Stochastic Differential Equations - Parameter Estimation

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We can equate the covariance function and expected value function of a stochastic differential equation and its corresponding time series model. Then we solve for the parameters of the SDE in terms of the time series model. This is know as the **principle of covariance equivalence**.

1 Ornstein-Uhlenbeck process

The AR(1) time series model corresponds to the OU model. The parameter estimates are shown below. Note that Δ is known.

1.1 AR(1) to OU

- Known parameters: ϕ_1, σ_a
- Unknown parameters: α, σ

$$\alpha = \frac{-\ln(\phi_1)}{\Delta} \tag{1}$$

$$\sigma^2 = \frac{2\alpha\sigma_a^2}{1-\phi_1^2} \tag{2}$$

1.2 OU to AR(1)

- Known parameters: α, σ
- Unknown parameters: ϕ_1, σ_a

$$\phi_1 = e^{-\alpha\Delta} \tag{3}$$

$$\sigma_a^2 = \frac{\sigma^2}{2\alpha}(1-\phi_1^2) \tag{4}$$

2 Second Order SDE

- Let μ_1 and μ_2 be the eigenvalues of $A = \begin{pmatrix} \alpha_1 & \alpha_0 \\ 1 & 0 \end{pmatrix}$, i.e. $\mu_1 = \frac{1}{2}\left(\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_0}\right)$ and $\mu_2 = \frac{1}{2}\left(\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_0}\right)$
- Let λ_1 and λ_2 be the roots of the AR(2) characteristic equation, i.e. $\lambda_1 = \frac{1}{2}\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)$ and $\lambda_2 = \frac{1}{2}\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)$

2.1 ARMA(2,1) to Second Order SDE

The simplified estimation procedure is described below.

- Known parameters: $\phi_1, \phi_2 \implies \lambda_1, \lambda_2$ are known
- Unknown parameters: $\alpha_1, \alpha_2 \implies \mu_1, \mu_2$ are unknown

First compute the eigenvalues

$$\mu_1 = \ln(\lambda_1)/\Delta \tag{5}$$
$$\mu_2 = \ln(\lambda_2)/\Delta \tag{6}$$

Next, compute the unknown SDE parameters

$$\alpha_1 = \mu_1 + \mu_2 \tag{7}$$
$$\alpha_0 = -\mu_1\mu_2 \tag{8}$$

2.2 Second Order SDE to ARMA(2, 1)

- Known parameters: $\alpha_0, \alpha_1, \sigma \implies \mu_1, \mu_2$ are known
- Unknown parameters: $\phi_1, \phi_2, \theta_1, \sigma_a \implies \lambda_1, \lambda_2$ are unknown

First, compute roots of AR(2) characteristic equation

$$\lambda_1 = e^{\mu_1\Delta} \tag{9}$$
$$\lambda_2 = e^{\mu_2\Delta} \tag{10}$$

Next, compute

$$\phi_1 = \lambda_1 + \lambda_2 \tag{11}$$
$$\phi_2 = -\lambda_1\lambda_2 \tag{12}$$

Next define $P = \dfrac{-\mu_1(1+\lambda_1^2)(1-\lambda_2^2)+\mu_2(1+\lambda_2^2)(1-\lambda_1^2)}{\mu_1\lambda_1(1-\lambda_2^2)-\mu_2\lambda_2(1-\lambda_1^2)}$. Then θ_1 is the root of the equation

$$\theta_1^2 + 2P\theta_1 + 1 = 0 \qquad \text{choose the root such that } |\theta_1| < 1 \tag{13}$$

Also, we have that

$$\sigma_a^2 = \frac{\sigma^2}{2\mu_1(\mu_1^2-\mu_2^2)} \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1-\theta_1)} \left(\frac{\lambda_1-\theta_1}{1-\lambda_1^2} - \frac{\lambda_2-\theta_1}{1-\lambda_1\lambda_2} \right)^{-1} \tag{14}$$