Acma 490 Exercises

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Contents

2 Deterministic interest (Pure insurance risk)

2.1. Find $E[Z_{k+1}]$ and $Var[Z_{k+1}]$ for a whole-life insurance payable at the end of the year of death if the future lifetime of the life insured is exponentially distributed with parameter μ (mean $1/\mu$). Use a constant force of interest of δ.

Solution:

Since ${}_{k|}q_x = {}_kp_x + {}_{k+1}p_x = e^{-\mu k} - e^{-\mu(k+1)} = e^{-\mu k}(1-e^{-\mu})$ we have that

$$\begin{aligned} E[Z_{k+1}] &= \sum_{k=0}^{\infty} e^{-\delta k} e^{-\delta}(1-e^{-\mu})e^{-\mu k} \\ &= \sum_{k=0}^{\infty} e^{-k(\delta+\mu)}(1-e^{-\mu})e^{-\delta} \\ &= \frac{1}{1-e^{-(\delta+\mu)}}(1-e^{-\mu})e^{-\delta} \end{aligned}$$

The second moment is

$$\begin{aligned} E[Z_{k+1}^2] &= \sum_{k=0}^{\infty} e^{-2\delta k} e^{-2\delta}(1-e^{-\mu})e^{-k(2\delta+u)} \\ &= \frac{1}{1-e^{-(2\delta+\mu)}}(1-e^{-\mu})e^{-2\delta} \end{aligned}$$

2.2. Find A_x and $Var[Z_{k+1}]$ if ${}_{k|}q_x = pq^k$ for $k = 0, 1, 2, \ldots, \infty$. Compare with question 2.1.

Solution:

The first moment is

$$\begin{aligned} E[Z_{k+1}] &= \sum_{k=0}^{\infty} {}_{k|}q_x \exp\left(-\delta(k+1)\right) \\ &= \sum_{k=0}^{\infty} pq^k(e^{-\delta(k)})e^{-\delta} \\ &= \frac{pe^{-\delta}}{1-qe^{-\delta}} \end{aligned}$$

Note that this is the same as in Exercise 2.1 if $p = (1 - e^{-\mu})$ and $q = e^{-\mu}$. The second moment is

$$
\begin{aligned}
E[Z_{k+1}^2] &= \sum_{k=0}^{\infty} e^{-2\delta} p (qe^{-2\delta})^k \\
&= \frac{pe^{-2\delta}}{1 - qe^{-2\delta}}
\end{aligned}
$$

2.3. Using De Moivre ($l_x = l_0(1 - x/\omega)$ for $0 < x < \omega$), find $E[Z_{k+1}]$ and $Var[Z_{k+1}]$ for a discrete whole-life insurance.

Solution:

For $T_x \sim U(0, \omega - x)$ so ${}_{k|}q_x = \dfrac{1}{\omega - x}$ and we have that

$$
\begin{aligned}
E[Z_{k+1}] &= \sum_{k=0}^{\omega-x-1} {}_{k|}q_x e^{-\delta k} \\
&= \sum_{k=0}^{\omega-x-1} \frac{1}{\omega - x} e^{-\delta k} \\
&= \frac{1 - e^{-\delta(\omega - x)}}{(\omega - x)(1 - e^{-\delta})}
\end{aligned}
$$

and the second moment is

$$
\begin{aligned}
E[Z_{k+1}^2] &= \sum_{k=0}^{\omega-x-1} \frac{1}{\omega - x} e^{-2\delta k} \\
&= \frac{1 - e^{-2\delta(\omega - x)}}{(\omega - x)(1 - e^{-2\delta})}
\end{aligned}
$$

2.4. Consider 100 independent lives insured aged x, subject to a constant force of mortality ($\mu = 0.04$) and all covered by a single premium 10-year temporary insurance contract of \$2,000,000 payable at the end of the year of death (if within 10 years, of course). The death benefits will be paid from a fund earning $\delta = 0.06$. Calculate the amount needed in that fund at time 0 such that the probability of being able to pay all the benefits is 0.95.

