Estimating Models of Supply and Demand: Instruments and Covariance Restrictions*

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Abstract

We consider the identification of empirical models of supply and demand with imperfect competition. As is well known, a supply-side instrument can resolve price endogeneity in demand estimation. We demonstrate that, under common assumptions, two other approaches also yield consistent estimates of the joint model: (i) a demand-side instrument, or (ii) a covariance restriction between unobserved demand and cost shocks. Covariance restrictions can obtain identification even the absence of instruments. Further, supply and demand assumptions alone may bound the structural parameters without additional restrictions. We illustrate the covariance restriction approach with applications to ready-to-eat cereal, cement, and airlines.

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1 Introduction

A fundamental challenge in identifying models of supply and demand is that firms can adjust markups in response to demand shocks. Even if marginal costs are constant, this source of price endogeneity generates upward-sloping supply curves in settings with imperfect competition. Thus, the empirical relationship between prices and quantities does not represent a demand curve but rather a mixture of demand and supply. Researchers typically address this challenge by using supply-side instruments to estimate demand, and then using the supply model to recover marginal costs and simulate counterfactuals (e.g., Berry et al., 1995; Nevo, 2001).

In this paper, we demonstrate how the supply model can be used to identify models of imperfect competition in the absence of supply-side instruments. We recast equilibrium as a system of simultaneous semi-linear equations that correspond to supply and demand schedules. Through the presence of markups, the endogenous coefficient on price appears in both equations. We use this relationship to formalize two new approaches to identification. First, we show that *demand-side* instruments can also resolve price endogeneity and identify the joint model. Second, we show that a covariance restriction between unobserved demand and cost shocks can fully identify the model without any valid instruments.

Putting our results in context, early research at the Cowles Foundation examined identification in linear systems of equations, including supply and demand models of perfect competition (e.g., Koopmans, 1949; Koopmans et al., 1950).[1] With perfect competition, the supply curve may be upward-sloping due to increasing costs of production. With upward-sloping supply and downward-sloping demand, two separate restrictions are required for identification—one per equation (Hausman and Taylor, 1983). Our contribution lies in the extension to oligopoly models of imperfect competition, in which markups also affect the slope of the supply curve. Thus, markups generate an additional source of price endogeneity beyond the case of increasing costs. We demonstrate that demand and markup adjustments are linked theoretically, which eliminates the need for an additional restriction. When price endogeneity arises through markups only, a single restriction is sufficient for identification.

We provide formal econometric results for the covariance restrictions approach, which, in the context of imperfect competition, has not previously been examined. In particular, we establish a link between the endogenous price coefficient and the covariance of unobservable cost and demand shocks. Thus, a covariance restriction on unobserved shocks achieves identification. The core intuition is that the supply-side model dictates how prices respond to demand shocks, shaping the relative variation of quantities and prices in the data. This information can be exploited to resolve endogeneity bias and recover the causal structural parameters. There is no relevance condition that must be satisfied—i.e., no "first-stage" empirical requirement—

[1]Many articles advanced this research agenda in subsequent decades (e.g., Fisher, 1963, 1965; Wegge, 1965; Rothenberg, 1971; Hausman and Taylor, 1983; Hausman et al., 1987). More recently, Matzkin (2016) examines covariance restrictions in semi-parametric models.

because the endogenous data are interpreted directly through the lens of the model.[2] At a high level, this approach relates to Petterson et al. (2021), who show how to bound structural parameters based on beliefs about the magnitudes of unobserved shocks.

The results address one of the largest obstacles to research in empirical economics—that of finding valid instruments—and may allow researchers to push ahead along new frontiers. For example, Döpper et al. (2021) use our covariance restriction approach to estimate demand and markups across more than 100 consumer product categories. In that study, product and time fixed effects absorb the correlations that pose the most obvious threats to validity, such as the possibility that higher quality products are more expensive to produce. The residual variation in marginal costs (the "cost shock") then can be conceptualized as incorporating the contribution of instruments used elsewhere in the literature, such as input prices fluctuations that affect products differentially (Backus et al., 2021). In a setting where finding valid instruments would be difficult, the covariance restriction approach provides a theory-based path to recovering causal parameters. Covariance restrictions can also be combined with instruments or micro-moments (e.g., Berry and Haile, 2020) to jointly identify different sets of parameters or provide overidentifying restrictions using the generalized method of moments (GMM).

The strategy of using supply-side restrictions to reduce identification requirements has parallels in a handful of other articles. A simple linear example is provided in Koopmans (1949). Leamer (1981) examines a linear model of perfect competition, and provides conditions under which the price parameters can be bounded using only the endogenous variation in prices and quantities. Feenstra (1994) considers the case of monopolistic competition with constant markups, and a number of application in the trade literature extend this approach (e.g., Broda and Weinstein, 2006, 2010; Soderbery, 2015).[3] Zoutman et al. (2018) return to perfect competition and show that, under a standard assumption in models of taxation, both supply and demand can be estimated with exogenous variation in a single tax rate. Our research builds on these articles by focusing on imperfect competition with adjustable markups.

For much of the paper, we focus on the special case of constant marginal costs, wherein price endogeneity only arises through adjustable markups. This assumption is widespread in empirical models of imperfect competition (e.g., Nevo, 2001; Villas-Boas, 2007; Miller and Weinberg, 2017; Backus et al., 2021). If marginal costs vary with output, then an extra restriction is needed to address simultaneity bias and identify the model. By modeling the relationship between demand and markup adjustments, our results reduce the number of restrictions nec-

[2]The international trade literature provides identification results for the special case where markups do not respond to demand shocks (e.g., Feenstra, 1994). We discuss this literature in more detail later. For applications in industrial organization that impose supply-side assumptions, see Thomadsen (2005), Cho et al. (2018), and Li et al. (2021). Thomadsen (2005) assumes no unobserved demand shocks, and Cho et al. (2018) assume no unobserved cost shocks; both are special cases of the covariance restriction approach.

[3]There are interesting historical antecedents to this trade literature. Leamer attributes an early version of his results to Schultz (1928). The identification argument of Feenstra (1994) is also proposed in Leontief (1929). Frisch (1933) provides an important econometric critique.

essary for identification from three to two.

In addition to our formal analysis, we demonstrate how to apply the approach in three distinct empirical settings. In each, we discuss the credibility of covariance restrictions. The applications also illustrate useful extensions, such as how to generate additional covariance restrictions and how to account for increasing marginal costs.

We organize the paper as follows. Section 2 describes the three approaches to identification, using demand and supply assumptions that are commonly employed in empirical studies of imperfect competition. Section 3 develops an identification strategy using covariance restrictions alone. Section 4 provides numerical simulations to compare the three approaches to identification. In small samples, the covariance restriction approach performs well, even when an instrument-based approach suffers from the weak instruments problem. In Section 5, we present three empirical applications.

2 Model

2.1 Data-Generating Process

The model examines supply and demand in a number of markets that can be conceptualized as geographically or temporally distinct. In each market t, there is a set $\mathcal{J}_t = \{0, 1, \ldots, J_t\}$ products available for purchase. The market t is defined by $(\mathcal{J}_t, \boldsymbol{\chi}_t)$, where

$$\boldsymbol{\chi}_t = (\boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}_t, \boldsymbol{\eta}_t)$$

contains characteristics of the product and market. Among these, $\boldsymbol{p}_t = (p_{1t}, \ldots, p_{J_t t})$ are prices, $\boldsymbol{x}_t = (\boldsymbol{x}_{1t}, \ldots, \boldsymbol{x}_{J_t t})$, are non-price product and market characteristics that are observable to the econometrician, and $\boldsymbol{\xi}_t = (\xi_{1t}, \ldots, \xi_{J_t t})$ and $\boldsymbol{\eta}_t = (\eta_{1t}, \ldots, \eta_{J_t t})$ are demand-side and supply-side structural error terms, respectively, that represent unobservable product-level or market-level characteristics. Let each $p_{jt}, \xi_{jt}, \eta_{jt} \in \mathbb{R}$, each $\boldsymbol{x}_{jt} \in \mathbb{R}^K$, and the support of χ_t be χ. We assume that non-price characteristics are exogenous, in that $\mathbb{E}[\xi_{jt} | \boldsymbol{x}_t] = \mathbb{E}[\eta_{jt} | \boldsymbol{x}_t] = 0$ for all $j = 1, \ldots, J_t$, but that prices may be correlated with the structural error terms. Without loss of generality, we assume that $\mathcal{J}_t = \mathcal{J} = \{0, 1, \ldots, J\}$ going forward.

We place restrictions on both demand and supply. For demand, we assume that the quantity sold of each product is determined by $q_{jt} = \sigma_{jt}(\chi_t; \boldsymbol{\theta})$, where each $\sigma_{jt}(\cdot)$ is a differentiable, invertible demand function and $\boldsymbol{\theta}$ is a vector of parameters. We place the following restriction on inverse demand:

$$h_{jt}(\boldsymbol{q}_t, \boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}_t; \boldsymbol{\theta}) \equiv \sigma_{jt}^{-1}(\boldsymbol{q}_t, \boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}_t; \boldsymbol{\theta}) = \beta p_{jt} + \boldsymbol{x}_{jt}' \boldsymbol{\alpha} + \xi_{jt} \tag{1}$$

where $\boldsymbol{q}_t = (q_{1t}, \ldots, q_{Jt})$. We assume downward-sloping demand (i.e., $\beta < 0$). Models that satisfy this restriction are used regularly in the empirical literature of industrial organization. For example, with logit demand, we have $\sigma_{jt}^{-1}(\cdot) \equiv \ln(s_{jt}) - \ln(s_{0t})$, where we follow convention and use markets shares, s_{jt}, in place of quantities.

On the supply side, prices can be decomposed into markups and marginal costs according to

$$p_{jt} = \mu_{jt}(\chi_t; \boldsymbol{\theta}) + mc_{jt}(\chi_t; \boldsymbol{\theta}).$$

We assume that marginal costs are a linear function of covariates:

$$mc_{jt}(\chi_t; \boldsymbol{\theta}) = \boldsymbol{x}_{jt}' \boldsymbol{\gamma} + \eta_{jt}. \tag{2}$$

The restriction on marginal costs in (2) implies that any correlation between prices and the demand-side structural error term arises only through markup adjustments or a correlation between demand-side and supply-side structural error terms. As we develop, if the latter can be ruled out then price endogeneity can be resolved in estimation using the implications of the model for markup adjustments.

Finally, to allow for a broad class of oligopoly models, we assume that markups take the form

$$\mu_{jt}(\chi_t; \boldsymbol{\theta}) = -\frac{1}{\beta} \lambda_{jt}(\boldsymbol{q}_t, \boldsymbol{p}_t, D(\chi_t), \boldsymbol{\eta}_t; \boldsymbol{\theta}), \tag{3}$$

where $D(\chi_t)$ denotes the $J \times J$ matrix of partial derivatives $\left[\frac{\partial \sigma_{kt}(\chi_t)}{\partial p_{lt}}\right]_{k,l}$. For example, with single-product Bertrand pricing, $\mu_{jt}(\cdot) = -\frac{1}{dq_{jt}/dp_{jt}} q_{jt} = -\frac{1}{\beta} \frac{dh_{jt}}{dq_{jt}} q_{jt}$. When demand is logit, $\lambda_{jt}(\cdot) \equiv \frac{dh_{jt}}{dq_{jt}} q_{jt} = \frac{1}{1-s_{jt}}$.

