1. (a) Find the general solution to the following differential equation.

$$y'' + 10y' + y = 0$$

$r^2 + 10r + 1 = 0 \qquad r = \frac{-10 \pm \sqrt{100-4}}{2} = -5 \pm \sqrt{24}$

$$\boxed{y = c_1 e^{(-5+\sqrt{24})t} + c_2 e^{(-5-\sqrt{24})t}}$$

(b) Find the solution to the following initial value problem.

$$2y'' + 6y' + 17y = 0, \quad y(0) = 3, \quad y'(0) = -1$$

$2r^2 + 6r + 17 = 0 \qquad r = \frac{-6 \pm \sqrt{36-136}}{4} = -\frac{3}{2} \pm 10i$

general solution $\quad y = e^{-3/2\,t}(c_1 \cos 10t + c_2 \sin 10t)$

$y(0) = 3 \Rightarrow e^0(c_1 \cos 0 + c_2 \sin 0) = 3 \Rightarrow \quad c_1 = 3$

$y' = -\frac{3}{2} e^{-3/2\,t}(3\cos 10t + c_2 \sin 10t) + e^{-3/2\,t}(-10(3)\sin 10t + 10c_2 \cos 10t)$

$y'(0) = -1 \Rightarrow -\frac{3}{2}(3) + (10c_2) = -1 \Rightarrow \quad 10c_2 = \frac{7}{2} \Rightarrow c_2 = \frac{7}{20}$

so

$$\boxed{y = e^{-3/2\,t}\left(\cancel{3\cos 10t + \frac{7}{20}\sin 10t}\right)}$$

(3 cos(5t/2)+(7/5)sin(5t/2))

2. (a) Solve the following initial value problem.

$$y'' + y = t^3, \quad y(0) = 0, \quad y'(0) = 2$$

$r^2+1=0 \quad r=\pm i \quad$ so $y_h = c_1\cos t + c_2 \sin t$.

$Y = At^3+Bt^2+Ct+D$

$Y' = 3At^2+2Bt+C$

$Y'' = 6At+2B$

$Y''+Y = At^3+Bt^2+Ct+D+6At+2B$

$t^3: A=1$

$t^2: B=0$

$t: C+6A=0 \Rightarrow C+6=0 \Rightarrow C=-6$

$1: D+2B=0 \Rightarrow D=0$.

So $Y = t^3-6t$

and $y = c_1\cos t + c_2\sin t + t^3 - 6t$

$y(0)=0 \Rightarrow \quad c_1 = 0$

$y' = \quad c_2\cos t + 3t^2 - 6 = 2$

$c_2 - 6 = 2$

$c_2 = 8$

So $\boxed{y = 8\sin t + t^3 - 6t}$

(b) Find the general solution to the following differential equation.

$$y'' + 2y' + y = 15e^t$$

$r^2+2r+1=0 \Rightarrow r=1,1$ so $y_{hom} = c_1e^{-t} + tc_2e^{-t}$

$Y = Ae^t$

$Y' = Ae^t$

$Y'' = Ae^t$

so $A+2A+A=15$

$A = 15/4$.

$$\boxed{y = c_1e^{-t} + tc_2e^{-t} + \frac{15}{4}e^t}$$

3. Consider the following differential equation

$$t^2y'' + 2ty' - 2y = 0$$

The function $y_1 = t$ is a solution to this equation. Use reduction of order to find a solution that is not a constant multiple of y_1.

$y = vt \qquad y' = v't + v \qquad y'' = v''t + v' + v'$

$= v''t + 2v'.$

$t^2y'' + 2ty' - 2y = v''t^3 + 2v't^2 + 2v't^2 + 2tv - 2vt$

$= t^3v'' + 4t^2v'$

Let $u = v'$

$t^3u' + 4t^2u = 0$

$\frac{u'}{u} = \frac{-4t^2}{t^3}$

$\int \frac{du}{u} = \int -\frac{4}{t}\,dt$

$\ln u = -4\ln t$

$u = t^{-4}$

So $v = \int u = t^{-3}$ (constant doesn't matter)

So $y = tt^{-3} = \boxed{t^{-2}}$

4. A spring of length 50 cm has a spring constant of 41 N/m. It has a damper on it that exerts a force proportional to and in the opposite direction of the velocity. When an object is moving 1 m/s, the damper exerts a force of 2 N. An object of mass 2 kg is attached to the unstretched spring and dropped. You are on the planet earth, so gravity induces an acceleration of 9.8 m/s^2.

