RIEMANN'S METHOD OF INTEGRATION: ITS EXTENSIONS WITH AN APPLICATION

GIVEN BY

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SUMMARY

The classical method of integration of hyperbolic differential equations, due to Riemann, is developed and extended. The cases in which the Cauchy data is discontinuous, and the initial curve is cut by a characteristic in more than one point, are considered. An application of the extensions to onedimensional gas dynamics is made.

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PART I

THE METHOD AND ITS EXTENSION

§ 1. The canonical or normal form of the second order hyperbolic differential equation in two independent variables

The general linear partial differential equation of second order in two independent variables x and y may be written in the form

$$A w_{xx} + 2 B w_{xy} + C w_{yy} + D w_x + E w_y + F w = G, \tag{1}$$

where the coefficients, A, B, C, D, E, F and G are functions of x and y only.

We shall assume that $AC - B^2 < 0$ in some region the x-y plane. The equation (1) is then said to be of «hyperbolic» type in this region. It is shown in standard textbooks on partial differential equations [e. g. see [2], Chapter III] that if (1) is hyperbolic, then there is a *real* coordinate transformation

$$x = x(r, s); \qquad y = y(r, s), \tag{2}$$

which carries (1) into the canonical form

$$w_{rs} + a w_r + b w_s + c w = d, \tag{3}$$

where the coefficients a, b, c, d are functions of r and s. It is convenient and is no great restriction to assume $d \equiv 0$.

The lines, $r =$ constant, $s =$ constant, in the r-s plane play an important part in the solution of (3) and accordingly are known as the *characteristics* of that equation. Likewise, the curves in the x-y plane corresponding to these lines under the transformation (2) are called the *characteristics of* (1).

§ 2. The adjoint operator

We shall define the operator L by the identity

$$L\,(w) \equiv w_{rs} + a\,w_r + b\,w_s + c\,w.$$

The operator M defined by the identity

$$M\,(v) \equiv v_{rs} - (a\,v)_r - (b\,v)_s + c\,v$$

is known as the *adjoint* of L.

It is easy to verify that

$$v\,L\,(w) - w\,M\,(v) = \frac{\partial X}{\partial r} + \frac{\partial Y}{\partial s},$$

where

$$\begin{cases} X \equiv \dfrac{1}{2}(v\,w_s - w\,v_s) + a\,v\,w, \\ Y \equiv \dfrac{1}{2}(v\,w_r - w\,v_r) + b\,v\,w. \end{cases}$$

If the functions v and w are such that $L\,(w) = M\,(v) = 0$ throughout a region G bounded by a sufficiently smooth closed curve Γ, then an applicaton of Green's theorem yields

$$(4) \qquad \int_\Gamma [X\,n_r + Y\,n_s]\,d\gamma = \iint_G [v\,L\,(w) - w\,M\,(v)]\,d\,r\;d\,s = 0,$$

where n_r and n_s are the components in the r and s directions respectively of the outer normal on Γ and γ denotes arc length on Γ. It is assumed that v and w have no singularities in G.

§ 3. THE CAUCHY PROBLEM OF THE FIRST KIND

In this problem the values of w and $\frac{\partial w}{\partial n}$ are prescribed on an arc C which is nowhere tangential to a characteristic. It is required to find a solution w of the differencial equation (3) which is valid for all points $P : (\xi, \eta)$ such that the lines $r = \xi$ and $s = \eta$ both intersect C (see fig. 1). If these points of intersection are denoted by P_1 and P_2 respectively, then the condition that C is nowhere tangential to a characteristic ensures that P_1 and P_2 are uniquely determined.

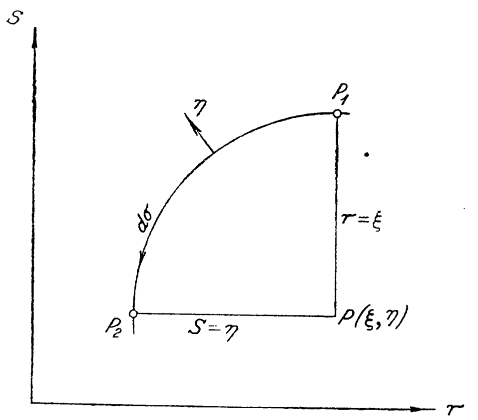

FIG. 1

Now, for the region G of (4) we take the interior of the curvilinear triangle $P P_1 P_2$. Its boundary is $\Gamma = P P_1 + P_1 P_2 + P_2 P$. On $P P_1$ $n_r = 1$, $n_s = 0$. On $P P_2$, $n_r = 0$, $n_s = -1$. Thus,

$$I_1 \equiv \int_P^{P_1} [X n_r + Y n_s] \, d\gamma = \int_P^{P_1} \left[\frac{1}{2} (v w_s - w v_s) + a v w\right] d s$$
$$= \int_P^{P_1} w (a v - v_s) \, d s + \frac{1}{2} v w \Big|_P^{P_1}.$$

Similarly,

$$I_2 \equiv \int_{P_2}^{P} [X n_r + Y n_s] \, d\gamma = - \int_{P_2}^{P} w (b v - v_r) \, d r - \frac{1}{2} v w \Big|_{P_2}^{P}.$$

If v is required not only to satisfy the equation

$$(5) \qquad M(v) = 0,$$

but also the conditions

$$(6) \qquad \begin{cases} a\,v - v_s = 0: & \text{on } r = \xi, \\ b\,v - v_r = 0: & \text{on } s = \eta, \\ \quad v = 1: & \text{at the point } P: (\xi, \eta), \end{cases}$$

then (4) yields the formula of Riemann

$$(7) \quad w(P) = \int_{P_2}^{P_1} [X n_r + Y n_s]\, d\sigma + \frac{1}{2}\left\{v(P_1)\, w(P_1) + v(P_2)\, w(P_2)\right\},$$

where σ denotes arc length on C. The right-hand member of (7) depends only on the values of w, w_r and w_s on the arc C and these are determined by w and $\frac{\partial w}{\partial n}$ on C, using the relation $\frac{\partial w}{\partial \sigma} = w_r \frac{d r}{d t} + w_s \frac{d s}{d t}$.

It is important to observe that the preceding argument shows only that if the Cauchy problem of the first kind has a solution, w, then w must be given by (7). Hence, if a solution exists, it must be unique. To verify that (7) actually provides a solution of (3) taking the prescribed initial values on C, one differentiate it to obtain w_r, w_s and w_{rs} and substitute these expressions in (3). For these calculations, see [3]. Having established that formula (7) satisfies (3), one must then show that it takes the specified initial values (again see [3]). Alternatively, the existence of a solution to the problem can be demonstrated by the Picard iteration method, see [6], and it then follows from the above discussion, that such a solution is unique and given by the formula (7).

§ 4. Discussion of the solution

It is clear that the value of w, as given by (7), at any point within the curvilinear triangle $P P_1 P_2$ is determined by the initial values on that part of the curve C which lies between P_1 and P_2. Hence the initial values of w and $\frac{\partial w}{\partial n}$ on the part of C exterior to the arc $P_1 P_2$ may be changed arbitrarily without affecting the solution in $PP_1 P_2$. This means that there can be no unique continuation of a solution across a *characteristic* line, however many derivatives are prescribed to be continuous across it.

The crux of Riemann's method is the replacement of the original initial value problem by a *standard problem* which depends only on the coefficients in equation (3). Once obtained, the solution, v, of this standard problem provides us with the solution, (7), to every Cauchy problem of the first kind for any arc C and any initial values of w and $\frac{\partial w}{\partial n}$. Such a function, v, i. e. a solution of (5) which satisfies (6), is called a Riemann function. The argument of the preceding section depends entirely on the existence of such a function, which can be established by the Picard iteration method. We shall obtain it explicitly in two special cases.

