FACTORIZATION OF TROPICAL POLYNOMIALS IN ONE AND SEVERAL VARIABLES

by

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ABSTRACT

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This paper discusses the theory of factoring tropical polynomials as well as some factoring techniques. I give an elementary proof of the Fundamental Theorem of Tropical Algebra and describe an algorithm to factor single-variable polynomials with rational coefficients. For multi-variable tropical polynomials I discuss the corner locus and a correspondence between polynomials and weighted balanced complexes. I prove that tropical factorization is NP-complete, and I give an algorithm that is efficient for many classes of polynomials.

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Chapter 1

Introduction

Tropical algebra, also called "min-plus" or "max-plus" algebra, is a relatively new topic in mathematics that has recently caught the interest of algebraic geometers, computer scientists, combinatorists, and other mathematicians. According to Andreas Gathmann [5], tropical algebra was pioneered by mathematician and computer scientist Imre Simon in the 1980s but did not receive widespread attention until a few years ago. Interestingly, the title "tropical algebra" is nothing more than a reference to Simon's home country—Brazil.

Tropical algebra is a useful tool in solving a wide variety of problems. For example, it can be used to solve complicated phylogenetic problems [2, 14], to simplify mathematical physics problems [8], and to study implications of statistical models [11]. In addition, tropical algebra often provides an easier way to solve complex enumerative geometry problems [6, 10]. It can even be used to find and assess algorithms for solving the classic traveling salesman problem [1].

The tropical semi-ring, which is the basis of this problem-solving tool and our major object of study, is formed by replacing the standard addition operation with the minimum function and replacing multiplication with addition (see Definition 1).

Although many difficult problems can be translated into simpler tropical problems, even these simpler problems can be difficult if we do not understand the tropical algebraic system. Since tropical algebra is so new, several things about it are unknown. Our purpose in studying it, then, is to understand it better, since the more we understand about the mechanics of the tropical algebraic system, the more effective this tool becomes.

To provide a better understanding of this system, this thesis concentrates on factoring tropical polynomials. There are two reasons we want to factor polynomials. First, it helps us to better study other areas of tropical algebra, which makes it easier to solve some of the problems mentioned above. Second, factoring tropical polynomials helps us to factor classical polynomials [4, 5].

In this paper I discuss the factorization of polynomials over the rational tropical semi-ring. I discuss and present an elementary proof of the Fundamental Theorem of Tropical Algebra. I show that every tropical polynomial of a single variable can be factored completely into linear factors by a simple and straightforward process. I also discuss the ideas of functional equivalence of tropical polynomials and least-coefficient polynomials, two very important themes in tropical algebra.

I also discuss the factorization of polynomials of several variables. I prove that factorization is NP-complete, and give an algorithm that runs in polynomial time for some classes of polynomials. Since this algorithm depends on the properties of a polynomial's zero locus, I also discuss the theory of tropical zero loci. I discuss weighted balanced complexes, as defined by Mikhalkin in [9].

1.1 The Rational Tropical Semi-ring

Definition 1. The *rational tropical semi-ring* is $\mathcal{Q} = (\mathbb{Q} \cup \{\infty\}, \oplus, \odot)$, where

$$\begin{aligned} a \oplus b &:= \min(a, b)\,, \text{ and} \\ a \odot b &:= a + b\,. \end{aligned}$$

We note that the additive identity of $\mathcal{Q}$ is ∞, in the sense that $a \oplus \infty = a$ for all $a \in \mathcal{Q}$. Similarly, the multiplicative identity is 0 , since $a \odot 0 = a$ for all $a \in \mathcal{Q}$. Elements of $\mathcal{Q}$ do not have additive inverses, but the multiplicative inverse of a is the classical *negative* a . The commutative, associative, and distributive properties hold.

Notation. To simplify our notation, we will generally write tropical multiplication $a \odot b$ as just ab , and we will write repeated multiplication $a \odot a$ as a^2 . Since the multiplicative inverse of an element a is *negative* a, we will write it as either a^{-1} or $-a$. We will write classical addition, subtraction, multiplication, and division as $a+b$, $a-b$, $a \cdot b$, and $\dfrac{a}{b}$, respectively.

For example, in a tropical polynomial a monomial ax^b corresponds to the classical function $a + b \cdot x$. A general polynomial $a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_1 x \oplus a_0$ corresponds to the classical function $\min\big\{a_n + n \cdot x, a_{n-1} + (n-1) \cdot x, \ldots, a_1 + x, a_0\big\}$. If we write x^n without a coefficient, we mean $0x^n$, since 0 is the multiplicative identity. Also note that $1x^n \neq x^n$, so if we mean $1x^n$, we must write $1x^n$.

Chapter 2

Tropical Polynomials of One Variable

We begin by discussing the semi-ring $\mathcal{Q}[x]$—tropical polynomials of one variable. The goal of this chapter is to present an elementary proof of the Fundamental Theorem of Tropical Algebra,[1] which states

Fundamental Theorem of Tropical Algebra. *Every tropical polynomial in one variable with rational coefficients can be factored uniquely as a product of linear tropical polynomials with rational coefficients, up to functional equivalence.*[2]

Although the authors of some papers refer to this theorem, they only mention it to confirm it as true or dismiss it as trivial. Nevertheless, our proof of this theorem is key to understanding vital components of the tropical algebraic structure. Note that one version of the proof has been published by Izhakian in [7], but Izhakian gives his proof over an "extended" tropical semi-ring that is substantially different

[1]This theorem is referred to as the Fundamental Theorem of Tropical Algebra because it is the tropical analogue of the Fundamental Theorem of Algebra. Clearly, however, this theorem is separate from (and not a consequence of) the classical Fundamental Theorem.

[2]This theorem is only true if we consider two polynomials to be equivalent if they define the same function, as we do in this thesis. See Section 2.1.

from the standard tropical semi-ring that most others study. Hence, there is merit in discussing this elementary proof and the underlying ideas it addresses.

To prove this theorem, we will first discuss the idea of functional equivalence, in Section 2.1. Tropically, it is most useful to consider two polynomials as equivalent if the functions they define are equivalent. In Section 2.2, we will discuss a best representative for each functional equivalence class, called a *least-coefficient polynomial*. We will prove that every tropical polynomial is equivalent to a least-coefficient polynomial and that each least-coefficient polynomial can be easily factored using the formula given in Section 2.3.

2.1 Polynomials and Functions

A polynomial $f(x) \in \mathcal{Q}[x]$ is defined to be a formal sum

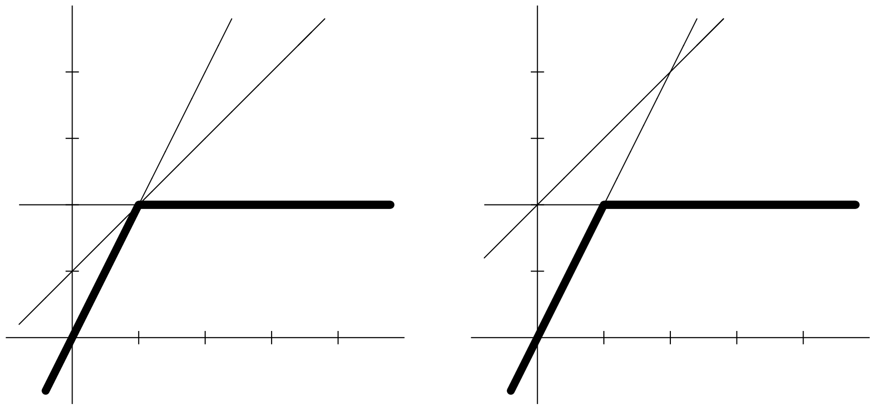
$$f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_0 \,.$$

For two polynomials f and g, we write $f = g$ if each pair of corresponding coefficients of f and g are equal.

We can also think of a tropical polynomial as a function. Two polynomials f and g are *functionally equivalent* (written $f \sim g$) if for each $x \in \mathcal{Q}$, $f(x) = g(x)$. It is easy to check that functional equivalence is in fact an equivalence relation. Notice that functional equivalence does not imply equality. For example, the polynomials $x^2 \oplus 1x \oplus 2$ and $x^2 \oplus 2x \oplus 2$ are functionally equivalent,[3] but not equal as polynomials. In general, functional equivalence is a more useful equivalence relation to use with tropical polynomials than equality of coefficients.

[3]That is, the classical functions that these polynomials correspond to—$\min(2 \cdot x, 1 + x, 2)$ and $\min(2 \cdot x, 2 + x, 2)$ respectively—are equal as functions.

The function graph of a polynomial. For a polynomial of one variable, we can plot the input x versus the output $f(x)$ to better visualize the difference between equality and functional equivalence. For example, the polynomial $x^2 \oplus 1x \oplus 2$ mentioned above can be described classically as $\min\{2 \cdot x, 1 + x, 2\}$. So each term in the polynomial can be represented by a classical line, and a polynomial's function corresponds to the lowest of these lines at each point, as shown in Figure 2.1.



$f(x) = x^2 \oplus 1x \oplus 2$ $\qquad$ $g(x) = x^2 \oplus 2x \oplus 2$

Figure 2.1: Although $f(x)$ and $g(x)$ are not equal as polynomials, they are functionally equivalent. Also, the $1x$ in $f(x)$ is a least-coefficient term, while the $2x$ in $g(x)$ is not.

When dealing with equivalence classes, it is useful to have a representative of each equivalence class. One useful representative of the equivalence class $[f(x)]$ is the polynomial formed by making each of the coefficients of $f(x)$ as small as possible without changing the function defined by $f(x)$.

Definition 2. A coefficient a_i of a polynomial $f(x)$ is a *least coefficient* if for any $b \in \mathcal{Q}$ with $b < a_i$, the polynomial $g(x)$ formed by replacing a_i with b is not functionally equivalent to $f(x)$.

Note. If $f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_r x^r$, where $a_n, a_r \neq \infty$, then a_n and a_r are least coefficients. Additionally, if $r < i < n$ and $a_i = \infty$, then a_i is not a least coefficient.

Example. Let $f(x) = x^2 \oplus 1x \oplus 2$ and let $g(x) = x^2 \oplus 2x \oplus 2$, as in Figure 2.1. Then the $2x$ in $g(x)$ is not a least-coefficient term, since we can replace the 2 with a 1 and still have the same function. The $1x$ in $f(x)$, however, *is* a least-coefficient term, since replacing the 1 with any number less than 1 would form a polynomial with a different function than $f(x)$.

Lemma 1 (Alternate definition of least coefficient)**.** *Let $a_i x^i$ be a term of a polynomial $f(x)$, with a_i not equal to infinity. Then a_i is a least coefficient of $f(x)$ if and only if there is some $x_0 \in \mathbb{Q}$ such that $f(x_0) = a_i x_0^i$.*

Proof. For all $x \in \mathbb{Q}$, note that $f(x) \leq a_i x^i$. Suppose that there is no x such that $f(x) = a_i x^i$. Then $f(x) < a_i x^i$ for all x. Now, let $\varphi(x) = f(x) - (i \cdot x + a_i)$. Note that φ is a piecewise-linear, continuous function that is linear over a finite number of intervals. Thus, there is an interval large enough to contain all the pieces of φ. By applying the extreme value theorem to this interval, we see that $\sup \varphi \in \varphi(\mathbb{R})$, and hence $\sup \varphi < 0$. Let $\epsilon = |\sup \varphi|$ and $b \in \mathcal{Q}$ be such that $a_i - \epsilon < b < a_i$. Then

$$f(x) - (i \cdot x + b) < f(x) - (i \cdot x + a_i) + \epsilon \leq 0$$

and therefore $f(x) < i \cdot x + b$ for all $x \in \mathbb{Q}$. Since the polynomial created by replacing a_i with b is functionally equivalent to $f(x)$, a_i is not a least coefficient.

