Question 1: Suppose $R \in \mathbb{C}^{m\times m}$ is upper triangular and invertible. Prove that R^{-1} is also upper triangular.

Proof. Let R_k denote the $k \times k$ submatrix of R consisting of the first k columns and k rows of R. Construct the matrix X as follows: Let e_k^j denote the kth standard unit vector in $\mathbb{C}^j$.

$$X = \begin{pmatrix} X_1 & \cdots & X_m \end{pmatrix} \text{ such that } X_k = \begin{pmatrix} x_k \\ 0 \end{pmatrix} \text{ with } x_k = R_k^{-1} e_k^k,\ 0 \in \mathbb{C}^{m-k}$$

The vector x_k exists since R_k is invertible. Notice that X is upper-triangular. Then,

$$RX = R\begin{pmatrix} X_1 & \cdots & X_m \end{pmatrix} = \begin{pmatrix} RX_1 & \cdots & RX_m \end{pmatrix} = \begin{pmatrix} e_1^m & \cdots & e_m^m \end{pmatrix} = I_m$$

Thus, $X = R^{-1}$ and we are done. ■