Solution:

The first moment is

$$E[Z(c)] = c \times \frac{be^{-\delta}(1-e^{-\mu})(1-e^{-(\delta+\mu)n})}{1-e^{-(\delta+\mu)}}$$

The second moment is

$$E[Z(c)^2] = c^2 \times \frac{b^2e^{-2\delta}(1-e^{-\mu})(1-e^{-(2\delta+\mu)n})}{1-e^{-(2\delta+\mu)}}$$

The variance is

$$Var[Z(c)] = c^2\left\{E[Z(c)^2] - E[Z(c)]^2\right\}$$

Using the normal approximation, the amount needed is

$$1.645\sqrt{Var[Z(c)]} + E[Z(c)]$$

This is calculated using R below.

```
> mu = 0.04
> delta = 0.06
> n = 10
> b = 2e+06
> c = 100
> first = b * exp(-delta) * (1 - exp(-mu)) * (1 - exp(-(delta + mu)*n)) /
+         (1 - exp(-(delta + mu)))
> second = b^2 * exp(-2*delta) * (1 - exp(-mu)) *
+         (1 - exp(-(2*delta + mu)*n)) / (1 - exp(-(2*delta + mu)))
> amount = 1.645 * sqrt(c * (second - first^2)) + first * c
```

This gives an amount of 60807943.

2.5. Show that $Var[Z(c)]$ obtained from using the result for the variance of iid random variables is equivalent to using (2.8) and (2.9).

Solution:

Using first principles, the variance of $Z(c)$ is

$$\begin{aligned} Var[Z(c)] &= Var\left[\sum_{i=1}^{c} Z_i\right] \\ &= \sum_{i=1}^{c} Var[Z_i] \\ &= c[E[Z^2] - E[Z]^2] \end{aligned}$$

2.6. Find an expression for $E[Z_1^2 Z_2]$.

Solution:

We have that

$$E[Z_1^2 Z_2] = \sum_{k_1}\sum_{k_2}(b_{k_1+1}v^{k_1+1})^2 b_{k_2+1}v^{k_2+1}{}_{k_1|}q_{x\,k_2|}q_x$$

2.7. Rework section 2 for n-year endowment contracts.

Solution:

The functions used for the endowment insurance are shown below.

```
> library(stocins) # see github.com/nathanesau
> irm = iratemodel(list(delta = 0.06), "determ")
> mort = mortassumptions(list(x=50, table="MaleMort82"))
> endow1 = insurance(list(n = 1, d = 1, e = 1), "isingle", "endow")
> endow10 = insurance(list(n = 10, d = 1, e = 1), "isingle", "endow")
> port1 = insurance(list(single = endow1, c = 10), "iport", "endow")
> port10 = insurance(list(single = endow10, c = 10), "iport", "endow")
> first = c(z.moment(1, endow1, mort, irm),
+          z.moment(1, endow10, mort, irm),
+          z.moment(1, port1, mort, irm) / port1$c,
+          z.moment(1, port10, mort, irm) / port1$c)
> sdev = c(0, z.sd(endow10, mort, irm), 0,
+          z.sd(port10, mort, irm) / port1$c)
> skew = c(0, z.sk(endow10, mort, irm),
+          0, z.sk(port10, mort, irm))
```

c	n	$E[Z(c)/c]$	$sd[Z(c)/c]$	$sk[Z(c)/c]$
1	1	0.94176	0	0
1	10	0.56337	0.0582	4.57037
10	1	0.94176	0	0
10	10	0.56337	0.01841	1.44528

3 Life Insurance with Random Interest and Mortality

3.1. It has been suggested that the interest rate risk and the mortality risk can be studied independently since K and δ are assumed independent. More precisely, the suggestion is that the total "uncertainty" in Z is the sum of

(i) the variance of the random discounted values when the mortality is assumed to follow exactly the life table, and

(ii) the variance of the present value of the benefit discounted at the deterministic expected interest rates

(a) Formulate this mathematically for a temporary insurance contract.

(b) Illustrate the approach with a 10-year temporary insurance contract issued to someone aged 50, using an Ornstein-Uhlenbeck process with $\delta = 0.05$, $\delta_0 = 0.08$, $\alpha = 0.1$ and $\sigma = 0.02$. (Find the variance of Z, the interest rate risk and the mortality risk).