For the other direction, suppose that there is an $x_0 \in \mathbb{Q}$ such that $f(x_0) = a_i x_0^i$. Given $b < a_i$, let $g(x)$ be $f(x)$ with a_i replaced by b. Then $g(x_0) \leq b x_0^i < a_i x_0^i = f(x_0)$, so g is not functionally equivalent to f. Therefore a_i is a least coefficient. □

2.2 Least-coefficient Polynomials

Definition 3. A polynomial is a *least-coefficient polynomial* if all its coefficients are least coefficients.

Lemma 2 (Uniqueness of least-coefficient polynomials)**.** *Let f and g be tropical least-coefficient polynomials. Then f is equal to g if and only if f is functionally equivalent to g .*

Proof. It is clear that $f = g$ implies $f \sim g$, since this is true for all tropical polynomials. Now suppose that $f \neq g$. Then for some term $a_i x^i$ of $f(x)$ and the corresponding term $b_i x^i$ of $g(x)$, we have $a_i \neq b_i$. Without loss of generality, suppose $a_i < b_i$. Since g is a least-coefficient polynomial, $g(x_0) = b_i x_0^i$ for some x_0 , by Lemma 1. Now,

$$f(x_0) \leq a_i x_0^i < b_i x_0^i = g(x_0)\,,$$

so f is not functionally equivalent to g . □

The least-coefficient representative is often the most useful way to represent a functional equivalence class of tropical polynomials.

Lemma 3. *Let $f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_r x^r$. There is a unique least-coefficient polynomial $g(x) = b_n x^n \oplus b_{n-1} x^{n-1} \oplus \cdots \oplus b_r x^r$ such that $f \sim g$. Furthermore, each coefficient b_j of $g(x)$ is given by*

$$b_j = \min\left(\left\{a_j\right\} \cup \left\{\frac{a_i \cdot (k-j) + a_k \cdot (j-i)}{k-i} \middle| r \leq i < j < k \leq n\right\}\right)\,. \qquad (2.1)$$

Proof. First we will show that $f \sim g$. Given x_0 , note that $f(x_0) = a_s x_0^s = a_s + s \cdot x_0$ for some s . Also,

$$\begin{aligned} g(x_0) &= \min_{r \leq j \leq n} \{b_j + j \cdot x_0\} \\ &= \min_{r \leq i < j < k \leq n} \left\{a_j + j \cdot x_0, \frac{a_i \cdot (k-j) + a_k \cdot (j-i)}{k-i} + j \cdot x_0\right\}\,. \qquad (2.2) \end{aligned}$$

So for any $i, j,$ and k such that $r \leq i < j < k \leq n$, if $x_0 \geq \frac{a_i - a_k}{k-i}$ then

$$\begin{aligned} a_s + s \cdot x_0 &\leq a_i + i \cdot x_0 \\ &= \frac{a_i \cdot (k-j) + a_k \cdot (j-i)}{k-i} + (j-i) \cdot \left(\frac{a_i - a_k}{k-i} \right) + i \cdot x_0 \\ &\leq \frac{a_i \cdot (k-j) + a_k \cdot (j-i)}{k-i} + (j-i) \cdot x_0 + i \cdot x_0 \\ &= \frac{a_i \cdot (k-j) + a_k \cdot (j-i)}{k-i} + j \cdot x_0 \, . \end{aligned}$$

A similar argument shows that if $x_0 \leq \frac{a_i - a_k}{k-i}$, then

$$a_s + s \cdot x_0 \leq a_k + k \cdot x_0 \leq \frac{a_i \cdot (k-j) + a_k \cdot (j-i)}{k-i} + j \cdot x_0 \, .$$

Since this is true for all i, j, and k, the equation in (2.2) evaluates to $g(x_0) = a_s x_0^s$, so $g(x_0) = f(x_0)$ and $f \sim g$, as desired.

Secondly, we must show g is a least-coefficient polynomial. Given a coefficient b_j in g, suppose that a_j is a least coefficient of f. From Equation (2.1) we see that $b_j \leq a_j$. Since a_j is a least coefficient, there is some x_0 such that $f(x_0) = a_j x_0^j$, so $b_j x_0^j \geq g(x_0) = f(x_0) = a_j x_0^j$. Therefore $b_j = a_j$ and $g(x_0) = b_j x_0^j$.

Now suppose that a_j is not a least coefficient. Then since a_r and a_n are least-coefficient, we can choose $u < j$ and $v > j$ such that a_u and a_v are least coefficients and for any t such that $u < t < v$, a_t is not a least coefficient.

Let $x_0 = \frac{a_u - a_v}{v-u}$ and suppose, by way of contradiction, that $f(x_0) \neq a_u x_0^u$. Then $f(x_0) = a_w x_0^w < a_u x_0^u$ for some w. Note that a_w is a least coefficient, so it cannot be that $u < w < v$ by our assumption on u and v. If $w < u$ then for $x \geq x_0$,

$$\begin{aligned} a_w + w \cdot x &= a_w + u \cdot x - (u-w) \cdot x \\ &\leq a_w + u \cdot x - (u-w) \cdot x_0 \\ &= a_w + w \cdot x_0 + u \cdot (x - x_0) \\ &< a_u + u \cdot x_0 + u \cdot (x - x_0) \\ &= a_u + u \cdot x \end{aligned}$$

For $x < x_0$,

$$\begin{aligned} a_v + v \cdot x &= a_v + u \cdot x + (v-u) \cdot x \\ &< a_v + u \cdot x + (v-u) \cdot x_0 \\ &= a_v + v \cdot x_0 + u \cdot (x - x_0) \\ &= a_u + u \cdot x_0 + u \cdot (x - x_0) \\ &= a_u + u \cdot x \end{aligned}$$

So there is no x such that $f(x) = a_u x^u$ and thus a_u is not a least coefficient, which contradicts our assumption. If $w > v$, a similar argument shows that a_v is not a least coefficient, again contradicting our assumption. Therefore,

$$\begin{aligned} f(x_0) &= a_u + u \cdot \left(\frac{a_u - a_v}{v-u}\right) \\ &= \frac{a_u \cdot (v-j) + a_v \cdot (j-u)}{v-u} + j \cdot \left(\frac{a_u - a_v}{v-u}\right) \\ &= c + j \cdot x_0 \text{, where } c = \frac{a_u \cdot (v-j) + a_v \cdot (j-u)}{v-u}. \end{aligned} \tag{2.3}$$

Again, from (2.1) we see that $b_j \leq c$, and from (2.3) we see $cx_0^j = f(x_0) = g(x_0) \leq b_j x_0^j$. So $b_j = c$ and $g(x_0) = b_j x_0^j$.

Finally, g is the only such polynomial by Lemma 2. □

Note. The use of a least-coefficient polynomial as a best representative for a functional equivalence class is one of the key ideas of this chapter. We cannot develop well-defined algebraic transformations of tropical polynomials without unique representatives for functional equivalence classes. While Izhakian discusses in [7] what he calls an "effective" coefficient (similar to a least coefficient), the idea of using least-coefficient polynomials to represent functional equivalence classes has not been discussed.

When dealing with polynomials of a single variable, we have a simple way to determine whether or not a polynomial is a least coefficient:

Lemma 4. *Let $f(x) = a_n x^n \oplus a_{n-1}x^{n-1} \oplus \cdots \oplus a_r x^r$, where each a_i is not infinity. Let $d_i = a_{i-1} - a_i$ be the difference between two consecutive coefficients. Then $f(x)$ is a least-coefficient polynomial if and only if the difference between consecutive coefficients is non-decreasing, that is, if $d_n \leq d_{n-1} \leq \cdots \leq d_{r+1}$.*

Proof. Suppose that f has a set of consecutive coefficients whose differences are *decreasing*, that is, ax^{i+1}, bx^i, and cx^{i-1} are consecutive terms of $f(x)$ such that $b-a > c-b$. Then $b > \frac{1}{2}\cdot(a+c)$. We will show that $f(x_0) < bx_0^i$ for all x_0, meaning that b is not a least coefficient.

Given x_0, if $x_0 \leq \frac{1}{2}\cdot(c-a)$ then

$$\begin{aligned}
ax_0^{i+1} &= (i+1)\cdot x_0 + a \\
&\leq i\cdot x_0 + \frac{1}{2}\cdot(c-a) + a \\
&= i\cdot x_0 + \frac{1}{2}\cdot(c+a) \\
&< i\cdot x_0 + b = bx_0^i,
\end{aligned}$$

so $f(x_0) \leq ax_0^{i+1} < bx_0^i$. Similarly, if $x_0 \geq \frac{1}{2}\cdot(c-a)$,

$$\begin{aligned}
cx_0^{i-1} &= (i-1)\cdot x_0 + c \\
&\leq i\cdot x_0 - \frac{1}{2}\cdot(c-a) + c \\
&= i\cdot x_0 + \frac{1}{2}\cdot(c+a) \\
&< i\cdot x_0 + b = bx_0^i,
\end{aligned}$$

so $f(x_0) \leq cx_0^{i-1} < bx_0^i$. Therefore b is not a least coefficient, and f is not a least-coefficient polynomial.

For the other direction, suppose that the differences between the coefficients of $f(x)$ are non-decreasing. Since $a_n, a_r \neq \infty$, a_n and a_r are least coefficients. Let a_i be a coefficient of f, with $r < i < n$, and let $x_0 = \frac{a_{i-1}-a_{i+1}}{2}$. We will show that $f(x_0) = a_i x_0^i$, so a_i is a least coefficient. We must show for all k that $i\cdot x_0 + a_i \leq k\cdot x_0 + a_k$.

This is certainly true for $i = k$. Suppose $k > i$. Since $(a_t - a_{t+1}) \leq (a_s - a_{s+1})$ for $t \geq s$, we have

$$\begin{aligned}(a_i - a_{i+2}) = (a_i - a_{i+1}) + (a_{i+1} - a_{i+2}) &\leq 2 \cdot (a_i - a_{i+1}) \\ (a_i - a_{i+3}) = (a_i - a_{i+2}) + (a_{i+2} - a_{i+3}) &\leq 3 \cdot (a_i - a_{i+1})\end{aligned}$$

And in general we get

$$\begin{aligned}(a_i - a_k) &\leq (a_i - a_{i+1}) \cdot (k - i) = \frac{1}{2} \cdot \big(2 \cdot (a_i - a_{i+1})\big) \cdot (k - i) \\ &\leq \frac{1}{2} \cdot \big((a_{i-1} - a_i) + (a_i - a_{i+1})\big) \cdot (k - i) = x_0 \cdot (k - i)\,.\end{aligned}$$

Thus, $i \cdot x_0 + a_i \leq k \cdot x_0 + a_k$. A similar argument holds for $k < i$. So (tropically) $a_i {x_0}^i \leq a_s {x_0}^s$ for all s. This means that $f(x_0) = a_i {x_0}^i$, so a_i is a least coefficient. Therefore, f is a least-coefficient polynomial. $\square$

Note. If $f(x)$ has a coefficient a_i such that $a_i = \infty$ for $r < i < n$, then f is not a least-coefficient polynomial; but of course, $a_j = \infty$ for all $j > n$ and all $j < r$, even in a least-coefficient polynomial.

2.3 The Fundamental Theorem of Tropical Algebra

We are now ready to state and prove the Fundamental Theorem of Tropical Algebra.

Theorem 5 (Fundamental Theorem)**.** *Let $f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_r x^r$ be a least-coefficient polynomial. Then $f(x)$ can be written uniquely as the product of linear factors*

$$a_n x^r \,(x \oplus d_n)\,(x \oplus d_{n-1}) \cdots (x \oplus d_{r+1})\ , \tag{2.4}$$

where $d_i = a_{i-1} - a_i$. In other words, the roots of $f(x)$ are the differences between consecutive coefficients. Since every polynomial is equivalent to a least-coefficient polynomial, this proves that every tropical polynomial can be factored into linear terms.

