1 Geometry of Complex Variables

We first develop some geometric understanding of how the complex numbers work. Recall that there are two main ways to think of complex numbers:

- In Cartesian coordinates: $z = x + iy$ where $x, y \in \mathbb{R}$
- In Polar coordinates: $z = re^{i\theta}$ where $r > 0$ and $\theta \in \mathbb{R}$.

We can relate these two forms through Euler's Formula.

Theorem 1 (Euler's Formula)**.** $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

A full relation is done through

$$z = x + iy \to z = |z|e^{i\theta}, \text{ where } \theta = \begin{cases} \arctan y/x, & x > 0 \\ \arctan y/x + \pi, & x < 0, y \geq 0 \\ \arctan y/x - \pi, & x < 0, y < 0 \\ \frac{\pi}{2}, & x = 0, y > 0 \\ -\frac{\pi}{2}, & x = 0, y < 0 \end{cases}$$

$$z = re^{i\theta} \to r\cos\theta + ir\sin\theta$$

where we have to take limits for $\arctan(y/x)$ when $x = 0$.

Problems

(1) Recall the definition of the conjugate of a complex number, $z = x + iy$, is $\overline{z}$, where $\overline{z} = x - iy$.

(a) Find and sketch the complex conjugates of the following complex numbers

i. 1

ii. $3 + 4i$

iii. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

iv. $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

v. $8 + \frac{\sqrt{5}}{3}i$

vi. $-3 + \frac{\pi}{6}i$

(b) Give a geometric explanation for taking the conjugate of z.

(c) We define $\text{Re}(z)$ as $\text{Re}(x + iy) = x$. Find a formula for $\text{Re}(z)$ in terms of z and $\overline{z}$.

(d) We define $\text{Im}(z)$ as $\text{Im}(x + iy) = y$. Find a formula for $\text{Im}(z)$ in terms of z and $\overline{z}$.

(e) Find a formula for $|z|$ using z and $\overline{z}$.

(f) Find a polar formula for $\overline{z}$. *Hint: Use Euler's Formula.*

(2) We look at how different rational functions affect the complex plane. For sketching the image of $\mathbb{C}$ under different functions you can do so in two ways: (I) Draw the boundary of the new region (all of our functions will be nice enough that you can do this safely) or (II) indicate where a few key points will end up.

(a) Sketch the image of $\mathbb{C}$ under z^2.

(b) Sketch the image of $\mathbb{C}$ under $z^{1/2}$.

(c) Sketch the image of $\mathbb{C}$ under z^α where $0 < \alpha < 1$. Give a geometric explanation for how this function affects $\mathbb{C}$.

(d) Sketch the image of $\mathbb{C}$ under z^n where $n \in \mathbb{N}$. Give a geometric explanation for how this function affects $\mathbb{C}$.

(e) Sketch the image of $\mathbb{C}$ under $\frac{1}{z}$. Give a geometric explanation for how this function affects $\mathbb{C}$.

(f) Sketch the image of $\mathbb{C}$ under $z + z^2$.

(g) Sketch the image of $\mathbb{C}$ under $e^{i\theta}z$ where $0 \leq \theta < 2\pi$.