§ 5. Examples of the Riemann function

(*a*) In equation (3), let $a = b = \frac{\lambda}{r+s}$, ($\lambda = \text{const.}$), $C = 0$. Thus $L(w) \equiv w_{rs} + \frac{\lambda}{r+s}(w_r + w_s) = 0$ and the Riemann function must satisfy

$$5\,(a) \qquad M(v) \equiv v_{rs} - \frac{\lambda}{r+s}(v_r + v_s) - \frac{2\lambda}{(v+v^2)}\, v = 0,$$

and the conditions

$$6\,(a) \qquad \begin{cases} v_s = \dfrac{\lambda}{r+s}\, v: & \text{on } r = \xi, \\ v_r = \dfrac{\lambda}{r+s}\, v: & \text{on } s = \eta, \\ v = 1 \quad : & \text{at } r = \xi, \quad s = \eta. \end{cases}$$

If v is to satisfy the first and third conditions, we must have

$$v(\xi, s;\ \xi, \eta) = \exp \int_\eta^s \frac{\lambda}{\xi + s}\, ds = \left(\frac{\xi + s}{\xi + \eta}\right)^\lambda.$$

Similarly to satisfy the second and third conditions

$$v(r, \eta;\ \xi, \eta) = \left(\frac{r+\eta}{\xi+\eta}\right)^\lambda.$$

Both of these requirements are satisfied by the function

$$\left(\frac{r+s}{\xi+\eta}\right)^{\lambda}.$$

Therefore, it is reasonable to attempt to find a solution of (5) in the form $v = \left(\frac{r+s}{\xi+\eta}\right)^{\lambda} V$, where

(8) $$V = 1 \; ; \quad \text{on} \quad r = \xi \quad \text{and on} \quad s = \eta.$$

For v to be a solution of (5), V must satisfy

$$V_{rs} + \frac{\lambda(1-\lambda)}{(r+s)^2} V = 0.$$

Now, if V is assumed to be a function of $z = \frac{-(r-\xi)(s-\eta)}{(r+s)(\xi+\eta)}$, this reduces to the ordinary differential equation

$$z(1-z)\frac{d^2 V}{dz^2} + (1-2z)\frac{dV}{dz} - \lambda(1-\lambda) V = 0.$$

The latter is a hypergeometric equation having a solution (*)

$$V = F(1-\lambda, \lambda, 1 \; ; z)$$

with the required property (8). Thus the Riemann function for this example is

(9) $$v = \left(\frac{r+s}{\xi+\eta}\right)^{\lambda} F(1-\lambda, \lambda, 1 \; ; z).$$

When λ is a positive or negative integer, the hypergeometric function reduces to a polynomial in z.

(b) As a second example we consider equation (3) with $a = b = m$, where m is a constant, and $c = 0$. Thus $L(w) \equiv w_{rs} + m(w_r + w_s) = 0$,

(*) The general hypergeometric equation is

$$z(1-z)\frac{d^2 V}{dz^2} + [c - (a + b + 1)z]\frac{dV}{dz} - abV = 0,$$

with one solution given by

$$V = F(a, b, c \; ; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots$$

an equation which can be reduced to the «telegarphist's equation». In this case, the Riemann function must satisfy the equation

$$5\ (b) \qquad M(v) \equiv v_{rs} - m\ (v_r + v_s) = 0$$

and the conditions

$$6\ (b) \qquad \begin{cases} v_s = m v: & \text{on } r = \xi, \\ v_r = m v: & \text{on } s = \eta, \\ v = 1 \ \ : & \text{at } r = \xi, \quad s = \eta. \end{cases}$$

If v is to satisfy the first and third conditions, we must have

$$v(\xi,\ s\ ;\ \xi,\ \eta) = \exp \int_{\eta}^{s} m\, d s = e^{m(s-\eta)}.$$

Similarly, to satisfy the second and third conditions,

$$v(r,\ \eta\ ;\ \xi,\ \eta) = e^{m(r-\xi)}.$$

Both of these requirements are satisfied by the function

$$e^{m[(r-\xi)+(s-\eta)]}.$$

Hence, we attempt a solution of 5 (b) in the form

$$v\ (r,\ s\ ;\ \xi,\ \eta) = e^{m[(r-\xi)+(s-\eta)]}\ V,$$

where V must then satisfy the equation

$$V_{rs} - m^2\, V = 0.$$

Now, if V is assumed to be a function of $z = -m^2\,(r - \xi)\ (s - \eta)$, the preceding equation becomes

$$z \frac{d^2 V}{d z^2} + \frac{d V}{d z} + V = 0.$$

This can be reduced to Bessel's equation and one of its solutions, $V = J_0\left(2\sqrt{z}\right)$, satisfies condition (8). Thus the Riemann function for this example is $\exp\left\{m\,[(r - \xi) + (s - \eta)]\right\}.\ J_0\left(2\sqrt{z}\right)$.

§ 6. The Cauchy problem of the second kind

In this problem the values of w only are prescribed on two segments QA, QB of characteristic lines of different families (see fig. 2). It is required to show that there is a solution of (3) valid at any point $P: (\xi, \eta)$ such that the lines $r = \xi$ and $s = \eta$ intersect QA and QB respectively. Again applying Green's theorem as in the above, we obtain the same formula (7) with the following obvious modifications. On

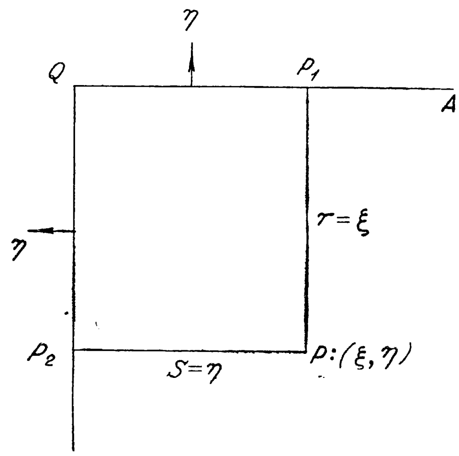

FIG. 2

QP_1, $n_r = 0$, $n_s = 1$. On QP_2, $n_r = -1$, $n_s = 0$. The first term on the righthand side of (7) becomes

$$\int_Q^{P_1} \left\{ \frac{1}{2} (v w_r - w v_r) + b v w \right\} dr - \int_{P_2}^{Q} \left\{ \frac{1}{2} (v w_s - w v_s) + a v w \right\} ds$$
$$= \int_Q^{P_1} v (w_r + b w) \, dr - \frac{1}{2} v w \Big|_Q^{P_1} - \int_{P_2}^{Q} v (w_s + a w) \, ds + \frac{1}{2} v w \Big|_{P_2}^{Q},$$

so that formula (7) becomes

$$w(P) = \int_Q^{P_1} v (w_r + b w) \, dr - \int_{P_2}^{Q} v (w_s + a w) \, ds + v(Q) \, w(Q).$$

Since w_r can be computed from the values of w on QP_1 and w_s from the values of w on P_2Q, $w(P)$ is determined.

§ 7. An extension of Riemann's method to the case where the curve C is tangential to a characteristic at some point (see [4] and [8]).

As a preliminary, we parametrize C, say

$$(10) \qquad r = \varphi(x), \qquad s = \psi(x).$$

where $\varphi'(x)$ and $\psi'(x)$ are assumed to be sectionally continuous. Thus X and Y can be considered as functions of x on C and formula (7) may be written as

$$(11) \quad w(P) = \int_{X_1}^{X_2} [X\psi' - Y\varphi']\, dx + \frac{1}{2}[v(P_1)\, w(P_1) + v(P_2)\, w(P_2)].$$

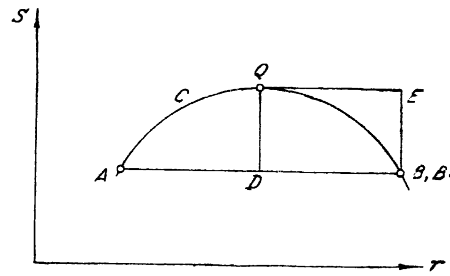

FIG. 3

where X_1 and X_2 are the values of the parameter x at P_1 and P_2 respectively. Formula (11) does not depend on the orientation of C with respect to P nor on the direction of increasing x on C.

