

A Note on Integration of Banach-Valued Functions and Applications to Parabolic PDEs

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Abstract

While studying partial differential equations (PDEs) during my master's course, I encountered Banach-valued Sobolev spaces such as $W^{1,p}(0, T; X)$ and Bochner integrals in the analysis of parabolic equations. Our lectures followed Evans [Ev], where Bochner integration is introduced briefly in an appendix, and the Fundamental Theorem of Calculus for Banach-valued functions is invoked without a complete proof. To understand these results rigorously, I consulted Folland [F2] and Yosida [Y], but found that the treatment of Riemann and Bochner integrals is often concise and notationally inconsistent, making it difficult to trace precisely what is used in Evans [Ev].

This work consolidates the key results on integration in Banach spaces within a consistent framework. Where the literature omits or sketches arguments, I have tried to provide and reference complete and self-contained proofs. In doing so, I clarify the link between the Riemann and Bochner integrals and establish the Fundamental Theorem of Calculus rigorously in the setting of $W^{1,p}(0, T; X)$ spaces. The exposition is intended as a concise reference for readers seeking a clear analytic foundation for the weak formulation of parabolic PDEs.

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1 Preliminaries

Throughout this exposition, we establish a theory for integrating functions $f : [0, T] \rightarrow X$ taking values in the Banach space $(X, \|\cdot\|_X)$ over the field of real numbers. We endow the interval $[0, T]$ with the Lebesgue measure and consider the measure space $([0, T], \mathcal{L}([0, T]), \lambda)$, where λ denotes the Lebesgue measure on $[0, T]$ and $\mathcal{L}([0, T])$ is the λ -completion of the Borel σ -algebra $\mathcal{B}([0, T])$. In particular, $\mathcal{L}([0, T])$ consists of all sets of the form $B \cup N$, where $B \in \mathcal{B}([0, T])$ and $N \subseteq N_0$ for some λ -null set N_0 . Functions are identified when equal λ -almost everywhere.

2 Riemann Integration in Banach Spaces

In this section we lift results and their proofs from the fantastic article by Chernysh [Ch]. For a more comprehensive text on the properties of the Riemann integral for Banach-valued functions you are encouraged to refer to their article. We only highlight the relevant results needed to develop the fundamental theorem of calculus for continuous Banach-valued functions.

2.1 Definitions

Definition 2.1 (Partitions). *Consider the interval $[a, b]$ and partition of it $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$. A set of tags for \mathcal{P} is a collection $\{t_1, \dots, t_n\}$ of points in $[a, b]$ such that $t_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$). A tagged partition is the pairing of a partition together with a set of tags e.g. $\dot{\mathcal{P}} = (\mathcal{P}, \{t_i\}_{i=1}^n)$. We will use the convention that a dotted partition is assumed to be equipped with tags. We also define the mesh of a partition \mathcal{P} as*

$$|\mathcal{P}| = \max_{1 \leq i \leq n} (x_i - x_{i-1}).$$

Definition 2.2 (Refinements).

Definition 2.3 (Riemann Sum). *Let $f : [a, b] \rightarrow X$ be a function and $\dot{\mathcal{P}}$ a tagged partition of $[a, b]$. The Riemann sum of f over $\dot{\mathcal{P}}$ is defined as*

$$S(f, \mathcal{P}) := \sum_{i=1}^n f(t_i)(x_i - x_{i-1}).$$

Definition 2.4. *A function $f : [a, b] \rightarrow X$ is Riemann integrable if $\exists \Gamma \in X$ so that $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that*

$$\|S(x, \dot{\mathcal{P}}) - \Gamma\|_X < \varepsilon,$$

for all tagged partitions $\dot{\mathcal{P}}$ of $[a, b]$ with $|\mathcal{P}| < \delta$. If f is Riemann integrable we call Γ the Riemann integral of f over $[a, b]$ and denote it by

$$\int_a^b f(t) dt.$$

We will also refer to $\mathcal{R}([a, b]; X)$ as the space of Riemann integrable functions.

2.2 Uniqueness of the Riemann integral

Proposition 2.1 (Uniqueness). *Let $f \in \mathcal{R}([a, b]; X)$, then its Riemann integral is unique.*

Proof.

