Solutions to Homework 5

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Section 5, Exercises 1, 3, 4, 7, 10, and 11

Exercise 1. Let $K = \mathbb{Q}(\alpha)$, where α is a root of the polynomial $x^3 + 2x + 1$, and let $g(x) = x^3 + x + 1$. Does $g(x)$ have a root in K?

Solution. The answer is no, which we will prove by contradiction. Let L denote the splitting field of f. Assume that $\beta \in K$ is a root of $g(x)$. Then $\beta \in L$. But g is irreducible, which means that $\beta \notin \mathbb{Q}$. Since $L/\mathbb{Q}$ is Galois, there must be some element of $G(L/\mathbb{Q})$ that does not fix β. But this automorphism must take β to some other root of g, which means that this other root of g must be an element of L. So in fact all the roots of g are in L.

In particular, the square roots of the discrimants of f and g (i.e. $\sqrt{-59}$ and $\sqrt{-31}$) are in L. But this means that L has two intermediate fields of degree 2 over $\mathbb{Q}$, which is impossible since $G(L/\mathbb{Q}) = S_3$, which has only one subgroup of index 2. So we have a contradiction, which means that our assumsion was not true. So g has no roots in K. □

Exercise 3. Let G be a finite group. Prove that there exists a field F and a Galois extension K of F whose Galois group is G.

Proof. First note that by Caylee's theorem, $G \cong H$, where H is a subgroup of S_n for some n. Let $K = \mathbb{Q}(x_1, \dots, x_n)$ be the field of rational functions in n variables. For each $\sigma \in S_n$ there is an automorphism $\tilde{\sigma}$ of K that sends x_i to $x_{\sigma(i)}$ for each i. It is clear that $\tilde{\sigma}\tilde{\tau}(\alpha) = \widetilde{\sigma\tau}(\alpha)$ for all $\alpha \in K$. Also, $\tilde{\sigma} = \tilde{\tau}$ implies $\sigma = \tau$, so S_n is a subgroup of the group of automorphisms of K. Let L be the fixed field of S_n, then K/L is Galois, with $G(K/L) = S_n$. Finally, let $F = K^H$. By the main Galois theorem, $G(K/F) = H$, which is isomorphic to G, as required. □

Exercise 4. Assume it is known that π and e are transcendental numbers. Let K be the plitting field of the polynomial $f(x) = x^3 + \pi x + 6$ over the field $F = \mathbb{Q}(\pi)$. Prove that $[K : F] = 6$ and that K is isomorphic to the splitting field of $f(x) = x^3 + ex + 6$ over $\mathbb{Q}(e)$.

Proof. First, we show that $f(x)$ is irreducible over $\mathbb{Q}(\pi)$. If f is reducible then since it is cubic it must have a root α. This gives us an equation $\alpha^3 + \pi\alpha + 6 = 0$, where $\alpha \in \mathbb{Q}(\pi)$. We can take this equation, change all the π's to x's, and clear denominators, and we get a polynomial in $\mathbb{Q}[x]$ that has π as a root, which is a contradiction since π is transcendental. Therefore f is irreducible. Finally, the square root of the discrimant is $\sqrt{-972 - 4\pi^3}$, which is not in $\mathbb{R}$, much less in $\mathbb{Q}(\pi)$. So the Galois group of f is S_3, which means $[K : F] = 6$.

Next, there is an isomorphism of the fields $\mathbb{Q}(\pi)$ and $\mathbb{Q}(e)$ which takes π to e, which means it takes f to g. Then by Proposition (5.2), the splitting field of f is isomorphic to the splitting field of g, as desired. $\square$

Exercise 7. Let f be an irreducible cubic polynomial over $\mathbb{Q}$ whose Galois group is S_3. Determine the possible Galois groups of the polynomial $(x^3 - 1) \cdot f(x)$.

Solution. Let K be the splitting field of f, and let ω be a non-real root of $x^3 - 1$. We are asked to determine the possibilities for $G = G(K(\omega)/\mathbb{Q})$. First, note that if $\omega \in K$, then we have $G = S_3$.

If $\omega \notin K$, then we have the following field diagram:

$$\begin{tikzcd}
& K(\omega) \arrow[dl, dash, "6"'] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\omega) \arrow[dr, dash, "2"'] & & K \arrow[dl, dash, "6"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Now, $\mathbb{Q}(\omega)/\mathbb{Q}$ is a Galois extension, since $\mathbb{Q}(\omega)$ is the splitting field of a polynomial. By the Galois correspondence, there is a normal subgroup H of G such that $G/H \cong G(\mathbb{Q}(\omega)/\mathbb{Q}) \cong C_2$. For the same reasons, $K/\mathbb{Q}$ is Galois, and there is a normal subgroup H' such that $G/H' \cong G(K/\mathbb{Q}) \cong S_3$. A basic group theory result tells us that since H and H' are normal, the product HH' is a subgroup of G, so it must correspond to a subfield of $K(\omega)$. To find out which, we compute the fixed field of HH'. But H and H' are both subgroups of HH', so if α is fixed by HH' it must be fixed by H and H' as well, which means it must be in $\mathbb{Q}(\omega)$ (the fixed field of H) and K (the fixed field of H'), which means it must be in $\mathbb{Q}$. But $\mathbb{Q}$ is the fixed field of G, which means that $HH' = G$. Now we know that $|H \cap H'| = |H||H'|/|HH'| = 1$. This fact, together with the fact that H and H' are normal subgroups, means that $G = HH' \cong H \times H'$. But then $H \cong G/H' \cong S_3$ and $H' \cong G/H \cong C_2$, so we have $G \cong S_3 \times C_2$. This is isomorphic to the dihedral group of order 12. $\square$

Exercise 10. Let K be a splitting field of an irreducible cubic polynomial $f(x)$ over a field F whose Galois group is S_3. Determine the group $G = G(F(\alpha)/F)$.

Solution. Since $F(\alpha) \subseteq K$, with $[K : F(\alpha)] = 2$, $F(\alpha)$ corresponds to a subgroup of S_3 of order 2. We know that the subgroups of S_3 of order 2 are not normal subgroups, so therefore $F(\alpha)/F$ is not Galois, which implies that

$|G| \neq 3$. But we know that the order of G divides 3, which means that $|G| = 1$, and G is the trivial group. $\square$

Exercise 11. Let K/F be a Galois extension whose Galois group is the symmnetric group S_3. Is it true that K is the splitting field of an irreducible cubic polynomial over F?

Solution. Yes. Let H be a subgroup of S_3 of order 2. Let K^H be the fixed field of H, and let α be such that $K^H = F(\alpha)$. Then $[F(\alpha) : F] = 3$, so α is the root of a degree 3 polynomial in $F[x]$, which I will call $f(x)$. I claim that K is the splitting field of f.

Since f has one root of K, it has all its roots in K by the argument we gave in Exercise 1. Also, H is not normal in S_3, so $F(\alpha)/F$ is not Galois, therefore $F(\alpha)$ is not the splitting field of f. So the splitting field of f contains but is not equal to $F(\alpha)$ and is contained in K. By the Galois theorem, the splitting field of f corresponds to a subgroup of S_3 that is contained in but not equal to H. But $|H| = 2$, so the splitting field must correspond to the trivial subgroup, which means the fixed field is K, as desired. $\square$