Find the general solution to the following problem

$$8u'' + 2u' + 5u = 0$$

Solution

Use the Quadratic Formula to find two solutions:

$$8r^2 + 2r + 5 = 0$$

$$\frac{-2 \pm \sqrt{4-160}}{16} = \frac{-1}{8} \pm \frac{\sqrt{39}}{8} i$$

Use the two solutions to write out the general solution:

$$y = e^{\frac{-t}{8}} \left(c_1 \cos \frac{\sqrt{39}}{8} t + c_2 \sin \frac{\sqrt{39}}{8} t \right)$$

2. Solve the initial value problem.

y'' + 3y' +2y = 5t^2 + 6 y(0) = 0, y'(0) = 0

Solution:

r^2 + 3r +2 = 0

(r – 1)(r – 2) = 0

y_h = c1e^-t + c2e^-2t

Y = At^2 + Bt + C

Y' = 2At + B

Y'' = 2A

t^2: 2A = 5 A = 5/2

t: 2B + 6A = 0 B = -15/2

Constant: 2C + 3B + 2A = 6 C = 47/4

y = c_1e^-t + c_2e^-2t + 5/2t^2 – 15/2t + 47/4

y(0) = 0 = c_1 + c_2 + 47/4

y'(0) = 0 = -c_1 – 2c_2 – 1547/4/2

0 = -c_2 + 17/4 c_2 = 17/4

-c_1 = 17/4 + 47/4 c_1 = -16

y = -16e^-t + 17/4e^-2t + 5/2t^2 – 15/2t + 47/4

A small cow with a mass of 5 kg stretches a spring 1 m. The cow is acted upon by an external force of 1
in a viscous fluid that imparts a force of 6 N when the speed of the cow is $30\frac{cm}{s}$. The cow is set into moti
from the equilibrium point with an initial positive velocity of $1\frac{m}{s}$. Find the position of this poor cow at ti

$$m = 5kg \quad L = 1m \quad g(t) = 17e^{-t} \qquad mg = KL \rightarrow \quad K = 49\frac{kg}{s^2} \qquad \gamma = \frac{6N}{.3\frac{m}{s}} = 20\frac{kg}{s}$$

$$mu'' + \gamma u' + Ku = 17e^{-t} \qquad \rightarrow \qquad 5u'' + 20u' + 49u = 17e^{-t}$$

Solve homogeneous solution first: $\frac{-20 \pm \sqrt{400-980}}{10} \quad \rightarrow \quad -2 \pm \frac{i\sqrt{145}}{5}$

$$e^{-2t}\left(\alpha\cos\left(\frac{\sqrt{145}}{5}t\right) + \beta\sin\left(\frac{\sqrt{145}}{5}t\right)\right) = y\ homogeneous$$

Solve characteristic equation: $Y = Ae^{-t} \qquad Y' = -Ae^{-t} \quad Y'' = Ae^{-t}$

$$5Ae^{-t} - 20Ae^{-t} + 49Ae^{-t} = 17e^{-t} \qquad \rightarrow \qquad 5A - 20A + 49A = 17$$

$$A = \frac{1}{2} \qquad \rightarrow \qquad Y = \frac{1}{2}e^{-t}$$

$$y = e^{-2t}\left(\alpha\cos\left(\frac{\sqrt{145}}{5}t\right) + \beta\sin\left(\frac{\sqrt{145}}{5}t\right)\right) + \frac{1}{2}e^{-t}$$

Use initial conditions u(0)=0 and u'(0)=1 to solve for coefficients:

$$0 = e^0(\alpha\cos(0) + \beta\sin(o)) + \frac{1}{2}e^0 \qquad \rightarrow \qquad \alpha + \frac{1}{2} = 0 \qquad \rightarrow \qquad \alpha = -\frac{1}{2}$$

$$y' = -\frac{\sqrt{145}}{5}\alpha\, e^{(-2t)}\, sin\left(\frac{\sqrt{145}}{5}t\right) - 2\alpha\, e^{(-2t)}\, cos\left(\frac{\sqrt{145}}{5}t\right) + \frac{\sqrt{145}}{5}\beta\, e^{(-2t)}\, cos\left(\frac{\sqrt{145}}{5}t\right)$$

$$- 2\beta\, e^{(-2t)}\, sin\left(\frac{\sqrt{145}}{5}t\right) - \frac{1}{2}e^{-t}$$