□

2.3 Properties of the Riemann integral

Proposition 2.2 (Linearity). *Let $f, g \in \mathcal{R}([a, b]; X)$ and $\alpha, \beta \in \mathbb{R}$ then*

$$\int_a^b \alpha f(t) + \beta g(t) dt = \alpha \int_a^b f(t) dt + \beta \int_a^b g(t) dt$$

Proof.

□

2.4 Continuous functions and Riemann integrability

Definition 2.5. *space of continuous functions. with norm*

Proposition 2.3. *A function $f : [a, b] \rightarrow X$ is Riemann integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})\|_X < \varepsilon$$

for all tagged partitions $\dot{\mathcal{P}}, \dot{\mathcal{Q}}$ of $[a, b]$ whose meshes are strictly less than δ .

Proof.

□

Lemma 2.1. *Let $f \in C([0, T]; X)$. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ whose mesh is strictly less than δ , and any refinement $\dot{\mathcal{Q}}$ of $\dot{\mathcal{P}}$ then*

$$\|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})\|_X < \varepsilon$$

Proof.

□

Theorem 2.1. *Let $f \in C([0, T]; X)$, then f is Riemann integrable on $[a, b]$.*

Proof.

□

2.5 Fundamental theorem of calculus

Define $C^1([a, b]; X)$

Definition 2.6. *A function $f : [a, b] \rightarrow X$ is differentiable at a point $c \in [a, b]$ if there exists $\lambda \in X$ such that*

$$f(t) - f(c) = \lambda(t - c) + o(|t - c|)$$

as $t \rightarrow c$ with $t \in [a, b]$. That is, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(t) - f(c) - \lambda(t - c)\|_X \leq |t - c|, \quad \text{for all } t \in [a, b] \text{ with } |t - c| < \delta.$$

In this case we write $f'(c) = \lambda$ can call this the derivative of f at c . f is called differentiable on $[a, b]$ if it is differentiable at all points in $[a, b]$.

Comment about functional analytic definition and how there is an equivalent expression that looks more like the tradition calculus derivative i.e. next lemma.

Lemma 2.2. *Fix a point $c \in [a, b]$, a function $f : [a, b] \rightarrow X$ is differentiable at c , with derivative $f'(c) \in X$ if and only if*

$$\lim_{\substack{t \rightarrow c \\ t \in [a, b]}} \frac{f(t) - f(c)}{t - c}$$

exists and is equal to $f'(c)$.

Proof. □

Lemma 2.3. *Let $f : [a, b] \rightarrow X$ be differentiable at a point $c \in [a, b]$ then f is continuous at c .*

Proof. □

Theorem 2.2. *Let $f \in C([a, b]; X)$ and define*

$$F : [a, b] \rightarrow X, \quad F(x) := \int_a^x f(t) dt,$$

then $F \in C^1([a, b]; X)$ and satisfies $F' \equiv f$ on $[a, b]$.

Proof. □

Proposition 2.4. *Let $f \in C([a, b]; X)$ and differentiable on (a, b) . If $f'(t) = 0$ for all $t \in (a, b)$ then f is constant.*

Proof. □

Theorem 2.3. *Let $f \in C^1([a, b]; X)$ then the following holds*

$$\int_a^b f'(t) dt = f(b) - f(a)$$

Proof. □

3 Bochner Integration

3.1 Definition via simple functions

Definition 3.1. *A function $f : [0, T] \rightarrow X$ is said to be simple if it can be written as:*

$$f(t) = \sum_{i=1}^n \mathbb{1}_{E_i}(t) u_i(t),$$

where $E_1, \dots, E_n \in \mathcal{L}([0, T])$ and $u_1, \dots, u_n \in X$

Definition 3.2. *Let $f : [0, T] \rightarrow X$ be a simple function then we define:*

$$\int_0^T f(t) \lambda(dt) := \sum_{i=1}^n \lambda(E_i) u_i(t).$$

Lemma 3.1 (Linearity).