Now suppose C is tangential to a characteristic at some point Q. Suppose $\psi'(x) = 0$ at Q (see fig. 3). Then in general there is no solution of the Cauchy problem for C; that is, given arbitrary initial values for w and $\frac{\partial w}{\partial n}$ on C, one cannot assert the existence o fa function w which

(*i*) satisfies the differential equation $L(w) = 0$.

(*ii*) assumes the prescribed boundary values together with its derivatives and

(*iii*) is continuous together with w_r and w_s in the region $A\,Q\,B\,D\,A$.

For suppose such a function did exist. Then by the uniqueness of the solution to the problem developed in § 3, the value of w at points to the right and left of D is given by the Riemann formula applied to the arcs $Q\,B$ and $Q\,A$ respectively. Since w is continuous its value at D is equal to the limit approached by $w(P)$ as P approaches D either from the right or left. Clearly, it is possible to prescribe initial values for w and $\frac{\partial w}{\partial n}$ on C in such a way as to make the limit from the right different from the limit from the left, since the position of P_2 changes abruptly from B to A as P passes through D. This is a contradiction of the continuity. Hence there is not always a solution of the Cauchy problem.

It should be remarked that the Riemann procedure could be applied to the whole region $A\,Q\,B\,D\,A$ if we were to adopt the rather obvious rule that we shall choose for the point P_2 the point on C which is on the arc $A\,Q$. The Riemann formula would then yield a single-valued function $w(P)$ at all points P in this region. Furthermore, this single-valued function would satisfy the differential equation $L(w) = 0$. However, the proof that it assumes the prescribed initial values on the arc QB of C would break down, for in this proof (see [3] or [6]) we require that both of the points P_1 and P_2 shall approach a point P^* on the curve C whenever P approaches P^*. This property obviously does not subsist for points P^* to the right of Q when we follow the rule that P_2 is always to be chosen to lie on the arc AQ.

The preceding remark indicates how it is possible to extend Riemann method to a curve which is tangential to a characteristic. First, we obtain a function w_1 which satisfies conditions (*i*) and (*iii*) above but not condition (*ii*) in the region $ABEQA$ by adopting the rule whereby P_2 is chosen to lie on the arc $A\,Q$. Then using the values of this *constructed solution*, w_1, and its normal derivative, $\frac{\partial w_1}{\partial n}$, as initial values on the characteristic $Q\,E$, we can obtain a new solution of our differential equation, valid in the region QBE, and taking the prescribed initial values on $Q\,E$. This is a classical result (see [7]). In fact, there are infinitely many such solutions. We assert that there is one and only one such solution w_2 which takes the prescribed initial values on the arc $Q\,B$. Granting this for the moment, we see that Riemann's method is extended in the following sense : We have obtained a function $w(P)$ which is double-valued in $Q\,B\,E$. There is a single-valued branch w_1 of this function in the region $A\,Q\,E\,B\,A$ which yields a solution of our differencial equation with the correct initial values on arc $A\,Q$. Another single valued branch w_2 of $w(P)$ yields a solution in $Q\,B\,E$ with the correct initial values on the arc $Q\,B$. The two branches agree, together with their partial derivatives, on the line $Q\,E$. One may introduce the concept of a two-sheeted surface to investigate these single-valued branches; however, we shall employ a slightly different conceptual device as it leads toward our ultimate objective.

Consider fig. 3 again und let $Q\,B'\,E$ be the reflection of $Q\,B\,E$ in the line $Q\,E$. Now let the resulting figure be divorced from the coordinate system, and define the coordinates of a point in $Q\,B'\,E$ to be those which the image point in $Q\,B\,E$ had in the original coordinate system.

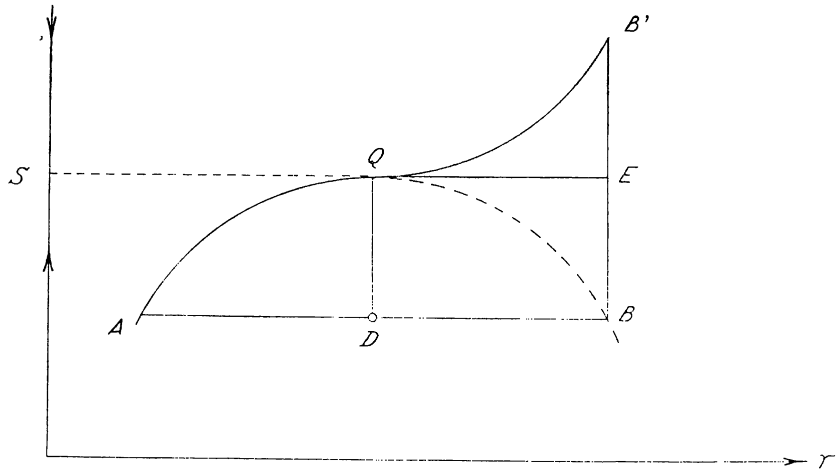

Fig. 4

This « unfolded » diagram no longer represents a coordinate plane since the s-coordinate increases up to line $Q\,E$ and then decreases. Thus it is now $A\,Q\,B'$ rather than $A\,Q\,B$ which is regarded as the initial curve, in the sense that it is on the former arc in the unfolded diagram that we shall expect w to take the prescribed initial values. If we use the original parametric equations of the curve C, then the points on QB' correspond to the values of the parameter x between x_Q and x_B, the values of x at Q and B respectively in the original coordinate plane and we may regard the ecuations $r = \varphi(x)$, $s = \psi(x)$ as defining the curve $A\,Q\,B'$ in fig. 4. We may then apply formula (11), the parametric form of Riemann's formula, to *the curve* $A\,Q\,B'$ using the *prescribed initial values.* In the region $A\,Q\,E\,B\,A$ of figure 4 this yields the same functional values w_1 as before, the only difference being that it is no longer necessary to specify how P_2 is to be chosen since the curve $A\,Q\,B'$

replaces C. Now, formula (11) is applicable to the triangular region $Q\,E\,B'$ as well. Thus, Riemann's method yields a solution, $w(P)$, in the region $A\,Q\,B'\,E\,B\,A$ which assumes the prescribed initial values on the curva $A\,Q\,B'$. This solution, $w(P)$, is a single-valued function of *points* in the diagram of figure 4, However, as a function of coordinates (r, s) it is obviously multiple-valued. If we consider the part of $w(P)$ in the triangular region $Q\,E\,B'$ as a function of coordinates, we see that it yields a solution in the region $Q\,E\,B$ of the r - s plane (fig. 3) which takes the prescribed values on the arc $Q\,B$ and the constructed values on the characteristic $Q\,E$, thus proving the assertion on (*) page 303. The diagram of fig. 4, although actually unnecessary to the argument, is, like a Riemann surface, a convenient device for distinguishing between the different values of $w(P)$. We shall find it very useful in the application which we shall consider next.

In conclusion, it may be stated that this «unfolding» may be applied to an initial curve which has any number of points of tangency to characteristics, or more generally, of points at which one or both of the coordinates r, s have extrema. One merely «unfolds» at each such point in turn. For more details see § 15.

PART II

APPLICATION OF THE METHOD

§ 8. One-dimensional gas dynamics

The theory of one-dimensional flow of gases in which the velocity, w, density, ϱ, and pressure, p, depend on only one space variable, say x, and the time t can be based on the following equations :

$$(12) \qquad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{1}{\varrho}\,\frac{\partial p}{\partial x} = 0. \qquad \text{(Conservation of Momentum)}$$

$$(13) \qquad u\,\frac{\partial \varrho}{\partial x} + \varrho\,\frac{\partial u}{\partial x} + \frac{\partial \varrho}{\partial t} = 0. \qquad \text{(Continuity).}$$

$$(14) \qquad p = f(\varrho)$$

(*) The precise verification that the $w(P)$ defined by Riemann's formula satisfies the differential equation and takes the prescribed initial values on the initial curve in the «unfolded» diagram follows the same lines as in § 3.

