Modeling Problems and solutions

I have not verified if these solutions are correct

2.) A tank contains 200 gal. of water, and 100 oz of salt. Water containing a salt concentration given by $\frac{1}{2}(1+\frac{1}{2}\cos 2t)\frac{oz}{gal}$ flows into the tank at a rate of $4\frac{gal}{min}$ and the mixture flows out at the same rate.

Find the differential equation, and set up but do not evaluate the integral for the solution.

$S(t)$ = amount of salt at time t

$$\frac{dS}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dS}{dt} = \frac{1}{2}\left(1+\frac{1}{2}\cos 2t\right)\frac{oz}{gal}\left(\frac{4\,gal}{min}\right) - \frac{S(t)\,\frac{4\,gal}{min}}{200\,gal}$$

$$\frac{dS}{dt} = 2\left(1+\frac{1}{2}\cos 2t\right) - \frac{S(t)}{50}$$

$$S' + S(t)/50 = 2\left(1+\frac{1}{2}\cos 2t\right)$$

$$\mu(t) = e^{\int \frac{1}{50}dt} \qquad \mu(t) = e^{1/50\,t}$$

$$\boxed{Se^{1/50\,t} = \int e^{1/50\,t}\left(2\left(1+\frac{1}{2}\cos 2t\right)\right)dt} \qquad S(0) = 100\,oz$$

At time $t=0$ a tank contains S_0 lb of salt dissolved in 200 gal of water. Water entering 1/2 lb of salt / gal is entering at rate r gal/min and that mixture is draining from the tank at the same rate. Set up the initial value problem. As t goes to infinity (long time) find the amount of salt in the tank.

$S(0) = S_0 \qquad \frac{ds}{dt} =$ Flow in - Flow out

concentration = $S/200$ gal $\Leftarrow$ because flow in and out is the same.

Flow in = concentration of flow in × rate

Flow out = concentration of flow out × rate

$$\frac{ds}{dt} = \left(\frac{1}{2}\frac{\text{lb}}{\text{gal}} \cdot \frac{r\text{ gal}}{\text{min}}\right) - \left(\frac{S\text{ lb}}{200\text{ gal}} \cdot \frac{r\text{ gal}}{\text{min}}\right) = \quad S' = \frac{r}{2} - \frac{Sr}{200}$$

$$S' + \frac{Sr}{200} = \frac{r}{2}$$

$$\mu(t) = e^{\int r/200 \, dt} = e^{rt/200} \quad \text{integrating factor}$$

$$e^{rt/200} S = \int \frac{r}{2} \cdot e^{rt/200} dt \qquad \int \frac{r}{2} \cdot e^{rt/200} dt = 100 e^{rt/200} + c$$

$$e^{rt/200} S = 100 e^{rt/200} + c$$

$$S = 100 + c e^{-rt/200} \quad \text{using initial condition } S(0) = S_0$$

$$S_0 = 100 + c e^{0(t)/200} \qquad S_0 = 100 + c \qquad c = S_0 - 100$$

$$\boxed{S(t) = 100 + (S_0 - 100) e^{-rt/200}} \quad \text{Limiting amount is } S_L = 100 \quad t \to \infty$$

Problem 2. A 200L tank of water has 300 grams of dye in it. Water flows out of the tank at a rate of 20 L/min and in at the same rate. How long until only 1% of the dye is left in the tank? The tank is well-mixed at all times

t: minutes

Volume: Constant $\quad D(0) = 300$

$D(t)$ = Dye in tank (grams)

$D'(t)$ = $-$Dye Flowing out of the tank (grams/min)

$= -\frac{D\,20}{200}$

$D' = \frac{-D}{10}$

$\int \frac{1}{D}\,dD = \int \frac{1}{10}\,dt$

$\ln|D| = \frac{-t}{10} + C$

$D = Ce^{\frac{-t}{10}}$

$D(0) = 300 = Ce^0$

$= C$

$D = 300e^{\frac{-t}{10}}$

$3 = 300e^{\frac{-t}{10}}$

$0.01 = e^{\frac{-t}{10}}$

$\ln 0.01 = \frac{-t}{10}$

$-10\ln 0.01 = t$

$20\ln 10 = t$

$t \approx 46.05$ minutes

Question #2: Consider a 300L tank of hydrofluoric acid at a concentration of $\frac{2}{3}{}^{g}/_{L}$ that needed to be cleaned out and refilled by deionized water flowing in at a rate of 2 ${}^{L}/_{min}$ and an even mixture of the substances flowing out at the same rate. Find the time when the amount of hydrofluoric acid is at 5% of its original value.

Answer: Q(0) = 200 g of HF $\qquad$ what is t when Q(t) = 10 ?

