EXAMPLE MODELING PROBLEMS

NATHAN GRIGG, MATH 307

Credit Card Interest. You have \$2000 on a credit card with an annual interest rate of 15%, compounded continuously. Suppose that you pay \$30 per month (paid continuously) and make no new purchases. When will the bill be paid off? How much money will you pay?

Solution. Let $y(t)$ be the account balance, with positive meaning I owe money. Then the initial condition is $y(0) = 2000$. I will measure time in years. I need to describe the rate at which the account balance is changing. There are two ways that this is happening: I am paying it off, and it is growing due to interest.

First, the balance is decreasing as a pay it off. I am paying at a *constant* rate of \$30/month, which is \$360/year. So, if this were the only thing happening to my account, I would have a differential equation of $y' = -360$.

The second thing that is happening is interest. The more money on the account, the faster the balance grows due to interest. In other words, the rate of change is proportional to the balance, $y' = ky$. The k is the interest rate, so my differential equation would be $y' = 0.15y$ if I weren't making payments. I can double-check my reasoning by asking myself what would happen in a year if I weren't making payments. The balance would increase by 15% of 2000, which is 0.15×2000, just like my equation says. (Of course, this is just to help me think, actually the balance would increase by more than that because it is being compounded)

Now we put these together to get $y' = 0.15y - 360$. We solve this (using either separation of variables or an integrating factor) to get $y = ce^{0.15t} + 2400$. Using our initial condition, we find that $c = -400$, so our equation for the account balance is $y = 2400 - 400e^{0.15t}$.

This will be paid off when the balance is zero, which happens when $t = (\ln \frac{2400}{400})/0.15 = 11.95$ years. Over this time you will pay $(11.95)(360) = \$4302$

Salty water. A 50-L vat contains a salt dissolved in water at a concentration of 0.5 g/L. Water flows in at a rate of 3 L/min with a concentration of 10 g/L. It is mixed and flows out at the same rate. How long will it take until the concentration is 9 g/L?

Solution. Let $Q(t)$ be the quantity of salt (in grams) dissolved in the water. Then $Q(0) = 25$ There are two ways that the quantity is changing. Salt is coming in at a constant rate of $(10)(3) = 30$ g/min. It is flowing out at a rate of (current concentration)(3). At any given time, the concentration of salt in the tank is $Q(t)/50$. So the salt flows out at a rate of $(3/50)Q(t)$.

So our differential equation is $Q' = 30 - (3/50)Q$. We solve this to get $Q = 500 + ce^{-3t/50}$. The constant is -475, so we have $Q(t) = 500 - 475e^{-3t/50}$. Now would be a good time to check what happens as $t \to \infty$. Then $Q \to 500$, which is a concentration of 10 g/L. Good, because that is what we expect should happen.

Let's find when the concentration hits 9 g/L. A concentration of 9 g/L corresponds to $Q(t) = 450$. So solve $450 = 500 - 475e^{-3t/50}$ for t to get $t = \ln(\frac{50}{475})(-50/3) \approx 37.52$ minutes.