(c) Compare the numerical results given by the 2 components of (3.16).

(d) Redo this problem when (ii) above is replaced by the variance of the present value of the benefit discounted at the deterministic expected discount factors, $E[e^{-y(t)}]$.

Solution:

(a) Formulas for the two components are derived below.

(i) The investment risk is

$$Var_y\left[\sum_k {}_{k|}q_x e^{-y(k+1)}\right]$$

(ii) The insurance risk is

$$Var_k[e^{-\delta(k+1)}]$$

(b) The mth moment of Z is

$$E[Z^m] = \sum_{k=0}^{n-1} E[e^{-my(k+1)}]_{k|}q_x$$

from which we can easily calculate the variance of Z. This is done using R below.

```
> mort = mortassumptions(params = list(x = 50,
+         table = "MaleMort82"))
> oumodel = iratemodel(params = list(delta = 0.05,
+         delta0 = 0.08, alpha = 0.10, sigma = 0.02), "ou")
> term = insurance(params = list(n = 10, d = 1),
+ "isingle", "term")
> VarZ = z.moment(2, term, mort, oumodel) -
+   z.moment(1, term, mort, oumodel)^2
```

Some functions to calculate the two components are shown below.

```
> insrisk <- function(ins, mort, irm)
+ {
+   second = 0
+   first = 0
+
+   for(k in seq(0,ins$n-1,1))
+   {
+     second = second + exp(-y.ev(k+1,irm)*2) *
+     kdeferredqx(k, mort)
+     first = first + exp(-y.ev(k+1,irm)) *
+     kdeferredqx(k, mort)
+   }
+
+   second - first^2
+ }
> invrisk <- function(ins, mort, irm)
+ {
+   total = 0
+
+   for(k in seq(0,ins$n-1,1))
+   {
+     for(j in seq(0,ins$n-1,1))
+     {
+       total = total + pv.cov(k+1,j+1,irm) * kdeferredqx(k,mort) *
+       kdeferredqx(j, mort)
+     }
+   }
```

```
+
+   total
+ }
> InsuranceRisk = insrisk(term, mort, oumodel)
> InvestmentRisk = invrisk(term, mort, oumodel)
```

This gives $Var[Z] = 0.04069478$.

(i) The variance of the random discounted values is $6.753e - 05$.

(ii) The variance of the present value of the benefit is 0.03901077.

(c) The components of 3.16 are calculated below.

```
> InsuranceRisk = z.insrisk(term, mort, oumodel)
> InvestmentRisk = z.invrisk(term, mort, oumodel)
```

This gives

(i) The insurance risk is 0.00090981.

(ii) The investment risk is 0.03978498.

(d) Omit

3.2. Modeling the force of interest by an Ornstein-Uhlenbeck process with $\delta = 0.05$, $\delta_0 = 0.03$, $\alpha = 0.1$ and $\sigma = 0.02$, study the discount function, $e^{-y(t)}$.

(a) For which value of t is its standard deviation (or variance) maximum?

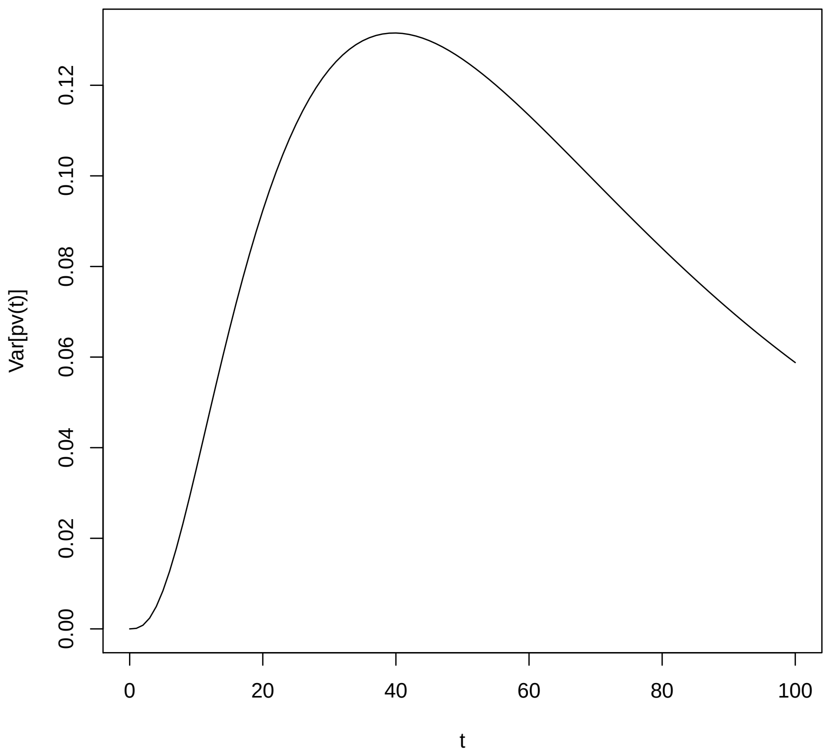
(b) Calculate the standard deviation of Z for temporary insurance contracts for different values of n and x. Referring to section 3.2, could you have "roughly predicted" those results?

