ADVANCED NONLINEAR PHYSICS

Duration: 3 hours. Problem A corresponds to the first part of the course, problem B to the second part. They are independent of each other. Please use different sheets for writing up the solution of each problem.

Dictionaries and handwritten notes on the courses are allowed. Printed and scanned notes on the first part of the teaching are also allowed. Books as well as computers, telephones and other electronic devices are forbidden.

A One dimensional inviscid gas dynamics

A gas in a cylindrical pipe[1] is supposed to experience only one-dimensional motion along the axis x of the pipe. Its state is then described by two real fields: the velocity field $u(x,t)$ and the density[2] $n(x,t)$. When viscosity effects can be discarded the gas dynamics is governed by continuity and Euler equations:

$$n_t + (nu)_x = 0 \, , \qquad u_t + u\, u_x + \frac{1}{m\, n}\, P_x = 0 \, . \tag{A1}$$

where m is the mass of the particles constituting the gas and $P(x,t)$ is the pressure within the gas. For a barotropic equation of state the pressure is a function of n only: $P = P(n)$. In this problem we consider the special case of a polytropic equation of state for which this function is a power law. It is then customary to write $P = \frac{m\,K}{\gamma}\, n^\gamma$ where K (> 0) is a constant of proportionality and the constant γ (> 1) is known as the adiabatic index. In the following it will appear more convenient to introduce the quantity $\alpha = \gamma - 1$ and to work with α (> 0) instead of γ:

$$P = \frac{m\,K}{\alpha+1}\, n^{\alpha+1} \, . \tag{A2}$$

A.1 Riemann invariants

Let's define the real fields r_1 and r_2 by

$$r_1(x,t) = \frac{u(x,t)}{2} + \frac{\sqrt{K}}{\alpha}\left[n(x,t)\right]^{\alpha/2} \, , \quad r_2(x,t) = \frac{u(x,t)}{2} - \frac{\sqrt{K}}{\alpha}\left[n(x,t)\right]^{\alpha/2} \, . \tag{A3}$$

For historical reasons r_1 and r_2 are called "Riemann invariants" although they obviously vary in time and position.

Show that the Riemann invariant $r_i(x,t)$ ($i = 1$ or 2) obeys an equation of the form

$$\partial_t r_i + V_i\, \partial_x r_i = 0 \, , \tag{A4}$$

where you will express the quantity V_i first in terms on $u(x,t)$ and $n(x,t)$ and then in terms on $r_1(x,t)$ and $r_2(x,t)$. In particular, show that when $\alpha = 2$, V_i only depends on r_i and takes the form: $V_i = 2\, r_i$.
<u>Indication</u>: If you cannot answer this question you may admit the result (A4) and the fact that $V_i = 2\, r_i$ ($i = 1$ or 2) when $\alpha = 2$, and continue.

[1] pipe= tuyau in french and tubo in italian.
[2] Here "density" means "number of particles per unit volume".

A.2 The dam break problem

One considers a configuration where a plane wall at $x = 0$ separates an empty half space ($x > 0$) from the half space $x < 0$ which is occupied by a gas at rest with density n_0. Hence

$$n(x,0) = \begin{cases} n_0 & \text{if } x \le 0\,, \\ 0 & \text{if } x > 0\,, \end{cases} \qquad \text{and} \qquad u(x,0) = 0\,. \tag{A5}$$

At $t = 0$ one the wall is taken away. One wishes to study the subsequent evolution of the gas and to determine $n(x,t)$ and $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$.

For simplicity we consider here only the case $\alpha = 2$.

1. Draw the characteristic lines for r_1 in the plane (x,t) and argue that r_1 undergoes a shock in the region $x > 0$: $r_1(x,t)$ is piece-wise constant in regions $]-\infty, x^{(+)}(t)]$ and $[x^{(+)}(t), +\infty[$ with $x^{(+)}(t) > 0$.[3]

2. Draw the characteristic lines for r_2 in the plane (x,t) and argue that r_2 undergoes a rarefaction wave (do not try to proceed further in the analysis).

