1a. $y(t) = \dfrac{c}{t^2} + \dfrac{1}{t}$

1b. $y(x) = -\dfrac{1}{2}\ln\left(\dfrac{2}{3}e^{-3x} - c\right)$

2. The differential equation should be $Q'(t) = 2\gamma - 2Q(t)/3, Q(0) = 1$. The solution is $Q(t) = 3\gamma(1 - e^{-2t/3}) + e^{-2t/3}$. Using the condition $Q(\ln 8) = 13/16$, we get $\gamma = 1/4$.

3a. $y(t) = c_1 e^{5t}t + c_2 e^{5t} + \dfrac{1}{2}e^{5t}t^2$

3b. After a substitution $y = v(t)t^3$, we get $t^2v'' - tv' = 0$. Then substituting $u(t) = v'(t)$, we get $t^2u' - tu = 0$. We solve to get $u = c_1 t$. We integrate to get $v = c_1 t^2 + c_2$. Therefore $y = c_1 t^5 + c_2 t^3$. So one (simplest) possible choice for the second solution is t^5.

4. The differential equation is $\dfrac{1}{4}u'' + ku = 0, u(0) = 1, u'(0) = 1$. The solution is $u(t) = \dfrac{1}{2\sqrt{k}}\sin\left(2\sqrt{k}t\right) + \cos\left(2\sqrt{k}t\right)$, which has amplitude $\sqrt{\dfrac{1}{4k} + 1}$. Equating this with 5/4 and solving for k, we get $k = 4/9$.

5. The equation is $\dfrac{1}{8}u'' + \dfrac{1}{2}u' + 5u = 0, u(0) = -\dfrac{1}{2}, u'(0) = -2$, which has the solution $-\frac{1}{2}e^{-2t}\left(\sin(6t) + \cos(6t)\right)$. This is zero when $\sin(6t) + \cos(6t) = 0$, which happens when $\tan(6t) = -1$, which first happens when $6t = 3\pi/4$, that is, when $t = \pi/8$.

6. [Assuming the exam writer meant $y'' + 2y' + 2y = g(t)$]
We write $g(t) = -1 + (t-1)u_1(t) + (-\frac{4}{3}t + 4)u_3(t) + (\frac{1}{3}t - 2)u_6(t)$. Using $Y = \mathcal{L}\{y\}$, we have as the Laplace transform of the differential equation

$$Y(s^2 + 2s + 2) = -\frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{(4/3)e^{-3s}}{s^2} + \frac{(1/3)e^{-6s}}{s^2}.$$

Setting $F(s) = \dfrac{1}{s(s^2 + 2s + 2)}$ and $H(s) = \dfrac{1}{s^2(s^2 + 2s + 2)}$, we can write this as

$$Y = -F(s) + e^{-s}H(s) - (4/3)e^{-3s}H(s) + (1/3)e^{-6s}H(s).$$

Using partial fractions, we get

$$F(s) = \frac{1/2}{s} + \frac{(-1/2)s - 1}{(s+1)^2 + 1} = \frac{1}{2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2 + 1} - \frac{1}{(s+1)^2 + 1}\right].$$

The inverse Laplace transform is $f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)$ Repeating the process, we get

$$H(s) = \frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{s+1}{(s+1)^2 + 1}\right),$$

which has inverse Laplace transform $h(t) = \frac{1}{2}(t - 1 + e^{-t}\cos t)$. So the inverse Laplace transform of Y is

$$y(t) = -f(t) + h(t-1)u_1(t) - \frac{4}{3}h(t-3)u_3(t) + \frac{1}{3}h(t-6)u_6(t).$$