Acma 490 Exercises

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2 Introduction

2.1. Verify the portfolio values for the example in Section 2.1.

Solution:

We purchase N_1 bonds with time to maturity 1, N_2 bonds with time to maturity 4 and N_3 bonds with time to maturity 10. N_1 can be set arbitrarily, in this case to $N_1 = 707.891$.

We want the time 0 value of our portfolio to be zero, i.e.

$$N_1 e^{-0.06(1)} + N_2 e^{-0.06(4)} + N_3 e^{-0.06(10)} = 0$$

Also, we want the duration of our portfolio (which is proportional to the first derivative with respect to the interest rate r) to equal 0, i.e.

$$\begin{aligned} \frac{d}{dr}\left[N_1 e^{-r(1)} + N_2 e^{-r(4)} + N_3 e^{-r(10)}\right|_{r_0} &= 0 \\ \left[N_1 e^{-r(1)} + 4N_2 e^{-r(4)} + 10N_3 e^{-0.06(10)}\right|_{r_0} &= 0 \\ N_1 e^{-0.06(1)} + 4N_2 e^{-0.06(4)} + 10N_3 e^{-0.06(10)} &= 0 \end{aligned}$$

Setting $N_1 = 707.891$ gives the following two equations

$$\begin{aligned} N_2 e^{-0.06(4)} + N_3 e^{-0.06(10)} &= -707.891 e^{-0.06(1)} \\ 4N_2 e^{-0.06(4)} + 10N_3 e^{-0.06(10)} &= -707.891 e^{-0.06(1)} \end{aligned}$$

Subtracting four times the first second into the second equation gives

$$6N_3 e^{-0.06(10)} = 3(707.891)e^{-0.06(1)} \implies N_3 = 607.373$$

and $N_2 = \dfrac{-707.891 e^{-0.06(1)} - 607.373 e^{-0.06(10)}}{e^{-0.06(4)}} = -1271.25$. The portfolio value at time 1 is

- Up case: $707.891 - 1271e^{-0.09(3)} + 607.373 + e^{-0.09(9)} = 7.64$
- Down case: $707.891 - 1271.25e^{-0.04(3)} + 607.373e^{-0.04(9)} = 4.14$

2.2. Consider a par 5-year bond with 4% annual coupons and a redemption value of \$100 priced to yield 5%.

(a) Calculate the price of the bond.

(b) Calculate the duration of the bond.

Solution:

(a) The price is

$$P = 100(0.04)a_{\overline{5}|5\%} + 100v^5_{5\%} = 95.67$$

(b) The first derivative of the price with respect to the interest rate is

$$\frac{dP}{di} = \frac{d}{di}\left[4a_{\overline{5}|} + 100v^5\right]$$

Note that

$$\begin{aligned}
\frac{d}{di}v^n &= -n(1+i)^{-n-1} \\
\frac{d}{di}a_{\overline{n}|} &= \frac{d}{di}\frac{1-v^n}{i} \\
&= \frac{1}{i}n(1+i)^{-n-1} - \frac{1}{i^2}[1-(1+i)^{-n}]
\end{aligned}$$

Applying these formulas gives

$$\begin{aligned}
\left.\frac{dP}{di}\right|_{i=5\%} &= 4\left[\frac{1}{0.05}5(1.05^{-6} - \frac{1}{0.05^2}[1-(1.05)^{-5}]\right] - 100(5)(1.05)^{-6} \\
&= -420.98
\end{aligned}$$

The (Macaulay) duration is

$$\frac{\frac{dP}{di}(1+i)}{P} = \frac{-420.98(1.05)}{95.67} = 4.62$$

2.3. The continuous spot rates for different terms T are:

T	1	2	3
$R(0,T)$	6%	6.5%	7%

Find the forward rates $F(0,1,2)$, $F(0,1,3)$ and $F(0,2,3)$.

Solution:

The bond prices are calculated using $P(0,T) = \exp(-TR(0,T))$ as shown below:

T	1	2	3
$P(0,T)$	0.9417645	0.8780954	0.8105842

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The forward prices can be calculated using $F(0,T,S) = (S-T)^{-1} log[P(0,T)/P(0,S)]$ for $T < S$, so

$$F(0,1,2) = (2-1)^{-1} log[0.9417645/0.8780954] = 0.07$$
$$F(0,1,3) = (3-1)^{-1} log[0.9417645/0.8105842] = 0.075$$
$$F(0,2,3) = (3-2)^{-1} log[0.8780954/0.8105842] = 0.08$$

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3 Time Series

3.1. Find $E[X_t X_{t-k}|X_0]$ for an AR(1) model.

Solution:

