1 Solutions to Quadratic Equations

The first activity we'll do is deriving the general solution to quadratic equations. First, we'll start with the definition of a quadratic equation.

Definition 1. A quadratic equation is an equation of the form $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

Definition 2. $s \in \mathbb{R}$ is a solution to a quadratic equation $f(x) = ax^2 + bx + c$ if $f(s) = 0$.

When solving equations there are two main questions we can ask.

(I) How many solutions are there?

(II) What are the solutions?

Over the next couple of exercises we will answer both of those for every quadratic equation.

(1) Show that $s = \sqrt{a}$ and $s = -\sqrt{a}$ are solutions to $f(x) = x^2 - a$. What happens if $a < 0$?

(2) Show that $s = 0$ and $s = -b$ are solutions to $f(x) = x^2 + bx$.

(3) Show that $s = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $s = \frac{-b-\sqrt{b^2-4ac}}{2a}$ solve $f(x) = ax^2 + bx + c$.

(4) Show that $a\left(x - \frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{-b-\sqrt{4ac}}{2a}\right) = ax^2 + bx + c$. This shows that, if $f(x) = ax^2 + bx + c$ has a solution, then it can be factored as a product of linear equations. The y-intercepts of these linear equations are precisely the solutions to the quadratic.

(5) How many solutions can a quadratic equation have? *Hint: Draw some graphs.*

(6) Let $f(x) = ax^2 + bx + c$. For which values of $a, b, c \in \mathbb{R}$ does $f(x)$ have:

   (a) 2 solutions?

   (b) 1 solution?

   (c) 0 solutions?

   Hint: Use (3) and (4).

2 Functions

The second activity we'll do is looking at some properties functions have so we have more intuition about them. First, we'll start with the definition of a function.

Definition 3. A function f with domain A and codomain B is $f : A \to B, x \mapsto f(x)$ where if $a \in A$ and $f(a) = b_1 \in B$ and $f(a) = b_2 \in B$, then $b_1 = b_2$. Intuitively, this means that every input has a unique output.

We can combine functions through composition.

Definition 4. Let $f : A \to B$ and $g : B \to C$. Then, the **composition** of g with f is $g \circ f : A \to C, x \mapsto g(f(x))$.

Now, functions can have some special properties.

Definition 5. We say $f : A \to B$ is **injective** if, whenever $f(x) = f(y)$, we have $x = y$.

Definition 6. We say $f : A \to B$ is **surjective** if, whenever $b \in B$, there exists some $a \in A$ such that $f(a) = b$. Intuitively, everything in the codomain is "reached" by the function.

Definition 7. We say $f : A \to B$ is **bijective** if it is both injective and surjective.

The following exercises will help develop our understanding of these properties.

(1) State whether y can be written as a function of x in the following equations. If so, state the domain, codomain, and if you can, the range.

   (a) $ax + by + c = 0$

   (b) $\frac{x}{y} = 42$

   (c) $x^2 + y^2 = 1$

(2) State whether the following functions are injective, surjective, bijective, or none of those.

   (a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x$

   (b) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$

   (c) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

   (d) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^n$ where $n \in \mathbb{N}$

   (e) $f : \{x \in \mathbb{R} : x \geq 0\} \to \{x \in \mathbb{R} : x \geq 0\}, f(x) = x^2$

(3) If $f : A \to B$ is injective and $g : B \to C$ is injective, must $g \circ f$ be injective?

(4) If $f : A \to B$ is surjective and $g : B \to C$ is surjective, must $g \circ f$ be surjective?

(5) Suppose we have $f : A \to B$, $g : B \to C$, and we know $g \circ f$ is injective.

   (a) Must f be injective? If not, find a counterexample.

   (b) Must g be injective? If not, find a counterexample.