The above restrictions yield the following inverse supply equation:

$$\lambda_{jt}(\boldsymbol{q}_t, \boldsymbol{p}_t, D(\chi_t); \boldsymbol{\theta}) = -\beta p_{jt} + \beta \boldsymbol{x}_{jt}' \boldsymbol{\gamma} + \beta \eta_{jt} \tag{4}$$

Supply is upward sloping (as $\beta < 0$) due to market power: higher prices are needed to induce firms to supply greater quantities, even with marginal costs that are constant in output.

Together, (1) and (4) constitute a system of supply and demand equations that characterizes equilibrium for many empirical oligopoly models. They obtain, for example, with multi-product Bertrand pricing and either nested logit, random coefficients logit, or constant elasticity demands, as well as with some models of Cournot competition and collusion (Appendix A). Thus, the results we obtain are general to a number of widely-used models.

2.2 Three Identification Strategies

The most common approach in applied research is to estimate (1) using instruments taken from the supply-side of the model (Berry and Haile, 2021; Gandhi and Nevo, 2021). For example, if $\alpha^{(k)} = 0$ and $\gamma^{(k)} \neq 0$, where $\alpha^{(k)}$ and $\gamma^{(k)}$ are the k^{th} elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, then the characteristic $x^{(k)}$ is a cost-shifter that satisfies the exclusion restriction (in demand) and the relevance condition (in supply). Similarly, "markup-shifters" that create variation in $\mu_{jt}(\cdot)$ but do not enter (1) can be valid supply-side instruments. These might include functions of other products' characteristics (Berry et al., 1995; Gandhi and Nevo, 2021) or competitive events such as mergers, entry, or exit (e.g., Miller and Weinberg, 2017).

An important property of inverse demand and supply equations, (1) and (4), is that the slopes with respect to price are exact opposites. This provides a second path to identification: estimating the model through (4), using instruments taken from the demand-side of the model. For example, if $\alpha^{(k)} \neq 0$ and $\gamma^{(k)} = 0$, then $x^{(k)}$ is a demand-shifter that satisfies the exclusion restriction (in supply) and the relevance condition (in demand). Markup-shifters also can be valid instruments for this purpose. To our knowledge, the idea that exogenous demand-side variation identifies the parameters of the model—potentially without any exogenous supply-side variation—has not previously been recognized in the literature. Berry et al. (1995) use markup-shifters to jointly estimate the supply and demand equations, and the simultaneous equations framework helps clarify why this improves efficiency.

The third path to identification is to place a direct restriction on the relationship of the structural error terms, along the lines of $Cov(\xi_{jt}, \eta_{jt}) = 0$. Such covariance restrictions have been studied in the context of perfect competition, where the slope of the supply equation is generated by increasing marginal cost functions (e.g., Hausman and Taylor, 1983; Hausman et al., 1987; Matzkin, 2016). We consider the case in which the slope of the supply equation may be affected market power, which we develop in the next section. We offer a formal comparison to the case of perfect competition in Appendix B.

These three approaches all impose an orthogonality condition involving one or more structural error terms. There are nonetheless important distinctions. In particular, supply-side instruments allow for identification with an informal understanding of supply and with nonparametric demand (Berry and Haile, 2014), whereas demand-side instruments and covariance restrictions require a formal supply-side model and parametric assumptions on demand. Thus, when valid supply-side instruments are available, the standard approach to estimation can proceed under weaker assumptions. Another distinction is that the instrument-based approaches require a relevance condition to be satisfied. In contrast, the covariance restriction exploits all of the endogenous variation in the data and there is no "first-stage" relevance condition that must be met; estimation can proceed even when valid instruments are unavailable or weak.

3 Covariance Restrictions

3.1 Identification

We now formalize the identification argument under a covariance restriction between the unobserved demand and marginal cost shocks. Let β^{OLS} denote the probability limit of the OLS estimate of the price coefficient obtained from a regression of $h(\cdot)$ on p and $\boldsymbol{x}$, where the absence of subscripts indicates random variables. Then,

$$\beta^{OLS} \equiv \frac{Cov(p^*, h)}{Var(p^*)} = \beta + \frac{Cov(p^*, \xi)}{Var(p^*)}, \tag{5}$$

where p^* is the component of price orthogonal to $\boldsymbol{x}$, i.e., $p^* = p - \boldsymbol{x}\mathbb{E}[\boldsymbol{x}'\boldsymbol{x}]^{-1}\mathbb{E}[\boldsymbol{x}'p]$. The residuals of a regression of p on $\boldsymbol{x}$ provide approximate realizations of p^*.

Our main identification connects the price coefficient to the covariance of the structural error terms and empirical variation present in the data:

Proposition 1. *The probability limit of the OLS estimate can be written as a function of the β, the residuals from an OLS regression, prices and quantities, and a covariance term:*

$$\beta^{OLS} = \beta - \frac{1}{\beta + \frac{Cov(p^*,\lambda)}{Var(p^*)}} \frac{Cov\left(\xi^{OLS}, \lambda\right)}{Var(p^*)} + \beta \frac{1}{\beta + \frac{Cov(p^*,\lambda)}{Var(p^*)}} \frac{Cov(\xi, \eta)}{Var(p^*)} \tag{6}$$

Therefore, β solves the following quadratic equation:

$$\begin{aligned} 0 = \; & \beta^2 \\ & + \left(\frac{Cov\left(p^*, \lambda\right)}{Var(p^*)} - \beta^{OLS} + \frac{Cov(\xi, \eta)}{Var(p^*)} \right) \beta \\ & + \left(-\beta^{OLS} \frac{Cov\left(p^*, \lambda\right)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \lambda\right)}{Var(p^*)} \right) \end{aligned} \tag{7}$$

All proofs are in Appendix D. Aside from β and $Cov(\xi, \eta)$, all of the terms in (7) can be constructed from data. In particular, (7) links the reduced-form OLS coefficient to the true causal parameter.

There are at most two solutions for β for any given value of $Cov(\xi, \eta)$. Further, in most empirical models, β is likely to be the lower root of (7). Our second identification result provides formal conditions under which this is the case:

Proposition 2. *The parameter β is the lower root of* (7) *if and only if*

$$-\frac{1}{\beta} Cov(\xi, \eta) \leq Cov\left(p^*, \eta - \frac{1}{\beta}\xi\right) \tag{8}$$

and, furthermore, β is the lower root of (7) *if*

$$0 \leq \beta^{OLS} Cov\left(p^*, \lambda\right) + Cov\left(\xi^{OLS}, \lambda\right). \tag{9}$$

In the first condition, it is helpful to think of $-\frac{1}{\beta}\xi$ as the demand-side structural error term, scaled so that units are equivalent to those of marginal costs (and price). If $Cov(\xi, \eta) = 0$, the condition holds as long as prices tend to increase with demand and marginal costs, as occurs in most empirical models. Thus, β is likely the lower root of (7) in most applications. The second condition is derived using properties of the quadratic formula. Because the terms in (9) are constructed from data, the sufficient condition may be estimated and used to test (and possibly reject) the null hypothesis that multiple negative roots exist. Henceforth, we assume that β is the lower root of (7).

We have shown that the unknown price coefficient β can be solved for as a function of $Cov(\xi, \eta)$. Because of this, we focus on covariance restrictions in first moments, though higher-order restrictions can work. Estimation can use the method-of-moments or employ the analytical solution to equation (7). See Appendix C.1.

The empirical variation that identifies β is the relative variance in quantity and price in the data. For the case in which $Cov(\xi, \eta) = 0$, we obtain a formal result:

Proposition 3. *If $Cov(\xi, \eta) = 0$, then a first-order approximation to lower root of* (7) *is*

$$\tilde{\beta}^{Approx} = -\sqrt{\frac{Var(h^*)}{Var(p^*)}} \tag{10}$$

where h^* is the component of h orthogonal to $\boldsymbol{x}$, i.e., $h^* = h - \boldsymbol{x}\mathbb{E}[\boldsymbol{x}'\boldsymbol{x}]^{-1}\mathbb{E}[\boldsymbol{x}'h]$. The residuals of a regression of h on $\boldsymbol{x}$ provide approximate realizations of h^*.

Intuition for this result can be gleaned from the simultaneous equations representation of the model (Section 2.2), in which β determines the slope of both demand and supply. A large β corresponds to a flatter inverse demand schedule (i.e., price sensitive consumers) and a flatter inverse supply schedule (i.e., less market power). Uncorrelated shifts in such schedules tend to generate more variation in quantity than price. By contrast, a small β corresponds to steeper inverse demand and inverse supply schedules, such that uncorrelated shifts generate more variation in price than quantity. Connecting these observations formally generates an approximation of the lower root based on the ratio of variances. Estimation with a covariance restriction converts the endogenous variation in prices and quantity into consistent estimates.

3.2 Analysis of Bounds

In settings where the use of covariance restrictions to point identify demand may be inappropriate, it may nonetheless be possible to sign $Cov(\xi, \eta)$. This allows for bounds to be place on β. The reason is that there is a one-to-one mapping between the value of $Cov(\xi, \eta)$ and the lower root of equation (7):

Lemma 1. ***(Monotonicity)*** *Under assumptions 1 and 2, a valid lower root of equation* (7) *(i.e., one that is negative) is decreasing in* $Cov(\xi, \eta)$*. The range of the function is* $(0, -\infty)$.

Thus, if higher quality products are more expensive to produce ($Cov(\xi, \eta) \geq 0$) or firms invest to lower the marginal costs of their best-selling products ($Cov(\xi, \eta) \leq 0$), then one-sided bounds can be placed on β. More generally, let $r(m)$ be the lower root of the quadratic in (7), evaluated at $Cov(\xi, \eta) = m$. Then $Cov(\xi, \eta) \geq m$ produces $\beta \in (-\infty, r(m)]$, and $Cov(\xi, \eta) \leq m$ produces $\beta \in [r(m), 0)$. The lower root, $r(m)$, can be estimated with the method-of-moments.[4]

Even if prior knowledge does not sign $Cov(\xi, \eta)$, some values may be possible to rule out if they imply that the data are incompatible with the model. To see why, represent the quadratic of (7) as $az^2 + bz + c$, keeping in mind that one root is $\beta < 0$. As $a = 1$, the quadratic forms a $\cup$-shaped parabola. If $c < 0$ then the existence of a negative root is guaranteed. However, if $c > 0$ then b must be positive and sufficiently large for a negative root to exist. This places restrictions on $Cov(\xi, \eta)$. From the monotonicity result (Lemma 1), we can use the excluded values of $Cov(\xi, \eta)$ from this result to rule out values of β.

We now state the result formally:

Proposition 4. *The model and data may bound* $Cov(\xi, \eta)$ *from below. The bound is given by:*

$$Cov(\xi, \eta) > Var(p^*)\beta^{OLS} - Cov(p^*, \lambda) + 2Var(p^*)\sqrt{\left(-\beta^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \lambda\right)}{Var(p^*)}\right)}$$

The bound exists if and only if the term inside the radical is non-negative. Further, through (7)*, this lower bound on* $Cov(\xi, \eta)$ *provides an upper bound on* β.