The conditional AR(1) model can be written as

$$
\begin{aligned}
X_t &= \sum_{j=0}^{t-1} G_j a_{t-j} + G_t a_0 \\
&= \sum_{j=0}^{t-1} \phi_1^j a_{t-j} + \phi_1^t a_0
\end{aligned}
$$

so we have that

$$
\begin{aligned}
E[X_t X_{t-k}|X_0] &= E\left[\left(\sum_{j=0}^{t-1} \phi_1^j a_{t-j} + \phi_1^t a_0\right)\left(\sum_{i=0}^{t-k-1} \phi_1^j a_{t-k-i} + \phi_1^{t-k} a_0\right)\right] \\
&= E\left[\sum_{j=0}^{t-1} \phi_1^j a_{t-j} \sum_{i=0}^{t-k-i} a_{t-k-i}\right] + E\left[\phi_1^{2t-k} a_0 a_0\right] \\
&= \sigma_a^2 \left(\phi_1^k + \phi_1^{k+2} + \cdots + \phi_1^{k+2(t-k-1)}\right) + E(a_0^2)\phi_1^{2t-k} \\
&= \sigma_a^2 \phi_1^k \left(\frac{1-\phi_1^{2(t-k)}}{1-\phi_1^2}\right) + E(a_0^2)\phi_1^{2t-k}
\end{aligned}
$$

where $a_0 = X_0$.

3.2. Find the conditional expectation, variance, and autocovariance function at lag k given X_0 and X_{-1} for an AR(2) model.

Solution:

The conditional AR(2) model can be written as

$$
\begin{aligned}
X_t &= \sum_{j=0}^{t-1} G_j a_{t-j} + G_t a_0 + G_{t+1} a_{-1} \\
&= \sum_{j=0}^{t-1} \left(g_1 \lambda_1^j + g_2 \lambda_2^j\right) a_{t-j} + G_t a_0 + G_{t+1} a_{-1}
\end{aligned}
$$

where $g_1 = \dfrac{\lambda_1}{\lambda_1 - \lambda}$ and $g_2 = \dfrac{\lambda_2}{\lambda_2 - \lambda_1}$. The expected value is

$$\begin{aligned} E(X_t) &= G_t E(a_0) + G_{t+1} E(a_{-1}) \\ &= \left[\frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_1^t + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_2^t \right] E(a_0) + \left[\frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_1^{t+1} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_2^{t+1} \right] E(a_{-1}) \end{aligned}$$

where $a_0 = X_0 - \phi_1 X_{-1}$ and $a_{-1} = X_{-1}$.

For the autocovariance function, we have that

$$\begin{aligned} Cov(X_t, X_{t-k}) &= Cov\left[\sum_{j=0}^{t-1} \left(g_1 \lambda_1^j + g_2 \lambda_2^j \right) a_{t-j}, \sum_{i=0}^{t-k-1} (g_1 \lambda_1^i + g_2 \lambda_2^i) a_{t-k-i} \right] \\ &= \sigma_a^2 \Big[(g_1 \lambda_1^k + g_2 \lambda_2^k)(g_1 + g_2) + (g_1 \lambda_1^{k+1} + g_2 \lambda_2^{k+1})(g_1 \lambda_1 + g_2 \lambda_2) + \ldots + \\ &\qquad (g_1 \lambda_1^{k+(t-k-1)} + g_2 \lambda_2^{k+(t-k-1)})(g_1 \lambda_1^{t-k-1} + g_2 \lambda_2^{t-k-1}) \Big] \\ &= \sigma_a^2 \Big[g_1^2 (\lambda_1^k + \lambda_1^{k+2} + \cdots + \lambda_1^{k+2(t-k-1)}) + \\ &\qquad g_1 g_2 \left[(\lambda_1^k + \lambda_1^{k+1} \lambda_2 + \cdots + \lambda_1^{k+(t-k-1)} \lambda_2^{t-k-1}) + (\lambda_2^k + \lambda_2^{k+1} \lambda_1 + \cdots + \lambda_2^{k+(t-k-1)} \lambda_1^{t-k-1}) \right] + \\ &\qquad g_2^2 (\lambda_2^k + \lambda_2^{k+2} + \cdots + \lambda_2^{k+2(t-k-1)}) \Big] \\ &= \sigma_a^2 \left[g_1^2 \lambda_1^k \left(\frac{1 - \lambda_1^{2(t-k)}}{1 - \lambda_1^2} \right) + g_1 g_2 (\lambda_1^k + \lambda_2^k) \left(\frac{1 - (\lambda_1 \lambda_2)^{t-k}}{1 - \lambda_1 \lambda_2} \right) + g_2^2 \lambda_2^k \left(\frac{1 - \lambda_2^{2(t-k)}}{1 - \lambda_2^2} \right) \right] \end{aligned}$$

Note that the variance can be found by setting $k = 0$ in $Cov(X_t, X_{t-k})$.

3.3. Find the expected value and standard deviation of a 5-year annuity-immediate if the force of interest per period (as defined in example 3 above) is modeled by a conditional AR(1) model of the form: $(\delta_t - \delta) = \phi_1(\delta_{t-1} - \delta) + a_t$.

Solution:

The R code is below.

```
> library(stocins) # see github.com/nathanesau
> ar1model = iratemodel(params = list(delta = 0.05, delta0 = 0.08,
+   phi1 = 0.90, sigma = 0.01), "ar1")
> annev = ann.ev(5, ar1model)
> annvar = ann.var(5, ar1model)
```

Using this code gives $E[a_{\overline{5}|}] = 4.00004844$ and $\sqrt{Var[a_{\overline{5}|}]} = 0.08162954$.

