Solutions to Homework 1

Section 1, Exercises 1, 2, 3, and 5.

Exercise 1. Determine the irreducible polynomial for $i + \sqrt{2}$ over $\mathbb{Q}$.

Solution. The irreducible polynomial is $f(x) = x^4 - 2x^2 + 9$. To show that f is indeed the irreducible polynomial of $i + \sqrt{2}$, we must show that $i + \sqrt{2}$ is a root of f and that f is irreducible over $\mathbb{Q}$. An easy calculation shows that $f(i + \sqrt{2}) = 0$, so the first part is done.

Suppose that f is reducible. There are two possibilities: either f is the product of a linear polynomial and a cubic polynomial, or f is the product of two quadratic polynomials. In the first case, $f(x) = 0$ for some $x \in \mathbb{Q}$. This cannot be, since $f(x) = (x^2 - 1)^2 + 8$, which is positive for all real x. In the second case, $i + \sqrt{2}$ must be a root of one of the factors of f. But $i + \sqrt{2}$ cannot be the root of a quadratic polynomial, since by the quadratic formula, all roots of rational quadratic polynomials are of the form $a + \sqrt{b}$ for some $a, b \in \mathbb{Q}$. Hence f must be irreducible. $\square$

Exercise 2. Prove that the set $(1, i, \sqrt{2}, i\sqrt{2})$ is a basis for $\mathbb{Q}(i, \sqrt{2})$ over $\mathbb{Q}$.

Proof. Consider the set $S = \operatorname{Span}(1, i, \sqrt{2}, i\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, i)$. We see that S is an integral domain and a finite dimensional vector space over $\mathbb{Q}$, so it is a field containing $\sqrt{2}$ and i. Since $\mathbb{Q}(\sqrt{2}, i)$ is the smallest field containing $\sqrt{2}$ and i, we have $\mathbb{Q}(\sqrt{2}, i) \subseteq S$, so the two are equal. This shows that the potential basis is a spanning set.

On the other hand, since $i \notin \mathbb{Q}(\sqrt{2})$, the degree of $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}(\sqrt{2})$ is 2. The degree of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is also 2, so the degree of $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$ is 4. But 4 vectors can only span a 4-dimensional vector space if they are linearly independent, so the potential basis is indeed a basis. $\square$

Exercise 3. Determine the intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Solution. There are five. The Galois group G of the extension is $\{\mathrm{id}, \sigma, \tau, \sigma\tau\}$, where

$$\begin{aligned} \mathrm{id} :&\sqrt{2} \mapsto \sqrt{2} & \sigma :&\sqrt{2} \mapsto -\sqrt{2} & \tau :&\sqrt{2} \mapsto \sqrt{2} & \sigma\tau :&\sqrt{2} \mapsto -\sqrt{2} \\ &\sqrt{3} \mapsto \sqrt{3} & &\sqrt{3} \mapsto \sqrt{3} & &\sqrt{3} \mapsto -\sqrt{3} & &\sqrt{3} \mapsto -\sqrt{3}. \end{aligned}$$

Since the degree of the field extension is 4 and we have 4 elements in the Galois group, the extension is Galois. So we can use the main Galois theorem, which says that there is a correspondence between the subgroups of G and the intermediate fields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Now G is isomorphic to the Klein group, and has 5 subgroups. Each subgroup corresponds to an intermediate field which is the set of all elements fixed by everything in the subgroup:

$$\begin{aligned}
\{\mathrm{id}\} &\rightsquigarrow \mathbb{Q}(\sqrt{2}, \sqrt{3}) \\
\{\mathrm{id}, \sigma\} &\rightsquigarrow \mathbb{Q}(\sqrt{3}) \\
\{\mathrm{id}, \tau\} &\rightsquigarrow \mathbb{Q}(\sqrt{2}) \\
\{\mathrm{id}, \sigma\tau\} &\rightsquigarrow \mathbb{Q}(\sqrt{6}) \\
\{\mathrm{id}, \sigma, \tau, \sigma\tau\} &\rightsquigarrow \mathbb{Q}
\end{aligned}$$

□

Exercise 5. Prove that the automorphism of $\mathbb{Q}(\sqrt{2})$ sending $\sqrt{2}$ to $-\sqrt{2}$ is discontinuous.

Proof. If we call this automorphism f, note that $f(x) = x$ for all $x \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}(\sqrt{2})$, f must be the identity on all of $\mathbb{Q}(\sqrt{2})$ if it is to be continuous. But $f(\sqrt{2}) = -\sqrt{2}$, so f is not continuous.

Another way to think about this is to consider a sequence x_i of rational numbers that approach $\sqrt{2}$. Then $f(x_i)$ approaches $\sqrt{2}$ which is not equal to $f(\sqrt{2})$. Since f does not preserve limits of sequences, it must not be continuous.

□