggestion, let us formulate the problem of solving the Tsarev equations in more general form. It is natural to look for w_i in the form

$$w_i = \left(1 - (\partial_i \ln L)^{-1}\partial_i\right) W. \tag{6.29}$$

With the use of Eqs. (6.11),

$$v_i = 2s_1 - 2(\partial_i \ln L)^{-1}, \quad s_1 = r_1 + r_2 + r_3, \tag{6.30}$$

we rewrite Eqs. (6.29) in the form

$$w_i = W + \left(\tfrac{1}{2}v_i - s_1\right)\partial_i W.$$

Then simple calculations give

$$\begin{aligned} w_i - w_j &= \left(\tfrac{1}{2}v_j - s_1\right)(\partial_i W - \partial_j W) + \tfrac{1}{2}(v_i - v_j)\partial_i W, \\ \partial_j w_i &= \partial_j W - \partial_i W + \left(\tfrac{1}{2}v_i - s_1\right)\partial_{ij} W + \tfrac{1}{2}\partial_j v_i \cdot \partial_i W, \end{aligned}$$

where $\partial_{ij} = \partial^2/\partial r_i \partial r_j$. Substitution of these expressions into Eq. (6.26) gives

$$\partial_j W - \partial_i W + \left(\tfrac{1}{2}v_i - s_1\right)\partial_{ij} W = \left(\tfrac{1}{2}v_j - s_1\right)(\partial_i W - \partial_j W)\tfrac{\partial_j v_i}{v_i - v_j}. \qquad (6.31)$$

To calculate the right hand side of Eq. (6.26), let us use an easily verified identity

$$\partial_{ij}\frac{1}{\sqrt{P(\lambda)}} = \frac{1}{2(r_i - r_j)}\left(\partial_i \frac{1}{\sqrt{P(\lambda)}} - \partial_j \frac{1}{\sqrt{P(\lambda)}}\right), \quad i \neq j. \qquad (6.32)$$

Since the wavelength L is equal (up to inessential here constant factor) to $L \propto \oint d\mu/\sqrt{P(\mu)}$, integration of Eq. (6.32) over the contour around the forbidden gap yields the relation

$$\frac{\partial_{ij} L}{\partial_i L - \partial_j L} = \frac{1}{2(r_i - r_j)}. \qquad (6.33)$$

Substitution of Eq. (6.30) into the right hand side of Eq. (6.26) gives the expression

$$\frac{\partial_j v_i}{v_i - v_j} = \frac{\partial_j L}{L} + \frac{\partial_j L}{\partial_i L} \cdot \frac{\partial_{ij} L}{\partial_i L - \partial_j L},$$

which with the help of Eq. (6.33) and the relation $L/\partial_i L = s_1 - \frac{1}{2}v_i$ reduces to

$$\frac{\partial_j v_i}{v_i - v_j} = \frac{1}{s_1 - (1/2)v_j}\left[1 + \frac{s_1 - \frac{1}{2}v_j}{2(r_i - r_j)}\right].$$

Substitution of this expression into Eq. (6.31) yields after simple transformations the equations for the potential W,

$$\partial_{ij} W - (\partial_i W - \partial_j W)/2(r_i - r_j) = 0, \quad i \neq j. \qquad (6.34)$$

These equations generalize the Euler-Poisson equation (1.186) on the situations with the number of Riemann invariants greater than two.

Comparison of Eq. (6.34) with the identity (6.32) shows that $W = 1/\sqrt{P(\lambda)}$ is a solution of Eq. (6.34), so we conclude at once that the generating function (6.28) leads indeed to the velocities of the commuting flows (6.27). Their linear combinations in the form of integrals over λ with arbitrary weights lead to a general solution of Eqs. (6.34) depending on arbitrary functions. However, we shall not consider here this general solution and confine ourselves to particular solutions given by the generating function (6.28).

Note that if the appropriate functions w_i are found, then the boundary conditions (6.21) and (6.22) can be presented in the form

$$w_1|_{r_1=r_2} = w_2|_{r_1=r_2}, \quad w_3|_{r_1=r_2} = \bar{x}(r_3) \tag{6.35}$$

and

$$w_1|_{r_2=r_3} = \bar{x}(r_1), \quad w_2|_{r_2=r_3} = w_3|_{r_2=r_3}, \tag{6.36}$$

correspondingly. Here we have taken into account that at the edge points of the oscillation region the solutions (6.25) have to match with the solution (6.23) of the Hopf equation.

One should note also that the KdV equation is invariant with respect to the Galileo transformation

$$x \to x + 6Ct, \quad t \to t, \quad u \to u + C, \tag{6.37}$$

and with respect to the scaling transformation

$$x \to x/C^{1/2}, \quad t \to t/C^{3/2}, \quad u \to Cu, \tag{6.38}$$

where in both cases C is a constant parameter. Besides that, as follows from homogeneity of the Whitham equations, they have self-similar solutions in the form

$$r_i(x,t) = t^{\gamma} R_i(xt^{-1-\gamma}), \tag{6.39}$$

where γ is an arbitrary index of self-similarity and $R_i(z)$ are the solutions of the system of ordinary differential equations

$$[(1+\gamma)z - v_i(R)]R_i' = \gamma R_i, \quad i = 1,2,3, \tag{6.40}$$

where $z = xt^{-1-\gamma}$, $R_i' \equiv dR_i/dz$, and $v_i(R) = t^{-\gamma}v(r)$, that is $v_i(R)$ are expressed in terms of R_i by the same formulas as $v_i(r)$ are expressed in terms of r_i.

Now we may turn to consideration of some typical situations.

6.3 Decay of an initial discontinuity

We shall start with the simplest example analogous to the dam problem (see Sec. 1.3.1) when $u_0(x)$ has a finite discontinuity, that is, $u = u^- = \text{const}$ for $x < 0$ and $u = u^+ = \text{const}$ for $x > 0$.

Let us assume at first that $u^- < u^+$, that is, the initial distribution does not lead to the wave-breaking phenomenon. By means of transformations (6.37) and (6.38) the initial distribution can be reduced to the form

$$u_0(x) = u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{6.41}$$

If we neglect the dispersion effects, then it is easy to obtain a self-similar solution of the Hopf equation,

$$u(x,t) = \begin{cases} 0, & x < 0, \\ x/6,t & 0 \le x \le 6t, \\ 1, & x > 6t, \end{cases} \tag{6.42}$$

corresponding to the initial condition (6.41); see Fig 6.1. This solution is similar to the solution of the dam problem found in Sec. 1.3.1 for the case of a dust matter (with the pressure equal identically to zero). At the end points $x = 0$, $x = 6t$ of the non-uniform region this solution has jumps of the derivative. These singularities have to be smoothed by the dispersion effects due to the last term in the KdV equation (6.1), but they do not lead to formation of regions of strong oscillations. Thus, for the step-like initial condition with $u^- < u^+$, when there is no wave breaking, the dispersion effects are relatively small and the exact solution does not differ much from the solution of the Hopf equation where the dispersion effects are not taken into account at all.

Let now $u_0(x)$ have at the initial moment the step-like discontinuity with $u^- > u^+$. Then we can transform it with the use of Eqs. (6.37) and

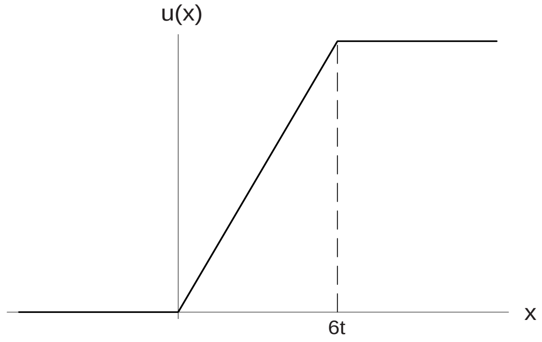

Fig. 6.1 The solution of the Hopf equation for the step-like initial conditions (6.41).

(6.38) to a convenient form

$$u_0(x) = u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{6.43}$$

In the dispersionless approximation we obtain the solution of the Hopf equation

$$u(x,t) = \begin{cases} 1, & x < 6t, \\ x/6t, & 0 \leq x \leq 6t, \\ 0, & x > 6t, \end{cases}$$

which is multi-valued in the region $0 < x < 6t$ (see Fig. 6.2). According to our approach we assume that instead of this multi-valued region there must appear a modulated nonlinear periodic wave which evolution is governed by the Whitham equations. Since the initial conditions do not include any parameters with a dimension of length, the solution of the Whitham equations must be self-similar (see Eq. (6.39)) with $\gamma = 0$, that is,

$$r_i = r_i(z), \quad z = x/t, \tag{6.44}$$

where $r_i(z)$ satisfy the differential equations (see Eqs. (6.40))

$$(v_i - z)(dr_i/dz) = 0. \tag{6.45}$$

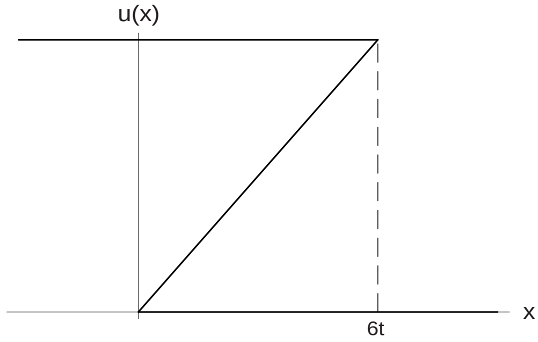

Fig. 6.2 A multi-valued solution of the Hopf equation for the step-like initial conditions (6.43).

At the trailing edge $z = z^-$, where oscillations disappear, that is, $r_1 = r_2$ and a mean value $\overline{u}$ coincides with $u = 1$ (see Eq. (6.16)), we have the boundary condition

$$r_1(z^-) = r_2(z^-), \quad r_3(z^-) = 1. \tag{6.46}$$

At the leading edge $z = z^+$, where $r_2 = r_3$ and the mean value $\bar{u} = r_1$ vanishes, we have another boundary condition

$$r_2(z^+) = r_3(z^+), \quad r_1(z^+) = 0. \tag{6.47}$$

It is easy to see that we obtain the solution of Eqs. (6.45), which satisfies the necessary boundary conditions, if we take

$$r_1 \equiv 0, \quad r_3 \equiv 1, \quad v_2 = z. \tag{6.48}$$

Then we have $m = (r_2 - r_1)/(r_3 - r_1) = r_2$ and the last equation (6.48) determines the dependence of the self-similar variable $z = x/t$ on r_2,

$$z = \frac{x}{t} = 2(1 + r_2) - \frac{4r_2(1 - r_2)\mathrm{K}(r_2)}{\mathrm{E}(r_2) - (1 - r_2)\mathrm{K}(r_2)}. \tag{6.49}$$

Taking the limit $r_2 \to 0$, we find the value of the self-similar variable at the trailing edge,

$$z^- = -6 \quad \text{or} \quad x^- = -6t, \tag{6.50}$$

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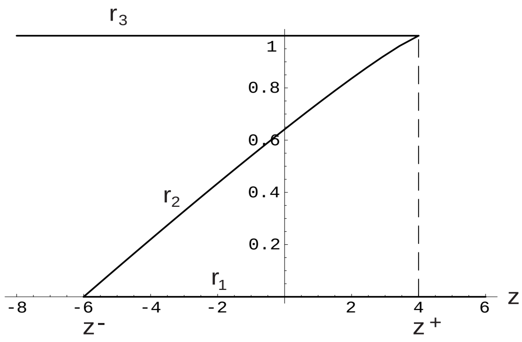

Fig. 6.3 The dependence of the Riemann invariants on the self-similar variable $z = x/t$ for the step-like initial conditions (6.43).

that is the region of oscillations expands into the undisturbed region with the velocity $v_g = -6$ equal to the group velocity of linear waves of modulations propagating on the background $u = 1$ (when $u = 1 + u'$, $u'_t + 6u'_x + u'_{xxx} = 0$) with the dispersion law $\Omega = 6K - K^3$. Indeed, the group velocity $d\Omega/dK = 6 - 3K^2$ equals to $\Omega' = v_g = -6$ at the wavelength (6.7) equal to $L(0) = \pi$ (or at the wavevector $K = 2\pi/L = 2$).

At the leading edge we have $r_2 \to 1$ and Eq. (6.49) gives

$$z^+ = 4 \quad \text{or} \quad x^+ = 4t, \tag{6.51}$$

that is, this edge moves with the soliton velocity $V_s = 4r_3 = 4$. The dependence of $r_2 = m$ on the coordinate $z'' = 4 - z$, $|z''| \ll 1$ near the leading edge is determined by the equation

$$z'' \cong 2(1 - m)\ln(16/(1 - m))$$

which gives with logarithmic accuracy

$$1 - m \cong z''/2\ln(1/z'').$$

Hence, the distance between solitons near the leading edge (where $4t - x \sim 1$, or $4 - z = z'' \sim 1/t$) grows with time according to the law

$$L = 2\mathrm{K}(m)/\sqrt{r_3 - r_1} \cong \pi \ln(1/z'') = \pi \ln t. \tag{6.52}$$

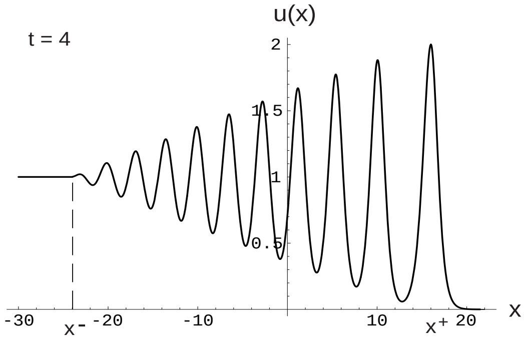

Fig. 6.4 The dissipationless shock wave the for step-like initial conditions (6.43) at the moment $t = 4$.

The whole dependence of $r_2 = m$ on z is shown in Fig. 6.3. Substitution of the found values of the Riemann invariants into Eq. (6.5) yields the parametric expression for $u(x,t)$ in the dissipationless shock wave

$$u(x,t) = 1 + r_2 - 2r_2 \operatorname{sn}^2(x(r_2) - 2(1+r_2)t, r_2), \tag{6.53}$$

where the dependence $x(r_2)$ at given t is defined by Eq. (6.49). An example of the dependence $u(x)$ on x at the moment $t = 4$ is shown in Fig. 6.4. Thus, we have obtained the analytic description of the evolution of the dissipationless shock wave arising from the initial step-like discontinuity.

6.4 Wave breaking of a parabolic initial profile

A step-like initial profile (6.43) is an idealization of a pulse with a very sharp leading front. Let us suppose now that at the wave-breaking moment $t = 0$ the pulse has a parabolic initial profile

$$u_0(x) = u(x,0) = \begin{cases} \sqrt{-x}, & x < 0, \\ 0, & x > 0. \end{cases} \tag{6.54}$$

that is, the wave profile goes to zero according to a square root law. Such a profile arises in the piston problem considered in Sec. 1.3.3. We can use the transformations (6.37) and (6.38) to reduce a general profile ($x|_{t=0} \propto -u^2$) to a simple form (6.54).

The solution of the Hopf equation with the initial condition (6.54) is given by the formula (see Eq. (6.3))

$$x - 6ut = -u^2, \tag{6.55}$$

that is, we have a multi-valued region $0 < x < 9t^2$ for $t > 0$. This means that due to the dispersion effects a dissipationless shock wave must appear instead of this region. At the trailing edge it matches with the solution (6.55), that is (see Eq. (6.35)),

$$w_3|_{r_1=r_2} = -u^2, \quad u = r_3^-, \tag{6.56}$$

and, hence, we are interested in such a solution (6.25) in which the velocities w_i are quadratic functions of the Riemann invariants in the limit $m \to 0$. Fortunately, higher symmetries of the KdV equation yield such velocities w_i which are power functions of r_i in the limit $m \to 0$. In particular, the coefficient of λ^{-2} in the generating function (6.28) gives just the required quadratic dependence. Thus, we take as $w_i(r)$ the functions

$$\begin{aligned} &w_i = C\left(1 - (L/\partial_i L)\partial_i\right) W_2, \quad W_2 = 2s_2 - \tfrac{3}{2}s_1^2, \\ &s_2 = r_1r_2 + r_2r_3 + r_3r_1, \quad s_1 = r_1 + r_2 + r_3, \end{aligned} \tag{6.57}$$

where C is some constant. At the leading edge x^+ the mean amplitude vanishes and this condition gives $r_1 = 0$ and $r_2 = r_3$. Thus, in this solution only one Riemann invariant r_1 is constant and equal to zero, whereas the two other Riemann invariants are some functions of x and t coinciding with each other at the leading edge,

$$r_2 = r_3, \quad \text{that is,} \quad m = r_2/r_3 = 1 \quad \text{at} \quad x = x^+. \tag{6.58}$$

At the trailing edge we have

$$r_2 = r_1 = 0, \quad \text{that is,} \quad m = r_2/r_3 = 0 \quad \text{at} \quad x = x^-. \tag{6.59}$$

The Whitham equation for r_1 is satisfied identically, whereas r_2 and r_3 are determined by the formulas

$$x - v_2t = w_2, \quad x - v_3t = w_3, \tag{6.60}$$

where v_2 and v_3 are the usual Whitham velocities with $r_1 = 0$,

$$\begin{aligned} v_2 &= 2r_3\left(1 + m - \frac{2m(1-m)\mathrm{K}(m)}{\mathrm{E}(m) - (1-m)\mathrm{K}(m)}\right), \\ v_3 &= 2r_3\left(1 + m + \frac{2(1-m)\mathrm{K}(m)}{\mathrm{E}(m)}\right); \end{aligned} \tag{6.61}$$

w_2 and w_3 are the velocities defined by Eqs. (6.57) with $r_1 = 0$,

$$\begin{aligned} w_2 &= Cr_3^2\left(2m - \tfrac{3}{2}(1+m)^2 + \frac{2m(1-m^2)\mathrm{K}(m)}{\mathrm{E}(m) - (1-m)\mathrm{K}(m)}\right), \\ w_3 &= Cr_3^2\left(2m - \tfrac{3}{2}(1+m)^2 - \frac{2(1-m)(3+m)\mathrm{K}(m)}{\mathrm{E}(m)}\right), \end{aligned} \tag{6.62}$$

and the constant C is to be found from the boundary condition (6.56). Since at $m \to 0$ we have $w_3 = -\frac{15}{2}Cr_3^2$, hence, $C = 2/15$, and we arrive at the final formulas for the solution of the Whitham equations

$$\begin{aligned} x - v_2 t &= \tfrac{2}{15}\left[W + \left(\tfrac{1}{2}v_2 - s_1\right)\partial W/\partial r_2\right], \\ x - v_3 t &= \tfrac{2}{15}\left[W + \left(\tfrac{1}{2}v_3 - s_1\right)\partial W/\partial r_3\right], \end{aligned} \tag{6.63}$$

where $W = 2r_2r_3 - \frac{1}{2}(r_2 + r_3)^2$, $s_1 = r_2 + r_3$.

At the trailing edge these two equations reduce to

$$x^- + 6r_3^- t = \tfrac{1}{3}(r_3^-)^2, \quad x^- - 6r_3^- t = -(r_3^-)^2,$$

which give at once the parametric representation,

$$x^- = -\tfrac{1}{3}(r^-)^2, \quad t = \tfrac{1}{9}r^-,$$

of the trailing edge motion law, that is,

$$x^- = -27t^2. \tag{6.64}$$

At the leading edge, where $r_2 = r_3$, both equations (6.63) tend to the same one,

$$x^+ - 4r_3 t = -\tfrac{8}{15}r_3^2,$$

and the value of r^+ is determined by the maximum value of x in the oscillation region, that is, $dx^+/dr_3|_{t=\mathrm{const}} = 0$, hence, $r_3 = 15t/4$, and

$$x^+ = \tfrac{15}{2}t^2. \tag{6.65}$$

This is the motion law of the leading (soliton) edge.

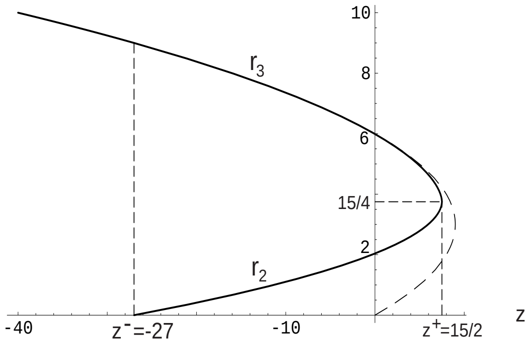

Fig. 6.5 The dependence of the Riemann invariants r_3 and r_2 on the self-similar variable $z = x/t^2$ for the parabolic initial condition ($r_1 \equiv 0$). A dashed curve shows the solution of the Hopf equation with the same initial conditions. The vertical dashed lines indicate the region of modulated oscillations.

From $v \propto r\tilde{v}(m)$, $w \propto r^2\tilde{w}(m)$ it follows that this is a self-similar solution (see Eq. (6.39)) with $\gamma = 1$. Dependence of R_2, R_3 on the self-similar variable $z = x/t^2$ is determined by the equations

$$\begin{aligned} z - v_2(R) &= \tfrac{2}{15}\left[\mathrm{W}(R) + \left(\tfrac{1}{2}v_2(R) - s_1\right)\partial\mathrm{W}(R)/\partial R_2\right], \\ z - v_3(R) &= \tfrac{2}{15}\left[\mathrm{W}(R) + \left(\tfrac{1}{2}v_3(R) - s_1\right)\partial\mathrm{W}(R)/\partial R_3\right], \end{aligned} \tag{6.66}$$

where $W(R)$, $v_i(R)$, s_1, s_2 are determined in terms of R_i by the same formulas as usual variables do in terms of r_i. The dependence of the Riemann invariants on z is shown in Fig. 6.5. When v_i, w_i are known, we can find the dependence of x and t on r_2 and r_3 by means of Eqs. (6.60),

$$x = \frac{w_3 v_2 - w_2 v_3}{v_2 - v_3}, \quad t = -\frac{w_2 - w_3}{v_2 - v_3}. \tag{6.67}$$

Substitution of the found expressions into Eqs. (6.5) and (6.6) yields the parametric form of the function $u(x,t)$ in the oscillation region. Example of such a plot $u(x,t)$ at fixed value of t is shown in Fig. 6.6.

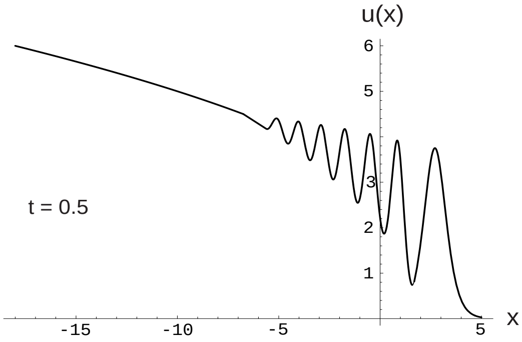

Fig. 6.6 The dissipationless shock wave the for the parabolic initial conditions.

6.5 Wave breaking of a cubic initial profile (Gurevich-Pitaevskii problem)

The problems considered in the preceding two sections correspond to quite specific initial conditions with either step-like profile $u_0(x)$ or its parabolic dependence $u_0 \propto (-x)^{1/2}$. However, in typical cases the evolution of an initial pulse leads in the dispersionless approximation to appearance of the wave-breaking point with vertical tangent line (see Sec. 1.3.2) at some moment, and after this moment the profile has a form of a cubic parabola in vicinity of the wave-breaking point. Indeed, at the wave-breaking point $t = t_b$ the general solution

$$x = 6ut + \bar{x}(u) \tag{6.68}$$

of the Hopf equation satisfies, besides the condition

$$\partial x/\partial u|_{t=t_b} = 0$$

that $u(x, t_b)$ has a vertical tangent line, also the condition

$$\partial^2 x/\partial u^2|_{t=t_b} = 0,$$

that this is an inflexion point. We may choose the wave-breaking moment t_b, space coordinate x, and the transformation (6.37) so that at this point we have $t_b = 0$, $x_b = 0$, $u(x_b, t_b) = 0$, that is,

$$\bar{x}(u) \cong -\mathrm{const} \cdot u^3.$$

Transformation (6.38) permits us to make the proportionality constant equal to unity. As a result, we arrive at the solution of the Hopf equation

$$x - 6ut = -u^3 \tag{6.69}$$

corresponding to the initial condition

$$\bar{x}(u) = -u^3 \quad \text{at} \quad t = 0. \tag{6.70}$$

Typical initial profiles lead just to this distribution in vicinity of the wave-breaking point. Our aim is to calculate the oscillatory structure arising due to the dispersion effects after such a wave breaking.