Lemma 3.2 (Triangle Inequality). *Let $f : [0, T] \rightarrow X$ be a simple function then:*

$$\left\| \int_0^T f(t) \lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X \lambda(dt),$$

Proof. Suppose

$$f(t) = \sum_{i=1}^n \mathbb{1}_{E_i}(t) u_i(t),$$

for some $E_1, \dots, E_n \in \mathcal{L}([0, T])$ and $u_1, \dots, u_n \in X$. We apply the norm and apply the usual triangle inequality to get the result,

$$\begin{aligned} \left\| \int_0^T f(t) \lambda(dt) \right\|_X &= \left\| \sum_{i=1}^n \lambda(E_i) u_i(t) \right\|_X \\ &\leq \sum_{i=1}^n \lambda(E_i) \|u_i(t)\|_X \\ &= \int_0^T \sum_{i=1}^n \mathbb{1}_{E_i}(t) \|u_i(t)\|_X \lambda(dt) \\ &= \int_0^T \|f(t)\|_X \lambda(dt) \end{aligned}$$

□

3.2 Extension by completion

Definition 3.3. A function $f : [0, T] \rightarrow X$ is said to be strongly measurable if there exists a sequence of simple functions $(s_n : [0, T] \rightarrow X)_{n \in \mathbb{N}}$ such that:

$$\|s_n(t) - f(t)\|_X \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{a.e. } 0 \leq t \leq T.$$

Definition 3.4. A function $f : [0, T] \rightarrow X$ is said to be summable if there exists a sequence of simple functions $(s_n : [0, T] \rightarrow X)_{n \in \mathbb{N}}$ such that:

$$\int_0^T \|s_n(t) - f(t)\|_X \lambda(dt) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

for a summable function $f : [0, T] \rightarrow X$ we define:

$$\int_0^T f(t) \lambda(dt) := \lim_{n \rightarrow \infty} \int_0^T s_n(t) \lambda(dt), \tag{1}$$

Lemma 3.3 (Linearity). *Let $f, g : [0, T] \rightarrow X$ be summable and $\alpha, \beta \in \mathbb{R}$ then:*

$$\int_0^T \alpha f(t) + \beta g(t) \lambda(dt) = \alpha \int_0^T f(t) \lambda(dt) + \beta \int_0^T g(t) \lambda(dt)$$

Proof. Since f and g are summable there exists respective approximating sequences of simple functions $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$, so we write:

$$\int_0^T \alpha f(t) + \beta g(t) \lambda(dt) = \lim_{n \rightarrow \infty} \int_0^T \alpha u_n(t) + \beta v_n(t) \lambda(dt),$$

we note that $\alpha u_n(t) + \beta v_n(t)$ is a simple function as the sum of simple functions so we apply linearity of the Bochner integral for simple functions 3.1 to obtain:

$$\int_0^T \alpha u_n(t) + \beta v_n(t) \lambda(dt) = \alpha \int_0^T u_n(t) \lambda(dt) + \beta \int_0^T v_n(t) \lambda(dt),$$

taking the limit as $n \rightarrow \infty$ gives the result. \square

Lemma 3.4 (Triangle Inequality). *Let $f : [0, T] \rightarrow X$ be summable then:*

$$\left\| \int_0^T f(t) \lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X \lambda(dt).$$

Proof. Since f is summable there exists a sequence of approximating simple functions $(s_n)_{n \in \mathbb{N}}$ so we can write:

$$\left\| \int_0^T f(t) \lambda(dt) \right\|_X = \lim_{n \rightarrow \infty} \left\| \int_0^T s_n(t) \lambda(dt) \right\|_X \leq \lim_{n \rightarrow \infty} \int_0^T \|s_n(t)\|_X \lambda(dt),$$

where we have used continuity of $\|\cdot\|_X$ and the triangle inequality for Bochner integrals of simple functions 3.2. It remains to show that:

$$\lim_{n \rightarrow \infty} \int_0^T \|s_n(t)\|_X \lambda(dt) = \int_0^T \|f(t)\|_X \lambda(dt).$$

Indeed, by linearity of the Lebesgue integral, triangle and reverse triangle inequalities and the dominated convergence theorem we obtain:

$$\begin{aligned} \left| \int_0^T \|s_n(t)\|_X \lambda(dt) - \int_0^T \|f(t)\|_X \lambda(dt) \right| &\leq \int_0^T |\|s_n(t)\|_X - \|f(t)\|_X| \lambda(dt) \\ &\leq \int_0^T \|s_n(t) - f(t)\|_X \lambda(dt) \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ \square