Proof. Since $f(x)$ is a least-coefficient polynomial, the differences between consecutive coefficients is non-decreasing, i.e., $d_n \leq d_{n-1} \leq \cdots \leq d_{r+1}$. Knowing these inequalities, we can expand (2.4) to get

$$a_n x^n \oplus a_n d_n x^{n-1} \oplus a_n d_n d_{n-1} x^{n-2} \oplus \cdots \oplus a_n d_n d_{n-1} \cdots d_{r+1}\,. \tag{2.5}$$

But the coefficient of the x^i term in this polynomial is

$$a_n d_n d_{n-1} \cdots d_{i+1} = a_n + d_n + d_{n-1} + \cdots + d_{i+1}\,.$$

A straightforward computation shows that this is equal to a_i, so the polynomial in (2.5) is equal to $f(x)$, as desired.

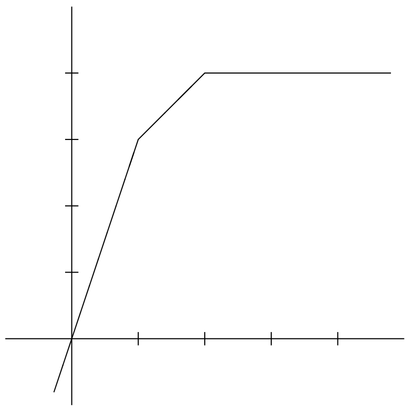
Now suppose that there is another way of writing $f(x)$ as a product of linear factors. Call this product g and note that it must have the same degree as f. Additionally, the smallest non-infinite term of g must have the same degree as the smallest non-infinite term of f. Hence, we are able to write

$$g(x) = a'_n x^r \left(x \oplus d'_n\right)\left(x \oplus d'_{n-1}\right) \cdots \left(x \oplus d'_{r+1}\right)\,,$$

with each d'_i chosen, after reindexing, if necessary, such that $d'_n \leq d'_{n-1} \leq \cdots \leq d'_{r+1}$. Expanding this product shows that the differences between consecutive coefficients of g are non-decreasing, so g is a least-coefficient polynomial by Lemma 4. Also, it is clear that $f \neq g$, so by Lemma 2, f is not functionally equivalent to g. Therefore, the factorization is unique. □

Factoring: an example. To better understand how factoring one-variable polynomials works, consider the polynomial $f(x) = x^3 \oplus 3x^2 \oplus 2x \oplus 4$. First we use

Equation 2.1 to find a least coefficient polynomial that is functionally equivalent to f. This polynomial is $g(x) = x^3 + 1x^2 + 2x + 4$. Then the "roots" of $f(x)$ are the differences between consecutive coefficients of $g(x)$: 1,1,2. So $f(x) \sim (x \oplus 1)^2(x \oplus 2)$. Note that these "roots" correspond to the corners of the function graph of $f(x)$ (see Figure 2.3)



$f(x) = x^3 \oplus 3x^2 \oplus 2x \oplus 4$

Figure 2.2: The function graph of $f(x)$. Note that $f(x)$ has "roots" at 1,1,2, corresponding to the corners of the graph.

2.4 The Corner Locus of a Polynomial

Finally we note that tropical factoring gives us a slightly different result than classical factoring. Classically, the set of roots (or zero locus) of a polynomial is the set of points at which the polynomial evaluates to the additive identity. Unfortunately, tropical polynomials have either no roots or trivial roots in this sense. In fact, if $f(x) \neq \infty$, then $f(x_0)$ never evaluates to the additive identity ∞ when $x_0 \neq \infty$. However, as we have seen, polynomials in $\mathcal{Q}[x]$ can be factored and seem to have "roots," although they do not evaluate to the additive identity at these points. Clearly, we must use a different, more meaningful definition. In [12] motivation is given for the following definition of zero locus.

Definition 4. Let $f(x) \in \mathcal{Q}[x]$. The *zero locus* (or *corner locus*) $\mathcal{Z}(f)$ is defined to be the set of points x_0 in $\mathcal{Q}$ for which at least two monomials of f attain the minimum value.

The d_i in (2.4) are precisely that points of $\mathcal{Z}(f)$, as we now show.

Theorem 6. *Given a point $d \in \mathcal{Q}$ and a least-coefficient polynomial $f(x)$, $x \oplus d$ is a factor of $f(x)$ if and only if $f(d)$ attains its minimum on at least two monomials.*

Proof. First, suppose that $x \oplus d$ is a factor of $f(x)$. If we write f as a product of linear factors as in (2.4), $d_i = d$ for some d_i .For all $j < i$,

$$\begin{aligned} a_i + i \cdot d_i &= a_i + (i-j) \cdot d_i + j \cdot d_i \\ &\leq a_i + \underbrace{d_i + d_{i-1} + \cdots + d_{j+1}}_{i-j \text{ terms}} + j \cdot d_i \\ &= a_j + j \cdot d_i \,. \end{aligned}$$

A similar calculation shows that for $j > i$, we have $a_i + i \cdot d_i \leq a_j + j \cdot d_i$. So $f(d_i) = a_i d_i^i$. Also, $a_i d_i^i = a_n d_n d_{n-1} \cdots d_{i+1} d_i^i = a_n d_n d_{n-1} \cdots d_i d_i^{i-1} = a_{i-1} d_i^{i-1}$, so the minimum is attained by at least two monomials of $f(x)$ at $x = d$.

For the other direction, suppose that the minimum is attained by two monomials at $f(d)$. By way of contradiction, suppose that these monomials are not consecutive. Then for some $j < i < k$, we have $a_j d^j = a_k d^k < a_i d^i$. If $x \leq d$ then

$$\begin{aligned} a_k + k \cdot x &= a_k + i \cdot x + (k-i) \cdot x \\ &\leq a_k + i \cdot x + (k-i) \cdot d \\ &= a_k + k \cdot d + i \cdot (x-d) \\ &< a_i + i \cdot d + i \cdot (x-d) \\ &= a_i + i \cdot x. \end{aligned}$$

Similarly, if $x \geq d$, then $a_j x^j < a_i x^i$. Thus there is no x such that $f(x) = a_i x^i$, so a_i is not a least coefficient, which is a contradiction. Therefore there is some i such that $a_i d^i = a_{i-1} d^{i-1}$. Thus we have

$$0 = a_{i-1} + (i-1) \cdot d - (a_i + i \cdot d) = a_{i-1} - a_i - d\,.$$

So $d = a_{i-1} - a_i$, the difference between two consecutive coefficients. Since f is a least-coefficient polynomial, $x \oplus d$ is a factor of f by the Fundamental Theorem. $\square$

Thus, as in the classical case, the unique factorization of a polynomial in $\mathcal{Q}[x]$ gives us what could be considered the roots of the polynomial. It is clear that all of the arguments and results of this chapter hold if we replace the rationals $\mathbb{Q}$ with any classical ordered field. Thus any ordered field, together with ∞, can be said to be tropically algebraically closed.

Chapter 3

Tropical Polynomials of Several Variables

Although we can factor every single-variable polynomial into linear factors, we have no hope of being able to do the same for polynomials of two or more variables. In fact, tropical polynomials in more than one variable do not always have unique factorizations.[1] Nevertheless, the structure of the tropical zero locus makes it possible for us to learn a lot more about tropical factorization than we would expect to be able to. To make use of this structure, we discuss the tropical zero locus in detail. We discuss polyhedral complexes and a correspondence between weighted balanced complexes and tropical polynomials. We prove that factorization of tropical polynomials in two or more variables is NP-complete. We then present a method in which we use the structure of the tropical zero locus to find factorizations of polynomials.

Notation. In Chapter 2, we were very careful to distinguish between formal equality and functional equivalence. In this chapter, we will not be so careful. We will now consider two polynomials the same polynomial if they define the same function. In

[1]For example, the cubic polynomial $x^2y \oplus xy^2 \oplus x^2 \oplus xy \oplus y^2 \oplus x \oplus y$ factors into irreducible polynomials as $(x \oplus 0)(y \oplus 0)(x \oplus y)$ or as $(x \oplus y \oplus 0)(xy \oplus x \oplus y)$.

other words, we will write $f = g$ to mean that f and g are functionally equivalent.

We will also use multi-index notation throughout this chapter. In other words, we will write a polynomial $f \in \mathcal{Q}[x_1, \dots, x_n]$ as

$$f(x) = \bigoplus_{I \in \Delta} a_I x^I,$$

where $I = (I_1, \dots I_n) \in (\mathbb{Z}^+)^n$, and $x^i = {x_1}^{I_1} {x_2}^{I_2} \cdots {x_n}^{I_n}$. The set Δ is called the support of f. It is important to note that a tropical monomial $a_I x^I$ is written classically as $\langle x, I \rangle + a_I$, where $\langle \cdot, \cdot \rangle$ is the scalar product and we think of x as $(x_1, \dots, x_n)$. Therefore, we have

$$f(x_1, \dots, x_n) = f(x) = \bigoplus_{I \in \Delta} a_I x^I = \min_{I \in \Delta} \{\langle x, I \rangle + a_I\}.$$

It is also important to note that since our base field is $\mathcal{Q} = \mathbb{Q} \cup \{\infty\}$, our calculations will sometimes include multiplication by or subtraction of infinity. We will use the convention that $\infty \cdot 0 = 0$ and $\infty - \infty = 0$.

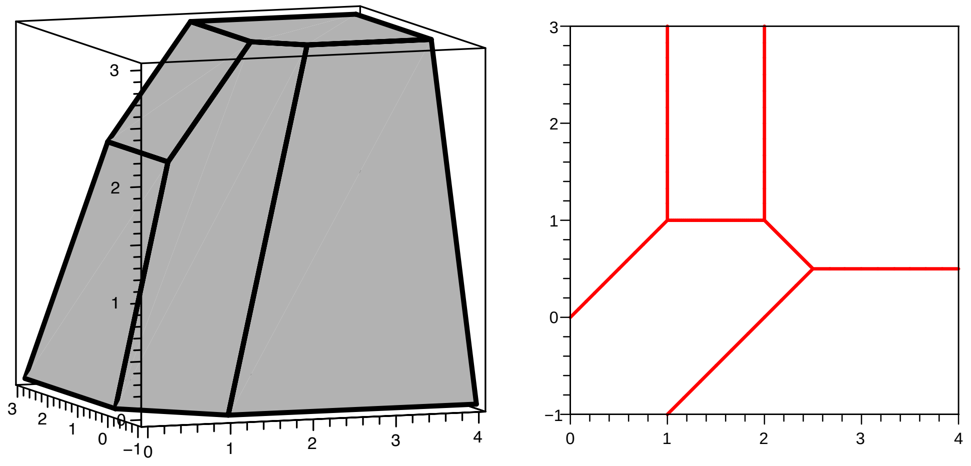
3.1 The Corner Locus and Weighted Balanced Complex

3.1.1 The Corner Locus

Definition 5. If $f \in \mathcal{Q}[x_1, \dots, x_n]$ is a polynomial then the *tropical zero locus* or *corner locus* of f is the set

$$\mathcal{Z}(f) := \{\alpha \in \mathcal{Q}^n : \text{two or more monomials of } f \text{ attain the minimum at } \alpha\}.$$

Even though it is not technically a zero locus, we call $\mathcal{Z}(f)$ the tropical zero locus because it is analogous to the classical zero locus. We often refer to $\mathcal{Z}(f)$ as the corner



$$f(x,y) = x^2 \oplus xy \oplus 2y^2 \oplus 1x \oplus 3$$

Figure 3.1: Left: the surface $z = f(x,y)$. Right: the tropical zero locus of f.

locus because it is in fact the projection of corners of the surface $z = f(x_1, \ldots, x_n)$ into the hyperplane $z = 0$ (see Figure 3.1).

Let $f(x_1, \ldots x_n)$ be a polynomial, and let $a_I x^I$ and $a_J x^J$ be two monomials of f. Then if the two monomials attain the minimum together at some $y \in \mathcal{Q}^n$, they are equal, and thus satisfy the equation

$$\langle y, I - J \rangle = a_J - a_I.$$

In other words, the set of all points where $a_I x^I$ and $a_J x^J$ simultaneously attain the minimum of f is contained in a hyperplane that is perpendicular to the integer vector $I - J$.

