A VARIATION ON THE STEINER PROBLEM: EQUALLY SPACED POINTS ON A WIDE CONE

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Abstract. In this paper we provide solutions to a family of Steiner problems. We prove that if n points lie on the unit circle on a wide cone of angle $(60n)^\circ$, then the least-length network necessary to connect the points is formed by connecting $n-1$ sides for small n and connecting points to the center in pairs for large n, with one triple included if n is odd.

1. Introduction

The Steiner problem is a widely known problem that is stated very simply: given n points, find the least-length network required to connect the points. An algorithm due to Melzak ([1]) is known for finding this least-length network, but its complexity grows quickly as the number of points grows. It is interesting, then, to study special cases of the Steiner problem whose solutions have a closed form. Graham's conjecture, which was proved in 1992 by Rubinstein and Thomas in [2], focuses on points on a circle. One consequence of the conjecture is that for n points equally spaced around a circle, $n \geq 6$, the shortest network connecting the points is formed by drawing $n-1$ sides of the n-gon.

In the 2005 summer REU program at Brigham Young University, Colleen Hughes and Christine Truesdell worked with Gary Lawlor on a slight variation of Graham's problem. Given a wide cone of angle 480° (see Definition 2.4), they tried to find the least-length network to connect eight points equally spaced around the unit circle centered at the cone's vertex. Since this problem is closely analogous to Graham's problem with six points, one might expect the shortest network to consist of seven edges of the octagon. However, Hughes and Truesdell showed that the least length network consists of four Y-shaped networks connecting to the origin. Unfortunately, their proof required the analysis of more than 90 different cases, and the complicated details prevented them from publishing their proof.

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In the BYU summer REU of 2006, the authors of this paper took an interest in the results of Hughes and Truesdell and looked for a way to generalize them. We tried to answer the question: given a regular n-gon centered at the origin on a wide cone of angle $(60n)^\circ$, what is the minimal network necessary to connect the vertices of the n-gon? We found (Theorem 5.1) that for $n \leq 7$, the shortest network connecting the points is formed by drawing $n-1$ sides of the n-gon; when $n = 6$ this coincides with the known planar result. For $n \geq 8$, the shortest network is formed by connecting Y-shaped networks to the origin, as Hughes and Truesdell showed for eight points. Furthermore, we discovered a simple proof that only requires consideration of two subcases.

2. Background and Definitions

Definition 2.1. Given points $p_1, \ldots, p_n$, a Steiner tree T is a network of least possible length connecting the points. We will generally refer to "a Steiner tree" for $p_1, \ldots, p_n$ rather than "*the* Steiner tree," since the least-length tree may not be unique.

All nonterminal nodes of the tree will be points (called "Steiner points") where three edges meet at 120° angles. We call $p_1, \ldots, p_n$ the *regular points* of T. We adopt the convention that subscripts are taken modulo n when necessary, so that, for example, p_{i+1} means p_1 if $i = n$.

Definition 2.2. Let T be a given Steiner tree, and P a polygonal region with vertices (in order) $p_i, s_1, s_2, \ldots, s_m, p_j$, where p_i and p_j are regular points of T and each s_i is a Steiner point of T, and where every edge of P except $\overline{p_j p_i}$ is an edge of T.

Then we call P a *Steiner polygon* of T, also called the *Steiner polygon of* p_i *and* p_j.

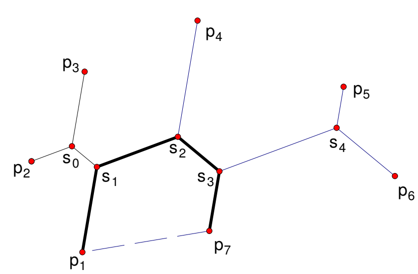

Figure 1. The 5-sided Steiner polygon of p_1 and p_7 includes the edge $\overline{p_1 p_7}$.

Definition 2.3. Two regular points of a Steiner tree are called *siblings* if they are connected to the same Steiner point, i.e. if their Steiner polygon has three edges and three vertices. In Figure 1, p_2 and p_3 are siblings, as are p_5 and p_6.