The nature of (14) depends on the particular type of motion we wish to consider. Thus, if the motion is isothermal, $f(\varrho) = \text{constant}\cdot\varrho$. If the motion is isentropic, $f(\varrho) = \text{constant}\cdot\varrho^{\gamma}$. In other cases, $f(\varrho)$ may be considered to be an approximation to the empirical behavior of a gas. We shall make the assumption that $f'(\varrho)$ is positive, that is,

$$\frac{dp}{d\varrho} = f'(\varrho) = a^2 > 0. \tag{15}$$

Using (15), equation (12) may be rewritten as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{a^2}{\varrho}\frac{\partial \varrho}{\partial x} = 0. \tag{16}$$

The problem of gas dynamics is, of course, to determine u and ϱ as functions of x and t which satisfy (16) and (13), and certain boundary and initial conditions. This system is non-linear and consequently presents difficulties.

Now, by introducing new variables, the non-linear system can be replaced by an equivalent linear one. First, (13) and (16) may be combined to give the equivalent pair of equations

$$\frac{\partial u}{\partial t} + (u \pm a)\frac{\partial u}{\partial x} \pm \frac{a}{\varrho}\left\{\frac{\partial \varrho}{\partial t} + (u \pm a)\frac{\partial \varrho}{\partial x}\right\} = 0. \tag{17}$$

Then, if we define σ by

$$\sigma = \int_{\varrho_0}^{\varrho} \frac{a}{\varrho}\, d\varrho, \tag{18}$$

where ϱ_0 is some standard density, (17) may be written as

$$\frac{\partial}{\partial t}(u \pm \sigma) + (u \pm a)\frac{\partial}{\partial x}(u \pm \sigma) = 0. \tag{19}$$

Now since $d(u \pm \sigma) = \frac{\partial}{\partial x}(u \pm \sigma)\, dx + \frac{\partial}{\partial t}(u \pm \sigma)\, dt,$

we observe that along curves in the (x, t)-plane defined by the relations $dx = (u \pm a)\, dt$ we have

$$d(u \pm \sigma) = \left\{(u \pm a)\frac{\partial (u \pm \sigma)}{\partial x} + \frac{\partial (u \pm \sigma)}{\partial t}\right\} dt = 0,$$

by (19). Thus,

$$(20)\quad \begin{cases} u + \sigma = \text{constant along curves on which } dx = (u + a)\,dt. \\ u - \sigma = \text{constant along curves on which } dx = (u - a)\,dt. \end{cases}$$

This suggests the introduction of a new pair of variables, r and s, defined by

$$(21)\qquad r = \frac{1}{2}(u + \sigma), \qquad s = -\frac{1}{2}(u - \sigma).$$

Having the solution for u and σ as functions of x and t, (21) defines a transformation from the (x, t) — plane into the (r, s) — plane. In any region G in which this transformation is one-to-one we may consider the inverse transformation and regard x and t as functions of r and s. Then using (20), the partial derivatives of x with respect to r and s may be written in the form.

$$(22)\qquad \begin{cases} x_s = (u + a)\,t_s, \\ x_r = (u + a)\,t_r, \end{cases}$$

where a is expressed in terms of r and s by means of (15), (18) and (21), i. e., by (15) a is a function of ϱ which is a function of σ by (18) and $\sigma = r + s$ by (21).

By differentiating the first of equations (22) with respect to r, we obtain

$$x_{sr} = (u + a)\,t_{sr} + (u_r + a_r)\,t_s.$$

Now, $u = r - s$ and $\sigma = r + s$. Hence $u_r = \sigma_r = \sigma_s = 1$ and $u_s = -1$.

Further, $$a_r = \frac{da}{d\varrho}\cdot\frac{d\varrho}{d\sigma}\cdot\frac{\partial\sigma}{\partial r} = \frac{a'}{\sigma'},$$

where $$a' = \frac{da}{d\varrho}, \qquad \sigma' = \frac{d\sigma}{d\varrho}.$$

Thus, $$x_{sr} = (u + a)\,t_{sr} + \left(1 + \frac{a'}{\sigma'}\right)t_s.$$

Similarly, differentiating the second of equations (22) with respect to s, we obtain

$$x_{rs} = (u - a)\,t_{rs} + \left(-1 - \frac{a'}{\sigma'}\right)t_r.$$

Elimination of x_{rs} yields the linear hyperbolic differential equation

$$t_{rs} + \frac{1}{2a}\left(1 + \frac{a'}{\sigma'}\right)(t_r + t_s) = 0, \tag{23}$$

where $\frac{1}{2a}\left(1 + \frac{a'}{\sigma'}\right)$ is a function of ϱ and therefore of $\sigma = r + s$.

The equation (22) and (23) constitute the linear system which replaces the original non-linear equations. Assuming that (23) can be solved, this yields t as a function of r and s. Using (22) and the expressions obtained for t_s and t_r, by differentiating the solution t, we can find x as a function of r and s. But r and s are known functions of u and ϱ. Thus, t and x are known functions of u and ϱ. These functions determine the desired relationship between x, t and u, ϱ, thereby solving the flow problem for the region G.

For example, in the isothermal case, $f(\varrho) = A^2 \varrho$, where A is a constant. Hence, $a^2 = f'(\varrho) = A^2$ or $a = A$, and $\sigma = A \log \frac{\varrho}{\varrho_0}$. Thus

$$\frac{1}{2a}\left(1 + \frac{a'}{\sigma'}\right) = \frac{1}{2A},$$

and equation (23) is of the form of the example in § 5 (b) with $m = \frac{1}{2} A$ and $w = t$.

For the isentropic case, $f(\varrho) = A^2 \varrho^\gamma$, where A and $\gamma + 1$ are constants. Thus, $a^2 = \gamma A^2 \varrho^{\gamma-1}$ and $a = A \gamma^{\frac{1}{2}} \varrho^{\frac{\gamma-1}{2}}$, $\sigma = \frac{2a}{\gamma - 1}$ (if $\varrho_0 = 0$ is used), so that

$$\frac{1}{2a}\left(1 + \frac{a'}{\sigma'}\right) = \frac{\gamma + 1}{4a} = \frac{\gamma + 1}{2(\gamma - 1)} \cdot \frac{1}{r + s}.$$

Equation (23) becomes that of § 5 (a) with $\lambda = \frac{\gamma + 1}{2(\gamma - 1)}$, $w = t$. (Note that the integral values of λ correspond to $\gamma = \frac{2n + 1}{2n - 1}$. Thus for $n = 2, 3$ we have the values $\gamma = 5/3$, $7/5$ of monatomic and diatomic gases respectively).

§ 9. The initial values

The linear system (22), (23) is in itself an incomplete formulation of the flow problem, for the solution is required to satisfy certain boundary and initial conditions. The initial conditions which will be associated with the original equations of motion (12), (13) are that u and a (or ϱ) are prescribed functions of x at time $t = 0$. Since r and s depend on u and a, they must likewise be prescribed functions of x at time $t = 0$, say $r = \varphi(x)$, $s = \psi(x)$. This puts initial conditions on the solution t of (23), namely, $t = 0$ on the curve C in the $r - s$ plane given parametrically by the equations $r = \varphi(x)$, $s = \psi(x)$. We assume that $\varphi(x)$ and $\psi(x)$ are continuous, have sectionally continuous derivatives and neither is constant (*) for a finite range of x.

To solve (23) by Riemann's method, the initial values of t_r and t_s on C are also required. Now, *along the curve* C, $dt = \left(t_r \dfrac{dr}{dx} + t_s \dfrac{ds}{dx}\right) dx$ and we have

$$dt = (t_r \varphi' + t_s \psi')\, dx = 0, \tag{24}$$

since $t = 0$ on C, Likewise, $dx = x_r \dfrac{dr}{dx} + x_s \dfrac{ds}{dx}$, and we have, using (22),

$$dx = \left\{(u - a)\, t_r \varphi' + (u + a)\, t_s \psi'\right\} dx. \tag{25}$$

It follows from (24) and (25) that t_r and t_s on C satisfy the equations

$$t_r \varphi' + t_s \psi' = 0, \tag{24'}$$

$$t_r (u - a)\, \varphi' + t_s (u + a)\, \psi' = 1, \tag{25'}$$

whence

$$t_r = -\frac{1}{2 a \varphi'}, \tag{26}$$

$$t_s = +\frac{1}{2 a \psi'}. \tag{27}$$

(*) If either r or s is constant for all x initially, then a so-called «simple wave» results. If either r or s is constant for a finite range of x there are no changes in principle in what follows. These cases are omitted here for convenience.