3.3 Assessment

The covariance restrictions approach to estimation can be both useful and credible when exogenous variation in marginal costs exists, yet the cost-shifters that give rise to the variation are unobserved by the econometrician. As an illustrative example, Döpper et al. (2021) estimate demand for consumer products using covariance restrictions. Given the specification of the marginal cost function, the structural error includes two sources of variation that have been exploited as instruments in recent research: product-specific changes in input costs (Backus et al., 2021) and product-specific changes in distribution costs (Miller and Weinberg, 2017). Döpper et al. (2021) employ a rich set of fixed effects to better isolate exogenous variation in costs, and they obtain demand elasticities that are similar to those reported in the literature.

Credibility also depends on whether the supply-side structural error term contains additional, confounding variation. In some applications, it can be possible to absorb the most

[4] Nevo and Rosen (2012) develop similar bounds for estimation with imperfect instruments, defined as instruments that are less correlated with the structural error term than the endogenous regressor.

obvious sources of confounding variation using fixed effects or other controls. To make this explicit, suppose that the demand and cost functions are given by:

$$\begin{aligned} h(q_{jt}) &= \beta p_{jt} + \boldsymbol{x}'_{jt}\boldsymbol{\alpha} + D_j + F_t + E_{jt} \\ mc_{jt} &= \boldsymbol{x}'_{jt}\boldsymbol{\gamma} + U_j + V_t + W_{jt} \end{aligned}$$

where, again, the subscripts j and t refer to products and markets, respectively. The unobserved error terms are $\xi_{jt} = D_j + F_t + E_{jt}$ and $\eta_{jt} = U_j + V_t + W_{jt}$. If products with higher quality have higher marginal costs then $Cov(U_j, D_j) > 0$. The econometrician can account for the relationship by estimating D_j for each firm; the residual $\xi^*_{jt} = \xi_{jt} - D_j$ is uncorrelated with U_j. Similarly, if costs are higher (or lower) in markets with high demand then $Cov(F_t, V_t) \neq 0$, but market fixed effects can be incorporated to absorb the confounding variation. Thus, it can be possible to isolate components of the error terms over which a covariance restriction is credible.

4 Small-Sample Performance

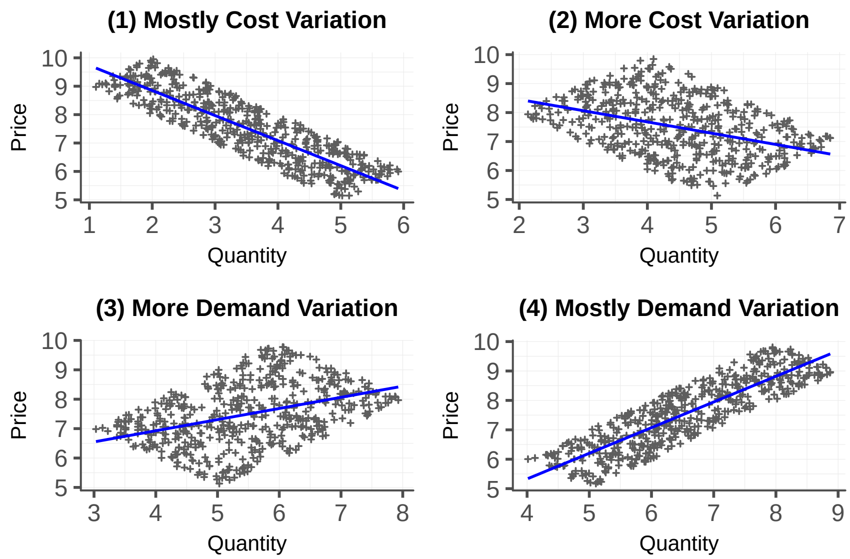
We use Monte Carlo simulations to examine the small sample performance of the three estimation strategies discussed in Section 2.2. We focus on the distinction that instrument-based approaches require a valid instrument, whereas the covariance restriction approach uses all of the endogenous variation in the data. For simplicity, we examine a monopolist with a marginal cost of $mc_t = \eta_t$ and a linear demand schedule of $q_t = 10 - p_t + \xi_t$. Thus, $\beta = -1$. We let ξ_t and η_t have independent uniform distributions. The four specifications that we consider are: (i) $\xi \sim U(0, 2)$ and $\eta \sim U(0, 8)$, (ii) $\xi \sim U(0, 4)$ and $\eta \sim U(0, 6)$, (iii) $\xi \sim U(0, 6)$ and $\eta \sim U(0, 4)$, and (iv) $\xi \sim U(0, 8)$ and $\eta \sim U(0, 2)$. Moving from (i) to (iv), demand-side variation increases and supply-side variation decreases. Figure 1 illustrates the different samples.

We consider sample sizes of 25, 50, 100, and 500 observations. For each specification and sample size, we randomly draw 10,000 datasets, and with each we estimate the model with a covariance restriction, $Cov(\xi, \eta) = 0$, with a supply-side instrument, and with a demand-side instrument. We use η_t as the supply-side instrument and ξ_t as the demand-side instrument. The estimate of β under the covariance restriction is simply $-\sqrt{Var(q)/Var(p)}$ because the first-order approximation of (10) is exact in this context. In this experimental design, all three approaches rely on the same orthogonality condition: $\mathbb{E}[\xi'_t \eta_t] = 0$.

Table 1 summarizes the results.[5] Panel (a) shows that the covariance restriction approach to estimation yields estimates that are consistently close to the true value. Panel (b) shows that, with supply-side instruments, small sample bias becomes substantial with smaller datasets and less variance in the cost shock. This is due to a weak instrument—for example, the mean first-

[5]To avoid outliers, arising from the weak instrument problem, we bound the estimates of β on the range $[-100, 100]$. For specifications that suffer from weak instruments, this will bias the standard errors toward zero. This affects specifications where the estimated standard error is greater than one, i.e., in 9 of 48 specifications.

Figure 1: Prices and Quantities in the Monopoly Model



Notes: This figure displays equilibrium prices and quantities under four different specifications for the distribution of unobserved shocks to demand and marginal costs. The line in each figure indicates the slope obtained by OLS regression.

stage F-statistics in specification (iv) are 2.6, 4.2, 7.3, and 32.6 for markets with 25, 50, 100, and 500 observations, respectively. Panel (c) shows that, with demand-side instruments, small sample bias becomes substantial with smaller datasets and less variance in the demand shock, which also is due to a weak instruments problem. Thus, in settings where instruments perform poorly, a covariance restriction may still provide a precise estimate because it exploits all of the endogenous price and quantity variation in the data.

Of course, other differences between the estimators also can be important. Instrument-based approaches require an orthogonality condition between one observable and one unobservable, whereas the covariance restriction approach require an orthogonality condition between two unobservables; this can complicate an assessment of credibility. Estimation with a covariance restriction and demand-side instruments also can produce biased estimates due to supply-side misspecification. Using simulations, we find that covariance restrictions—which also incorporate demand—can mitigate this potential bias. See Appendix C.2.

Table 1: Small-Sample Properties: Relative Variation in Demand and Supply Shocks

(a) Covariance Restrictions

Observations	(i) $Var(\eta) \gg Var(\xi)$		(ii) $Var(\eta) > Var(\xi)$		(iii) $Var(\eta) < Var(\xi)$		(iv) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.098)	-1.017	(0.201)	-1.018	(0.206)	-1.005	(0.099)
50	-1.001	(0.068)	-1.008	(0.136)	-1.007	(0.135)	-1.001	(0.068)
100	-1.001	(0.047)	-1.003	(0.094)	-1.004	(0.093)	-1.001	(0.047)
500	-1.000	(0.021)	-1.001	(0.041)	-1.001	(0.042)	-1.000	(0.021)

(b) Supply Shifters (IV-1)

Observations	(i) $Var(\eta) \gg Var(\xi)$		(ii) $Var(\eta) > Var(\xi)$		(iii) $Var(\eta) < Var(\xi)$		(iv) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.105)	-1.039	(0.303)	-1.310	(2.629)	-0.899	(13.921)
50	-1.001	(0.072)	-1.018	(0.201)	-1.113	(1.135)	-1.392	(10.890)
100	-1.001	(0.050)	-1.008	(0.138)	-1.048	(0.332)	-1.432	(5.570)
500	-1.000	(0.022)	-1.001	(0.060)	-1.009	(0.138)	-1.061	(0.411)

(c) Demand Shifters (IV-2)

Observations	(i) $Var(\eta) \gg Var(\xi)$		(ii) $Var(\eta) > Var(\xi)$		(iii) $Var(\eta) < Var(\xi)$		(iv) $Var(\eta) \ll Var(\xi)$	
25	-0.881	(12.794)	-1.295	(3.087)	-1.040	(0.312)	-1.006	(0.106)
50	-1.448	(10.980)	-1.112	(0.596)	-1.016	(0.198)	-1.001	(0.073)
100	-1.597	(5.837)	-1.045	(0.333)	-1.009	(0.136)	-1.001	(0.050)
500	-1.070	(0.414)	-1.008	(0.137)	-1.002	(0.060)	-1.000	(0.022)

Notes: Results are based on 10,000 simulations of data for each specification and number of observations. The demand curve is $q_t = 10 - p_t + \xi_t$, so $\beta = -1$, and marginal costs are $c_t = \eta_t$. IV-1 estimates are calculated using 2SLS with marginal costs (η) as an instrument in the demand equation. Analogously, IV-2 estimates are calculated using 2SLS with demand shocks (ξ) as an instrument in the supply equation. In specification (i), $\xi \sim U(0,2)$ and $\eta \sim U(0,8)$. In specification (ii), $\xi \sim U(0,4)$ and $\eta \sim U(0,6)$. In specification (iii), $\xi \sim U(0,6)$ and $\eta \sim U(0,4)$. In specification (iv), $\xi \sim U(0,8)$ and $\eta \sim U(0,2)$.

5 Empirical Applications

5.1 Ready-to-Eat (RTE) Cereals

We choose RTE cereals for our first application because, with panel data and appropriate fixed effects, a covariance assumption appears credible, for reasons that we explain below. We use the pseudo-real cereals data of Nevo (2000) and compare estimates obtained with a covariance restriction to those obtained with the instruments provided with the data.[6]

The model features random coefficients logit demand and Bertrand competition. The indirect utility that consumer i receives from product j in region m and period t is given by

$$u_{ijmt} = \boldsymbol{x}_j' \boldsymbol{\alpha}_i^* + \beta_i^* p_{jmt} + \xi_j + \Delta\xi_{jmt} + \epsilon_{ijmt}$$

where ξ_j are product fixed effects, $\Delta\xi_{jmt}$ is the demand-side structural error term, and ϵ_{ijmt} is a logit shock. The indirect utility provided by the outside good, $j = 0$, is $u_{i0mt} = \epsilon_{i0mt}$. The consumer-specific coefficients take the form $[\boldsymbol{\alpha}_i^* \ \beta_i^*] = [\boldsymbol{\alpha} \ \beta] + \boldsymbol{\Pi}\boldsymbol{D}_i + \boldsymbol{\Sigma}\boldsymbol{\nu}_i$ where $\boldsymbol{D}_i$ is a vector of observed demographics and $\boldsymbol{\nu}_i$ is vector of unobserved demographics with standard normal distributions. Marginal costs are given by

$$mc_{jmt} = \eta_j + \Delta\eta_{jmt}$$

where η_j is a product fixed effect, and $\Delta\eta_{jmt}$ is the supply-side structural error term.

