Generalized hodograph method for the case where 2 Riemann invariant vary while the other(s) is (are) constant

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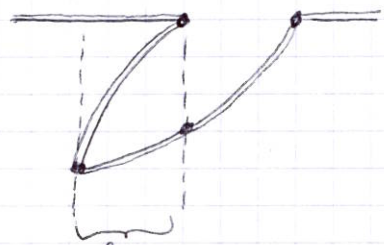
one considers KdV (3 Riemann invariants) or NLS (4 Riemann invariants) for a DSW in which 2 R.I. vary while the other(s) is(are) constant

For example for KdV this could correspond to:



this region

let's denote by 1 and 2 the indices of the varying R.I.s.

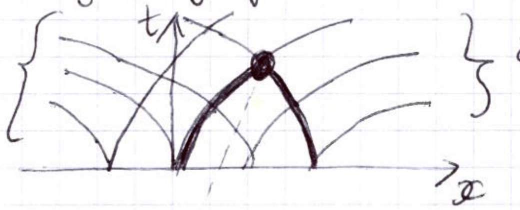
one has:

$$(1)\quad \boxed{\partial_t \lambda_{\binom{1}{2}} + V_{\binom{1}{2}}\, \partial_x \lambda_{\binom{1}{2}} = 0}$$

where V_i is a function of λ_1 and λ_2 and also of the other(s) constant λ_3^0 (and λ_4^0).

if one of the $\lambda_{\binom{1}{2}}$ is constant (say λ_1) this defines a family of characteristics in the (x,t) plane. along each characteristic λ_1 is constant. Similarly one has an other family of characteristics along which $\lambda_2 = C^{st}$.

$\mathcal{E}_2$ = family of characteristics where $\lambda_2 = C^{st}$ along each curve



$\mathcal{E}_1$ = family of characteristics where $\lambda_1 = C^{st}$ along each curve.

here, for a given (x,t) one can express (x,t) as a function of λ_1 and λ_2.

So, in principle one can write:

$$(2)\quad \begin{cases} x = x(\lambda_1, \lambda_2) \\ t = t(\lambda_1, \lambda_2) \end{cases}$$

let's transform eqs (1) into eqs. for x and t. Suppose $\phi(x,t) = \phi(x(\lambda_1,\lambda_2), t(\lambda_1,\lambda_2))$ then one has:

$$\begin{cases} \partial_1 \phi = \partial_x \phi\, \partial_1 x + \partial_t \phi\, \partial_1 t \\ \partial_2 \phi = \partial_x \phi\, \partial_2 x + \partial_t \phi\, \partial_2 t \end{cases} \qquad \text{where } \partial_i = \frac{\partial}{\partial \lambda_i}$$

this is a 2×2 system for $\partial_x \phi$ and $\partial_t \phi$. The determinant is:

$$\det = \partial_1 x \cdot \partial_2 t - \partial_2 x \cdot \partial_1 t \quad \text{and} \quad \begin{cases} \partial_x \phi = (\partial_1 \phi \cdot \partial_2 t - \partial_2 \phi \cdot \partial_1 t)/\det \\ \partial_t \phi = (\partial_2 \phi \cdot \partial_1 x - \partial_1 \phi \cdot \partial_2 x)/\det \end{cases}$$

considering the particular cases $\phi = \lambda_1$ and $\phi = \lambda_2$ one gets:

$$\begin{cases} \partial_x \lambda_1 = \partial_2 t/\det \\ \partial_t \lambda_1 = -\partial_2 x/\det \end{cases} \quad \text{and} \quad \begin{cases} \partial_x \lambda_2 = -\partial_1 t/\det \\ \partial_t \lambda_2 = \partial_1 x/\det \end{cases}$$

inserting this back into (1), the det goes away, and one gets:

$$(3)\begin{cases} -\partial_2 x + V_1\,\partial_2 t = 0 \\ +\partial_1 x - V_2\,\partial_1 t = 0 \end{cases}$$

where V_1 and V_2 are known functions of λ_1 and λ_2. Note that this is now a linear system of PDEs for the unknown functions x and t.