(a) Find a function that describes the position of the object at any given time.

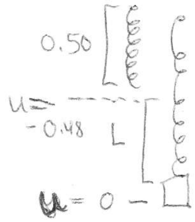
$k = 41$ N/m

$F_{damp} = \gamma v$

$2 = \gamma 1$

$2 = \gamma \frac{Ns}{m}$

$m = 2kg$



$mg = kL$

$2(9.8) = 41L$

$0.48 = L$

solve: $2u'' + 2u' + 41u = 0 \quad u(0) = -.48 \quad u'(0) = 0$

$r = \frac{-2 \pm \sqrt{4-328}}{4} = -\frac{1}{2} \pm i\frac{9}{2}$

general soln $y = e^{-\frac{1}{2}t}(c_1 \cos\frac{9}{2}t + c_2 \sin\frac{9}{2}t)$

from $u(0) = -0.48$ we get $c_1 = -0.48$

$y' = -\frac{1}{2}e^{-\frac{1}{2}t}(c_1\cos\frac{9}{2}t + c_2\sin\frac{9}{2}t) + e^{-\frac{1}{2}t}(-\frac{9}{2}c_1\sin\frac{9}{2}t + \frac{9}{2}c_2\cos\frac{9}{2}t)$

$-\frac{1}{2}(-0.48) + \frac{9}{2}c_2 = 0 \qquad c_2 = -0.05$

$$\boxed{y = e^{-\frac{1}{2}t}(-0.48\cos\frac{9}{2}t - 0.05\sin\frac{9}{2}t)}$$

(b) Determine the amount of time it takes before the object's motion is confined to within 1 cm of the equilibrium.

quasi-Amplitude $= \sqrt{.48^2 + .05^2} = .48$

$.48e^{-\frac{1}{2}t} = .01$

$-\frac{1}{2}t = \ln\frac{.01}{.48}$

$t = -2\ln\frac{.01}{.48} =$

$\boxed{7.74 \text{ sec}}$

there is definitely some rounding error.

(c) Determine the amount of time it takes for the object to reach the equilibrium the first time.

$-.48\cos\frac{9}{2}t - 0.05\sin\frac{9}{2}t = 0$

$-0.48\cos\frac{9}{2}t = 0.05\sin\frac{9}{2}t$

$-9.6 = \tan\frac{9}{2}t$

$-1.467 = \frac{9}{2}t$

$t = -0.326$

add $\frac{1}{2}$ period $\left(\frac{2\pi}{9/2} = \frac{4\pi}{9}\right)$ = period

$\frac{2\pi}{9}$ = half period

$\boxed{t = .372}$

(d) Bonus[1]: Your physics teacher, who owns this spring, told you that it will be irreparably damaged if it is stretched to a total length of more than 135 cm. Did you break it?

email me if you want to check your answer.

[1] There will be no bonus questions on the actual exam

5. A ball weighing 16 lbs stretches a spring 16 ft. When the ball is moving 1 ft/s, the damping force is 1 lb. An external force of $2\cos t$ is applied to the ball.

   (a) Find a function describing the steady state of the system.

$mg = kL$ $\quad$ $16 = 16k$ $\quad$ $\underline{k=1}$

$\underline{m = \frac{16}{32} = \frac{1}{2}}$

$F_{damp} = \gamma v$ $\quad$ $1 = \gamma 1$ $\quad$ $\underline{\gamma = 1}$

$\underline{\frac{1}{2}u'' + u' + u = 2\cos t}$

$r = \frac{-1 \pm \sqrt{1-2}}{1} = -1 \pm i$

$u_h = e^{-t}(c_1 \cos t + c_2 \sin t)$

$Y = A\cos t + B\sin t$

$Y' = -A\sin t + B\cos t$

$Y'' = -A\cos t - B\sin t$

	cos	sin
Y''	$-A$	$-B$
Y'	B	$-A$
Y	A	B
add to	2	0

$-\frac{1}{2}A + A + B = 2$ $\quad$ $-\frac{1}{2}B - A + B = 0$

$\frac{1}{2}A + B = 2$ $\quad$ $\frac{1}{2}B - A = 0$

$A = \frac{4}{5}$ $\quad$ $B = \frac{8}{5}$

Steady state: $\boxed{\frac{4}{5}\cos t + \frac{8}{5}\sin t}$

   (b) What is the amplitude of the steady state response?

$\sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = \frac{4}{\sqrt{5}} = \underline{1.78}$