Definition 2.4. The *wide cone of angle* α, denoted C_α, is a space locally isometric to $\mathbb{R}^2$ except at the origin, where it has angle $\alpha > 2\pi$. An initial representation of C_α can be created from the subset $(r, \theta) \in [0, \infty) \times [0, \alpha]$ of $\mathbb{R}^2$ by identifying the segment $\{0\} \times [0, \alpha]$ to a single point, called the vertex V of C_α, and identifying each pair of points $(r, 0)$ and (r, α), for all $r \geq 0$. The metric is given, as in polar coordinates, by $r\,ds$.

A more intuitive realization of C_α in $\mathbb{R}^3$ (with the standard metric) is as a "ruffled cone," a union of straight rays out from the origin. This can be formed by selecting a simple closed curve U of length α on the unit sphere (perhaps one that oscillates above and below the equator), and mapping the rays $[0, \infty) \times \{\theta\}$ to rays in $\mathbb{R}^3$ from the origin out through points of U.

A special case of this, a "gathered cone," stretches a portion of U having length nearly 2π around the sphere's equator, and then bunches the remainder of the length of U (just over $\alpha - 2\pi$) into very tight oscillations in a small neighborhood of one point of the equator. Thus, the gathered cone representation (used in Figures 3 and 4) spreads a portion of C_α out flat into $\mathbb{R}^2 \subset \mathbb{R}^3$, gathering the remainder of the cone into tight ruffles near, say, the negative y axis in $\mathbb{R}^2$. This allows us to analyze wedge-shaped portions of C_α using familiar geometry in $\mathbb{R}^2$.

Lemma 2.5. *Let $n \geq 6$ and $\alpha = \frac{n\pi}{3}$, and consider a wide cone C_α with vertex V, with n points $p_1, \ldots, p_n$ on C_α all a unit distance from V, equally spaced around the central unit circle of C_α. Let D be the region on C formed as the union of n equilateral triangles each having vertices at V and at two consecutive points p_i and p_{i+1}.*

Then any Steiner tree connecting these n regular points

(1) is contained in D,

(2) contains no edge directly connecting the origin to a regular point, and

(3) contains no edge directly connecting two regular points unless the entire network consists of $n - 1$ outer edges.

Proof. Fact (1) is standard; if the network went outside D then we could project it inward to the boundary of D and decrease length.

We will prove fact (3) next. Recall that a Steiner tree never has two edges meeting at an angle less than 120 degrees, since a small deformation would then decrease its length. Therefore, in the present case a Steiner tree cannot contain an edge connecting two *nonconsecutive* regular points, since this edge could not connect to any other edges without creating an angle less than 120 degrees.

Now suppose a Steiner tree contains an edge between two consecutive regular points. Then the only way to connect another edge to this one without creating an angle smaller than 120 degrees is for the next edge to connect directly to the next adjacent regular point. The third edge would likewise have to connect to the following regular point, and so on. This process cannot stop before filling out the whole tree; otherwise the tree would be disconnected. Thus, the tree must consist of $n-1$ outer edges.

Finally, to prove (2), suppose that a Steiner tree T contains an edge e from the origin out to a regular point p_i. No further edge can connect to p_i without creating a small angle in the tree, so the rest of the tree must be connected to the origin. This means that we could delete e and replace it with an edge from p_i to p_{i+1}; the resulting tree T' would still be connected and would have the same length as T. But now by the argument above for fact (3), since T' contains an outer edge, it could be shortened unless it consisted (entirely) of $n-1$ outer edges. This is not the case, however, since T (and thus T') contained an edge to the origin besides e.

$\square$

Note: From this point forward we will consider the Steiner problem with n points equally spaced around the central unit circle of C_α, with α measured in *degrees*. With the exception of the following lemma, we will restrict our attention to the case where $\alpha = 60n$ degrees, with $n \geq 6$.

Lemma 2.6. *Given a Steiner tree T of the wide cone C_α, if some Steiner polygon contains the origin in its interior, then we have*

$$\frac{\alpha(n-1)}{60n} - 1 < k \leq \frac{\alpha(n-2)}{60n} + 2\,, \tag{1}$$

where k is the number of vertices in the Steiner polygon. In the special case where $\alpha/n = 60°$, then $k = n$ or $k = n-1$.

