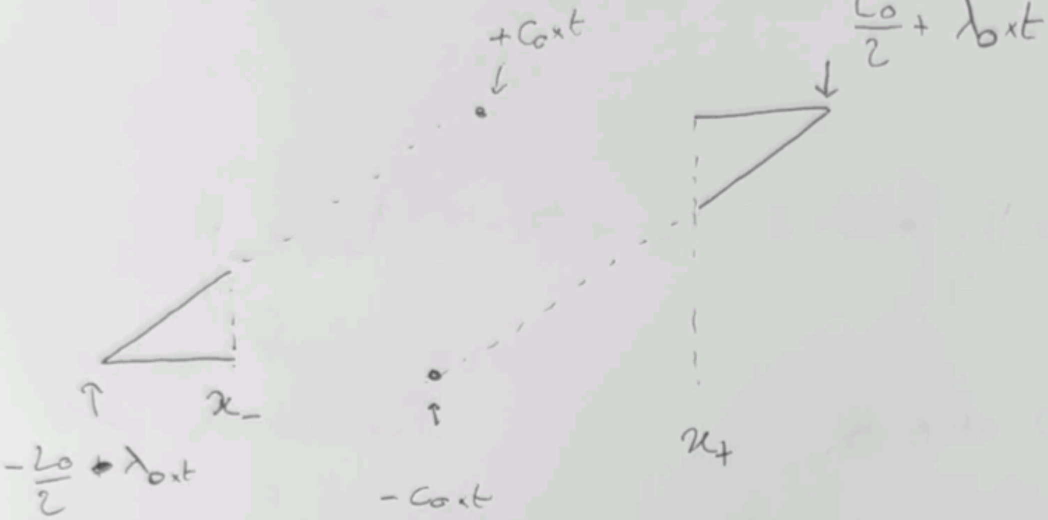
The lambdas look like:



Remember $V_1 - V_2 = \frac{1}{2}(\gamma-1)\left[\lambda_+ - \lambda_-\right]$

So $V_1 > V_2$ always, so x_+ will be travelling at $V_1(\lambda_-(x_+))$, x_- at $V_2(\lambda_+(x_-))$.

We can thus get differential equations for them.

(linear interpolation)

$$\lambda_-(x_+) = 2\lambda_0\left(\frac{x_+ + c_0 t}{(\lambda_0 + c_0)t + \frac{L_0}{2}}\right) - \lambda_0$$

$$\lambda_+(x_-) = 2\lambda_0\left(\frac{x_- + \lambda_0 t + \frac{L_0}{2}}{(\lambda_0 + c_0)t + \frac{L_0}{2}}\right) - \lambda_0 = \left(\frac{x_- - c_0 t}{(\lambda_0 + c_0)t + \frac{L_0}{2}}\right)\times 2\lambda_0$$

$+ \lambda_0$!

$$\frac{dx_+}{dt} = V_1(\lambda_o(x_+)) = \frac{1}{4}\left((1+\gamma)\lambda_o + (3-\gamma)\lambda_o(x_+)\right)$$

$$\frac{dx_-}{dt} = V_2(\lambda_+(x_-)) = \frac{1}{4}\left((1+\gamma)\lambda_0 + (3-\gamma)\lambda_+(x_-)\right)$$

And Both are 0 at $t=0$.

Reminder $\lambda_0 = \frac{2}{\gamma-1}c_0$

$L_0 = \frac{\gamma+1}{\gamma-1}L$

Focusing on x^+:

$$\frac{dx_+}{dt} = \frac{1}{4}\left[(1+\gamma)\lambda_0 + (3-\gamma)\left[2\lambda_0\left(\frac{x_+ + c_0t}{(\lambda_0+c_0)t + \frac{L_0}{2}}\right) - \lambda_0\right]\right]$$

$$= c_0 + \frac{3-\gamma}{4}\left[2\lambda_0\left(\frac{x_+ + c_0t}{(\lambda_0+c_0)t + \frac{L_0}{2}}\right)\right]$$