We use the covariance restriction $Cov(\Delta\xi_{jmt}, \Delta\eta_{jmt}) = 0$ in estimation. With our specification, the supply-side structural error term incorporates some of the cost-shifter instruments that have been used in the recent literature, including time-varying, product-specific shipping costs (Miller and Weinberg, 2017) and the time-varying prices of product-specific ingredients (Backus et al., 2021). Given the fixed effects, these cost-shifters can be conceptualized as providing the variation that is exploited in estimation. Furthermore, it is reasonable to think that marginal costs may be roughly constant in output for consumer products, as is often maintained in the literature (Villas-Boas, 2007; Chevalier et al., 2003; Hendel and Nevo, 2013; Miller and Weinberg, 2017; Backus et al., 2021).

The parameters for estimation include $(\beta, \boldsymbol{\alpha})$, as in our baseline model from Section 2, and also $(\boldsymbol{\Pi}, \boldsymbol{\Sigma})$. Therefore, additional identifying assumptions are needed. Some recent applications use so-called "micro-moments" constructed from the observed behavior of individual consumers (e.g., Backus et al., 2021; Döpper et al., 2021), following the theoretical results of Berry and Haile (2020). An alternative strategy is to use instruments constructed from competitor characteristics (e.g., Berry et al., 1995; Gandhi and Houde, 2020). As neither option is available to us given the data and specification, we extend the covariance assumption such

[6] See also Dubé et al. (2012), Knittel and Metaxoglou (2014), and Conlon and Gortmaker (2020). Our implementation adds time fixed effects to the "restricted" specification of Conlon and Gortmaker (2020).

Table 2: Point Estimates for Ready-to-Eat Cereal

(a) Available Instruments

Variable	Means	Standard Deviations	Interactions with Demographics: Income	Age	Child
Price	-32.019	1.803	4.187	–	11.755
	(2.304)	(0.920)	(4.638)		(5.198)
Constant	–	0.120	3.101	1.198	–
		(0.163)	(1.105)	(1.048)	
Sugar	–	0.004	-0.190	0.028	–
		(0.012)	(0.035)	(0.032)	
Mushy	–	0.086	1.495	-1.539	–
		(0.193)	(0.648)	(1.107)	

(b) Covariance Restrictions

Variable	Means	Standard Deviations	Interactions with Demographics: Income	Age	Child
Price	-36.230	1.098	14.345	–	26.906
	(1.122)	(1.067)	(1.677)		(1.384)
Constant	–	0.051	-0.156	1.072	–
		(0.230)	(0.286)	(0.240)	
Sugar	–	0.003	-0.084	-0.004	–
		(0.014)	(0.018)	(0.010)	
Mushy	–	0.130	0.301	-0.845	–
		(0.162)	(0.196)	(0.103)	

Notes: This table reports point estimates for the random-coefficients logit demand system estimated using the Nevo (2000) dataset. Panel (a) employs the available instruments. Panel (b) employs covariance restrictions.

that $Cov(\Delta\xi_{jmt}, \Delta\eta_{kmt}) = 0$ for all j, k. The restrictions are valid if the demand shock of each product is orthogonal to its own marginal cost shock and those of all other products. As there are 24 products in each market, the full covariance matrix of demand and cost shocks provides sufficient moments to estimate the 12 nonlinear parameters in the specification.

Table 2 summarizes the results of estimation based on the instruments (panel (a)) and covariance restrictions (panel (b)). Both identification strategies yield similar mean own-price demand elasticities: -3.70 with instruments and -3.61 with covariance restrictions. Overall, the different approaches produce similar patterns for the coefficients. Most of the point estimates under covariance restrictions fall in the 95 percent confidence intervals implied by the specification with instruments, including that of the mean price parameter. The standard errors are noticeably smaller with covariance restrictions, which likely reflects that the covariance restrictions approach to estimation more fully exploits the variation that is present in the data. We conclude that in this setting—where a covariance restriction appears credible—estimation with covariance restrictions and with instruments indeed produce similar results.

5.2 The Portland Cement Industry

Our second empirical application considers a setting in which marginal costs increase with output. Given the marginal cost specification in (2), the baseline model accommodates increasing marginal costs only if $Cov(\xi, \eta) > 0$. However, if the upward-sloping component of marginal costs can be modeled more explicitly, then the covariance restriction approach to estimation may remain valid. To illustrate, we consider the setting and data of Fowlie et al. (2016) ["FRR"], which examines market power in the cement industry.

The model features Cournot competition among cement plants facing capacity constraints. Let the market demand curve in region r and year t have a logit form:

$$h_{rt}(Q_{rt}) \equiv \ln(Q_{rt}) - \ln(M_r - Q_{rt}) = \boldsymbol{x}'_{rt}\boldsymbol{\alpha} + \beta p_{rt} + \xi_{rt} \tag{11}$$

where $Q_{rt} = \sum_{j \in \mathcal{J}} q_{jrt}$ is total quantity and M_r is the "market size" of the region.[7] Further, we allow marginal costs to vary with quantity according to

$$mc_{jrt} = \boldsymbol{x}'_{jrt}\boldsymbol{\gamma} + g_{jrt}(q_{jrt}) + \eta_{jrt} \tag{12}$$

In particular, we follow FRR and assume that that g is a "hockey stick" function, $g_{jrt}(q_{jrt}) = 2\psi 1\{q_{jrt}/k_{jr} > 0.9\}(q_{jrt}/k_{jr} - 0.9)$, where k_{jr} and q_{jrt}/k_{jr} are capacity and utilization, respectively. Marginal costs are constant if utilization is less than 90%, and increasing linearly at rate determined by $\psi \geq 0$ otherwise.

As in our baseline model, correlation between price and the demand-side structural error term can arise both due to markup adjustments and the effect of demand on marginal costs. However, due to the presence of $g_{jrt}(\cdot)$ in the cost function, the latter channel exists even under the covariance restriction $Cov(\xi_{rt}, \bar{\eta}_{rt}) = 0$, where $\bar{\eta}_{rt} = \frac{1}{J}\eta_{jrt}$. If $g_{jrt}(\cdot)$ is known or can identified with additional moments, then the covariance restriction is sufficient to resolve price endogeneity, as the model informs the markup adjustments. In estimation, we maintain the covariance restriction at the market level.

Our demand and supply framework of equations (1) and (4) readily admit Cournot competition. As only market-level price and costs measures are observed, one must use the mean firm-level quantity $\overline{q}_{rt} = \frac{1}{J}Q_{rt}$ to obtain an expression for mean market-level markups and λ. In particular, when firms compete in quantities, we obtain $\lambda_{rt} = \frac{1}{J}\frac{dh}{dq}Q_{rt}$. We now extend our main identification result to allow for increasing marginal costs, where $\overline{g}_{rt} = \frac{1}{J}\sum_{j=1}^{J} g_{jrt}(q_{jrt})$.

Proposition 5. *With increasing marginal costs following equation* (12)*, the OLS estimates of β is:*

$$\beta^{OLS} = \beta - \frac{Cov\left(\xi, \lambda\right)}{Var(p^*)} + \frac{Cov(\xi, \overline{\boldsymbol{g}})}{Var(p^*)}$$

[7] We use logit demand rather than the constant elasticity demand of FRR to allow for adjustable markups. The 2SLS results are unaffected by the choice. In our implementation, we assume $M_r = 2 \times \max_t\{Q_{rt}\}$.

Therefore, β solves the following quadratic equation:

$$\begin{aligned}0 = &\left(1 - \frac{Cov(p^*, \overline{\boldsymbol{g}})}{Var(p^*)}\right)\beta^2 \\ &+ \left(\frac{Cov\,(p^*, \lambda)}{Var(p^*)} + \frac{Cov(\xi, \overline{\boldsymbol{\eta}})}{Var(p^*)} - \beta^{OLS} + \frac{Cov(p^*, \overline{\boldsymbol{g}})}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\hat{\xi}^{OLS}, \overline{\boldsymbol{g}})}{Var(p^*)}\right)\beta \\ &+ \left(-\frac{Cov\,(p^*, \lambda)}{Var(p^*)}\beta^{OLS} - \frac{Cov\,(\xi^{OLS}, \lambda)}{Var(p^*)}\right)\end{aligned}$$

In our implementation, we assume that $\psi = 800$, such that our $g_{jrt}(\cdot)$ function is close to what is used in Fowlie et al. (2016). Whether the covariance restrictions approach to estimation produces consistent estimates then hinges on the assumption $Cov(\xi_{rt}, \bar{\eta}_{rt}) = 0$. Here, whether the covariance restriction is reasonable depends primarily on the relationship between construction activity (unobserved demand) and the prices of coal and electricity (unobserved supply costs). In this context, there is a theoretical basis for orthogonality: for example, if coal suppliers have limited market power and roughly constant marginal costs, then coal prices should not respond much to demand for cement. Indeed, this is the identification argument of FRR, as coal and electricity prices are included in the set of excluded instruments.

We find that the covariance restrictions approach yields a demand elasticity of -1.15, with a standard error of 0.18.[8] This is nearly identical to the 2SLS estimate of -1.07 (standard error 0.19) that we obtain using the FRR instruments: coal prices, natural gas prices, electricity prices, and wage rates. This similarity reflects, we believe, that the identifying assumptions are actually quite similar, with the main difference being whether the cost shifters are treated as observed (2SLS) or unobserved (covariance restrictions). By contrast, we obtain a demand elasticity of -0.47 (standard error of 0.15) using OLS. And, if we use the covariance restriction without accounting for the presence of $g_{jrt}(\cdot)$, we obtain a demand elasticity of -0.90 (standard error of 0.13), which is in between the OLS and 2SLS estimates.

5.3 The Airline Industry

In our third empirical application, we examine demand for airline travel using the setting and data of Aguirregabiria and Ho (2012) ["AH"].[9] The economics of the industry suggest that the covariance restriction $Cov(\xi, \eta) = 0$ would not be credible. The reason is that airlines bear an opportunity cost when they sell a seat because it can no longer be sold at a higher price to another passenger (Williams, 2021). Thus, all else equal, greater demand generates more

[8] There are 520 region-year observations over 1984-2009. The demand specification includes region fixed effects. We obtain bootstrapped standard errors based on 200 random samples constructed by drawing from the data with replacement.

[9] We thank Victor Aguirregabiria for providing the data. Replication is not exact because the sample differs somewhat from what is used in the AH publication and because we employ a different set of fixed effects in estimation.

higher marginal costs, inclusive of the opportunity cost. Absent a model of these opportunity costs, it is difficult to achieve point identification using the covariance restriction. Instead, we use the industry to illustrate the bounds approach to identification, with multiple sets of bounds.

