# A Note on Integration of Banach-Valued Functions and Applications to Parabolic PDEs

Nathan Evans

November 2, 2025

**Abstract**

While studying partial differential equations (PDEs) during my master's course, I encountered Banach-valued Sobolev spaces such as $W^{1,p}(0,T;X)$ and Bochner integrals in the analysis of parabolic equations. Our lectures followed Evans [Ev], where Bochner integration is introduced briefly in an appendix, and the Fundamental Theorem of Calculus for Banach-valued functions is invoked without a complete proof. To understand these results rigorously, I consulted Folland [F2] and Yosida [Y], but found that the treatment of Riemann and Bochner integrals is often concise and notationally inconsistent, making it difficult to trace precisely what is used in Evans [Ev].

This work consolidates the main results on integration in Banach spaces within a consistent framework. Where the literature omits or sketches arguments, I provide complete and self-contained proofs. In doing so, I clarify the link between the Riemann and Bochner integrals and establish the Fundamental Theorem of Calculus rigorously in the setting of $W^{1,p}(0,T;X)$ spaces. The exposition is intended as a concise reference for readers seeking a clear analytic foundation for the weak formulation of parabolic PDEs.

## Contents

# 1 Preliminaries

Throughout this paper, we establish a theory for integrating functions $f : [0, T] \to X$ taking values in the Banach space $(X, \|\cdot\|_X)$ over the field of real numbers. We endow the interval $[0, T]$ with the Lebesgue measure and consider the measure space $([0, T], \mathcal{L}([0, T]), \lambda)$, where $\lambda$ denotes the Lebesgue measure on $[0, T]$ and $\mathcal{L}([0, T])$ is the $\lambda$-completion of the Borel $\sigma$-algebra $\mathcal{B}([0, T])$. In particular, $\mathcal{L}([0, T])$ consists of all sets of the form $B \cup N$, where $B \in \mathcal{B}([0, T])$ and $N \subseteq N_0$ for some $\lambda$-null set $N_0$. Functions are identified when equal $\lambda$-almost everywhere.

# 2 Riemann Integration in Banach Spaces

In this section we mostly lift results and their proofs from the article by Chernysh [Ch]

## 2.1 Definition

## 2.2 Existence theorem

## 2.3 Linearity and dual compatibility

## 2.4 Fundamental theorem of calculus

# 3 Bochner Integration

## 3.1 Definition via simple functions

**Definition 3.1.** *A function $f : [0, T] \to X$ is said to be* simple *if it can be written as:*

$$f(t) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(t) u_i(t),$$

*where $E_1, \dots E_n \in \mathcal{L}([0, T])$ and $u_1, \dots, u_n \in X$*

**Definition 3.2.** *Let $f : [0, T] \to X$ be a simple function then we define:*

$$\int_0^T f(t)\, \lambda(dt) := \sum_{i=1}^{n} \lambda(E_i) u_i(t).$$

**Lemma 3.1** (Linearity)**.**

**Lemma 3.2** (Triangle Inequality)**.** *Let $f : [0, T] \to X$ be a simple function then:*

$$\left\| \int_0^T f(t)\, \lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X\, \lambda(dt),$$

*Proof.* Suppose

$$f(t) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(t) u_i(t),$$

for some $E_1, \ldots, E_n \in \mathcal{L}([0,T])$ and $u_1, \ldots, u_n \in X$. We apply the norm and apply the usual triangle inequality to get the result,

$$\begin{aligned}
\left\| \int_0^T f(t)\, \lambda(dt) \right\|_X &= \left\| \sum_{i=1}^n \lambda(E_i) u_i(t) \right\|_X \\
&\leq \sum_{i=1}^n \lambda(E_i) \|u_i(t)\|_X \\
&= \int_0^T \sum_{i=1}^n \mathbb{1}_{E_i}(t) \|u_i(t)\|_X \, \lambda(dt) \\
&= \int_0^T \|f(t)\|_X \, \lambda(dt)
\end{aligned}$$