$Q' = (-2\ {}^{L}/_{min})(\frac{Q}{300}) = \frac{-Q}{150}$

$\ln|Q| = \frac{-t}{150} + C$

$Q = Ce^{-t/150} \qquad 200 = C$

$Q = 200e^{-t/150}$

$5 = 200e^{-t/150}$

$\boxed{t = 150\ln(40)}$

2. Suppose we have a 1000-L tank that contains ununuriium at a concentration of 272g/L. Fresh water is poured into the tank at a rate of 40 L/sec, mixed, and water drains at the same rate. Find an expression in terms of y for the amount of Ununuriium in the tank at any time t.

Solution

$y(t)$ = amount of Unununiium

$y(0) = (272 g/L)(1000 L) = 272{,}000$ L ... so this is the initial value.

$y'(t)$ = rate in - rate out

$\therefore\ y'(t) = 0 - 25L \cdot$ concentration of ununurium

$y'(t) = -40L\left(\frac{y}{1000L}\right)$

$y'(t) = -\frac{y}{25}$

This is a separable function

$\therefore\ \frac{dy}{dt} = -\frac{y}{25}$

$\frac{dy}{y} = -\frac{dt}{25}$

$\ln|y| = -\frac{t}{25} + C$

y cannot be negative, so

$\ln y = -\frac{t}{25} + C$

$y = e^{-\frac{t}{25} + C}$

$y = Ce^{-\frac{t}{25}}$

We know that $y(0) = 272{,}000$, so $C = 272{,}000$

$\therefore\ \boxed{y = 272{,}000e^{-\frac{t}{25}}}$

Margaret Stark   Math 307I

1. Suppose that an object is falling in the atmosphere near the sea level. Assume that the drag is proportional to the velocity with the drag coefficient of 5 kg/sec. Then mass of the object is 50 kg and is dropped from a height of 275m, how long will it take for the object to hit the ground and how fast will it go at the time of the impact?

Solution. $F=ma$ $\quad a=dv/dt$ $\quad g=9.8\,m/s^2$

$F=m\,dv/dt$ $\quad \gamma=5v$

$m\,dv/dt=mg-5v$

$50\,dv/dt=(50)9.8-5v$

$dv/dt=9.8-v/10$

$v'+v/10=9.8$ $\quad \mu(t)=e^{t/10}$

$\int e^{t/10}\,v\,d/dt=9.8\int e^{t/10}\,dt$

$e^{t/10}v=9.8(10)e^{t/10}+C$

$v=98+Ce^{-t/10}$ $\quad v(0)=0$

$0=98+C$ $\quad C=-98$ $\quad$ 98 m/s → terminal velocity

$v=98-98e^{-t/10}$ $\quad v=dx/dt$

$\int dx=\int 98-98e^{-t/10}\,dt$

$x=98t+980e^{-t/10}+C$

$x(0)=0$ $\quad 0=0+980(1)+C$ $\quad C=-980$

$x=98t+980e^{-t/10}-980$

$275=98t+980e^{-t/10}-980 \rightarrow \boxed{t=8.56}$

$\boxed{v(8.56)=56.36\,m/s}$

2. $2y'+y=e^{t/2}$ $\quad y(0)=5$

2. A tank contains 1000 L of 10% concentration of salt. Well-stir solution flowing out at 100 liters per minute. Fresh water enters the tank at same rate

a) Find the formula for $Q(t)$

b) How long does it take for the salt concentration to reach 1%?

<u>Soltn</u>

a) $\frac{dQ}{dt} = \text{rate in} - \text{rate out}$

$$\frac{dQ}{dt} = 0 - \frac{Q}{1000} \cdot 100$$

$$\frac{dQ}{dt} = -\frac{Q}{10}$$

$$\frac{dQ}{Q} = -\frac{1}{10} dt$$

$$\int \frac{dQ}{Q} = \int -\frac{1}{10} dt$$

$$\ln Q = -\frac{1}{10} t + C$$

$$\boxed{Q = e^{-1/10 t + C}}$$

b) $Q(0) = 1000 \times .10\%$

$= 100$

$100\cancel{0} = Q(0) = C$

$Q(t) = 100 e^{-t/10}$

When will $Q(t) = 10 = (1\% \cdot 1000)$

$10 = 100 e^{-t/10}$

$\log(10/100) = -t/10$

$$\boxed{10 \log(10) = t}$$

1. Jake's parents take out a loan of \$9,000 to buy him a motorcycle for his birthday. The bank charges them interest compounded continuously at an annual rate of 12%. Assume that Jake's parents are making payments continuously at a constant annual rate expressed by K. From this, determine the payment rate K that is required to pay off the loan in 2 years, as well as the amount of interest paid during this 2-year period.

solution:

$g(t)$ will define the loan at time t.

$$\frac{dG}{dt} = rG - K$$

$G(0) = 9,000$ and $G(2) = 0$.