The AH model features differentiated-products Bertrand competition among firms facing a nested logit demand system. Products are classified into the following groups: nonstop flights, one-stop flights, and the outside good. The nested logit demand system can be expressed as

$$\ln s_{jmt} - \ln s_{0mt} - \sigma \ln \overline{s}_{jmt|g} = \beta p_{jmt} + x'_{jmt}\alpha + \xi_{jmt} \tag{13}$$

where s_{jmt} is the market share of product j in market m in period t. The conditional market share, $\overline{s}_{j|g} = s_j / \sum_{k \in g} s_k$, is the the choice probability of product j given that its "group" of products, g, is selected. The outside good is indexed as $j = 0$. Higher values of σ increase within-group consumer substitution relative to across-group substitution.[10]

Given the role of opportunity costs in the industry, we assume $Cov(\xi_{jmt}, \eta_{jmt}) \geq 0$. Under that assumption, we reject values (β, σ) that produce a negative correlation in product-specific shocks. We combine this with model-based bounds (Section 3.2). Finally, if the correlation in product-level shocks is weakly positive, it is reasonable to assume that the group-level shocks are also weakly positive, through a similar deduction. Thus, we apply the group-level inequality

$$E_{gmt}[\overline{\xi}_{gmt} \cdot \overline{\eta}_{gmt}] \geq 0, \tag{14}$$

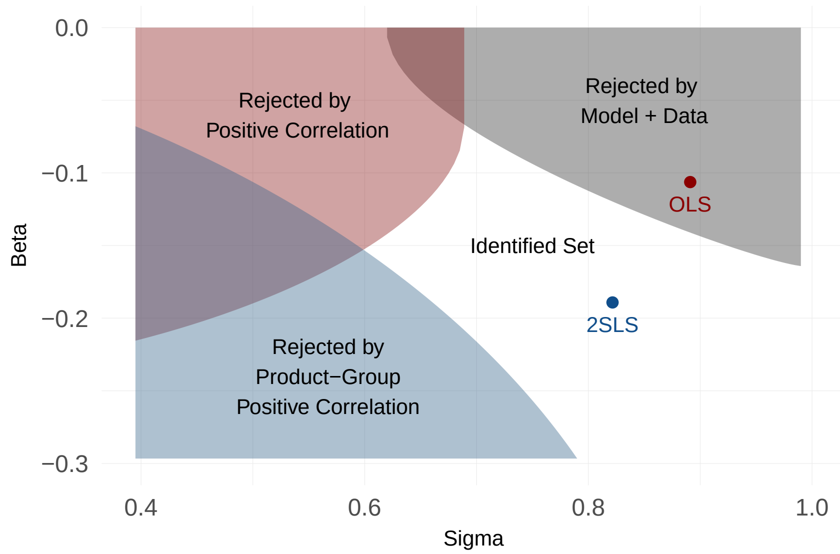
where $\overline{\xi}_{gmt} = \frac{1}{|g|} \sum_{j \in g} \xi_{jmt}$ and $\overline{\eta}_{gmt} = \frac{1}{|g|} \sum_{j \in g} \eta_{jmt}$ are the mean demand and cost shocks within a group-market-period. By rejecting parameter values that fail to generate the data or that deliver negative correlations between costs and demand, we narrow the identified set.

Figure 2 displays the rejected regions based on the model and our assumptions on unobserved shocks. The gray region corresponds to the parameter values rejected by the model-based bounds; the model itself rejects some values of β if $\sigma \geq 0.62$. As σ becomes larger, a more negative β is required to rationalize the data. The dark red region corresponds to parameter values that generate negative correlation between demand and supply shocks. The region is rejected under the prior that $Cov(\xi_{jmt}, \eta_{jmt}) \geq 0$. The dark blue region provides the corresponding set for the prior $Cov(\overline{\xi}_{gmt}, \overline{\eta}_{gmt}) \geq 0$ and is similarly rejected.

The three regions overlap, but no region is a subset of another. The non-rejected values provide the identified set. We rule out values of σ less than 0.599 for any value of β, as these lower values cannot generate positive correlation in both product-level and product-group-level shocks. Similarly, we obtain an upper bound on β of -0.067 across all values of σ. For context,

[10] The covariates include an indicator for nonstop itineraries, the distance between the origin and destination cities, and a measure of the airline's "hub sizes" at the origin and destination cities. We also include airline fixed effects and route×quarter fixed effects. Market size, which determines the market share of the outside good, is equal to the total population in the origin and destination cities.

Figure 2: Analysis of Bounds in the Airlines Industry



Notes: This figure displays candidate parameter values for (σ, β). The gray region indicates the set of parameters that cannot generate the observed data from the assumptions of the model. The red region indicates the set of parameters that generate $Cov(\xi, \eta) < 0$, and the blue region indicates parameters that generate $Cov(\overline{\xi}, \overline{\eta}) < 0$. The identified set is obtained by rejecting values in the above regions under the assumption of (weakly) positive correlation. For context, the OLS and the 2SLS estimates are plotted. The parameter σ can only take values on $[0, 1)$.

we plot the OLS and the 2SLS estimates in Figure 2. The OLS estimate falls in a rejected region and can be ruled out by the model alone. The 2SLS estimate falls within the identified set. This result is not mechanical, as these point estimates are generated with non-nested assumptions.

6 Conclusion

Our objective has been to evaluate the identifying power of supply-side assumptions in models of imperfect competition. Invoking the supply model in estimation expands the set of restrictions that obtain identification. In particular, we demonstrate that demand-side instruments and covariance restrictions between unobserved demand and cost shocks can resolve price endogeneity, providing a path for research in areas where strong supply-side instruments are unavailable. We provide three empirical applications to demonstrate how covariance restrictions can be applied and evaluated.

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Appendix

A Demand System Applications

The demand system of equation (1) is sufficiently flexible to nest monopolistic competition with linear demands (e.g., as in the motivating example) and the discrete choice demand models that support much of the empirical research in industrial organization. The demand assumption can also be modified to allow for semi-linearity in a transformation of prices, $f(p_{jt})$:

$$h_{jt}(\boldsymbol{q}_t, \boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}_t; \boldsymbol{\theta}) = \beta f(p_{jt}) + \boldsymbol{x}'_{jt}\boldsymbol{\alpha} + \xi_{jt} \tag{A.1}$$

Under this modified assumption, it is possible to employ a method-of-moments approach to estimate the structural parameters. When $f(p_{jt}) = \ln p_{jt}$, it is straightforward to extend our analytical identification results, under the modified assumptions that ξ is orthogonal to $\ln X$ and that $\ln \eta$ and ξ are uncorrelated. To see this, note that the optimal price for these demand systems takes the form $p_{jt} = \mu_{jt} c_{jt}$, where μ_{jt} is a markup that reflects demand parameters and (in general) demand shocks. It follows that the probability limit of an OLS regression of h on $\ln p$ is given by:

$$\beta^{OLS} = \beta - \frac{1}{\beta}\frac{Cov(\ln \mu, \xi)}{Var(\ln p^*)} + \frac{Cov(\ln \eta, \xi)}{Var(\ln p^*)}. \tag{A.2}$$

Therefore, the results developed in this paper extend in a straightforward manner.

We provide some typical examples below. We then show how multi-product firms fit within the framework of Section 2.

A.1 Nested Logit Demand

Following the exposition of Cardell (1997), let the firms be grouped into $g = 0, 1, \ldots, G$ mutually exclusive and exhaustive sets, and denote the set of firms in group g as $\mathscr{J}_g$. An outside good, indexed by $j = 0$, is the only member of group 0. Then the left-hand-side of equation (1) takes the form

$$h_{jt}(\boldsymbol{q}_t, \boldsymbol{p}_t, \boldsymbol{x}_t, \boldsymbol{\xi}_t; \boldsymbol{\theta}) \equiv \ln(s_j) - \ln(s_0) - \sigma \ln(\bar{s}_{j|g})$$

where $\bar{s}_{j|g,t} = \sum_{j \in \mathscr{J}_g} \frac{s_{jt}}{\sum_{j \in \mathscr{J}_g} s_{jt}}$ is the market share of firm j within its group. The parameter $\sigma \in [0, 1)$ determines the extent to which consumers substitute disproportionately among firms within the same group. If $\sigma = 0$ then the logit model obtains. We can construct the markup by calculating the total derivative of h with respect to s. For single-product firms at the Bertrand-Nash equilibrium,

$$\lambda_{jt} = \frac{dh_{jt}}{ds_{jt}} s_{jt} = \frac{1}{\frac{1}{1-\sigma} - s_{jt} - \frac{\sigma}{1-\sigma}\bar{s}_{j|g,t}}.$$

In our third application, we use the nested logit model to estimate bounds on the structural parameters (Section 5.3).

A.2 Random Coefficients Logit Demand

Modifying slightly the notation of Berry (1994), let the indirect utility that consumer $i = 1, \ldots, I$ receives from product j be

$$u_{ij} = \beta p_j + \boldsymbol{x}_j' \boldsymbol{\alpha} + \xi_j + \left[\sum_k x_{jk} \sigma_k \zeta_{ik} \right] + \epsilon_{ij}$$

where x_{jk} is the kth element of x_j, ζ_{ik} is a mean-zero consumer-specific demographic characteristic, and ϵ_{ij} is a logit error. We have suppressed market subscripts for notational simplicity. Decomposing the right-hand side of the indirect utility equation into $\delta_j = \beta p_j + \boldsymbol{x}_j' \boldsymbol{\alpha} + \xi_j$ and $\phi_{ij} = \sum_k x_{jk} \sigma_k \zeta_{ik}$, the probability that consumer i selects product j is given by the standard logit formula

$$s_{ij} = \frac{\exp(\delta_j + \phi_{ij})}{\sum_k \exp(\delta_k + \phi_{ik})}.$$

Integrating yields the market shares: $s_j = \frac{1}{I} \sum_i s_{ij}$. Berry et al. (1995) prove that a contraction mapping recovers, for any candidate parameter vector $\tilde{\sigma}$, the vector $\delta(s, \tilde{\sigma})$ that equates these market shares to those observed in the data. This "mean valuation" is $h(\cdot)$ in our notation. An estimator can be applied to recover the price coefficient, again taking some $\tilde{\sigma}$ as given. For single-product firms at the Bertrand-Nash equilibrium, λ_j takes the form

$$\lambda_j = \frac{dh_j}{ds_j} s_j = \frac{s_j}{\frac{1}{I} \sum_i s_{ij}(1 - s_{ij})}.$$

Thus, with the uncorrelatedness assumption the linear parameters can be recovered given the candidate parameter vector $\tilde{\sigma}$. We demonstrate how to estimate these parameters using additional covariance restrictions in our first application (Section 5.1).

A.3 Constant Elasticity Demand

A special case that is often estimated in empirical work is when h and $f(p)$ are logarithms. With the modified demand assumption of equation (A.1), the constant elasticity of substitution (CES) demand model of Dixit and Stiglitz (1977) can be incorporated:

$$\ln(q_{jt}/q_t) = \alpha + \beta \ln\left(\frac{p_{jt}}{\Pi_t}\right) + \xi_{jt}$$

where q_t is an observed demand shifter, Π_t is a price index, and β provides the constant elasticity of demand. This model is often used in empirical research on international trade and firm productivity (e.g., De Loecker, 2011; Doraszelski and Jaumandreu, 2013). Due to the constant elasticity, profit-maximization and uncorrelatedness imply $Cov(p, \xi) = 0$, and OLS produces unbiased estimates of the demand parameters.[11] Indeed, this is an excellent illustration of our basic argument: so long as the data-generating process is sufficiently well understood, it is possible to characterize the bias of OLS estimates.

[11] The international trade literature following Feenstra (1994) consider non-constant marginal costs, which requires an additional restriction. See section 5.2 for an extension of our methodology to non-constant marginal costs.