For convenience we consider the isentropic case with $\gamma \neq 1$. Thus $2a = \sigma\ (\gamma - 1)$ by the last paragraph of the preceding section, and since $\sigma = r + s$ we have

$$(28) \qquad t_r = -\frac{1}{(\gamma - 1)\,\varphi'(\varphi + \psi)},$$

$$(29) \qquad t_s = \frac{1}{(\gamma - 1)\,\psi'(\varphi + \psi)}.$$

Now applying Riemann's formula (*) (11), we obtain

$$(30) \qquad t(P) = \frac{1}{\gamma - 1} \int_{X_1}^{X_2} \frac{v(\varphi,\ \psi;\ \xi,\ \eta)}{\varphi + \psi}\, d\,x.$$

The problem is to find the solution $r = r(x,\ t)$ and $s = s(x,\ t)$ satisfying (19) and taking the correct initial values $\varphi(x)$, $\psi(x)$ on $t = 0$. If in a region G of the $(x,\ t)$ — plane containing $t = 0$ this solution define a (1, 1) correspondence with a region G' of the $(r,\ s)$ — plane, then in G' the inverse functions $x = x(r,\ s)$, $t = t(r,\ s)$ will satisfy (22), and take the corresponding initial values (28) and (29) on the corresponding initial curve C. Conversely, if $x = x(r,\ s)$, $t = t(r,\ s)$ is *such* a solution in a region G' of the speedgraph plane, then the inverse functions $r = r(x,\ t)$, $s = s(x,\ t)$ constitute a solution of the original flow problem in the corresponding region G of the $(x,\ t)$ — plane. Further, if $x = x(r,\ s)$, $t = t(r,\ s)$ is *such* a solution in a region G' of the *unfolded* diagram (**), then again the inverse functions $r = r(x,\ t)$, $s = s(x,\ t)$ solve the flow problem. If one starts with the solution in the unfolded diagram, the region G' will certainly be limited by lines across which

$$J \equiv x_r t_s - x_s t_r = -2\,a\,t_r\,t_s,$$

changes sign, and the corresponding region G will certainly (***) be bounded by a limit line beyond which the inverse mapping is not defined. The uniqueness of this solution of the flow problem, follows from the uniqueness of the solution in the speedgraph plane.

)*) Although this formula was deduced under the (implicit) assumption that t_r and t_s are continuous on C, it applies equally well to the case when they have discontinuities, even infinite ones (φ' or ψ' zero), see § 15.

(**) The (1, 1) correspondence now being defined as one between points, and not coordinates.

(***) G' may be smaller since only local breakdown of one-one correspondence is indicated by change in sign of the Jacobian.

§ 10. Motion in a closed tube

The motion of gas enclosed in a tube with fixed end walls gives rise to a problem of mixed type, see [4]. The initial conditions $\varphi(x)$ and $\psi(x)$ in this case are prescribed only for $0 \leq x \leq l$. Since it will be assumed that there is no initial discontinuity, we must have

$$\varphi(0) - \psi(0) = \varphi(l) - \psi(l) = 0,$$

i. e. the initial velocity at each end $[u(0) = \varphi(0) - \psi(0)$ and $u(l) = \varphi(l) - \psi(l)]$ must be zero. In addition, the boundary condi-

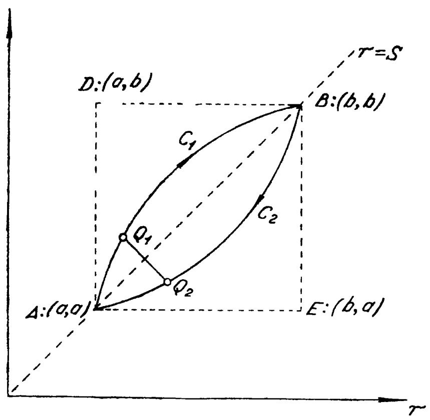

Fig. 5

tions that $u = 0$ at $x = 0$ and $x = l$ for all time t are imposed. This is the simplest kind of mixed problem and can be resolved into a pure initial values problem as follows :

$u(x) = \varphi(x) - \psi(x)$ is defined to be an odd function in $-l \leq x \leq l$ and to be periodic with period $2l$.

$\sigma(x) = \varphi(x) + \psi(x)$ is defined to be an even function in $-l \leq x \leq l$ and to be periodic with period $2l$. This implies $\varphi(x) = \psi(-x)$ and $\psi(x) = \varphi(-x)$ and ensures that $u = 0$ for $x = \pm nl$, $n = 0, 1, 2, \ldots$ for all t. Hence the boundary conditions are satisfied automatically by the solution of this initial value problem.

For convenience, a simple type of initial curve resulting from the above definitions is shown in figure 5 in which neither $\varphi'(x)$ nor $\psi'(x)$ is zero for $0 \leq x \leq l$. As x increases from $-l$ to 0, the corresponding point Q_1 passes along C_1 from A to B. As x increases from 0 to l, the corresponding point Q_2 passes along C_2, the reflection of C_1 with respect to the line $r = s$, from B to A.

Image points such as Q_1 and Q_2 correspond to values of x which differ only in sign. As x increases from l to ∞ and decreases from $-l$

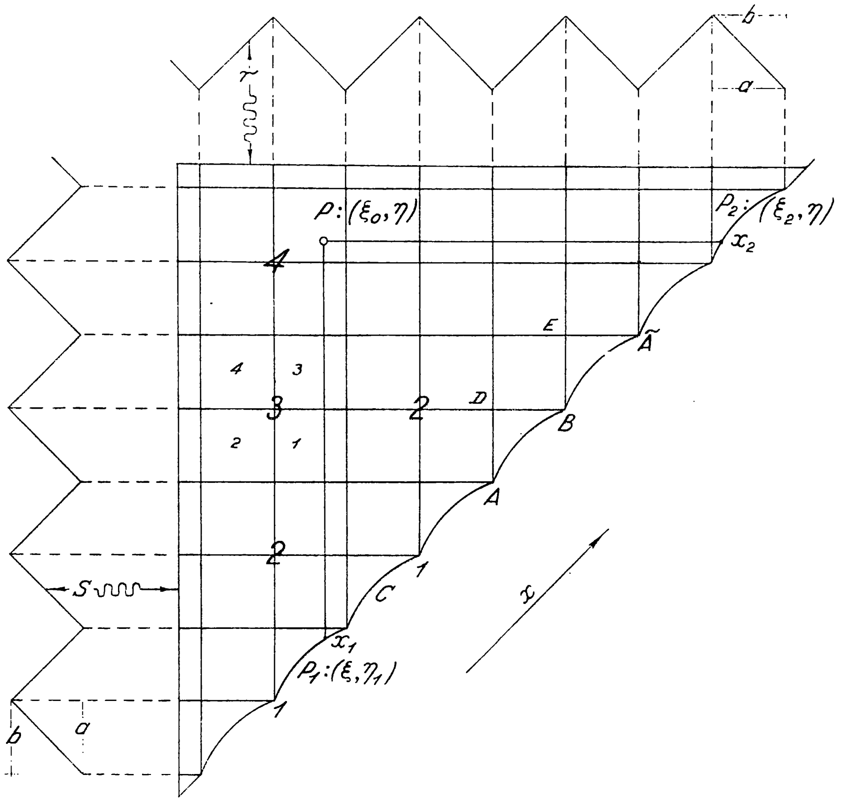

FIG. 6

to $-\infty$, the curves C_1 and C_2 are covered infinitely many times. Thus it is seen that the mapping from the physical plane to the (r, s)-plane is not one-to-one so that our solution $t(r, s)$ cannot be single-valued. However, by means of the generalization of the «unfolding» procedure noted at the end of § 7, we may select single-valued branches if we replace the (r, s)-plane by a multi-sheeted surface, or by its corresponding unfolded diagram (see fig. 6). Along one diagonal, the arc C_1 is drawn joining A to B, the arc C_2 joining B to $\tilde{A}$, and then arcs alternately congruent to C_1 and C_2. The resulting curve C corresponds to the total range $-\infty < x < \infty$ of the parameter x. The direction of increasing x is indicated by the arrow. Note that the curvilinear triangles ADB and $BE\tilde{A}$ correspond to $t > 0$. This follows from (28) and (29). The direction of increasing t is therefore up to the left.

