SOLUTIONS FOR THE FALL, 2001 SAMPLE FINAL EXAM

QUESTION 1.

(a) We will compute the determinant by the expansion along row 2.

$$det\left(\begin{bmatrix}1&3&4\\1&0&3\\1&1&3\end{bmatrix}\right) = -1(det\left(\begin{bmatrix}3&4\\1&3\end{bmatrix}\right)) + 0(det\left(\begin{bmatrix}1&4\\1&3\end{bmatrix}\right)) - 3(det\left(\begin{bmatrix}1&3\\1&1\end{bmatrix}\right))$$

$$= (-1)(3\cdot 3 - 4\cdot 1) - 3(1\cdot 1 - 3\cdot 1) = (-1)\cdot 5 - 3\cdot(-2) = 1$$

(b) We find A^{-1} by row-reduction of the augmented matrix $[A|I_3]$:

$$\begin{bmatrix}1&3&4&1&0&0\\1&0&3&0&1&0\\1&1&3&0&0&1\end{bmatrix}, \quad \begin{bmatrix}1&3&4&1&0&0\\0&-3&-1&-1&1&0\\0&-2&-1&-1&0&1\end{bmatrix}, \quad \begin{bmatrix}1&3&4&1&0&0\\0&1&1/3&1/3&-1/3&0\\0&-2&-1&-1&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&3&4&1&0&0\\0&1&1/3&1/3&-1/3&0\\0&0&-1/3&-1/3&-2/3&1\end{bmatrix}, \quad \begin{bmatrix}1&0&3&0&1&0\\0&1&1/3&1/3&-1/3&0\\0&0&-1/3&-1/3&-2/3&1\end{bmatrix}, \quad \begin{bmatrix}1&0&3&0&1&0\\0&1&1/3&1/3&-1/3&0\\0&0&1&1&2&-3\end{bmatrix},$$

$$\begin{bmatrix}1&0&0&-3&-5&9\\0&1&1/3&1/3&-1/3&0\\0&0&1&1&2&-3\end{bmatrix}, \quad \begin{bmatrix}1&0&0&-3&-5&9\\0&1&0&0&-1&1\\0&0&1&1&2&-3\end{bmatrix}$$

Hence, $A^{-1} = \begin{bmatrix}-3&-5&9\\0&-1&1\\1&2&-3\end{bmatrix}$

(c) To find a matrix B so that $BA = \begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}$, we multiply both sides by A^{-1}, placing the A^{-1} on the right:

$$(BA)A^{-1} = \begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\begin{bmatrix}-3&-5&9\\0&-1&1\\1&2&-3\end{bmatrix} = \begin{bmatrix}-3+0+3&-5-2+6&9+2-9\\-9+0+1&-15-2+2&27+2-3\end{bmatrix} = \begin{bmatrix}0&-1&2\\-8&-15&26\end{bmatrix}$$

Since $(BA)A^{-1} = B(AA^{-1}) = BI_3 = B$, we have $B = \begin{bmatrix}0&-1&2\\-8&-15&26\end{bmatrix}$.

To find C, we multiply both sides of the equation by A^{-1}, placing the A^{-1} on the left:

$$A^{-1}(AC) = \begin{bmatrix} -3 & -5 & 9 \\ 0 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -12-15+27 \\ 0-3+3 \\ 4+6-9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since $A^{-1}(AC) = (A^{-1}A)C = I_3C = C$, we have $C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(d) A is an invertible 3×3 matrix. Hence A is nonsingular and so rank$(A) = 3$. Since F is row-equivalent to A, we must have rank$(F) = 3$ also. Thus, F is also non-singular. This implies that

$$\mathcal{N}(F) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{R}(F) = \boldsymbol{R}^3$$

(e) We can recover G from the give information as follows: The matrix A has the vectors V_1, V_2, V_3 as its columns. Let D be the diagonal matrix which has the eigenvalues of G along the main diagonal (taken in the same order as the above eigenvectors):

