Print Your Name

Student ID #

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Directions

- Please check that your exam contains a total of 6 pages.
- Write complete solutions or you may not receive credit.
- This exam is closed book. You may use one 8.5×11 sheet of notes and a calculator.
- You may not share notes or calculators. You may not use a graphing calculator or any electronic device other than a calculator.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

Signature. Please sign below to indicate that you have not and will not give or receive any unauthorized assistance on this exam.

Signature: ______________________

1. (10 points) Let

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 4 & 1 \\ 9 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) What is the nullspace of A?

(b) Find a basis for the column space of A.

(c) Find a basis for the row space of A.

(d) What is the rank of A?

(e) What is the nullity of A?

2. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

(a) Find a least squares solution to the overdetermined system $A\mathbf{x} = \mathbf{b}$. (Your answer should be a single vector).

(b) Let $\mathbf{x}^*$ be your answer to part (a). In $\mathbb{R}^2$, draw the range of A, the point $\mathbf{b}$, and the point $A\mathbf{x}^*$.

3. (10 points) Let W be a subspace of $\mathbb{R}^4$, and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find an orthogonal basis for W.

(b) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be the basis you found in part a. Let

$$\mathbf{w} = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix}.$$

Find a, b, and c such that $\mathbf{w} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$.

4. (10 points) True or false. No justification is necessary.

(a) If U and V are subspaces of $\mathbb{R}^n$ and every vector of U is also in V (that is, $U \subseteq V$), then $\dim U \leq \dim V$

(b) If $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ is a linearly independent set of vectors, then so is the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1}\}$.

(c) If W is a subspace of $\mathbb{R}^n$ and B_1 and B_2 are two bases for W, then there is a vector of W that is in B_1 and B_2.

(d) If W is a subspace of $\mathbb{R}^n$ and $\dim(W) = k$, then W contains exactly k vectors.

(e) If A is any $(m \times n)$ matrix and $\mathbf{b}$ is an $(m \times 1)$ vector, then the equation $A^T A\mathbf{x} = A^T\mathbf{b}$ has at least one solution.

(f) Every set of orthogonal vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

(g) Every subspace of $\mathbb{R}^n$ contains the zero vector.

(h) The natural basis for $\mathbb{R}^n$ is an orthonormal basis.

(i) Every nonzero subspace of $\mathbb{R}^n$ has a unique basis.

(j) If $A\mathbf{x} = \mathbf{b}$ is a consistent system of equations, then every least squares solution to this system is also a solution to this system.

5. (10 points) Let V be a subspace of $\mathbb{R}^n$ and W be a subspace of $\mathbb{R}^m$. Suppose that $T_1 : V \to W$ and $T_2 : V \to W$ are linear transformations. Define a new function T by

$$T(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x}).$$

Prove that T is a linear transformation.