Definition 3.5. *We define the function space $L^1(0, T; X)$ to be the space of all Bochner summable functions, with:*

$$\|f\|_{L^1([0, T]; X)} := \int_0^T \|f(t)\|_X \lambda(dt) < \infty.$$

Remark 3.1. *Every Bochner summable function is strongly measurable, since convergence in the L^1 -norm implies convergence almost everywhere along a subsequence. Conversely, a function $f : [0, T] \rightarrow X$ is Bochner summable if and only if it is strongly measurable and the scalar function $t \mapsto \|f(t)\|_X$ is Lebesgue integrable:*

$$\int_0^T \|f(t)\|_X \lambda(dt) < \infty.$$

This equivalence identifies the abstractly constructed space of Bochner summable functions with the more familiar intrinsic form

$$L^1(0, T; X) = \{ f : [0, T] \rightarrow X \text{ strongly measurable} : \|f(\cdot)\|_X \in L^1(0, T) \}.$$

Remark 3.2. Now we have established a function space and a norm for Bochner summable functions it is useful to note that by the triangle inequality 3.4:

$$\left\| \int_0^T f(t) \lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X \lambda(dt) = \|f\|_{L^1(0,T;X)},$$

so the integral is a bounded linear operator, $I : L^1(0, T; X) \rightarrow X$

Remark 3.3. The definition of the Bochner integral for general summable functions (1) is well defined since $L^1(0, T; X)$ is the completion of the space of simple functions with respect to the norm $\|\cdot\|_{L^1(0,T;X)}$. Thus, the integral operator defined on simple functions extends uniquely and continuously to all Bochner summable functions.

Loosely speaking, let $f : [0, T] \rightarrow X$ be summable and let $(s_n)_{n \in \mathbb{N}}$ be an approximating sequence of simple functions. We denote the integral operator for simple functions by I , so for $n \in \mathbb{N}$

$$I(s_n) = \int_0^T s_n(t) \lambda(dt) \in X.$$

Then, by the triangle inequality 3.2,

$$\|I(s_n) - I(s_m)\|_X \leq \int_0^T \|s_n(t) - s_m(t)\|_X \lambda(dt).$$

Since I is a bounded linear operator 3.2 it is continuous, so we can pass the limits as $n, m \rightarrow \infty$ inside the integral to deduce that the sequence $(I(s_n))_{n \in \mathbb{N}}$ is Cauchy in X . By the completeness of X , this sequence converges to a limit in X , which we define by (1). For this to be completely rigorous we would need to show that this limit is independent of the choice of approximating sequence but this is routine.

3.3 Equivalence with Riemann

Theorem 3.1. Let $f \in C(0, T; X)$ then

- (i) f is Bochner summable,
- (ii) the Riemann and Bochner integrals coincide, i.e.

$$\int_0^T f(t) dt = \int_0^T f(t) \lambda(dt).$$

Proof. (i) Since f is continuous with compact domain $[0, T]$, it is uniformly continuous and bounded. We define the modulus of continuity which serves as a way to quantify the uniform continuity of a function by

$$\omega(\delta) := \sup\{\|f(t) - f(s)\|_X : |t - s| \leq \delta, t, s \in [0, T]\},$$

by uniform continuity of f it should be clear that as $\delta \rightarrow 0$ we have $\omega(\delta) \rightarrow 0$.

Let $\mathcal{P} = \{(t_{i-1}, t_i, \tau_i)\}_{i=1}^n$ be a tagged partition with mesh $|\mathcal{P}| := \max_{1 \leq i \leq n} (t_i - t_{i-1})$ and define the simple function

$$s_{\mathcal{P}}(t) := \sum_{i=1}^n \mathbb{1}_{(t_{i-1}, t_i]} f(\tau_i).$$

We apply the Bochner integral and note that this is equal to the Riemann sum,

$$\int_0^T s_{\mathcal{P}}(t) \lambda(dt) = \sum_{i=1}^n \lambda(t_{i-1}, t_i] f(\tau_i) = S(f, \mathcal{P}).$$