Let's recall that for a simple wave (when, say, $\lambda_2 = \lambda_2^0$) the formal solution of $\partial_t \lambda_1 + V_1(\lambda_1, \lambda_2^0)\,\partial_x \lambda_1 = 0$ was seeked under the implicit form: $x - V_1(\lambda_1, 0)\,t = W_1(\lambda_1)$

So, here, it is natural to look for a solution under the form

$$(4)\begin{cases} x - V_1(\lambda_1,\lambda_2)\,t = W_1(\lambda_1,\lambda_2) & (a) \\ x - V_2(\lambda_1,\lambda_2)\,t = W_2(\lambda_1,\lambda_2) & (b) \end{cases}$$

... by the way this ansatz could have been guessed directly after eq. (1). No need of the intermediate steps between (1) and (4). But these steps will become useful in a few lines.

let's compute $\partial_2(4a)$ = $\underbrace{\partial_2 x - V_1\,\partial_2 t}_{0 \text{ from } (3a)} - t\,\partial_2 V_1 = \partial_2 W_1$

thus $t = -\dfrac{\partial_2 W_1}{\partial_2 V_1}$

one has also (from (4a)-(4b)): $t = -\dfrac{W_1 - W_2}{V_1 - V_2}$

comparing the two equations one gets:

$$\boxed{\frac{\partial_2 W_1}{W_1 - W_2} = \frac{\partial_2 V_1}{V_1 - V_2} \text{ and also, of course, } \frac{\partial_1 V_2}{V_1 - V_2} = \frac{\partial_1 W_2}{W_1 - W_2}} \quad (5)$$

(obtained by computing $\partial_1(4b)$)

← Tsarev eqs.

Particular case of a (dispersionless) polytropic gas with $p = \rho^\gamma$. In this case the Riemann velocities associated with the dispersionless R.I. are:

$V_1 = 1/4\left((\gamma+1)\lambda_1 + (3-\gamma)\lambda_2\right)$ and $V_2 = 1/4\left((3-\gamma)\lambda_1 + (\gamma+1)\lambda_2\right)$

here $\partial_1 V_2 = \dfrac{3-\gamma}{4} = \partial_2 V_1$ and $\dfrac{\partial_1 V_2}{V_1 - V_2} = \dfrac{\partial_2 V_1}{V_1 - V_2} = \underbrace{\dfrac{3-\gamma}{2(\gamma-1)}}_{\text{let's denote this constant by } n} \cdot \dfrac{1}{\lambda_1 - \lambda_2}$

from (5) this shows that $\partial_2 W_1 = \partial_1 W_2$

It is then legitimate to assume that $W_{\binom{1}{2}}$ derive from a "potential" χ, that is that:

$$W_{\binom{1}{2}} = \partial_{\binom{1}{2}}\chi$$

substituting this in one of the eqs. (5) one gets (using the above result):

$$\frac{\partial_{12}\chi}{W_1 - W_2} = \frac{n}{\lambda_1 - \lambda_2} \quad \text{or,} \quad \boxed{\partial_{12}\chi - \frac{n}{\lambda_1 - \lambda_2}\left(\partial_1\chi - \partial_2\chi\right) = 0} \quad (6)$$

this is Euler-Poisson-Darboux eq. it is linear, and has been studied in great details by mathematicians.

if $n = 0$ (case $\gamma = 3$) the solution is easy $\chi = F(\lambda_1) + G(\lambda_2)$, F and G arbitrary

if $n = 1$ (case $\gamma = 5/3$) one can show that the solution should be of the form:

$$\chi = \frac{F(\lambda_1) + G(\lambda_2)}{\lambda_1 - \lambda_2}$$

• end of particular case

I now return to the more difficult case of Whitham-Riemann eqs. where 2 R.I. vary.
Let's first obtain a general formula very useful in the following: from the "phase conservation law" $k_t + \omega_x = 0$ one gets, assuming that $k = k(\lambda_i(x,t))$ and $\omega = V \times k$ = $\sum_{i=1}^{3(4)} \left[\partial_i k \, \partial_t \lambda_i + (k \partial_i V + V \partial_i k) \partial_x \lambda_i \right] = 0$

thus $\sum_i \left[\partial_t \lambda_i + \left(V + \frac{k}{\partial_i k} \partial_i V\right) \partial_x \lambda_i \right] = 0$ where (I forgot to say) V is the phase velocity. In all the systems (KdV or NLS) one has

$$\boxed{V = \alpha \sum_j \lambda_j = \alpha s_1} \quad (7)$$

$\alpha = c^{st} = 1/2$ or $1/3$, depending on the model (KdV or NLS) and on the convention.