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Then, as explained in class, $GA = AD$ and so $G = ADA^{-1}$. Thus,

$$G = \left(\begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & 3 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \right) \begin{bmatrix} -3 & -5 & 9 \\ 0 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 9 & 28 \\ 2 & 0 & 21 \\ 2 & 3 & 21 \end{bmatrix} \begin{bmatrix} -3 & -5 & 9 \\ 0 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -6+0+28 & -10-9+56 & 18+9-84 \\ -6+0+21 & -10+0+42 & 18+0-63 \\ -6+0+21 & -10-3+42 & 18+3-63 \end{bmatrix} = \begin{bmatrix} 22 & 37 & -57 \\ 15 & 32 & -45 \\ 15 & 29 & -42 \end{bmatrix}$$

QUESTION 2 The characteristic polynomial of this matrix is

$$p(t) = det(A - tI_3) = det\left(\begin{bmatrix} 3-t & 4 & 2 \\ 0 & 5-t & 1 \\ 0 & 0 & 3-t \end{bmatrix} \right) = (3-t)(5-t)(3-t) = -(t-3)^2(t-5)$$

The eigenvalues are $\lambda = 3$ and $\lambda = 5$. The algebraic multiplicity for $\lambda = 3$ is 2. The algebraic multiplicity for $\lambda = 5$ is 1.

We will find a basis for the eigenspace for each eigenvalue.

For $\lambda = 5$, the eigenspace is

$$\mathrm{E}_5 = \mathcal{N}(A - 5I_3) = \mathcal{N}\left(\begin{bmatrix} -2 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 2y = 0, \quad z = 0\right\} = \left\{\begin{bmatrix} 2y \\ y \\ 0 \end{bmatrix}\right\} = Sp\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

The geometric multiplicity for $\lambda = 5$ is $\dim(E_5) = 1$.

For $\lambda = 3$, the eigenspace is

$$E_3 = \mathcal{N}(A - 3I_3) = \mathcal{N}\left(\begin{bmatrix} 0 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 & 1/2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : y - (1/2)z = 0\right\} = \left\{\begin{bmatrix} x \\ (1/2)z \\ z \end{bmatrix}\right\}$$

$$= Sp\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}\right\}$$

The geometric multiplicity for $\lambda = 3$ is $\dim(E_3) = 2$.

The geometric multiplicity is equal to the algebraic multiplicity for each of the eigenvalues. Hence A is a diagonalizable matrix. In fact, A is similar to each of the following matrices:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Finally, here is a description of the eigenvectors for each eigenvalue:

For $\lambda = 5$, the eigenvectors are of the form $a \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, where $a \neq 0$.

For $\lambda = 3$, the eigenvectors are of the form $a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$, where at least one of the numbers a or b is nonzero.

QUESTION 3. Before doing this question, we recall some facts:

(1) $AX = \boldsymbol{b}$ has at least one solution if $\boldsymbol{b}$ is in $\mathcal{R}(A)$ and has no solutions if $\boldsymbol{b}$ is not in $\mathcal{R}(A)$.

(2) Let n denote the number of columns of A. Let r denote the rank of A. If $\boldsymbol{b}$ is in $\mathcal{R}(A)$, then $AX = \boldsymbol{b}$ has exactly one solution if $r = n$ and $AX = \boldsymbol{b}$ has infinitely many solutions if $r < n$.

(3) Let m denote the number of rows of A. Let r denote the rank of A. Then $\mathcal{R}(A)$ is the subspace of $\boldsymbol{R}^m$ spanned by the columns of A. We have $\dim(\mathcal{R}(A)) = r$.

Now we can answer the question.

(a). All three possibilities can occur. Here are examples to illustrate each possibility:

No solution: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Then $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a spanning set for $\mathcal{R}(A)$. There are no solutions to $AX = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ because the vector $\begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and so can't be in $\mathcal{R}(A)$.