Solution:

(a) The graph is shown below. The maximum variance occurs for $t = 39.7624$.

```
> oumodel = iratemodel(params = list(delta = 0.05, delta0 = 0.03,
+          alpha = 0.10, sigma = 0.02), "ou")
> pv.var <- function(t) {
+          pv.moment(t,2,oumodel) - pv.moment(t,1,oumodel)^2
+ }
> plot(pv.var, 0, 100, ylab = "Var[pv(t)]", xlab = "t")
> maxt = optim(par=c(20), fn=function(t)-pv.var(t),
+               method="Brent", lower=0, upper=50)$par
```

(b) The graph is shown below.




```
> library(lattice)
> library(latex2exp)
> # tweak g params
> g <- expand.grid(y = seq(1,70,20), x = seq(20,70,25))
> g$z <- numeric(nrow(g))
> counter = 1
> for(i in 1:nrow(g))
+ {
+   term = insurance(params = list(n = g$y[counter], d = 1),
+   "isingle", "term")
+   mort = mortassumptions(params = list(x = g$x[counter],
+   table = "MaleMort82"))
+
+   g$z[counter] = z.sd(term, mort, oumodel)
+   counter = counter + 1
+ }
> wireframe(z ~ y * x, data = g,
+           drape = TRUE,
+           col = 'black',
+           col.regions = 'white',
```

```
+           aspect = c(1.0, 0.8),
+           colorkey = FALSE,
+           xlab = "n",
+           ylab = "issue age",
+           zlab = "",
+           screen = list(z = 340, x = -70),
+           scales = list(arrows = FALSE, col="black", font = 10,
+                         cex= 1.0),
+           par.settings = list(regions=list(alpha = 0.3),
+                               axis.line = list(col = "transparent")),
+           zoom = 0.90)
```