Note for any $t \in (t_{i-1}, t_i]$ we have

$$\|s_{\mathcal{P}}(t) - f(t)\|_X = \|f(\tau_i) - f(t)\|_X \leq \omega(|\mathcal{P}|),$$

so now applying the Lebesgue integral and taking the limit as the mesh gets finer we show that $s_{\mathcal{P}} \rightarrow f$ in $L^1(0, T; X)$,

$$\int_0^T \|s_{\mathcal{P}}(t) - f(t)\|_X \lambda(dt) \leq \int_0^T \omega(|\mathcal{P}|) \lambda(dt) \leq T \cdot \omega(|\mathcal{P}|),$$

since $\omega(|\mathcal{P}|) \rightarrow 0$ as $|\mathcal{P}| \rightarrow 0$ we get that indeed $s_{\mathcal{P}} \rightarrow f$ in $L^1(0, T; X)$. We can conclude that f is Bochner summable.

- (ii) Appealing to the fact the Bochner integral is a bounded linear operator 3.2 we have that

$$\left\| \int_0^T s_{\mathcal{P}}(t) \lambda(dt) - \int_0^T f(t) \lambda(dt) \right\|_X \leq \int_0^T \|s_{\mathcal{P}}(t) - f(t)\|_X \lambda(dt),$$

from the proof of part (i) we know the right hand side tends to 0 as the mesh gets finer, rewriting this reads

$$S(f, \mathcal{P}) \longrightarrow \int_0^T f(t) \lambda(dt)$$

in X as $|\mathcal{P}| \rightarrow 0$.

Finally applying the definition of the Riemann integral and uniqueness of limits we conclude that for $f \in C(0, T; X)$

$$\int_0^T f(t) dt = \int_0^T f(t) \lambda(dt)$$

□

4 Fundamental Theorem of Calculus in $W^{1,p}(0, T; X)$

In this section we define the Sobolev space $W^{1,p}(0, T; X)$ and annotate the proof of theorem 2 (calculus in an abstract space) in section 5.9 of [Ev] using the results we have established.

Definition 4.1. We define the function space $L^p(0, T; X)$ to be the space of all strongly measurable functions $u : [0, T] \rightarrow X$, with:

$$\|u\|_{L^p([0,T];X)} := \left(\int_0^T \|u(t)\|_X^p \lambda(dt) \right)^{\frac{1}{p}} < \infty$$

for $1 \leq p < \infty$, and

$$\|u\|_{L^\infty([0,T];X)} := \operatorname{ess\,sup}_{t \in [0,T]} \|u(t)\|_X < \infty.$$

Definition 4.2. Let $u \in L^1(0, T; X)$, we say $v \in L^1(0, T; X)$ is the weak derivative of u , written

$$u' = v,$$

provided that for all scalar test functions $\phi \in C_c^\infty(0, T)$

$$\int_0^T \phi'(t) u(t) \lambda(dt) = - \int_0^T \phi(t) v(t) \lambda(dt).$$

Definition 4.3. We define the Sobolev space $W^{1,p}(0, T; X)$ to consist of all functions $u \in L^p(0, T; X)$ such that u' exists in the weak sense and belongs to $L^p(0, T; X)$, with norm:

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|_X^p + \|u'(t)\|_X^p \right)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \operatorname{ess\,sup}_{t \in [0,T]} \|u(t)\|_X + \|u'(t)\|_X & (p = \infty) \end{cases}$$

Theorem 4.1 (Calculus in an abstract space [Ev]). Let $u \in W^{1,p}(0, T; X)$ for some $1 \leq p \leq \infty$, then

(i) $u \in C([0, T]; X)$, after possibly being redefined on a set of measure zero,

(ii) for all $0 \leq s \leq t \leq T$

$$u(t) = u(s) + \int_s^t u'(\tau) \lambda(d\tau),$$

(iii) for a constant C depending on T

$$\|u\|_{C([0,T];X)} \leq C \|u\|_{W^{1,p}(0,T;X)}$$

Proof. (i)

(ii)

(iii)

□

A Additional Results

For completeness we state other well known results for the Bochner integral but omit their proofs as they are not the primary focus of this paper.

References

- [Ev] L. C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, AMS, Providence, RI, 2010.
- [F2] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Wiley-Interscience, New York, 1999.
- [Y] K. Yosida, *Functional Analysis*, 6th ed., Springer-Verlag, Berlin, 1980.
- [Ch] E. Chernysh, *The Riemann Integral for Functions Mapping to Banach Spaces*, the article can be found [here](#).