$$1 = -\frac{\sqrt{145}}{5}\alpha\, e^{(0)}\, sin(0) - 2\alpha\, e^{(0)}\, cos(0) + \frac{\sqrt{145}}{5}\beta\, e^{(0)}\, cos(0) - 2\beta\, e^{(0)}\, sin(0) - \frac{1}{2}e^0$$

$$1 = 1 + \frac{\sqrt{145}}{5}\beta - \frac{1}{2} \qquad \rightarrow \qquad \beta = \frac{5}{2\sqrt{145}}$$

$$y = e^{(-2t)}\left(-\frac{1}{2}\cos\left(\frac{\sqrt{145}}{5}t\right) + \frac{5}{(2\sqrt{145})}\sin\left(\frac{\sqrt{145}}{5}t\right)\right) + \frac{1}{2}e^{-t}$$

Solve the differential equation: $5y'' + 7y' + 9y = 0$ when y'(0)=11 and y(0)=9

$$5y'' + 7y' + 9y = 0 \quad \rightarrow \quad 5r^2 + 7r + 9 = 0$$

$$\frac{-7 \pm \sqrt{49 - 180}}{10} \quad \rightarrow \quad -\frac{7}{10} \pm \frac{i\sqrt{131}}{10}$$

$$e^{-\frac{7}{10}t}\left(\alpha\cos\left(\frac{\sqrt{131}}{10}t\right) + \beta\sin\left(\frac{\sqrt{131}}{10}t\right)\right) = y$$

$$11 = e^0(\alpha\cos(0) + \beta\sin(0)) \quad \rightarrow \quad \alpha = 11$$

$$y' = e^{-\frac{7}{10}t}\left(-\frac{\sqrt{131}}{10}\alpha\sin\left(\frac{\sqrt{131}}{10}t\right) + \frac{\sqrt{131}}{10}\beta\cos\left(\frac{\sqrt{131}}{10}t\right)\right)$$

$$-\frac{7}{10}e^{-\frac{7}{10}t}\left(\alpha\cos\left(\frac{\sqrt{131}}{10}t\right) + \beta\sin\left(\frac{\sqrt{131}}{10}t\right)\right)$$

$$9 = e^0\left(-\frac{\sqrt{131}}{10}\alpha\sin(0) + \frac{\sqrt{131}}{10}\beta\cos(0)\right) - \frac{7}{10}e^0(\alpha\cos(0) + \beta\sin(0))$$

$$9 = \frac{\sqrt{131}}{10}\beta - \frac{7}{10}\alpha$$

$$9 = \frac{\sqrt{131}}{10}\beta - \frac{77}{10} \quad \rightarrow \quad \frac{167}{10} = \frac{\sqrt{131}}{10}\beta$$

$$\beta = \frac{167}{\sqrt{131}}$$

$$y = e^{-\frac{7}{10}t}\left(11\cos\left(\frac{\sqrt{131}}{10}t\right) + \frac{167}{\sqrt{131}}\sin\left(\frac{\sqrt{131}}{10}t\right)\right)$$

1.) A spring has a mass of 16 lbs, and is stretched to equilibrium of 9 inches. An outside force stretches it an extra 12 inches, and sends it downward with a speed of 5 ft/sec. Calculate the period, frequency, amplitude and the phase.

A:

$m = 2\ lb*s^2\ /ft$ $\quad k(3/4\ ft) = 16\ lb$ $\quad k = 64/3\ lb/ft$ $\quad u(0) = 1\ ft$ $\quad u'(0) = 5$

$2u'' + \frac{64}{3}u = 0$ $\quad r = \pm(32/3)^{1/2}\, i$

$U = c_1\cos(t(32/3)^{1/2}) + c_2\sin(t(32/3)^{1/2})$

$C_1 = 1$

$U' = -(32/3)^{1/2}\sin(t(32/3)^{1/2}) + c_2(32/3)^{1/2}\cos(t(32/3)^{1/2})$

$5 = c_2(32/3)^{1/2}$ $\quad C_2 = 5/(32/3)^{1/2}$

$$\boxed{U = \cos(t(32/3)^{1/2}) + 5/(32/3)^{1/2}\sin(t(32/3)^{1/2})}$$

$\omega_0 = (32/3)^{1/2}$ $\quad T = 2\pi/\omega_0$ $\quad T = 2\pi/(32/3)^{1/2}$

$R = ((\frac{32}{3})^1 + 1)^{1/2}$ $\quad R = (35/3)^{1/2}$

$\tan(\delta) = B/A$ $\quad \delta = \arctan(5/(32/3)^{1/2})$

Problem #1: Using the method of reduction order, find the second solution of the given differential equation.