A.4 Other Demand Systems

The demand assumption in equation (1) accommodates many rich demand systems. Consider the linear demand system, $q_{jt} = \alpha_j + \sum_k \beta_{jk} p_k + \xi_{jt}$, which sometimes appears in identification proofs (e.g., Nevo, 1998) but is seldom applied empirically due to the large number of price coefficients. In principle, the system could be formulated such that $h(q_{jt}, w_{jt}; \sigma) \equiv q_{jt} - \sum_{k \neq j} \beta_{jk} p_k$. In addition to the own-product uncorrelatedness restrictions that could identify β_{jj}, one could impose cross-product covariance restrictions to identify β_{jk} ($j \neq k$). We discuss these cross-product covariance restrictions in our first application (Section 5.1). A similar approach could be used with the almost ideal demand system of Deaton and Muellbauer (1980).

A.5 Multi-Product Firms

We illustrate how our framework incorporates multi-product firms with the case of Bertrand pricing. Let K^m denote the set of products owned by multi-product firm m. When the firm sets prices on each of its products to maximize joint profits, there are $|K^m|$ first-order conditions, which can be expressed as

$$\sum_{k \in K^m} (p_k - mc_k) \frac{\partial q_k}{\partial p_j} = -q_j \;\; \forall j \in K^m.$$

The market subscript, t, is omitted to simplify notation. For demand systems satisfying (1), $\frac{\partial q_k}{\partial p_j} = \beta \frac{1}{\frac{dh_j}{dq_k}}$, where the derivative $\frac{dh_j}{dq_k}$ is calculated holding the prices of other products fixed. Therefore, the set of first-order conditions can be written as

$$\sum_{k \in K^m} (p_k - mc_k) \frac{1}{dh_j / dq_k} = -\frac{1}{\beta} q_j \;\; \forall j \in K^m.$$

For each firm, stack the first-order conditions, writing the left-hand side as the product of a matrix A^m of loading components and a vector of markups, $(p_j - mc_j)$, for products owned by the firm. The loading components are given by $A^m_{i(j),i(k)} = \frac{1}{dh_j / dq_k}$, where $i(\cdot)$ indexes products within a firm. Next, invert the loading matrix to solve for markups as function of the loading components and $-\frac{1}{\beta} \boldsymbol{q}^m$, where $\boldsymbol{q}^m$ is a vector of the multi-product firm's quantities. Equilibrium prices equal marginal costs plus a markup, where the markup is determined by the inverse of A^m ($(A^m)^{-1} \equiv \Lambda^m$), quantities, and the price parameter:

$$p_j = mc_j - \frac{1}{\beta} \left(\Lambda^m \boldsymbol{q}^m \right)_{i(j)}. \tag{A.3}$$

Here, $\left(\Lambda^m \boldsymbol{q}^m \right)_{i(j)}$ provides the entry corresponding to product j in the vector $\Lambda^m \boldsymbol{q}^m$. As the matrix Λ^m is not a function of the price parameter after conditioning on observables, this form of the first-order condition allows us to solve for β. Letting $\lambda \equiv \left(\Lambda^m \boldsymbol{q}^m \right)_{i(j)}$, we see that multi-product Bertrand fits in the class of models specified by equation (3).

B Relationship to the Simultaneous Equations Literature

To help place these results in context, we provide an overview of the existing literature on the identification of simultaneous equations with covariance restrictions. The subject received early attention in research at the Cowles Foundation (e.g., Koopmans et al., 1950); we focus on the more recent articles of Hausman and Taylor (1983) and Hausman et al. (1987). Adopting their notation, the model of supply and demand is given by:

$$\begin{aligned} y_1 &= \beta_{12} y_2 + \gamma_{11} z_1 + \epsilon_1 \quad \text{(Supply)} \\ y_2 &= \beta_{21} y_1 + \epsilon_2 \qquad\qquad\quad \text{(Demand)} \end{aligned}$$

This system is analogous to equations (1) and (4) in the present paper. The two key differences are: by assumption, the covariate z_1 is excluded from the second equation, and there are two "slope" parameters (β_{12} and β_{21}). The linearity of the system is less consequential; for example, linearity also obtains in our setting with monopoly and linear demands.

Hausman et al. (1987) show that the coefficients β_{12}, β_{21}, and γ_{11} are identified by invoking exogeneity of z_1, the exclusion restriction, and a covariance restriction: $E[z_1 \cdot \epsilon_1] = 0$, $E[z_1 \cdot \epsilon_2] = 0$, and $Cov(\epsilon_1, \epsilon_2) = 0$.[12] The parameters can be estimated jointly via GMM by iterating over candidate parameter values. Alternatively, the demand equation can be estimated with 2SLS using z_1 as an instrument, and then the supply equation can be estimated with 2SLS using z_1 and the residual $\hat{\epsilon}_2 = y_2 - \hat{\beta}_{21 y_1}$ as instruments (Hausman and Taylor, 1983). Thus, the covariance restriction can be recast as an orthogonality condition involving a residual instrument.[13] Consistent estimation in this context requires that a valid and relevant instrument (z_1) exists, in addition to the covariance restriction on unobserved shocks.[14]

By contrast, the approach that we introduce does not require the presence of an instrument. We use the structure of demand and supply to link the price coefficients β_{12} and β_{21} across the two equations. This reduces the number of endogenous parameters from two to one. Thus, fewer moments are needed for identification, and there is no instrument relevance condition that must be satisfied.

C Additional Results for Covariance Restrictions

C.1 Two Approaches to Estimation

Equation (7) of Proposition 2 yields a quadratic equation for β. In the typical case when the true parameter is the lower root, the equation can be used to construct an analytical solution,

[12]These three equations appear as (2.10a), (2.10b), and (2.10c) in that article.

[13]This interpretation of covariance restrictions as allowing for residual instruments has been influential. For example, see the lecture notes of Professor Daniel McFadden, dated 1999, which state that:

> Even covariance matrix restrictions can be used in constructing instruments. For example, if you know that the disturbance in an equation you are trying to estimate is uncorrelated with the disturbance in another equation, then you can use a consistently estimated residual from the second equation as an instrument.

The notes can be obtained at `https://eml.berkeley.edu/~mcfadden/e240b_f01/ch6.pdf`, last accessed July 17, 2019.

[14]Matzkin (2016) provides extensions of this approach to semi-parametric models. Hausman et al. (1987) also consider the identification of simultaneous equations with covariance restrictions alone. This requires at least three equations, however, and thus is not applicable to models of supply and demand.

which we employ to develop a three-stage estimator for β.

Corollary 1. ***(Three-Stage Estimator)*** *When β is the lower root of* (7) *and $Cov(\xi, \eta) = 0$, a consistent estimate is given by*

$$\hat{\beta}^{\text{3-Stage}} = \frac{1}{2}\left(\hat{\beta}^{OLS} - \frac{Cov\,(p^*, \lambda)}{Var(p^*)} - \sqrt{\left(\hat{\beta}^{OLS} + \frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\hat{\xi}^{OLS}, \lambda\right)}{Var(p^*)}}\right) \tag{C.1}$$

The estimator is consistent for the lower root of equation (7). It can be calculated in three stages: (i) regress h on p and $\boldsymbol{x}$ with OLS, (ii) regress p on $\boldsymbol{x}$ with OLS and obtain the residuals p^*, and (iii) construct the estimator as shown. The computational burden is trivial, which may be beneficial if nested inside a search for θ.[15]

The identifying restriction $Cov(\xi, \eta) = 0$ can be recast as a GMM estimator that exploits the orthogonality condition $E[\xi \cdot \eta] = 0$. Thus, an alternative approach to estimation is to search numerically for a $\tilde{\beta}$ that satisfies the corresponding empirical moment, yielding

$$\hat{\beta}^{GMM} = \arg\min_{\tilde{\beta}<0}\left[\frac{1}{T}\sum_{t}\frac{1}{|J_t|}\sum_{j\in J_t}\xi(\tilde{\beta}; h_{jt}, x_{jt}, \boldsymbol{\alpha}) \cdot \eta(\tilde{\beta}; \lambda_{jt}, x_{jt}, \boldsymbol{\gamma})\right]^2 \tag{C.2}$$

where $\xi(\tilde{\beta}; h, x, \boldsymbol{\alpha})$ and $\eta(\tilde{\beta}; \lambda, x, \boldsymbol{\gamma})$ are the estimated residuals given the data and the candidate parameter, and the firms present in each market t are indexed by the set J_t. Note that h and λ will often depend on additional parameters, $\boldsymbol{\theta}$. We see two main situations in which the numerical approach may be preferred despite its greater computational burden. First, additional moments can be incorporated in estimation, allowing for efficiency improvements and specification tests (e.g., Hausman, 1978; Hansen, 1982). Second, the three-stage estimator requires orthogonality between ξ and all regressors (i.e., $E[\boldsymbol{x}\xi] = \boldsymbol{0}$), whereas the numerical approach can be pursued under a weaker assumption that allows for correlation between ξ and covariates that enter the cost function only.

C.2 Supply-Side Misspecification: Evidence from Numerical Simulations

To illustrate how supply-side misspecification may affect the performance of the estimators, we simulate duopoly markets in which the standard assumption of Bertrand price competition may not match the data-generating process.[16] We assume the demand system is logit, providing consumers with a differentiated discrete choice, and we allow them to select an outside option in addition to a product from each firm. The quantity demanded of firm j in market t is

$$q_{jt} = \frac{\exp(2 - p_{jt} + \xi_{jt})}{1 + \sum_{k=j,i}\exp(2 - p_{kt} + \xi_{kt})}$$

On the supply side, marginal costs are $c_{kt} = \eta_{kt}$ $(k = j, i)$. Firm j sets price to maximize $\pi_j + \kappa\pi_i$, and likewise for firm i, where $\kappa \in [0, 1]$ is a conduct parameter (e.g., Miller and

[15]A more precise two-stage estimator can be constructed for special cases in which the observed cost and demand shifters are uncorrelated.

[16]Another form of misspecification could arise if prices or quantities are measured with error, in which case the demand and cost residuals might be correlated even if the underlying shocks are uncorrelated.

Table C.1: Small-Sample Properties: Supply-Side Misspecification

Estimation Method	(1) $\kappa = 0.0$	(2) $\kappa = 0.2$	(3) $\kappa = 0.4$	(4) $\kappa = 0.6$	(5) $\kappa = 0.8$	(6) $\kappa = 1.0$
Covariance Restrictions	-1.001	-1.002	-1.000	-1.003	-1.016	-1.038
	(0.050)	(0.052)	(0.053)	(0.054)	(0.053)	(0.051)
IV-1: Supply Shifters	-1.002	-1.000	-1.001	-1.001	-1.001	-1.002
	(0.076)	(0.077)	(0.077)	(0.076)	(0.073)	(0.071)
IV-2: Demand Shifters	-1.015	-1.017	-1.012	-1.025	-1.082	-1.220
	(0.153)	(0.155)	(0.159)	(0.178)	(0.213)	(0.298)
IV-1: First-stage *F*-statistic	1079.9	1335.3	1424.9	1277.8	1027.1	801.9
IV-2: First-stage *F*-statistic	99.0	108.4	111.4	100.8	77.8	50.8

Notes: Results are based on 10,000 simulations of 200 duopoly markets for each specification. The demand curve is $h_{jt} = 2 - p_{jt} + \xi_{jt}$, so that $\beta = -1$, and marginal costs are $c_{jt} = \eta_{jt}$. Demand is logit: $h(q_{jt}) = \ln(q_{jt}) - \ln(q_{0t})$, where q_{0t} is consumption of the outside good. IV-1 estimates are calculated using two-stage least squares with marginal costs (η) as an instrument in the demand equation. Analogously, IV-2 estimates are calculated using two-stage least squares with demand shocks (ξ) as an instrument in the supply equation. Across all specifications, $\xi \sim U(0, 0.5)$ and $\eta \sim U(0, 0.5)$. The data-generating process varies in the nature of competition across specifications, indexed by the conduct parameter κ. The coefficients are estimated under the (misspecified) assumption of Bertrand price competition ($\kappa = 0$).

