A Note on Integration of Banach-Valued Functions and Applications to Parabolic PDEs

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Abstract

While studying partial differential equations (PDEs) during my master's course, I encountered Banach-valued Sobolev spaces such as $W^{1,p}(0,T;X)$ and Bochner integrals in the analysis of parabolic equations. Our lectures followed Evans [Ev], where Bochner integration is introduced briefly in an appendix, and the Fundamental Theorem of Calculus for Banach-valued functions is invoked without a complete proof. To understand these results rigorously, I consulted Folland [F2] and Yosida [Y], but found that the treatment of Riemann and Bochner integrals is often concise and notationally inconsistent, making it difficult to trace precisely what is used in Evans [Ev].

This work consolidates the main results on integration in Banach spaces within a consistent framework. Where the literature omits or sketches arguments, I provide complete and self-contained proofs. In doing so, I clarify the link between the Riemann and Bochner integrals and establish the Fundamental Theorem of Calculus rigorously in the setting of $W^{1,p}(0,T;X)$ spaces. The exposition is intended as a concise reference for readers seeking a clear analytic foundation for the weak formulation of parabolic PDEs.

Contents

1 Preliminaries

Throughout this paper, we establish a theory for integrating functions $f : [0,T] \to X$ taking values in the Banach space $(X, \|\cdot\|_X)$ over the field of real numbers. We endow the interval $[0,T]$ with the Lebesgue measure and consider the measure space $([0,T], \mathcal{L}([0,T]), \lambda)$, where λ denotes the Lebesgue measure on $[0,T]$ and $\mathcal{L}([0,T])$ is the λ-completion of the Borel σ-algebra $\mathcal{B}([0,T])$. In particular, $\mathcal{L}([0,T])$ consists of all sets of the form $B \cup N$, where $B \in \mathcal{B}([0,T])$ and $N \subseteq N_0$ for some λ-null set N_0. Functions are identified when equal λ-almost everywhere.

We will use the notions of summable and integrable from Evans

2 Riemann Integration in Banach Spaces

2.1 Definition

2.2 Existence theorem

2.3 Linearity and dual compatibility

2.4 Fundamental theorem of calculus

3 Bochner Integration

3.1 Definition via simple functions

Definition 3.1. *A function $f : [0,T] \to X$ is said to be* simple *if it can be written as:*

$$f(t) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(t) u_i(t),$$

where $E_1, \ldots E_n \in \mathcal{L}([0,T])$ and $u_1, \ldots, u_n \in X$

Definition 3.2. *Let $f : [0,T] \to X$ be a simple function then we define:*

$$\int_0^T f(t)\, \lambda(dt) \coloneqq \sum_{i=1}^{n} \lambda(E_i) u_i(t).$$

Lemma 3.1 (Linearity)**.**

Lemma 3.2 (Triangle Inequality)**.** *Let $f : [0,T] \to X$ be a simple function then:*

$$\left\| \int_0^T f(t)\, \lambda(dt) \right\|_X \le \int_0^T \|f(t)\|_X \, \lambda(dt),$$

Proof. Suppose

$$f(t) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(t) u_i(t),$$

for some $E_1, \dots, E_n \in \mathcal{L}([0,T])$ and $u_1, \dots, u_n \in X$. We apply the norm and apply the usual triangle inequality to get the result,

$$\begin{aligned}\left\|\int_0^T f(t)\,\lambda(dt)\right\|_X &= \left\|\sum_{i=1}^n \lambda(E_i)u_i(t)\right\|_X \\ &\leq \sum_{i=1}^n \lambda(E_i)\|u_i(t)\|_X \\ &= \int_0^T \sum_{i=1}^n \mathbb{1}_{E_i}(t)\|u_i(t)\|_X\,\lambda(dt) \\ &= \int_0^T \|f(t)\|_X\,\lambda(dt)\end{aligned}$$

□

3.2 Extension by completion

Definition 3.3. *A function $f : [0,T] \to X$ is said to be* strongly measurable *if there exists a sequence of simple functions $(s_n : [0,T] \to X)_{n\in\mathbb{N}}$ such that:*

$$\|s_n(t) - f(t)\|_X \to 0 \quad \textit{as } n \to \infty \quad \textit{a.e. } 0 \leq t \leq T.$$

Definition 3.4. *A function $f : [0,T] \to X$ is said to be* summable *if there exists a sequence of simple functions $(s_n : [0,T] \to X)_{n\in\mathbb{N}}$ such that:*

$$\int_0^T ||s_n(t) - f(t)||_X\,\lambda(dt) \longrightarrow 0 \quad \textit{as } n \to \infty,$$

for a summable *function $f : [0,T] \to X$ we define:*

$$\int_0^T f(t)\,\lambda(dt) \coloneqq \lim_{n\to\infty} \int_0^T s_n(t)\,\lambda(dt), \tag{1}$$