3. Since there is no characteristic length in the initial condition (A5) it is legitimate to assume that all quantities (n, u, r_1 and r_2) depend on one single scaling variable: $\xi = x/t$.

   (a) Express ∂_x and ∂_t in terms of ξ, t and $\frac{\mathrm{d}}{\mathrm{d}\xi}$.

   (b) Show that r_1 and r_2 both satisfy the same equation:

   $$\Big(-\xi + 2\,r_i\Big)\,\frac{\mathrm{d}r_i}{\mathrm{d}\xi} = 0\,, \quad \text{where} \quad i = 1 \text{ or } 2\,, \tag{A6}$$

   meaning that either r_i is constant, either $r_i = \xi/2$.

4. On the basis of the previous questions 1, 2 and 3, it is legitimate to assume that r_2 is piece-wise constant in regions $]-\infty, x^{(-)}(t)]$ and $[x^{(+)}(t), +\infty[$ and depends on ξ in the region $[x^{(-)}(t), x^{(+)}(t)]$.

   • The quantities $x^{(+)}(t)$ and $x^{(-)}(t)$ are still unknown, but from the analysis of the previous questions it is clear that $x^{(+)}(t) > 0$ and $x^{(-)}(t) < 0$.

   • Also, both r_1 and r_2 are continuous at $x = x^{(-)}(t)$ and discontinuous at $x = x^{(+)}(t)$.

   (a) Represent $r_1(x,t)$ and $r_2(x,t)$ at fixed $t > 0$. It is convenient to introduce the quantity $c_0 = \sqrt{K}\,n_0$. Determine $x^{(-)}(t)$ from the continuity of r_2 at $x = x^{(-)}(t)$.

   (b) Represent $n(x,t)$ and $u(x,t)$ at fixed $t > 0$.

   (c) Deduce the value of $x^{(+)}(t)$ by imposing $n(x^{(+)}(t),t) = 0$.

[3] Beware, $x^{(+)}(t)$ is not straightforwardly deduced from the usual shock condition seen in the course. The reason is that we have a shock in two quantities, r_1 and r_2. The value of $x^{(+)}(t)$ will be determined in question 4(c) below.

B Gierer-Meinhardt model for biological pattern formation

Notations : the quantities written with **bold** fonts are two-dimensional vectors. ∇^2 denotes the Laplacian operator.

Two chemical species A and I freely diffuse in a solvent with diffusion constant D_A and D_I respectively, and undergo chemical reactions. We consider a two dimensional system and denote by $A(\mathbf{r},t)$ and $I(\mathbf{r},t)$ the local concentrations of the species A and I with the position vector $\mathbf{r}=(x,y)$. The production rates of A and I due to chemical reactions are,

$$f_A = \sigma_A - k_A A + \frac{\alpha A^2}{(1+K_A A^2)I} \quad \text{and} \quad f_I = -k_I I + \frac{\beta A^2}{1+K_I A^2} \tag{B1}$$

where σ_A, σ_I, k_A, k_I, α, and β are reaction rates and, K_A and K_I are parameters.

B.1 Dimensionless reaction-diffusion equations

1. Explain why the chemical species A is called an "activator" of I, and I is called an "inhibitor" of A.

2. Write the equations satisfied by $A(\mathbf{r},t)$ and $I(\mathbf{r},t)$.

3. Show that these equations can be written in a dimensionless form as,

$$\frac{\partial A}{\partial t} = a - A + \frac{A^2}{(1+cA^2)I} + \nabla^2 A \quad \text{and} \quad \frac{\partial I}{\partial t} = -bI + \frac{A^2}{1+eA^2} + d\,\nabla^2 I \tag{B2}$$

   (*Warning: this is not exactly the same dimensionless form as in the exercise sheet n*$^{\text{o}}$ 2). What are the characteristic (concentrations, time and length) scales used to redefine A, I, t and x,y? What are the definitions of the dimensionless parameters a, b, c, d and e?

In all the following we set $a=c=e=0$.