Also note that $\mathcal{Z}(f)$ partitions $\mathcal{Q}^n$ into connected regions, and on each of these connected regions, one of the monomials $a_I x^I$ is strictly less than all of the others.

3.1.2 The Weighted Balanced Complex

Corner loci of tropical polynomials have several nice properties. They can be written as the union of a finite number of super-convex polyhedra (that is, convex polyhedra whose faces of every dimension are also convex) that intersect each other only along entire faces. The polyhedra can also be assigned weights and are balanced in a sense that we will discuss later. We formalize all of this in the definition of a weighted balanced complex. We also give the definition of a weighted balanced complex. Our definitions are based on Mikhalikin's definitions, given in [9].

Definition 6. A subset $\Pi \subset \mathcal{Q}^{n+1}$ is called a *rational polyhedral complex* (herein a *complex*) if it can be presented as a finite union of closed sets in $\mathcal{Q}^{n+1}$ called *cells* with the following properties.

- Each cell is a closed convex (possibly semi-infinite or infinite) polyhedron. We call a cell of dimension k a k-cell.
- Each cell has rational slope (i.e. its affine span is the kernel of a rational matrix).
- The boundary of a k-cell (that is, the boundary in the corresponding affine span) is a union of $(k-1)$-cells.
- Different open cells (that is, the interiors of the cells in the corresponding affine spans) do not intersect.

For simplicity, we require all cells to be maximal in the sense that the union of two distinct k-cells is not a k-cell. The dimension of a complex is the maximal dimension of its cells. We call a complex of dimension n an n-complex.

Definition 7. A complex $\Pi \subset \mathcal{Q}^{n+1}$ is called *weighted* if there is a natural number assigned to each n-cell of Π.

We are now ready to discuss what it means for a weighted complex to be balanced. First, it is important to distinguish between boundary cells and interior cells.

Definition 8. In a complex $\Pi \subset \mathcal{Q}^{n+1}$, a *boundary cell* is a cell A that is contained in a boundary hyperplane of $\mathcal{Q}^{n+1}$. In other words $A \subset \{(x_1, \ldots, x_n) : x_i = \infty\}$ for some i. An *interior cell* is a cell not contained in any boundary hyperplane.

Definition 9. In a complex $\Pi \subset \mathcal{Q}^{n+1}$, the *integer covector* of an interior n-cell F is a vector $c \in \mathbb{Z}^n$ such that c is normal to F, with the entries of c relatively prime. Note an integer covector is only well-defined up to its sign. The covector becomes well defined when we associate it with one of the regions of $\mathcal{Q}^{n+1}$ on either side of it. The integer covector of an n-cell with respect to a region is the integer covector of the n-cell that points from the n-cell into the region.

Definition 10 (Balancing Condition)**.** Let $\Pi \subset \mathcal{Q}^{n+1}$ be a weighted n-complex with the property that for $k < n$, every $(k-1)$-cell is contained in the boundary of some k-cell. Let r be an interior $(n-1)$-cell of Π. Let $\{F_1, \ldots, F_k\}$ be the set of n-cells in Π containing r in their boundary. Then Π is *balanced at* r if for the weights $\omega_1, \ldots, \omega_k$ corresponding to the F_i, we have

$$\sum_{i=1}^{k} \omega_i c_i = 0,$$

where each c_i is an integer covector corresponding to F_i and no two c_i correspond to the same region of the complement of Π.

We say that Π is *balanced at the boundary* if each boundary hyperplane either contains no n-cells or is the union of a set of n-cells, all of which have the same weight.

We say that Π is balanced (i.e. Π is a *weighted balanced complex*) if Π is balanced at each of its interior $(n-1)$-cells and balanced at the boundary.

Note. Much work has been done regarding so-called tropical graphs and tropical weighted balanced graphs that correspond to tropical curves (see [5, 13]). In $\mathcal{Q}^2$, a

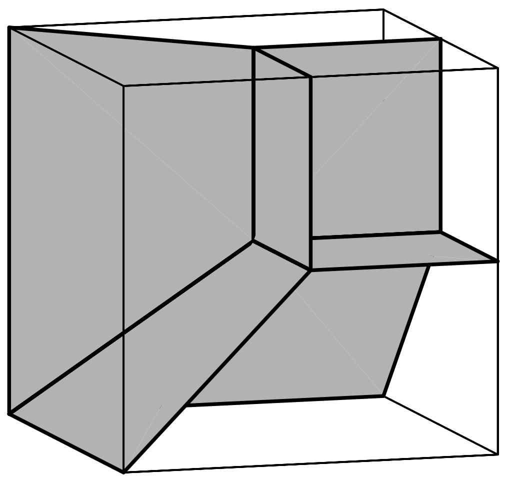


Figure 3.2: Corner locus of $x \oplus y \oplus z \oplus 0$: A weighted balanced 2-complex. Note that the 2-cells are balanced around the 1-cells

weighted balanced 1-complex is a weighted balanced graph. The 1-cells of a weighted balanced 1-complex correspond to the edges of a weighted balanced graph, and the 0-cells correspond to vertices. In this paper, we do not consider tropical graphs in $\mathcal{Q}^n$ for $n > 2$, since they do not correspond to the corner locus of a single polynomial. For the same reason, we do not consider polyhedral complexes of codimension greater than one.

Also, note that the balancing condition (together with our requirement that cells of a complex be maximal) implies that each interior $(n-1)$-cell is in fact the boundary of three or more n-cells.

3.1.3 Polynomial-Complex Correspondence

Theorem 7. *There is a bijective correspondence between weighted balanced complexes and tropical polynomials, up to functional equivalence and multiplication of the polynomial by a scalar.*

This theorem is proved by Mikhalkin in [9], although he does not use the language of tropical polynomials. We should also note that Mikhalkin only proves the correspondence up to multiplication by an indeterminant. It is easy to see that by using $\mathcal{Q}^n$, we do not have this ambiguity. We also note that Mikhalkin proves the theorem for $\mathbb{R}^n$, while we only use $\mathcal{Q}^n$. Since $\mathcal{Q}$ is algebraically closed, $\mathcal{Q}^n$ is sufficient.

Using this theorem, we can prove things about polynomials by proving things about their corresponding weighted balanced complexes. For convenience, we will give the correspondence explicitly.

The Weighted Balanced Complex Corresponding to a Polynomial. Let f be a tropical polynomial. Mikhalkin has shown that $\mathcal{Z}(f)$ is a complex. We make $\mathcal{Z}(f)$ into a weighted complex by assigning each n-cell F a weight as follows:

- If F a boundary n-cell, contained in the hyperplane corresponding to $x_i = \infty$, then x_i divides f. Let m be the largest power of x_i dividing f. The we assign the weight m to F.

- If F is an interior n-cell, then F is the boundary between two $(n+1)$-dimensional regions in $\mathcal{Q}^{n+1}$. If $a_I x^I$ and $a_J x^J$ are the monomials corresponding to these regions, then we define the weight of F to be the lattice length of the vector $I - J$.[2]

Definition 11. The weighted balanced complex formed above is called the weighted balanced complex of f and is denoted $\mathcal{W}(f)$.

The Polynomial Corresponding to a Weighted Balanced Graph Given a weighted balanced complex $\Pi \subset \mathcal{Q}^{n+1}$, we construct a polynomial as follows.

[2]This weight is easily calculated to be the GCD of the entries of $I - J$.

1. Choose a region R_0 of the complement of Π. Some monomial attains the minimum on this region. Call it $a_I x^I$, where a_I is arbitrary.

2. For every other region R of the complement of Π, choose a sequence of regions $R_0, R_1, \ldots, R_{k-1}, R_k = R$ such that each R_{i-1} is adjacent to the region R_i. For $1 \leq i \leq k$, let F_i be the n-cell that separates R_{i-1} from R_i. Let ω_i be the weight of F_i, and let c_i be the integer covector corresponding to F_i and R_i. Assign the monomial $a_J x^J$ to R, where
$$J = I - \sum_{i=1}^{k} \omega_i c_i \,.$$

3. Adjust the entries of I to be all positive, but as small as possible so that ${x_i}^m$ divides f but ${x_i}^{m+1}$ does not, where m is the weight of the n-cells in the x_i boundary, if there are any, and 0 otherwise.

4. Assign coefficients using the $(n-1)$-cells.

3.2 Factors and Balanced Subcomplexes

Lemma 8. *Let $a, b, c, d \in \mathbb{Z}^2$ with $b - a$ and $d - c$ multiples of the same primitive integral vector e. Then the lattice length from $a + c$ to $b + d$ is the sum of the lattice length from a to b and the lattice length from c to d.*

Proof. Let $b - a = \omega_1 e$ and $d - c = \omega_2 e$. Then ω_1 and ω_2 are the lattice lengths from a to b and c to d respectively. It is clear that $(b + d) - (a + c) = (b - a) + (d - c) = \omega_1 e + \omega_2 e = (\omega_1 + \omega_2)e$. So the lattice length from $a + c$ to $b + d$ is $\omega_1 + \omega_2$, as desired. $\square$

Lemma 9 (Aaron Hill)**.** *If f and g are tropical polynomials, then the corner locus of fg is equal to the union of the corner locus of f and the corner locus of g. In other words, $\mathcal{Z}(fg) = \mathcal{Z}(f) \cup \mathcal{Z}(g)$.*

Proof. The proof is straightforward, and is given in [3]. $\square$

Lemma 10. *Let $f, g \in \mathcal{Q}[x_1, \ldots, x_{n+1}]$, and let F be an n-cell of $\mathcal{W}(fg) \subset \mathcal{Q}^{n+1}$. Let ω_f be the weight of the n-cell of $\mathcal{W}(f)$ that contains F, if such an n-cell exists, or 0 if no such n-cell exists. Define ω_g the same way. Then the weight of F is equal to $\omega_f + \omega_g$.*

Proof. If F is a boundary n-cell, then this follows directly from Definition 11. So we may assume that F is an interior n-cell.

We will now prove that F is either contained in an n-cell of $\mathcal{W}(f)$ or the interior of F is disjoint from $\mathcal{W}(f)$. Note that every point of a weighted balanced complex is either in the interior of an n-cell or in the boundary of an n-cell. Small enough neighborhoods of an interior point of an n-cell are contained in a hyperplane, while neighborhoods of points on n-cell boundaries are not. Suppose that there is a point $\alpha \in F^\circ \cap \mathcal{W}(f)$. Since α is in the interior of F, it must be in the interior of some n-cell G of $\mathcal{W}(f)$. Let β be any other point of the interior of F. Since F is convex, every point on the line segment from α to β is in the interior of F, and therefore not in the boundary of G. So in fact, β must be in the interior of G, and we have $F \subset G$.

If F is contained in an n-cell of $\mathcal{W}(f)$, then there is one monomial of f for each side of F that attains the minimum of f in that region. If the interior of F is disjoint from $\mathcal{W}(f)$, then the same monomial attains the minimum on each side of F. The same is of course true for $\mathcal{W}(g)$.