in this eq. the terms for different i should cancel separately, because it often occurs that one of the λ_i depends on (x,t) while the others are constant.
Hence one has $\partial_t \lambda_i + \left(V + \frac{k}{\partial_i k} \partial_i V\right) \partial_x \lambda_i = 0$
and comparing with the standard Whitham equations one has:

$$\boxed{V_i = \left(1 + \frac{k}{\partial_i k} \partial_i\right) V \underset{\uparrow}{=} \left(1 - \frac{L}{\partial_i L} \partial_i\right) V \underset{\uparrow}{=} V - \alpha L / \partial_i L} \quad (8)$$

using $L = 2\pi/k$
L is the period of the nonlinear wave

using (7) which implies that $\partial_i V = \alpha$

one wants to solve the Tsarev equations (5). From (8) it is natural to look for solutions under the form:

$$\left.\begin{aligned} W_i &= \left(1 - \frac{L}{\partial_i L} \partial_i\right) W \\ W_i &= W + \left(\frac{V_i}{\alpha} - s_1\right) \partial_i W \end{aligned}\right\} \quad (9)$$

which, from (8) can be written under the form (since $-L/\partial_i L = (V_i - V)/\alpha$)

one wants to insert the ansatz (9) into Tsarev eqs (5). Before that let's do some intermediate computations:

$* \; W_i - W_j \overset{\text{(from 9)}}{=} \left(\frac{V_i}{\alpha} - s_1\right) \partial_i W - \left(\frac{V_j}{\alpha} - s_1\right) \partial_j W = \left(\frac{1}{\alpha} V_j - s_1\right)(\partial_i W - \partial_j W) + \frac{1}{\alpha}(V_i - V_j) \partial_i W$

$* \;$ again from (9) one also gets: $\partial_j W_i = \partial_j W + \left(\frac{1}{\alpha} \partial_j V_i - 1\right) \partial_i W + \left(\frac{V_i}{\alpha} - s_1\right) \partial_{ij} W$
$= \partial_j W - \partial_i W + \frac{1}{\alpha} \partial_j V_i \, \partial_i W + \left(\frac{V_i}{\alpha} - s_1\right) \partial_{ij} W$

inserting these two results into Tsarev's eq. (5) (i.e. assuming the existence of W which permits to compute W_i according to (9)) one gets, after writing Tsarev's equations under the generic form:

$$\boxed{\frac{\partial_j W_i}{W_i - W_j} = \frac{\partial_j V_i}{V_i - V_j} \quad \text{where } i \neq j} \quad (10)$$

this yields: $\partial_j W_i = (W_i - W_j) \frac{\partial_j V_i}{V_i - V_j}$ or, using the above $(*)$ formulae:

$$\partial_j W - \partial_i W + \cancel{\frac{1}{\alpha} \partial_j V_i \, \partial_i W} + \left(\frac{V_i}{\alpha} - s_1\right) \partial_{ij} W = \frac{\partial_j V_i}{V_i - V_j} \left\{ \left(\frac{V_j}{\alpha} - s_1\right)(\partial_i W - \partial_j W) + \frac{V_i - V_j}{\alpha} \partial_i W \right\}$$
$$= \frac{V_j/\alpha - s_1}{V_i - V_j} \partial_j V_i (\partial_i W - \partial_j W) + \cancel{\frac{1}{\alpha} \partial_j V_i \, \partial_i W}$$

hence: $\partial_j W - \partial_i W + \left(\frac{V_i}{\alpha} - s_1\right)\partial_{ij} W = \left(\frac{1}{\alpha}V_j - s_1\right)(\partial_i W - \partial_j W)\frac{\partial_j V_i}{V_i - V_j}$ (11)

▨ before continuing, let's derive an important intermediate result.
let's denote by P the "unknown polynomial":

$$* \quad P = \prod_{j=1}^{3(4)} (\lambda - \lambda_j) \quad (12)$$

it is clear that: $\partial_i \frac{1}{\sqrt{-P}} = \frac{1}{2\sqrt{-P}} \times \frac{1}{\lambda - \lambda_i}$
and, for $j \neq i$

$$\partial_{ij} \frac{1}{\sqrt{-P}} = \frac{1}{4\sqrt{-P}} \times \frac{1}{(\lambda-\lambda_i)(\lambda-\lambda_j)}$$

hence

$$\partial_i \frac{1}{\sqrt{-P}} - \partial_j \frac{1}{\sqrt{-P}} = \frac{1}{2\sqrt{-P}} \times \left(\frac{1}{\lambda-\lambda_i} - \frac{1}{\lambda-\lambda_j}\right) = \frac{1}{2\sqrt{-P}}\frac{\lambda_i - \lambda_j}{(\lambda-\lambda_i)(\lambda-\lambda_j)}$$

and one can write

$$\partial_{ij}\frac{1}{\sqrt{-P}} = \frac{1}{2(\lambda_i - \lambda_j)}\left(\partial_i \frac{1}{\sqrt{-P}} - \partial_j \frac{1}{\sqrt{-P}}\right) \quad (13)$$