Proof. For the first inequality above, we note that the sum of the angles of a k-gon containing the origin of a wide cone must be $180k - \alpha$ degrees. Each of the $k-2$ angles with vertices at Steiner points must measure $120°$. Finally, since the origin is contained in the interior of the Steiner polygon, each of the two angles with vertices at regular points must be larger than $90 - \alpha/(2n)$ degrees. This gives us the following inequality:

$$180k - \alpha \quad > \quad 120(k-2) + 180 - \frac{\alpha}{n}$$

$$k \quad > \quad \frac{\alpha(n-1)}{60n} - 1 \tag{2}$$

Similarly, for the second inequality in (1), we note that the sum of the two angles with vertices at regular points must be less than or equal to $360 - 2\alpha/n$ degrees, or the Steiner tree will not be contained in our n-gon. This gives us

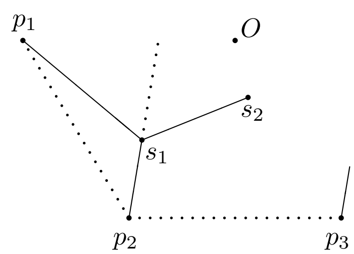

FIGURE 2. Configuration of points in Lemma 3.1

the following inequality:

$$\begin{aligned} 180k - \alpha &\leq 120(k-2) + 360 - \frac{2\alpha}{n} \\ k &\leq \frac{\alpha(n-2)}{60n} + 2 \end{aligned} \tag{3}$$

Combining Equations (2) and (3) , we get Equation (1), as desired.

Finally, if we let $\alpha = 60n$, Equation (1) reduces to $n - 2 < k \leq n$. □

3. Analyzing a Steiner Polygon Containing the Origin in its Interior

We will show that if the origin lies in the interior of some Steiner polygon, then there is only one possible topology for the Steiner tree. To do this, we will use Lemma 2.6 to show that there are initially only two ways that the origin could lie in the interior of a Steiner polygon. We will then work by contradiction to eliminate one of these. The following lemma will be of use:

Lemma 3.1. *Let p_1, p_2, and p_3 be adjacent regular points such that p_1 and p_2 are siblings connected to a Steiner point s_1. If $\measuredangle s_1p_2p_3 \leq 90°$, then the Steiner polygon of p_2 and p_3 does not contain the origin in its interior.*

Proof. Figure 2 contains an illustration of p_1, p_2, p_3, and s_1. Note that $\triangle p_1p_2O$ is an equilateral triangle, and that $\overrightarrow{s_1s_2}$ is simply the reflection of $\overrightarrow{s_1p_1}$ across $\overrightarrow{p_2s_1}$. Since $\measuredangle s_1p_2p_3 \leq 90°$, then $\overrightarrow{s_1s_2}$ must pass through or below O. □

We are now ready for the main lemma of this section.

Lemma 3.2. *If a Steiner polygon contains the origin in its interior, then it has n vertices.*

Proof. Refer to Figure 3, a gathered representation of C_α in which the part of the cone from p_1 around to p_6 is laid out flat in the plane, and the remainder is gathered around the negative y axis. Use standard coordinates so that, for example, $p_2 = (-1, 0)$ and $p_5 = (1, 0)$.

Let P be an arbitrary Steiner polygon containing the origin in its interior. Number the vertices p_i so that P is the Steiner polygon of p_1 and p_n. Let k be the number of vertices of P, so that we can write $P = p_1 s_0 s_1 \cdots s_{k-2} p_n$. By Lemma 2.6, we have that either $k = n-1$ or $k = n$. Recall that $n \geq 6$, so that, in particular, s_4 is one of the vertices of P.

Assume, by way of contradiction, that $k = n - 1$. We will see that this forces the ray going downward from s_3 (toward the necessary Steiner point s_4) in Figure 3 to pass on the outside of p_6, an impossibility.