Lemma 3.3 (Linearity)**.** *Let $f, g : [0,T] \to X$ be summable and $\alpha, \beta \in \mathbb{R}$ then:*

$$\int_0^T \alpha f(t) + \beta g(t)\,\lambda(dt) = \alpha \int_0^T f(t)\,\lambda(dt) + \beta \int_0^T g(t)\,\lambda(dt)$$

Proof. Since f and g are summable there exists respective approximating sequences of simple functions $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$, so we write:

$$\int_0^T \alpha f(t) + \beta g(t)\,\lambda(dt) = \lim_{n\to\infty} \int_0^T \alpha u_n(t) + \beta v_n(t)\,\lambda(dt),$$

we note that $\alpha u_n(t) + \beta v_n(t)$ is a simple function as the sum of simple functions so we apply linearity of the Bochner integral for simple functions 3.1 to obtain:

$$\int_0^T \alpha u_n(t) + \beta v_n(t)\,\lambda(dt) = \alpha \int_0^T u_n(t)\,\lambda(dt) + \beta \int_0^T v_n(t)\,\lambda(dt),$$

taking the limit as $n \to \infty$ gives the result. □

Lemma 3.4 (Triangle Inequality)**.** *Let $f : [0, T] \to X$ be summable then:*

$$\left\| \int_0^T f(t)\, \lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X \, \lambda(dt).$$

Proof. Since f is summable there exists a sequence of appoximating simple functions $(s_n)_{n\in\mathbb{N}}$ so we can write:

$$\left\| \int_0^T f(t)\, \lambda(dt) \right\|_X = \lim_{n\to\infty} \left\| \int_0^T s_n(t)\, \lambda(dt) \right\|_X \leq \lim_{n\to\infty} \int_0^T \|s_n(t)\|_X \, \lambda(dt),$$

where we have used continuity of $\|\cdot\|_X$ and the triangle inequality for Bochner integrals of simple functions 3.2. It remains to show that:

$$\lim_{n\to\infty} \int_0^T \|s_n(t)\|_X \, \lambda(dt) = \int_0^T \|f(t)\|_X \, \lambda(dt).$$

Indeed, by linearity of the Lebesgue integral, triangle and reverse triangle inequalities and the dominated convergence theorem we obtain:

$$\begin{aligned} \left| \int_0^T \|s_n(t)\|_X \, \lambda(dt) - \int_0^T \|f(t)\|_X \, \lambda(dt) \right| &\leq \int_0^T \big| \|s_n(t)\|_X - \|f(t)\|_X \big| \, \lambda(dt) \\ &\leq \int_0^T \|s_n(t) - f(t)\|_X \, \lambda(dt) \longrightarrow 0 \end{aligned}$$

as $n \to \infty$ □

Definition 3.5. *We define the function space $L^1(0, T; X)$ to be the space of all Bochner summable functions, with the norm:*

$$\|f\|_{L^1([0,T];X)} := \int_0^T \|f(t)\|_X \, \lambda(dt).$$

Remark 3.1. *Now we have established a function space and a norm for Bochner summable functions it is useful to note that by the triangle inequality 3.4:*

$$\left\| \int_0^T f(t)\, \lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X \, \lambda(dt) = \|f\|_{L^1(0,T;X)},$$

so the integral is a bounded linear operator, $I : L^1(0, T; X) \to X$

Remark 3.2. *The definition of the Bochner integral for general summable functions* (1) *is well defined since $L^1(0, T; X)$ is the completion of the space of simple functions with respect to the norm $\|\cdot\|_{L^1(0,T;X)}$. Thus, the integral operator defined on simple functions extends uniquely and continuously to all Bochner summable functions.*

Loosely speaking, let $f : [0, T] \to X$ be summable and let $(s_n)_{n\in\mathbb{N}}$ be an approximating sequence of simple functions. We denote the integral operator for simple functions by I, so for $n \in \mathbb{N}$

$$I(s_n) = \int_0^T s_n(t)\, \lambda(dt) \in X.$$

Then, by the triangle inequality 3.2,

$$\|I(s_n) - I(s_m)\|_X \leq \int_0^T \|s_n(t) - s_m(t)\|_X \, \lambda(dt).$$

Since I is a bounded linear operator 3.1 it is continuous, so we can pass the limits as $n, m \to \infty$ inside the integral to deduce that the sequence $(I(s_n))_{n\in\mathbb{N}}$ is Cauchy in X. By the completeness of X, this sequence converges to a limit in X, which we define by (1). *For this to be completely rigorous we would need to show that this limit is independent of the choice of approximating sequence but this is routine.*

3.3 Equivalence with Riemann

4 Fundamental Theorem of Calculus in $W^{1,p}(0,T;X)$

References

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[F2] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Wiley-Interscience, New York, 1999.

[Y] K. Yosida, *Functional Analysis*, 6th ed., Springer-Verlag, Berlin, 1980.