Weinberg, 2017). The first-order conditions take the form

$$\begin{bmatrix} p_j \\ p_i \end{bmatrix} = \begin{bmatrix} c_j \\ c_i \end{bmatrix} - \left[\begin{pmatrix} 1 & \kappa \\ \kappa & 1 \end{pmatrix} \circ \left(\frac{\partial q}{\partial p} \right)^T \right]^{-1} \begin{bmatrix} q_j \\ q_i \end{bmatrix}$$

where $\frac{\partial q}{\partial p}$ is a matrix of demand derivatives and $\circ$ denotes element-by-element multiplication. The model nests Bertrand competition ($\kappa = 0$) and joint price-setting behavior ($\kappa = 1$), as well as capturing (non-micro-founded) intermediate cases by scaling markups. Of course, there are other ways to change the equilibrium concept; these simulations are not fully general.

We generate data with different conduct parameters: $\kappa \in \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$. For each specification, we simulate datasets with 400 observations (200 markets $\times$ two firms), and estimate the model under the erroneous assumption of Bertrand price competition ($\kappa = 0$), thus generating supply-side misspecification. Table C.1 displays the results.

As expected, supply-side misspecification introduces bias into the covariance restrictions approach, with a bias of -3.8 percent when the true nature of conduct is $\kappa = 1$. The bias does not appear to be meaningful for modest values of κ (i.e., 0.4 or less). Likewise, the demand-side instruments (IV-2), which invoke the formal assumption about conduct in estimation, perform worse when the true κ is farther from the assumed value. The demand-side instruments perform poorly when the true conduct is $\kappa = 1$, with a mean bias of over 20 percent.

By contrast, supply-side instruments do not use a formal assumption about conduct in estimation and provide consistent estimates across the specifications (IV-1). Consistent with the earlier simulations, the three-stage estimator outperforms IV-1 when conduct is correctly specified ($\kappa = 0$). These results illustrate a key trade-off to the econometrician: if the supply-side assumptions are to be maintained, then covariance restrictions can offer better precision relative to instrument-based approaches. However, supply-side instruments are robust to misspecification of firm conduct, whereas covariance restrictions are not.

We note that the covariance restriction approach, which uses both demand-side and supply-side variation, is not as susceptible to misspecification bias as demand-side instruments in our simulations. The estimator appears to place greater weight on the source of variation with more power. In specification (6), the mean coefficient of -1.038 is much closer to the supply-shifter mean of -1.002 than the demand-shifter mean of -1.220. Indeed, it is approximately equal to the IV-1 and IV-2 estimates weighted by the square root of the respective F-statistics. By placing greater weight on supply-side shocks as the demand-side instruments degrade, the covariance restriction approach receives some protection against bias from model misspecification.

D Proofs

D.1 A Consistent and Unbiased Estimate for ξ

Our proofs make use of the following lemma, which identifies a consistent and unbiased estimate for the unobserved term in a linear regression when one of the covariates is endogenous. Though demonstrated in the context of semi-linear demand, the proof also applies for any endogenous covariate, including when (transformed) quantity depends on a known transformation of price, as no supply-side assumptions are required. For example, we may replace p with $\ln p$ everywhere and obtain the same results.

Lemma D.1. *A consistent and unbiased estimate of ξ is given by $\xi_1 = \xi^{OLS} + \left(\beta^{OLS} - \beta\right) p^*$*

For some intuition, note that we can construct both the true demand shock and the OLS residuals as:

$$\begin{aligned} \xi &= h(q) - \beta p - \boldsymbol{x}'\boldsymbol{\alpha} \\ \xi^{OLS} &= h(q) - \beta^{OLS} p - \boldsymbol{x}'\boldsymbol{\alpha}^{OLS} \end{aligned}$$

where this holds even in small samples. Without loss of generality, we assume $E[\xi] = 0$. The true demand shock is given by $\xi_0 = \xi^{OLS} + (\beta^{OLS} - \beta)p + \boldsymbol{x}'(\boldsymbol{\alpha}^{OLS} - \boldsymbol{\alpha})$. We desire to show that an alternative estimate of the demand shock, $\xi_1 = \xi^{OLS} + (\beta^{OLS} - \beta)p^*$, is consistent and unbiased. (This eliminates the need to estimate the true α parameters). It suffices to show that $(\hat{\beta}^{OLS} - \beta)p^* = (\hat{\beta}^{OLS} - \beta)p + \boldsymbol{x}'(\hat{\boldsymbol{\alpha}}^{OLS} - \boldsymbol{\alpha}) + \Upsilon$, where Υ is such that $E[\Upsilon = 0]$ and $\Upsilon \to 0$ as N gets large. It is straightforward to show this using the projection matrices for p and x.[17]

D.2 Proof of Proposition 1 (Set Identification)

From equation (5), we have $\hat{\beta}^{OLS} \xrightarrow{p} \beta + \frac{Cov(p^*,\xi)}{Var(p^*)}$. The general form for a firm's first-order condition is $p = mc + \mu$, where mc is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = \boldsymbol{x}'\boldsymbol{\gamma} + \eta$. If we substitute the first-order condition $p^* = \boldsymbol{x}'\boldsymbol{\gamma} + \eta + \mu - \hat{p}$ into the

[17] Please contact the authors for the full proof.

bias term from the OLS regression, we obtain

$$\begin{aligned}\beta^{OLS}-\beta=\frac{Cov(p^*,\xi)}{Var(p^*)}&=\frac{Cov(\xi,\boldsymbol{x}'\boldsymbol{\gamma}+\eta+\mu-\hat{p})}{Var(p^*)}\\&=\frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{Cov(\xi,\mu)}{Var(p^*)}\end{aligned}\tag{D.1}$$

where the second line follows from the exogeneity assumption ($E[X\xi]=0$).

From Lemma D.1, we can construct a consistent estimate of the unobserved demand shock as $\xi=\xi^{OLS}+\left(\beta^{OLS}-\beta\right)p^*$. We substitute this expression into $\frac{Cov(\xi,\mu)}{Var(p^*)}$, along with the above expression for $\left(\beta^{OLS}-\beta\right)$ to obtain

$$\begin{aligned}\frac{Cov\left(\xi,\mu\right)}{Var(p^*)}=&\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\left(\frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\right)\frac{Cov(p^*,\mu)}{Var(p^*)}\\ \frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\left(1-\frac{Cov(p^*,\mu)}{Var(p^*)}\right)=&\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)}\\ \frac{Cov\left(\xi,\mu\right)}{Var(p^*)}=&\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)}\end{aligned}$$

Plugging this into (D.1) yields

$$\begin{aligned}\beta^{OLS}&=\beta+\frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{\frac{Cov(p^*,\mu)}{Var(p^*)}}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\\ \beta^{OLS}&=\beta+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\end{aligned}$$

Thus, we obtain an expression for the OLS estimator in terms of the OLS residuals, the residualized prices, the markup, and the correlation between unobserved demand and cost shocks. If the markup can be parameterized in terms of observables and the correlation in unobserved shocks can be calibrated, we have a method to estimate β from the OLS regression. Under our supply and demand assumptions, $\mu=-\frac{1}{\beta}\lambda$, and plugging in obtains the first equation of the proposition:

$$\beta^{OLS}=\beta-\frac{1}{\beta+\frac{Cov(p^*,\lambda)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}+\beta\frac{1}{\beta+\frac{Cov(p^*,\lambda)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}.$$

The second equation in the proposition is obtained by rearranging terms. QED.

D.3 Proof of Proposition 2 (Point Identification)

Part (1). We first prove the sufficient condition, i.e., that under assumptions 1 and 2, β is the lower root of equation (7) if the following condition holds:

$$0\leq\beta^{OLS}\frac{Cov(p^*,\lambda)}{Var(p^*)}+\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}\tag{D.2}$$

Consider a generic quadratic, ax^2+bx+c. The roots of the quadratic are $\frac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$. Thus, if $4ac<0$ and $a>0$ then the upper root is positive and the lower root is negative. In equation (7), $a=1$, and $4ac<0$ if and only if equation (D.2) holds. Because the upper root is positive, $\beta<0$ must be the lower root, and point identification is achieved given knowledge of $Cov(\xi,\eta)$. QED.

Part (2). In order to prove the necessary and sufficient condition for point identification, we first state and prove a lemma:

Lemma D.2. *The roots of equation* (7) *are* β *and* $\frac{Cov(p^*,\xi)}{Var(p^*)}-\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov(\xi,\eta)}{Var(p^*)}$.

Proof of Lemma D.2. We first provide equation (7) for reference:

$$\begin{aligned} 0 &= \beta^2 \\ &+ \left(\frac{Cov(p^*,\lambda)}{Var(p^*)}+\frac{Cov(\xi,\eta)}{Var(p^*)}-\beta^{OLS}\right)\beta \\ &+ \left(-\beta^{OLS}\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}\right)\end{aligned}$$

To find the roots, begin by applying the quadratic formula

$$\begin{aligned}(r_1,r_2) &= \frac{1}{2}\left(-B\pm\sqrt{B^2-4AC}\right)\\ &= \frac{1}{2}\left(\beta^{OLS}-\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov(\xi,\eta)}{Var(p^*)}\right) \\ &\pm\frac{1}{2}\sqrt{\left(\beta^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}+\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS}-\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)}\end{aligned} \tag{D.3}$$

Looking inside the radical, consider the first part: $\left(\beta^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}$

$$\begin{aligned}&\left(\beta^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}\\ &=\left(\beta^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi-p^*(\beta^{OLS}-\beta),\lambda\right)}{Var(p^*)}\\ &=\left(\beta^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi,\lambda\right)}{Var(p^*)}-4\frac{Cov(p^*,\xi)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\ &=\left(\beta^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi,\lambda\right)}{Var(p^*)}-4\left(\frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{Cov(\xi,-\frac{1}{\beta}\lambda)}{Var(p^*)}\right)\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\ &=\left(\beta^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi,\lambda\right)}{Var(p^*)}\left(1+\frac{1}{\beta}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)-4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\end{aligned} \tag{D.4}$$

To simplify this expression, it is helpful to use the general form for a firm's first-order condition, $p=c+\mu$, where c is the marginal cost and μ is the markup. We can write $p=p^*+\hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c=\boldsymbol{x}'\boldsymbol{\gamma}+\eta$. It follows that

$$\begin{aligned} p^* &= \boldsymbol{x}'\boldsymbol{\gamma} + \eta + \mu - \hat{p} \\ &= \boldsymbol{x}'\boldsymbol{\gamma} + \eta - \frac{1}{\beta}\lambda - \hat{p} \end{aligned}$$

