Math 308B
Spring, 2004
Final Exam
9 June 2004

Name: ________________________________
Student Number: __________________________

- There are five problems; read the instructions to each problem **carefully**.
- Show all your work, and justify all your answers. If you're asked to give an example of a Thingy which has property Whatsit, demonstrate explicitly that your example does indeed have property Whatsit.
- You may not use the textbook or your lecture notes; you may use a card of notes.
- You may use a scientific calculator if you wish, but you won't need one.
- Read the exam carefully; if you have any questions about the wording, raise your hand and I'll explain.
- If you get stuck on one problem, go on and do the rest, and come back to it later.
- There are 100 points possible in all.
- Have fun.

For grader's use only

Problem	Points
1	
2	
3	
4	
5	
Total	

1. (a) Let

$$A = \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & 6 \\ 0 & 6 & 3 \end{bmatrix}.$$

(10pts.) Compute $\det(A)$. Find the eigenvalues of A, and for each eigenvalue λ find a basis for the eigenspace E_λ. What is the sum of the dimensions of all the eigenspaces?

(b) Let

$$B = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(10pts.) Compute $\det(B)$. Find the eigenvalues of B, and for each eigenvalue λ find a basis for the eigenspace E_λ. What is the sum of the dimensions of all the eigenspaces?

2. You have a sample of a certain long-lived radioactive substance in your laboratory. You have been detecting and recording decay events with your instruments; for several times t, you have recorded the total number $y(t)$ of decay events up to time t. Your data looks like this, in appropriate units:

t	1	2	3	4
$y(t)$	2	2	5	5

(a) **(15pts.)** Because the substance has such a long half-life, you expect $y(t)$ to be approximately linear over reasonable time-scales. Find a least-squares linear fit $y(t) = at + b$ to the given data.

(b) Now drop some of the data points:

t	1	4
$y(t)$	2	5

(5pts.) Find a least-squares linear fit $y(t) = ct + d$ to this new data.

3. Let

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 11 \\ -15 \\ 5 \end{bmatrix}, \vec{u} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}.$$

(a) **(5pts.)** Find scalars c_1, c_2, c_3 with $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{u}$.

(b) **(5pts.)** A certain 3×3 matrix C satisfies $C\vec{v_1} = \vec{0}, C\vec{v_2} = \vec{0}, C\vec{v_3} = \vec{v_2} + \vec{v_3}$. What is $C^4\vec{u}$?

(c) **(10pts.)** Find a 3×3 matrix D such that $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ is a basis for the nullspace of D.

4. Let

$$\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

(a) **(5pts.)** Find scalars a_1, a_2, b_1, b_2 such that $\overrightarrow{e_1} = a_1\overrightarrow{u_1} + a_2\overrightarrow{u_2}$ and $\overrightarrow{e_2} = b_1\overrightarrow{u_1} + b_2\overrightarrow{u_2}$.

(b) **(15pts.)** A certain 2×2 matrix F satisfies $F\overrightarrow{u_1} = 5\overrightarrow{u_1}, F\overrightarrow{u_2} = -10\overrightarrow{u_2}$. Find F and F^{-1}.

5. We say an $n \times n$ matrix N is **nilpotent** if for some positive integer k, $N^k = 0$. We say an $n \times n$ matrix P is **idempotent** if $P^2 = P$.

   (a) **(3pts.)** Show that any nilpotent matrix N is singular.

   (b) **(5pts.)** Suppose that λ is **any** eigenvalue of a nilpotent matrix N. Find all possible values of λ. For each possible λ, find a nilpotent matrix having that λ as an eigenvalue.

   (c) **(4pts.)** Show that, if N is any nilpotent matrix, $I - N$ is nonsingular.

   (d) **(6pts.)** Suppose that λ is **any** eigenvalue of an idempotent matrix P. Find all possible values of λ. For each possible λ, find an idempotent matrix having that λ as an eigenvalue.

   (e) **(2pts.)** Show that, if P is idempotent, $I - P$ is also idempotent.