Answers to Math 308 Sample Midterm 2

1. (a) The nullspace of A is $\{\mathbf{0}\}$.

   (b) One possible solution is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$. Another solution is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 34/43 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 39/43 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -12/43 \end{bmatrix} \right\}$.

   (c) The rowspace is $\mathbb{R}^3$, so any basis for $\mathbb{R}^3$ will work. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix} \right\}$.

   (d) 3 (e) 0.

2. (a) Your answer should be some vector of the form $\begin{bmatrix} 4/5 - 2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$.

   One possible solution is $\begin{bmatrix} 4/5 \\ 0 \\ 0 \end{bmatrix}$.

   (b) No matter what answer you put for part (a), $A\mathbf{x}^* = \begin{bmatrix} 4/5 \\ 8/5 \end{bmatrix}$. The range of A should be a line through the origin with slope 2. $A\mathbf{x}^*$ should be on the line, and $\mathbf{b}$ should be off the line, but if you connect $\mathbf{b}$ and $A\mathbf{x}^*$, you should get a line perpendicular to the range of A.

3. (a) One possible answer (using Gram-Schmidt with the basis in the given order) is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 1 \\ -1/3 \end{bmatrix} \right\}$.

   (b) $a = 3, b = 5/3, c = 1$, assuming your answer to (a) was the same as mine.

4. (a) True
   (b) False. For example $\mathbf{x}_{p+1}$ could be the same as x_p.
   (c) False. For example $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\}$ are two bases for $\mathbb{R}^2$ that don't share any vectors.
   (d) False. If W is nonzero, it has infinitely many vectors.
   (e) True. There is always a least squares solution to a system of equations.
   (f) False. This is only true if the set contains only nonzero vectors, and the set is a spanning set for $\mathbb{R}^n$.
   (g) True.
   (h) True.
   (i) False.
   (j) True.

5. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in V and let a be a scalar. Then we check the two conditions of a linear transformation:

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) &= T_1(\mathbf{x}+\mathbf{y}) + T_2(\mathbf{x}+\mathbf{y}) \\
&= T_1(\mathbf{x}) + T_1(\mathbf{y}) + T_2(\mathbf{x}) + T_2(\mathbf{y}) \\
&= (T_1(\mathbf{x}) + T_2(\mathbf{x})) + (T_1(\mathbf{y}) + T_2(y)) \\
&= T(\mathbf{x}) + T(\mathbf{y}). \\
T(a\mathbf{x}) &= T_1(a\mathbf{x}) + T_2(a\mathbf{x}) \\
&= aT_1(\mathbf{x}) + aT_2(\mathbf{x}) \\
&= a(T_1(\mathbf{x}) + T_2(\mathbf{x})) \\
&= aT(\mathbf{x}).
\end{aligned}$$

Therefore T is a linear transformation.