Mapping from the (r, s) to the (x, t) — plane, we obtain fig. 7 wherein the curved characteristics corresponding to the lines $r =$ const. and $s =$ const. are drawn as straight lines for the sake of clarity. Note that the periodicity in x of the initial conditions produces a periodic condition in the characteristics of fig. 7 in that the curves which are a horizontal distance $2l$ apart correspond to the same values of r in the

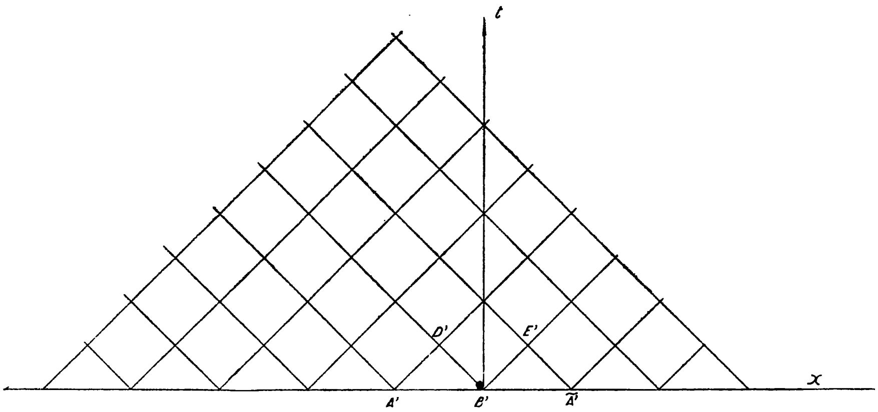

FIG. 7

family $dx = (u + a)\,dt$ and to the same value of s in the family $dx = (u - a)\,dt$. With this in mind it may perhaps be clearer to regard corresponding squares in figures 6 and 7 as being those which have the same position relative to $ADBE\tilde{A}$ and $A'D'B'E'\tilde{A}'$ respectively.

Now, fig. 6 is divided into congruent squares (thick lines) by means of the two characteristic lines through each of the points on the curve C corresponding to $x = \pm (2n + 1)\,l$, $n = 0, 1, 2, \ldots$ These squares are numbered 1, 2, 3, ... according to their distance from C, as shown. Each of these large squares is subdivided into four equal squares by the two characteristics through each of the points $x = \pm 2nl$, and these are numbered 1, 2, 3, 4 as indicated. Then $t_{ij}(r, s)$ will denote the unique value assumed by t at the point (r, s) in the small square $j = 1, 2, 3, 4$ of the large square $i = 1, 2, 3, \ldots$. We now consider, how the solution is constructed.

The initial values on C determine $t_{12}(r, s)$ and $t_{13}(r, s)$ directly by means of (30). Taking the values of these functions on the appropriate intersecting characteristics as initial values for $t_{14}(r, s)$, we obtain an initial value problem of the second kind to determine $t_{14}(r, s)$ Similarly, we obtain $t_{21}(r, s)$ and continuing in this manner, we extend the solution to squares at successively greater distances from C. In general, having obtained $t_{i2}(r, s)$ and $t_{i3}(r, s)$, one can construct t_{i4} and $t_{i+1,1}$. These, in turn, permit us to find $t_{i+1,2}$ and $t_{i+1,3}$.

Actually, we need not resort to the above step-by-step procedure which involves the second kind of initial value problem, for we shall prove (*) that formula (30) holds throughout the diagram of fig. 6.

Now, on the curve C of fig. 6, the initial values of t_r and t_s, see (28) and (29), have discontinuities for $x = \pm n l$, $n = 0, 1, 2, \ldots$ since at these points φ' and ψ' change alternatively from positive values to negative values. Further, φ' or ψ' may be 0 on C, giving rise to other points of discontinuity. Excluding characteristics through such points of C, we may verify that (30) is a solution directly. Thus, consider

$$t(P) = \frac{1}{\gamma - 1} \int_{X_1}^{X_2} \frac{v(\varphi, \psi; \xi, \eta)}{\varphi + \psi} \, dx.$$

The function $v(r, s; \xi, \eta)$ is continuous, see (9), in all its arguments in the region shown in fig. 6 if we assume $r + s > 0$ in this region (**). Hence, v is bounded, as is also $1/(r + s)$. This implies that, as P approaches a point Q on C, $t(P)$ approaches zero, which is the prescribed initial value for t on C. Further,

$$(32) \quad t_\xi(P) = \frac{1}{\gamma - 1} \int_{X_1}^{X_2} \frac{v_\xi(\varphi, \psi; \xi, \eta)}{\varphi + \psi} \, dx - \frac{d X_1}{d \xi} \cdot \frac{v(\xi, \eta_1; \xi, \eta)}{(\gamma - 1)(\xi + \eta_1)},$$

and since v_ξ is continuous, as $P \to Q$ we have

$$t_\xi(P) \to - \left[\frac{1}{(\gamma - 1) \varphi' (\varphi + \psi)} \right]_Q,$$

demonstrating that t_ξ attains the prescribed initial values on C, and similarly for t_η.

(*) This proof is covered by the general theorem quoted in § 15, but a proof for the present case is included for illustration.

(**) This implies that the velocity of sound, a, is strictly positive throughout the flow.

Further,

$$t_{\xi\eta}(P)=\frac{1}{\gamma-1}\int_{X_1}^{X_2}\frac{v_{\xi\eta}(\varphi,\psi;\xi,\eta)}{\varphi+\psi}\,dx+\frac{dX_2}{d\eta}\frac{v_\xi(\xi_2,\eta;\xi,\eta)}{(\gamma-1)(\xi_2+\eta)}-$$
$$-\frac{dX_1}{d\xi}\frac{v_\eta(\xi,\eta_1;\xi,\eta)}{(\gamma-1)(\xi+\eta_1)}.$$

Therefore,

$$(33)\left\{\begin{aligned}&t_{\eta\xi}+\frac{\lambda}{\xi+\eta}(t_\xi+t_\eta)=\\&=\frac{1}{(\gamma-1)}\int_{X_1}^{X_2}\frac{1}{\varphi+\psi}\left\{v_{\xi\eta}(\varphi,\psi;\xi,\eta)+\frac{\lambda}{\xi+\eta}(v_\xi+v_\eta)\right\}dx+\\&+\left[\frac{v_\xi(\xi_2,\eta;\xi,\eta)}{(\gamma-1)\psi'(\xi_2+\eta)}+\frac{\lambda v(\xi_2,\eta;\xi,\eta)}{(\gamma-1)\psi'(\xi_2+\eta)(\xi+\eta)}\right]\\&-\left[\frac{\lambda v(\xi,\eta_1;\xi,\eta)}{(\gamma-1)\varphi'(\xi+\eta_1)(\xi+\eta)}+\frac{v_\eta(\xi,\eta_1;\xi,\eta)}{(\gamma-1)\varphi'(\xi+\eta_1)}\right].\end{aligned}\right.$$

We must show that the right of (34) is zero. To do this we require (*) a property of the Riemann function which we shall state without proof, see [2]. If $v(r, s; \xi, \eta)$ is the Riemann function of the operator L, and $V(r, s; \xi, \eta)$ is that of its adjoint M, so that it satisfies $L(V)=0$, then

$$v(r, s; \xi, \eta) = V(\xi, \eta; r, s).$$

Thus in this case

$$v_{\xi\eta}(\varphi, \psi; \xi, \eta)+\frac{\lambda}{\xi+\eta}(v_\xi+v_\eta)=0.$$

Referring to ecuations 6 (b), it is seen that the quantities in each of the square brackets are also zero. This establishes that $t(P)$ satisfies the differential equation at each interior point of each amall square except on the characteristic passing through a point on C at which either φ' or ψ' is zero. Further, from the fact that $t(P)$ is continuous throughout the unfolded diagram it follows that it is just the solution constructed by the step-by-step procedure given above. Its uniqueness follows from an adaptation of the uniqueness proof of § 3, see § 15.

