Sample problems and solutions

I have not verified if these solutions are correct

① Solve the initial value problem

$$\frac{dy}{dx} + \frac{1}{2}y = 5e^{4x} \qquad y(1)=0$$

Solution:

$$\mu = \exp\int \tfrac{1}{2}\,dx$$

$$\mu = e^{x/2}$$

$$\frac{dy}{dx}\left(e^{x/2}\,y\right) = 5e^{4x}$$ ~ integrate both sides ~

$$e^{x/2}y = \int 5e^{4x}$$

$$e^{x/2}y = \frac{5}{4}e^{4x} + c$$

$$y = \frac{5}{4}e^{4x}e^{\frac{2}{x}} + c$$

$$y = \frac{5}{4}e^{4x+\frac{2}{x}} + c$$ ~ initial value $y(1)=0$ ~

$$0 = \frac{5}{4}e^{(4)(1)+\frac{2}{(1)}} = e^{6} + c \qquad c = -e^{6}$$

~ Final solution ~ $$\boxed{y = \frac{5}{4}e^{4x+\frac{2}{x}} - e^{6}}$$

Problem 1: Find the solution of the given initial value problem.

$5ty' + 8y = 7t^3 - 5t + \frac{1}{t}$, $y(1) = 3$

Solution:

1) $5ty' + 8y = 7t^3 - 5 + \frac{1}{t}$ $\quad y(1) = 3$

$y' + \frac{8}{5t} y = \frac{7}{5} t^2 - \frac{1}{t} + \frac{1}{5t^2}$

$\mu(t) = \exp \int \frac{8}{5t} dt = e^{\frac{8}{5}\ln|t|} = t^{8/5}$

$t^{8/5} y = \int \frac{7}{5} t^{18/5} - t^{3/5} + \frac{1}{5t^{2/3}} dt$

$t^{8/5} y = \frac{7}{23} t^{23/5} - \frac{5}{8} t^{8/5} + \frac{1}{5} \ln|t| + c$

$y = \frac{7}{23} t^3 - \frac{5}{8} + \frac{\ln|t|}{5t^{8/5}} + c$

$3 = \frac{7}{23}(1)^3 - \frac{5}{8} + \frac{\ln|1|}{5(1)^{8/5}} + c$

$c = \frac{611}{184}$

$y = \frac{7}{23} t^3 - \frac{5}{8} + \frac{\ln|t|}{5t^{8/5}} + \frac{611}{184}$

1. Find the solution of the given initial value problem.

$$ty' - 3y = 8 + \frac{t^2}{2} \qquad t > 0 \qquad y(1) = 0$$

Solution :

First divide the whole equation so that the coefficient of y' is 1.

$$ty' - 3y = 8 + \frac{t^2}{2}$$

$$y' - \frac{3}{t}y = \frac{8}{t} + \frac{t}{2} \quad \cdots ①$$

Then the integrating factor is

$$\mu(t) = \text{expo} \int -\frac{3}{t}dt$$

$$= \text{expo} - 3\ln|t|$$

$$= \text{expo} -3\ln(t) \quad \text{since } t > 0$$

$$= t^{-3}$$

Then multiply ① by the integrating factor so that

$$\frac{d}{dt}(t^{-3}y) = 8t^{-4} + \frac{1}{2}t^{-2}$$

Integrate both sides to get

$$t^{-3}y = -\frac{8}{3}t^{-3} - \frac{1}{2}t^{-1} + c$$

divide both sides by t^{-3}

$$y = -\frac{8}{3} - \frac{1}{2}t^2 + ct^3$$

Then solve for the initial value $y(1) = 0$

$$0 = -\frac{8}{3} - \frac{1}{2} + c \qquad \frac{16}{6} - \frac{3}{6}$$

$$c = \frac{19}{6}$$

so, the solution is

$$\boxed{y = -\frac{8}{3} - \frac{1}{2}t^2 + \frac{19}{6}t^3}$$

Solve the Initial value problem

(2) $y' = 8y + e^{16t}$ $y(0) = 1$

$y' - 8y = e^{16t}$ find Integration factor e^{-8t}

$e^{-8t}y = e^{-8t}e^{16t}$ Simplify

$e^{-8t}y = \int e^{2t}\,dt$ Integrate Right side

$e^{-8t}y = \frac{e^{2t}}{2} + c$ Solve for y

$y = \frac{e^{2t}e^{8t}}{2} + ce^{8t}$ Simplify

$y = \frac{e^{10t}}{2} + ce^{8t}$ Solve for C useing initial values

$1 = \frac{\cancel{e^{10(0)}}^{1}}{2} + c\cancel{e^{8(0)}}^{1}$

$1 - \frac{1}{2} = c$ $c = \frac{1}{2}$ restate general solution with arbitrary constiant C replaced with ½