Let a and b correspond to the exponents of the monomials of f that attain the minimum on either side of A, and let c and d correspond to the exponents of the monomials of g that attain the minimum on either side of A (It is possible that $c = d$). Now $b - a = \omega_f e$ for some primitive integer vector e. It is clear that $d - c = \omega_g e$ for the same vector e. So by Lemma 8, the weight of F is the lattice length from $a + c$

to $b + d$ which is simply the lattice length from a to b plus the lattice length from c to d, which is $\omega_f + \omega_g$, as desired. □

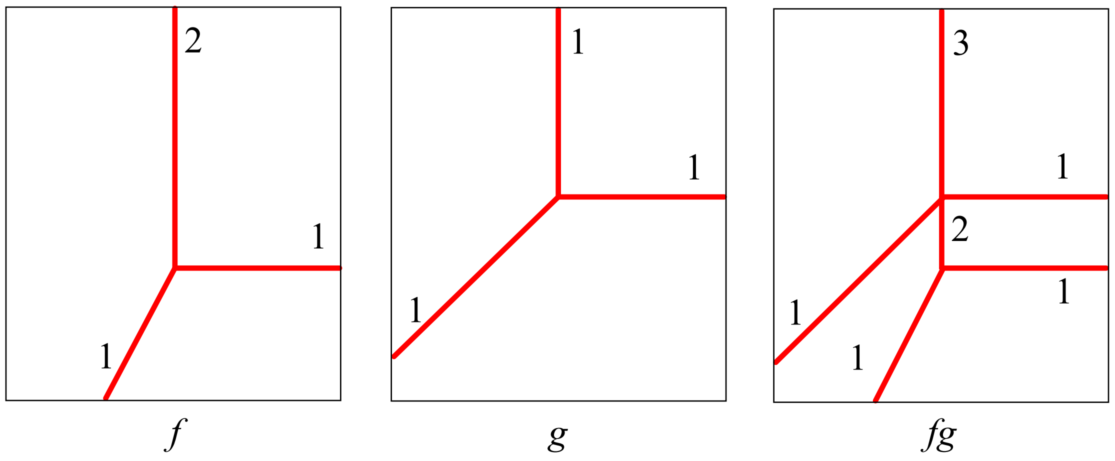


Figure 3.3: Weighted balanced graphs of f, g, and fg. Note that when a segment overlaps in the product, the weights add (see Lemma 10).

Definition 12. If G is a weighted complex, then a subset H of G is a *subcomplex* of G if H is a complex, and each n-cell F of H has weight less than or equal to the weight of the n-cell of G that contains F.

Lemma 11. *If f is a factor of g then $\mathcal{W}(f)$ is a balanced subcomplex of $\mathcal{W}(g)$.*

Proof. By definition, $\mathcal{W}(f)$ is a weighted balanced complex. Lemma 10 guarantees that $\mathcal{W}(f)$ is a subcomplex of $\mathcal{W}(g)$. □

Theorem 12. *If G is a subcomplex of $\mathcal{W}(f) \subset \mathcal{Q}^{n+1}$ that satisfies the balancing condition, then the polynomial corresponding to G is a factor of f.*

Proof. Consider $\mathcal{W}(f)$ as a collection of n-cells $\{F_1, \ldots, F_k\}$, with each F_i assigned a weight ω_i. We can think of G as the same collection of n-cells, with each F_i assigned a weight σ_i, with $0 \leq \sigma_i \leq \omega_i$. Note that we assign an n-cell F_i a weight of $\sigma_i = 0$ in

G if F_i is not in G, otherwise σ_i corresponds to the weight of F_i in G. Also note that while G may not be maximal in the sense that we required in Definition 6, this is of little importance. Let H be the weighted complex given by $\{F_1, \ldots, F_k\}$ with each F_i assigned a weight $\omega_i - \sigma_i$.

We first prove that H satisfies the balancing condition. Let r be an interior $(n-1)$-cell of H and let $\{F_{n_1}, \ldots, F_{n_m}\}$ be the set of n-cells adjacent to r. Let $\{c_{n_1}, \ldots c_{n_m}\}$ be the integer covectors of the F_{n_i}, no two with respect to the same region of the complement of H. Then we have

$$\sum_{i=1}^{m}(\omega_{n_1} - \sigma_{n_i})c_{n_i} = \underbrace{\sum_{i=1}^{m}\omega_{n_i}c_{n_i} - \sum_{i=1}^{m}\sigma_{n_i}c_{n_i}}_{\substack{\text{both zero since } \mathcal{W}(f) \text{ and } G \\ \text{are respectively balanced}}} = 0 - 0 = 0.$$

Thus, H is balanced at all of its interior $(n-1)$-cells. It is clear that H is also balanced at the boundary.

Let f_1 be the polynomial corresponding to G and f_2 the polynomial corresponding to H. By Lemma 10, $\mathcal{W}(f_1 f_2) = \mathcal{W}(f)$. So in fact $cf_1 f_2 = f$ for some constant c. Therefore, the polynomial f_1 corresponding to G is a factor of f. □

3.3 A Factoring Algorithm

3.3.1 NP-completeness

Theorem 13. *The problem of factoring tropical polynomials in two or more variables is NP-complete.*[3]

[3] NP stands for nondeterministic polynomial time, and we say that a problem is NP if it is possible to *verify* a solution in polynomial time (though it may not be possible to *find* a solution to such a problem in polynomial time). To say that factoring is *NP-complete* means that if we could find factors in polynomial time (as opposed to just being able to verify them), then we could solve every NP problem in polynomial time. In other words, this theorem asserts that it is extremely unlikely that a polynomial-time factoring algorithm exists. For more information, see http://mathworld.wolfram.com/NP-CompleteProblem.html

Proof. It is clear that the problem of factoring tropical polynomials is NP, since the weighted balanced complex of a polynomial can clearly be found in polynomial time, and verifying that one polynomial is a factor of another is a matter of checking that one weighted balanced complex is a subcomplex of the other.

In [4], Gao and Lauder show that the problem of decomposing Newton polyhedra is NP-complete. We prove that any algorithm that would factor tropical polynomials would decompose Newton polyhedra as well.

Let P be a two-dimensional Newton polyhedron, let v_0 be a vertex of P and for $1 \leq i \leq k$, let e_i be a primitive integer vector and ω_i a weight such that for $1 \leq j \leq k$, $v_k = v_0 + \sum_{i=1}^{j} \omega_i e_i$. Since P is a polygon, $\sum_{i=1}^{k} \omega_i e_i = 0$. So there is a weighted balanced 1-complex with k edges (1-cells), each of which has weight ω_i and is perpendicular to e_i. If we factor the polynomial corresponding to this complex then we find a subcomplex, which corresponds to a summand of P by Lemma 13 of [4] (see Figure 3.4). Since decomposing Newton polyhedra is NP-complete, so is factoring tropical polynomials. $\square$

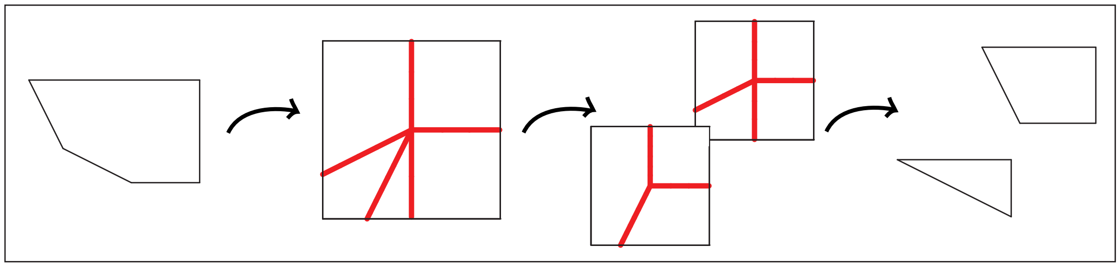


Figure 3.4: If there were a polynomial-time algorithm to factor tropical polynomials, then the same algorithm would decompose Newton polyhedra. Since decomposition of Newton polyhedra is NP-complete, so is tropical factoring.

3.3.2 A factoring algorithm

Let $f \in \mathcal{Q}[x_1, \ldots x_{n+1}]$ be a tropical polynomial. Let $G = \mathcal{W}(f) \subset \mathcal{Q}^{n+1}$.

1. Choose an n-cell F of G. Let $S = \{F\}$.

2. For each element F_i of S, analyze the $(n-1)$-cells in the boundary of F_i to determine if and how each $(n-1)$-cell can be decomposed. If necessary, choose one of the decompositions. Add all the n-cells of the decomposition to S.[4]

3. Repeat Step 2 until no new n-cells are added to S.

When the algorithm is finished, S will be a balanced subcomplex of G. So by Theorem 12, the polynomial corresponding to S is a factor of f. It is possible, however, that $S = G$, in which case we have found a trivial factor. Even if $S \neq G$, there may be multiple (distinct) factorizations of f. It is necessary to repeat the whole process, using all possible combinations of choices in Step 2 to determine all possible factorizations of f or to determine that f is irreducible.

Example To see how the algorithm works, we will look at an example (see Figure 3.5). Let $f = x^3 \oplus x^2y \oplus 1xy^2 \oplus 1y^3 \oplus x^2 \oplus xy \oplus y^2 \oplus x \oplus y \oplus 2$. The weighted balanced graph G of f consists of 8 segments (1-cells) of weight 1 and one segment of weight 2. There are two 3-valent vertices (0-cells) and one 5-valent vertex (shown in part 1 of the figure).

We first choose one segment of G, as indicated in part 2 of the figure. By choosing the segment, we have also chosen two vertices. We analyze each of these vertices to decompose them. The upper right vertex cannot be decomposed, so we must add the other two adjacent segments to our set. The lower left vertex can be decomposed, in two different ways. We choose one of these decompositions and add the corresponding segments to our set (shown in part 3). It is important to note that if we had chosen the other decomposition, we would have gotten a different factorization, as

[4]Decomposing an $(n-1)$-cell is equivalent to decomposing a two-dimensional Newton polyhedron. For an algorithm that can do this, see [4].

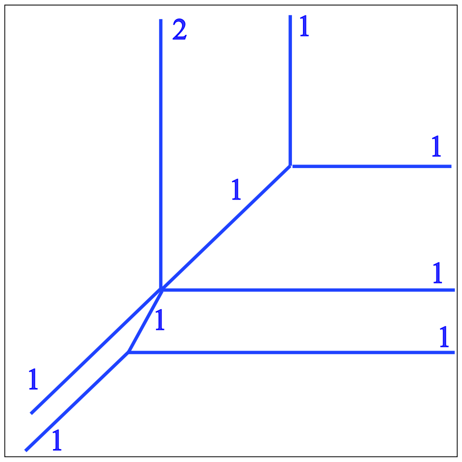
this polynomial can be factored in two different ways. Finally, this step has given us one more vertex. It cannot be decomposed, so we add two more segments to our set (part 4).

Since all of the vertices are now balanced, we have found a balanced subgraph, which by Theorem 12 corresponds to a factor of f. This (red) balanced subgraph corresponds to the polynomial $x^2 \oplus 1y^2 \oplus x \oplus y \oplus 2$, while the remaining (blue) part corresponds to the polynomial $x \oplus y \oplus 0$. The reader can check that these two factors indeed multiply to give f.

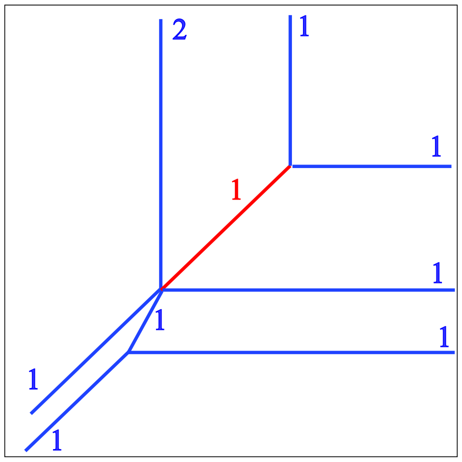
3.3.3 Analysis

The efficiency of this algorithm depends on the number of decompositions of a the $(n-1)$-cells of $\mathcal{W}(f)$. For polynomials of a fixed degree, the number of decompositions of the vertices of the corner loci of these polynomials is inversely proportional to the number of vertices. So in general, for polynomials of a fixed degree, polynomials whose corner loci have more vertices can be factored more efficiently.

