Print Your Name

Student ID #

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Directions

- Please check that your exam contains a total of 6 pages.
- Write complete solutions or you may not receive credit.
- This exam is closed book. You may use one 8.5×11 sheet of notes and a calculator.
- You may not share notes or calculators. You may not use a graphing calculator or any electronic device other than a calculator.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

Signature. Please sign below to indicate that you have not and will not give or receive any unauthorized assistance on this exam.

Signature: ______________________

1. (10 points) Let

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 4 & 1 \\ 9 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

   (a) Find a basis for the nullspace of A.

   (b) Find a basis for the column space of A.

   (c) Find a basis for the row space of A.

   (d) What is the rank of A?

   (e) What is the nullity of A?

2. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

(a) Find a least squares solution to the overdetermined system $A\mathbf{x} = \mathbf{b}$. (Your answer should be a single vector).

(b) Let $\mathbf{x}^*$ be your answer to part (a). In $\mathbb{R}^2$, draw the range of A, the point $\mathbf{b}$, and the point $A\mathbf{x}^*$.

3. (10 points) Let W be a subspace of $\mathbb{R}^4$, and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find an orthogonal basis for W.

(b) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be the basis you found in part a. Let

$$\mathbf{w} = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix}.$$

Find a, b, and c such that $\mathbf{w} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$.

4. (10 points) True or false. No justification is necessary. Your score on this problem will be the number of questions you get right minus the number of questions you get wrong.

   (a) If U and V are subspaces of $\mathbb{R}^n$ and every vector of U is also in V (that is, $U \subseteq V$), then $\dim U \leq \dim V$

   (b) If $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ is a linearly independent set of vectors, then so is the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1}\}$.

   (c) If W is a subspace of $\mathbb{R}^n$ and B_1 and B_2 are two bases for W, then there is a vector of W that is in B_1 and B_2.

   (d) If W is a subspace of $\mathbb{R}^n$ and $\dim(W) = k$, then W contains exactly k vectors.

   (e) If A is any $(m \times n)$ matrix and $\mathbf{b}$ is an $(m \times 1)$ vector, then the equation $A^T A\mathbf{x} = A^T\mathbf{b}$ has at least one solution.

   (f) Every set of orthogonal vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

   (g) Every subspace of $\mathbb{R}^n$ contains the zero vector.

   (h) The natural basis for $\mathbb{R}^n$ is an orthonormal basis.

   (i) Every nonzero subspace of $\mathbb{R}^n$ has a unique basis.

   (j) If $A\mathbf{x} = \mathbf{b}$ is a consistent system of equations, then every least squares solution to this system is also a solution to this system.

5. (10 points) Let V be a subspace of $\mathbb{R}^n$ and W be a subspace of $\mathbb{R}^m$. Suppose that $T_1 : V \to W$ and $T_2 : V \to W$ are linear transformations. Define a new function T by

$$T(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x}).$$

Prove that T is a linear transformation.