$\rightarrow$ $G(t) = G(0)e^{rt} - \frac{K(e^{rt}-1)}{r}$ ⟩ set-up

$\rightarrow$ $\frac{rG(0)e^{rt}}{e^{rt}-1} = K$ ⟩ manipulation

$\rightarrow$ $\frac{r(9000)}{1-e^{-rt}}$

⟩ manipulate & plug-in values

$\rightarrow$ $\frac{(0.12)(9000)}{1-e^{(-12\%)(2)}}$

$\rightarrow$ 5061.58 per year

pt 2:

$2K - G(0) =$ ⟩ initial loan + interest paid

$2(5061.58) - 9000$

$= 1123.16$

1) A tank contains 50 liters of sweetened pink lemonade that has 60 grams of sugar in it. Pure (unsweetened) pink lemonade enters the tank at a rate of 1/2 liter per minute. The mixture leaves the tank at a rate of 1 liter per minute. How much sugar is in the tank at the time when the tank contains 30 liters of sweetened pink lemonade?

ANSWER:

$S(t)$ = sugar at time t in grams

$S(0) = 60g$

t is in minutes

$\frac{ds}{dt}$ = (rate sugar in) − (rate sugar out)

rate sugar in $= 0$ (incoming pink lemonade is unsweetened)

rate sugar out $= \frac{s(t)}{v(t)}$ L/min

$v(t)$ = volume of liquid in tank at time t

$= 50 - \frac{1}{2}t$

When $v(t) = 30$:

$30 = 50 - \frac{1}{2}t$

$\frac{1}{2}t = 20$

$t = 40$ minutes

$= \frac{s(t)}{(50 - \frac{1}{2}t)}$

so $\frac{ds}{dt} = \frac{-s}{(50 - \frac{1}{2}t)}$

$\frac{ds}{s} = \frac{-dt}{(50 - \frac{1}{2}t)} = \frac{2dt}{(t - 100)}$

So Now We Have: $\ln|s| = 2\ln|t - 100| + C$

$s = A(t - 100)^2$

Using the IC: $60 = A(100)^2$

$A = \frac{60}{(100)^2} = \frac{6}{1000}$

SNWH: $s(t) = \frac{6}{1000}(t - 100)^2$

$s(40) = \frac{6}{1000}(40 - 100)^2$

$\boxed{= 21.6 \text{ g sugar}}$

1. The population of a rather small town grows at a rate proportional to the population at any given time. The initial population of 1,000 increases by 20% in 10 years. How large will the population be in 30 years?

Solution: 1. $\frac{dx}{dt} = kx$ where k is a growth constant

2. Separate Equations and Integrate $\rightarrow \int \frac{dx}{x} = \int k\,dt \rightarrow \ln|x| = kt + C$

3. $x(0) = 1000 \rightarrow \ln|1000| = C$

Solve for C

4. Put C back into the equation

$\ln|x| = kt + \ln(1,000)$

$x = e^{kt + \ln(1000)}$

$x = 1000e^{kt}$

5. Solve for k

In 10 years... $1000 \cdot .20 = 200$

$200 + 1000 = 1,200$

$1,200 = 1000e^{k \cdot 10}$

$k = .0182$

6. Solve original question

$x = 1000e^{.0182t} = x = 1000e^{.0182 \cdot 30}$

$\boxed{x \approx 1,727.}$

Jeffy wants to paint his house green. He finds out that it will cost \$3000, but Jeffy is broke. He really wants to paint it, so he goes to a man who offered to lend him money on the subway. The man seemed professional and was friendly to Jeffy. Jeffy was pleased to hear that he would only have to pay 20% interest compounded continuously. The man told him he had to pay it off in exactly 4 years, no more, no less. Jeffy didn't know it, but if he didn't meet this requirement, his arms would be broken in the back alley. If K is the annual payment rate, determine what K must be for Jeffy to avoid getting his arms broken. Also determine how much interest he must pay.

2) $y(0)=3000$ $y=$ money owed
$t=$ years

$y'=.2y-k$

$y'-.2y=-k$

$\mu(t)=e^{\int -.2dt}$

$\mu(t)=e^{-.2t}$ $e^{-.2t}y=-ke^{-.2t}$

$\int\left(\frac{d}{dt}e^{-.2t}y\,dt\right)=\int -ke^{-.2t}$

$u=-.2t$
$du=-.2dt$
$dt=\frac{du}{-.2}$

$\frac{k}{.2}=5k$

$\frac{e^{-.2t}y}{e^{-.2t}}=\frac{5ke^{-.2t}+c}{e^{-.2t}}$

$e^{-.2(0)}(3000)=5ke^{-.2(0)}$
$3000=5k+c$
$3000-5k=c$

$y=5k+ce^{.2t}$
$y=5k+(3000-5k)e^{.2t}$
$y=5k+3000e^{.2t}-5ke^{.2t}$
$y=3000e^{.2t}-5k(e^{.2t}-1)$
$(0)=3000e^{.2(4)}-5k(e^{.2(4)}-1)$
$-6676.62=-6.1277k$
$1089.579278=k$ $\rightarrow$ $\boxed{\$1089.60/yr}$
$4k=4358.32$ total money spent
$\$4358.32-\$3000=$ $\boxed{\$1358.32 \text{ interest paid}}$