HERE ARE SOME solutions, partial solutions, or just final answers for most of the sample midterm.

Problem 1. (a) $\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx$.

(b) $\bar{x} = \dfrac{1}{m} \iint_D xke^y\,dA = \dfrac{k}{m} \iint_D xe^y\,dA$. The integral is easiest using the answer from (a):

$$\int_0^2 \int_0^{x^2} xe^y\,dy\,dx = \int_0^2 xe^y\Big|_{y=0}^{y=x^2}\,dx = \int_0^2 x(e^{x^2}-1)\,dx = \frac{e^{x^2}}{2} - \frac{x^2}{2}\Big|_0^2 = \frac{e^4-5}{2}.$$

(It can also be done in the original order:

$$\int_0^4 \int_{\sqrt{y}}^2 xe^y\,dx\,dy = \int_0^4 \frac{x^2}{2}e^y\Big|_{x=\sqrt{y}}^{y=2}\,dy = \int_0^4 (2e^y - \frac{ye^y}{2})\,dy = 2e^y - \frac{ye^y}{2} + \frac{e^y}{2}\Big|_0^4 = \frac{e^4-5}{2},$$

using integration by parts to get $\int ye^y dy = ye^y - e^y + C$.) Either way, $\bar{x} = \dfrac{k(e^4-5)}{2m}$.

Problem 2. If you don't know the polar equation for this circle, find it as follows. The xy-equation for the circle is $(x-2)^2 + y^2 = 4$, or $x^2+y^2 = 4x$, so $r^2 = 4r\cos\theta$, or $r = 4\cos\theta$.

$$\int_{\pi/4}^{\pi/2} \int_0^{4\cos\theta} \frac{k}{r}\, r\,dr\,d\theta = k\int_{\pi/4}^{\pi/2} r\Big|_{r=0}^{r=4\cos\theta}\,d\theta = 4k\int_{\pi/4}^{\pi/2} \cos\theta\,d\theta = 4k\sin\theta\Big|_{\pi/4}^{\pi/2} = 4k(1-\frac{1}{\sqrt{2}}).$$

(It's not too hard to set up the integral for this mass in xy-coordinates but the integrand is $(x^2+y^2)^{-1/2}$, which is rather complicated to integrate.)

Problem 3. (a) I'm giving a detailed explanation of how to find the limits – more than was required for full credit – because this problem seemed to be the hardest problem on the test.

To find the limits y and z, look at region D in the yz-plane which is the projection of the region E to this plane and is bounded by $y = z$, $y^2+z^2 = 1$, and the y-axis. (Draw a picture!) In D, the largest value of z occurs where the surfaces $y = z$ and $y^2+z^2 = 1$ intersect, so we get $2z^2 = 1$, or $z = 1/\sqrt{2}$, and the lower limit for z is 0. Still considering D, for fixed z, the lowest value of y occurs on $y = z$ and the highest value on the circle $y^2+z^2 = 1$. Finally, for the limits for x we must think about the three dimensional region E, pictured on the handout. For any point (y,z) in D, the x values are bounded below by the yz-plane, and bounded above by the sphere. Putting all this together, we get

$$\int_0^{1/\sqrt{2}} \int_z^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2-y^2}} f(x,y,z)\,dx\,dy\,dz.$$

Originally, I was going to give you this iterated integral, and ask you to rewrite it as an iterated integral with respect to $dy\,dz\,dx$. Try this, and for even more practice, try all the other orders of integration; answers for most of these are on p. 3.

(b) If we integrate first with respect to z, then we have to set limits for z for each choice of (x,y) possible in E. The lower limit for z is always 0, but the upper limit is on the sphere some places and on the plane $y = z$ others. We can see this even just looking at the intersection of E with the yz-plane, which happens to be the same as the projection onto the yz-plane in this example: for $x = 0$ and $y \leq 1/\sqrt{2}$, the upper bound is $z = y$, and for $x = 0$ and larger y it's $z = \sqrt{1-y^2}$. More precisely, we have an upper limit of $z = y$ in region R_1 in the xy-plane and an upper limit $z = \sqrt{1-x^2-y^2}$ in region R_2. Thus we'd have to set up two separate integrals if we want the integration in this order.

Problem 4. The sphere $x^2 + y^2 + z^2 = 2z$ is centered at $z = 1$ on the z-axis and has radius 1. In spherical coordinates, it's

$$\rho^2 = x^2 + y^2 + z^2 = 2z = 2\rho\cos\varphi,$$

and we can safely cancel one factor of ρ, because the origin remains a solution of the resulting equation $\rho = 2\cos\varphi$.

Next we find the ρ-coordinate where the spheres intersect. The other sphere has the equation $\rho = 1$, so they intersect where $\rho = 1 = 2\cos\varphi$. This implies $\varphi = \pi/3$. Thus the integral is

$$\int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

(You can change the order of integration by putting $d\theta$ last, first, or in the middle, but must put $d\rho$ before $d\varphi$ unless you want to deal with very messy limits of integration.)

Originally, this problem included computing the volume, but the computation was deleted because the test was too long. In case you compute it for practice, the answer is on p. 3.

Problem 5. By the change of variable formula in §15.9,

$$\iint_R \left(\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2\right) dA = \iint_S \left(\left(\frac{2u}{2}\right)^2 + \left(\frac{3v}{3}\right)^2\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA,$$

where we need to figure out what region S is in the uv-plane and compute the Jacobian. Substituing, we find S is bounded by

$$1 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \left(\frac{2u}{2}\right)^2 + \left(\frac{3v}{3}\right)^2 = u^2 + v^2$$

so S is the interior of the unit circle centered at the origin. Because R and S are *different regions*, you should not use the same letter for both of them! The Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y\ \partial u & \partial y\ \partial v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6.$$

To do the integral on S in the uv-plane, it is easiest to use polar coordinates:

$$\iint_S (u^2 + v^2) 6 du\, dv = \int_0^{2\pi}\int_0^1 r^2\, r\, dr\, d\theta = 6\int_0^{2\pi} d\theta \int_0^1 r^3\, dr = 12\pi \frac{r^4}{4}\bigg|_0^1 = 3\pi.$$