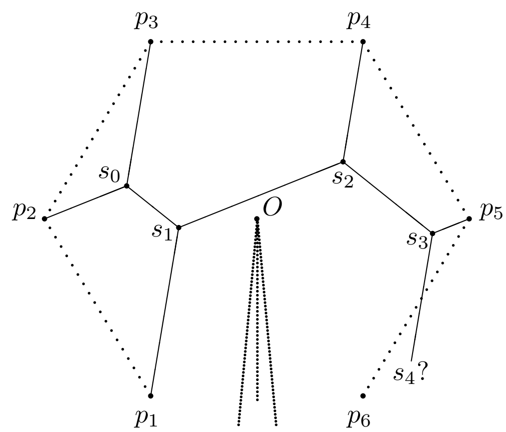

FIGURE 3. Configuration of points $p_1, p_2, \ldots, p_6$ and s_0, s_1, s_2, s_3 in Lemma 3.2

Since the interior of P contains the origin, the sum of its angles is $180°k - \alpha = 180°(n-1) - 60°n$. Note that all but two of its vertices are at Steiner points, whose angles measure $120°$. So the sum of the other two angles is

$$180°(n-1) - 60°n - 120°(n-3) = 180°.$$

Certainly one of these angles is less than or equal to $90°$, so by Lemma 3.1, either p_2 and p_1 are not siblings or p_n and p_{n-1} are not siblings.

Suppose, without loss of generality, that p_1 and p_2 are not siblings. Since the Steiner polygon of p_1 and p_n has $n-1$ sides, it has $n-3$ of the tree's total $n-2$ Steiner points as vertices. This forces p_2 and p_3 to be siblings, as shown in Figure 3.

We now use Melzak's construction (see [1]) of equilateral triangles related to the Steiner points of the tree. Note that although Melzak's idea can be used to actually construct Steiner trees, we will only use it here to help analyze a tree that is already given.

Suppose that s_0 is the Steiner point of p_2 and p_3. Since $\measuredangle p_2 s_0 p_3 = 120°$ and $\overrightarrow{s_1 s_0}$ bisects this angle, the point $E_0 = (-3/2, \sqrt{3}/2)$ (that makes $\triangle p_2 p_3 E_0$ equilateral) lies on $\overrightarrow{s_1 s_0}$, as in Figure 4(a). Similarly, the point $E_1 = (-5/2, -\sqrt{3}/2)$ that makes $\triangle E_1 E_0 p_1$ an equilateral triangle lies on $\overrightarrow{s_2 s_1}$; see Figure 4(b). Continuing, (Figure 4(c)) we get the point $E_2 = (-5/2, 3\sqrt{3}/2)$ such that $\triangle E_2 E_1 p_4$ is an equilateral triangle and E_2 lies on $\overrightarrow{s_3 s_2}$. Finally, (Figure 4(d)) we find that the point $E_3 = (3/2, 5\sqrt{3}/2)$ lies on $\overrightarrow{s_4 s_3}$ and makes $\triangle E_3 E_2 p_5$ equilateral.

As a side note (to help with a later lemma), Melzak's method also tells us that the total length of the three edges between the vertices p_2, p_3, s_0 and s_1 equals the distance from s_1 to E_0. Likewise, the portion of the tree to the left of s_2 has length equal to the distance from s_2 to E_1. To the left of s_3 the tree's length equals the distance from s_3 to E_2, and so forth.

