# A Circuit

A series circuit has a $1 \times 10^{-7}$ F capacitor, a 1 H inductor and a $2 \times 10^{3}\,\Omega$ resistor. The capacitor initially has a charge of $3 \times 10^{-6}$ C, and the circuit initially has no current. Find a formula for the charge at time $t$.

$$Q'' + 2\times10^3 Q' + 10^7 Q = 0$$

$$r = \frac{-2\times10^3 \pm \sqrt{4\times10^6 - 40\times10^6}}{2} = -10^3 \pm 3\times10^3 i$$

$$Q = c_1 e^{-1000t}\cos 3000t + c_2 e^{-1000t}\sin 3000t$$

$$3\times10^{-6} = c_1$$

$$0 = -1000 c_1 e^{-1000t}\cos 3000t - c_1 \sim \sin \sim \; - c_2 e^{-} \sin \sim + 3000 c_2 e^{-1000t}\cos 3000t$$

$$1000 c_1 = 3000 c_2 \Rightarrow \quad c_2 = \tfrac{1}{3} c_1 = 10^{-6}$$

Answer: $Q = \boxed{3\times10^{-6} e^{-1000t}\cos 3000t + 10^{-6} e^{-1000t}\sin 3000t}$

# A Spring

A 500 g mass hangs from a spring. The weight of the mass stretches the spring 9.8 cm. The damping coefficient $\gamma$ can be adjusted.

$$mg = kL$$
$$(0.5)(9.8) = k(0.098)$$
$$k = 50$$

Find the value for $\gamma$ that makes the system critically damped.

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(0.5)(50)}}{2(0.5)} = -\gamma \pm \sqrt{\gamma^2 - 100}$$

critically damped: $\gamma^2 - 100 = 0 \Rightarrow \boxed{\gamma = 10}$

Assume that the system is underdamped. The spring is pulled down a certain amount and released. Find the quasi-frequency and phase shift in terms of $\gamma$. (It does not matter how far the spring was pulled.)

$(\gamma < 10)$ $\quad r = -\gamma \pm \sqrt{100 - \gamma^2}\, i$

$$u = c_1 e^{-\gamma t} \cos\sqrt{100-\gamma^2}\, t + c_2 e^{-\gamma t} \sin\sqrt{100-\gamma^2}\, t$$

$u(0) = a \quad u'(0) = 0$

$c_1 = a$

$$u' = -\gamma c_1 e^{-\gamma t} \cos\sim + \sqrt{100-\gamma^2}\, c_1 e^{\sim} \sin\sim - \gamma c_2 e^{\sim} \sin\sim + \sqrt{100-\gamma^2}\, c_2 e^{\sim} \cos\sim$$

$$0 = -\gamma c_1 + \sqrt{100-\gamma^2}\, c_2$$

$$c_2 = \frac{\gamma}{\sqrt{100-\gamma^2}}\, a$$

quasi-frequency: $\sqrt{100 - t^2}$

phase shift: $\tan^{-1} \frac{c_2}{c_1} = \tan^{-1}\left(\frac{\gamma}{\sqrt{100-\gamma^2}}\right)$

What are the quasi-frequency and phase shift when $\gamma = 0$? What about as $\gamma \to 10$?

| | quasi-frequency | phase shift |
|---|---|---|
| $\gamma = 0$: | 10 | 0 |
| $\gamma \to 10$: | 0 | $\pi/2$ |