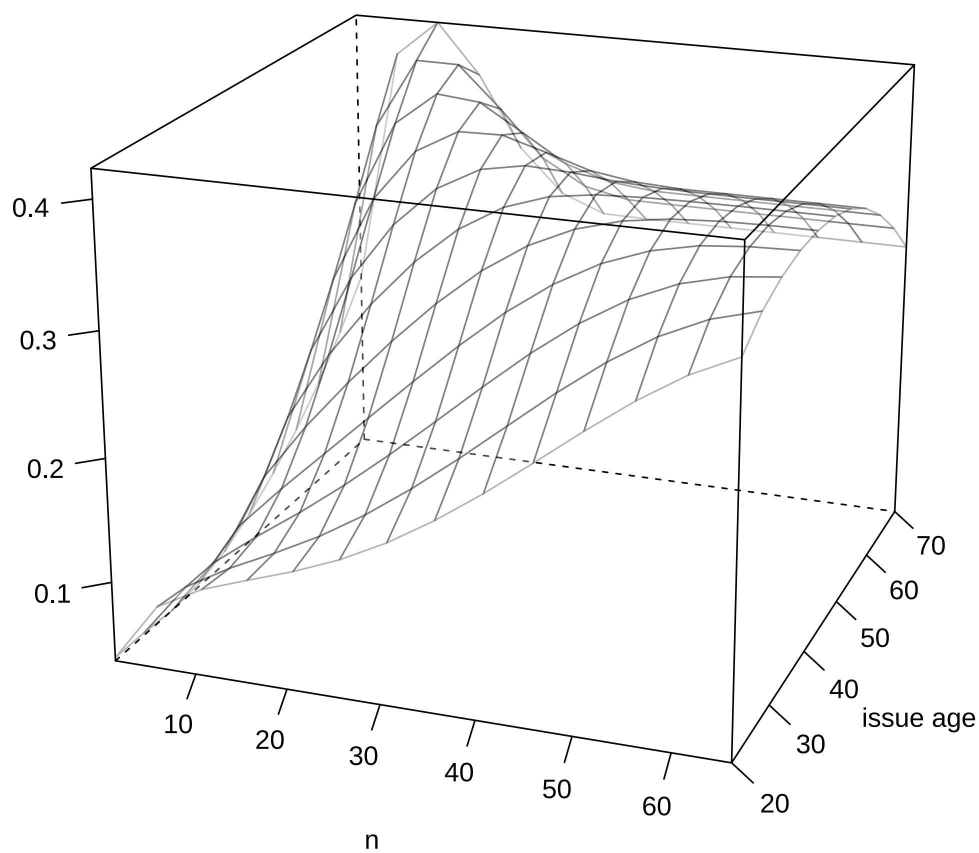


Figure 1: Standard deviation for Exercise 3.2

4 A Portfolio of Policies with Random Interest and Mortality

4.1. Prove (4.10).

Solution:

We have that

$$\begin{aligned} E[Z(c)^2] &= c(c-1)E[Z_1Z_2] + cE[Z^2] \\ E[Z(c)]^2 &= c^2E[Z]^2 \end{aligned}$$

so the variance per policy is

$$\frac{Var[Z(c)]}{c^2} = \frac{c(c-1)}{c^2}E[Z_1Z_2] + \frac{cE[Z^2]}{c^2} - \frac{c^2}{c^2}E[Z]^2$$

and the limiting variance is

$$\lim_{c\to\infty}\frac{Var[Z(c)]}{c^2} = E[Z_1Z_2] - E[Z]^2$$

4.2. (a) Show that $\frac{d}{dc}Var[Z(c)/c] = -\frac{1}{c^2}({}^2A - E[Z_1Z_2])$ if it is treated as a continuous function of c.

(b) Show that under assumptions 4.2, the above derivative if $-E[Var(Z|y(t))]/c^2$

Solution:

(a) Using the expression for $Var[Z(c)]$ from Exercise 4.1,

$$\begin{aligned} \frac{Var[Z(c)]}{c^2} &= \frac{c(c-1)}{c^2}E[Z_1Z_2] + \frac{cE[Z^2]}{c^2} - \frac{c^2}{c^2}E[Z]^2 \\ &= (1-c^{-1})E[Z_1Z_2] + \frac{1}{c}E[Z^2] - E[Z]^2 \end{aligned}$$

so the derivative is

$$\begin{aligned}\frac{d}{dc}\frac{Var[Z(c)]}{c^2} &= c^{-2}E[Z_1Z_2] - c^{-2}E[Z^2] \\ &= -\frac{1}{c^2}[-E[Z_1Z_2] + E[Z^2]]\end{aligned}$$

(b) Using the result of (a), we just need to show that

$$E[Var[Z|\{y(t)\}]] = E[Z^2] - E[Z_1Z_2]$$

using Equation (4.8) we have that

$$\begin{aligned}E[Var[Z|\{y(t)\}]] &= E\Big[E[Z^2|\{y(t)\}] - E[Z|\{y(t)\}] \cdot E[Z|\{y(t)\}]\Big] \\ &= E[Z^2] - E\Big[E[Z_1|\{y(t)\}] \cdot E[Z_2|\{y(t)\}]\Big] \\ &= E[Z^2] - E\Big[E[Z_1Z_2|\{y(t)\}]\Big] \\ &= E[Z^2] - E[Z_1Z_2]\end{aligned}$$

4.3. Prove (4.20).

Solution:

The third central moment is

$$\frac{E[Z(c)^3]}{c^3} - \frac{3E[Z(c)^2]E[Z(c)]}{c^3} + \frac{2E[Z(c)]^3}{c^3}$$