4 Stochastic Differential Equations

4.1. Derive $cov(Y_s, Y_t)$ when X_t is an Ornstein-Uhlenbeck process, see (4.39, 4.40).

Solution:

The covariance of Y_s, Y_t for an OU process is

$$Cov(Y_s, Y_t) = Var(Y_0) + \int_0^s \int_0^r Cov(X_r, X_u) dudr + \int_0^s \int_r^t Cov(X_r, X_u) dudr \qquad \text{for } s \leq t$$

The first double integral is

$$\begin{aligned}
&\int_0^s \int_0^r Cov(X_r, X_u) dudr \\
&= \int_0^s \int_0^r e^{-\alpha(u+r)} \left[Var(c) + \frac{e^{2\alpha u} - 1}{2\alpha} \sigma^2 \right] dudr \\
&= Var(c) \int_0^s \int_0^r e^{-\alpha(u+r)} dudr + \frac{\sigma^2}{2\alpha} \int_0^s \int_0^r (e^{-\alpha(u-r)} - e^{\alpha(u+r)} du) dr \\
&= \int_0^s Var(c) \left[\frac{e^{-\alpha r} - e^{-\alpha(2r)}}{\alpha} \right] + \frac{\sigma^2}{2\alpha} \left[\left(\frac{1 - e^{-\alpha r}}{\alpha} \right) - \left(\frac{e^{-\alpha r} - e^{-\alpha(2r)}}{\alpha} \right) \right] dr \\
&= Var(c) \left[\frac{(1 - e^{-\alpha r})}{\alpha^2} - \frac{(1 - e^{-2\alpha s})}{2\alpha^2} \right] + \frac{\sigma^2}{2\alpha} \left[\frac{s}{\alpha} - \frac{(1 - e^{-\alpha s})}{\alpha^2} - \frac{(1 - e^{-\alpha s})}{\alpha^2} + \frac{(1 - e^{-2\alpha s})}{2\alpha^2} \right] \\
&= Var(c) \left[\frac{(1 - e^{-\alpha s})}{\alpha^2} - \frac{(1 - e^{-2\alpha s}}{2\alpha^2} \right] + \frac{\sigma^2}{2\alpha} \left[\frac{s}{\alpha} - \frac{2(1 - e^{-\alpha s})}{\alpha^2} + \frac{(1 - e^{-2\alpha s})}{2\alpha^2} \right]
\end{aligned}$$

The second double integral is

$$\begin{aligned}
&\int_0^s \int_r^t Cov(X_r, X_u) dudr \\
&= \int_0^s \int_r^t e^{-\alpha(u+r)} \left[Var(c) + \frac{(e^{2\alpha r} - 1)}{2\alpha} \sigma^2 \right] dudr \\
&= \int_0^s Var(c) \left[\frac{e^{-2\alpha r} - e^{-\alpha(r+t)}}{\alpha} \right] dr + \frac{\sigma^2}{2\alpha} \left[\int_0^s \int_r^t \left(e^{\alpha(r-u)} - e^{-\alpha(u+r)} \right) dudr \right] \\
&= \int_0^s Var(c) \left[\frac{e^{-2\alpha r} - e^{-\alpha(r+t)}}{\alpha} \right] dr + \frac{\sigma^2}{2\alpha} \left[\int_0^s \frac{1 - e^{-\alpha(t-r)}}{\alpha} - \left(\frac{e^{-2\alpha r} - e^{-\alpha(r+t)}}{\alpha} \right) ds \right] \\
&= Var(c) \left[\frac{1 - e^{-2\alpha s}}{2\alpha^2} - \frac{(e^{-\alpha t} - e^{-\alpha(r+t)})}{\alpha^2} \right] + \frac{\sigma^2}{2\alpha} \left[\frac{e^{-\alpha(t-s)} - e^{-\alpha t} + sa}{\alpha^2} - \frac{(1 - e^{-2\alpha s})}{2\alpha^2} + \frac{(e^{-\alpha t} - e^{-\alpha(t+s)})}{\alpha^2} \right]
\end{aligned}$$

So the covariance is

$$
\begin{aligned}
&Cov(Y_s, Y_t) \\
&= Var(Y_0) + Var(c)\left[\frac{(1-e^{-\alpha s})}{\alpha^2} - \cancel{\frac{(1-e^{-2\alpha s})}{2\alpha^2}} + \cancel{\frac{(1-e^{-2\alpha s})}{2\alpha^2}} - \frac{(e^{-\alpha t} - e^{-\alpha(s+t)})}{\alpha^2}\right] + \\
&\frac{\sigma^2}{2\alpha}\left[\frac{s}{\alpha} - \frac{2(1-e^{-\alpha s})}{\alpha^2} + \cancel{\frac{(1-e^{-2\alpha s})}{2\alpha^2}} + \frac{2e^{-\alpha t}}{\alpha^2} - \frac{e^{-\alpha(t-s)}}{\alpha^2} - \frac{e^{-\alpha(t+s)}}{\alpha^2} - \cancel{\frac{(1-e^{-2\alpha s})}{\alpha^2}} + \frac{s\alpha}{\alpha^2}\right] \\
&= Var(Y_0) + \left[\frac{1-e^{-\alpha s} - e^{-\alpha t} + e^{-\alpha(s+t)}}{\alpha^2}\right] Var(c) + \frac{\sigma^2}{\alpha^2}s + \frac{\sigma^2}{2\alpha^2}\left(-2 + 2e^{\alpha t} + 2e^{-\alpha s} - e^{-\alpha(t-s)} - e^{-\alpha(t+s)}\right)
\end{aligned}
$$

