For these two questions, consider the differential equation $y' = \dfrac{e^x(y-1)}{e^x+1}$

1. Determine the function $y(x)$ that satisfies the differential equation above and the initial condition $y(0) = y_0$, where y_0 is some constant.

Solution: This is a separable and linear, so you can use either method.

The separable method gives the equation

$$\frac{1}{y-1}\frac{dy}{dx} = \frac{e^x}{e^x+1}$$

which integrates to

$$\ln|y-1| = \ln|e^x+1| + c.$$

Exponentiating both sides, replacing the constant with a new one, and carefully removing the absolute value signs, we get

$$y - 1 = c(e^x+1)$$

To use the integrating factor method, we first rewrite the equation as

$$y' - \frac{e^x}{e^x+1}y = -\frac{e^x}{e^x+1}$$

The integrating factor is

$$\mu(x) = \exp\int\left(-\frac{e^x}{e^x+1}\right)dx = \exp(-\ln|e^x+1|) = \frac{1}{e^x+1}$$

Then we get the same thing as before:

$$\frac{1}{e^x+1}y = \int\left(-\frac{e^x}{(e^x+1)^2}\right)dx = \frac{1}{e^x+1} + c$$

Using the initial condition, we find that $y_0 = 1 + 2c$, giving a final answer of

$$y = 1 + \frac{1}{2}(y_0-1)(e^x+1)$$

2. Find all constant solutions to the differential equation above.

Solution: $y \equiv 1$