$t^2y''-2ty'+2y=0$; $y_1(t)=t^2$

Solution:

$y_2(t)=t^2v(t)$

$y_2'(t)=2tv(t)+t^2v'(t)$

$y_2''(t)=2v(t)+2tv'(t)+2tv'(t)+t^2v''(t)$

$t^2(2v(t)+2tv'(t)+2tv'(t)+t^2v''(t))-2t(2tv(t) + t^2v'(t))+6t^2v(t)=0$

$2t^2v(t)+4t^3v'(t)+t^4v''(t)-4t^2v(t)-2t^3v'(t)+2t^2v(t)=0$ ← v(t)'s drop out

$t^4v''(t)+2t^3v'(t)=0$ → $v''(t)=w'$ & $v'(t)=w$

$t^4w'+2t^3w=0$

$dw/dt=-2w/t$

$dw/w=-2dt/t$

$\int dw/w=\int -2dt/t$

$\ln|w|=-2\ln|t|$

w=eln|t|^-2

$w=t^{-2}$

$\int w=v(t)$

$\int w=\int t^{-2}\, dt$

$v(t)=-1/t$

$y_2(t)=t^2v(t)$

$y_2(t)=-t^2/t$

$\mathbf{y_2(t)=-t}$

2) A mass weighing 4 lb stretches a spring 6". If the mass is pulled down an additional 3" and then released, and if there is no damping, determine the position u of the mass at any time. Plot u vs t. Find the frequency, period, and amplitude of the motion.

Answer:

mass = 4 lb

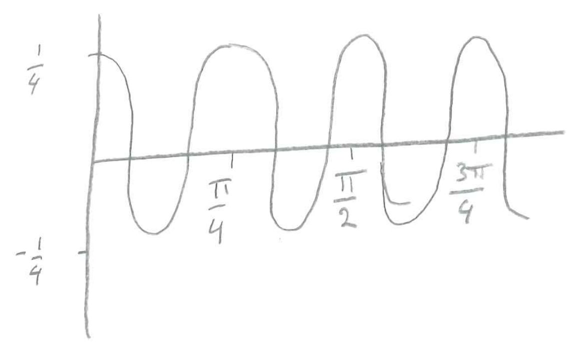
$L = 6''$ or $\frac{1}{2}$ ft

$m u'' + K u = 0$

$u(0) = \frac{1}{4}$ ft $\quad u'(0) = 0$

$m = \frac{F}{a} = \frac{4 \text{ lb}}{32 \text{ ft/s}^2} = \frac{1}{8} \frac{\text{lb s}^2}{\text{ft}}$

$K = \frac{F}{L} = \frac{4 \text{ lb}}{\frac{1}{2} \text{ ft}} = 8 \frac{\text{lb}}{\text{ft}}$



$\frac{1}{8} u'' + 8u = 0$

$\frac{1}{8} r^2 + 8 = 0$

$r^2 = -64$

$r = 8i$

$u = (C_1 \cos 8t + C_2 \sin 8t)$

$u' = -8C_1 \sin 8t + 8C_2 \cos 8t$

$\frac{1}{4} = C_1 \cos 0 + C_2 \sin 0 \quad C_1 = \frac{1}{4}$

$0 = -8C_1 \sin 0 + 8C_2 \cos 0 \quad C_2 = 0$

$u = \frac{1}{4} \cos 8t$

Amplitude $= \frac{1}{4}$

Frequency $= 8$

Period $= \frac{\pi}{4}$

$2t^2y'' + 3ty' - y = 0 \qquad t>0,\ y_1(t) = t^{-1}$

We set $y = v(t)t^{-1}$

$y' = v't^{-1} - vt^{2}$, $y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$

So: $2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1}$

$= 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v$

$= 2tv'' - v' = 0.$

So: $v'(t) = ct^{\frac{1}{2}}$

$v(t) = \frac{2}{3}ct^{\frac{3}{2}} + K$

$y = \frac{2}{3}ct^{\frac{1}{2}} + Kt^{-1}$

1. A spring of spring constant 100 N/m sits at rest at a length of 1 meter when a 4.5 kg block is attached. The spring is compressed to a distance of 35 cm and released from rest. If the spring is damped by a force of 11 Ns/m, what is the position function of the object u(t)?