As in the preceding section, we notice that the velocities $w_i(r)$ corresponding to the third commuting flow in the expansion (6.28) give at the end points $m = 0$ and $m = 1$ functions cubic in r_i. More definitely, it is easy to find that the formulas

$$w_i = (1 - (L/\partial_i L)\partial_i)\, W_3, \tag{6.71}$$

where

$$\begin{aligned} &W_3 = -\tfrac{5}{4}s_1^3 + 3s_1 s_2 - 2s_3, \\ &s_1 = r_1 + r_2 + r_3, \quad s_2 = r_1 r_2 + r_2 r_3 + r_3 r_1, \quad s_3 = r_1 r_2 r_3, \end{aligned} \tag{6.72}$$

give

$$w_3 = -\tfrac{35}{4} r_3^3 \quad \text{at} \quad m \to 0 \tag{6.73}$$

and

$$w_1 = -\tfrac{35}{4} r_1^3 \quad \text{at} \quad m \to 1. \tag{6.74}$$

Thus, we find that the solution of the Whitham equations (6.10) which matches with the solution (6.69) of the Hopf equation at the end points is given by the formulas

$$x - 6v_i t = \tfrac{4}{35} w_i, \quad i = 1, 2, 3, \tag{6.75}$$

where functions w_i, $i = 1, 2, 3$, are defined by Eqs. (6.71,6.72). It is clear that this is a self-similar solution (see Eq. (6.39))

$$r_i = t^{1/2} R_i \left(x/t^{3/2} \right) \tag{6.76}$$

of Eqs. (6.40) corresponding to $\gamma = 1/2$, that is,

$$\frac{dR_i}{dz} = \frac{R_i}{3z - v_i(R)}. \tag{6.77}$$

Let us consider some its properties. At the trailing edge we have $r_1 = r_2$ $(m = 0)$ and Eq. (6.75) with $i = 3$ becomes

$$x - 6r_3 t = -r_3^3 \quad (\text{at } r_1 = r_2), \tag{6.78}$$

and at the leading edge we have $r_2 = r_3$ and Eq. (6.75) with $i = 1$ becomes

$$x - 6r_1 t = -r_1^3 \quad (\text{at } r_2 = r_3). \tag{6.79}$$

Thus, the boundary matching conditions are

$$r_3 = u \quad \text{at} \quad r_1 = r_2 \quad \text{and} \quad r_1 = u \quad \text{at} \quad r_2 = r_3, \tag{6.80}$$

where u is the solution (6.69) of the Hopf equation.

In vicinity of the trailing edge we introduce a local coordinate x',

$$x = x^- + x', \tag{6.81}$$

and small deviations r_i' of the Riemann invariants from their limiting values,

$$r_1 = r_1^- + r_1', \quad r_2 = r_2^- + r_2', \quad r_3 = r_3^- + r_3'. \tag{6.82}$$

The expansions of Eqs. (6.75) in powers of r_i' at some fixed value of t are given by

$$\begin{aligned}
&x^- + x' - (12r_1 - 6r_3)t - (9r_1' + 3r_2' - 6r_3')t \\
&\quad = \tfrac{1}{5}(-16r_1^3 + 8r_1^2 r_3 + 2r_1 r_3^2 + r_3^3) - \tfrac{3}{10}(24r_1^2 - 8r_1 r_3 - r_3^2)r_1' \\
&\quad - \tfrac{1}{10}(24r_1^2 - 8r_1 r_3 - r_3^2)r_2' + \tfrac{1}{5}(8r_1^2 + 4r_1 r_3 + 3r_3^2)r_3', \\
&x^- + x' - (12r_1 - 6r_3)t - (3r_1' + 9r_2' - 6r_3')t \\
&\quad = \tfrac{1}{5}(-16r_1^3 + 8r_1^2 r_3 + 2r_1 r_3^2 + r_3^3) - \tfrac{1}{10}(24r_1^2 - 8r_1 r_3 - r_3^2)r_1' \\
&\quad - \tfrac{3}{10}(24r_1^2 - 8r_1 r_3 - r_3^2)r_2' + \tfrac{1}{5}(8r_1^2 + 4r_1 r_3 + 3r_3^2)r_3', \\
&x^- + x' - 6r_3 t - 6r_3' t = -r_3^3 - 3r_3^2 r_3',
\end{aligned} \tag{6.83}$$

where we denote temporarily $r_1 \equiv r_1^-$, $r_3 \equiv r_3^-$. Subtraction of the second equation from the first one yields the relation

$$t = \tfrac{1}{30}(24r_1^2 - 8r_1r_3 - r_3^2). \tag{6.84}$$

This means that the coefficients of r_1' and r_2' in the first two equations (6.83) are equal to zero and, hence, x' is a quadratic function of r_1', r_2'

$$x' \propto {r_1'}^2, {r_2'}^2, r_3'.$$

At the point x^- these two equations give

$$x^- - (12r_1 - 6r_3)t = \tfrac{1}{5}(-16r_1^3 + 8r_1^2r_3 + 2r_1r_3^2 + r_3^3), \tag{6.85}$$

and the third equation (6.83) reduces, as it was noted above, to the solution

$$x^- - 6r_3t = -r_3^3 \tag{6.86}$$

of the Hopf equation. Equations (6.84–6.86) allow us to determine the law of motion of the trailing edge. Subtraction of Eq. (6.86) from Eq. (6.85) and division of the result by $(r_1 - r_3)$ gives the relation

$$t = \tfrac{1}{30}(8r_1^2 + 4r_1r_3 + 3r_3^2),$$

so that its comparison with Eq. (6.84) yields the relationship between the values of the Riemann invariants at the trailing edge

$$r_1^- = r_2^- = -\tfrac{1}{4}r_3^-. \tag{6.87}$$

Then Eqs. (6.84) and (6.86) yield

$$t = \tfrac{1}{12}(r_3^-)^2, \quad x^- = -\tfrac{1}{2}(r_3^-)^3 \tag{6.88}$$

and the self-similar variable is equal to

$$z^- = x^-/t^{3/2} = -12\sqrt{3}. \tag{6.89}$$

The solution of Eqs. (6.77) satisfies at this point the relation

$$z^- - 6R_3^- = -(R_3^-)^3, \tag{6.90}$$

which follows from Eq. (6.86). Taking into account Eq. (6.89), we find

$$R_3^- = 2\sqrt{3} \tag{6.91}$$

and, hence,

$$R_1^- = R_2^- = -\tfrac{1}{4}R_3^- = -\tfrac{1}{2}\sqrt{3}. \tag{6.92}$$

From the third Eq. (6.83) we find the correction r_3' as a function of x',

$$x' = 6r_3' - 3r_3^2 r_1' = -\tfrac{5}{2}r_3^2 r_3',$$

that is,

$$r_3' = -\tfrac{2}{5}(r_3^-)^{-2}x' \tag{6.93}$$

or in terms of the self-similar variable we have

$$R_3' = -\frac{2}{5(R_3^-)^2}z' = -\tfrac{1}{30}z'. \tag{6.94}$$

To find analogous relations for R_1' and R_2', it is necessary to expand the global solution (6.75) up to quadratic terms in r_1' and r_2'. However, it is easier to resort for such a local investigation to the differential equations (6.77). In vicinity of $z = z^- = -12\sqrt{3}$ we have ($z = z^- + z'$)

$$\begin{aligned} v_1 &= -18\sqrt{3} + 9R_1' + 3R_2' - 6R_3', \\ v_2 &= -18\sqrt{3} + 3R_1' + 9R_2' - 6R_3', \\ v_3 &= 12\sqrt{3} + 6R_3', \end{aligned} \tag{6.95}$$

so that in view of $R_2' = -R_1' \propto (z')^{1/2}$ we obtain

$$dR_2'/dz' = (1/8\sqrt{3})(1/R_3'),$$

and, hence,

$$R_2' = -R_1' = \sqrt{z'/4\sqrt{3}}. \tag{6.96}$$

The third Eq. (6.77) with $i = 3$ reduces to $dR_3'/dz' = -1/30$ in agreement with Eq. (6.94).

Now let us turn to investigation of the leading front where

$$x = x^+ - x'', \quad x'' > 0, \tag{6.97}$$

and

$$r_1 = r_1^+ + r_1'', \quad r_2 = r_2^+ + r_2'', \quad r_3 = r_3^+ + r_3''. \tag{6.98}$$

Expansions of Eq. (6.34) with taking into account only the main correction terms are given by

$$\begin{aligned}
&x^+ - x'' - 6r_1 t + [8(r_3 - r_1)/\ln(16/(1-m))]\, t \\
&\quad = -r_1^3 + \tfrac{4}{35}(15r_1^2 + 12r_1r_3 + 8r_3^2)\,[(r_3 - r_1)/\ln(16/(1-m))]\,, \\
&x^+ - x'' - (2r_1 + 4r_3)t + 2\ln(16/(1-m))(r_3'' - r_2'')t \\
&\quad = -\tfrac{1}{35}(5r_1^3 + 6r_1^2 r_3 + 8r_1 r_3^2 + 16r_3^2) \\
&\quad + \tfrac{1}{35}(3r_1^2 + 8r_1r_3 + 24r_3^2)\ln(16/(1-m))(r_3'' - r_2''), \\
&x^+ - x'' - (2r_1 + 4r_3)t - 2\ln(16/(1-m))(r_3'' - r_2'')t \\
&\quad = -\tfrac{1}{35}(5r_1^3 + 6r_1^2 r_3 + 8r_1 r_3^2 + 16r_3^2) \\
&\quad - \tfrac{1}{35}(3r_1^2 + 8r_1r_3 + 24r_3^2)\ln(16/(1-m))(r_3'' - r_2''),
\end{aligned} \tag{6.99}$$

where

$$1 - m = (r_3'' - r_2'')/(r_3 - r_1), \tag{6.100}$$

and again we denote temporarily $r_1 \equiv r_1^+$ and $r_3 \equiv r_3^+$. Subtraction of the third Eq. (6.99) from the second one gives the relation

$$t = \tfrac{1}{70}(3r_1^2 + 8r_1r_3 + 24r_3^2), \tag{6.101}$$

which together with the zero order approximations of Eq. (6.99),

$$\begin{aligned}
&x^+ - 6r_1 t = -r_1^3, \\
&x^+ - (2r_1 + 4r_3)t = -\tfrac{1}{35}(5r_1^3 + 6r_1^2 r_3 + 8r_1 r_3^2 + 16r_3^3),
\end{aligned} \tag{6.102}$$

determine the law of motion of the leading edge. Indeed, the difference of Eqs. (6.102) yields another relation

$$t = \tfrac{1}{70}(15r_1^2 + 12r_1r_3 + 8r_3^2), \tag{6.103}$$

which after comparison with Eq. (6.101) gives

$$r_3^+ = -\tfrac{3}{4}r_1^+, \quad (r_1^+ < 0), \tag{6.104}$$

so that

$$t = \tfrac{3}{20}(r_1^+)^2, \quad x^+ = \tfrac{1}{10}|r_1^+|^3, \tag{6.105}$$

and the limiting value of the self-similar variable $z = x/t^{3/2}$ is equal to

$$z^+ = \tfrac{4}{9}\sqrt{15}. \tag{6.106}$$

From the first Eqs. (6.88) and (6.105) we find the relationship between r_3^- and r_3^+, and, consequently, the values of the Riemann invariants,

$$R_1^+ = -\tfrac{2}{3}\sqrt{15}, \quad R_2^+ = -R_3^+ = -\tfrac{1}{2}\sqrt{15}. \tag{6.107}$$

The dependence of R_i'' on z'' can be found again with the use of the differential equations (6.77) which give

$$dR_1''/dz'' = 1/14, \quad \text{that is,} \quad R_1'' = \tfrac{1}{14}z'', \tag{6.108}$$

and

$$\frac{dR_3''}{dz''} = -\frac{\sqrt{15}}{16}\frac{1}{R_3''\ln R_3''},$$

which integration yields the relation

$$(R_3'')^2\ln(1/R_3'') = (\sqrt{15}/8)z'', \tag{6.109}$$

so that with the logarithmic accuracy we have

$$R_3'' \simeq \frac{\sqrt{15}}{4}\frac{z''}{\ln(1/z'')}. \tag{6.110}$$

The global dependence of R_i on z is determined implicitly by the expressions

$$z = 6v_1 - w_1, \quad z = 6v_2 - w_2, \quad z = 6v_3 - w_3, \tag{6.111}$$

where

$$\begin{aligned} v_1 &= 2(R_1+R_2+R_3) + \frac{4(R_2-R_1)\mathrm{K}(m)}{\mathrm{E}(m)-\mathrm{K}(m)}, \\ v_2 &= 2(R_1+R_2+R_3) - \frac{4(R_2-R_1)(1-m)\mathrm{K}(m)}{\mathrm{E}(m)-(1-m)\mathrm{K}(m)}, \\ v_3 &= 2(R_1+R_2+R_3) + 4(R_3-R_1)(1-m)\mathrm{K}(m)/\mathrm{E}(m); \end{aligned} \tag{6.112}$$

and

$$w_i = W + (\tfrac{1}{2}v_i - R_1 - R_2 - R_3)W_{R_i}, \tag{6.113}$$

$$\begin{aligned} W =& \tfrac{4}{35}[-\tfrac{5}{4}(R_1+R_2+R_3)^3 \\ &+ 3(R_1+R_2+R_3)(R_1R_2+R_2R_3+R_3R_1) - 2R_1R_2R_3], \\ W_{R_1} =& -\tfrac{1}{35}[15R_1^2 + 3(R_2^2+R_3^2) + 6R_1(R_2+R_3) + 2R_2R_3], \end{aligned}$$

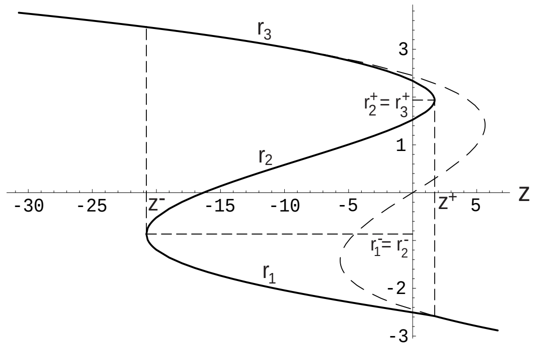

Fig. 6.7 The dependence of the Riemann invariants r_1, r_2 and r_2 on the self-similar variable $z = x/t^{3/2}$ for a cubic initial profile. A dashed curve shows the solution of the Hopf equation with the same initial conditions. Vertical dashed lines indicate the region of modulated oscillations.

W_{R_2} and W_{R_3} are obtained from W_{R_1} by cyclic permutations of R_i. This dependence is shown in Fig. 6.7, where the dashed line denotes the cubic curve

$$z = 6R - R^3, \tag{6.114}$$

which matches with the Riemann invariants R_3 and R_1 at the points z^- and z^+, correspondingly.

When the invariants $r_i = t^{1/2} R_i(x/t^{1/2})$ are found, their substitution into Eq. (6.5) yields the description of the oscillatory structure arising in vicinity of the wave-breaking point due to dispersion effects. The edge points of this structure propagate according to the laws

$$x^- = -12\sqrt{3}\, t^{3/2}, \quad x^+ = \tfrac{4}{9}\sqrt{15}\, t^{3/2}, \tag{6.115}$$

the amplitude of oscillations tends to zero at the trailing edge according to the law

$$a = r_2 - r_1 \simeq 2r_2' \propto \sqrt{x'}, \tag{6.116}$$

and the distance between solitons at the leading edge depends on x'' as

$$L \propto \ln(1/|x''|). \tag{6.117}$$

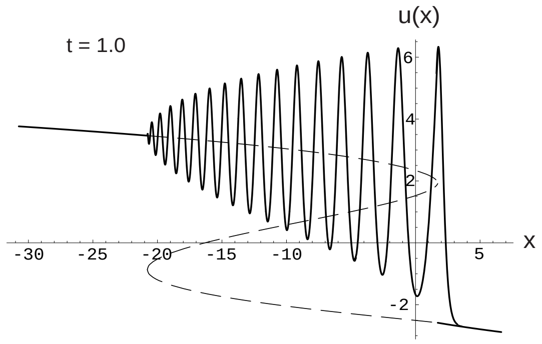

Fig. 6.8 The dissipationless shock wave the for a cubic initial profile. A dashed curve shows the behaviour of the Riemann invariants inside the oscillatory region.

This oscillatory structure is illustrated in Fig. 6.8. The considered here self-similar solution is valid as long as the pulse beyond the oscillatory structure is given with the sufficient accuracy by the cubic curve (6.69). At larger values of time it is necessary to use more general solution which can also be found by the generalized hodograph method, but we shall not consider this generalization here.

Thus, we have succeeded in development of the analytic theory of dissipationless shock wave in dispersive wave systems whose behaviour is described by the KdV equation. Analogous approach may be used for the defocusing NLS equation and other evolution equations describing nonlinear waves in modulationally stable systems. However, the Whitham approach turns out to be useful also for description of modulationally unstable systems which will be considered in the last Chapter of the book.

Bibliographic remarks

The idea that wave breaking leads in dissipationless case to formation of oscillatory structures belongs to Sagdeev (1964). Gurevich and Pitaevskii (1973) showed that this problem can be solved by the Whitham method. They found the solution for the step-like initial data and studied local behaviour near the leading and trailing edges for a cubic initial profile.

The global behavior was investigated numerically. The analytic solution (6.71,6.75) was found by Potemin (1988) by the method of Krichever (1988). The parabolic initial profile was considered by Gurevich, Krylov, and Mazur (1989) and Kudashev and Sharapov (1990). The generalized hodograph transform method was developed by Tsarev (1985, 1990) (see also Dubrovin and Novikov, 1993), and it was applied to the general initial value problem by Gurevich, Krylov and El (1991, 1992) and Tian (1993,1994). Similar theory for the defocusing NLS equation wave trains was developed by Gurevich and Krylov (1987), El, Geogjaev, Gurevich and Krylov (1995), El and Krylov (1995).

Exercises on Chapter 6

Exercise 6.1
Show that in vicinity of singular points of the solution (6.42) of the Hopf equation the solutions of the KdV equation have a self-similar form. Derive the corresponding ordinary differential equations for $u(z)$, z being an appropriate self-similar variable.

Exercise 6.2
Show that the Whitham equations for the KdV wave train with small ($m \ll 1$) but finite amplitude and with zero mean value $\overline{u}$ can be reduced to the shallow water equations (1.272).

Exercise 6.3
Let the Whitham equations for moderately small amplitude wave train be presented in a hydrodynamical form indicated in the preceding exercise and transformed to the reference frame moving with characteristic velocity v_0 connected with the phase velocity of the modulated wave train,

$$\rho_t + \rho v_x + v\rho_x = 0, \quad v_t + vv_x + \rho_x = 0,$$

where we have omitted primes in variables x', t', v'. Find the solution of this system which satisfies the initial condition with a parabolic 'density of vibrations',

$$\rho(x, t = 0) = \begin{cases} 1 - x^2, & |x| \leq 1, \\ 0, & |x| > 1, \end{cases}$$

and

$$v(x, t = 0) = 0.$$

Exercise 6.4
(Karpman, 1967a) As we know, the soliton solution of the KdV equation corresponds to the single eigenvalue $\lambda = -a$ of the equation

$$\psi_{xx} + u_s(x, t)\psi = -\lambda\psi$$

with the potential

$$u_s(x, t) = \frac{2a}{\cosh^2\left[\sqrt{a}(x - 4at)\right]}, \qquad a > 0.$$

On the other hand, evolution of the potential $u(x, t)$ according to the KdV equation preserves the spectrum of the equation $\psi_{xx} + u(x, t)\psi = -\lambda\psi$. One can show that higher degree polynomials $P(\lambda) = (\lambda - \lambda_{2N+1}) \prod_{i=1}^{N} (\lambda - \lambda_i)^2$ with degenerate zeros λ_i correspond to the multi-soliton solutions $u(x, t)$ of the KdV equation with N eigenvalues. Therefore one can expect that an initial distribution $u_0(x)$ with N eigenvalues will evolve ultimately into a multi-soliton solution of the KdV equation. Find asymptotic distribution of soliton amplitudes and density of solitons for an initial pulse profile $u_0(x)$ with a large number of eigenvalues $N \gg 1$.

Chapter 7

Nonlinear theory of modulational instability

7.1 The physical nature of the modulational instability

We have already discussed the problem of modulational instability of a plane wave described by the NLS equation. As it was shown in Sec. 1.6.3, linearization of this equation with respect to small amplitude of modulation leads to the dispersion relation which gives complex values of the frequency at small enough values of the wavevector. This means that such a wave of modulation grows with time exponentially. In this Chapter we shall try to clarify what happens at larger values of time when the amplitude of modulation cannot be considered as small. But at first we shall consider the physical nature of the modulational instability, and this consideration will serve us as a basis for the nonlinear theory.

So, we return to a linear theory of the modulational instability. As we know, the NLS equation describes evolution of a wave packet made of a large number of harmonics with frequencies in vicinity of the frequency of the so called 'carrier wave'. From this point of view, the modulational instability means that in the wave packet, which represents a slowly modulated plane wave, some side harmonics with wavevectors $k \pm K$ and frequencies $\omega(k \pm K)$ grow on account of the carrier wave with the wavevector k and frequency $\omega(k)$. Thus, from a physical point of view, there is a transformation of the carrier wave into two side harmonics caused by a nonlinear interaction of these three waves. For possibility of this transformation, the dispersion law $\omega(k)$ must be such that the conservation laws of the energy and the momentum are fulfilled. When the amplitude of the carrier wave is much greater than the amplitudes of the two side harmonics, this carrier wave may be considered as an infinite source of energy and one can neglect its change. Then we have to see an exponential growth of the side harmonics amplitudes.

To explain this approach to modulational instability, let us consider a linear theory from this point of view in some more detail. To be definite, we shall consider modulation of a plane wave governed by the nonlinear Klein-Gordon equation

$$q_{tt} - q_{xx} + q - \gamma q^3 = 0, \tag{7.1}$$

when the envelope $u(x,t)$ of the wave packet $q = u(x,t)\exp[i(kx - \omega t)]$ obeys the NLS equation

$$i(u_t + \omega_0'(k)u_x) + \tfrac{1}{2}\omega_0''(k)u_{xx} + \tfrac{3}{2}\left(\gamma/\omega_0(k)\right)|u|^2 u = 0, \tag{7.2}$$

where

$$\omega_0(k) = \sqrt{k^2+1} \tag{7.3}$$

is the frequency of the carrier wave in the linear approximation.

The wave with constant amplitude a_0 has a nonlinear frequency shift,

$$q(x,t) = a_0 \exp\left(-i\omega_2 a_0^2 t\right)\exp\left[i(kx - \omega_0(k)t)\right], \quad \omega_2 = -3\gamma/2\omega_0(k), \tag{7.4}$$

that is, the envelope amplitude is given by

$$u(x,t) = a_0 \exp(-i\omega_2 a_0^2 t). \tag{7.5}$$

Due to the nonlinear self-interaction this wave can decay into the side harmonics

$$\begin{aligned} q_\pm &\propto \exp[i(k_0 \pm K)x - i\omega_0(k \pm K)t] \\ &\cong \exp[i(k_0 x - \omega_0(k)t)]\exp\left[\pm i(Kx - \omega_0'(k)Kt) - \tfrac{i}{2}\omega_0''(k)K^2 t\right], \end{aligned}$$

where we have assumed that K is small compared to the wavevector k of the carrier wave. Hence, the envelope amplitudes of the side harmonics are given by

$$u_\pm = a_\pm \exp(-i\omega_2 a_0^2 t)\exp\left[\pm i(Kx - \omega_0'(k)Kt) - \tfrac{i}{2}\omega_0''(k)K^2 t\right].$$

Thus, the modulated envelope function has the form

$$u = e^{-i\omega_2 a_0^2 t}\left[a_0 + a_+ e^{iK(x-\omega_0' t) - \frac{i}{2}\omega_0'' K^2 t} + a_- e^{-iK(x-\omega_0' t) - \frac{i}{2}\omega_0'' K^2 t}\right] \tag{7.6}$$

This envelope amplitude obeys Eq. (7.2), hence the coefficients a_0, $a_\pm$ are the slow functions of time t. Substitution of Eq. (7.6) into Eq. (7.2) gives in a linear with respect to $a_\pm$ approximation the equation

$$\begin{aligned} &i\dot{a}_0 + e^{iK(x-\omega_0' t)}\left[i\dot{a}_+ e^{-\frac{i}{2}\omega_0'' K^2 t} - \omega_2 a_0^2\left(a_+ e^{-\frac{i}{2}\omega_0'' K^2 t} + a_-^* e^{\frac{i}{2}\omega_0'' K^2 t}\right)\right] \\ &\quad + e^{-iK(x-\omega_0' t)}\left[i\dot{a}_- e^{-\frac{i}{2}\omega_0'' K^2 t} - \omega_2 a_0^2\left(a_- e^{-\frac{i}{2}\omega_0'' K^2 t} + a_+^* e^{\frac{i}{2}\omega_0'' K^2 t}\right)\right] = 0, \end{aligned} \tag{7.7}$$

where the overdot stands for the derivative with respect to time t. In this approximation the amplitude a_0 is constant and the amplitudes $a_\pm$ of the side harmonics satisfy the equations

$$i\dot{a}_+ = \omega_2 a_0^2\left(a_+ + a_-^* e^{i\omega_0'' K^2 t}\right), \quad i\dot{a}_- = \omega_2 a_0^2\left(a_- + a_+^* e^{i\omega_0'' K^2 t}\right). \tag{7.8}$$

Introducing new variables

$$A_+ = a_+ e^{-\frac{i}{2}\omega_0'' K^2 t}, \quad A_- = a_- e^{-\frac{i}{2}\omega_0'' K^2 t}, \tag{7.9}$$

we arrive at the system

$$\begin{aligned} i\dot{A}_+ &= \tfrac{1}{2}\omega_0'' K^2 A_+ + \omega_2 a_0^2 (A_+ + A_-^*), \\ i\dot{A}_- &= \tfrac{1}{2}\omega_0'' K^2 A_- + \omega_2 a_0^2 (A_- + A_+^*), \end{aligned} \tag{7.10}$$

with constant coefficients. It has a solution in the form

$$A_+ = \tilde{A}_+ e^{-i\Omega t}, \quad A_- = \tilde{A}_- e^{i\Omega t}, \tag{7.11}$$

where $\tilde{A}_\pm$ are real constants and frequency Ω is given by

$$\Omega = K\sqrt{\left(\tfrac{1}{2}\omega_0''(k)\right)^2 K^2 + \omega_0''(k)\omega_2 a_0^2}. \tag{7.12}$$

This expression coincides actually with the dispersion relation for the modulational waves (see Eq. (3.57)).