(*) Actually since v is known explicitly, the following result can be verified by differentation without appeal to this general theorem.

It has just been shown that for any point $P : (\xi, \eta)$ in the square (i, j) the value of $t_{ij}(P)$ is given by

(34) $$t_{ij}(\xi, \eta) = \frac{1}{\gamma - 1} \int_{X_1}^{X_2} \frac{v(\varphi, \psi ; \xi, \eta)}{\varphi + \psi} \, d x.$$

Clearly, the Riemann function v does not depend on the square (i, j), whereas the values x_1, x_2 corresponding to the points P_1, P_2 in which the characteristic lines through P intersect C (fig. 6) do depend on (i, j). Using (34), we obtain the following recursion formula :

(35) $$t_{ij}(r, s) = t_{i-1, j}(r, s) + K(r, s),$$

where

$$K(r, s) = \frac{1}{\gamma - 1} \int_{X_1}^{X_1 + 2l} \frac{v(\varphi, \psi ; r, s)}{\varphi + \psi} \, d x = \frac{1}{\gamma - 1} \int_{-l}^{l} \frac{v(\varphi, \psi ; r, s)}{\varphi + \psi} \, d x,$$

since $\varphi(x)$ and $\psi(x)$ are of period $2l$ in x. Clearly, $K(r, s)$ is independent of (i, j). Also, $K(r, s) = K(s, r)$. since $\varphi(x) = \psi(-x)$, $\psi(x) = \varphi(-x)$ by construction, and $v(\xi, \eta ; r, s) = v(\eta, \xi ; s, r)$ by § 5 (*b*). From (35) we get

(36) $$t_{ij}(r, s) = t_{1j}(r, s) + (i - 1) K(r, s).$$

Further, it follows from (34) that t_r, t_s are continuous functions of distance along a characteristic $r =$ constant and $s =$ constant respectively, in the unfolded diagram. Now the function $t_{ij}(r, s)$ determines a function $x_{ij}(r, s)$ and the pair together define a mapping from the unfolded diagram to the (x, t) — plane. Thus provided the Jacobian $J \equiv x_r t_s - x_s t_r = -2 a t_r t_s$ does not change sign, we obtain a flow pattern in the (x, t) — plane defined for all values of $t \geq 0$. It will now be shown that for arbitrary initial conditions, the Jacobian does change sign, that is the solution breaks down an a limit line occurs in the flow pattern. An estimate of the time of first occurrence of this breakdown will be given.

K_r cannot be zero everywhere (*) in the squares $i = 1$, $j = 2$. Suppose it is positive in some region, Π, of each of these squares. By (35), $\dfrac{\partial t_{i2}}{\partial r} > 0$ in the corresponding region, Π_{i2}, of each of the i^{th}

(*) The case $\gamma = -1$ is exceptional since then $v(r, s ; \xi \; \eta) \equiv 1$ and $K_r = K_s \equiv 0$.

large squares provided i is sufficiently large. But by (28), $\dfrac{\partial t_{i2}}{\partial r} > 0$ on C since $\varphi' > 0$ for $0 < x < l$. Since t_r is continuous on any vertical line, t_r must therefore change sign at some point on every vertical line between C and each Π_{i2}. If $K_r < 0$ in Π then it is also negative in the image region Π' of the square $i = 1$, $j = 3$ since K is independent of the squares. For i sufficiently large $\dfrac{\partial t_{i3}}{\partial r} < 0$ in Π'_{i3}. However, since $\varphi' < 0$ for $l < x < 2l$, $\dfrac{\partial t_{i3}}{\partial r} > 0$ on C. Hence, as before t_r must change sign at some point on every vertical line between C and each Π'_{i3}. This means that the solution breaks down (*) somewhere in the strip $-l \leq x \leq l$ of the (x, t) — plane. However, the symmetry of the solution implies breakdown in the strip $0 \leq x \leq l$ i. e. in the closed tube. This symmetry may expressed mathematically by easily derived relations such a

$$t_{i2}(r, s) = t_{i3}(s, r)\,; \quad t_{i4}(r, s) = t_{i4}(s, r)\,; \quad t_{i1}(r, s) = t_{i1}(s, r).$$

Thus if $t_r = 0$ along an arc Γ, then $t_s = 0$ along any arc Γ' which is the reflection of Γ in a line $r + s =$ constant, passing through the center of a large square, und vice versa.

§ 11. Estimate of the time of breakdown

If it is assumed that the variations of $\varphi(x)$ and $\psi(x)$ are small, then an estimate of the time of first occurrence of breakdown can be made. In this case, neglecting terms of degree more than one in $(r - \varphi)$ and $(s - \psi)$, we may write

$$v(\varphi, \psi\,; r, s) \doteq \left(\frac{\varphi + \psi}{r + s}\right)^{\lambda}.$$

Hence,

$$K(r, s) \doteq \frac{\alpha}{(r + s)^{\lambda}},$$

with

$$\alpha = \frac{1}{\gamma - 1}\int_0^l (\varphi + \psi)^{\lambda - 1}\, dx > 0.$$

(*) Even if J does not change sign with t_r (i. e. t_s changes sign also) there will be «physical» breakdown in an extended sense since infinite pressure or velocity gradients still occur in the physical plane.

Thus

$$K_r = \doteq \frac{\lambda \alpha}{(r + s)^{\lambda+1}} < 0,$$

and since on the curve C

$$t_r = - \frac{1}{(\gamma - 1)\, \varphi' (\varphi + \psi)},$$

breakdown takes place before the i^{th} square, where

$$i - 1 \doteq \frac{1}{(\gamma - 1)} \frac{(\varphi + \psi)^{\lambda}}{\varphi' \lambda \alpha}.$$

($r + s$ is replaced by $\varphi + \psi$ in K_r, thereby effecting a comparison between the values of t_r at the point x on the initial curve and its value at the corresponding point in the i^{th} square.) The time t in the i^{th} square is approximately $(i - 1)\; K(\varphi, \psi)$. Hence, breakdown will occur first at time (*)

$$t_0 = \frac{2}{(\gamma + 1)\beta}, \tag{37}$$

where β is the maximum value of $- \varphi'$ (or ψ') on C.

§ 12. A CHECK ON THE ESTIMATE

For : $\gamma = 3$, we have $\sigma = a$.

Hence,

$$\begin{cases} 2r = u + a, \\ -2s = u - a, \end{cases}$$

so that (22) may be integrated to give the characteristics

$$\begin{cases} x = -2\psi(\varepsilon) \cdot t + \varepsilon, \\ x = 2\varphi(\eta)\, t + \eta, \end{cases} \tag{38}$$

where ε, η are parameters. Thus the characteristics in the physical

(*) This estimate indicates the exceptional nature of $\gamma = -1$.

plane are straight line families. Their envelopes are obtained by eliminatign ε, η between (38) and

$$\begin{cases} 0 = -2\psi'(\varepsilon)\, t + 1, \\ 0 = \quad 2\varphi'(\eta)\, t + 1, \end{cases}$$

Thus the smallest value of t on either of the two envelopes is $1/2\beta$ in agreement with (37).

It may be stressed that the result of the last section is true whatever the shape of the initial curve C. The discussion is complicated by the squares in fig. 6 being replaced by rectangles in the general case.

§ 13. Arbitrary periodic initial conditions

The preceding discussion shows that the inevitability of the breakdown of the motion depends solely upon two facts. First, K_r and K_s must be non-zero almost everywhere in the square of figure 5. Secondly, for any value of r there is at least one point on $C_1 + C_2$ with that abscissa, for which $\varphi' > 0$ and one for which $\varphi' < 0$. Likewise, given any value of s, there is at least one point on $C_1 + C_2$ with that ordinate for which $\psi' > 0$ and one for which $\psi' < 0$. The latter fact follows from the closed nature of the curve. (If $C_1 + C_2$ has a vertical or horizontal tangent, the above holds for almost any r and s.)