Solutions*:

1. $u(t) = \frac{-77}{180\sqrt{1679}}\sin\sqrt{1679}t - \frac{7}{20}\cos\sqrt{1679}t$

Find the solution to the initial value problem.

$4y'' - 8y' + 29y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

$4r^2 - 8r + 29 = 0 \qquad r = \dfrac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2(a)}$

$r = \dfrac{8 \pm \sqrt{64 - 4(4)(29)}}{2(4)} = \dfrac{8 \pm \sqrt{-400}}{8} = \dfrac{8}{8} \pm \dfrac{20i}{8}$

roots $= 1 \pm 5/2\, i$

$y = c_1 e^t \cos 5/2\, t + c_2 e^t \sin 5/2\, t \implies$ Equation 1

Apply the first condition $y(0) = 1$

$1 = c_1 e^0 \cos(5/2\,(0)) + \cancel{c_2 e^0 \sin(5/2 \cdot 0)}^{\to 0}$

$1 = c_1$

Find the derivative of equation 1 to apply second condition

$y' = c_1 e^t \cos 5/2\, t - \frac{5}{2} c_1 e^t \sin 5/2\, t + c_2 e^t \sin 5/2\, t + 5/2\, c_2 e^t \cos 5/2\, t$

$0 = (1) e^0 \cos(5/2 \cdot 0) - \cancel{5/2 (1) e^0 \sin(5/2 \cdot 0)}^{\to 0} + \cancel{c_2 e^0 \sin(5/2 (0))}^{\to 0} + 5/2\, c_2 e^0 \cos(5/2 \cdot 0)$

$0 = 1 + 5/2\, c_2 \;, \quad c_2 = -2/5$

Final answer $\quad \boxed{y = e^t \cos 5/2\, t - 2/5\, e^t \sin 5/2\, t}$

2.) Suppose a 4lb weight stretches a spring 4 inches. The weight moves through a viscous fluid that imparts a force of 4lb when the speed of the mass is 2 ft/sec, is given an initial velocity of 7in/sec, and is acted on by an external force of 6cos2t lb. Find the **general solution** for *u* at any time *t*.

Solution: First find m, k, and the damping coefficient ("d" in this problem)

Weight – 4lb = force => 4lb = mg g=32ft/sec => m = 4/32 = 1/8 = 0.125

k = mg/L = 4lb/(4/12) (since you put inches into feet to have the same units for everything) => k = 12

Since du'(t) = damping force and we know that the damping force is 4lb when u'(t) = 2ft/sec, then d = force/u'(t) = 4/2 = 2

Finally, we get: 0.125u'' + 2u' + 12u = 6cos2t

Then, solve for the homogeneous equation *uh*: 0.125r^2 + 2r + 12 = 0

Quadratic formula = $\dfrac{-2 \pm (2 - 4(0.125)12)\text{^}(1/2)}{2(0.125)}$

= -8 ± 4(-2)^(1/2) = -8 ± 4(2)^(1/2)***i***

=> *uh* = exp(-8t)·(C1cos4√(2)t + C2sin4√(2)t)

Now we must find the solution to U(t) by the method of undetermined coefficients:

U = Asin2t + Bcos2t

U' = 2Acos2t – 2Bsin2t

U'' = -4Asin2t – 4Bcos2t

Then, plug those equations back into the original equation:

=> -0.5Asin2t – 0.5Bcos2t + 4Acos2t - 4Bsin2t + 12Asin2t + 12Bcos2t = 6cos2t

=> Combining sines and cosines:

11.5Asin2t – 4Bsin2t = 0 => B = 11.5A/4

11.5Bcos2t + 4Acos2t = 6cos2t => 11.5B + 4A = 6

11.5(11.5A/4) + 4A = 6 => A = 24/148.25

B = 11.5(24/148.25)/4 = 276/593

U(t) = (24/148.25)sin2t + (276/593)cos2t

Finally, combining *uh*(t) and U(t):

u(t) = exp(-8t)·(C1cos4√(2)t + C2sin4√(2)t) + (24/148.25)sin2t + (276/593)cos2t

(Although we know u(0) = 0 and u'(0) = (7/12) ft/sec. and we could solve for C1 and C2, I only asked for the general solution and I am too busy/really don't want to have to solve it and type more……) ☺