$\lambda_0 + c_0 = \frac{\gamma+1}{\gamma-1}c_0$

$$= c_0 + \underbrace{\frac{3-\gamma}{4}\times\frac{\gamma-1}{\gamma+1}\times 2\lambda_0}\times\left[\frac{x_+ + c_0t}{c_0t + \frac{L}{2}}\right]$$

$$= c_0 + \frac{3-\gamma}{\gamma+1}\times c_0\times\left[\frac{x_+ + c_0t}{c_0t + \frac{L}{2}}\right]$$

So $\dfrac{dx_+}{dt} - \dfrac{3-\gamma}{\gamma+1} c_0 \times \dfrac{x_+}{c_0 t + \frac{L}{2}} = c_0 + \dfrac{3-\gamma}{\gamma+1} \dfrac{c_0^2 t}{c_0 t + \frac{L}{2}}$

Solution homogene:

$$\exp\left(\int \frac{3-\gamma}{\gamma+1} c_0 \times \frac{1}{c_0 t + \frac{L}{2}} dt \right) = \exp\left(\frac{3-\gamma}{\gamma+1} \times \frac{\cancel{c_0}}{\cancel{c_0}} \times \ln\left(c_0 t + \frac{L}{2}\right) \right)$$

$$= \left(c_0 t + \frac{L}{2}\right)^{\frac{3-\gamma}{\gamma+1}}$$

$u(t) = \left(c_0 t + \frac{L}{2}\right)^{\frac{3-\gamma}{\gamma+1}}$

Solution particuliere : $\lambda(t) \times u(t)$.

$$\lambda'(t) \times u(t) = c_0 + \frac{3-\gamma}{\gamma+1} \frac{c_0^2 t}{c_0 t + \frac{L}{2}}$$

calling $\alpha = \dfrac{3-\gamma}{\gamma+1}$

$$\lambda'(t) = \frac{(1+\alpha) c_0^2 t + c_0 \frac{L}{2}}{\left(c_0 t + \frac{L}{2}\right)^{\alpha+1}}$$

Chgt de variable : $V = c_0 t + \frac{L}{2}$

$$\lambda' = \int \frac{1}{c_0} \frac{dV}{V^{\alpha+1}} \left((1+\alpha) c_0 \left(V - \frac{L}{2}\right) + c_0 \frac{L}{2} \right)$$

$$= \int \frac{dV}{V^{\alpha+1}} \left((1+\alpha) V - \alpha \frac{L}{2} \right).$$

$$= \int dV \; (1+\alpha) V^{-\alpha} - \alpha \frac{L}{2} V^{-\alpha-1}$$

$$= \frac{1+\alpha}{1-\alpha} V^{1-\alpha} + \frac{L}{2} V^{-\alpha} + C$$

So Because $x_+ = V^{\alpha} \times \lambda$:

$$x_+ = \frac{1+\alpha}{1-\alpha} V + \frac{L}{2} + C V^{\alpha}$$

And we want $x_+(t=0) = 0$, then, Because $V(t=0) = \frac{L}{2}$

$$C = \frac{-1}{\left(\frac{L}{2}\right)^{\alpha}} \times \left(\cancel{\frac{L}{2}} \rightarrow \frac{2}{1-\alpha} \right) \times \frac{L}{2}$$

So, to conclude:

$$x_+(t) = \frac{1+\alpha}{1-\alpha}\left(c_0 t + \frac{L}{2}\right) + \frac{L}{2} + C_\alpha \left(c_0 t + \frac{L}{2}\right)^\alpha$$

$$\alpha = \frac{3-\gamma}{1+\gamma}, \quad C_\alpha = -\left(\frac{L}{2}\right)^{1-\alpha} \cdot \left(\frac{2}{\alpha - 1}\right)$$

A few remarks: $(-1 \leqslant \alpha < 1)$

$$\frac{dx_+}{dt} = \frac{1+\alpha}{1-\alpha} c_0 + C_\alpha \frac{c_0}{\left(c_0 t + \frac{L}{2}\right)^{1-\alpha}}$$

thus, because $C_\alpha < 0$ $\quad \dfrac{dx_+}{dt} < \dfrac{1+\alpha}{1-\alpha} c_0 = \lambda_0$

speed of sides.

So x_+ is slower than sides, stays physical!

You can also check that $\dfrac{dx_+}{dt} > 0$, thus no backtracking.