Thus, we have arrived at the following picture of the modulational instability of the plane wave—the nonlinear frequency shift $\omega_2 a_0^2$ compensates the discrepancy of the frequencies $2\omega_0(k) - \omega_0(k+K) - \omega_0(k-K)$ caused by dispersion, and this makes possible the transformation of waves

$$(k,\ k;\ \omega_0(k),\ \omega_0(k)) \to (k+K,\ k-K;\ \omega_0(k+K),\ \omega_0(k-K)).$$

This is an example of the so-called 'parametric process' well known in physics. In this case the existence of the carrier wave leads to a parametric coupling of two side-band harmonics described by Eqs. (7.8) or (7.10). At early stage, when decrease of the carrier wave is negligibly small, the side harmonics increase exponentially. It is convenient to represent this growth in somewhat different form.

We shall start now from the canonical form of the NLS equation,

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{7.13}$$

Let at the initial moment the wave with constant amplitude equal to unity ($u = e^{2it}$, $a_0 \equiv 1$) be slightly modulated

$$u(x,t) = [1 + A(t)\cos Kx]e^{2it}, \tag{7.14}$$

where $A(t)$ is a complex-valued function of time. Substitution of Eq. (7.14) into Eq. (7.13) yields in a linear approximation the system for $A(t)$ and its complex conjugate A^* (cf. Eqs. (7.10)),

$$idA/dt = (K^2 - 2)A - 2A^*, \quad idA^*/dt = -(K^2 - 2)A^* + 2A. \tag{7.15}$$

Hence, we find

$$id(A + A^*)/dt = K^2(A - A^*), \quad id(A - A^*)/dt = (K^2 - 4)(A + A^*),$$

that is,

$$d^2(A + A^*)/dt^2 + K^2(K^2 - 4)(A + A^*) = 0,$$

and we obtain again the dispersion law for waves of modulation $(A \pm A^*) \propto e^{\pm i\Omega t}$,

$$\Omega = K\sqrt{K^2 - 4}$$

(cf. Eq. (1.365)), so that for waves of modulation with wavevectors $K < 2$ we have

$$\begin{aligned} A + A^* &= 2\tilde{\alpha}e^{K\sqrt{4-K^2}\,t} + 2\tilde{\beta}e^{-K\sqrt{4-K^2}\,t}, \\ A - A^* &= 2i\left(\sqrt{4 - K^2}/K\right)\left(\tilde{\alpha}e^{K\sqrt{4-K^2}\,t} - \tilde{\beta}e^{-K\sqrt{4-K^2}\,t}\right), \end{aligned}$$

and

$$A = \tilde{\alpha}\left(1 + i\sqrt{4 - K^2}/K\right)e^{K\sqrt{4-K^2}\,t} + \tilde{\beta}\left(1 - i\sqrt{4 - K^2}/K\right)e^{-K\sqrt{4-K^2}\,t}.$$

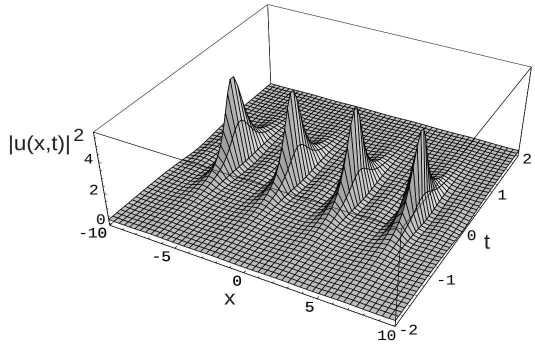

Fig. 7.1 Surface plot of the intensity $|u(x,t)|^2$ for the flattening-steepening solution of the focusing NLS equation.

As a result, we obtain the solution

$$u(x,t) = \left[1 + \left(\alpha e^{i\phi+\Gamma t} + \beta e^{-i\phi-\Gamma t}\right)\cos Kx\right] e^{2it} \tag{7.16}$$

of the NLS equation at $t \ll 1/\Gamma$, where

$$\tan\phi = \sqrt{4-K^2}/K, \quad \Gamma = K\sqrt{4-K^2}, \quad (K<2), \tag{7.17}$$

and α and β are some constants. This wave grows with increment Γ which has the maximum value $\Gamma_{max} = 2$ at $K = \sqrt{2}$. This solution is valid as long as $e^{\Gamma t} \ll 1$. At larger values of time it is necessary to take into account the decrease of the amplitude of the carrier wave, so that growth of $a_\pm$ slows down and stops at some moment after which the process reverses and develops in the opposite direction, that is, the side harmonics transform into the carrier wave. This is the recurrence phenomenon discovered numerically in the pioneering paper of Fermi, Pasta and Ulam (1955). It turns out that for the case of the NLS equation one can find the exact solution which describes the whole process. We shall write here the final result

$$u(x,t) = \left[1 - \frac{K^2\cosh\Gamma t + i\Gamma\sinh\Gamma t}{(\Gamma/K)\cos Kx + 2\cosh\Gamma t}\right] e^{2it}, \tag{7.18}$$

which can be verified by a direct calculation.

The surface plot of the intensity $|u(x,t)|^2$ is shown in Fig. 7.1. As we see, this is a periodic function of x with the wavevector K. At $t \to -\infty$ it goes to the initial plane wave solution with infinitely small modulation,

$$u(x, -\infty) = e^{2i \arccos \sqrt{1-K^2/4}} \cdot e^{2it}, \quad (t \to -\infty). \tag{7.19}$$

With increase of time the amplitude of modulation grows taking its maximum value at $t = 0$, when

$$u(x, 0) = 1 - \frac{K^2}{(\Gamma/K)\cos Kx + 1}, \quad (t = 0), \tag{7.20}$$

and after that it decreases returning to infinitely small values at $t \to +\infty$,

$$u(x, +\infty) = e^{-2i \arccos \sqrt{1-K^2/4}} \cdot e^{2it}, \quad (t \to +\infty). \tag{7.21}$$

Thus, the modulational instability is a manifestation of the process of wave transformation from some modes of the wave packet to others. For special initial conditions of a periodically modulated uniform wave this transformation is reversible and is followed by analogous transformation in the opposite direction. However, if the wave packet or disturbance of the plane wave is localized in space, then such a transformation of the wave is accompanied by the spreading of the packet due to the dispersion effects and the process in the opposite direction becomes impossible. In what follows we shall turn to this kind of problems with localized initial disturbances.

7.2 Formation of solitons at the sharp front of a pulse

As we have found in the preceding section, in the case of a periodic modulation of a uniform wave solution of the NLS equation there is a region of wavevectors of modulation, $0 < K < 2$, where a small modulation grows with time with the increment

$$\Gamma = K\sqrt{4 - K^2} \tag{7.22}$$

which has the maximum value

$$\Gamma_{max} = 2 \quad \text{at} \quad K = \sqrt{2}. \tag{7.23}$$

This means that if the initial disturbance is localized, then its Fourier harmonics will behave differently depending on the value of the corresponding wavevector. Short wavelength components with $K > 2$ will propagate in accordance with a usual linear theory of dispersive waves leading to spreading of the initial disturbance. But long wavelength harmonics will increase with time and one may expect that this growth will result in an oscillatory structure with a wavelength close to the value corresponding to the maximum of the increment Γ. However, it should be noted that the initial value problems for unstable systems are ill-posed in sense that they are very sensitive to a choice of initial conditions, and their small change can lead to considerable difference in a long time behaviour of the solution. Nevertheless, one may get some qualitative picture of what takes place during the evolution of a local disturbance in modulationally unstable systems from the analysis of typical examples.

Let us suppose, for instance, that the initial distribution has a form of a step-like pulse with a very sharp front. Then the Fourier expansion has harmonics with all values of K, so that one may expect that the oscillatory region will arise near the front, and this region may be represented as a modulated periodic solution of the NLS equation. Then evolution of the complex Riemann invariants λ_i, $i = 1, 2, 3, 4$, is governed by the Whitham equations. Since there is no parameters with a dimension of length, we may suppose that λ_i depend only on the self-similar variable $z = x/t$. Since $\lambda_3 = \lambda_1^*$ and $\lambda_4 = \lambda_2^*$, it is sufficient to use only two Whitham equations (5.45), which in our self-similar case take the form

$$(d\lambda_1/dz)\,(v_1 - z) = 0, \quad (d\lambda_2/dz)\,(v_2 - z) = 0. \tag{7.24}$$

From (5.42) it is clear that δ varies with z, so that the solution corresponding to our initial data is given by $\lambda_1 = \text{const}$, $v_2 = z = x/t$, or

$$\alpha + i\gamma = \text{const}, \tag{7.25}$$

$$\begin{aligned} &-2\,(\alpha+\beta) \\ &- \frac{4\delta\,[\gamma - \delta + i\,(\beta-\alpha)]\,\mathrm{K}\,(m)}{(\beta-\alpha)\,[\mathrm{K}\,(m) - \mathrm{E}\,(m)] + i\,[(\delta-\gamma)\,\mathrm{K}\,(m) + (\delta+\gamma)\,\mathrm{E}\,(m)]} = \frac{x}{t}. \end{aligned} \tag{7.26}$$

Separating in the above equation the real and imaginary parts, we obtain the equations

$$\frac{\mathrm{E}\,(m)}{\mathrm{K}\,(m)} = \frac{(\alpha-\beta)^2 + (\gamma-\delta)^2}{(\alpha-\beta)^2 + \gamma^2 - \delta^2}, \tag{7.27}$$

$$-4\beta - \frac{2\left(\gamma^2 - \delta^2\right)}{\beta - \alpha} = \frac{x}{t}, \tag{7.28}$$

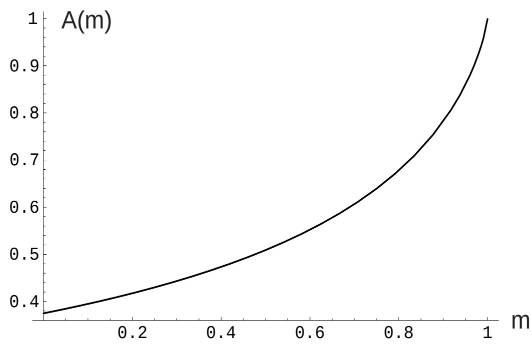

Fig. 7.2 The plot of the function $A(m)$.

which together with $\alpha = \text{const}$, $\gamma = \text{const}$, and the equation

$$m = \frac{4\gamma\delta}{(\alpha-\beta)^2 + (\gamma+\delta)^2}, \tag{7.29}$$

determine implicitly the dependence of β and δ on $z = x/t$.

It is convenient to express β and δ as functions of m,

$$\beta(m) = \alpha - \gamma\sqrt{4A(m) - (1 + mA(m))^2}, \tag{7.30}$$

$$\delta(m) = \gamma m A(m), \tag{7.31}$$

where we have introduced the function

$$A(m) = \frac{(2-m)\,\mathrm{E}(m) - 2(1-m)\,\mathrm{K}(m)}{m^2\mathrm{E}(m)}. \tag{7.32}$$

The function $A(m)$ has the limiting behaviour

$$A(m) \simeq \tfrac{3}{8} + \tfrac{27}{64}m, \qquad m \ll 1, \tag{7.33}$$

and

$$A(m) \simeq 1 - (1-m)\ln(16/(1-m)), \qquad (1-m) \ll 1. \tag{7.34}$$

Its plot is shown in Fig. 7.2.

Let us investigate the region of fast oscillations at both its edges. If $m \to 1$, then we have

$$\beta \simeq \alpha - 2\gamma\sqrt{1-m}, \quad \delta \simeq \gamma\left[1 - (1-m)\ln(16/(1-m))\right],$$

and according to Eq. (7.28) this edge moves with the soliton velocity (see (5.41))

$$v_s = -4\alpha. \tag{7.35}$$

If $m \to 0$, then β and δ tend to the limiting values

$$\beta = \alpha - \gamma/\sqrt{2}, \quad \delta = 0, \tag{7.36}$$

and Eq. (7.28) takes the form

$$v = x/t = -4\alpha + 4\sqrt{2}\gamma. \tag{7.37}$$

In this limit of a small modulation the Whitham theory must reproduce the linear theory, that is, v must coincide with the corresponding group velocity of the modulation wave. The dispersion law of the modulation wave is given by (see Eq. (5.44))

$$\Omega(K) = K\left(-4\alpha + \sqrt{K^2 - 4\gamma}\right). \tag{7.38}$$

Calculation of the group velocity gives

$$v_g = \frac{d\Omega}{dK} = -4\alpha + \frac{2K^2 - 4\gamma^2}{\sqrt{K^2 - 4\gamma^2}}, \tag{7.39}$$

which after substitution of K from Eqs. (5.43) takes the form

$$v_g = -4\beta - 2\gamma^2/(\beta - \alpha), \tag{7.40}$$

coinciding with the result of the Whitham theory (7.28) at $\delta = 0$, that is, with the limit of a small modulation. At the values of β and δ from (7.36) we have

$$K = \sqrt{6}\,\gamma, \quad v_g = -4\alpha + 4\sqrt{2}\,\gamma, \tag{7.41}$$

that is, this edge of the small modulation region moves with the group velocity corresponding to $K = \sqrt{6}\gamma$. It is interesting that the expression (7.41) gives the minimal absolute value of the group velocity. This result has clear physical meaning following from the matching conditions. Only in this case the slightly modulated periodic wave matches at the edge $x = v_g t$ with

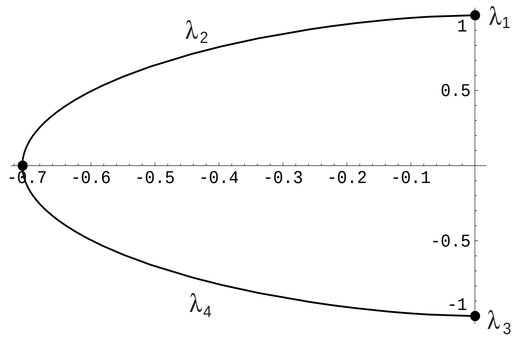

Fig. 7.3 The loci of the Riemann invariants λ_2 and λ_4 in the case of the self-similar solution of the Whitham equations for the NLS periodic wave with $\alpha = 0,\ \gamma = 1$.

only one linear wave propagating outward the non-uniform region. Indeed, for $|x| > |v_g|\, t$ at each point x there are two linear waves propagating with the same group velocity which corresponds to two different values of K, but at the edge point these two waves coincide with each other and propagate with the minimal group velocity which permits them to match with only one modulated periodic wave corresponding to the inside region.

The solution obtained can be applied to description of two different situations. At first, let us take $\alpha = 0$, that is, the 'soliton' region does not move, which corresponds to the asymptotic stage of the evolution of an initially local disturbance. Oscillations have the form of solitons in the central region, and with approaching the edges the amplitude of the wave modulation decreases and vanishes at the points

$$|x_{\pm}| = 4\sqrt{2}\,\gamma t. \tag{7.42}$$

The whole region expands with time. In Fig. 7.3 we have shown the loci of the Riemann invariants λ_2 and λ_4 in the λ plane (for $\alpha = 0$, $\gamma = 1$). We see that a pair of the complex Riemann invariants arises at $\lambda_2 = \lambda_4 = \beta\,(0) = -1/\sqrt{2}$ on the real axis, after that they move in the complex plane and coalesce with the constant pair $\lambda_1 = i$, $\lambda_3 = -i$ at $m = 1$. The initial configuration of λ_i corresponds to the matching point with the uniform region, and the final one to the central soliton region. Substitution of

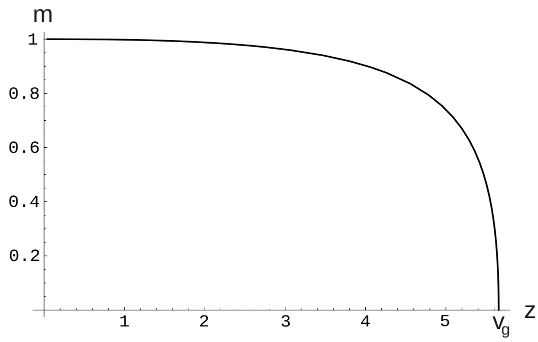

Fig. 7.4 The dependence of the parameter m on the self-similar variable $z = x/t$ for the solution of the Whitham equations for the modulated wave train of the focusing NLS equation. The parameter m vanishes at the point $v_g = 4\sqrt{2}$ where the modulated wave matches with the unmodulated one.

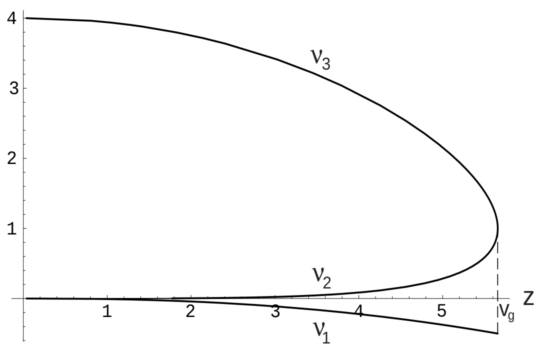

Fig. 7.5 The dependence of the resolvent's zeros ν_1, ν_2, ν_3 on the self-similar variable $z = x/t$ corresponding to the self-similar solution of the Whitham equations.

Eqs. (7.30), (7.31) into (7.28) gives us the dependence of the parameter m on the self-similar variable $z = x/t$. The corresponding plot is shown in Fig. 7.4. Note that the solution obtained is stable at both its edges—in the soliton region and in the region of small modulation with $K > 2\gamma$. The

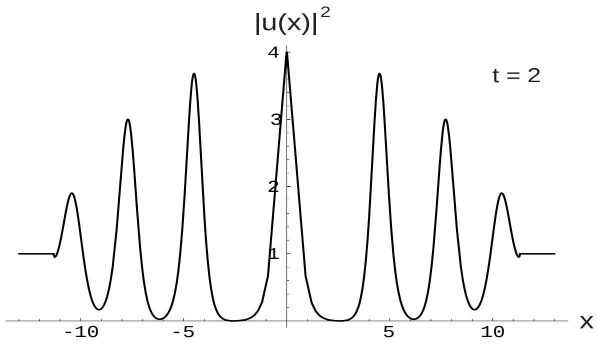

Fig. 7.6 The dependence of the intensity $|u(x)|^2$ on the space coordinate x at the moment $t = 2$ in the region of oscillations arising from a local disturbance at $x = 0$. The edge points move with group velocity into the unmodulated regions of the plane wave solution.

dependence of ν_1, ν_2, ν_3 on z is shown in Fig. 7.5. At $x = x_\pm = v_g t$ (the boundaries with the uniform plane wave) we have $\nu_2 = \nu_3$, and at $x = 0$ (the central soliton region) we have $\nu_1 = \nu_2$. Note that this plot looks like the plot for the real Riemann invariants in the Gurevich-Pitaevskii type problems. This oscillatory region is shown in Fig. 7.6. This plot illustrates the process of formation of such a non–uniform region from an initially local disturbance in the modulationally unstable systems described by the NLS equation.

The situation with $\alpha \neq 0$ corresponds to the process of the solitons creation at the pulse front. This generalization of the above results is straightforward and leads to the picture shown in Fig. 7.7. The leading edge of the oscillatory region moves with the soliton velocity (7.35) and the trailing edge propagates into the uniform region with the group velocity (7.39) of the small modulation wave.

7.3 Formation of solitons on the sharp front of the SIT pulse

The periodic solution (5.85) of the SIT equations is determined with the use of the same resolvent as in the NLS equation case. The corresponding Whitham velocities are given by Eq. (5.94). Again for the step-like initial

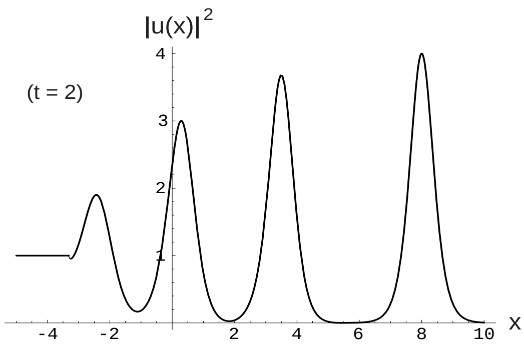

Fig. 7.7 The dependence of the intensity $|u(x)|^2$ on the space coordinate x at the moment $t = 2$ in the region of oscillations arising at the initially sharp front. The right edge point moves with the soliton velocity and the left edge point with the group velocity of linear waves of modulation propagating into the unmodulated region of the plane wave solution.

pulse the Whitham equations reduce to Eqs. (7.24) where $z = \xi/\tau$ is a self-similar variable and the calculation similar to that for the NLS equation case leads instead of Eqs. (7.26,7.27) to the solution

$$\frac{\mathrm{E}(m)}{\mathrm{K}(m)} = \frac{\beta(\alpha^2+\beta^2+\gamma^2+\delta^2) - 2\beta(\alpha\beta+\gamma\delta) - \Delta[(\alpha-\beta)^2+(\gamma-\delta)^2]}{\beta(\alpha^2+\beta^2+\gamma^2+\delta^2) - 2\alpha(\beta^2+\delta^2) - \Delta[(\alpha-\beta)^2+\gamma^2-\delta^2]}, \tag{7.43}$$

$$\begin{aligned} &-\frac{1}{4\sqrt{((\alpha-\Delta)^2+\gamma^2)((\beta-\Delta)^2+\delta^2)}} \\ &\times\frac{\alpha(\beta^2+\delta^2)-\beta(\alpha^2+\gamma^2)+\Delta(\alpha^2-\beta^2+\gamma^2-\delta^2)-\Delta^2(\alpha-\beta)}{(\alpha-\beta)[(\beta-\Delta)^2+\delta^2]} = \frac{\tau}{\xi}, \end{aligned} \tag{7.44}$$

which together with $\alpha = \text{const}$, $\gamma = \text{const}$ and Eq. (7.29) determine implicitly the dependence of β and δ on $z = \xi/\tau$. In the limit $\Delta \to \infty$ Eqs. (7.43) and (7.44) yield

$$\frac{\mathrm{E}(m)}{\mathrm{K}(m)} = \frac{(\alpha-\beta)^2+(\gamma-\delta)^2}{(\alpha-\beta)^2+\gamma^2-\delta^2}, \quad 2\beta + \frac{\gamma^2-\delta^2}{\beta-\alpha} = \frac{4\Delta^3\tau}{\xi} + \Delta, \tag{7.45}$$

which actually coincides (after appropriate redefinition of space and time variables) with the solution (7.27), (7.28) of the Whitham equations for the focusing NLS equation.

