Laplace Transform Practice

Solve the following differential equation

$$y'' + 4y = \begin{cases} \sin t & t \le 9\pi \\ 0 & t > 9\pi \end{cases}, \quad y(0) = 0, \quad y'(0) = -1$$

Determine the amplitude of the oscillation when $t > 9\pi$.

Answer The differential equation can be written

$$y'' + 4y = \sin t - u_{9\pi}(t) \sin t$$

Applying the Laplace transform, we get

$$s^2 Y + 1 - 4Y = \frac{1}{s^2+1} + \frac{1}{s^2+1} e^{-9\pi s}$$

Note the plus instead of minus between the two fractions, which comes because the final Laplace transform is computed as

$$\mathcal{L}\{u_{9\pi}(t) \sin t\} = e^{-9\pi s} \mathcal{L}\{\sin(t + 9\pi)\} = e^{-9\pi s} \mathcal{L}\{-\sin t\} = -e^{-9\pi s} \frac{1}{s^2+1}$$

Solving for Y, we get

$$\frac{1}{(s^2+1)(s^2+4)}(1 + e^{-9\pi s}) - \frac{1}{s^2+4} = \left(\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right)(1 + e^{-9\pi s}) - \frac{1}{s^2+4}$$

Taking inverse Laplace transforms, we get

$$\begin{aligned} &\frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \left(\frac{1}{3}\sin(t - 9\pi) - \frac{1}{6}\sin(2(t - 9\pi))\right) u_{9\pi}(t) - \frac{1}{2}\sin 2t \\ &\qquad = \frac{1}{3}\sin t - \frac{2}{3}\sin 2t + \left(-\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right) u_{9\pi}(t) \end{aligned}$$

If $t > 9\pi$ then $u_{9\pi}(t) \equiv 1$, and the equation is $-(5/6)\sin 2t$, which has amplitude $5/6$.

Solve the initial value problem

$$y'' + y = \delta(t - \pi) + \delta(t - 3\pi), \quad y(0) = 1, \quad y'(0) = 0.$$

Answer The Laplace transform of the equation is $s^2 Y - s + Y = e^{-\pi s} + e^{-3\pi s}$. So we have

$$Y = \frac{s}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1}$$

Take the inverse Laplace transform to get

$$\begin{aligned} y(t) &= \cos(t) + u_{\pi}(t) \sin(t - \pi) + u_{3\pi}(t) \sin(t - 3\pi) \\ &= \cos(t) - u_{\pi}(t) \sin t - u_{3\pi}(t) \sin t. \end{aligned}$$