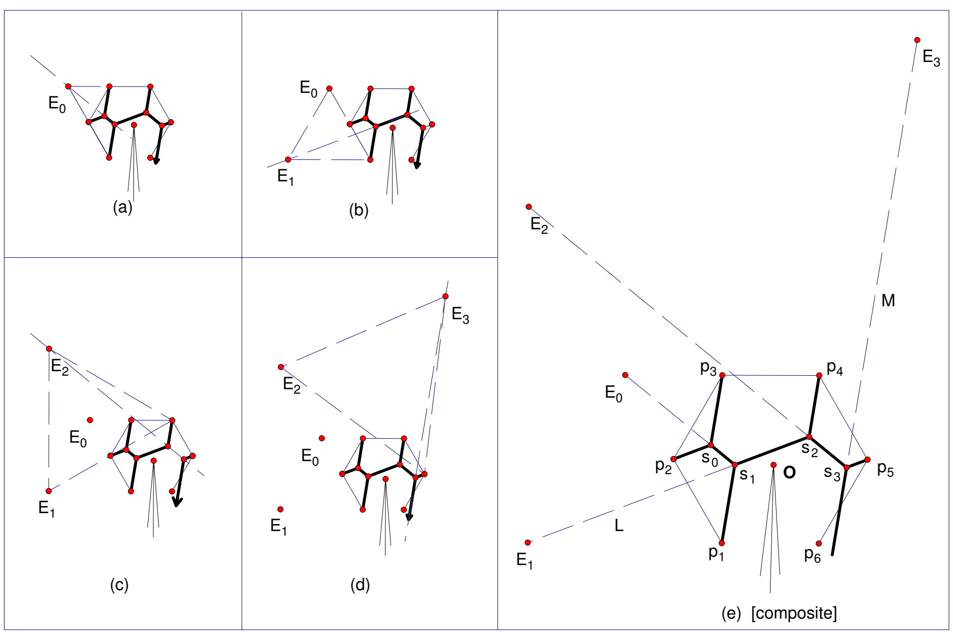

FIGURE 4. Melzak's construction applied to part of a Steiner tree

Now the line through E_1 and $(0,0)$ has slope $\sqrt{3}/5$, and we know that the line L through E_1, s_1 and s_2 must be above the origin, so its slope is greater than $\sqrt{3}/5$. Notice that the line M through E_3 and s_3 is rotated $60°$ up from L. Thus, the slope of M must be greater than $\tan(\tan^{-1}(\sqrt{3}/5)+60°) = 3\sqrt{3}$. But that would make M pass to the right of the point p_6, causing the Steiner tree to go outside the polygon D, which is a contradiction.

This means it is impossible for k to equal $n-1$. Thus, $k = n$, so that any Steiner polygon containing the origin in its interior has n vertices.

□

4. Analyzing a Steiner Polygon Containing the Origin On an Edge

Lemma 4.1. *Let p_1 and p_2 be regular points such that the origin is not contained in the interior of their Steiner polygon P, and let k be the number of sides of P. Then*

(1) $k \leq 5$, *and*
(2) *if* $\measuredangle p_1 \leq 60°$ *then* $k = 4$ *or* $k = 3$

Proof. Since P does not contain the origin in its interior it is a Euclidean polygon, so the sum of the measures of its angles is $180k - 360$ degrees. Since $k-2$ of the angles measure $120°$, the two remaining angles measure

$$\measuredangle p_1 + \measuredangle p_2 = 180°k - 360° - 120°(k-2) = 60°(k-2)$$

Since $\measuredangle p_1$ and $\measuredangle p_2$ are both smaller than $120°$, $\measuredangle p_1 + \measuredangle p_2 < 240°$, giving us $60°(k-2) < 240°$. Therefore, $k < 6$, as desired.

In the case that $\measuredangle p_1 \leq 60°$, then $\measuredangle p_1 + \measuredangle p_2 < 180°$. This gives us $60°(k-2) < 180°$, so $k < 5$, as desired. □

Lemma 4.2. *If the origin is part of a Steiner minimal tree then the origin is a sibling with some regular point.*

Proof. By Lemma 2.5 the origin does not connect directly to any regular point. Suppose now that the origin is not a sibling with any regular point. Then the origin connects to a Steiner point s_1 which in turn connects to two Steiner points s_2 and s_3 . By Lemma 4.1, no Steiner polygon can have more than five sides, so s_2 and s_3 must connect to regular points p_2 and p_3 , as shown in Figure 5 (otherwise, the Steiner polygon that has s_1, s_2, s_3, and two regular points as vertices would have more than five sides). Since the sum of the angles $\angle p_2$ and $\angle p_3$ in the Steiner polygon of p_2 and p_3 is $180°$, we can assume without loss of generality that $\measuredangle p_2 \geq 90°$.