□

## 3.2 Extension by completion

**Definition 3.3.** *A function $f : [0,T] \to X$ is said to be* strongly measurable *if there exists a sequence of simple functions $(s_n : [0,T] \to X)_{n \in \mathbb{N}}$ such that:*

$$\|s_n(t) - f(t)\|_X \to 0 \quad \text{as } n \to \infty \quad \text{a.e. } 0 \leq t \leq T.$$

**Definition 3.4.** *A function $f : [0,T] \to X$ is said to be* summable *if there exists a sequence of simple functions $(s_n : [0,T] \to X)_{n \in \mathbb{N}}$ such that:*

$$\int_0^T ||s_n(t) - f(t)||_X \, \lambda(dt) \longrightarrow 0 \quad \text{as } n \to \infty,$$

*for a* summable *function $f : [0,T] \to X$ we define:*

$$\int_0^T f(t)\, \lambda(dt) \; := \; \lim_{n \to \infty} \int_0^T s_n(t)\, \lambda(dt), \tag{1}$$

**Lemma 3.3** (Linearity)**.** *Let $f, g : [0,T] \to X$ be summable and $\alpha, \beta \in \mathbb{R}$ then:*

$$\int_0^T \alpha f(t) + \beta g(t)\, \lambda(dt) = \alpha \int_0^T f(t)\, \lambda(dt) + \beta \int_0^T g(t)\, \lambda(dt)$$

*Proof.* Since $f$ and $g$ are summable there exists respective approximating sequences of simple functions $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$, so we write:

$$\int_0^T \alpha f(t) + \beta g(t)\, \lambda(dt) = \lim_{n \to \infty} \int_0^T \alpha u_n(t) + \beta v_n(t)\, \lambda(dt),$$

we note that $\alpha u_n(t) + \beta v_n(t)$ is a simple function as the sum of simple functions so we apply linearity of the Bochner integral for simple functions 3.1 to obtain:

$$\int_0^T \alpha u_n(t) + \beta v_n(t)\, \lambda(dt) = \alpha \int_0^T u_n(t)\, \lambda(dt) + \beta \int_0^T v_n(t)\, \lambda(dt),$$

taking the limit as $n \to \infty$ gives the result. □

**Lemma 3.4** (Triangle Inequality). *Let $f : [0,T] \to X$ be summable then:*

$$\left\| \int_0^T f(t)\,\lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X\,\lambda(dt).$$

*Proof.* Since $f$ is summable there exists a sequence of appoximating simple functions $(s_n)_{n\in\mathbb{N}}$ so we can write:

$$\left\| \int_0^T f(t)\,\lambda(dt) \right\|_X = \lim_{n\to\infty} \left\| \int_0^T s_n(t)\,\lambda(dt) \right\|_X \leq \lim_{n\to\infty} \int_0^T \|s_n(t)\|_X\,\lambda(dt),$$

where we have used continuity of $\|\cdot\|_X$ and the triangle inequality for Bochner integrals of simple functions 3.2. It remains to show that:

$$\lim_{n\to\infty} \int_0^T \|s_n(t)\|_X\,\lambda(dt) = \int_0^T \|f(t)\|_X\,\lambda(dt).$$

Indeed, by linearity of the Lebesgue integral, triangle and reverse triangle inequalities and the dominated convergence theorem we obtain:

$$\begin{aligned} \left| \int_0^T \|s_n(t)\|_X\,\lambda(dt) - \int_0^T \|f(t)\|_X\,\lambda(dt) \right| &\leq \int_0^T \big| \|s_n(t)\|_X - \|f(t)\|_X \big|\,\lambda(dt) \\ &\leq \int_0^T \|s_n(t) - f(t)\|_X\,\lambda(dt) \longrightarrow 0 \end{aligned}$$

as $n \to \infty$ □

**Definition 3.5.** *We define the function space $L^1(0,T;X)$ to be the space of all Bochner summable functions, with:*

$$\|f\|_{L^1([0,T];X)} := \int_0^T \|f(t)\|_X\,\lambda(dt) < \infty.$$

**Remark 3.1.** *Every Bochner summable function is strongly measurable, since convergence in the $L^1$-norm implies convergence almost everywhere along a subsequence. Conversely, a function $f : [0,T] \to X$ is Bochner summable if and only if it is strongly measurable and the scalar function $t \mapsto \|f(t)\|_X$ is Lebesgue integrable:*

$$\int_0^T \|f(t)\|_X\,\lambda(dt) < \infty.$$

*This equivalence identifies the abstractly constructed space of Bochner summable functions with the more familiar intrinsic form*

$$L^1(0,T;X) = \{\, f : [0,T] \to X \text{ strongly measurable} : \|f(\cdot)\|_X \in L^1(0,T) \,\}.$$