The parameters β and δ can be expressed as functions of m,

$$\beta = \Delta + \frac{\alpha - \Delta}{(\alpha-\Delta)^2 + \gamma^2 m^2 A^2(m)} \cdot \Big((\alpha-\Delta)^2 + (2-m)\,\gamma^2 A(m) + \gamma\sqrt{4(\alpha-\Delta)^2 A(m) + 4\gamma^2 A^2(m)(1-m) - (\alpha-\Delta)\,(1+mA(m))^2} \Big),$$

$$\delta = \frac{\gamma}{\alpha-\Delta} m\, A(m)(\beta - \Delta), \tag{7.46}$$

which permit us to investigate the region of fast oscillations at both its edges. The edge with $m = 1$ moves with the soliton velocity

$$v_s = \xi/\tau|_{m\to 1} = 4((\alpha-\Delta)^2 + \gamma^2). \tag{7.47}$$

If $m \to 0$, then β and δ go to the values

$$\begin{aligned} &\beta = \Delta + (\alpha-\Delta)\left[1 + \frac{3\gamma^2}{4(\alpha-\Delta)^2}\left(1 + \sqrt{1 + \frac{8(\alpha-\Delta)^2}{9\gamma^2}}\right)\right], \\ &\delta = 0, \end{aligned} \tag{7.48}$$

and Eq. (7.44) takes the form

$$\frac{1}{v} = \frac{\tau}{\xi} = \frac{\alpha^2 + \gamma^2 - \alpha\beta - \Delta(\alpha-\beta)}{4(\beta-\Delta)^2(\alpha-\beta)\sqrt{(\alpha-\Delta)^2+\gamma^2}}. \tag{7.49}$$

As usual, in this limit of small modulation the Whitham theory must reproduce the linear approximation. Indeed, Eq. (5.85) at small δ gives

$$|\mathcal{E}|^2 \simeq 4\gamma^2 + 4\gamma\delta\left[\exp\left(i2\sqrt{(\alpha-\beta)^2+\gamma^2}\left(\tau - \frac{\xi}{4(\beta-\Delta)\sqrt{(\alpha-\Delta)^2+\gamma^2}}\right)\right) + \text{c.c.}\right], \tag{7.50}$$

that is, the frequency Ω and the wavenumber K of the modulational wave are equal to

$$\Omega = 2\sqrt{(\alpha-\beta)^2+\gamma^2}, \qquad K = \frac{\Omega}{4(\beta-\Delta)\sqrt{(\alpha-\Delta)^2+\gamma^2}}. \tag{7.51}$$

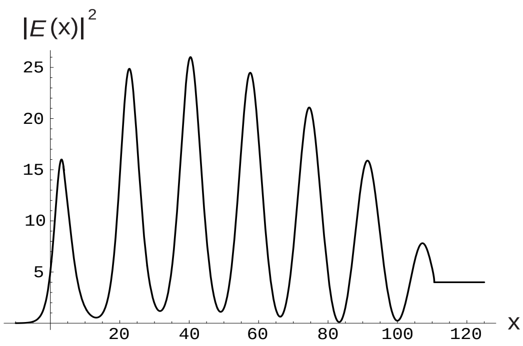

Fig. 7.8 The dependence of the intensity $|\mathcal{E}(x)|^2$ on the space coordinate x for the self-similar solution of the Whitham equations for a SIT wave train.

Elimination of β leads to the dispersion relation

$$K(\Omega) = \frac{\Omega(\sqrt{\Omega^2 - 4\gamma^2} - 2(\alpha - \Delta))}{2\sqrt{(\alpha-\Delta)^2 + \gamma^2}\,[\Omega^2 - 4((\alpha-\Delta)^2 + \gamma^2)]}, \tag{7.52}$$

which can also be obtained by means of linearization of the SIT equations (5.67) with respect to small modulation of the plane wave solution

$$\begin{aligned} \mathcal{E}(\tau,\xi) &= 2\gamma \exp\left[-2i\alpha\tau - \frac{i\xi}{2\sqrt{(\alpha-\Delta)^2+\gamma^2}}\right] \\ d(\tau,\xi) &= -\frac{i\gamma}{\sqrt{(\alpha-\Delta)^2+\gamma^2}} \exp\left[-2i\alpha\tau - \frac{i\xi}{2\sqrt{(\alpha-\Delta)^2+\gamma^2}}\right], \\ n &= -\frac{\alpha-\Delta}{\sqrt{(\alpha-\Delta)^2+\gamma^2}}. \end{aligned} \tag{7.53}$$

The calculation of the group velocity $v_g = (dK/d\Omega)^{-1}$ at the value of Ω given by Eq. (7.51) reproduces Eq. (7.49).

An example of the whole oscillatory region is shown in Fig. 7.8. The slower edge of this region moves with the soliton velocity and consists of the train of solitons. The faster edge propagates with the group velocity $v_g > v_s$ of the small modulation wave. The solution found corresponds to the step-like initial data

$$|\mathcal{E}(\xi,0)| = 2\gamma \quad \text{for } \xi \geq 0, \qquad |\mathcal{E}(\xi,0)| = 0 \quad \text{for } \xi < 0, \tag{7.54}$$

and it is stable at both its edges—in the soliton region and in the region of small modulation (where $\Omega > 2\gamma$, as one can see from Eq. (7.51)). The velocity $v = \xi/\tau$ ($v > 0$) is connected with the velocity v_{phys} in physical coordinates x and t (see (5.69)) by the relation

$$v_{phys} = \frac{v}{1+v}.$$

We see that $v_{phys} < 1$ (the dimensionless speed of light equals to unity), and solitons move slower than the perturbation of the uniform region. Thus, the solitons are created at the back region of the long light pulse propagating through the resonant medium.

Similar technique can be applied to other equations discussed in Chapter 2 and they lead to similar results.

7.4 Formation of singularity due to modulational instability

The considered above solutions of the Whitham equations correspond to formation of a region of oscillations near the sharp front of a step-like pulse, that is, the initial conditions are singular. However, one might expect that singularity can be developed from an initially smooth pulse as it takes place in the modulationally stable case after the wave-breaking process. We shall consider here an important example of formation of such a singularity in a modulationally unstable system.

It is clear that a singularity can arise in a solution, if one neglects the dispersion effects. Therefore, let us investigate solutions of the focusing NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{7.55}$$

in the dispersionless limit. In Sec. 1.6.1 it was shown that in this case it is convenient to make a substitution

$$u(x,t) = \sqrt{\rho(x,t)}\exp[i\theta(x,t)], \tag{7.56}$$

and then, neglecting the dispersion terms, we arrive at the hydrodynamical system (see Eqs. (1.326))

$$\tfrac{1}{2}\rho_t + v\rho_x + \rho v_x = 0, \quad \tfrac{1}{2}v_t + vv_x - \rho_x = 0, \tag{7.57}$$

where $v = \theta_x$ and the modulational instability manifests itself in the 'negative pressure'

$$p = -\tfrac{1}{2}\rho^2$$

and the complex-valued 'sound velocity'

$$c = \sqrt{dp/d\rho} = i\sqrt{\rho}. \tag{7.58}$$

In a usual way the system (7.57) can be written in terms of the Riemann invariants

$$\lambda_+ = \tfrac{1}{2}v + i\sqrt{\rho}, \quad \lambda_- = \tfrac{1}{2}v - i\sqrt{\rho}, \tag{7.59}$$

so that $\lambda_\pm$ satisfy the equations

$$\frac{\partial \lambda_+}{\partial t} + v_+(\lambda_+,\lambda_-)\frac{\partial \lambda_+}{\partial x} = 0, \quad \frac{\partial \lambda_-}{\partial t} + v_-(\lambda_+,\lambda_-)\frac{\partial \lambda_-}{\partial x} = 0, \tag{7.60}$$

where

$$v_+(\lambda_+,\lambda_-) = 3\lambda_+ + \lambda_-, \quad v_-(\lambda_+,\lambda_-) = \lambda_+ + 3\lambda_-. \tag{7.61}$$

The hodograph transform of Eqs. (7.60) yields the system

$$\frac{\partial x}{\partial \lambda_-} - v_+(\lambda_+,\lambda_-)\frac{\partial t}{\partial \lambda_-} = 0, \quad \frac{\partial x}{\partial \lambda_+} - v_-(\lambda_+,\lambda_-)\frac{\partial t}{\partial \lambda_+} = 0, \tag{7.62}$$

whose solution has the form

$$x - v_+(\lambda_+,\lambda_-)t = \partial\chi/\partial\lambda_+, \quad x - v_-(\lambda_+,\lambda_-)t = \partial\chi/\partial\lambda_-, \tag{7.63}$$

where χ satisfies the equation

$$\frac{\partial^2\chi}{\partial\lambda_+\partial\lambda_-} - \frac{1}{2(\lambda_+ - \lambda_-)}\left(\frac{\partial\chi}{\partial\lambda_+} - \frac{\partial\chi}{\partial\lambda_-}\right) = 0. \tag{7.64}$$

In the real variables p, q, which are defined by the relations

$$\lambda_+ = p + iq, \quad \lambda_- = p - iq, \tag{7.65}$$

that is,

$$p = \tfrac{1}{2}(\lambda_+ + \lambda_-) = \tfrac{1}{2}v, \quad q = \tfrac{1}{2i}(\lambda_+ - \lambda_-) = \sqrt{\rho}, \tag{7.66}$$

the solution (7.63) takes the form

$$2x - 8pt = \partial\chi/\partial p, \quad 4qt = \partial\chi/\partial q. \tag{7.67}$$

Let at the initial moment $t = 0$ we have $v = 0$. Then, if we put $t = 0$ into Eqs. (7.67) and take into account Eqs. (7.66), we shall obtain the boundary conditions for the potential χ,

$$\left.\frac{\partial\chi}{\partial p}\right|_{p=0} = 2x_0(q), \quad \left.\frac{\partial\chi}{\partial q}\right|_{p=0} = 0, \tag{7.68}$$

where $x_0(q)$ is a function connected with the initial distribution of the intensity by the relation

$$q = \sqrt{\rho_0(x)}. \tag{7.69}$$

Equation (7.64) in the real variables p and q takes the form

$$\frac{\partial^2\chi}{\partial p^2} + \frac{\partial^2\chi}{\partial q^2} + \frac{1}{q}\frac{\partial\chi}{\partial q} = 0, \tag{7.70}$$

which coincides with the Laplace equation $\Delta\chi = 0$ written in cylindrical coordinates for the case when the potential χ does not depend on a polar angle.

We have arrived at the electrostatic-like problem of finding the potential created by a distribution of the 'electric charge' on the plane $p = 0$ with the density $2x_0(q)$, where q is a radius-vector of a polar system of coordinates in this plane. It is convenient to solve such a problem in the ellipsoidal system of coordinates (ξ, η), which are connected with p and q by the relations (see, e.g., Morse and Feshbach, 1953)

$$p = \xi\eta, \quad q^2 = (\xi^2 + 1)(1 - \eta^2), \quad 0 \le \xi < \infty, \quad -1 \le \eta \le 1. \tag{7.71}$$

Exclusion of η from these formulas gives the equation

$$\frac{q^2}{\xi^2 + 1} + \frac{p^2}{\xi^2} = 1$$

of the ellipsoid in cylindrical coordinates, and at $\xi = 0$ we obtain $p = 0$, $q^2 = (1 - \eta^2) \le 1$, that is, the ellipsoid reduces to the disk with the radius $q = 1$ lying in the plane $p = 0$.

Formulas (7.69) and (7.71) indicate that a relatively simple analytic solution can be found for the initial intensity

$$\rho_0(x) = \cosh^{-2} x, \tag{7.72}$$

that is,

$$x_0(q) = \ln\left[\left(1 + \sqrt{1-q^2}\right)\Big/ q\right], \tag{7.73}$$

and in coordinates (7.71) this distribution is given by

$$x_0(\eta) = -\tfrac{1}{2}\ln\frac{1+\eta}{1-\eta}, \tag{7.74}$$

where we have chosen a sign before square root so that $\eta = -\sqrt{1-q^2}$ at $\xi = 0$, and in what follows the positive values of $t > 0$ will correspond to the positive values of $\xi > 0$. Then the boundary conditions (7.68) in coordinates (ξ, η) take the form

$$\left.\frac{\partial\chi}{\partial\eta}\right|_{\xi=0} = 0, \quad \left.\frac{\partial\chi}{\partial\xi}\right|_{\xi=0} = -\eta\ln\frac{1+\eta}{1-\eta}. \tag{7.75}$$

Transformation of Eq. (7.70) to coordinates (ξ, η) yields

$$\frac{\partial}{\partial\xi}\left[\left(\xi^2+1\right)\frac{\partial\chi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\left(1-\eta^2\right)\frac{\partial\chi}{\partial\eta}\right] = 0. \tag{7.76}$$

This is the equation with separable variables so that its solution is given by a linear combination

$$\chi = \Phi_0(\xi) + \Psi_0(\eta) + \sum_{n=1}\Phi_n(\xi)\Psi_n(\eta), \tag{7.77}$$

where $\Phi_n(\xi)$, $\Psi_n(\eta)$ satisfy the equations

$$\frac{d}{d\xi}\left[\left(\xi^2+1\right)\frac{d\Phi_n}{d\xi}\right] = c_n\Phi_n, \quad \frac{d}{d\eta}\left[\left(1-\eta^2\right)\frac{d\Psi_n}{d\eta}\right] = -c_n\Psi_n, \tag{7.78}$$

c_n are some constants (eigenvalues), and the functions Φ_0 and Ψ_0 corresponding to $c_0 = 0$ may be added into the superposition separately since they make the first two terms in the sum (7.77) equal to zero. These are the Legendre equations with imaginary and real arguments and their solutions are well known. For our aim it is sufficient to write down several simplest solutions of Eqs. (7.78). The eigenvalue $c_0 = 0$ corresponds to the solutions

$$\Phi(\xi) = Q_0(i\xi) = \tfrac{1}{2}\arctan\xi, \quad \Psi_0(\eta) = Q_0(\eta) = \tfrac{1}{2}\ln\frac{1+\eta}{1-\eta}, \tag{7.79}$$

up to some constant factors, and the eigenvalue $c_1 = 2$ corresponds to the solutions

$$\Phi_1(\xi) = P_1(\xi) = \xi, \quad \Psi_1(\eta) = Q_1(\eta) = \tfrac{1}{2}\eta\ln\frac{1+\eta}{1-\eta} - 1 \tag{7.80}$$

again up to some constant factors. These solutions can be readily verified by means of their substitution into Eqs. (7.78). Then the superposition (7.77) can be written in the form

$$\chi = C_1\left(\xi\eta\ln\frac{1+\eta}{1-\eta} - 2\xi\right) + C_2\arctan\xi + C_3\ln\frac{1+\eta}{1-\eta},$$

where constants C_1, C_2, C_3 must be chosen so that χ satisfies the boundary conditions (7.75). We find at once that $C_1 = -1$, $C_2 = -2$ and $C_3 = 0$, so that

$$\chi = -\xi\eta\ln\frac{1+\eta}{1-\eta} + 2\xi - 2\arctan\xi, \tag{7.81}$$

After transformation of Eqs. (7.67) to the (ξ,η)-coordinates, we obtain the solution of the Whitham equation in the form

$$x = -\tfrac{1}{2}\ln\frac{1+\eta}{1-\eta} + \frac{2\xi^2\eta}{(\xi^2+1)(1-\eta^2)}, \quad t = \frac{\xi}{2(\xi^2+1)(1-\eta^2)}. \tag{7.82}$$

These formulas define implicitly the dependence of ξ and η on x and t. If $\xi(x,t)$ and $\eta(x,t)$ are found, then the physical variables can be calculated by means of equations

$$v = 2\xi\eta, \qquad \rho = (\xi^2+1)(1-\eta^2). \tag{7.83}$$

The variables ξ and η can be eliminated from Eqs. (7.82) and (7.83), and then we obtain the solution in terms of ρ and v. Indeed, from Eqs. (7.82) we find $2(x-2vt) = -\ln[(1+\eta)/(1-\eta)]$, hence, $\eta = -\tanh(x-2vt)$ and since $\rho = (\xi^2+1)(1-\eta^2) = \xi/2t$, we have $\xi = 2\rho t$. On substitution of these expressions for ξ and η into Eqs. (7.83), we obtain

$$\begin{aligned} v &= -2t\rho\tanh(x-2vt), \\ \rho &= \left(1+4\rho^2t^2\right)\cosh^{-2}(x-2vt). \end{aligned} \tag{7.84}$$

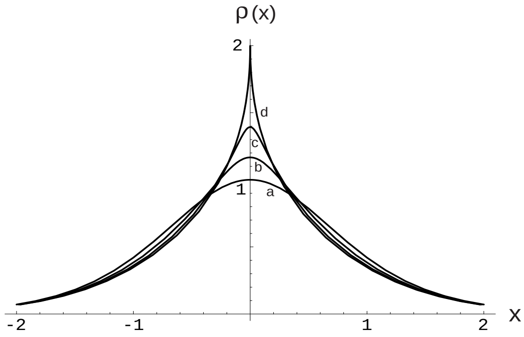

Fig. 7.9 The dependence of the intensity ρ on the space coordinate x at several values of time t for the self-focusing solution of the equations of the geometric optics approximation for the NLS equation; the curves a, b, c, d correspond to the values of time $t = 0$, 0.175, 0.225, 0.25, respectively.

It is evident that this solution satisfies the initial conditions $v|_{t=0} = 0$ and $\rho|_{t=0} = \cosh^{-2}(x)$.

The formulas obtained permit us to calculate the functions $\rho(x, t)$ and $v(x, t)$. The results for $\rho(x)$ and $v(x)$ at several values of t are presented in Figs. 7.9 and 7.10. As we see, the distribution of the intensity $\rho(x)$ sharpens with growth of t, and in the limit $t \to 1/4$ the function $\rho(x)$ acquires a cusp-like singularity, so that for $t > 1/4$ it is not defined at all. In a similar way, the derivative $v'(x)$ at $x = 0$ becomes infinite in the limit $t \to 1/4$.

I am grateful to the reader reached this last page of the book. I hope that he/she has leaned something new and will be able to make his/her own contribution into this fascinating field of theoretical and mathematical physics.

Bibliographic remarks

The possibility of modulational instability of nonlinear waves was discovered by many people in the middle of sixties. Probably, for the first time it was indicated by Vedenov and Rudakov (1964) for the Langmuir waves in plasma. Later it was considered in the context of the self-focusing process of light beams in nonlinear media by Bespalov and Talanov (1966), Ostrovsky

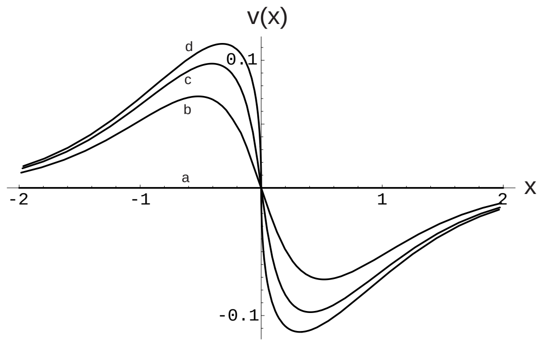

Fig. 7.10 The dependence of $v(x)$ on the space coordinate x at several values of time t for the self-focusing solution of the equations of the geometric optics approximation for the NLS equation; the curves a, b, c, d correspond to the values of time $t = 0$, 0.175, 0.225, 0.25, respectively.

(1967), Zakharov (1967). Modulational instability of a deep water Stokes wave was indicated by Lighthill (1965), Benjamin and Feir (1967), Zakharov (1968). Its experimental confirmation by Benjamin and Feir (1967) made a sensation and attracted much attention to this phenomenon. Geometrical optics (hydrodynamical) approach to the modulationally unstable (self-focusing) systems and formation of singularity was developed by Lighthill (1965) and Akhmanov, Sukhorukov and Khokhlov (1966). The latter authors found the solution (7.84) which was derived by means of a more systematic method by Gurevich and Shvartsburg (1970). The recurrent solution of the NLS equation was found independently by Akhmediev, Eleonsky and Kulagin (1985, 1987) and by Steudel and Meinel (1985). The idea that a local disturbance of a modulationally unstable plane wave transforms into a modulated nonlinear periodic or quasi-periodic wave was introduced by Karpman (1967b) and it was supported by numerical simulations by Karpman and Krushkal (1968). The Whitham approach was applied to this problem for the focusing NLS equation case by El, Gurevich, Khodorovsky and Krylov (1993), Bikbaev and Kudashev (1994a,b) and to many other equations by Kamchatnov (1992a,1995,1996,1997). Note that this theory is ill posed for the unstable systems, since a small change of initial conditions can lead to considerable changes in the resulting behaviour.

Nevertheless, one may hope that it provides some qualitative picture of how local disturbance develops in modulationally unstable systems. Modulational instability of finite pulses was studied by Bronski and McLaughlin (1994) and Jin, Levermore and McLaughlin (1994).

Exercises on Chapter 7

Exercise 7.1
Show that the NLS equation can be written in a Hamiltonian form

$$iu_t = \delta H/\delta u^*, \qquad H = \int \left(|u_x|^2 - |u|^4\right) dx,$$

and prove that in addition to the Hamiltonian H the momentum

$$P = \tfrac{i}{2} \int (u_x u^* - u u_x^*)\, dx$$

and the 'intensity'

$$N = \int |u|^2 dx$$

are also conserved.

Exercise 7.2
(Kuznetsov, Rubenchik and Zakharov, 1986) Show that the soliton solution of the NLS equation is modulationally stable by proof that it minimizes the Hamiltonian H at fixed values of the momentum P and intensity N defined in the preceding exercise.

Exercise 7.3
(Talanov, 1965) Find solution of Eqs. (7.59) for a parabolic initial intensity distribution ρ and zero 'velocity' v (cf. Exercise 6.3).

Appendix A

Some formulas from the theory of elliptic functions

As it was shown in Sec. 1.5.2, the integral

$$\int_z^{z_3} \frac{dz}{\sqrt{(z_3-z)(z-z_2)(z-z_1)}}, \quad z_1 < z_2 < z < z_3,$$

can be transformed with the help of the substitution

$$z = z_3 - (z_3 - z_2)\sin^2\varphi, \quad \sin\varphi = \sqrt{\frac{z_3 - z}{z_3 - z_2}} \tag{A.1}$$

to a standard form

$$\int_z^{z_3} \frac{dz}{\sqrt{(z_3-z)(z-z_2)(z-z_1)}} = \frac{2}{\sqrt{z_3 - z_1}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - m\sin^2\varphi}}, \tag{A.2}$$

where

$$m = \frac{z_3 - z_2}{z_3 - z_1}. \tag{A.3}$$

In a similar way one can prove the identity

$$\begin{aligned} &\int_z^{z_4} \frac{dz}{\sqrt{(z_4-z)(z-z_3)(z-z_2)(z-z_1)}} \\ &\qquad = \frac{2}{\sqrt{(z_4-z_2)(z_3-z_1)}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-m\sin^2\varphi}}, \end{aligned} \tag{A.4}$$

where $z_1 < z_2 < z_3 < z < z_4$, by means of the substitution

$$z = \frac{z_4(z_3 - z_1) + z_1(z_4 - z_3)\sin^2\varphi}{z_3 - z_1 + (z_4 - z_3)\sin^2\varphi}, \quad \sin\varphi = \sqrt{\frac{(z_3-z_1)(z_4-z)}{(z_4-z_3)(z-z_1)}}, \tag{A.5}$$

where now

$$m = \frac{(z_4 - z_3)(z_2 - z_1)}{(z_4 - z_2)(z_3 - z_1)}. \tag{A.6}$$

Analogous formulas can be obtained for other values of the integration limits.