4.2. Obtain (4.42) and use it to find $E\begin{pmatrix} X_t \\ Y_t \end{pmatrix}$ and check your result with (4.35) and (4.38). Find $cov\left(\begin{pmatrix} X_s \\ Y_s \end{pmatrix}, \begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right)$ and verify that the element (2,2) is the same as (4.40).

Solution:

For an OU process define A as

$$A = \begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of A are found by setting the determinant of $A - \lambda I = 0$, i.e.

$$det\begin{pmatrix} -\alpha - \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0 \implies \lambda(\alpha + \lambda) = 0$$

This gives eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -\alpha$. The first eigenvector v_1 is found by setting $(A - \lambda_1 I)v_1 = 0$, i.e.

$$\begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = 0 \implies \begin{aligned} \alpha v_{1,1} &= 0 \\ v_{1,1} &= 0 \end{aligned}$$

so $v_1 = (0, 1)$. Similarly we can find v_2 by setting $(A - \lambda_2 I)v_2 = 0$, i.e.

$$\begin{pmatrix} 0 & 0 \\ 1 & \alpha \end{pmatrix}\begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = 0 \implies v_{2,1} + \alpha v_{2,2} = 0$$

so $v_2 = (-\alpha, 1)$. Thus, if P is matrix with column 1 equal to v_1 and column 2 equal to v_2 and D is a matrix with λ_1 and λ_2 on the diagonal than the diagonalization of A is

$$
\begin{aligned}
A &= PDP^{-1} \\
&= \begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 1 \\ -\alpha^{-1} & 0 \end{pmatrix}
\end{aligned}
$$

Furthermore, the matrix exponentiation $\Phi(t) = e^{tA}$ is

$$\begin{aligned}
e^{tA} &= Pe^{tD}P^{-1} \\
&= \begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-tA} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 1 \\ -\alpha^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\alpha e^{-t\alpha} \\ 1 & e^{-t\alpha} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 1 \\ -\alpha^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} e^{-t\alpha} & 0 \\ \dfrac{1-e^{-t\alpha}}{\alpha} & 1 \end{pmatrix}
\end{aligned}$$

Then, for the expected value we have that

$$\begin{aligned}
E\begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= Pce^{\lambda t} \\
&= \begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 e^{-\alpha t} \end{pmatrix}
\end{aligned}$$

Evaluating this expression at $t = 0$ gives

$$\begin{aligned}
\begin{pmatrix} E(X_0) \\ E(Y_0) \end{pmatrix} &= \begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= \begin{pmatrix} -\alpha c_2 \\ c_1 + c_2 \end{pmatrix} \implies \begin{matrix} c_2 = -E(X_0)/\alpha \\ c_1 = E(Y_0) + E(X_0)/\alpha \end{matrix}
\end{aligned}$$

So we have that

$$\begin{aligned}
E\begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} 0 & -\alpha \\ 1 & 1 \end{pmatrix} \begin{pmatrix} E(Y_0) + \dfrac{E(X_0)}{\alpha} \\ \dfrac{-E(X_0)}{\alpha} e^{-\alpha t} \end{pmatrix} \\
&= \begin{pmatrix} E(X_0)e^{-\alpha t} \\ E(Y_0) + \dfrac{E(X_0)}{\alpha}(1 - e^{-\alpha t}) \end{pmatrix}
\end{aligned}$$

Next, the covariance is

$$Cov\left(\begin{pmatrix} X_s \\ Y_s \end{pmatrix}, \begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right) = \Phi(s)\left[Var(c) + I(s)\right]\Phi(t)^T \qquad \text{for } s \le t$$

The first term is

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$$
\begin{aligned}
\Phi(s)Var(c)\Phi(t)^T &= Var(c)\begin{pmatrix} e^{-\alpha s} & 0 \\ \dfrac{1-e^{-\alpha s}}{\alpha} & 1 \end{pmatrix}\begin{pmatrix} e^{-\alpha t} & \dfrac{1-e^{-\alpha t}}{\alpha} \\ 0 & 1 \end{pmatrix} \\
&= Var(c)\begin{pmatrix} e^{-\alpha(s+t)} & \dfrac{e^{-\alpha s}(1-e^{-\alpha t})}{\alpha} \\ \dfrac{(1-e^{-\alpha s})}{\alpha}e^{-\alpha t} & \dfrac{(1-e^{-\alpha s})(1-e^{-\alpha t})}{\alpha^2}+1 \end{pmatrix}
\end{aligned}
$$

where the 2x2 element is $Var(c)\left(\dfrac{1-e^{-\alpha s}-e^{-\alpha t}+e^{-\alpha(s+t)}}{\alpha^2}+1\right)$. For the second term in the covariance, we need to compute $I(s)$ where