Now suppose $C_1 + C_2$ is replaced by an arbitrary closed curve. This curve corresponds to *arbitrary* periodic initial conditions in the physical plane. Then the diagram which replaces fig. 6 will have the same periodic structure as fig. 6, although the large squares are replaced by rectangles (numbered in the same way) and there are no small squares at all. Again, (34) will hold (the subscript j now taking more values) and hence a relation like (35) will subsist, $K(r, s)$ being given by the same formula as before. Further, t_r will still be continuous on almost all lines $r =$ constant and t_s on almost all lines $s =$ constant.

The same discussion of the vanishing of t_r and t_s will apply since the two conditions described in the first paragraph of this section will be satisfied in the rectangle circumscribing the closed curve. Hence, for arbitrary initial conditions the solution eventually breaks down. Also, the same estimate for the time of first occurrence of a limit line will hold.

PART III

REMARKS AND RIGOR

§ 14. Remarks concerning the application

It is well-known that a finite initial disturbance breaks up eventually into two simple waves, each of which finally breaks down (see for example [1]). Riemann conjectured that for a infinite inital disturbance, the motion eventually breaks down. This also can be proved by the above methods, see [4].

In the above discussion the isothermal case $\gamma = 1$ was omitted for convenience. However, the theory applies equally well for this case and indeed the estimate (37) with $\gamma = 1$ still holds. Nevertheless, it appears that for $\gamma \div 1$, that is $\lambda \div \pm \infty$, the approximation used to obtain (37) is not valid, see (9). This is not the case because $z \div 0$ and indeed $\lambda^2 z$ tends to a limit as $\lambda \to \pm \infty$, see [5].

Just as the problem of periodic initial conditions has been discussed by forming the equation (23) for t, finding the corresponding Riemann function, and then setting up the simple formula (30), so the problem of periodicity in time might be considered by forming the corresponding equation in x, finding its Riemann function, and setting up a simple formula. This latter equation is; in the isentropic case ($\gamma \neq 1$),

$$x_{rs} + \frac{\lambda}{r+s}\left[\left(\frac{u+a}{u-a}\right) x_r + \left(\frac{u-a}{u+a}\right) x_s\right] = 0,$$

and the Riemann function is, see [5],

$$v(r, s; \xi, \eta) = \left(\frac{r+s}{\xi+\eta}\right)^{\lambda} \frac{1}{(\alpha r - \beta s)(\beta r - \alpha s)} \left\{ (\alpha\xi - \beta s)(\beta r - \alpha\eta) F(1-\lambda, \lambda, 1; z) \right.$$
$$\left. - \frac{\alpha\beta\lambda(\lambda-1) z (\xi - r)(\eta - s)}{2} F(2-\lambda, 1+\lambda, 3; z) \right\}$$

where $\alpha = (\gamma+1)/2$, $\beta = (3-\gamma)/2$, and the notation is as before. The isothermal case follows similar lines.

§ 15. Statement of the mathematical theorems

The treatment throughout has been a formal one and but vague reference has been made to the underlying assumptions. In addition there have been no precise definitions of the terms used. This approach has been followed in order to keep the ideas behind this extension and application of Riemann's method free from unnecessary detail. It will be found that the following statement of results obtained in [8] is sufficiently general to justify all that has gone before.

We return to the general operator $L(w)$, defined in § 2. Then under the assumptions that a, b, c as well as a_r, b_s are continuous functions of (r, s) in any given characteristic rectangle R (taken as closed), it may be shown, see [6, 8], that not only $v(r, s ; \xi, \eta)$ exists and is continuous in all its variables in R, but also that its derivatives v_r, v_s, v_ξ, v_η ; $v_{\xi r}$, $v_{\xi s}$, $v_{\eta r}$, $v_{\eta s}$, $v_{\xi\eta}$; $v_{\xi\eta r}$, $v_{\xi\eta s}$ exist and are continuous in R. This fact is sufficient to prove the two theorems which follow (*).

Definition : A function $v(r, s)$ will be called a *regular solution* of $L(w) = 0$ in R if is continuous and has continuous derivatives of the first order at every interior point of R, except possibly at the points of a finite number of segments of characteristics, across which the normal derivative of w alone may have a (finite or infinite) discontinuity, and satisfies $L(w) = 0$ at all other interior points.

Theorem I : Let C be an initial curve which is intersected in at most one point by any characteristic of either family and which has a tangent which turns continuously except at a finite number of points of C. Let the initial values (**) $f(\sigma)$, $g(\sigma)$, $h(\sigma)$ of w, w_r, w_s be continuous functions of arc length σ, except at a finite number of points of C, where g, h may have finite or infinite discontinuities. Then there exists a unique regular solution of $L(w) = 0$ in R, the rectangle having the end points of C for opposite vertices, which takes the prescribed initial value f, g, h on C, provided certain integrals of these values converge uniformly (***).

The solution is given by Riemann's formula (7). The restriction concerning the integrals is clearly satisfied in the application discussed previously.

(*) Rademacher uses this fact (without proof) in the verification of Riemann's formula (7) given in [3], though his proof is not quite correct.

(**) These values being obtained from prescribed values of w and $\frac{\partial w}{\partial n}$.

(***) These integrals are the one in (7) together with those obtained by differentiating the integrand with respect to ξ alone, η alone, and ξ, η successively.

Suppose now that the initial curve is such that r, s as functions of $\sigma\,(\sigma_0 < \sigma < \sigma_{p+1})$ have a finite number (*) p of extrema

$$\sigma = \sigma_1,\ \sigma_2, \ldots, \sigma_p \qquad (\sigma_1 < \sigma_2 < \ldots < \sigma_p),$$

the arcs between these extrema having a piecewise continuous tangent direction and no segment coincident with a segment of characteristic.

Definition : Let (x_n, y_n) be the coordinates of the point $\sigma = \sigma_n$ of the initial curve C, and R_{mn} $(m,\ n = 1, 2, \ldots, p+1)$ the rectangle enclosed by the characteristics $x = x_{m-1},\ x_m\,;\ y = y_{n-1},\ y_n$. Then the *sheeted surface* R associated with C is that surface formed when R_{mn} is joined to $R_{m+1,n}$ along their common side $x = x_m$ and to $R_{m,n+1}$ along their common side $y = y_n$, the arc (σ_{n-1}, σ_n) of C being considered to lie in the same sheet as R_{nn}.

Definition : The sheeted surface R may be represented diagrammatically by an array of $(p+1)^2$ rectangles, such that the rectangle in the m^{th} column and n^{th} row is congruent coordinate-wise with R_{mn} and is joined to its neighbors in a similar manner. This diagram, which is simply-connected, will be called the *unfolded diagram* associated with C.

The term « continuity » which occurs in the definition of a regular solution, is now defined with respect to position on the sheeted surface (or in the unfolded diagram). The values of w_ξ, w_η, $w_{\xi\eta}$ on a fold of the surface R are defined as the common limit of these functions, respectively, as the fold is approached on the two sheets having the fold in common, provided these limits exist.

Theorem II : Let C be an initial curve which has a finite number of extrema in the coordinates r, s and which has a tangent which turns continuously except at a finite number of points of C. Let the initial values $f(\sigma)$, $g(\sigma)$, $h(\sigma)$ of w, w_r, w_s be continuous functions of arc length σ, except at a finite number of points of C, where g and h may have finite or infinite discontinuities. Then there exists a unique regular solution of $L(w) = 0$ in R, the sheeted surface defined above, which takes the prescribed initial values f, g, h on C, provided certain integrals of these initial values converge uniformly.

The integrals in question are those mentioned in Theorem *I*, and again the solution is given by Riemann's formula. Theorem *II* covers the problem of motion in a closed tube, since that problem may be considered as a sequence of problems, in which the initial conditions are prescribed on the finite segment $|x| < (2n+1)\,l$, $n = 0, 1, 2, 3, \ldots$, of the x-axis.

(*) The points may be extrema of r, s simultaneously.