The standard integral

$$F(\varphi, m) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - m\sin^2\varphi}} = \int_0^{\sin\varphi} \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}}, \tag{A.7}$$

appeared in Eqs. (A.2) and (A.4), is called the elliptic integral of the first kind. The elliptic integral of the second kind is defined by the formula

$$E(\varphi, m) = \int_0^{\varphi} \sqrt{1 - m\sin^2\varphi}\, d\varphi = \int_0^{\sin\varphi} \sqrt{\frac{1-mz^2}{1-z^2}}\, dz. \tag{A.8}$$

If $\varphi = \pi/2$, then these integrals are called complete elliptic integrals of the first and second kind, respectively,

$$\mathrm{K}(m) = F(\pi/2, m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - m\sin^2\varphi}} = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}}, \tag{A.9}$$

$$\mathrm{E}(m) = E(\pi/2, m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\varphi} d\varphi = \int_0^1 \sqrt{\frac{1-mz^2}{1-z^2}} dz. \tag{A.10}$$

We need asymptotic formulas for these functions at $m \ll 1$ and $1-m \ll 1$:

$$\begin{aligned}
&\mathrm{K}(m) \cong \tfrac{1}{2}\pi\left(1 + \tfrac{1}{4}m + \tfrac{9}{64}m^2 + \ldots\right), \qquad m \ll 1,\\
&\mathrm{K}(m) \cong \ln\frac{4}{\sqrt{1-m}} + \frac{1}{4}\left(\ln\frac{4}{\sqrt{1-m}} - 1\right)(1-m) + \ldots, \quad 1-m \ll 1,\\
&\mathrm{E}(m) \cong \tfrac{1}{2}\pi\left(1 - \tfrac{1}{4}m - \tfrac{3}{64}m^2 - \ldots\right), \qquad m \ll 1,\\
&\mathrm{E}(m) \cong 1 + \tfrac{1}{2}\left(\ln\frac{4}{\sqrt{1-m}} - \tfrac{1}{2}\right)(1-m) + \ldots, \qquad 1-m \ll 1.
\end{aligned} \tag{A.11}$$

The complete elliptic integrals satisfy the following differential equations,

$$\frac{d\mathrm{K}}{dm} = \frac{\mathrm{E} - (1-m)\mathrm{K}}{2m(1-m)}, \qquad \frac{d\mathrm{E}}{dm} = \frac{\mathrm{E} - \mathrm{K}}{2m}. \tag{A.12}$$

Elliptic functions can be defined as functions inverse to the elliptic integrals. We consider the upper integration limit φ in the formula

$$u = \int_0^{\varphi} \frac{dz}{\sqrt{1 - m\sin^2 z}} = F(\varphi, m) \tag{A.13}$$

as a function of u and, following to Jacobi, call it 'the amplitude' and denote

$$\varphi = \operatorname{am} u. \tag{A.14}$$

Then the main Jacobi elliptic functions are defined by the formulas

$$\begin{aligned} \operatorname{sn} u &= \sin\varphi = \sin \operatorname{am} u, \\ \operatorname{cn} u &= \cos\varphi = \cos \operatorname{am} u, \\ \operatorname{dn} u &= \sqrt{1 - m\sin^2\varphi} = d\varphi/du. \end{aligned} \tag{A.15}$$

They are periodic functions of u: $\operatorname{sn}(u, m)$ and $\operatorname{cn}(u, m)$ have the period $4\mathrm{K}(m)$, and $\operatorname{dn}(u, m)$ the period $2\mathrm{K}(m)$. These periods are real for real values of m. Besides that, they have also complex periods: $\operatorname{sn}(u, m)$ has the period $2i\mathrm{K}'(m) \equiv 2i\mathrm{K}(m')$ $(m' = 1 - m)$, $\operatorname{cn}(u, m)$ has the period $2\mathrm{K}(m) + 2i\mathrm{K}'(m)$, and $\operatorname{dn}(u, m)$ has the period $4i\mathrm{K}'(m)$. Thus, the Jacobi elliptic functions are double-periodic functions.

The functions sn and cn have the following Fourier expansions at $m \to 0$:

$$\begin{aligned} \operatorname{sn}(u, m) &= \left(1 + \tfrac{1}{16}m\right)\sin\frac{u}{1 + m/4} + \frac{m}{16}\sin\frac{3u}{1 + m/4} + \ldots, \quad m \ll 1, \\ \operatorname{cn}(u, m) &= \left(1 - \tfrac{1}{16}m\right)\cos\frac{u}{1 + m/4} + \frac{m}{16}\cos\frac{3u}{1 + m/4} + \ldots, \quad m \ll 1, \end{aligned} \tag{A.16}$$

that is, at $m = 0$ they become usual sine and cosine functions (and complex periods become infinite in this limit).

At $m = 1$, when real periods are infinitely large, we have

$$\operatorname{sn}(u, 1) = \tanh u, \quad \operatorname{cn}(u, 1) = \operatorname{dn}(u, 1) = 1/\cosh u, \tag{A.17}$$

and these functions have only complex periods.

Jacobi elliptic functions satisfy the differential equations

$$\begin{aligned}
\left(\frac{d\,\mathrm{sn}(u,m)}{du}\right)^2 &= \left(1-\mathrm{sn}^2(u,m)\right)\left(1-m\,\mathrm{sn}^2(u,m)\right), \\
\left(\frac{d\,\mathrm{cn}(u,m)}{du}\right)^2 &= \left(1-\mathrm{cn}^2(u,m)\right)\left(1-m+m\,\mathrm{cn}^2(u,m)\right), \\
\left(\frac{d\,\mathrm{dn}(u,m)}{du}\right)^2 &= \left(1-\mathrm{dn}^2(u,m)\right)\left(\mathrm{dn}^2(u,m)-1+m\right).
\end{aligned} \tag{A.18}$$

The formula

$$\int_0^{\mathrm{K}(m)} \mathrm{dn}^2(u,m)du = \mathrm{E}(m) \tag{A.19}$$

is convenient for calculations of mean values of squared elliptic functions.

For theoretical calculations another form of elliptic functions introduced by Weierstrass is more convenient. The main role is played here by the Weierstrass $\wp$-function which satisfies the differential equation

$$\begin{aligned}
\left(\wp'(z)\right)^2 &= 4\wp^3(z) - g_2\wp(z) - g_3 \\
&= 4\left(\wp(z)-e_1\right)\left(\wp(z)-e_2\right)\left(\wp(z)-e_3\right).
\end{aligned} \tag{A.20}$$

The zeros of the polynomial in the right hand side satisfy the relations

$$e_1+e_2+e_3=0, \quad e_1e_2+e_2e_3+e_3e_1=-\tfrac{1}{4}g_2, \quad e_1e_2e_3=\tfrac{1}{4}g_3. \tag{A.21}$$

Periods of the function $\wp(z)$ are denoted as 2ω and $2\omega'$, and if we introduce

$$\omega_1=\omega, \quad \omega_2=-\omega-\omega', \quad \omega_3=\omega', \tag{A.22}$$

then the zeros e_1, e_2, e_3 are equal to

$$e_\alpha=\wp(\omega_\alpha), \qquad \alpha=1,2,3. \tag{A.23}$$

It is an even function having the pole of the second order at $z\to 0$,

$$\wp(z)=\frac{1}{z^2}+\frac{g_2}{20}z^2+\dots. \tag{A.24}$$

Its relationships with the Jacobi elliptic functions are given by the formulas

$$\begin{aligned} \mathrm{sn}^2(u,m) &= \frac{e_1-e_3}{\wp(z)-e_3}, \\ \mathrm{cn}^2(u,m) &= \frac{\wp(z)-e_1}{\wp(z)-e_3}, \\ \mathrm{dn}^2(u,m) &= \frac{\wp(z)-e_2}{\wp(z)-e_3}, \end{aligned} \tag{A.25}$$

where

$$z = \frac{u}{\sqrt{e_1-e_3}}, \quad e_1 : e_2 : e_3 = (2-m) : (2m-1) : -(1+m), \tag{A.26}$$

or

$$u = \sqrt{e_1-e_3}\, z, \quad m = \frac{e_2-e_3}{e_1-e_3}. \tag{A.27}$$

The half-periods ω, ω' are connected with $\mathrm{K}(m)$ and $\mathrm{K}'(m) \equiv \mathrm{K}(m')$ ($m' = 1-m$) by the relations

$$\mathrm{K}(m) = \sqrt{e_1-e_3}\,\omega, \quad i\mathrm{K}'(m) = \sqrt{e_1-e_3}\,\omega'. \tag{A.28}$$

$\wp$-function satisfies the algebraic addition theorem

$$\wp(x+y)+\wp(x)+\wp(y) = \frac{1}{4}\left\{\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}\right\}^2. \tag{A.29}$$

It leads to various formulas, in particular, to

$$\wp(z+\omega_\alpha) = e_\alpha + \frac{(e_\alpha-e_\beta)(e_\alpha-e_\gamma)}{\wp(z)-e_\alpha}, \quad \alpha = 1,2,3. \tag{A.30}$$

Correspondingly, the Jacobi functions also have the algebraic addition formulas and we shall write here several important particular cases following from (A.25) and (A.30),

$$\begin{aligned} \mathrm{sn}(u+\mathrm{K},m) &= \frac{\mathrm{cn}(u,m)}{\mathrm{dn}(u,m)}, & \mathrm{sn}(u+i\mathrm{K}',m) &= \frac{1}{\sqrt{m}\,\mathrm{sn}(u,m)}; \\ \mathrm{cn}(u+\mathrm{K},m) &= -\sqrt{m'}\cdot\frac{\mathrm{sn}(u,m)}{\mathrm{dn}(u,m)}, & \mathrm{cn}(u+i\mathrm{K}',m) &= -\frac{i}{\sqrt{m}}\frac{\mathrm{dn}(u,m)}{\mathrm{sn}(u,m)}; \\ \mathrm{dn}(u+\mathrm{K},m) &= \frac{\sqrt{m'}}{\mathrm{dn}(u,m)}, & \mathrm{dn}(u+i\mathrm{K}',m) &= -i\frac{\mathrm{cn}(u,m)}{\mathrm{sn}(u,m)}. \end{aligned} \tag{A.31}$$

Integration of $\wp$-function leads to the Weierstrass ζ-function:

$$\begin{aligned} \wp(z) &= -\zeta'(z), \\ \zeta(z) &= \frac{1}{z} - \frac{g_2}{60}z^3 - \dots. \end{aligned} \tag{A.32}$$

This is an odd function; it is not double periodic (elliptic) but satisfies the relations

$$\zeta(z+2\omega) = \zeta(z) + 2\eta, \quad \zeta(z+2\omega') = \zeta(z) + 2\eta', \tag{A.33}$$

where

$$\eta = \zeta(\omega), \qquad \eta' = \zeta(\omega'), \tag{A.34}$$

and ω, ω' and η, η' are connected by the Legendre relation

$$\eta\omega' - \eta'\omega = \tfrac{1}{2}\pi i. \tag{A.35}$$

Further, the ζ-function is a logarithmic derivative of the Weierstrass σ-function:

$$\begin{aligned} \zeta(z) &= \frac{\sigma'(z)}{\sigma(z)}, \\ \sigma(z) &= z - \frac{g_2}{240}z^5 - \dots. \end{aligned} \tag{A.36}$$

If one changes the argument by a period $2\omega_\alpha$, $\alpha = 1, 2, 3$, then the σ-function transforms according to the rule

$$\sigma(z+2\omega_\alpha) = -\sigma(z)\exp\left[2\eta_\alpha(z+\omega_\alpha)\right], \tag{A.37}$$

where

$$\eta_1 = \eta, \quad \eta_2 = -\eta - \eta', \quad \eta_3 = \eta'. \tag{A.38}$$

Three additional σ-functions are defined by the formulas

$$\sigma_\alpha = -e^{\eta_\alpha z}\frac{\sigma(z-\omega_\alpha)}{\sigma(\omega_\alpha)}, \qquad \alpha = 1, 2, 3, \tag{A.39}$$

and they transform under addition of the periods according to

$$\begin{aligned} \sigma_\alpha(z+2\omega_\alpha) &= -e^{2\eta_\alpha(z+\omega_\alpha)}\sigma_\alpha(z), \quad \alpha = 1, 2, 3; \\ \sigma_\alpha(z+2\omega_\beta) &= e^{2\eta_\beta(z+\omega_\beta)}\sigma_\alpha(z), \quad \alpha = 1, 2, 3, \ (\alpha \neq \beta). \end{aligned} \tag{A.40}$$

The Jacobi elliptic functions are related with σ-functions by the formulas

$$\frac{\sigma(z)}{\sigma_3(z)} = \frac{1}{\sqrt{e_1 - e_3}}\,\mathrm{sn}(u,m), \quad \frac{\sigma_1(z)}{\sigma_3(z)} = \mathrm{cn}(u,m), \quad \frac{\sigma_2(z)}{\sigma_3(z)} = \mathrm{dn}(u,m), \tag{A.41}$$

where $u = \sqrt{e_1 - e_3}\,z$.

Functions ζ and σ satisfy the addition theorems

$$\zeta(x+y) = \zeta(x) + \zeta(y) + \frac{1}{2}\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)} \tag{A.42}$$

and

$$\wp(x) - \wp(y) = -\frac{\sigma(x+y)\sigma(x-y)}{\sigma^2(x)\sigma^2(y)}. \tag{A.43}$$

If the imaginary period is infinitely large, $\omega' \to i\infty$, that is,

$$\begin{aligned} &e_1 = 2a, \quad e_2 = e_3 = -a, \\ &g_2 = 12a^2, \quad g_3 = 8a^3, \quad \omega = \pi/\sqrt{12a}, \end{aligned} \tag{A.44}$$

then

$$\begin{aligned} \wp(z; 12a^2, 8a^3) &= -a + \frac{3a}{\sin^2(\sqrt{3a}\,z)}, \\ \zeta(z; 12a^2, 8a^3) &= az + \sqrt{3a}\cot(\sqrt{3a}\,z), \\ \sigma(z; 12a^2, 8a^3) &= \frac{1}{\sqrt{3a}}\sin(\sqrt{3a}\,z)\exp\left(\tfrac{1}{2}az^2\right). \end{aligned} \tag{A.45}$$

If the real period is infinitely large, $\omega \to \infty$, that is,

$$\begin{aligned} &e_1 = e_2 = a, \quad e_3 = -2a, \\ &g_2 = 12a^2, \quad g_3 = -8a^3, \quad \omega' = \pi i/\sqrt{12a}, \end{aligned} \tag{A.46}$$

then

$$\begin{aligned} \wp(z; 12a^2, -8a^3) &= a + \frac{3a}{\sinh^2(\sqrt{3a}\,z)}, \\ \zeta(z; 12a^2, -8a^3) &= -az + \sqrt{3a}\coth(\sqrt{3a}\,z), \\ \sigma(z; 12a^2, -8a^3) &= \frac{1}{\sqrt{3a}}\sinh(\sqrt{3a}\,z)\exp\left(-\tfrac{1}{2}az^2\right). \end{aligned} \tag{A.47}$$

The complete elliptic integrals of the second kind are expressed in terms of the Weierstrass functions by the formulas

$$\mathrm{E}(m) = \frac{e_1\omega + \eta}{\sqrt{e_1 - e_3}}, \quad \mathrm{E}'(m) \equiv \mathrm{E}(m') = \frac{e_3\omega' + \eta'}{\sqrt{e_1 - e_3}}, \tag{A.48}$$

The formula

$$\int_0^z \frac{dz}{\wp(z) - \wp(\kappa)} = \frac{1}{\wp'(\kappa)}\left[\ln\frac{\sigma(\kappa - z)}{\sigma(\kappa + z)} + 2\zeta(\kappa)z\right]. \tag{A.49}$$

is often used in calculations.

At last, Jacobi and Weierstrass functions can be expressed in terms of ϑ-functions which are defined by the fast converging series expansions

$$\begin{aligned}
\vartheta_1(v) &= 2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin[(2n+1)\pi v],\\
\vartheta_2(v) &= 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)}\cos[(2n+1)\pi v],\\
\vartheta_3(v) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2n\pi v),\\
\vartheta_4(v) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2n\pi v),
\end{aligned} \tag{A.50}$$

where

$$v = \frac{z}{2\omega} = \frac{u}{2\mathrm{K}(m)}, \quad u = \sqrt{e_1 - e_3}\,z, \tag{A.51}$$

and

$$\begin{aligned}
&\tau = \omega'/\omega = i\mathrm{K}'/\mathrm{K}, \qquad \Im\,\tau > 0;\\
&q = e^{i\pi\tau} = e^{i\pi\omega'/\omega} = \exp\left(-\frac{\pi\mathrm{K}'}{K}\right), \quad |q| < 1.
\end{aligned} \tag{A.52}$$

The σ-function is given by

$$\sigma(z) = 2\omega\exp\left(\frac{\eta z^2}{2\omega}\right)\frac{\vartheta_1(v)}{\vartheta_1'(0)}, \tag{A.53}$$

and the formulas

$$\begin{aligned}\mathrm{sn}(u,m) &= \frac{\vartheta_3(0)\vartheta_1(v)}{\vartheta_2(0)\vartheta_4(v)},\\ \mathrm{cn}(u,m) &= \frac{\vartheta_4(0)\vartheta_2(v)}{\vartheta_2(0)\vartheta_4(v)},\\ \mathrm{dn}(u,m) &= \frac{\vartheta_4(0)\vartheta_3(v)}{\vartheta_3(0)\vartheta_4(v)},\end{aligned} \tag{A.54}$$

where

$$m = \left(\frac{\vartheta_2(0)}{\vartheta_3(0)}\right)^4, \tag{A.55}$$

are also often used.

For more information about elliptic functions theory see, for example, Akhiezer (1970), Bateman and Erdélyi (1955), Whittaker and Watson (1927).

Appendix B

Algebraic resolvents of fourth degree polynomials

B.1 Zakharov-Shabat case (Ferrari resolvent)

In this case the main identity $f^2 - gh = P(\lambda) = \prod(\lambda - \lambda_i)$ has the form

$$\begin{aligned} P(\lambda) = &\left[(\lambda - s/4)^2 + (p - \nu)/2\right]^2 \\ &+ \nu\left(\lambda - s/4 + \left(q + \sqrt{\mathcal{R}(\nu)}\right)/2\nu\right)\left(\lambda - s/4 + \left(q - \sqrt{\mathcal{R}(\nu)}\right)/2\nu\right) \end{aligned} \tag{B.1}$$

where all notations are the same as in Sec. 5.1. This identity is valid for all values of ν. If we put ν equal to one of the zeros of the resolvent $\mathcal{R}(\nu)$, then Eq. (B.1) can be transformed to

$$\begin{aligned} P(\lambda) = &\left[(\lambda - s/4)^2 + (p - \nu)/2 + i\sqrt{\nu}(\lambda - s/4 + q/2\nu)\right] \\ &\times\left[(\lambda - s/4)^2 + (p - \nu)/2 - i\sqrt{\nu}(\lambda - s/4 + q/2\nu)\right]. \end{aligned} \tag{B.2}$$

Hence, $P(\lambda)$ vanishes, if λ is equal to one of the zeros of two quadratic equations

$$(\lambda - s/4)^2 + (p - \nu)/2 = \pm i\sqrt{\nu}(\lambda - s/4 + q/2\nu). \tag{B.3}$$

Just this method of solving a fourth degree algebraic equation $P(\nu) = 0$ by means of its relation to the third degree equation $\mathcal{R}(\nu) = 0$ and subsequent solution of two quadratic equations was the reason for introduction of the resolvent $\mathcal{R}(\nu)$ into algebra.

To find the relationship between a zero ν of the resolvent $\mathcal{R}(\nu)$ and four zeros of Eqs. (B.3), let us enumerate them so that λ_1 and λ_2 correspond to the upper sign in Eq. (B.3), and λ_3 and λ_4 to the lower sign. Then we subtract the equation for λ_2 from the equation for λ_1, the equation for λ_3 from the equation for λ_4, and add the results. After simple transformations we obtain

$$\begin{aligned}(\lambda_1-\lambda_2)(\lambda_1+\lambda_2-s/2)+&(\lambda_4-\lambda_3)(\lambda_3+\lambda_4-s/2)\\ &=i\sqrt{\nu}(\lambda_1-\lambda_2+\lambda_3-\lambda_4).\end{aligned}\tag{B.4}$$

Since

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=s,\tag{B.5}$$

we have $\lambda_1+\lambda_2-s/2=-(\lambda_3+\lambda_4-s/2)$, and, hence, Eq. (B.4) gives

$$i\sqrt{\nu}=\lambda_1+\lambda_2-s/2.$$

With the use of Eq. (B.5) this expression can be written in the form

$$\nu=-\tfrac{1}{2}(\lambda_1+\lambda_2-\lambda_3-\lambda_4)^2.$$

Let us denote this zero of $\mathcal{R}(\nu)$ as ν_3. Other enumeration of the zeros λ_i leads to other relation of ν_k with λ_i, and one can easily find that there are three possibilities leading to different expressions.

As a result we arrive at the three formulas

$$\begin{aligned}&\nu_1=-\tfrac{1}{4}(\lambda_1-\lambda_2+\lambda_3-\lambda_4)^2,\quad \nu_2=-\tfrac{1}{4}(\lambda_1-\lambda_2-\lambda_3+\lambda_4)^2,\\ &\nu_3=-\tfrac{1}{4}(\lambda_1+\lambda_2-\lambda_3-\lambda_4)^2,\end{aligned}\tag{B.6}$$

for the three zeros of the resolvent $\mathcal{R}$.

B.2 Heisenberg model case

Now the identity $f^2-gh=P(\lambda)$ after substitution of Eqs. (5.98) and (5.106) reduces to

$$\begin{aligned}4s_4P(\lambda)=&\left(\nu\lambda^2-s_3\lambda+2s_4\right)^2-\left(\nu^2-4s_4\right)\lambda^2\\ \times&\left(\lambda-\frac{2s_1s_4-s_3\nu+2i\sqrt{\mathcal{R}(\nu)}}{4s_4-\nu^2}\right)\left(\lambda-\frac{2s_1s_4-s_3\nu-2i\sqrt{\mathcal{R}(\nu)}}{4s_4-\nu^2}\right),\end{aligned}\tag{B.7}$$

where the resolvent $\mathcal{R}(\nu)$ is defined by Eq. (5.107) and it is the third degree polynomial in ν. If ν denotes one of the zeros of $\mathcal{R}(\nu)$, then four zeros of

$P(\lambda)$ can be found as roots of two quadratic equations

$$\nu\lambda - s_3 + 2s_4/\lambda = \pm\sqrt{\nu^2 - 4s_4}\left[\lambda - (2s_1s_4 - s_3\nu)/(4s_4 - \nu^2)\right]. \quad (B.8)$$

Let us enumerate λ_i, $i = 1, 2, 3, 4$, so that $\lambda_{1,2}$ correspond to the upper sign in Eq. (B.8), and $\lambda_{3,4}$ to the lower sign. Adding all four equations (B.8) and taking into account that $\lambda_1\lambda_2\lambda_3\lambda_4 = s_4$, we obtain

$$\begin{aligned}\nu[(\lambda_1 + \lambda_2)+&(\lambda_3 + \lambda_4)] - 2s_3\\ &= \sqrt{\nu^2 - 4s_4}\,[(\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4)].\end{aligned} \quad (B.9)$$

The form of this equation prompts that it is convenient to introduce new unknown variables x and y so that

$$\nu = x + y, \quad \sqrt{\nu^2 - 2s_4} = x - y. \quad (B.10)$$

These definitions give $xy = s_4$ and another equation is obtained by substitution of Eqs. (B.10) into Eq. (B.9). Thus, we arrive at the system

$$\begin{cases}(\lambda_3 + \lambda_4)x + (\lambda_1 + \lambda_2)y = s_3 \equiv (\lambda_3 + \lambda_4)\lambda_1\lambda_2 + (\lambda_1 + \lambda_2)\lambda_3\lambda_4,\\ xy = s_4 \equiv \lambda_1\lambda_2\lambda_3\lambda_4,\end{cases}$$

whose solution is evident without any calculations: $x = \lambda_1\lambda_2$, $y = \lambda_3\lambda_4$, and, hence,

$$\nu = \lambda_1\lambda_2 + \lambda_3\lambda_4.$$

We denote this zero as ν_3, and the other zeros are obtained by permutations of λ_i. Thus, we have the relationships

$$\nu_1 = \lambda_1\lambda_3 + \lambda_2\lambda_4, \quad \nu_2 = \lambda_1\lambda_4 + \lambda_2\lambda_3, \quad \nu_3 = \lambda_1\lambda_2 + \lambda_3\lambda_4 \quad (B.11)$$

between the zeros of the resolvent $\mathcal{R}(\nu)$ and the polynomial $P(\lambda)$.