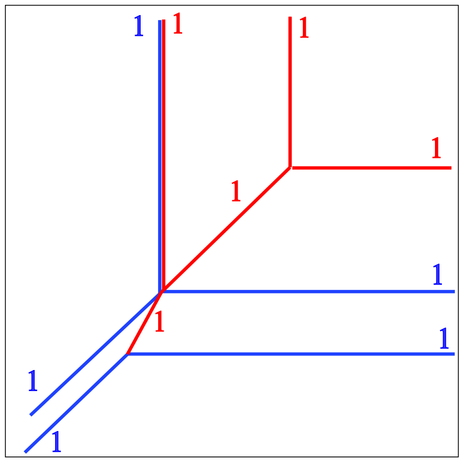
One interesting attribute of this factoring algorithm is that regardless of the dimension of the polyhedral complex, the problem of decomposing $(n-1)$-cells is the same. Decomposing $(n-1)$ cells is equivalent to the problem of decomposing a two-dimensional Newton polyhedron. In [4] it was shown that decomposing higher dimensional Newton polyhedron can help factor classical polynomials. Since decomposing an n-dimensional Newton polyhedron is equivalent to factoring a tropical polynomial that has a single 1-cell, we can decompose an n-dimensional Newton polyhedron simply by decomposing its two-dimensional faces.



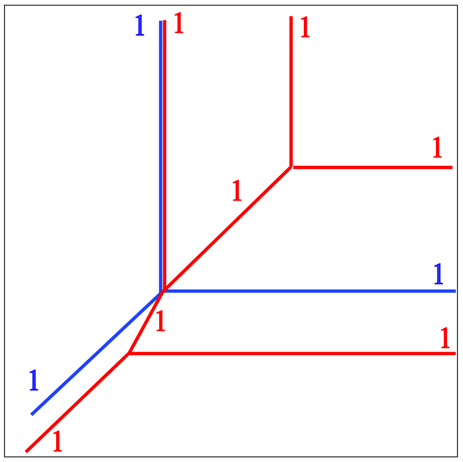
1. Start with the weighted balanced graph of f.



2. Choose one segment of the weighted balanced graph.



3. Decompose each of the chosen vertices and add the resulting segments to S.



4. Continue until there are no more unbalanced vertices.

Figure 3.5: The process of factoring a polynomial by decomposing its weighted balanced graph. The numbers correspond to the weights of the segments. It is especially important to note that in 3. we have arbitrarily chosen one of the two non-trivial decompositions of the 5-valent vertex.

Chapter 4

Conclusion

I have shown that tropical polynomials with rational coefficients can be factored into a product of linear polynomials. My elementary proof explains some of the nuances of functional equivalence and introduces least-coefficient polynomials as representatives of functional equivalences classes. I have given an algorithm to convert a polynomial into least-coefficient form and show that least-coefficient polynomials can be factored by inspection. Overall, my research provides a basis by which others can better understand one-variable polynomials.

I have also shown that the problem of factoring tropical polynomials of two or more variables is NP-complete. I have discussed weighted balanced complexes and their usefulness in factoring tropical polynomials. I have also given an algorithm that can factor tropical polynomials. My research on multi-variable polynomials provides a better view at the geometry behind tropical polynomials.

I have also written some Maple programming code that can graph and manipulate tropical polynomials. This code is included in Appendix 4, and has been very useful to the BYU Tropical Algebra research group and will also be useful to others who study tropical polynomials.

Appendix: The `Tropical` Maple Package

During my research for this thesis and other related problems in tropical geometry, I produced the `Tropical` Maple Package. It is compatible with Maple versions 10 and higher. It is capable of graphing and otherwise manipulating tropical polynomials. It can also compute tropical resultants and determinants of tropical matrices.