$y = \frac{e^{10t}}{2} + \frac{e^{8t}}{2}$

$\boxed{y = \frac{e^{8t}}{2}\left(e^{2t} + 1\right)}$ Simplify

1. (a) Solve the given differential Equation.

$$\frac{1}{2x}y' - \ln(x)\,y = 0$$

(b) Find the solution of the given initial value.

y(1) = 3

1. (a)

$$\frac{1}{2x}\frac{dy}{dx} = \ln(x)\,y$$

$$\int \frac{dy}{y} = \int 2x * \ln\,(x)dx$$

Then, u = ln(x); du = 1/x dx; dv = 2x dx; v = x^2.

$$\int 2x * \ln\,(x)dx = x^2\ln(x) - \int x^2 * \frac{1}{x}dx = x^2\ln(x) - \int xdx = \mathbf{x^2\ln(x)} - \frac{\mathbf{x^2}}{\mathbf{2}} + \boldsymbol{c}$$

$$\int \frac{dy}{y} = \int 2x * \ln\,(x)dx$$

$$\ln y = \mathbf{x^2\ln(x)} - \frac{\mathbf{x^2}}{\mathbf{2}} + \boldsymbol{c}$$

$$y = e^{\mathbf{x^2\ln(x)} - \frac{\mathbf{x^2}}{\mathbf{2}} + \boldsymbol{c}} = ce^{\mathbf{x^2\ln(x)} - \frac{\mathbf{x^2}}{\mathbf{2}}}$$

1. (b) Initial value y(1) = 3.

$$y = ce^{\mathbf{x^2\ln(x)} - \frac{\mathbf{x^2}}{\mathbf{2}}}$$

Plug in y(1) = 3

$$3 = ce^{-1/2}$$

$$c = 3e^{\frac{1}{2}}$$

$$y = 3e^{\frac{1}{2}} * e^{\mathbf{x^2\ln(x)} - \frac{\mathbf{x^2}}{\mathbf{2}}}$$

Solve the following initial value problem.

$4y' + \cos(t)y = 16\cos(t) \qquad y(0) = 6$

Solution:

$y' + (1/4)\cos(t)y = 4\cos(t)$

$\mu(t) = e \wedge \int(1/4)\cos(t)\, dt$

$\mu(t) = e \wedge (\sin(t)/4)$

$ye\wedge(\sin(t)/4) = \int 4\cos(t)e\wedge(\sin(t)/4)\, dt$

$ye\wedge\sin(t)/4) = 16e\wedge(\sin(t)/4) + C$

$y = 16 + C/(e\wedge(\sin(t)/4))$

$y(0) = 6$

$6 = 16 + C$

$C = -10$

$y = 16 – 10/e\wedge(\sin(t)/4)$

Find the solution of the given initial value problem.

$$xy' + y = x^3 + x^{-1}, \quad y(1) = \frac{3}{4}, \quad x > 0.$$

Solution:

$$y' + \frac{1}{x}y = x^2 + x^{-2}$$

$$\mu(x) = \exp \int \frac{1}{x} dx$$

$$= e^{\ln|x|}$$

$$= |x| \qquad x > 0.$$

$$= x$$

$$x \cdot y = \int x \cdot (x^2 + x^{-2}) \, dx$$

$$= \int x^3 + x^{-1} \, dx$$

$$= x^4/4 + \ln x + C.$$

$$y = \frac{x^3}{4} + \frac{\ln x}{x} + \frac{c}{x}$$

$$y(1) = \frac{3}{4}$$

$$\therefore \frac{1}{4} + \ln 1/1 + c = \frac{3}{4}$$

$$c = \frac{1}{2}$$

$$y = \frac{x^3}{4} + \frac{\ln x}{x} + \frac{1}{2x}$$