Now consider the regular point p_1 , as shown in Figure 5. The ray $\overrightarrow{s_2 s_4}$ is a reflection of the ray $\overrightarrow{s_2 s_1}$ across $\overrightarrow{p_2 s_2}$. Since $\measuredangle p2 \geq 90°$, $\overrightarrow{s_2 s_4}$ must pass below p_1 . This cannot be, so the center must be a sibling with some regular point. □

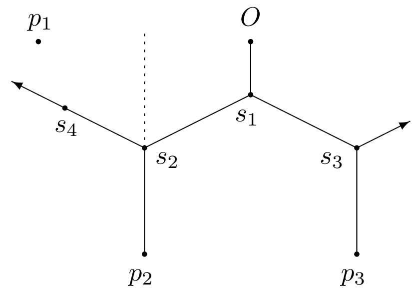

FIGURE 5. Topology for Lemma 4.2.

Lemma 4.3. *If the Steiner polygon of p_1 and p_2 has four vertices (call them, in order, p_1, s_1, s_2 and p_2), then $\measuredangle s_1p_1p_2 = \measuredangle s_2p_2p_3$.*

Proof. By Lemma 3.2, the origin is not the interior of the Steiner polygon, so the measures of $\angle s_1p_1p_2$ and $\angle s_2p_2p_1$ sum to $360° - 2 \cdot 120° = 120°$ (see Figure 6). Also, $\measuredangle s_2p_2p_3 + \measuredangle s_2p_2p_1 = \measuredangle p_1p_2p_3 = 120°$. Combining these equations gives us that $\measuredangle s_1p_1p_2 = \measuredangle s_2p_2p_3$. □

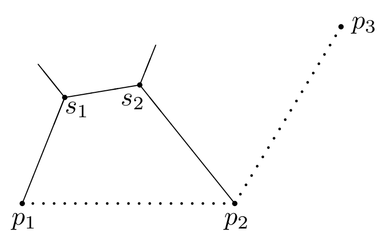

FIGURE 6. Steiner configuration for Lemma 4.3.

Definition 4.4. If T is a tree with a "spine" consisting of all the Steiner points of T connected linearly, we will say that T has the "snail topology" (so named since in the current context it will spiral around the origin.)

Lemma 4.5. *Any connected subtree whose terminal points are the origin and a number of regular points must have the snail topology.*

Proof. By Lemma 4.2, the origin O is a sibling with some regular point. Call this point p_1 , and let s_1 be the Steiner point that connects to the origin and to p_1 . Orient the graph so that s_1 lies counterclockwise of $\overline{Op_1}$ as shown in Figure 7 . Clearly, $\measuredangle s_1p_1p_2 < 60°$. By Lemma 4.1, the Steiner polygon of p_1

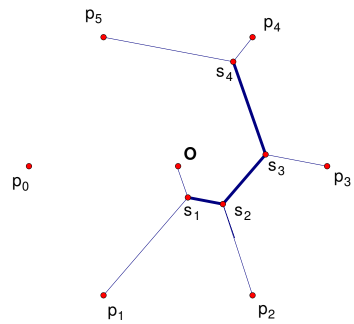

FIGURE 7. A snail with spine shown in bold

and p_2 has 3 or 4 sides. If it had 3 sides, the Steiner tree would end, since all Steiner points would be connected to three other points. Then by Lemma 4.3, $\measuredangle s_1 p_1 p_2 = \measuredangle s_2 p_2 p_3$. By induction, the angles $\measuredangle s_i p_i p_{i+1}$ are all equal (and thus less than $60°$), and every Steiner polygon of p_i and p_{i+1} has four sides until the last, which has three sides. In particular, none of them can have five sides, so the points are connected with a snail topology.

□

Lemma 4.6. *The total length of a snail-topology subtree whose terminal points are the origin and k regular points is $\sqrt{k^2 - k + 1}$.*

Proof. Use Melzak's construction as outlined in the proof of Lemma 3.2. The equilateral triangles that arise have all their vertices at points p_i or integer multiples (that is, dilations out from the origin) of the p_i.

Specifically, (referring to Figure 7) the first triangle has vertices at 0, p_0 and p_1, with p_0 lying on the line through s_1 and s_2. The distance from s_2 to p_0 equals the sum of lengths of the three edges between vertices 0, p_1, s_1 and s_2.

