308B Final Solutions, Spring 2004

1. (a) A contains a column all of zeroes, so the columns of A are certainly linearly dependent: A is singular, so we know at once that $\det(A) = 0$. $\det(A - \lambda I) = -\lambda(\lambda - 9)(\lambda + 3)$ so A has eigenvalues $\lambda = 0, 9, -3$. The eigenspace E_λ is the nullspace of $A - \lambda I$; we can find a basis by plugging in the appropriate eigenvalue and row-reducing, and obtain

$$E_0 = Span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, E_{-3} = Span\left\{\begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix}\right\}, E_9 = Span\left\{\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

The sum of the dimensions of the eigenspaces is 3.

(b) B is upper triangular, so its eigenvalues are its diagonal entries $\lambda = 1, 2$, and its determinant is the product of its diagonal entries: $\det(B) = 4$.

$$E_1 = Span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}, E_2 = Span\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

The sum of the dimensions of the eigenspaces is 2.

2. (a) $a = \frac{6}{5}, b = \frac{1}{2}$.

(b) We don't need to worry about doing anything least-squaresy here: the straight line $y(t) = t + 1$ passes though both data points on the nose.

3. (a) Recall that the vector equation $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{u}$ can be rephrased as the matrix equation

$$\begin{bmatrix} -1 & -2 & 11 \\ 1 & 3 & -15 \\ 0 & -1 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}.$$

We can solve this by setting up an augmented matrix and row-reducing; we find that $c_1 = 1, c_2 = 3, c_3 = 1$.

(b) $\vec{v_1} + 3\vec{v_2} + \vec{v_3} = \vec{u}$ so $C\vec{v_1} + 3C\vec{v_2} + C\vec{v_3} = C\vec{u} = C\vec{v_3} = \vec{v_2} + \vec{v_3}$. So $C^2\vec{u} = C\vec{v_3} = \vec{v_2} + \vec{v_3}$, and, continuing on in like manner, $C^4\vec{u} = \vec{v_2} + \vec{v_3}$.

Note that $\vec{v_2} + \vec{v_3}$ is a 1-eigenvector for C.

(c) There are many to choose from; one is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

4. (a) $\{\vec{u_1}, \vec{u_2}\}$ is an orthogonal basis of $\mathbb{R}^2$, so we can find our coefficients quickly and painlessly using inner products: $\vec{e_1} = \frac{1}{5}\vec{u_1} - \frac{2}{5}\vec{u_2}$ and $\vec{e_2} = \frac{2}{5}\vec{u_1} + \frac{1}{5}\vec{u_2}$.

(b) $F\vec{e_1} = \vec{u_1} + 4\vec{u_2}$ and $F\vec{e_2} = 2\vec{u_1} - 2\vec{u_2}$ so

$$F = \begin{bmatrix} -7 & 6 \\ 6 & 2 \end{bmatrix}, F^{-1} = \frac{1}{-50}\begin{bmatrix} 2 & -6 \\ -6 & -7 \end{bmatrix}.$$

5. (a) $\det(N^k) = (\det(N))^k = \det(0) = 0$ so $\det(N) = 0$ and N is singular.

(b) $N^k = 0$ implies that $\lambda^k = 0$ so the only possibility is $\lambda = 0$.

(c) Suppose $(I - N)v = 0 = 0v$ for some vector v. Then $Iv = v = Nv$ so v is a 1-eigenvector of N. But 1 is not an eigenvalue of N, so $v = 0$. Therefore the nullspace of $I - N$ is $\{0\}$ so $I - N$ is nonsingular. Alternatively, one could say that 0 is not an eigenvalue of $I - N$, so $I - N$ is nonsingular.

(d) $P^2 = P$ implies $\lambda^2 = \lambda$, so $\lambda = 0, 1$. The following idempotent matrix has both of these as eigenvalues:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(e) $(I - P)^2 = (I - P)(I - P) = I - 2P + P^2 = I - 2P + P = I - P$.