The limit for each of these terms is

$$\begin{aligned}\lim_{c\to\infty}\frac{E[Z(c)^3]}{c^3} &= \frac{c(c-1)(c-2)E[Z_1Z_2Z_3]}{c^3} + \frac{3c(c-1)E[Z_1^2Z_c]}{c^3} + \frac{cE[Z^3]}{c^3} \\ &= E[Z_1Z_2Z_3] \\ \lim_{c\to\infty}\frac{E[Z(c)^2]E[Z(c)]}{c^3} &= \frac{c(c-1)E[Z_1Z_2]cE[Z]}{c^3} + \frac{cE[Z^2]cE[Z]}{c^3} \\ &= E[Z_1Z_2] \\ \lim_{c\to\infty}\frac{E[Z(c)]^3}{c^3} &= \frac{c^3E[Z^3]}{c^3} \\ &= E[Z^3]\end{aligned}$$

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so the limit of the third central moment is

$$\lim_{c\to\infty} \frac{E[Z(c)^3]}{c^3} - \frac{3E[Z(c)^2]E[Z(c)]}{c^3} + \frac{2E[Z(c)]^3}{c^3} = E[Z_1 Z_2 Z_3] - 3E[Z_1 Z_2] + 2E[Z^3]$$

from Exercise 4.1, $\lim_{c\to\infty} Var[Z(c)]/c^2 = E[Z_1 Z_2] - E[Z^2]$ so the limiting skewness is

$$\lim_{c\to\infty} sk[Z(c)/c] = \frac{E[Z_1 Z_2 Z_3] - 3E[Z_1 Z_2] + 2E[Z^3]}{(E[Z_1 Z_2] - E[Z^2])^{3/2}}$$

4.4. (a) Show that for n-year temporary insurance contracts

$$\lim_{c\to\infty} Var[Z(c)/c] = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} {}_{k_1|}q_x \cdot {}_{k_2|}q_x cov(e^{-y(k_1+1)}, e^{-y(k_2+1)})$$

(b) Show that this limiting variance is increasing with n.

Solution:

(a) The variance is

$$\frac{Var[Z(c)]}{c^2} = \frac{c(c-1)}{c^2} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} E[e^{-y(k_1+1)-y(k_2+1)}]_{k_1|}q_{x\,k_2|}q_x + \frac{c}{c^2} \sum_{k=0}^{n-1} E[e^{-y(k+1)^2}]_{k|}q_x - \frac{c^2}{c^2}\left[\sum_{k=0}^{n-1} E[e^{-y(k+1)}]_{k|}q_x\right]^2$$

so the limiting variance is

$$\lim_{c\to\infty} \frac{Var[Z(c)]}{c^2} = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} E[e^{-y(k_1+1)-y(k_2+1)}]_{k_1|}q_{x\,k_2|}q_x - \sum_{k=0}^{n-1} E[e^{-y(k+1)}]^2{}_{k|}q_x^2$$

The covariance of two present values is

$$Cov[e^{-y(k_1+1)}, e^{-y(k_2+1)}] = E[e^{-y(k_1+1)-y(k_2+1)}] - E[e^{-y(k_1+1)}]E[e^{-y(k_2+1)}]$$

we can rearrange the limiting variance, i.e.

$$\begin{aligned}\lim_{c\to\infty}\frac{Var[Z(c)]}{c^2} &= \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} E[e^{-y(k_1+1)-y(k_2+1)}]_{k_1|}q_{x\,k_2|}q_x - \\ &\quad \sum_{k_1=0}^{n-1} E[e^{-y(k+1)}]_{k_1|}q_x \sum_{k_2=0}^{n-1} E[e^{-y(k_2+1)}]_{k_2|}q_x \\ &= \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} {}_{k_1|}q_{x\,k_2|}q_x cov[pv(k_1+1), pv(k_2+1)]\end{aligned}$$

4.5. (a) Calculate the standard deviation of $Z(c)/c$ for $c = 10$ and ∞ for the 10-year temporary insurance contract of exercise 3.1 (b).

(b) Is the limiting variance equal to the interest risk defined in (3.17)? Is it equal to component (1) in exercise 3.1?