Perhaps the most useful of these procedures are `plotgraph` and graph, which together with the internal procedure `dual` determine the corner locus of a polynomial in two variables. Note that there is not an implementation of our factoring algorithm, since this implementation was not completed at the time of printing.

```
Tropical:= module()
option package;
local trop,endpoints,Add,Multiply,Det,Isect,t,symcheck,
   maxmin,cv,bestrange,LATEX,dual;
export Latex,tropical,coefs,graph,plotgraph,dualgraph,multiply,
   add,deriv,minfunc,plotgraphs,Poly,toLCP,isLCP,det,
   SylvesterMatrix,resultant,makeparametric,makepoints,setCoordinates,
   setMaxMin,maxfunc,colorlist,isect,toCCP;
maxmin:=-1; #change this to 1 for max or -1 for min
cv:=(a)->[a[1],a[2]];
colorlist:=[red,green,blue,magenta,cyan,black,tan,yellow,navy,
   coral,plum,grey,aquamarine];

trop:=proc(poly)
 description "Convert a tropical polynomial into a list of classic
```

```
    polynomials. This allows the user to enter tropical polynomials in
    standard polynomial form.";
 if type(poly,`+`) then  op(map(trop,[op(poly)]))
  elif type(poly,`*`) then  `+`(op(map(trop,[op(poly)])))
  elif type(poly,`^`) then  `*`(op(map(trop,[op(poly)])))
  elif type(poly,list(list)) then
     seq(a[1]+a[2]*x+a[3]*y+`if`(nops(a)>3,a[4]*z,0), a in poly)
  elif type(poly,Array)  then
     seq(poly[a,1]+poly[a,2]*x+poly[a,3]*y+`if`(op([2,2,2],poly)>3,
       poly[a,4]*z,0),a=op([2,1],poly))
  elif type(poly,list(algebraic)) then op(poly)
  else poly
 fi;
end proc:

tropical:=poly->[eval(trop(poly),[`0`=0,`1`=1,zero=0,one=1,
  `-1`=-1,minusone=-1])];
#The tropical function is what the user uses. It calls trop and then
   makes certain substitutions to allow the user to enter polynomials
   such as `1`*x+`0`.

minfunc:=proc(poly)
 local s1,s2;
 description "Returns a functional representation of a polynomial";
 if nops([args])>1 then
  if type(op(2,[args]),symbol) then
   s1:=convert(op(2,[args]),string)
  else
   s1:=convert(op(2,[args]),string)[2..-2]
  fi;
  s2:=convert(min(op(tropical(poly))),string);
  parse("("||s1|| ")->" ||s2);
 else
  min(op(tropical(poly)))
 fi;
 end proc;

maxfunc:=proc(poly)
 local s1,s2;
 description "Returns a functional representation of a polynomial";
 if nops([args])>1 then
  if type(op(2,[args]),symbol) then
```

```
   s1:=convert(op(2,[args]),string)
  else
   s1:=convert(op(2,[args]),string)[2..-2]
  fi;
  s2:=convert(max(op(tropical(poly))),string);
  parse("("||s1|| ")->" ||s2);
 else
  max(op(tropical(poly)))
 fi;
 end proc;

coefs:=proc(poly)
 local f,g,clist,a,b,templist;
 description "Returns an array of the coefficients and powers of
   polynomial terms for a polynomial in x,y,and z. Used internally,
   but also useful to the user.";
 clist:=[];
 f:=tropical(poly);
 for g in f do
  b:=[`if`(type(g,`+`),op(g),g)];
  templist:=[0,0,0,0];
  for a in b do
   if member('x',indets(a)) then
    templist[2]:=a/x
   elif member('y',indets(a)) then
    templist[3]:=a/y
   elif member('z',indets(a)) then
    templist[4]:=a/z
   else templist[1]:=a
   fi
  od;
  clist:=[op(clist),templist];
od;
if nops([args])>1 then
 clist
else
 Array(1..nops(clist),1..4,clist);
fi;
end proc:

graph:=proc(poly)
 local duallines,lines,l1,l2,dir,p1,p2,p3,proj,t,opts,temp,s;
```

```
description "Returns a symbolic representation of the corner
   locus of a polynomial";
if convert([seq(a in {Ray,Segment,Line,Point},a in
   {seq(op(0,b),b in poly)})],‘and‘) then
 lines:=poly; #if poly is a symbolic graph, then we are already done.
else
 if type(poly,‘+‘) and #some basic error checking
   not type(poly,polynom(extended_numeric,[x,y,zero,one,‘1‘,‘0‘]))
   and nops({seq(degree(i,[x,y,z]),i in poly)})>1 then
     WARNING("Recieved a non-homogenous polynomial in three or more
     variables. I will try to scale out z to form a polynomial in
     two variables.");
    fi;
 duallines:=dual(poly);
   #the dual procedure does the actual work. The rest of this is
   #just converting the result into a symbolic tropical graph.
 lines:=[];
 while nops(duallines)>0 do
  l1:=duallines[1];
  duallines:=duallines[2..-1];
  l2:=NULL;
  for s from 1 to nops(duallines) while l2=NULL do
   if evalb(l1[1,1] in duallines[s,1] and l1[1,2] in duallines[s,1])
     then
    l2:=duallines[s];
    duallines:=[op(duallines[1..s-1]),op(duallines[s+1..-1])];
   fi;
  od;
  if l2=NULL then #we use projections to find a direction vector.
   p1:=op([1,1],l1);
   p2:=op([1,2],l1);
   p3:=l1[3];
   proj:=((p3[1]-p1[1])*(p2[1]-p1[1])+(p3[2]-p1[2])*(p2[2]-p1[2]))/
     ((p2[1]-p1[1])^2+(p2[2]-p1[2])^2);
   dir:=[proj*(p2[1]-p1[1])-(p3[1]-p1[1]),proj*
      (p2[2]-p1[2])-(p3[2]-p1[2])];
   if dir[1]=0 and dir[2]=0 then
    lines:=[op(lines),Line(cv(l1[2]),cv([p1[2]-p2[2],p2[1]-p1[1]]),
      igcd(l1[1,1,1]-l1[1,2,1],l1[1,1,2]-l1[1,2,2]))];
   else
    lines:=[op(lines),Ray(cv(l1[2]),cv(dir),
      igcd(l1[1,1,1]-l1[1,2,1],l1[1,1,2]-l1[1,2,2]))];
   fi
```

```
    else #that is, if l2 is not NULL
     lines:=[op(lines),Segment(cv(l1[2]),cv(l2[2]-l1[2]),
       igcd(l1[1,1,1]-l1[1,2,1],l1[1,1,2]-l1[1,2,2]))];
    fi;
   od;
  fi;
  opts:=[args[2..-1]];
  temp:='default';
  hasoption(opts,type,'temp','opts');
  #return section
  if temp='parametric' then
   makeparametric(lines)
  else
   lines;
  fi;
end proc:

plotgraph:=proc(poly)
  local ff,gra,temp,opts;
  description "Draws the tropical zero locus of a tropical polynomial";
  #some basic error checking
  if type(poly,list) and not (type(poly,list(list)) or
     type(poly,list(anything))) then
   WARNING("First argument of plotgraph must be a single polynomial.
     Use 'plotgraphs' for multiple polynomials.");
  elif not type(poly,polynom) and not type(poly,list(list)) and
     not type(poly,list(anything)) then
   WARNING("Unexpected first argument.");
  fi;
  opts:=[args[2..-1]];
  gra:=graph(poly);
  #view
   temp:=bestrange(gra);
   hasoption(opts,view,'temp','opts');
   opts:=[op(opts),view=temp];
  #thickness
   temp:=2;
   hasoption(opts,thickness,'temp','opts');
   if temp='weights' then
    temp:=[seq(op(3,a),a in gra)];
   fi;
   opts:=[op(opts),thickness=temp];
  #color
```

```
 temp:=red;
 hasoption(opts,color,'temp','opts');
 if evalb(temp<>'default') then
  if evalb(temp='weights') then
   temp:=[seq(colorlist[((op(3,a)-1) mod nops(colorlist))+1],
      a in gra)];
  fi;
  opts:=[op(opts),color=temp];
 fi;
 if nops(gra)>0 then
  temp:=makeparametric(gra);
  if nops(temp)>0 then
   plots[display](plot(makeparametric(gra),op(opts)),makepoints(gra),
     symbol=cross,symbolsize=20);
  else
   plots[display](makepoints(gra),symbol=cross,symbolsize=20,
     op(opts));
  fi;
 else
  WARNING("empty corner locus. Returned NULL");
  return NULL;
 fi;
end proc:

plotgraphs:=proc(polylist)
 local s,opts,graphs,f,thicklist,sglist,cl;
 description "Plots multiple graphs together, each a different color";
 cl:=colorlist;
 opts:=[args[2..-1]];
 #some error checking
 if nops(opts)>0 and type(opts[1],polynom) then
  error "Invalid 2nd argument. Put all polynomials in a list
  first argument: plotgraphs([f,g]);"
 fi;
 thicklist:=[2];
 hasoption(opts,color,'cl','opts');
 hasoption(opts,thickness,'thicklist','opts');
 if cl='weights' then error "Invalid option: color=weights
   can be used with plotgraph command only."; fi;
 if not type(cl,list) then cl:=[cl] fi;
 if thicklist='weights' then error "Invalid option: thickness=
   weights can be used with plotgraph command only."; fi;
 if not type(thicklist,list) then thicklist:=[thicklist] fi;
 s:=0;
 sglist:=map(graph,polylist); #make symbolic graphs
 if not(hasoption(opts,view)) then opts:=
   [op(opts),view=bestrange([seq(op(o),o in sglist)])] fi;
 graphs:=[];
 for f in sglist do
  graphs:=[op(graphs),plotgraph(f,op(opts),color=
    cl[(s mod nops(cl))+1],thickness=thicklist[(s mod
    nops(thicklist))+1])];
  s:=s+1;
 od;
 plots[display](graphs);
end proc;

dual:=proc(poly)
 local f,g,u,v,w,vertices,f0,fmin,t,points,h,hh,lines,xx,yy,det,
   vertices2,ncc;
 description "Analyzes the dual of a polynomial in order to
   draw the dual graph or the corner locus of a polynomial.";
 f:=eval(tropical(poly),z=0);
 g:=coefs(poly);
 vertices:={}:
 vertices2:={};
 ncc:={};
 for u from 1 to nops(f) do
 if u in ncc then next fi;
 for v from u+1 to nops(f) do
 if v in ncc then next fi;
 for w from v+1 to nops(f) do
   if w in ncc then next fi;
   det:=(g[u,2]-g[v,2])*(g[u,3]-g[w,3])-(g[u,2]-g[w,2])*
     (g[u,3]-g[v,3]);
   if det<>0 then
    xx:=(g[u,3]-g[w,3])/det*(g[v,1]-g[u,1])+(g[v,3]-g[u,3])/det*
       (g[w,1]-g[u,1]);
    yy:=(g[w,2]-g[u,2])/det*(g[v,1]-g[u,1])+(g[u,2]-g[v,2])/det*
       (g[w,1]-g[u,1]);
    if not [xx,yy] in vertices2 then
    f0:=eval(f,[x=xx,y=yy]);
    fmin:=maxmin*max(op(maxmin*f0));
    if fmin=f0[u] then
```

```
     points:={};
     for t from 1 to nops(f0) do
      if f0[t]=fmin then points:=points union {t} fi
     od;
     hh:=simplex[convexhull]([seq([maxmin*g[a,2],maxmin*g[a,3]],
        a in points)]);
     vertices:=vertices union {[hh,[xx,yy]]};
     vertices2:=vertices2 union {[xx,yy]};
     for h in points do
      if not [maxmin*g[h,2],maxmin*g[h,3]] in hh then
       ncc:=ncc union {h}
       fi
      od;
     fi
    fi
   fi
 od
 od
 od;
 lines:=[];
 for v in vertices do
  hh:=v[1];
  lines:=[op(lines),seq([{hh[t],hh[t+1]},v[2],hh[`if`(t+2>nops(hh),
    t-1,t+2)]],t=1..nops(hh)-1),[{hh[1],hh[-1]},v[2],hh[2]]];
 od;
 if nops(lines)=0 then
  #this is in case there are no polygons in the dual (only edges)
  vertices:={};
  for u from 1 to nops(f) do
  for v from u+1 to nops(f) do
   if g[u,2]=g[v,2] then
    xx:=0;
    yy:=(g[v,1]-g[u,1])/(g[u,3]-g[v,3]);
   else
    xx:=(g[v,1]-g[u,1])/(g[u,2]-g[v,2]);
    yy:=0;
   fi;
   f0:=eval(f,[x=xx,y=yy]);
   fmin:=maxmin*max(op(maxmin*f0));
   points:={};
   for t from 1 to nops(f0) do
    if f0[t]=fmin then
```

```
     points:=points union {maxmin*[g[t,2],g[t,3]]}
    fi;
   od;
   if nops(points)>1 then
    vertices:=vertices union {[points,[xx,yy]]}
   fi;
  od
 od;
 lines:=[];
 for v in vertices do
  hh:=[op(endpoints(v[1]))];
  lines:=[op(lines),[{hh[1],hh[2]},v[2],hh[2]]];
  od;
fi;
lines;
end proc:

dualgraph:=proc(poly)
local d,g,pairedlines,lines,side,dotlist,pair;
description "Graphs a tropical polynomial dual graph";
g:=coefs(poly);
pairedlines:=dual(poly);
if nops(pairedlines)=0 then
 error "Empty corner locus"
else
 lines:={};
 for pair in pairedlines do
   lines:=lines union {pair[1]};
 od;
 if member('dots',[args]) then
   d:=degree(Poly(poly));
   dotlist:=seq(seq(plottools[point]([maxmin*xx,maxmin*yy]),
     xx=0..(d-yy)),yy=0..d);
 else
   dotlist:=NULL
 fi;
 plots[display]([seq(plottools[line](op(1,l),op(2,l)),l in lines),
   dotlist],tickmarks=[[-1="1"],[-1="1"]],axes=none,
   scaling=constrained);
fi;
end proc:
```

```
Multiply:=proc(poly)
 local ff,p,f1,f2;
 description "Multiplies two or more polynomials
   [internal for faster computing]";
 if nargs=0 then return 0 fi;
 ff:=Add(poly);
 for p in args[2..-1] do
  ff:=Add([seq(seq(a+b,a in coefs(p,list)),b in coefs(ff,list))]);
 od;
 ff;
end proc;

multiply:=proc()
 description "Multiplies two polynomials [for user]"
 Poly(Multiply(args));
end proc;

Add:=proc(poly)
 local t,ff,s,w;
 description "Adds a number of polynomials";
 w:=false;
 ff:=[seq(op(coefs(p,list)),p in [args])];
 for t from 1 to nops(ff) while t<=nops(ff) do
 if type(ff[t,1],infinity) then
 #if ff[t,1]<>infinity then
  #w:=true fi;
  ff:=[op(ff[1..t-1]),op(ff[t+1..nops(ff)])];
  t:=t-1;
 else
  for s from t+1 to nops(ff) while s<=nops(ff) do
   if ff[t,2]=ff[s,2] and ff[t,3]=ff[s,3] and ff[t,4]=ff[s,4] then
    ff:=[op(ff[1..t-1]),[maxmin*max(maxmin*ff[t,1],maxmin*ff[s,1]),
     ff[t,2],ff[t,3],ff[t,4]],op(ff[t+1..s-1]),op(ff[s+1..nops(ff)])];
    s:=s-1;
   fi;
  od;