Exactly one solution. We must find A such that $\operatorname{rank}(A) = 3$ and such that $\begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ is in $\mathcal{R}(A)$. Here is one such example: $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 7 \\ 0 & 0 & 9 \end{bmatrix}$. The rank of A is clearly equal to 3 and the vector $\begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ is clearly in $\mathcal{R}(A)$.

Infinitely many solutions. Now we want an example where $\operatorname{rank}(A) < 3$ and where $\begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ is in $\mathcal{R}(A)$. Here is such an example: $A = \begin{bmatrix} 2 & 2 & 2 \\ 5 & 5 & 5 \\ 7 & 7 & 7 \\ 9 & 9 & 9 \end{bmatrix}$. Indeed, $\left\{ \begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix} \right\}$ is a basis for $\mathcal{R}(A)$ and so A has rank 1. Also, the vector $\begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$ is clearly in $\mathcal{R}(A)$.

(b) This question requires that $m = 3, \quad n = 4$. If r denotes the rank of A, then $r \leq 3$ and so it is not possible to have $r = n$. Hence, the only possibilities are: no solution or infinitely many solutions. Here are examples to illustrate those possibilities:

No solution. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Then $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a spanning set for $\mathcal{R}(A)$ and so the vector $\begin{bmatrix} 7 \\ 9 \\ 6 \end{bmatrix}$ is clearly not in $\mathcal{R}(A)$.

Infinitely many solutions. We just need to choose the matrix A so that the vector $\begin{bmatrix} 7 \\ 9 \\ 6 \end{bmatrix}$ is in $\mathcal{R}(A)$. Here is such an example: $A = \begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & 1 & 1 & 9 \\ 1 & 1 & 1 & 6 \end{bmatrix}$.

QUESTION 4

(a) The questions asks for an example of a spanning set for $\boldsymbol{R}^3$ which fails to be a linearly independent set. Here is one such example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

It is clear that this set is a spanning set for $\boldsymbol{R}^3$. It is also clear that this set is a linearly dependent set, hence not a basis for$\boldsymbol{R}^3$.

(b) A is a singular 4×4 matrix. Hence $\text{rank}(A) < 4$ and $\text{Det}(A) = 0$. Since B is row-equivalent to A, both A and B have the same rank. Hence $\text{rank}(B) < 4$. Therefore, B is a singular matrix and so $\text{Det}(B) = 0$. Therefore, it is true that $\text{Det}(B) = \text{Det}(A)$. Both determinants are equal to 0.

(c) Since C is diagonalizable, the geometric and algebraic multiplicities for each eigenvalue must be equal. The number 7 is an eigenvalue and has algebraic multiplicity equal to 2. Hence, the geometric multiplicity for the eigenvalue 7 is also equal to 2. That is, we have $\dim(\mathcal{N}(C - 7I_6) = 2$. The matrix $C - 7I_6$ is a 6×6 matrix. The dimension of the null space of this matrix is $6 - \text{rank}(C - 7I_6)$. Since this dimension is 2, it follows that $\text{rank}(C - 7I_6) = 4$.

(d) If A is a singular matrix, then A must have 0 as one of its eigenvalues. If a 2×2 matrix A has two different eigenvalues, then that matrix must be diagonalizable. Hence, a singular 2×2 matrix A which is not diagonalizable must only have one eigenvalue, namely $\lambda = 0$. Thus, the characteristic polynomial must be $p(t) = t^2$. That is, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we must have $a + d = 0$ and $ad - bc = 0$. There are many such matrices.

Here is one example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

The eigenvalue 0 has algebraic multiplicity 2, but the geometric multiplicity is 1 because $E_0 = \mathcal{N}(A)$ has dimension $2 - r$, where r denotes the rank of A. In this case, $r = 1$. Since the algebraic and geometric multiplicity for the eigenvalue 0 are not equal, it follows that A is not diagonalizable. Another example is the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$. There are infinitely many such matrices.