**Remark 3.2.** *Now we have established a function space and a norm for Bochner summable functions it is useful to note that by the triangle inequality 3.4:*

$$\left\| \int_0^T f(t)\,\lambda(dt) \right\|_X \leq \int_0^T \|f(t)\|_X\,\lambda(dt) = \|f\|_{L^1(0,T;X)},$$

*so the integral is a bounded linear operator, $I : L^1(0,T;X) \to X$*

**Remark 3.3.** *The definition of the Bochner integral for general summable functions* (1) *is well defined since $L^1(0,T;X)$ is the completion of the space of simple functions with respect to the norm $\|\cdot\|_{L^1(0,T;X)}$. Thus, the integral operator defined on simple functions extends uniquely and continuously to all Bochner summable functions.*

*Loosely speaking, let $f : [0,T] \to X$ be summable and let $(s_n)_{n\in\mathbb{N}}$ be an approximating sequence of simple functions. We denote the integral operator for simple functions by $I$, so for $n \in \mathbb{N}$*

$$I(s_n) = \int_0^T s_n(t)\,\lambda(dt) \in X.$$

*Then, by the triangle inequality 3.2,*

$$\|I(s_n) - I(s_m)\|_X \leq \int_0^T \|s_n(t) - s_m(t)\|_X\,\lambda(dt).$$

*Since $I$ is a bounded linear operator 3.2 it is continuous, so we can pass the limits as $n, m \to \infty$ inside the integral to deduce that the sequence $(I(s_n))_{n\in\mathbb{N}}$ is Cauchy in $X$. By the completeness of $X$, this sequence converges to a limit in $X$, which we define by* (1). *For this to be completely rigorous we would need to show that this limit is independent of the choice of approximating sequence but this is routine.*

## 3.3 Equivalence with Riemann

**Theorem 3.1.** *Let $f \in C(0,T;X)$ then*

*(i) $f$ is Bochner summable,*

*(ii) the Riemann and Bochner integrals coincide, i.e.*

$$\int_0^T f(t)\,dt \;=\; \int_0^T f(t)\,\lambda(dt).$$

*Proof.* (i) Since $f$ is continuous with compact domain $[0,T]$, it is uniformly continuous and bounded. We define the modulus of continuity which serves as a way to quantify the uniform continuity of a function by

$$\omega(\delta) \coloneqq \sup\{\|f(t) - f(s)\|_X \;: |t-s| \leq \delta, t, s \in [0,T]\},$$

by uniform continuity of $f$ it should be clear that as $\delta \to 0$ we have $\omega(\delta) \to 0$.

Let $\mathcal{P} = \{(t_{i-1}, t_i, \tau_i)\}_{i=1}^n$ be a tagged partition with mesh $|\mathcal{P}| \coloneqq \max_{1\leq i\leq n}(t_i - t_{i-1})$ and define the simple function

$$s_{\mathcal{P}}(t) \coloneqq \sum_{i=1}^n \mathbb{1}_{(t_{i-1},t_i]} f(\tau_i).$$

We apply the Bochner integral and note that this is equal to the Riemann sum,

$$\int_0^T s_{\mathcal{P}}(t)\,\lambda(dt) = \sum_{i=1}^n \lambda(t_{i-1}, t_i] f(\tau_i) = S(f, \mathcal{P}).$$

Note for any $t \in (t_{i-1}, t_i]$ we have

$$\|s_{\mathcal{P}}(t) - f(t)\|_X = \|f(\tau_i) - f(t)\|_X \le \omega(|\mathcal{P}|),$$

so now applying the Lebesgue integral and taking the limit as the mesh gets finer we show that $s_{\mathcal{P}} \to f$ in $L^1(0,T;X)$,

$$\int_0^T \|s_{\mathcal{P}}(t) - f(t)\|_X \,\lambda(dt) \le \int_0^T \omega(|\mathcal{P}|)\,\lambda(dt) \le T \cdot \omega(|\mathcal{P}|),$$

since $\omega(|\mathcal{P}|) \to 0$ as $|\mathcal{P}| \to 0$ we get that indeed $s_{\mathcal{P}} \to f$ in $L^1(0,T;X)$. We can conclude that $f$ is Bochner summable.