The second equilateral triangle has vertices at p_0, $2p_1$ and p_2. The distance from s_3 to $2p_1$ equals the sum of lengths of the five edges on the tree to the left of point s_3.

The third triangle has vertices at p_1, $3p_2$ and $2p_3$. In general, the j^{th} triangle has vertices at p_{j-2}, $j \cdot p_{j-1}$ and $(j-1) \cdot p_j$, and the length of the corresponding portion of the tree equals the distance from s_{k+1} to $j \cdot p_{j-1}$. In particular, letting $j = k-1$, we see that the length of the entire snail-topology subtree equals the distance from p_k to $(k-1) \cdot p_{k-2}$ (note that we have p_k in place of s_k because at the end of the snail we get the terminal point p_k instead of another intermediate Steiner point). Rotating, we see that this distance equals the distance from $(k-1, 0)$ to $(-1/2, -\sqrt{3}/2)$, which is $\sqrt{k^2 - k + 1}$, as desired.

□

5. Main Theorem

Theorem 5.1. *Given a cone of angle α and n points equally spaced on a circle centered at the origin such that $\alpha/n = 60°$, then the least length network to connect the n points is as follows:*

(1) *If $n \leq 7$ then the least length network is formed by directly connecting the n points with $n-1$ edges and no added points.*
(2) *If $n \geq 8$ then the least length network is formed by connecting Y's to the origin if n is even, or a triple and some Y's if n is odd.*

Proof. We have proven with Lemmas 3.2 and 4.5 that there are only two possibilities for our Steiner configuration: either we have an n-sided Steiner polygon or the union of several snail topologies. We will analyze the two options, assuming that the points are spaced one unit apart.

By Lemma 4.6, a snail connecting n points to the origin has total length $\sqrt{n^2 - n + 1}$. The function

$$\frac{\sqrt{n^2 - n + 1}}{n}$$

represents the cost per point to connect n points to the origin (see Figure 8). We see that it is most efficient to connect points to the origin two at a time. Lemma 2.5 (or direct calculation) shows that if there is a point left over (if n is odd) then rather than connect it directly to the origin (with a “1-snail”), it is better to connect the last three points to the origin with a 3-snail.

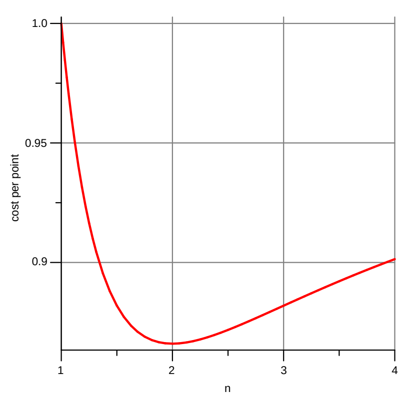

Figure 8. The cost per point of connecting n points to the origin using a snail.

Now if we connect n points using several 2-snails and a 3-snail if necessary, the total length of our network will be

$$\frac{n}{2}\sqrt{3} \quad \text{or} \quad \left(\frac{n-3}{2}\right)\sqrt{3}+\sqrt{7}$$

if n is even or odd respectively. Meanwhile, the length of the Steiner network in Case 1 is $n-1$. It is easy to see that for $n \geq 8$, the set of snails is the shortest, while for $n \leq 7$, the network of $n-1$ outer edges is more efficient. To further illustrate this, the cost per point of the two different methods is graphed in Figure 9. $\square$

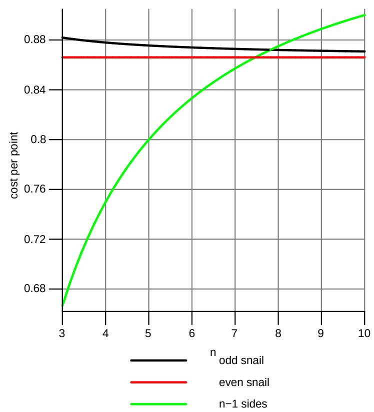

FIGURE 9. The cost per point of connecting n points to the origin using each of the possible methods

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