$$
\begin{aligned}
I(s) &= \int_0^s \Phi(u)^{-1}\sigma(u)\sigma(u)^T\left[\Phi(u)^{-1}\right]^T du \\
&= \int_0^s \begin{pmatrix} e^{u\alpha} & 0 \\ \dfrac{1-e^{-u\alpha}}{\alpha} & 1 \end{pmatrix}\begin{pmatrix} \sigma \\ 0 \end{pmatrix}\begin{pmatrix} \sigma & 0 \end{pmatrix}\begin{pmatrix} e^{u\alpha} & \dfrac{1-e^{u\alpha}}{\alpha} \\ 0 & 1 \end{pmatrix} du
\end{aligned}
$$

These integrals can be computed component-wise, i.e.

$$
I(s) = \begin{pmatrix} \int_0^s \sigma^2 e^{2\alpha u}du & \int_0^s \dfrac{\sigma^2 e^{u\alpha}(1-e^{\alpha u})du}{\alpha} \\ \int_0^s \sigma^2(1-e^{u\alpha})e^{u\alpha}du & \int_0^s \dfrac{\sigma^2(1-e^{u\alpha})(1-e^{\alpha u})du}{\alpha^2} \end{pmatrix}
$$

The elements of $I(s)$ are

$$
\begin{aligned}
I_{1,1}(s) &= \frac{\sigma^2}{2\alpha}(e^{2\alpha s}-1) \\
I_{1,2}(s) &= \frac{\sigma^2(2e^{2\alpha - e^{2\alpha s}-1})}{2\alpha^2} \\
I_{2,2}(s) &= \frac{\sigma^2(2s\alpha + e^{2\alpha s}+3-4e^{\alpha s})}{2\alpha^3}
\end{aligned}
$$

So the second term in the covariance is

$$
\begin{aligned}
&\Phi(s)I(s)\Phi(t)^T \\
&= \begin{pmatrix} e^{-\alpha s} & 0 \\ \dfrac{1-e^{-\alpha s}}{\alpha} & 1 \end{pmatrix}\begin{pmatrix} I_{1,1}(s) & I_{1,2}(s) \\ I_{1,2}(s) & I_{2,2}(s) \end{pmatrix}\begin{pmatrix} e^{-\alpha t} & \dfrac{1-e^{-\alpha t}}{\alpha} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-\alpha s}I_{1,1}(s) & e^{-\alpha s}I_{1,2}(s) \\ \dfrac{1-e^{-\alpha s}}{\alpha}I_{1,1}(s)+I_{1,2}(s) & \dfrac{1-e^{-\alpha s}}{\alpha}I_{1,2}(s)+I_{2,2}(s) \end{pmatrix}\begin{pmatrix} e^{-\alpha t} & \dfrac{1-e^{-\alpha t}}{\alpha} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-\alpha s}I_{1,1}(s)e^{-\alpha t} & \dfrac{e^{-\alpha s}I_{1,1}(s)(1-e^{-\alpha t})}{\alpha}+e^{-\alpha s}I_{1,2}(s) \\ \dfrac{(1-e^{-\alpha s})I_{1,1}(s)e^{-\alpha t}}{\alpha}+e^{-\alpha t}I_{1,2}(s) & \dfrac{(1-e^{-\alpha t})(1-e^{-\alpha s})}{\alpha^2}I_{1,1}(s)+\dfrac{(1-e^{-\alpha t})+(1-e^{-\alpha s})}{\alpha}I_{1,2}(s)+I_{2,2}(s) \end{pmatrix}
\end{aligned}
$$

where the 2x2 element is

$$
\begin{aligned}
&\frac{(1-e^{-\alpha t})(1-e^{-\alpha s})}{\alpha^2} I_{1,1}(s) + \frac{(1-e^{-\alpha t}) + (1-e^{-\alpha s})}{\alpha} I_{1,2}(s) + I_{2,2}(s) \\
&= \frac{(1-e^{-\alpha t})(1-e^{-\alpha s})}{\alpha^2} \frac{\sigma^2}{2\alpha}(e^{2\alpha s}-1) + \frac{(1-e^{-\alpha t}) + (1-e^{-\alpha s})}{\alpha} \frac{\sigma^2(2e^{2\alpha - e^{2\alpha s} - 1})}{2\alpha^2} + \frac{\sigma^2(2s\alpha + e^{2\alpha s} + 3 - 4e^{\alpha s})}{2\alpha^3} \\
&= \frac{\sigma^2}{\alpha^2}s + \frac{\sigma^2}{2\alpha^3}[\cancel{e^{2\alpha s}} - \cancel{e^{\alpha s}} - \cancel{e^{\alpha(2s-t)}} + e^{\alpha(s-t)} - 1 + e^{-\alpha s} + e^{-\alpha t} - e^{-\alpha(s+t)} + 2e^{\alpha s} - e^{2\alpha s} - 1 \\
&\qquad - 2e^{\alpha(s-t)} + \cancel{e^{\alpha(2s-t)}} + e^{-\alpha t} + 2e^{\alpha s} - \cancel{e^{2\alpha s}} - 1 - 2 + \cancel{e^{\alpha s}} + e^{-\alpha s}] \\
&= \frac{\sigma^2}{\alpha^2}s + \frac{\sigma^2}{2\alpha^3}[2e^{-\alpha t} - e^{-\alpha(s+t)} - e^{-\alpha(t-s)} - 2 + 2e^{-\alpha s}]
\end{aligned}
$$