(ii) Apealing to the fact the Bochner integral is a boudned linear operator 3.2 we have that

$$\left\| \int_0^T s_{\mathcal{P}}(t)\,\lambda(dt) - \int_0^T f(t)\,\lambda(dt) \right\|_X \le \int_0^T \|s_{\mathcal{P}}(t) - f(t)\|_X \,\lambda(dt),$$

from the proof of part (i) we know the right hand side tends to 0 as the mesh gets finer, rewriting this reads

$$S(f,\mathcal{P}) \longrightarrow \int_0^T f(t)\,\lambda(dt)$$

in $X$ as $|\mathcal{P}| \to 0$.

Finally applying the definition of the Riemann integral and uniqueness of limits we conclude that for $f \in C(0,T;X)$

$$\int_0^T f(t)\,dt = \int_0^T f(t)\,\lambda(dt)$$

$\square$

# 4 Fundamental Theorem of Calculus in $W^{1,p}(0,T;X)$

In this section we define the Sobolev space $W^{1,p}(0,T;X)$ and annotate the proof of theorem 2 (calculus in an abstract space) in section 5.9 of [Ev] using the results we have established.

**Definition 4.1.** *We define the function space $L^p(0,T;X)$ to be the space of all strongly measurable functions $u : [0,T] \to X$, with:*

$$\|u\|_{L^p([0,T];X)} \coloneqq \left( \int_0^T \|u(t)\|_X^p \,\lambda(dt) \right)^{\frac{1}{p}} < \infty$$

*for $1 \le p < \infty$, and*

$$\|u\|_{L^\infty([0,T];X)} \coloneqq \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_X < \infty.$$

**Definition 4.2.** *Let $u \in L^1(0,T;X)$, we say $v \in L^1(0,T;X)$ is the weak derivative of $u$, written*

$$u' = v,$$

*provided that for all scalar test functions $\phi \in C_c^\infty(0,T)$*

$$\int_0^T \phi'(t)u(t)\,\lambda(dt) = -\int_0^T \phi(t)v(t)\,\lambda(dt).$$

**Definition 4.3.** *We define the Sobolev space $W^{1,p}(0,T;X)$ to consist of all functions $u \in L^p(0,T;X)$ such that $u'$ exists in the weak sense and belongs to $L^p(0,T;X)$, with norm:*

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|_X^p + \|u'(t)\|_X^p\right)^{\frac{1}{p}} & (1 \le p < \infty) \\ \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_X + \|u'(t)\|_X & (p = \infty) \end{cases}$$

**Theorem 4.1** (Calculus in an abstract space [Ev])**.** *Let $u \in W^{1,p}(0,T;X)$ for some $1 \le p \le \infty$, then*

*(i) $u \in C([0,T];X)$, after possibly being redefined on a set of measure zero,*

*(ii) for all $0 \le s \le t \le T$*

$$u(t) = u(s) + \int_s^t u'(\tau)\,\lambda(d\tau),$$

*(iii) for a constant $C$ depending on $T$*

$$\|u\|_{C([0,T];X)} \le C\|u\|_{W^{1,p}(0,T;X)}$$

*Proof.* (i)

(ii)

(iii)

□

# A Additional Results

For completeness we state other well known results for the Bochner integral but omit their proofs as they are not the primary focus of this paper.

# References

[Ev] L. C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, AMS, Providence, RI, 2010.

[F2] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Wiley-Interscience, New York, 1999.

[Y] K. Yosida, *Functional Analysis*, 6th ed., Springer-Verlag, Berlin, 1980.

[Ch] E. Chernysh, *The Riemann Integral for Functions Mapping to Banach Spaces*, the article can be found here.