B.2 Turing instability

The goal of this part is to determine under which conditions a Turing instability, *i.e.* an instability leading to the emergence of a periodic stationary pattern, could occur in this two species mixture.

4. Find the stationary and homogeneous solution (A_s, I_s) of the reaction-diffusion equations (C2).

5. Write the linear equations satisfied by a small perturbation around the base state (A_s, I_s). Write the matrix $M(q)$ associated to the system of linear equations satisfied by the Fourier component (of wave vector $\mathbf{q}$ with $q=|\mathbf{q}|$) of the small perturbation.

6. Determine the necessary conditions regarding the values of b and d for a Turing instability to occur. What is the critical value of b for a given d? What is the critical wave vector q_c? (Provide detailed explanations based on the features of the determinant and trace of the matrix $M(q)$.).

B.3 Amplitude equations for hexagonal pattern near the instability threshold

We assume that near the threshold of the Turing instability, the concentration $A(\mathbf{r},t)$ and $I(\mathbf{r},t)$ can be written,

$$\begin{pmatrix} A(\mathbf{r},t) \\ I(\mathbf{r},t) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} u(\mathbf{r},t) \ , \tag{B3}$$

with

$$u(\mathbf{r},t) = A_1(t)\exp(i\,\mathbf{q}_1\cdot\mathbf{r}) + A_2(t)\exp(i\,\mathbf{q}_2\cdot\mathbf{r}) + A_3(t)\exp(i\,\mathbf{q}_3\cdot\mathbf{r}) + c.c.\ , \tag{B4}$$

where "$+c.c.$" means the addition of the complex conjugates of the three preceding terms. The functions $A_1(t)$, $A_2(t)$ and $A_3(t)$ are complex amplitudes. The 2D wave vectors have the same modulus $|\mathbf{q}_1| = |\mathbf{q}_2| = |\mathbf{q}_3|$. The angles between $\mathbf{q}_1$ and $\mathbf{q}_2$, and between $\mathbf{q}_2$ and $\mathbf{q}_3$, are $2\pi/3$.

7. What is the modulus of the wave vectors $|\mathbf{q}_j|$? What are V_1 and V_2? What is the vector (e_1, e_2)?

The equations satisfied by the amplitudes are,

$$\frac{dA_1}{dt} = \epsilon A_1 + \beta\,\overline{A_2}\,\overline{A_3} - \gamma_0|A_1|^2A_1 - \gamma_1\left(|A_2|^2 + |A_3|^2\right)A_1 \tag{B5}$$

$$\frac{dA_2}{dt} = \epsilon A_2 + \beta\,\overline{A_1}\,\overline{A_3} - \gamma_0|A_2|^2A_2 - \gamma_1\left(|A_1|^2 + |A_3|^2\right)A_2 \tag{B6}$$

$$\frac{dA_3}{dt} = \epsilon A_3 + \beta\,\overline{A_1}\,\overline{A_2} - \gamma_0|A_3|^2A_3 - \gamma_1\left(|A_1|^2 + |A_2|^2\right)A_3 \tag{B7}$$

where $\overline{A}$ denotes the complex conjugate of A.

8. Show that the functions $\{B_1, B_2, B_3\}$, defined by $B_j(t) = A_j(t)\exp(i\,\mathbf{q}_j\cdot\mathbf{r}_0)$ for $j \in \{1,2,3\}$, obey the same set of equations as the functions $\{A_1, A_2, A_3\}$, for any vector $\mathbf{r}_0$.
   Explain why the amplitude equations must have this invariance.

9. Show that the functions $\{B_1, B_2, B_3\}$, defined by $\{B_1 = A_2, B_2 = A_3, B_3 = A_1\}$, obey the same set of equations as $\{A_1, A_2, A_3\}$.
   Explain why the amplitude equations must have this invariance.

10. Express ϵ in terms of the parameters b and d.

11. Find the fixed points of the dynamical system (B5,B6,B7) corresponding, a) to stripes pattern, b) to hexagonal pattern. (We do not ask to the analysis of their stability).