Combining the 2x2 elements we have that

$$
Cov(Y_s, Y_t) = Var(c)\left(\frac{1 - e^{-\alpha s} - e^{-\alpha t} + e^{-\alpha(s+t)}}{\alpha^2} + 1\right) + \frac{\sigma^2}{\alpha^2}s + \frac{\sigma^2}{2\alpha^3}[2e^{-\alpha t} - e^{-\alpha(s+t)} - e^{-\alpha(t-s)} - 2 + 2e^{-\alpha s}]
$$

4.3. Derive the result for e^{At} in (4.52).

Solution:

For the second order SDE define A as

$$
A = \begin{pmatrix} \alpha_1 & \alpha_0 \\ 1 & 0 \end{pmatrix}
$$

The eigenvalues of A are found by setting the determinant of $A - \lambda I = 0$, i.e.

$$
det\begin{pmatrix} \alpha_1 - \lambda & \alpha_0 \\ 1 & -\lambda \end{pmatrix} = 0 \implies (\alpha_1 - \lambda)(-\lambda) - \alpha_0 = 0
$$

The eigenvalues are the solutions of the quadratic equation $\lambda^2 - \alpha_1\lambda - \alpha_0 = 0$, i.e.

$$
\begin{aligned}
\lambda_1 &= \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_0}}{2} \\
\lambda_2 &= \frac{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_0}}{2}
\end{aligned}
$$

The first eigenvector v_1 is found by setting $(A - -\lambda_1 I)v_1 = 0$, i.e.

$$
\begin{pmatrix} \alpha_1 - \lambda_1 & \alpha_0 \\ 1 & -\lambda_1 \end{pmatrix}\begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = 0 \implies \begin{matrix} (\alpha_1 - \lambda_1)v_{1,1} + \alpha_0 v_{1,2} = 0 \\ v_{1,1} - \lambda_1 v_{1,2} = 0 \end{matrix}
$$

so $v_1 = (\lambda_1, 1)$. Similarly we can find v_2 by setting $(A - \lambda_2 I)v_2 = 0$, i.e.

$$\begin{pmatrix} \alpha_1 - \lambda_2 & \alpha_0 \\ 1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = 0$$

which can be solved the same was as v_1 giving $v_2 = (\lambda_2, 1)$. Using the diagonalization of A (see Exercise 4.2)

$$\begin{aligned}
\Phi(t) &= e^{tA} \\
&= Pe^{tD}P^{-1} \\
&= \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\lambda_1 - \lambda_2} & \dfrac{-\lambda_2}{\lambda_1 - \lambda_2} \\ \dfrac{-1}{\lambda_1 - \lambda_2} & \dfrac{\lambda_1}{\lambda_1 - \lambda_2} \end{pmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \\
&= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t} & -\lambda_1 \lambda_2 e^{\lambda_1 t} + \lambda_1 \lambda_2 e^{\lambda_2 t} \\ e^{\lambda_1 t} - e^{\lambda_2 t} & -\lambda_2 e^{\lambda_1 t} + \lambda_1 e^{\lambda_2 t} \end{pmatrix}
\end{aligned}$$

4.4. Modeling the force of interest by the Brownian Motion centered at $\delta = 0.05$ and with $\sigma = 0.01$, find the expected value and variance of $pv(10)$. What is $cov(pv(5), pv(10))$?

Solution:

The R code below was used to compute the expected value and variance of $pv(10)$.

```
> wienermodel = iratemodel(list(delta0 = 0.05, sigma = 0.01),
+                          "gbm")
> pv10ev = pv.ev(10, wienermodel)
> pv10var = pv.var(10, wienermodel)
> pvcov = pv.cov(5,10,wienermodel)
```

This gives $E[pv(10)] = 0.61672421$, $Var[pv(10)] = 0.01289196$ and $cov(pv(5), pv(10)) = 0.00503982$.

4.5. Assume that the force of interest is an Ornstein-Uhlenbeck process with $\delta_0 = 0.08$, $\delta = 0.05$, $\alpha = 0.1$ and $\sigma = 0.01$.

(i) What is the expected value and variance of $pv(10)$?

(ii) What is $cov(pv(5), pv(10))$?

(iii) Find the expected value and variance of $a_{\overline{10|}}$.

Solution:

(i) The R code below was used to compute the expected value and variance of $pv(10)$.

```
> oumodel = iratemodel(list(delta0 = 0.08, delta = 0.05,
+                          alpha = 0.1, sigma = 0.01), "ou")
> pv10ev = pv.ev(10, oumodel)
> pv10var = pv.var(10, oumodel)
> pvcov = pv.cov(5, 10, oumodel)
```

This gives $E[pv(10)] = 0.50599342$ and $Var[pv(10)] = 0.00434$.