Solution:

(a) The following additional code to exercise 3.1 (b) was needed to calculate $sd[Z(c)/c]$.

```
> oumodel = iratemodel(list(delta = 0.05, delta0 = 0.08,
+         alpha = 0.10, sigma = 0.02), "ou")
> mort = mortassumptions(list(x = 50, table = "MaleMort82"))
> term = insurance(list(n = 10, d = 1), "isingle", "term")
> port10 = insurance(list(single = term, c = 10), "iport", "term")
> portInf = insurance(list(single = term, c = 1e18), "iport", "term")
> sd10 = z.sd(port10, mort, oumodel) / port10$c
> sdInf = z.sd(portInf, mort, oumodel) / portInf$c
```

This gives $sd[Z(c)/c] = 0.06426706$ for $c = 10$ and $sd[Z(c)/c] = 0.0082176$ for $c = \infty$.

(b) The limiting variance is $6.753e-05$. This is the same as component (1) in exercise 3.1.

5 Distribution of the Present Value of Benefits for a Portfolio

5.1. (a) Show that $\rho(y(n), y(n-1))$ does not depend on the parameters of the White Noise process for the force of interest.

(b) Show that $\rho(y(n), y(n-1))$ does not depend on the parameters δ, δ_0 and σ of the Ornstein-Uhlenbeck process for the force of interest.

Solution:

(a) For a white-noise process, $Var[y(t)] = \sigma^2 t$ and $Cov[y(s), y(t)] = \sigma^2 \min(s,t)$ so we have that

$$\begin{aligned}\rho(y(n), y(n-1)) &= \frac{Cov(y(n), y(n-1)}{\sqrt{Var(y(n))}\sqrt{Var(y(n-1))}} \\ &= \frac{\sigma^2(n-1)}{\sqrt{\sigma^2 n}\sqrt{\sigma^2(n-1)}} \\ &= \frac{(n-1)}{[n(n-1)]^{0.5}}\end{aligned}$$

this does not depend on σ^2.

(b) For an Ornstein-Uhlenbeck process,

$$\begin{aligned}Var[y(t)] &= \frac{\sigma^2}{\alpha^2}t + \frac{\sigma^2}{2\alpha^3}(-3 + 4\exp(-\alpha t) - \exp(-2\alpha t)) \\ Cov[y(s), y(t)] &= \frac{\sigma^2}{\alpha^2}\min(s,t) + \frac{\sigma^2}{2\alpha^3}(-2 + 2\exp(-\alpha s) + \\ &\quad 2\exp(-\alpha t) - \exp(-\alpha|t-s|) - \exp(-\alpha(t+s)))\end{aligned}$$

so we have that

$$\begin{aligned}\rho(y(n), y(n-1)) &= \frac{\frac{\sigma^2}{\alpha^2}(n-1) + \frac{\sigma^2}{2\alpha^3}(-2 + 2e^{-\alpha(n-1)} + 2e^{-\alpha(n)} - e^{-\alpha} - e^{-\alpha(2n-1)})}{\sqrt{\frac{\sigma^2}{\alpha^2}n + \frac{\sigma^2}{2\alpha^3}(-3 + 4e^{-\alpha n} - e^{-2\alpha n})}\sqrt{\frac{\sigma^2}{\alpha^2}(n-1) + \frac{\sigma^2}{2\alpha^3}(-3 + 4e^{-\alpha(n-1)} - e^{-2\alpha(n-1)})}} \\ &= \frac{\frac{1}{\alpha^2}(n-1) + \frac{1}{2\alpha^3}(-2 + 2e^{-\alpha(n-1)} + 2e^{-\alpha(n)} - e^{-\alpha} - e^{-\alpha(2n-1)})}{\sqrt{\frac{1}{\alpha^2}n + \frac{1}{2\alpha^3}(-3 + 4e^{-\alpha n} - e^{-2\alpha n})}\sqrt{\frac{1}{\alpha^2}(n-1) + \frac{1}{2\alpha^3}(-3 + 4e^{-\alpha(n-1)} - e^{-2\alpha(n-1)})}}\end{aligned}$$

which only depends on α.