B.3 Uniaxial ferromagnet, SRS, and other equations

We shall use here the notations of Sec. 5.5. We have to find the relations of the zeros $\nu_i, i = 1, 2, 3, 4$, of the resolvent (5.133) with the zeros λ_i of the polynomial $P\,(\lambda)$. The identity $f^2 - gh = P(\lambda)$ after substitution of

Eqs. (5.123), (5.127), (5.128), and (5.132) takes the form

$$P(\lambda) = \left(S\lambda^2 - f_1\lambda + (s_3 - s_1 J)/2f_1 + JS\right)^2 + \left(1 - S^2\right)\left(\lambda^2 + J\right) \\ \times \left[\lambda - \frac{s_1 - 2f_1 S + 2i\sqrt{J\mathcal{R}(S)}}{2(1-S^2)}\right] \cdot \left[\lambda - \frac{s_1 - 2f_1 S - 2i\sqrt{J\mathcal{R}(S)}}{2(1-S^2)}\right]. \tag{B.12}$$

If we assume that S is equal to one of the zeros ν of the resolvent, then the right-hand side of Eq. (B.12) becomes the difference of two squares, so that four zeros of the polynomial $P(\lambda)$ prove to be the roots of two equations,

$$\nu\lambda_i^2 - f_1\lambda_i + \frac{s_3 - s_1 J}{2f_1} + J\nu = \pm i\sqrt{1-\nu^2} \cdot \sqrt{\lambda_i^2 + J}\left[\lambda_i - \frac{s_1 - 2f_1\nu}{2(1-\nu^2)}\right]. \tag{B.13}$$

Suppose that the zeros λ_1 and λ_2 correspond to the upper sign on the right-hand side of Eq. (B.13) and the zeros λ_3 and λ_4 to the lower sign. We introduce the following notation

$$(i) = 2f_1\lambda_i'\nu - \left(2f_1^2\lambda_i + s_1 J - s_3\right)/\lambda_i', \tag{B.14}$$

where

$$\lambda_i' = \sqrt{\lambda_i^2 + J}. \tag{B.15}$$

Dividing four relations (B.13) by each other, we obtain six formulas of the type

$$\frac{(i)}{(j)} = \pm\frac{2\lambda_i\left(1-\nu^2\right) - s_1 + 2f_1\nu}{2\lambda_j\left(1-\nu^2\right) - s_1 + 2f_1\nu}, \tag{B.16}$$

where the upper sign corresponds to $(1)/(2)$ and $(3)/(4)$, and the lower sign to the other combinations. From (B.16) we can obtain six expressions of the type

$$1 - \nu^2 = \frac{s_1 - 2f_1\nu}{2}\frac{(i) \pm (j)}{\lambda_j(i) \pm \lambda_i(j)}, \tag{B.17}$$

where we have used the same convention about signs. Equating (B.17) to each other, we obtain four equations linear in (i),

$$\begin{aligned}
&(\lambda_1-\lambda_2)\,(3)+(\lambda_1-\lambda_3)\,(2)+(\lambda_3-\lambda_2)\,(1)=0,\\
&(\lambda_1-\lambda_2)\,(4)+(\lambda_1-\lambda_4)\,(2)+(\lambda_4-\lambda_2)\,(1)=0,\\
&(\lambda_1-\lambda_3)\,(4)+(\lambda_4-\lambda_1)\,(3)+(\lambda_4-\lambda_3)\,(1)=0,\\
&(\lambda_2-\lambda_3)\,(4)+(\lambda_4-\lambda_2)\,(3)+(\lambda_4-\lambda_3)\,(2)=0,
\end{aligned}\tag{B.18}$$

and three equations quadratic in (i),

$$\begin{aligned}
&(1)\,(2)\,(\lambda_3-\lambda_4)+(2)\,(3)\,(\lambda_1-\lambda_4)+(3)\,(4)\,(\lambda_1-\lambda_2)\\
&\qquad\qquad\qquad\qquad+(4)\,(1)\,(\lambda_3-\lambda_2)=0,\\
&(1)\,(2)\,(\lambda_3-\lambda_4)+(2)\,(4)\,(\lambda_3-\lambda_1)+(4)\,(3)\,(\lambda_2-\lambda_1)\\
&\qquad\qquad\qquad\qquad+(3)\,(1)\,(\lambda_2-\lambda_4)=0,\\
&(1)\,(3)\,(\lambda_2-\lambda_4)+(3)\,(2)\,(\lambda_4-\lambda_1)+(2)\,(4)\,(\lambda_1-\lambda_3)\\
&\qquad\qquad\qquad\qquad+(4)\,(1)\,(\lambda_3-\lambda_2)=0.
\end{aligned}\tag{B.19}$$

Only three equations (B.18) are linearly independent and the equations (B.19) follow from Eqs. (B.18). Since Eq. (B.14) is linear in ν, Eqs. (B.18) are also linear in ν equations convenient for calculation of ν. In particular, from Eqs. (B.18) it follows that

$$[(1)-(3)]\,(\lambda_2-\lambda_4)+[(2)-(4)]\,(\lambda_1-\lambda_3)=0,$$

so that ν can be found by solving the following equation:

$$\begin{aligned}
&2f_1\,[(\lambda_1-\lambda_3)\,(\lambda_2'-\lambda_4')+(\lambda_2-\lambda_4)\,(\lambda_1'-\lambda_3')]\,\nu=\\
&\quad=2f_1^2\,[(\lambda_1-\lambda_3)\,(\lambda_2/\lambda_2'-\lambda_4/\lambda_4')+(\lambda_2-\lambda_4)\,(\lambda_1/\lambda_1'-\lambda_3/\lambda_3')]\\
&\quad-(s_3-s_1J)\,[(\lambda_1-\lambda_3)\,(1/\lambda_2'-1/\lambda_4')+(\lambda_2-\lambda_4)\,(1/\lambda_1'-1/\lambda_3')]\,.
\end{aligned}$$

After simple transformations we find the root

$$\begin{aligned}
\nu_1=&\,(4f_1J)^{-1}[(\lambda_1-\lambda_3)(\lambda_2'-\lambda_4')+(\lambda_2-\lambda_4)(\lambda_1'-\lambda_3')]^{-1}\\
&\times\{(\lambda_1-\lambda_3)[2(\lambda_1+\lambda_3)(\lambda_2'-\lambda_4')J\\
&\qquad\qquad+(\lambda_2\lambda_4'-\lambda_4\lambda_2')((\lambda_1+\lambda_3)^2-(\lambda_1'-\lambda_3')^2)]\\
&\quad+(\lambda_2-\lambda_4)[2(\lambda_2+\lambda_4)(\lambda_1'-\lambda_3')J\\
&\qquad\qquad+(\lambda_1\lambda_3'-\lambda_3\lambda_1')((\lambda_2+\lambda_4)^2-(\lambda_2'-\lambda_4')^2)]\},
\end{aligned}\tag{B.20}$$

ν_2 and ν_3 are obtained from ν_1 by permutations $3 \leftrightarrow 4$ and $3 \leftrightarrow 2$, respectively, and ν_4 can be found by means of the formula

$$\nu_4 = (s_1 J - s_4)/f_1 J - (\nu_1 + \nu_2 + \nu_3). \tag{B.21}$$

The cases with $f_1 = 0$ require special consideration. Note that f_1^2 cannot be negative because of the identity

$$P_2(J) - \left(J^2 - s_2 J + s_4\right)^2 = J\left(s_1 J - s_3\right)^2.$$

Combining this with Eq. (5.128), we find that f_1 vanishes in two cases:

$$(a) \quad s_1 = s_3 = 0, \quad (b) \quad J = s_3/s_1. \tag{B.22}$$

We start with the case (a) with two pairs of zeros λ_i lying on the imaginary axis:

$$\lambda_1 = i\gamma_1, \quad \lambda_2 = i\gamma_2, \quad \lambda_3 = -i\gamma_1, \quad \lambda_4 = -i\gamma_2. \tag{B.23}$$

Let $\gamma_1 \geq \gamma_2$. Then we have to distinguish three cases,

$$(1) \quad J \leq \gamma_2^2, \qquad (2) \quad J \geq \gamma_1^2, \qquad (3) \quad \gamma_2^2 \leq J \leq \gamma_1^2, \tag{B.24}$$

and we have $f_1 = 0$ only in the cases (1) and (2). The zeros of the resolvent are equal to

$$\begin{aligned} \nu_1 =& \frac{1}{J}\left(\gamma_1\gamma_2 - \sqrt{\left(\gamma_1^2 - J\right)\left(\gamma_2^2 - J\right)}\right), \quad \nu_2 = 1, \\ \nu_3 =& -1, \quad \nu_4 = -\frac{1}{J}\left(\gamma_1\gamma_2 + \sqrt{\left(\gamma_1^2 - J\right)\left(\gamma_2^2 - J\right)}\right) \end{aligned} \tag{B.25}$$

for the case (1) $\left(J \leq \gamma_2^2\right)$, and

$$\begin{aligned} \nu_1 =& 1, \quad \nu_2 = \frac{1}{J}\left(\gamma_1\gamma_2 + \sqrt{\left(J - \gamma_1^2\right)\left(J - \gamma_2^2\right)}\right), \\ \nu_3 =& -\frac{1}{J}\left(\gamma_1\gamma_2 - \sqrt{\left(J - \gamma_1^2\right)\left(J - \gamma_2^2\right)}\right), \quad \nu_4 = -1, \end{aligned} \tag{B.26}$$

for the case (2) $\left(J \geq \gamma_1^2\right)$.

For the case (3) $\left(\gamma_2^2 \leq J \leq \gamma_1^2\right)$ we have

$$\sqrt{P_2(J)} = -\left(J^2 - s_2 J + s_4\right),$$

that is,

$$f_1 = \left(s_2 J - s_4 - J^2\right)/J \geq 0, \quad f_2 = JS,$$
$$\mathcal{R}(S) = S^4 - (s_2/J)\, S^2 + s_4/J, \tag{B.27}$$

and the resolvent's zeros are given by

$$\nu_1 = \frac{\gamma_1}{\sqrt{J}}, \quad \nu_2 = \frac{\gamma_2}{\sqrt{J}}, \quad \nu_3 = -\frac{\gamma_2}{\sqrt{J}}, \quad \nu_4 = -\frac{\gamma_1}{\sqrt{J}}. \tag{B.28}$$

Now we turn to the case (b) of Eq. (B.22) $(J = s_3/s_1)$. This case corresponds to the following choice of the zeros of $P(\lambda)$:

$$\lambda_1 = \alpha + i\gamma, \quad \lambda_2 = i\sqrt{J}, \quad \lambda_3 = \alpha - i\gamma, \quad \lambda_4 = -i\sqrt{J}, \tag{B.29}$$

so that $\lambda_2' = \lambda_4' = 0$. Simple calculations lead to the resolvent

$$\mathcal{R}(S) = S^4 - \frac{\alpha^2 + \gamma^2 + J}{J} S^2 + \frac{\gamma^2}{J} \tag{B.30}$$

with the zeros

$$\begin{aligned} \nu_1 = -\,\nu_4 &= \sqrt{\frac{\alpha^2 + \gamma^2 + J + \sqrt{(\alpha^2 + \gamma^2 + J)^2 - 4\gamma^2 J}}{2J}}, \\ \nu_2 = -\,\nu_3 &= \sqrt{\frac{\alpha^2 + \gamma^2 + J - \sqrt{(\alpha^2 + \gamma^2 + J)^2 - 4\gamma^2 J}}{2J}}. \end{aligned} \tag{B.31}$$

This completes the discussion of the particular cases with $f_1 = 0$.

Appendix C

Solutions to exercises

Exercise 1.1

The solution is given by Eq. (1.105) in implicit form,

$$x = \bar{x} + \frac{t}{1+\bar{x}^2}. \tag{C.1}$$

If $\bar{x}$ is expressed from this equation as a function of x and t, $\bar{x} = \bar{x}(x,t)$, then $u(x,t)$ is equal to

$$u(x,t) = \frac{1}{1+\bar{x}^2}, \tag{C.2}$$

since u is constant along the characteristics. The caustic curve is given by Eqs. (1.107)

$$x = \bar{x} - \frac{u_0(\bar{x})}{u_0'(\bar{x})} = \frac{1+3\bar{x}^2}{2\bar{x}}, \quad t = -\frac{1}{u_0'(\bar{x})} = \frac{(1+\bar{x}^2)^2}{2\bar{x}}. \tag{C.3}$$

From the condition $u_0''(\bar{x}) = 0$ we find the critical value of $\bar{x}$,

$$\bar{x}_b = 1/\sqrt{3}, \tag{C.4}$$

and, hence,

$$x_b = \sqrt{3}, \qquad t_b = 8\sqrt{3}/9. \tag{C.5}$$

The straight line $f(\bar{x}) = (x_b - \bar{x})/t_b$ intersects the curve $u_0(\bar{x}) = 1/(1+\bar{x}^2)$ in its inflection point. It is clear that for $t > t_b$ there is a region of values of x for which the line $(x-\bar{x})/t$ intersects the curve $u_0(\bar{x})$ in three points.

Exercise 1.3
It is evident that $x = 3t$ is a solution of the equation

$$\frac{dx}{dt} = \frac{x}{t} + \left[\frac{1}{2t}\left(1 - \frac{x^2}{9t^2}\right)\right]^{1/3}. \tag{C.6}$$

Therefore it is natural to introduce a new variable

$$z = x/3t \tag{C.7}$$

and define new functions $x(z)$, $t(z)$ related by the equation

$$x(z) = 3zt(z). \tag{C.8}$$

Then Eq. (C.6) reduces to

$$\frac{3 \cdot 2^{1/3} dz}{(1 - z^2)^{1/3}} = \frac{dt}{t^{4/3}}$$

with an obvious solution

$$t^{-1/3} = C - 2^{1/3} \int_0^z \frac{dz}{(1 - z^2)^{1/3}},$$

where C is the integration constant. The integral on the right hand side can be expressed in terms of the hypergeometric function, and, hence, the characteristic curves are given parametrically by the equations

$$t(z) = \left[C \pm 2^{1/3} z F(\tfrac{1}{2}, \tfrac{1}{3}; \tfrac{3}{2}, z^2)\right]^{-3}, \quad x(z) = 3zt(z), \quad -1 \le z \le 1, \tag{C.9}$$

where two signs correspond to the two signs in Eq. (1.203). The constants C can be chosen so that one branch of the characteristic curve becomes a continuation of another one. Just this method was used for plotting the characteristic curves in Fig. 1.14.

Exercise 1.4
We transform the Boussinesq equation (1.271),

$$V_{TT} = V_{XX} + \tfrac{3}{2}\alpha \left(V^2\right)_{XX} + \tfrac{1}{3}\beta V_{XXXX}, \tag{C.10}$$

to slow variables

$$\tau = \varepsilon^a T, \qquad \xi = \varepsilon^b (X - T)$$

corresponding to the propagation of a wave with the amplitude $V \sim \varepsilon$ in the frame of reference moving with the group velocity of linear waves in the positive X direction:

$$\varepsilon^{2a} V_{\tau\tau} - 2\varepsilon^{a+b} V_{\tau\xi} = \tfrac{3}{2}\alpha\varepsilon^{2b}\left(V^2\right)_{\xi\xi} + \tfrac{1}{3}\beta\varepsilon^{4b} V_{\xi\xi\xi\xi}. \tag{C.11}$$

The condition that the nonlinear and dispersion effects have the same order of magnitude, that is, $\varepsilon^{1+a+b} \sim \varepsilon^{1+4b} \sim \varepsilon^{2+2b}$, gives the values of exponents,

$$a = \tfrac{3}{2}, \qquad b = \tfrac{1}{2}.$$

Then we can neglect in Eq. (C.11) the term $V_{\tau\tau}\varepsilon^3 \sim \varepsilon^4$ and obtain for the main term of the series expansion $V = \varepsilon V^{(1)} + \dots$ the KdV equation

$$V_\tau^{(1)} + \tfrac{3}{2}\alpha V^{(1)} V_\xi^{(1)} + \tfrac{1}{6}\beta V_{\xi\xi\xi}^{(1)} = 0.$$

Exercise 1.5

We look for the solution of Eq. (1.361) in the form

$$u(x,t) = p\left[F(x - Vt) + iG(x - Vt)\right]\exp(2ip^2 t), \tag{C.12}$$

where F and G are unknown functions of the variable $\xi = x - Vt$ and p is a constant parameter connected with the soliton amplitude. After substitution of Eq. (C.12) into the focusing NLS equation (1.361) and separation of the real and imaginary parts we obtain the system

$$\begin{aligned} &F_{\xi\xi} + VG_\xi + 2p^2(F^2 + G^2 - 1)F = 0,\\ &G_{\xi\xi} - VF_\xi + 2p^2(F^2 + G^2 - 1)G = 0, \end{aligned} \tag{C.13}$$

This system has the first integrals

$$\begin{aligned} &F_\xi^2 + G_\xi^2 = p^2\left[1 - (1 - F^2 - G^2)^2\right],\\ &G_\xi F - F_\xi G = \tfrac{1}{2}V(F^2 + G^2), \end{aligned} \tag{C.14}$$

where the integration constants are chosen so that $F^2 + G^2 \to 0$ at $|\xi| \to \infty$. In polar coordinates

$$F = r\cos\phi, \qquad G = r\sin\phi \tag{C.15}$$

these integrals take the form

$$\phi_\xi = \tfrac{1}{2}V, \qquad r_\xi^2 + r^2\phi_\xi^2 = p^2(2r^2 - r^4), \tag{C.16}$$

hence, we obtain at once that

$$\phi = \tfrac{1}{2}V\xi = \tfrac{1}{2}V(x - Vt), \tag{C.17}$$

where the integration constant (the phase at $\xi = 0$) is taken equal to zero. The second equation (C.16) reduces to

$$\left[\frac{dr}{d(2p\phi/V)}\right]^2 = r^2\left(\frac{u_0^2}{p^2} - r^2\right), \tag{C.18}$$

where we have introduced a new constant parameter

$$u_0 = \sqrt{2p^2 - (V/2)^2}. \tag{C.19}$$

Elementary integration of Eq. (C.18) gives

$$r = \frac{u_0/p}{\cosh[u_0(x - Vt)]}, \tag{C.20}$$

where we have used also Eq. (C.16). Substitution of Eqs. (C.15) and (C.20) into (C.12) yields the soliton solution (1.368).

Exercise 2.1
As usual, we introduce dimensionless space (X) and time (T) coordinates with the use of their characteristic values in a linear approximation,

$$X = (\beta/r_D)x, \qquad T = (c_0/r_D)\beta t, \tag{C.21}$$

where the parameter $\beta \ll 1$ controls weak dispersion effects. We define the dimensionless variables N, U, Φ by the expressions

$$n = n_0(1 + \alpha N), \quad \phi = (T_e/e)\alpha\Phi, \quad u = c_0\alpha U, \tag{C.22}$$

where the parameter $\alpha \ll 1$ controls weak nonlinear effects. In these new variables Eqs. (2.2), (2.3) and (2.6) take the form

$$N_T + [(1 + \alpha N)U]_X = 0, \tag{C.23}$$

$$U_T + \alpha U U_X + \Phi_X = 0, \tag{C.24}$$

$$\alpha\beta\Phi_{XX} = \exp(\alpha\Phi) - 1 - \alpha N. \tag{C.25}$$

The series expansion of Eq. (C.25) with respect to small parameters α and β gives in the lowest nontrivial approximation the following equation

$$\beta\Phi_{XX} - \Phi + N - \tfrac{1}{2}\alpha\Phi^2 = 0. \tag{C.26}$$

Slow evolution of the wave is determined by the value of the amplitude $U \sim \varepsilon \ll 1$ and it is essential for the scales $T \sim \varepsilon^{-a}$ of time and $X \sim \varepsilon^{-b}$ of space coordinates. From linear approximation we find the following scaling exponents for the variables $N \sim \varepsilon^c$ and $\Phi \sim \varepsilon^d$:

$$
\begin{aligned}
&N_T \sim U_X, \quad \text{that is,} \quad \varepsilon^{c+a} \sim \varepsilon^{1+b}, \quad \text{hence,} \quad c + a = 1 + b; \\
&U_T \sim \Phi_X, \quad \text{that is,} \quad \varepsilon^{1+a} \sim \varepsilon^{d+b}, \quad \text{hence,} \quad 1 + a = d + b; \\
&\Phi \sim N, \quad \text{that is,} \quad \varepsilon^{c} \sim \varepsilon^{d}, \quad \text{hence,} \quad c = d.
\end{aligned}
$$

Thus, we have $c = d = 1$ and $U \sim N \sim \Phi \sim \varepsilon$. Now we demand that the nonlinear and dispersive effects have the same order of magnitude,

$$
\begin{aligned}
&U_T \sim UU_X, \quad \text{that is,} \quad \varepsilon^{1+a} \sim \varepsilon^{2+b}, \quad \text{hence,} \quad 1 + a = 2 + b; \\
&\Phi_{XX} \sim \Phi^2, \quad \text{that is,} \quad \varepsilon^{1+2b} \sim \varepsilon^{2}, \quad \text{hence,} \quad 1 + 2b = 2;
\end{aligned}
$$

and, thus,

$$
a = \tfrac{3}{2}, \qquad b = \tfrac{1}{2}.
$$

The sound velocity in the dimensionless variables X and T is equal to unity, and if we consider a wave pulse propagating in the positive x direction, then its evolution corresponds to the slow variables

$$
\tau = \varepsilon^{3/2} T, \qquad \xi = \varepsilon^{1/2}(T - X). \tag{C.27}
$$

On substitution of the series expansions

$$
\begin{aligned}
&N = \varepsilon N^{(1)} + \varepsilon^2 N^{(2)} + \dots, \quad U = \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \dots, \\
&\Phi = \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \dots,
\end{aligned}
$$

Eqs. (C.23), (C.24), (C.26) take the form

$$
\begin{aligned}
-N_\xi^{(1)} + U_\xi^{(1)} + \varepsilon[N_\tau^{(1)} + \alpha N_\xi^{(1)} U^{(1)} + \alpha N^{(1)} U_\xi^{(1)} \\
- N_\xi^{(2)} + U_\xi^{(2)}] + \dots = 0, \\
-U_\xi^{(1)} + \Phi_\xi^{(1)} + \varepsilon[U_\tau^{(1)} + \alpha U^{(1)} U_\xi^{(1)} - U_\xi^{(2)} + \Phi_\xi^{(2)}] + \dots = 0, \\
-\Phi^{(1)} + N^{(1)} + \varepsilon[\beta^2 \Phi_{\xi\xi}^{(1)} - \tfrac{1}{2}\alpha(\Phi^{(1)})^2 - \Phi^{(2)} + N^{(2)}] + \dots = 0.
\end{aligned}
$$

In the main approximation we obtain

$$
\Phi^{(1)} = N^{(1)} = U^{(1)},
$$

so that equations of the next approximation can be transformed to

$$\begin{aligned}
&U_\tau^{(1)} + 2\alpha U^{(1)} U_\xi^{(1)} - N_\xi^{(2)} + U_\xi^{(2)} = 0,\\
&U_\tau^{(1)} + \alpha U^{(1)} U_\xi^{(1)} - U_\xi^{(2)} + \Phi_\xi^{(2)} = 0,\\
&\beta^2 U_{\xi\xi\xi}^{(1)} - \alpha U^{(1)} U_\xi^{(1)} - \Phi_\xi^{(2)} + N_\xi^{(2)} = 0.
\end{aligned}$$

Their sum yields the evolution equation for $U^{(1)}$,

$$U_\tau^{(1)} + \alpha U^{(1)} U_\xi^{(1)} + \tfrac{1}{2}\beta^2 U_{\xi\xi\xi}^{(1)} = 0. \tag{C.28}$$

Returning to the variables

$$U^{(1)} = \frac{u}{c_0\alpha\varepsilon}, \quad t = \frac{r_D}{c_0\beta\varepsilon^{3/2}}\tau, \quad x = \frac{r_D}{\beta}\left(\frac{\xi}{\varepsilon^{1/2}} + \frac{\tau}{\varepsilon^{3/2}}\right),$$

we obtain the KdV equation

$$u_t + c_0 u_x + u u_x + \tfrac{1}{2} c_0 r_D^2 u_{xxx} = 0. \tag{C.29}$$

Exercise 2.2

Introduction of the continuous space coordinate x along the lattice and transition to the long wave limit yields the equation

$$v_{tt} = \left(\omega_0^2 dv + \tfrac{1}{12}\omega_0^2 d^4 v_{xx} + \gamma d^3 v^2\right)_{xx}, \tag{C.30}$$

which coincides with the Boussinesq equation. Hence, the KdV equation can be obtained as in the solution of Exercise 1.4.