(ii) The covariance is $cov(pv(5), pv(10)) = 0.00209571$.

(iii) The following code was used to compute the expected value and variance of $a_{\overline{10|}}$.

```
> annev = ann.ev(10, oumodel)
> annvar = ann.var(10, oumodel)
```

Using this code gives $E[a_{\overline{10|}}] = 6.8841889$ and $Var[a_{\overline{10|}}] = 0.13503997$.

4.6. Use Ito's formula to prove that:

(i) $dW_t^2 = 2W_t dW_t + dt$

(ii) $de^{W_t} = e^{W_t} dW_t + 0.5e^{W_t} dt$

(iii) $d(e^{\sigma W_t - 0.5\sigma^2 t}) = \sigma(e^{\sigma W_t - 0.5\sigma^2 t}) dW_t$

Solution:

(i) Ito's formula can be stated as

$$dY(t, X_t) = Y_X dX_t + 0.5Y_{XX} dX_t^2 + Y_t dt$$

In this case, $Y(t, W_t) = W_t^2$. So $Y_W = 2W_t$, $Y_{WW} = 2$, $Y_t = 0$. Using Ito's formula we have that

$$dY(t, W_t) = 2W_t dW_t + 0.5(2) dW_t^2 = 2W_t dW_t + dt$$

(ii) In this case, $Y(t, W_t) = e^{W_t}$. So $Y_W = e^{W_t}$, $Y_{WW} = e^{W_t}$, $Y_t = 0$. Using Ito's formula we have that

$$dY(t, W_t) = e^{W_t} dW_t + 0.5e^{W(t)} dW_t^2 = e^{W_t} dW_t + 0.5e^{W_t} dt$$

(iii) In this case, $Y(t, W_t) = e^{\sigma W_t - 0.5\sigma^2 t}$. So $Y_W = \sigma e^{\sigma W_t - 0.5\sigma^2 t}$, $Y_{WW} = \sigma^2 e^{\sigma W_t - 0.5\sigma^2 t}$, $Y_t = -0.5\sigma^2 e^{\sigma W_t - 0.5\sigma^2 t}$. Using Ito's formula, we have that

$$dY(t, W_t) = \sigma e^{\sigma W_t - 0.5\sigma^2 t} dW_t + 0.5\sigma^2 e^{\sigma W_t - 0.5\sigma^2 t} dW_t^2 - 0.5\sigma^2 e^{\sigma W_t - 0.5\sigma^2 t} dt = \sigma e^{\sigma W_t - 0.5\sigma^2 t} dW_t$$

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5 Discrete versus Continuous Models

5.1. Find the discrete representation of the second order SDE with $\alpha_1 = -0.5$, $\alpha_0 = -0.04$, $\sigma = 0.01$ and $\Delta = 1$. Is it stationary?

Solution:

The R code below was used to estimate the parameters of an ARMA(2,1).

```
> sdemodel = iratemodel(params = list(alpha1 = -0.5, alpha2 = -0.04,
+         sigma2 = 0.01), "second")
> armamodel = iratemodel.convert("second", "arma", sdemodel, 1)
```

This gives $\phi_1 = 1.57515746$, $\phi_2 = -0.60653066$, $\theta_1 = -0.26534494$ and $\sigma_a = 0.00619855$.

5.2. It is believed that some discrete data was obtained for a second order SDE sampled at intervals $\Delta = 5$. The discrete data was used to estimate the parameters of an ARMA(2,1). The estimates $\phi_1 = 1.05$, $\phi_2 = -0.095$, $\theta_1 = -0.05$.

(i) Determine α_1 and α_0.

(ii) Suppose that your answers in (i) are the true parameters of the continuous system. What should the values of ϕ_1, ϕ_2 and θ_1 be in order to have a covariance equivalent system? Compare them with your respective estimates.

Solution:

(i) The R code used is shown below.

```
> armamodel = iratemodel(params = list(phi1 = 1.05, phi2 = -0.095,
+         theta1 = -0.05), "arma")
> sdemodel = iratemodel.convert("arma", "second", armamodel, 5)
```

This gives $\alpha_0 = -0.00472429$ and $\alpha_1 = -0.47077568$.

(ii) The R code below was used to calculate ϕ_1, ϕ_2 and θ_1.

```
> sdemodel$sigma2 = 0.01 # arbitrary here
> armamodel = iratemodel.convert("second", "arma", sdemodel, 5)
```

This gives $\phi_1 = 1.05$, $\phi_2 = -0.095$ and $\theta_1 = -0.2066469$. Note that the value of θ_1 has been changed.

5.3. Any ARMA(2,1) for which (ϕ_1, ϕ_2) lies in the "stability region" above the parabola has corresponding μ_1 and μ_2 that are real and uniquely determined. This implies that a unique covariance equivalent second order SDE exists. Comment.

Solution:

If ϕ_1 and ϕ_2 lie in the stability region then $|\lambda_1| < 1$ and $|\lambda_2| < 1$. So we can calculate unique real values of μ_1 and mu_2 using

$$\mu_1 = \ln(\lambda_1)/\Delta$$
$$\mu_2 = \ln(\lambda_2)/\Delta$$

and then the parameters of the SDE are $\alpha_1 = \mu_1 + \mu_2$ and $\alpha_2 = -\mu_1\mu_2$. So a unique covariance equivalent SDE will exist in this case.