  fi;
 od;
 #if w then WARNING("w") fi;
 if nops(ff)=0 then
  infinity
 else
  ff
 fi;
end proc;

add:=proc()
 Poly(Add(args));
end proc;

deriv:=proc(poly,vars)
 description "Takes the derivative of a tropical polynomial";
 local d,a,var,dlist,onevar;
 if type(vars,symbol) then onevar:=true else onevar:=false fi;
 dlist:=[];
 for var in vars do
  d:=[];
  for a in tropical(poly) do
   if member(cat(var),indets(a)) then d:=[op(d),a-var] fi;
  od;
  dlist:=[op(dlist),d]
 od;
 if onevar then Poly(op(dlist)) else map(Poly,dlist) fi;
end proc;

endpoints:=proc(points)
local mx,Mx,my,My,p,ep;
description "Finds the endpoints of a line segment [internal]";
 mx:=min(seq(p[1],p in points));
 Mx:=max(seq(p[1],p in points));
 my:=min(seq(p[2],p in points));
 My:=max(seq(p[2],p in points));
ep:={};
for p in points do
 if (p[1]>=Mx or p[1]<=mx) and (p[2]>=My or p[2]<=my) then
  ep:=ep union {p}
 fi
od;
ep;
end proc;

Poly:=proc(poly)
 local g,t,c;
 description "Makes a polynonmial out of a tropical polynomial";
 if type(poly,polynom) then
```

```
  poly
 else
  g:=coefs(poly);
  c:=op([2,1,2],g);
  if c=0 then return infinity fi;
  for t from 1 to c do
   if g[t,1]=1 and (g[t,2]<>0 or g[t,3]<>0 or g[t,4]<>0) then
    g[t,1]:=`1`
   elif g[t,1]=0 then
    if g[t,2]=0 and g[t,3]=0 and g[t,4]=0 then
     g[t,1]:=`0`
    else
     g[t,1]:=1
    fi;
   fi;
  od;
  `+`(seq(g[t,1]*x^g[t,2]*y^g[t,3]*z^g[t,4],t=1..c));
 fi;
end proc;

isLCP:=proc(poly1)
 local g,corners,u,v,tf,notinf,vertices,t,poly;
 description "Checks if a polynomial is least coefficients.";
 tf:=true;
 poly:=eval(poly1,z=`0`);
 g:=coefs(poly);
 notinf:=[seq([g[a,2],g[a,3]],a=op([2,1],g))];
 corners:=simplex[convexhull](notinf);
 for u from min(seq(g[a,2],a=op([2,1],g))) to
   max(seq(g[a,2],a=op([2,1],g))) while tf do
  for v from min(seq(g[a,3],a=op([2,1],g))) to
    max(seq(g[a,3],a=op([2,1],g))) while tf do
   if not evalb([u,v] in simplex[convexhull]([op(corners),[u,v]]))
     and not evalb([u,v] in notinf) then
    tf:=false
   fi
  od #v
 od; #u
 if tf then
  vertices:={seq(a[2],a in dual(poly))};
  for t from 1 to op([2,1,2],g) while tf do
   if g[t,1]>min(g[t,1],max(seq(eval(minfunc(poly),[x=v[1],y=v[2]])-
     g[t,2]*v[1]-g[t,3]*v[2],v in vertices))) then
    tf:=false;
   fi
  od;
 fi;
 tf;
end proc;

toLCP:=proc(poly1)
local g,corners,u,v,notinf,vertices,monomial,t,inf,s,poly;
 description "Determines a least coefficient polynomial
  for a polynomial in two variables";
 poly:=eval(poly1,z=`0`);
 g:=coefs(poly);
 notinf:=[seq([g[a,2],g[a,3]],a=op([2,1],g))];
 inf:=[];
 corners:=simplex[convexhull](notinf);
 for u from min(seq(g[a,2],a=op([2,1],g))) to max(seq(g[a,2],
   a=op([2,1],g))) do
  for v from min(seq(g[a,3],a=op([2,1],g))) to max(seq(g[a,3],
   a=op([2,1],g))) do
   if not evalb([u,v] in simplex[convexhull]([op(corners),[u,v]])) and
     not evalb([u,v] in notinf) then
    inf:=[op(inf),[u,v,0]];
   fi
  od #v
 od; #u
 s:=op([2,1,2],g);
 g:=Array(1..s+nops(inf),1..4,g);
 for t from 1 to nops(inf) do
  g[t+s,1]:=infinity;
  g[t+s,2]:=inf[t,1];
  g[t+s,3]:=inf[t,2];
  g[t+s,4]:=inf[t,3];
 od;
 vertices:={seq(a[2],a in dual(poly))};
 for t from 1 to op([2,1,2],g) do
  g[t,1]:=min(g[t,1],max(seq(eval(minfunc(poly),[x=v[1],y=v[2]])-
    g[t,2]*v[1]-g[t,3]*v[2],v in vertices)));
 od;
 Poly(g);
end proc;
```

```
toCCP:=proc(poly)
 description "Converts a polynomial to Contributing Coefficient Form";
 local f,d,l,t;
 d:=dual(poly);
 l:={seq(op([maxmin*op([1,1],g),maxmin*op([1,2],g)]),g in d)};
 f:=coefs(poly);
 Poly([seq(`if`([f[t,2],f[t,3]] in l,f[t,1]+f[t,2]*x+f[t,3]*y+
  f[t,4]*z,NULL),t=op([2,1],f))]);
end proc;

Det:=proc(mat)
 #errorchecking
 if type(mat,Matrix) then
  if op([1,2],mat)<>op([1,1],mat) then
   error "expected nxn matrix but received %1x%2 matrix",
     op([1,1],mat),op([1,2],mat)
  fi;
  if op([1,1],mat)=1 then
   mat[1,1]
  elif op([1,1],mat)=2 then
   Add(Multiply(mat[1,1],mat[2,2]),Multiply(mat[1,2],mat[2,1]));
  else #n>2
   Add(seq(Multiply(mat[1,i],Det(mat[2..op([1,1],mat),
     [1..(i-1),(i+1)..op([1,1],mat)]])),i=1..op([1,1],mat)));
  fi;
 elif type(mat,list) and not type(mat,list(list)) then
  if not type(sqrt(nops(mat)),integer) then
   error "list length must be the square of an integer.
    (Received a list of length %1)",nops(mat);
  fi;
  Det(Matrix(sqrt(nops(mat)),sqrt(nops(mat)),[seq([seq(
    mat[sqrt(nops(mat))*(i-1)+j],j=1..sqrt(nops(mat)))],
    i=1..sqrt(nops(mat)))]));
 else
  if not type(mat,list(list)) and not type(mat,array) then
   WARNING("unrecognized input. Input should be of type
     Matrix or list");
  fi;
  Det(Matrix(mat));
 fi;
end proc;
```

```
det:=proc()
 Poly(Det(args));
end proc;

SylvesterMatrix:=proc(p1,p2,v)
 local M,t,s;
 M:=LinearAlgebra[SylvesterMatrix](p1,p2,v);
 for t from 1 to op([1,1],M) do
  for s from 1 to op([1,2],M) do
   if M[t,s]=0 then M[t,s]:=infinity fi;
   if M[t,s]=1 then M[t,s]:=0 fi;
  od;
 od;
 M;
end proc;

resultant:=proc(p1,p2,v)
 det(SylvesterMatrix(p1,p2,v));
end proc;


makeparametric:=proc(o)
 description "Returns a parametric version of a symbolic graph";
 if nops([args])>1 then map(makeparametric,[args])
 elif type(o,list) then map(makeparametric,o);
 elif op(0,o)='Line' then
  [op([1,1],o)+op([2,1],o)*t,op([1,2],o)+op([2,2],o)*t,
    t=-infinity..infinity]
 elif op(0,o)='Ray' then
  [op([1,1],o)+op([2,1],o)*t,op([1,2],o)+op([2,2],o)*t,t=0..infinity]
 elif op(0,o)='Segment' then
  [op([1,1],o)+op([2,1],o)*t,op([1,2],o)+op([2,2],o)*t,t=0..1]
 fi;
end proc:

makepoints:=proc(o)
 description "Returns a list of points extracted from a
   symbolic graph";
 if nops([args])>1 then map(makepoints,[args])
 elif type(o,list) then map(makepoints,o);
 elif op(0,o)='Point' then
  plottools[point](op(o));
 fi;
```

```
end proc:

bestrange:=proc(symbolicgraph)
 description "Determines automatic range for a symbolic graph";
 local r1,s;
  r1:=[min(seq(op([1,1],g),g in symbolicgraph)),max(seq(op([1,1],g),
    g in symbolicgraph)),min(seq(op([1,2],g),g in symbolicgraph)),
    max(seq(op([1,2],g),g in symbolicgraph))];
 for s from 1 to 4 by 2 do
  if r1[s]=r1[s+1] then
   r1[s]:=r1[s]-1;
   r1[s+1]:=r1[s+1]+1;
  else
   r1[s]:=r1[s]-.5*(r1[s+1]-r1[s]);
   r1[s+1]:=r1[s+1]+.5*(r1[s+1]-r1[s]);
  fi
 od;
 s:=r1[2]-r1[1]-(r1[4]-r1[3]);
 if evalf(s)>0 then
  r1[4]:=r1[4]+s/2;
  r1[3]:=r1[3]-s/2;
 else
  r1[2]:=r1[2]-s/2;
  r1[1]:=r1[1]+s/2;
 fi;
 return [r1[1]..r1[2],r1[3]..r1[4]];
end proc;

Isect:=proc(object1,object2)
 description "Internal procedure to determine intersection of two
   Point-Segment-Ray-Line objects";
 local oo,o1,o2,p,t,r;
 o1:=symcheck(object1); o2:=symcheck(object2);
 if op(0,o1)=Point then
  oo:=o1;o1:=o2;o2:=oo;
 fi;
 if op(0,o2)=Point then #o2 is a point
  if op(0,o1)=Point then #if both are points
   if o1=o2 then return o1 else return NULL fi;
  fi;
  if op([2,2],o1)*(op([1,1],o1)-op([1,1],o2))<>op([2,1],o1)*
    (op([1,2],o1)-op([1,2],o2)) then
   return NULL;
  fi;
  if op([2,1],o1)=0 then
   t:=(op([1,2],o2)-op([1,2],o1))/op([2,2],o1)
  else
   t:=(op([1,1],o2)-op([1,1],o1))/op([2,1],o1)
  fi;
  if op(0,o1)=Line or
     (op(0,o1)=Ray and t>=0) or
     (op(0,o1)=Segment and t>=0 and t<=1) then
   return o2
  else
   return NULL;
  fi;
 fi;
 if op([2,1],o1)*op([2,2],o2)=op([2,2],o1)*op([2,1],o2) then
  #(they are parallel)
  if op([2,2],o1)*(op([1,1],o1)-op([1,1],o2))<>op([2,1],o1)*
    (op([1,2],o1)-op([1,2],o2)) then
   return NULL;
  fi;
   if op(0,o2)=Line then
   oo:=o1; o1:=o2; o2:=oo;
  fi;
  if op(0,o1)=Line then #(one of them is a line)
   return o2;
  fi;
  if op(0,o1)=Ray and op(0,o2)=Ray then
   if op([2,1],o2)*op([2,1],o1)+op([2,2],o2)*op([2,2],o1)>0 then
    if (op([1,1],o1)-op([1,1],o2))*op([2,1],o1)+(op([1,2],o1)-
      op([1,2],o2))*op([2,2],o1)>0 then
     return o1
    else
     return o2
    fi;
   else
    if (op([1,1],o2)-op([1,1],o1))*op([2,1],o1)+(op([1,2],o2)-
      op([1,2],o2))*op([2,2],o2)>=0 then
     return symcheck(Segment(op(1,o1),op(1,o2)-op(1,o1),
       min(op(3,o1),op(3,o2))));
    else
     return NULL;
    fi;
   fi;
  fi;    #(end case: both rays)
  if op(0,o1)=Ray then
   oo:=o1; o1:=o2; o2:=oo;
  fi;
  if op([2,1],o1)=0 then
   t:=[(op([1,2],o2)-op([1,2],o1))/op([2,2],o1),
     op([2,2],o2)/op([2,2],o1)]
  else
   t:=[(op([1,1],o2)-op([1,1],o1))/op([2,1],o1),
     op([2,1],o2)/op([2,1],o1)]
  fi;
  if op(0,o2)=Ray then
   r:=[0,infinity]
  else
   r:=[0,1]
  fi;
  r:=sort([t[1],t[1]]+t[2]*r);
  r:=[max(0,r[1]),min(1,r[2])];
  if r[2]>=r[1] then
   return symcheck(Segment(op(1,o1)+op(2,o1)*r[1],op(2,o1)*
     (r[2]-r[1]),min(op(3,o1),op(3,o2))));
  else
   return NULL;
  fi;
 fi; #end case: they are parallel
 #find the point where they would intersect
 p:=[(op([2,2],o2)*(op([1,1],o1)-op([1,1],o2))-op([2,1],o2)*
   (op([1,2],o1)-op([1,2],o2)))/
     (op([2,2],o1)*op([2,1],o2)-op([2,1],o1)*op([2,2],o2)),
     (op([2,2],o1)*(op([1,1],o2)-op([1,1],o1))-op([2,1],o1)*
       (op([1,2],o2)-op([1,2],o1)))/
     (op([2,2],o2)*op([2,1],o1)-op([2,1],o2)*op([2,2],o1))];
     #(Cramer's rule)
 for t from 1 to 2 do
  if op(0,[o1,o2][t])=Ray and p[t]<0 then
   return NULL;
  elif  op(0,[o1,o2][t])=Segment and ((p[t])<0 or (p[t])>1) then
   return NULL;
  fi;
 od;
 return Point(op(1,o1)+p[1]*op(2,o1));
end proc;
```

```
isect:=proc(o1)
 description "Returns the intersection of two symbolic graphs";
 if o1=[] then return [] fi;
 if type(o1,list({polynom,list})) then
  isect(op(o1));
 elif nargs=1 then
  graph(o1);
 elif nargs=2 then
  [op({seq(seq(Isect(a,b),a in graph(o1)),b in graph(args[2]))})];
 else
  isect(isect(o1,args[2]),args[3..-1]);
 fi;
end proc;

symcheck:=proc(object)
 description "Checks a symbolic graph for validity";
 if op(0,object)=Point and nops(object)<>1 then
  error "Invalid Point Object";
 fi;
 if op(0,object) in {Ray,Segment,Line} then
  if nops(object)<>3 then
   error "Invalid "||(op(0,object));
  elif op(2,object)=[0,0] then
   return Point(op(1,object))
  fi;
 fi;
 object;
end proc;

setCoordinates:=proc()
 description "Sets the Coordinate System";
 local c;
 if nargs>1 then
  error "expected one argument of type procedure or symbol,
    received %1 arguments",nargs
 elif nargs=1 then
  c:=op(1,[args]);
  if type(c,procedure) then
   cv:=c;
  elif type(c,symbol) then
   if c=default then
    cv:=a->[a[1],a[2]];
   elif c=equilateral then
    cv:=a->[a[1]-a[2]/2,sqrt(3)/2*a[2]];
   else
    error "expected 'default' or 'equilateral' or user-defined
     procedure,received '%1'.",c;
   fi;
  else
   error "expected 'default' or 'equilateral' or user-defined
    procedure,received type %1",whattype(c);
  fi;
 fi;
 print(cv)
end proc;

setMaxMin:=proc(x)
 if x='max' then maxmin:=1; 'max';
 elif x='min' then maxmin:=-1; 'min';
fi;
end proc;

Latex:=proc(poly)
 printf(LATEX(poly));
end proc;

LATEX:=proc(poly);
 if type(poly,`+`) then
  cat("",seq(op([LATEX(op(i,poly))," \\oplus "]),i=1..
    (nops(poly)-1)),LATEX(op(-1,poly)));
 elif type(poly,`*`) then
   cat("",seq(LATEX(op(i,poly)),i=1..nops(poly)));
 elif type(poly,`^`) then
   cat("",LATEX(op(1,poly)),seq(op(["^{",op(i,poly),"}"]),
     i=2..nops(poly)));
 elif type(poly,{numeric,symbol}) then
  if type(poly,infinity) then
   "\\infty "
  else
   convert(poly,string)
  fi;
 elif type(poly,Matrix) then
  cat(
    "\\left[\\begin{array}{",seq("c",i=1..op([1,2],poly)),"}\n",
    seq(op([seq(op([LATEX(poly[i,j]),"&"]),j=1..op([1,2],poly)-1),
      LATEX(poly[i,-1]),"\\\\\n"]),i=1..op([1,1],poly)-1),
      seq(op([LATEX(poly[-1,j]),"&"]),j=1..op([1,2],poly)-1),
      LATEX(poly[-1,-1]),
    "\n\\end{array}\\right]");
 else
  WARNING("I don't know how do LaTeX that...I might have messed up.
    (Type: %1)",whattype(poly));
  convert(poly,string);
 fi;
end proc;

end module:
```

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