Exercise 2.3

The evolution of small amplitude shallow water waves is described by the KdV equation (2.105),

$$q_t + q_{xxx} = -6\varepsilon q q_x. \tag{C.31}$$

We look for its solution in the form of a series expansion

$$q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \dots, \quad q = q(\theta), \quad \theta = kx - \omega t, \tag{C.32}$$

where the frequency ω is also presented in the form a series in powers of ε,

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots. \tag{C.33}$$

Equations for the coefficients up to the third order approximation read

$$\omega_0 q_0' - k^3 q_0''' = 0, \tag{C.34}$$

$$\omega_0 q_1' - k^3 q_1''' = 6k q_0 q_0' - \omega_1 q_0', \tag{C.35}$$

$$\omega_0 q_2' - k^3 q_2''' = 6k(q_0 q_1)' - \omega_2 q_0', \tag{C.36}$$

where prime denotes differentiation with respect to θ. From Eq. (C.34) we find a harmonic wave solution

$$q_0 = 2a\cos\theta, \tag{C.37}$$

where definition of the amplitude agrees with Eq. (2.84). Exclusion of the secular term from the right hand side of Eq. (C.35) gives $\omega_1 = 0$, so that equation for q_1 transforms to

$$q_1' - q_1''' = (12a^2/k^2)\sin 2\theta$$

which gives the solution

$$q_1 = b + (2a^2/k^2)\cos 2\theta.$$

We consider a uniform wave with a constant amplitude. Therefore, a nonzero value of b means here the change of the basin's depth. Hence, we put $b = 0$ and obtain from Eq. (C.36) equation for q_2,

$$q_2' + q_2''' = \frac{12a^3}{k^4}(3\sin 3\theta + \sin\theta) - \frac{2a\omega_2}{k^3}\sin\theta. \tag{C.38}$$

Exclusion of the secular term gives the frequency shift,

$$\omega_2 = 6a^2/k,$$

and, then, the solution of Eq. (C.38) yields

$$q_2 = (3a^3/2k^4)\cos 3\theta.$$

As a result, we obtain the Stokes expansions for the wave,

$$u = \varepsilon q = 2\varepsilon a\cos\theta + \frac{2(\varepsilon a)^2}{k^2}\cos 2\theta + \frac{3(\varepsilon a)^3}{2k^4}\cos 3\theta + \dots, \quad \theta = kx - \omega t, \tag{C.39}$$

and for the frequency,

$$\omega = -k^3 + \frac{6(\varepsilon a)^2}{k} + \dots. \tag{C.40}$$

These expressions coincide with the small amplitude expansions of the exact cnoidal wave solution of the KdV equation with the zero mean value $\overline{u}$ (see Sec. 2.2.2).

Exercise 2.4
The propagation of a stationary electromagnetic wave $E(x, y, z)\exp(-i\omega t)$ is described by the equation

$$E_{xx} + E_{yy} + E_{zz} + (\omega^2/c^2)\epsilon E = 0,$$

where the nonlinearity of the dielectric constant is assumed to be of the Kerr type,

$$\epsilon = \epsilon_0 + \epsilon_1 |E|^2.$$

The wavenumber k is determined by the dispersion relation

$$k^2 = (\omega^2/c^2)\epsilon_0(\omega).$$

Making the ansatz

$$E(x, y, z) = A(x, y, z)e^{ikz}$$

and assuming the paraxial approximation, $|\partial^2 A/\partial z^2| \ll k|\partial A/\partial z|$, we obtain the propagation equation for the slowly varying envelope $A(x, y, z)$,

$$i\frac{\partial A}{\partial z} + \frac{1}{2k}\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right) + \frac{\epsilon_1}{2k\epsilon_0}|A|^2 A = 0.$$

This is the NLS equation with two space coordinates and the coordinate z along the direction of the beam propagation playing the role of time.

Exercise 2.5
By a straightforward calculation we obtain

$$\begin{aligned} |\mathcal{E}_1|^2 &= \tfrac{1}{2}I(1 + S_3) = \tfrac{1}{2}I(1 + \cos\psi) = \tfrac{1}{2}\left(1 - \tanh\left[\tfrac{1}{2}(\tau - \zeta)\right]\right), \\ |\mathcal{E}_2|^2 &= \tfrac{1}{2}I(1 - S_3) = \tfrac{1}{2}I(1 - \cos\psi) = \tfrac{1}{2}\left(1 + \tanh\left[\tfrac{1}{2}(\tau - \zeta)\right]\right). \end{aligned} \tag{C.41}$$

This ‘kink-like’ solution describes the transformation of the pumping wave $\mathcal{E}_1$ into the Raman wave $\mathcal{E}_2$.

Exercise 3.1
As it follows from equations $\mathcal{L} = D(\omega, k)A^2$, $D(\omega, k) = 0$ (see (3.83,3.84)), the averaged Lagrangian is equal to zero, $\mathcal{L} = 0$. Hence, the first equation

(3.77) shows that the averaged energy density is given by $E = \mathcal{L}_\omega \cdot \omega$, and, consequently,

$$\mathcal{L} = \frac{E}{\omega}, \quad \mathcal{L}_k = D_k A^2 = -D_\omega v_g A^2 = -v_g \mathcal{L} = -\frac{v_g E}{\omega}.$$

Thus, the equation

$$\frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial x}\mathcal{L}_k = 0$$

(see Eq. (3.75)) yields

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega}\right) + \frac{\partial}{\partial x}\left(\frac{v_g E}{\omega}\right) = 0.$$

This relation generalizes the notion of 'adiabatic invariants' on wave systems (Whitham, 1965b). It is clear that it holds for any linear system with a Lagrangian quadratic with respect to field variables.

Exercise 3.2

Since in the limit $m \to 1$ solitons are separated from each other by distances much longer than their widths, we have with a good accuracy

$$\overline{u} = k\int_{-\infty}^{\infty} u(x)dx, \quad \overline{u^2} = k\int_{-\infty}^{\infty} u^2(x)dx, \tag{C.42}$$

where

$$u(x) = \frac{2a}{\cosh^2(\sqrt{a}\,x)} \tag{C.43}$$

is the soliton solution with the omitted time dependence. Substitution of Eq. (C.43) into Eq. (C.42) gives

$$\overline{u} = 4ka^{1/2}, \qquad \overline{u^2} = \tfrac{16}{3}ka^{3/2},$$

and we obtain one Whitham equation

$$(ka^{1/2})_t + (4ka^{3/2})_x = 0. \tag{C.44}$$

Another Whitham equation follows from the condition $k_t + \omega_x = 0$, if we take into account that in the soliton limit we have $\omega = kV = 4ka$. Thus, we obtain

$$k_t + (4ka)_x = 0. \tag{C.45}$$

Equations (C.44) and (C.45) determine a slow evolution of the wavenumber k and the amplitude a of the wave train.

Exercise 3.3
From Eqs. (C.44) and (C.45) we find the equation for the amplitude

$$(a^{1/2})_t + 4a(a^{1/2})_x = 0. \tag{C.46}$$

This is the Hopf equation with the well-known solution

$$x - 4at = F(a),$$

$F(a)$ is an arbitrary function. At asymptotically large values of time t we have $x \cong 4at$, that is, solitons propagate independently from each other with velocities determined by their amplitudes. Thus, we obtain an asymptotic solution

$$a(x,t) = \frac{x}{4t}, \tag{C.47}$$

(corresponding to $F(a) = 0$) which we are interested in. Then substitution of Eq. (C.47) into Eq. (C.45) gives the equation for k,

$$tk_t + xk_x = -k. \tag{C.48}$$

It can be solved by the method of characteristics which satisfy the equations

$$\frac{dt}{t} = \frac{dx}{x} = -\frac{dk}{k}.$$

This system has two obvious first integrals

$$x/t = C_1, \qquad kt = C_2.$$

Hence, the general solution of Eq. (C.48) is given by

$$\Phi(x/t, kt) = 0,$$

where Φ is an arbitrary function (see Sec.1.3.4), or

$$k(x,t) = \frac{1}{4t} f\left(\frac{x}{4t}\right), \tag{C.49}$$

where f is again an arbitrary function to be determined by the initial conditions. (Note that Eq. (C.48) is nothing but the Euler identity for homogeneous functions, and this observation gives at once the solution (C.49).)

Exercise 3.4
After elimination of A^2 from the averaged Lagrangian

$$\mathcal{L} = \mathcal{L}(k, \omega, A^2) \tag{C.50}$$

with the help of the dispersion relation

$$\omega = \omega(k, A^2) \tag{C.51}$$

we obtain $\mathcal{L} = \mathcal{L}(k, \omega)$ as a function of only two variables k and ω. Then substitution of

$$k = \partial\theta/\partial x, \qquad \omega = -\partial\theta/\partial t$$

into the Whitham equation

$$\frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial x}\mathcal{L}_k = 0 \tag{C.52}$$

gives at once

$$\frac{\partial^2 \mathcal{L}}{\partial \omega^2}\frac{\partial^2\theta}{\partial t^2} + 2\frac{\partial^2 \mathcal{L}}{\partial\omega\partial k}\frac{\partial^2\theta}{\partial t\partial x} + \frac{\partial^2 \mathcal{L}}{\partial k^2}\frac{\partial^2\theta}{\partial x^2} = 0. \tag{C.53}$$

The frequency is approximated by

$$\omega = \omega_0(k) + \omega_2(k)A^2. \tag{C.54}$$

The averaged Lagrangian $\mathcal{L}$ is also an even function of A. However, the first term in its expansion in powers of A is of order A^4, because the leading term of order A^2 vanishes (see Exercise 3.1). Accordingly,

$$\mathcal{L} = \mathcal{L}_1(k, \omega)A^4, \tag{C.55}$$

and elimination of A^2 from this equation gives

$$\mathcal{L} = \frac{\mathcal{L}_1(k, \omega)}{(\omega_2(k))^2}\left(\omega - \omega_0(k)\right)^2. \tag{C.56}$$

We substitute this expression into

$$\mathcal{D} = \frac{\partial^2 \mathcal{L}}{\partial \omega^2}\frac{\partial^2 \mathcal{L}}{\partial k^2} - \left(\frac{\partial^2 \mathcal{L}}{\partial\omega\partial k}\right)^2 \tag{C.57}$$

and after that put ω equal to Eq. (C.54). As a result, after lengthy calculation we obtain

$$\mathcal{D} = -4\left(\frac{\mathcal{L}_1(k,\omega)}{\omega_2^2(k)}\right)^2 \cdot \omega_0''(k)\omega_2(k)\cdot A^2. \tag{C.58}$$

Thus, Eq. (C.53) is hyperbolic ($\mathcal{D} < 0$) and its characteristics are real, if

$$\omega_0''(k)\omega_2(k) > 0. \tag{C.59}$$

But if

$$\omega_0''(k)\omega_2(k) < 0, \tag{C.60}$$

then $\mathcal{D} > 0$ and Eq. (C.53) is elliptic and unstable with respect to small perturbations (Lighthill, 1965). For gravity waves on deep water we have (see Sec. 2.2.3)

$$\omega_0 = \sqrt{gk}, \qquad \omega_2 = 2\sqrt{gk}\,k^2,$$

hence, they are modulationally unstable.

Exercise 4.1
If $n = 2i+1$, $m = 2j+1$, then

$$\{I_{2i+1}, I_{2j+1}\} = \{I_{i+j+1}, I_{i+j+1}\} \equiv 0;$$

if $n = 2i$, $m = 2j$, then

$$\{I_{2i}, I_{2j}\} = \{I_{i+j}, I_{i+j}\} \equiv 0;$$

if $n = 2i+1$, $m = 2j$, then

$$\{I_{2i+1}, I_{2j}\} = \{I_{i+j+1}, I_{i+j}\} = \{I_{i+j}, I_{i+j+1}\},$$

and the last expression is equal to zero by a skew symmetry of the Poisson bracket, $\{I_n, I_m\} = -\{I_m, I_n\}$.

Exercise 4.2
Straightforward calculation gives $\partial\mathbb{L}/\partial t = -u_t$ and

$$[\mathbb{L},\mathbb{A}] = -3\left[\mathbb{D}^2, u_x + 2u\mathbb{D}\right] - 4\left[u,\mathbb{D}^3\right] - 3\left[u, 2u\mathbb{D}\right] = u_{xxx} + 6uu_x.$$

Exercise 4.3
We have to check that

$$[\mathbb{L}, \mathbb{A}_n] = -\frac{\partial}{\partial x}\frac{\delta I_n}{\delta u} = -\frac{\partial}{\partial x}\widehat{\frac{\delta}{\delta u}}\mathcal{P}_n.$$

Straightforward calculation gives

$$[\mathbb{L}, \mathbb{A}_n] = -\tfrac{1}{4}\sum_{k=1}^{n+1}\left[\mathbb{R}\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{k-2}\right) - 4\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{k-2}\right)'\mathbb{L}\right]\mathbb{L}^{n-k+1},$$

where $\mathbb{R}$ is the recursion operator (4.33) and prime denotes differentiation with respect to x. By Eq. (4.78) we obtain

$$[\mathbb{L}, \mathbb{A}_n] = -\sum_{k=1}^{n+1}\left[\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{k-2}\right)' - \left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_{k-2}\right)'\mathbb{L}\right]\mathbb{L}^{n-k+1} \equiv -\left(\widehat{\frac{\delta}{\delta u}}\mathcal{P}_n\right)',$$

and the proof is complete.

Exercise 4.4
Elementary integration of Eq. (4.112) with $\lambda_2 = \lambda_1$, that is,

$$\mu_x = 2(\mu - \lambda_1)\sqrt{\lambda_3 - \mu},$$

gives

$$\mu(x) = \frac{\lambda_3 - \lambda_1}{\cosh^2\left(\sqrt{\lambda_3 - \lambda_1}\, x\right)} + \lambda_1.$$

Taking into account that $V_s = -2\lambda_1 - \lambda_3$, we obtain from Eq. (4.111)

$$u(x,t) = \frac{2(\lambda_3 - \lambda_1)}{\cosh^2\left[\sqrt{\lambda_3 - \lambda_1}\,(x + (2\lambda_1 + \lambda_3)t)\right]} - \lambda_3$$

in agreement with the limit $\lambda_2 \to \lambda_1$ of Eqs. (3.151.3.152).

Exercise 5.1
Calculations are parallel to those of Sec. 5.6. As a result, we arrive at the equation

$$\frac{dS_3}{d\,(2\xi)} = \sqrt{-\sigma^2 R\,(S_3)}, \quad \xi = \tau - \zeta/V, \quad V = 4\sqrt{P(-\Delta/2)}, \qquad \text{(C.61)}$$

where $\mathcal{R}$ is the resolvent (5.136) with J replaced by $-\sigma^2$. The periodic solution has a standard form

$$S_3 = \frac{(\nu_1 - \nu_3)\,\nu_4 + (\nu_4 - \nu_1)\,\nu_3 \,\mathrm{sn}^2\left(\sqrt{\sigma^2(\nu_1 - \nu_3)(\nu_4 - \nu_2)}\xi, m\right)}{\nu_1 - \nu_3 + (\nu_4 - \nu_1)\,\mathrm{sn}^2\left(\sqrt{\sigma^2(\nu_1 - \nu_3)(\nu_4 - \nu_2)}\xi, m\right)}, \tag{C.62}$$

where

$$m = \frac{(\nu_2 - \nu_3)(\nu_4 - \nu_1)}{(\nu_1 - \nu_3)(\nu_4 - \nu_2)} = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)}, \tag{C.63}$$

In the limit $m = 1$, ($\lambda_1 = \lambda_2 = \alpha + i\gamma$, $\lambda_3 = \lambda_4 = \alpha - i\gamma$), we obtain the soliton solution of Steudel (1983)

$$S_3 - 1 = V\,(1 + R_3) = \frac{2\gamma}{\sigma}\frac{\sin 2\varphi}{\cosh(4\gamma W) + \cos 2\varphi}, \tag{C.64}$$

where

$$\tan 2\varphi = \frac{2\sigma\gamma}{\sigma^2 - \alpha^2 - \gamma^2}. \tag{C.65}$$

Exercise 5.2

The $\mathbb{U}$–$\mathbb{V}$-pair is the same as in the preceding Exercise, but this time the periodic solution corresponds to the third degree polynomial

$$P(\lambda) = \prod_{i=1}^{3}(\lambda - \lambda_i) = \lambda^3 - s_1\lambda^2 + s_2\lambda - s_3, \tag{C.66}$$

λ_i being real. Standard calculations lead to the equation

$$\frac{dS_3}{d(2\xi)} = \sqrt{-\sigma^2\mathcal{R}(S_3)}, \tag{C.67}$$

where the resolvent is given by

$$\mathcal{R}(\nu) = \nu^4 - \frac{s_2 + \sigma^2}{f_1\sigma^2}\nu^3 + \frac{s_1}{\sigma^2}\nu^2 - \frac{f_1}{\sigma^2}\nu + \frac{1}{4\sigma^2}, \tag{C.68}$$

$$f_1^2 = \left[s_1\sigma^2 + s_3 - \sqrt{P(\sigma)\,P(-\sigma)}\right]/2\sigma^2,$$

and it has the zeros

$$\begin{aligned}
\nu_1 &= \frac{(\lambda_2' - \lambda_3')\,\sigma^2 - (\lambda_1 + \lambda_1')\,(\lambda_2\lambda_3' - \lambda_3\lambda_2')}{2f_1\sigma^2\,(\lambda_2 - \lambda_3 + \lambda_2' - \lambda_3')}, \\
\nu_2 &= \frac{(\lambda_1' - \lambda_3')\,\sigma^2 - (\lambda_2 + \lambda_2')\,(\lambda_1\lambda_3' - \lambda_3\lambda_1')}{2f_1\sigma^2\,(\lambda_1 - \lambda_3 + \lambda_1' - \lambda_3')}, \\
\nu_3 &= \frac{(\lambda_1' - \lambda_2')\,\sigma^2 - (\lambda_3 + \lambda_3')\,(\lambda_1\lambda_2' - \lambda_2\lambda_1')}{2f_1\sigma^2\,(\lambda_1 - \lambda_2 + \lambda_1' - \lambda_2')},
\end{aligned} \tag{C.69}$$

where

$$\lambda_i' = \sqrt{\lambda_i^2 - \sigma^2},$$

and ν_4 can be obtained from the formula

$$\nu_4 = \frac{\sigma^2 + s_2}{f_1\sigma^2} - (\nu_1 + \nu_2 + \nu_3)\,. \tag{C.70}$$

The zeros ν_i are real, if λ_i are also real and greater than σ. If $\lambda_1 < \lambda_2 < \lambda_3$, then the zeros ν_i are ordered according to $\nu_1 < \nu_2 < \nu_3 < \nu_4$. Hence, the variable S_3 can oscillate either in the interval $\nu_1 \le S_3 \le \nu_2$ or in the interval $\nu_3 \le S_3 \le \nu_4$, where $\mathcal{R}\,(S_3) \le 0$.

In the case $\nu_3 \le S_3 \le \nu_4$ we obtain the periodic solution

$$S_3 = \frac{(\nu_3 - \nu_1)\,\nu_4 + (\nu_4 - \nu_3)\,\nu_1\ \mathrm{sn}^2\left(\sqrt{\lambda_3 - \lambda_1}\,\xi, m\right)}{\nu_3 - \nu_1 + (\nu_4 - \nu_3)\ \mathrm{sn}^2\left(\sqrt{\lambda_3 - \lambda_1}\,\xi, m\right)}, \tag{C.71}$$

$$\xi = \tau - \chi/V, \quad V = 4\sqrt{-P\,(-\Delta/2)},$$

which gives in the limit $\nu_2 = \nu_3$ (i.e.,$\lambda_2 = \lambda_3$) the bright soliton solution

$$S_3\,(\xi) - \nu_2 = 2\frac{(\nu_4 - \nu_2)(\nu_2 - \nu_1)/(\nu_4 - \nu_1)}{\cosh\left(2\sqrt{\lambda_2 - \lambda_1}\,\xi\right) - (\nu_4 + \nu_1 - 2\nu_2)/(\nu_4 - \nu_1)}, \tag{C.72}$$

where now

$$\xi = \tau - \frac{\zeta}{V}, \quad V = 4\left(\lambda_2 + \frac{\Delta}{2}\right)\sqrt{\lambda_1 + \frac{\Delta}{2}},$$

and the formulas (C.69) reduce to

$$\nu_1 = \frac{\lambda_1 + \lambda_2 + \lambda_1'}{2f_1\,(\lambda_2 + \lambda_2')}, \quad \nu_2 = \nu_3 = \frac{f_1 + \sqrt{f_1^2 - 2\lambda_2}}{2\lambda_2}. \tag{C.73}$$

In the case $\nu_1 \le S_3 \le \nu_2$ we obtain the periodic solution

$$S_3(\xi) = \frac{\nu_2(\nu_3-\nu_1) - \nu_3(\nu_2-\nu_1)\,\mathrm{sn}^2\left(\sqrt{\lambda_3-\lambda_1}\,\xi, m\right)}{\nu_3-\nu_1-(\nu_2-\nu_1)\,\mathrm{sn}^2\left(\sqrt{\lambda_3-\lambda_1}\,\xi, m\right)}, \tag{C.74}$$

and the corresponding bright soliton solution

$$S_3 - \nu_2 = -2\frac{(\nu_4-\nu_2)\,(\nu_2-\nu_1)\,/\,(\nu_4-\nu_1)}{\cosh\left(2\sqrt{\lambda_3-\lambda_1}\,\xi\right) - (\nu_4+\nu_1-2\nu_2)\,/\,(\nu_4-\nu_1)} \tag{C.75}$$

with the same values of ξ and ν_i as in Eq. (C.72).

Exercise 5.3
In the limit of low population of the upper level we have $R_3 \simeq -1$, that is $|R_\pm| \ll 1$ and, hence,

$$R_3 = -\sqrt{1-R_+R_-} \simeq -1 + \tfrac{1}{2}R_+R_-.$$

If $|\mathcal{E}_2| \ll |\mathcal{E}_1|$, then $S_+ = 2\mathcal{E}_1\mathcal{E}_2$ and $\mathrm{S}_- = 2\mathcal{E}_1^*\mathcal{E}_2^*$ are small compared to $S_3 = |\mathcal{E}_1|^2 + |\mathcal{E}_2|^2$ and, consequently,

$$S_3 = \sqrt{1+S_+S_-} \simeq 1 + \tfrac{1}{2}S_+S_-.$$

On substituting these approximations into the TPP equations (Exercise 4.5), we obtain the system of two equations

$$\begin{aligned} -i\partial R_+/\partial\tau &= \Delta R_+ - S_+ + \tfrac{1}{2}\Delta S_+S_-R_+ + \tfrac{1}{2}R_+R_-S_+, \\ i\partial S_+/\partial\zeta &= \Delta S_+ + R_+ - \tfrac{1}{2}\Delta R_+R_-S_+ + \tfrac{1}{2}S_+S_-R_+. \end{aligned} \tag{C.76}$$

The second pair of the TPP equations is satisfied automatically as a consequence of Eq. (C.76). The first terms on the right hand sides of equations (C.76) can be eliminated by means of the substitutions

$$S_+ \to S_+\exp\left[i\Delta\left(\zeta-\tau\right)\right], \quad R_+ \to R_+\exp\left[-i\Delta\left(\zeta-\tau\right)\right].$$

In addition, for the sake of comparison with standard notations, it is convenient to make the replacements

$$S_\pm \to -S_\pm, \quad \zeta \to -\zeta,$$

so that

$$\begin{aligned} i\partial R_+/\partial\tau + S_+ + \tfrac{1}{2}\Delta S_+S_-R_+ + \tfrac{1}{2}R_+R_-S_+ &= 0, \\ -i\partial S_+/\partial\zeta + R_+ + \tfrac{1}{2}\Delta R_+R_-S_+ + \tfrac{1}{2}S_+S_-R_+ &= 0. \end{aligned}$$

The last two terms in either of these two equations have the same order of magnitude with respect to small parameters $|R_\pm|, |S_\pm| \ll 1$. But in the limit of strong dynamical Stark shift,

$$\Delta \gg 1,$$

the last terms can be neglected and we come to the standard form of the so-called Thirring model equations

$$\begin{aligned} i\partial R_+/\partial\tau + S_+ + \tfrac{1}{2}\Delta S_+ S_- R_+ &= 0, \\ -i\partial S_+/\partial\zeta + R_+ + \tfrac{1}{2}\Delta R_+ R_- S_+ &= 0. \end{aligned} \tag{C.77}$$

The TPP equations can be presented as a compatibility condition of two linear systems,

$$\partial\psi/\partial\tau = \mathbb{U}\psi, \quad \partial\psi/\partial\zeta = \mathbb{V}\psi,$$

where

$$\mathbb{U} = \begin{pmatrix} F & G \\ H & -F \end{pmatrix}, \quad \mathbb{V} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}.$$

and

$$\begin{aligned} &F = -i\tilde{\lambda}S_3, \quad G = \left(\tilde{\lambda} + \sigma\right)S_+, \quad H = \left(\tilde{\lambda} - \sigma\right)S_-, \\ &A = \frac{i}{2}\left(\Delta + \frac{1}{2\tilde{\lambda} + \Delta}\right)R_3, \quad B = -\frac{\tilde{\lambda} + \sigma}{2\tilde{\lambda} + \Delta}R_+, \quad C = -\frac{\tilde{\lambda} - \sigma}{2\tilde{\lambda} + \Delta}R_-, \end{aligned} \tag{C.78}$$