8 Discount Function (or Present Value)

8.1. Show (8.2).

Solution:

For an OU process, we have that

$$E[y(t)] = \delta t + (\delta_0 - \delta)\left(\frac{1-e^{-\alpha t}}{\alpha}\right)$$
$$Var[y(t)] = \frac{\sigma^2}{\alpha^2}t + \frac{\sigma^2}{2\alpha^3}\left(-3 + 4e^{-\alpha t} - e^{-2\alpha t}\right)$$

So we have that

$$\begin{aligned}
E[e^{-y(t)}] &= e^{-E[y(t)]+0.5Var[y(t)]} \\
&= e^{-\delta t}e^{-(\delta_0-\delta)/\alpha(1-e^{-\alpha t})}e^{\sigma^2/(2\alpha^2)t}e^{\sigma^2/(4\alpha^3)(-3)}e^{\sigma^2/\left(4\alpha^3\right)(4e^{-\alpha t}-e^{-2\alpha t})} \\
&= e^{t\left(\sigma^2/(2\alpha^2)-\delta\right)}e^{-(\delta_0-\delta)/\alpha(1-e^{-\alpha t})}e^{-3\sigma^2/(4\alpha^3)}e^{\sigma^2/(4\alpha^3)(4e^{-\alpha t}-e^{-2\alpha t})} \\
\lim_{t\to\infty} E[e^{-y(t)}] &= e^{-(\delta_0-\delta)/\alpha}e^{-3\sigma^2/(4\alpha^3)}\lim_{t\to\infty} e^{t(\sigma^2/(2\alpha^2)-\delta)} \\
&= \begin{cases} 0 & \delta > \sigma^2/(2\alpha^2) \\ e^{-(\delta_0-\delta)/\alpha}e^{-3\sigma^2/(4\alpha^3)} & \delta = \sigma^2/(2\alpha^2) \\ \infty & \delta < \sigma^2/(2\alpha^2) \end{cases}
\end{aligned}$$

8.2. Show (8.3).

Solution:

For the second moment, we have that

$$\begin{aligned}
E[e^{-2y(t)}] &= e^{-2E[y(t)]+2Var[y(t)]} \\
&= e^{-2\delta t}e^{-2(\delta_0-\delta)/\alpha(1-e^{-\alpha t})}e^{2\sigma^2/\alpha^2 t}e^{\sigma^2/\alpha^3(-3+4e^{-\alpha t}-e^{-2t})} \\
&= e^{t\left(2\sigma^2/\alpha^2-2\delta\right)}e^{-2(\delta_0-\delta)/\alpha(1-e^{-\alpha t})}e^{-3\sigma^2/\alpha^3}e^{\sigma^2/\alpha^3(4e^{-\alpha t}-e^{-2\alpha t})} \\
\lim_{t\to\infty} E[e^{-2y(t)}] &= \begin{cases} 0 & \delta > \sigma^2/\alpha^2 \\ e^{-2(\delta_0-\delta)/\alpha}e^{-3\sigma^2/\alpha^3} & \delta = \sigma^2/\alpha^2 \\ \infty & \delta < \sigma^2/\alpha^2 \end{cases}
\end{aligned}$$

So $\lim_{t\to\infty} Var[e^{-y(t)}] = \lim_{t\to\infty} E[e^{-2y(t)}]$.

8.3. Show that $[cv(e^{-y(t)})]^2 = e^{Var(y(t))} - 1$.

Solution:

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The quantities we are interested in are

$$
\begin{aligned}
E[e^{-y(t)}] &= e^{-E[y(t)]+0.5Var[y(t)]} \\
E[e^{-y(t)}]^2 &= e^{-2E[y(t)]+Var[y(t)]} \\
E[e^{-2y(t)}] &= e^{-2E[y(t)]+2Var[y(t)]}
\end{aligned}
$$

So the coefficient of variation is

$$
\begin{aligned}
cv[e^{-y(t)}]^2 &= \frac{E[e^{-2y(t)}] - E[e^{-y(t)}]^2}{E[e^{-y(t)}]^2} \\
&= \frac{[e^{-2E[y(t)]+Var[y(t)]}][e^{Var[y(t)]} - 1]}{e^{-2E[y(t)]+Var[y(t)]}} \\
&= e^{Var[y(t)]} - 1
\end{aligned}
$$

8.4. Noting that (8.7) is not valid when the force of interest is modeled by a White Noise process, determine whether the coefficient of variation is an increasing function of t or not.

Solution:

For a white noise process, $\delta_t \sim N(\delta, \sigma^2)$ so $y(t) \sim N(\delta t, \sigma^2 t)$, i.e. $Var[y(t)] = \sigma^2 t$. So the derivative of the variance is

$$
\frac{d}{dt} Var[y(t)] = \frac{d}{dt}\sigma^2 t = \sigma^2 > 0
$$

so the coefficient variation is an increasing function of t.