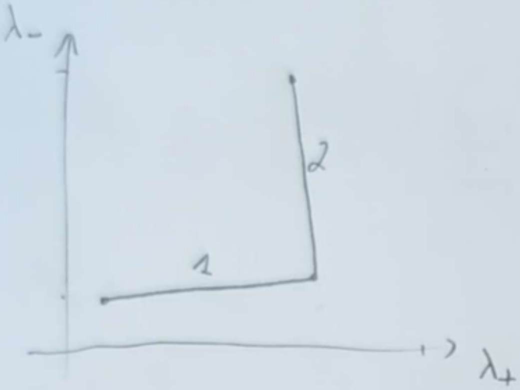
$\boxed{IC + \gamma = 3}$

Now, we want to solve for χ with the initial conditions on λ:



We can integrate χ along this path, as we know that at $t = 0$

$$x = \partial_1 \chi$$

$$x = \partial_2 \chi$$

So, along 1, $\dfrac{d\chi}{dx} = \dfrac{d\lambda_+}{dx} \times \partial_1 \chi = \dfrac{d\lambda_+}{dx} x$

$$= \frac{8c_0}{L_0(\gamma-1)} x$$

Same, along 2: $\dfrac{d\chi}{dx} = \dfrac{d\lambda_-}{dx} \partial_2 \chi = \dfrac{8c_0}{L_0(\gamma-1)} x$

Thus, setting $\chi\big|_{x=0} = 0$, $\chi(x) = \dfrac{4c_0}{L_0(\gamma-1)} x^2 = \dfrac{a_0}{2} x^2$

Calling $a_0 = \dfrac{8c_0}{L_0(\gamma-1)}$

$\lambda = \dfrac{2c_0}{\gamma-1}$

And along 1: $\lambda_- = -\lambda$

$\lambda_+ = a_0 x + \lambda$

along 2: $\lambda_- = a_0 x - \lambda$

$\lambda_+ = +\lambda$

Thus we get the initial conditions:

Along 1:

$$X(\lambda_+, -\lambda) = \frac{(\lambda_+ - \lambda)^2}{2a_0}$$

Along 2:

$$X(+\lambda, \lambda_-) = \frac{(\lambda + \lambda_-)^2}{2a_0}$$

~~Now, considering the specific case~~ $\cancel{\gamma = \frac{5}{3}}$

$$\cancel{X = \frac{F(\lambda_+) + G(\lambda_-)}{\lambda_+ - \lambda_-}}, \cancel{X}$$

So looking along 1.:

Consider the specific case $\gamma = 3$,

$$X = F(\lambda_+) + G(\lambda_-)$$

$X(\lambda, -\lambda) = 0$

$X(0) = 0$, $\quad F(\lambda) + G(-\lambda) = 0$, $F(\lambda) = C$, $G(-\lambda) = -C$

$$F(\lambda_+) = \frac{(\lambda_+ - \lambda)^2}{2a_0}$$

$$G(\lambda_-) = \frac{(\lambda_- + \lambda)^2}{2a_0}$$

$$X = \frac{1}{2a_0}\left[(\lambda_+ - \lambda)^2 + (\lambda_- + \lambda)^2\right]$$

Here $a_0 = \frac{8c_0}{L_0 \times 2} = \frac{4c_0}{L_0}$, $\lambda = c_0$

$V_+ = \lambda_+$

$V_- = \lambda_-$

Thus
$$\begin{cases} x - \lambda_+ t = \partial_1 X = \dfrac{\lambda_+ - \lambda}{a_0} \\ x - \lambda_- t = \partial_2 X = \dfrac{\lambda_- + \lambda}{a_0} \end{cases}$$

To invert, have λ_+, λ_- as a function of x, t:

$$\begin{cases} \lambda_+\left(\dfrac{1}{a_0} + t\right) = \dfrac{\lambda}{a_0} + x \\ \lambda_-\left(\dfrac{1}{a_0} + t\right) = x - \dfrac{\lambda}{a_0} \end{cases} \Rightarrow \quad \lambda_+ = \frac{a_0 x + \lambda}{1 + a_0 t}, \quad \lambda_- = \frac{a_0 x - \lambda}{1 + a_0 t}$$

(4)

So $u = \dfrac{\lambda_+ + \lambda_-}{2}$, $\rho = \dfrac{\lambda_+ - \lambda_-}{2}$

$= \dfrac{a_0 x}{1 + a_0 t}$. =

Now, there are bounds where this is true, and where it isn't.

We will always have $-\lambda \leq \lambda_+ \leq \lambda$, same for λ_-

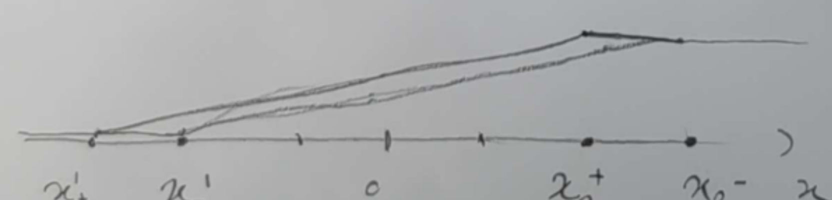
So, we calulate $x_+^1(t)$ $x_+^2(t)$ such that

$\lambda_+^t(x_+^1(t), t) = -\lambda$: $\lambda_+(x_+^2, t) = +\lambda$

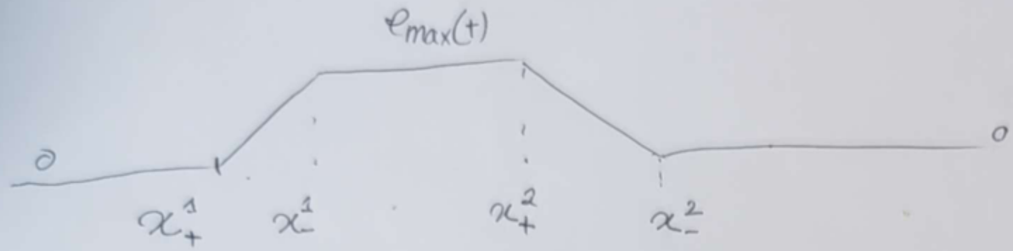
$x_+^1(t) = -\lambda\left(t + \dfrac{2}{a_0}\right)$ $x_+^2 = +c_0 t$

$= -c_0\left(t + \dfrac{L_0}{2c_0}\right) = \dfrac{L_0}{2} - c_0 t$

$x_-^1 = -c_0 t$ $x_-^2 = \dfrac{L_0}{2} + c_0 t$



So finally ρ looks like:



With $\rho_{max}(t) = \dfrac{c_0}{1 + \frac{4c_0}{L_0} t} = \dfrac{\rho_0}{1 + \frac{4\rho_0}{L_0} t}$

We can check that this solution conserves total density:

$$\rho_{max}(t) \times 2c_0 t + 2 \times \frac{L_0}{2} \times \rho_{max}(t) \times \frac{1}{2}$$

$$= \rho_{max} \times \frac{L_0}{2} \times \left(1 + \frac{4c_0 t}{L}\right) = \rho_0 \times \frac{L_0}{2}$$

Maybe error, check $c_0 = \sqrt{\gamma(\gamma-1)}\, \rho_0$