* then, one has to admit the following result which arises naturally in Toda's method for finding the single phase (nonlinear) periodic solutions: there exists a field $\mu(\xi = x - Vt)$ which has the same periodicity that the periodic solutions of KdV or NLS and which, for single phase solutions is solution of:

$$(\mu_\xi)^2 + P(\mu) = 0 \quad \text{where P is given by (12)}$$

hence $\mu_\xi = \sqrt{-P(\mu)}$ (14)

one can also show that, during the oscillations of the physical variables μ turns in the complex plane around 2 of the Whitham R.I
As a result on gets the following expression for the wavelength:

$$L = \oint \frac{d\mu}{\sqrt{-P(\mu)}} \quad (15)$$

applying this integration on formula (13) yields the useful formula:

$$\boxed{\partial_{ij} L = \frac{\partial_i L - \partial_j L}{2(\lambda_i - \lambda_j)}} \quad (16)$$

▨ as an other important intermediate step we use this result and eq.(8) to rewrite the term $\frac{\partial_j V_i}{V_i - V_j}$ appearing in (10) and, more important, in (11).

from (8) one gets $V_i - V_j = \alpha\left(\frac{L}{\partial_j L} - \frac{L}{\partial_i L}\right) = \alpha L \frac{\partial_i L - \partial_j L}{\partial_i L \cdot \partial_j L}$

and also $\partial_j V_i = \alpha - \alpha\frac{\partial_j L}{\partial_i L} + \alpha\frac{L\,\partial_{ij}L}{(\partial_i L)^2} = \frac{\alpha}{(\partial_i L)^2}\left[(\partial_i L)^2 - \partial_i L \cdot \partial_j L + L\,\partial_{ij} L\right]$

thus

$$\frac{\partial_j V_i}{V_i - V_j} = \frac{1}{L}\frac{\partial_i L \cdot \partial_j L}{\partial_i L - \partial_j L}\cdot\frac{1}{(\partial_i L)^2}\left[(\partial_i L)^2 - \partial_i L\cdot\partial_j L + L\,\partial_{ij}L\right]$$

$$= \frac{\partial_j L}{L} + \frac{\partial_j L}{\partial_i L}\cdot\frac{\partial_{ij} L}{\partial_i L - \partial_j L} \underset{\text{using (16)}}{=} \frac{\partial_j L}{L} + \frac{\partial_j L}{\partial_i L}\frac{1}{2(\lambda_i - \lambda_j)} \quad (17)$$

using eq. (8) one can also rewrite (17) (since from (8) $\frac{L}{\partial_i L} = s_1 - V_i/\alpha$) under the form =

$$\frac{\partial_j V_i}{V_i - V_j} = \frac{1}{s_1 - V_j/\alpha} \left(1 + \frac{s_1 - V_i/\alpha}{2(\lambda_i - \lambda_j)}\right) \quad (18)$$

☑ this is the end = the final step consists in re-inserting (18) into (11). this yields =

$$\cancel{\partial_j W - \partial_i W} + \left(\frac{V_i}{\alpha} - s_1\right) \partial_{ij} W = (\partial_j W - \partial_i W)\left(\cancel{1} + \frac{s_1 - V_i/\alpha}{2(\lambda_i - \lambda_j)}\right)$$

thus:

$$\boxed{\underset{(i \neq j)}{\partial_{ij} W} = (\partial_i W - \partial_j W) \times \frac{1}{2(\lambda_i - \lambda_j)}} \quad (19)$$

this is Euler Darboux Poisson eq., as eq. (6), with here $n = 1/2$

Important remark = one has only assumed here that $i \neq j$. But no hypothesis has been made on the constancy of the other λ's. Hence for each (i,j) with $i \neq j$ one gets an EDP equation. This gives for KdV 3 EDP eqs, and for NLS 6 EDP eqs. For the specific case we are interested in, only 2 of the λ's vary, say λ_1 and λ_2, and one has a single EDP equation.