$\tilde{\lambda}$ being here the spectral parameter. With the use of the above approximations and replacements, we obtain from (C.78) the matrix elements of the new $\mathbb{U}$–$\mathbb{V}$ pair,

$$\begin{aligned} &F = i\lambda^2 + \tfrac{1}{4}i\Delta S_+ S_-, \quad G = \lambda^2 S_+, \quad H = \Delta S_-, \\ &A = -\frac{i}{4\lambda^2} - \frac{i\Delta}{4}R_+ R_-, \quad B = \frac{1}{2}R_+, \quad C = \frac{\Delta}{2\lambda^2}R_-, \end{aligned} \tag{C.79}$$

where the new spectral parameter λ is introduced according to the definition

$$\tilde{\lambda} = -\tfrac{1}{2}\Delta - \lambda^2.$$

It is clear that the functions f, g, h can contain only even degrees of λ. The periodic solution corresponds to the eighth degree polynomial $P(\lambda)$

$$P(\lambda) = \prod_{i=1}^{4} \left(\lambda^2 - \lambda_i^2\right) = \lambda^8 - s_1\lambda^6 + s_2\lambda^4 - s_3\lambda^2 + s_4, \tag{C.80}$$

where $\pm\lambda_i$ are the zeros of the polynomial. Then the systems (4.140,4.141) give the solution

$$\begin{aligned} &f = \lambda^4 - f_1\lambda^2 + f_2, \quad g = -\lambda^2 S_+ \left(\lambda^2 - \mu\right), \\ &h = -\Delta S_- \left(\lambda^2 - \mu^*\right) \end{aligned} \tag{C.81}$$

provided f_1, f_2, μ, μ^* satisfy the equations

$$\frac{\partial S_+}{\partial \tau} = 2i\left(f_1 - \mu + \tfrac{1}{4}\Delta S_+ S_-\right) S_+, \quad \frac{\partial S_+}{\partial \zeta} = -iR_+ - \tfrac{i}{2}\Delta R_+ R_- S_+, \tag{C.82}$$

$$\frac{\partial f_1}{\partial \tau} = i\Delta S_+ S_- \left(\mu - \mu^*\right), \quad \frac{\partial f_1}{\partial \zeta} = -\tfrac{i}{2}\Delta\left(S_+ R_- - S_- R_+\right), \tag{C.83}$$

$$f_2 = \text{const}, \qquad \mu = 2f_2 R_+/S_+, \tag{C.84}$$

$$\begin{aligned} \frac{\partial\left(S_+\mu\right)}{\partial \tau} &= 2iS_+ f_2 + \tfrac{i}{2}\Delta S_+^2 S_- \mu, \\ \frac{\partial\left(S_+\mu\right)}{\partial \zeta} &= -iR_+ f_1 + \tfrac{i}{2} S_+ - \tfrac{i}{2}\Delta R_+ R_- S_+ \mu. \end{aligned} \tag{C.85}$$

On substituting Eqs. (C.82) and (C.83) into Eq. (C.85), we find the evolution equations for the parameter μ :

$$\frac{\partial \mu}{\partial \tau} = 2if\left(\mu^{1/2}\right) = 2i\sqrt{P\left(\mu^{1/2}\right)}, \quad \frac{\partial \mu}{\partial \zeta} = \frac{1}{4f_2}\frac{\partial \mu}{\partial \tau}. \tag{C.86}$$

Thus, μ depends only on the phase

$$\xi = \frac{1}{2}\left(\tau + \frac{\zeta}{4f_2}\right), \quad \frac{d\mu}{d\xi} = 4i\sqrt{P\left(\mu^{1/2}\right)}. \tag{C.87}$$

and moves along some curve in the complex plane with variation of ξ. This curve is determined implicitly by the constraint $f^2 - gh = P(\lambda)$ which after substitution of (C.80) and (C.81) yields the identity

$$\begin{aligned}\left(\lambda^4 - f_1\lambda^2 + f_2\right)^2 - &\Delta S_+ S_- \lambda^2 \left(\lambda^2 - \mu\right)\left(\lambda^2 - \mu^*\right) \\ &= \lambda^8 - s_1\lambda^6 + s_2\lambda^4 - s_3\lambda^2 + s_4.\end{aligned} \tag{C.88}$$

We see that the parameter

$$\nu = \Delta S_+ S_- = \Delta \left|S_+\right|^2 \tag{C.89}$$

can be chosen as a coordinate along the locus of μ. Comparing the coefficients of λ^k on both sides of (C.88), we obtain the system

$$\begin{aligned} &2f_1 + \nu = s_1, \qquad f_1^2 + 2f_2 + \nu\left(\mu + \mu^*\right) = s_2, \\ &2f_1 f_2 + \nu\mu\mu^* = s_3, \qquad f_2^2 = s_4, \end{aligned} \tag{C.90}$$

which coincides with the corresponding system for the derivative nonlinear Schrödinger equation (Sec. 5.8), so we can use the solution found there,

$$f_2 = \pm\sqrt{s_4}, \tag{C.91}$$

$$\mu\left(\nu\right) = \frac{1}{8\nu}\left[4s_2 \pm 8\sqrt{s_4} - \left(\nu - s_1\right)^2 + i\sqrt{-\mathcal{R}\left(\nu\right)}\right], \tag{C.92}$$

where the 'resolvent' $\mathcal{R}\left(\nu\right)$ is a fourth-degree polynomial in ν defined by Eq. (5.197). Equations (C.83) give

$$\frac{d\nu}{d\xi} = 4i\nu\left(\mu^* - \mu\right),$$

and with the use of Eq. (C.86) we obtain the equation

$$\frac{d\nu}{d\xi} = \sqrt{-\mathcal{R}\left(\nu\right)}. \tag{C.93}$$

Its periodic and soliton solution can be found in a usual way. For example, if

$$\lambda_1 = \lambda_4 = \alpha + i\gamma, \qquad \lambda_2 = \lambda_3 = \alpha - i\gamma, \tag{C.94}$$

then Eqs. (5.198) give

$$\nu_1 = 16\alpha^2, \qquad \nu_2 = \nu_3 = 0, \qquad \nu_4 = -16\gamma^2 \tag{C.95}$$

and it is convenient to define the parameter η by

$$\cos^2\frac{\eta}{2}=\frac{\gamma^2}{\alpha^2+\gamma^2} \tag{C.96}$$

and to introduce the parametrization

$$\alpha=D\sin\frac{\eta}{2},\qquad \gamma=D\cos\frac{\eta}{2}. \tag{C.97}$$

The corresponding solution of Eq. (C.93) is given by

$$|S_+|^2=\frac{8D^2\sin^2\eta}{\Delta}\frac{1}{\cosh 4\xi+\cos\eta}. \tag{C.98}$$

This soliton solution can be obtained as an appropriate limit of the TPP soliton solution (Exercise 5.1). Other choice of λ_i leads to bright and dark soliton solutions similar to those found in Sec. 5.8.

Exercise 5.4

We suppose that the coefficients A, B, C. have the form

$$B(\lambda)=\frac{b(\lambda)}{\lambda-\Delta},\quad C(\lambda)=\frac{c(\lambda)}{\lambda-\Delta},\quad A(\lambda)|_{pole}=\frac{a(\Delta)}{\lambda-\Delta}. \tag{C.99}$$

As we know, the solution of systems (4.140) and (4.141) is given by (4.144) where both $\mu(x,t)$ and $\mu^*(x,t)$ depend only on $\xi=x-Vt$, and the phase velocity V is equal to (4.147), that is,

$$V=\frac{b(\mu)}{G(\mu)(\mu-\Delta)}=\frac{c(\mu^*)}{H(\mu^*)(\mu^*-\Delta)}. \tag{C.100}$$

The condition for vanishing of the singular terms in Eqs. (4.140) at $\lambda=\Delta$ yields three equations two of which can be written as

$$\frac{ib(\Delta)}{g_0(\Delta)(\Delta-\mu)}=-\frac{a(\Delta)}{f(\Delta)},\qquad \frac{ic(\Delta)}{h_0(\Delta)(\Delta-\mu^*)}=-\frac{a(\Delta)}{f(\Delta)}, \tag{C.101}$$

and the third one is their corollary. We obtain from Eqs. (C.100) and (C.101) that

$$V^2=-\frac{g_0(\Delta)h_0(\Delta)b(\mu)c(\mu^*)}{G(\mu)H(\mu^*)b(\Delta)c(\Delta)}\cdot\frac{a^2(\Delta)}{f^2(\Delta)}. \tag{C.102}$$

Now let us substitute Eq. (4.144) into Eq. (4.142) at $\lambda=\Delta$:

$$f^2(\Delta)-g_0(\Delta)h_0(\Delta)(\Delta-\mu)(\Delta-\mu^*)=P(\Delta).$$

This equation and Eqs. (C.101) give

$$\frac{a^2(\Delta)}{f^2(\Delta)} = \frac{a^2(\Delta) + b(\Delta)c(\Delta)}{P(\Delta)},$$

so that the desired expression for the phase velocity has the form

$$V = \frac{Q}{\sqrt{P(\Delta)}}, \tag{C.103}$$

where

$$Q = \sqrt{-\frac{g_0(\Delta)h_0(\Delta)b(\mu)c(\mu^*)[a^2(\Delta) + b(\Delta)c(\Delta)]}{G(\mu)H(\mu^*)b(\Delta)c(\Delta)}}. \tag{C.104}$$

This expression for Q seems rather complicated but in practice one can easily calculate it for any particular case. Taking into account that the zeros of the polynomial $P(\lambda)$ are the Riemann invariants, we obtain a simple method of derivation of the Whitham equations. For example, in the SRS case we have $Q = 1/4$ and

$$V_{SRS} = \frac{1}{4\sqrt{P(-\Delta/2)}},$$

which coincides with Eq. (5.173) after taking into account that t should be replaced by ξ and x by τ. In the limit $\Delta \to \infty$ we have

$$\frac{1}{4\sqrt{P(-\Delta/2)}} \simeq \frac{1}{\Delta^2} - \frac{1}{\Delta^3}\sum \lambda_i,$$

that is, after appropriate redefinition of space and time variables

$$x \to x + \Delta t, \quad t \to \Delta^3 t$$

we reproduce the expression for the phase velocity (5.15) in the NLS equation case. Since, as it was mentioned above, λ_i are the Riemann invariants, we immediately obtain the corresponding Whitham equations.

Exercise 6.1
It is convenient to transform the solution (6.42) to the form

$$u(x,t) = \begin{cases} -1/2, & x < -3t, \\ x/6t, & -3t \le x \le 3t, \\ 1/2, & x > 3t, \end{cases} \tag{C.105}$$

by means of the Galileo transformation (6.37) and the scaling transformation (6.38). The trailing and leading edges can be considered separately, because at large values of time they practically do not influence on each other (indeed, the length of the region of dispersion oscillations can be estimated from $u/t \sim u/x^3$ and it grows with time as $x \sim t^{1/3}$ which is much less then the width of the non-uniform region proportional to t).

In the case of the trailing edge we have to find the solution of the KdV equation satisfying the boundary conditions

$$u \to \begin{cases} -1/2, & x \to -\infty, \\ x/6t, & x \to +\infty. \end{cases} \tag{C.106}$$

Let us look for the solution in a self-similar form

$$u = -\frac{1}{2} + \frac{1}{(\alpha t)^{2/3}}\chi(z), \quad z = \frac{x+3t}{(\alpha t)^{1/3}}.$$

Substitution of this expression into the KdV equation (6.1) shows that such a solution exists, if we choose $\alpha = 3$, that is,

$$u = -\frac{1}{2} + \frac{1}{(3t)^{2/3}}\chi(z), \quad z = \frac{x+3t}{(3t)^{1/3}}, \tag{C.107}$$

and $\chi(z)$ satisfies the ordinary differential equation

$$\chi''' + (6\chi - z)\chi' - 2\chi = 0 \tag{C.108}$$

and the boundary conditions

$$\chi \to \begin{cases} 0, & z \to -\infty, \\ z/2, & z \to +\infty. \end{cases} \tag{C.109}$$

This equation can be solved numerically (see Zakharov, Manakov, Novikov and Pitaevskii, 1980). From Eq. (C.107) it is clear that the amplitude of oscillations decreases with time as $\propto t^{-2/3}$.

In a similar way, in the case of the leading edge we have to find the solution satisfying the boundary conditions

$$u \to \begin{cases} x/6t, & x \to -\infty, \\ 1/2, & x \to +\infty. \end{cases} \tag{C.110}$$

Now the solution has the form

$$u = \frac{1}{2} + \frac{1}{(3t)^{2/3}}\chi(z), \quad z = \frac{x-3t}{(3t)^{1/3}}, \tag{C.111}$$

where $\chi(z)$ satisfies the same equation (C.108) and the boundary conditions

$$\chi \to \begin{cases} z/2, & z \to -\infty, \\ 0, & z \to +\infty. \end{cases} \tag{C.112}$$

In both cases dispersion effects lead to a small correction to the solution (6.42).

Exercise 6.2
For small $m \ll 1$ the Whitham equations (6.10,6.12) reduce to

$$\begin{aligned} &\partial r_1/\partial t + (9r_1 + 3r_2 - 6r_3)\partial r_1/\partial x = 0, \\ &\partial r_2/\partial t + (3r_1 + 9r_2 - 6r_3)\partial r_2/\partial x = 0, \\ &\partial r_3/\partial t + 6r_3 \partial r_3/\partial x = 0. \end{aligned} \tag{C.113}$$

Equation (6.15) in this limit gives $\overline{u} = r_3$, hence, the last equation disappears due to the condition that $\overline{u} = 0$, and the other two equations can be written for the variables

$$\lambda_+ = 12r_2, \qquad \lambda_- = 12r_1, \tag{C.114}$$

in the form (1.274,1.275)

$$\begin{aligned} \partial\lambda_+/\partial t + \tfrac{1}{4}(3\lambda_+ + \lambda_-)\partial\lambda_+/\partial x = 0, \\ \partial\lambda_-/\partial t + \tfrac{1}{4}(\lambda_+ + 3\lambda_-)\partial\lambda_-/\partial x = 0. \end{aligned} \tag{C.115}$$

Hence, they can be written as hydrodynamical equations

$$\rho_t + \rho v_x + v\rho_x = 0, \quad v_t + v v_x + \rho_x = 0, \tag{C.116}$$

for the variables

$$\rho = 9(r_2 - r_1)^2 = 9a^2, \quad v = 6(r_1 + r_2) = 3V, \tag{C.117}$$

where a is the amplitude of the nonlinear wave and V is its phase velocity. Note that the condition

$$m = \frac{r_2 - r_1}{-r_1} = \frac{4a}{2a - V} \ll 1 \tag{C.118}$$

means that V has to have large negative values. If the values of v for the wave packet are concentrated around some value v_0, ($|v_0| \gg a$), then it is convenient to make in Eqs. (C.116) the Galileo transformation

$$x' = x - v_0 t, \quad t' = t, \quad v' = v - v_0,$$

to the frame of reference moving with velocity v_0.

Exercise 6.3
We look for the solution in the form

$$\rho(x,t) = \frac{1}{f(t)}\left(1 - \frac{x^2}{f^2(t)}\right), \quad v(x,t) = \phi(t)x, \tag{C.119}$$

where functions $f(t)$ and $\phi(t)$ are to be determined. They satisfy the initial conditions

$$f(0) = 1, \qquad \phi(0) = 0. \tag{C.120}$$

Substitution of Eq. (C.119) into equations

$$\rho_t + \rho v_x + v\rho_x = 0, \quad v_t + vv_x + \rho_x = 0, \tag{C.121}$$

gives

$$\phi(t) = f'(t)/f(t) \tag{C.122}$$

and

$$f''(t) = 2/f^2. \tag{C.123}$$

The last equation can be readily integrated to give

$$f'(t) = 2\sqrt{(f-1)/f},$$

where the initial condition $f'(0) = \phi(0)f(0) = 0$ was taken into account. One more integration yields

$$2t = \sqrt{f(f-1)} + \ln(\sqrt{f-1} + \sqrt{f}). \tag{C.124}$$

This relation defines the function $f(t)$. When $f(t)$ is known, we find $\phi(t)$ by Eq. (C.122).

Exercise 6.4
We can apply the quasi-classical quantization method (Landau and Lifshitz, 1989) to the quantum-mechanical Schrödinger equation

$$\psi_{xx} + u_0(x)\psi = a\psi, \tag{C.125}$$

according to which eigenvalues are determined by the asymptotic Bohr-Sommerfeld quantization rule

$$\oint \sqrt{u_0(x) - a}\, dx = 2\pi \left(n + \tfrac{1}{2}\right). \tag{C.126}$$

Number of solitons N is equal to the maximum value of n and corresponds to the minimum value of $a = 0$,

$$N = \frac{1}{\pi} \int_{-\infty}^{\infty} \sqrt{u_0(x)}\, dx. \tag{C.127}$$

Maximal amplitude is equal to the maximum value u_m of the potential $u_0(x)$, that is, amplitudes are distributed inside the interval

$$0 < a < u_m = \max(u_0(x)). \tag{C.128}$$

Differentiation of Eq. (C.126) with respect to a gives the number of solitons with amplitudes in the interval $(a,\, a + da)$,

$$\left(\frac{1}{4\pi} \oint \frac{dx}{\sqrt{u_0(x) - a}} \right) da \equiv f(a)da. \tag{C.129}$$

These formulas specify the asymptotic solution of the Whitham equations found in Exercise 3.3. Indeed, amplitudes are distributed according to the law

$$a(x,t) = x/(4t), \qquad 0 < x < 4u_m t, \tag{C.130}$$

and the number of solitons in the interval $(x,\, x + dx)$ is given by

$$kdx = f(a)da = f\left(\frac{x}{4t}\right) \frac{dx}{4t},$$

that is, the distribution function equals to

$$k(x,t) = \frac{1}{4t} f\left(\frac{x}{4t}\right), \tag{C.131}$$

where the function $f(a)$ is defined by Eq. (C.129).

Exercise 7.1
An easy calculation of the variational derivative,

$$\delta H/\delta u^* = -u_{xx} - 2|u|^2 u,$$

shows at once that the NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{C.132}$$

can be presented in a Hamiltonian form

$$iu_t = \delta H/\delta u^*, \qquad H = \int \left(|u_x|^2 - |u|^4\right) dx. \tag{C.133}$$

Substitution of Eq. (C.132) into dP/dt and dN/dt, where

$$P = \tfrac{i}{2} \int (u_x u^* - u u_x^*)\, dx \tag{C.134}$$

is the momentum and

$$N = \int |u|^2 dx \tag{C.135}$$

the intensity of the wave, gives after integration by parts

$$\begin{aligned} dP/dt &= \int \left(|u_x|^2 - \tfrac{1}{2}(u u_{xx}^* + u_{xx} u^*) - |u|^4\right)_x dx, \\ dN/dt &= i \int (u_x u^* - u u_x^*)_x\, dx, \end{aligned}$$

and these expressions vanish as integrals of the total derivatives.

Exercise 7.2
By the Galileo invariance of the NLS equation it is sufficient to consider the stability of a soliton at rest, i.e., with $P = 0$, when the solution has the form

$$u(x,t) = \exp(i\lambda t)\, a(x), \tag{C.136}$$

so that $a(x)$ satisfies the equation

$$\lambda a - a_{xx} - 2a^3 = 0 \tag{C.137}$$

(we denote here the frequency shift by λ). This equation can be obtained as the Euler-Lagrange equation for a variational problem of minimization of the energy functional (Hamiltonian)

$$H = \int_{-\infty}^{\infty} \left(a_x^2 - a^4\right) dx \tag{C.138}$$

with the additional condition of a fixed value of

$$N = \int_{-\infty}^{\infty} a^2 dx, \tag{C.139}$$

that is,

$$\delta F/\delta a = 0, \qquad F = H + \lambda N, \tag{C.140}$$

λ is a Lagrange multiplier. Suppose that the solution $a(x)$, $a \to 0$ at $|x| \to \infty$, is found. Then it corresponds to some extremum of H at fixed N, and we have to show that this extremum point is not a 'saddle point' in the functional space. This will be proved, if we show that H is bounded from below. Let us introduce notations

$$I_1 = \int_{-\infty}^{\infty} a_x^2 dx, \qquad I_2 = \int_{-\infty}^{\infty} a^4 dx. \tag{C.141}$$

Then with the use of the Hölder inequality we have

$$\begin{aligned} I_2 = \int_{-\infty}^{\infty} a^4 dx &\le \max(a^2) \int_{-\infty}^{\infty} a^2 dx = \int_{-\infty}^{x_{\max}} \frac{da^2}{dx} dx \int_{-\infty}^{\infty} a^2 dx \\ &\le 2N \int_{-\infty}^{x_{\max}} |a||a_x| dx \le 2N \int_{-\infty}^{\infty} |a||a_x| dx \\ &\le 2N \left(\int_{-\infty}^{\infty} a^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} a_x^2 dx\right)^{1/2} = 2N^{3/2} I_1^{1/2}, \end{aligned}$$

that is,

$$I_2 \le 2N^{3/2} I_1^{1/2}, \tag{C.142}$$

and, hence,

$$H = I_1 - I_2 \ge I_1 - 2N^{3/2} I_1^{1/2}. \tag{C.143}$$

This function of I_1 has the minimum value at $I_1 = N^3$, hence, we obtain

$$H \ge -N^3. \tag{C.144}$$

It remains to show that H is negative at the soliton solution (otherwise the soliton could decay to linear waves with positive energy), and this can be done without knowledge of an explicit formula for $a(x)$. Indeed, let us multiply Eq. (C.137) by a_x and integrate over x to obtain

$$\lambda \int a^2 dx - \int a a_x dx - 2\int a^4 dx = 0,$$

or after integration of the second term by parts,

$$\lambda N + I_1 - 2I_2 = 0. \tag{C.145}$$

Another relation (analogous to a virial theorem) can be obtained after multiplication of Eq. (C.137) by xa_x and integration over x,

$$\lambda \int x a a_x dx - \int x a_x a_{xx} dx - 2\int x a^3 a_x dx = 0,$$

which after integration by parts gives

$$\lambda N - I_1 + I_2 = 0. \tag{C.146}$$

From Eqs. (C.145) and (C.146) we can express I_1 and I_2 in terms of N,

$$I_1 = \tfrac{1}{3}\lambda N, \qquad I_2 = \tfrac{2}{3}\lambda N, \tag{C.147}$$

and, consequently, the minimum value of H is equal to

$$H = -\tfrac{1}{3}\lambda N. \tag{C.148}$$

Thus, the NLS soliton minimizes the energy at given N, it has negative value and is bounded from below. Hence, it is stable with respect to modulations. Note that the actual soliton solution

$$a(x) = \frac{\sqrt{\lambda}}{\cosh(\sqrt{\lambda}x)} \tag{C.149}$$

satisfies the relations (C.147) and (C.148):

$$N = 2\lambda^{1/2}, \quad I_1 = \tfrac{2}{3}\lambda^{3/2}, \quad I_2 = \tfrac{4}{3}\lambda^{3/2}, \quad H = -\tfrac{2}{3}\lambda^{3/2}.$$

This approach to the soliton stability can be applied to other equations (see the review article by Kuznetsov, Rubenchik and Zakharov, 1986).

Exercise 7.3
Let the initial distributions be

$$\rho(x,t=0)=\begin{cases}1-x^2, & |x|\le 1,\\ 0, & |x|>1.\end{cases}$$

and

$$v(x,t=0)=0.$$

We look for the solution of Eqs. (7.58)

$$\tfrac{1}{2}\rho_t+v\rho_x+\rho v_x=0,\quad \tfrac{1}{2}v_t+vv_x-\rho_x=0, \tag{C.150}$$

in the form

$$\rho(x,t)=\frac{1}{f(t)}\left(1-\frac{x^2}{f^2(t)}\right),\quad v(x,t)=\phi(t)x, \tag{C.151}$$

where functions $f(t)$ and $\phi(t)$ satisfy the initial conditions

$$f(0)=1,\qquad \phi(0)=0. \tag{C.152}$$

Calculations similar to those of Exercise 6.3 give

$$\phi=\tfrac{1}{2}f'(t)/f(t), \tag{C.153}$$

and $f(t)$ is defined by the equation

$$t=\tfrac{1}{4}\left[\sqrt{f(1-f)}+\arccos\sqrt{f}\right]. \tag{C.154}$$

The considered here initial distribution leads to a more strong singularity then that considered in Sec. 7.3; namely, this time all ‘rays’ are focused at one point $x=0$ at $t_f=\pi/8$.

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