Therefore

$$Cov(p^*, \xi) = Cov(\xi, \eta) - \frac{1}{\beta} Cov(\xi, \lambda)$$

and

$$\begin{aligned} Cov(\xi, \lambda) &= -\beta \left(Cov(p^*, \xi) - Cov(\xi, \eta) \right) \\ \frac{Cov(\xi, \lambda)}{Var(p^*)} &= -\beta \left(\frac{Cov(p^*, \xi)}{Var(p^*)} - \frac{Cov(\xi, \eta)}{Var(p^*)} \right) \end{aligned} \tag{D.5}$$

Returning to equation (D.4), we can substitute using equation (D.5) and simplify:

$$\begin{aligned}
&\left(\beta^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\,(\xi, \lambda)}{Var(p^*)}\left(1 + \frac{1}{\beta}\frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right) - 4\frac{Cov(\xi, \eta)}{Var(p^*)}\frac{Cov\,(p^*, \lambda)}{Var(p^*)} \\
=&\left(\beta^{OLS}\right)^2 + \left(\frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right)^2 + 2\beta^{OLS}\frac{Cov\,(p^*, \lambda)}{Var(p^*)} - 4\frac{Cov(\xi, \eta)}{Var(p^*)}\frac{Cov\,(p^*, \lambda)}{Var(p^*)} \\
&+ 4\frac{Cov\,(\xi, \lambda)}{Var(p^*)} + 4\frac{1}{\beta}\frac{Cov\,(\xi, \lambda)}{Var(p^*)}\frac{Cov\,(p^*, \lambda)}{Var(p^*)} \\
=&\left(\beta + \frac{Cov\,(p^*, \xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right)^2 + 2\left(\beta + \frac{Cov\,(p^*, \xi)}{Var(p^*)}\right)\frac{Cov\,(p^*, \lambda)}{Var(p^*)} - 4\frac{Cov(\xi, \eta)}{Var(p^*)}\frac{Cov\,(p^*, \lambda)}{Var(p^*)} \\
&- 4\beta\left(\frac{Cov(p^*, \xi)}{Var(p^*)} - \frac{Cov(\xi, \eta)}{Var(p^*)}\right) - 4\left(\frac{Cov(p^*, \xi)}{Var(p^*)} - \frac{Cov(\xi, \eta)}{Var(p^*)}\right)\frac{Cov\,(p^*, \lambda)}{Var(p^*)} \\
=&\left(\beta + \frac{Cov\,(p^*, \xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right)^2 + 2\left(\beta + \frac{Cov\,(p^*, \xi)}{Var(p^*)}\right)\frac{Cov\,(p^*, \lambda)}{Var(p^*)} \\
&- 4\beta\left(\frac{Cov(p^*, \xi)}{Var(p^*)}\right) - 4\left(\frac{Cov(p^*, \xi)}{Var(p^*)}\right)\frac{Cov\,(p^*, \lambda)}{Var(p^*)} + 4\beta\frac{Cov(\xi, \eta)}{Var(p^*)} \\
=&\beta^2 + \left(\frac{Cov\,(p^*, \xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right)^2 + 2\beta\frac{Cov\,(p^*, \lambda)}{Var(p^*)} \\
&- 2\beta\frac{Cov\,(p^*, \xi)}{Var(p^*)} - 2\frac{Cov\,(p^*, \xi)}{Var(p^*)}\frac{Cov\,(p^*, \lambda)}{Var(p^*)} + 4\beta\frac{Cov(\xi, \eta)}{Var(p^*)} \\
=&\left(\left(\beta + \frac{Cov\,(p^*, \lambda)}{Var(p^*)}\right) - \frac{Cov\,(p^*, \xi)}{Var(p^*)}\right)^2 + 4\beta\frac{Cov(\xi, \eta)}{Var(p^*)}
\end{aligned}$$

Now, consider the second part inside of the radical in equation (D.3):

$$
\begin{aligned}
&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*,\lambda)}{Var(p^*)}\right)\\
=&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta + \frac{Cov(\xi,\eta)}{Var(p^*)} - \frac{1}{\beta}\frac{Cov(\xi,\lambda)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)}\right)\\
=&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 + 2\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(\xi,\lambda)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&-\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right) + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}
\end{aligned}
$$

Combining yields a simpler expression for the terms inside the radical of equation (D.3):

$$
\begin{aligned}
&\left(\left(\beta + \frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right) - \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)}\\
&+\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&\left(\left(\beta + \frac{Cov\,(p^*,\lambda)}{Var(p^*)}\right) - \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2\\
&+2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&\left(\beta + \frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2
\end{aligned}
$$

Plugging this back into equation (D.3), we have:

$$
\begin{aligned}
(r_1, r_2) \;&=\; \frac{1}{2}\left(\beta^{OLS} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right.\\
&\qquad\left.\pm\sqrt{\left(\beta + \frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\right)\\
&=\; \frac{1}{2}\left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right.\\
&\qquad\left.\pm\sqrt{\left(\beta + \frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\right)
\end{aligned}
$$

The roots are given by

$$
\begin{aligned}
&\frac{1}{2}\left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} + \beta + \frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)\\
=&\beta
\end{aligned}
$$

and

$$\frac{1}{2}\left(\beta+\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(p^{*},\lambda)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}-\beta-\frac{Cov\left(p^{*},\lambda\right)}{Var(p^{*})}+\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)$$
$$=\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(p^{*},\lambda)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}$$

which completes the proof of the intermediate result. QED.

Part (3). Consider the roots of equation (7), β and $\frac{Cov(p^{*},\xi)}{Var(p^{*})}-\frac{Cov(p^{*},\lambda)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}$. The price parameter β may or may not be the lower root.[18] However, β is the lower root iff

$$\begin{aligned}
\beta &< \frac{Cov(p^{*},\xi)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
\beta &< -\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}+\beta\frac{Cov(p^{*},-\frac{1}{\beta}\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
\beta &< -\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}+\beta\frac{Cov(p^{*},p^{*}-c)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
\beta &< \beta\frac{Var(p^{*})}{Var(p^{*})}-\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}-\beta\frac{Cov(p^{*},\eta)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
0 &< -\beta\frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}-\beta\frac{Cov(p^{*},\eta)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\\
0 &< \frac{Cov(p^{*},-\frac{1}{\beta}\xi)}{Var(p^{*})}+\frac{Cov(p^{*},\eta)}{Var(p^{*})}+\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^{*})}
\end{aligned}$$

The third line relies on the expression for the markup, $p-c=-\frac{1}{\beta}\frac{dh}{dq}q$. The final line holds because $\beta<0$ so $-\beta>0$. It follows that β is the lower root of (7) iff

$$-\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^{*})}\leq\frac{Cov\left(p^{*},-\frac{1}{\beta}\xi\right)}{Var(p^{*})}+\frac{Cov\left(p^{*},\eta\right)}{Var(p^{*})}$$

in which case β is point identified given knowledge of $Cov(\xi,\eta)$. QED.

[18] Consider that the first root is the upper root if

$$\beta+\frac{Cov\left(p^{*},\lambda\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}>0$$

because, in that case,

$$\sqrt{\left(\beta+\frac{Cov\left(p^{*},\lambda\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)^{2}}=\beta+\frac{Cov\left(p^{*},\lambda\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}$$

When $\beta+\frac{Cov\left(p^{*},\lambda\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}<0$, then $\sqrt{\left(\beta+\frac{Cov(p^{*},\lambda)}{Var(p^{*})}-\frac{Cov(p^{*},\xi)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)^{2}}=-\left(\beta+\frac{Cov\left(p^{*},\lambda\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)$, and the first root is then the lower root (i.e., minus the negative value).

D.4 Proof of Proposition 3 (Approximation)

The demand and supply equations are given by:

$$\begin{aligned} h =& \beta p + \boldsymbol{x}'\boldsymbol{\alpha} + \xi \\ p =& \boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\beta}\frac{dh}{dq}q + \eta \end{aligned}$$

Using an first-order expansion of h about q, $h \approx \overline{h} + \overline{\frac{dh}{dq}}\left(q - \overline{q}\right)$, we can solve for a reduced-form for p and h. It follows that

$$\begin{aligned} \overline{h} + \overline{\frac{dh}{dq}}\left(q - \overline{q}\right) \approx& \beta p + \boldsymbol{x}'\boldsymbol{\alpha} + \xi \\ \overline{\frac{dh}{dq}}q \approx& \beta p + \boldsymbol{x}'\boldsymbol{\alpha} + \xi - \overline{h} + \overline{\frac{dh}{dq}}\overline{q} \end{aligned}$$

Letting $\frac{dh}{dq}q = \frac{\tilde{dh}}{dq}q + \overline{\frac{dh}{dq}}q$, we have

$$\begin{aligned} p &\approx \boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\beta}\frac{\tilde{dh}}{dq}q - \frac{1}{\beta}\left(\beta p + \boldsymbol{x}'\boldsymbol{\alpha} + \xi - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\right) + \eta \\ 2p &\approx \boldsymbol{x}'\boldsymbol{\gamma} + \frac{1}{\beta}\boldsymbol{x}'\boldsymbol{\alpha} - \frac{1}{\beta}\overline{h} + \frac{1}{\beta}\overline{\frac{dh}{dq}}\overline{q} - \frac{1}{\beta}\frac{\tilde{dh}}{dq}q + \eta + \frac{1}{\beta}\xi \\ p &\approx \frac{1}{2}\left(\boldsymbol{x}'\boldsymbol{\gamma} + \frac{1}{\beta}\boldsymbol{x}'\boldsymbol{\alpha} - \frac{1}{\beta}\overline{h} + \frac{1}{\beta}\overline{\frac{dh}{dq}}\overline{q} - \frac{1}{\beta}\frac{\tilde{dh}}{dq}q + \eta + \frac{1}{\beta}\xi\right). \end{aligned}$$

Let H^* denote the residual from a regression of $\frac{\tilde{dh}}{dq}q$ on x. Then p^*, the residual from a regression of p on x, is

$$p^* \approx \frac{1}{2}\left(\eta + \frac{1}{\beta}\xi - \frac{1}{\beta}H^*\right). \tag{D.6}$$

Likewise, as $h - \overline{h} + \overline{\frac{dh}{dq}}\overline{q} \approx \overline{\frac{dh}{dq}}q$,

$$p \approx \boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\beta}\frac{\tilde{dh}}{dq}q - \frac{1}{\beta}\overline{\frac{dh}{dq}}q + \eta$$

$$\begin{aligned} h \approx& \beta\left(\boldsymbol{x}'\boldsymbol{\gamma} - \frac{1}{\beta}\frac{\tilde{dh}}{dq}q - \frac{1}{\beta}\overline{\frac{dh}{dq}}q + \eta\right) + \boldsymbol{x}'\boldsymbol{\alpha} + \xi \\ h \approx& \beta\boldsymbol{x}'\boldsymbol{\gamma} + \boldsymbol{x}'\boldsymbol{\alpha} - \frac{\tilde{dh}}{dq}q - \left(h - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\right) + \beta\eta + \xi \\ 2h \approx& \beta\boldsymbol{x}'\boldsymbol{\gamma} + \boldsymbol{x}'\boldsymbol{\alpha} - \frac{\tilde{dh}}{dq}q + \overline{h} - \overline{\frac{dh}{dq}}\overline{q} + \beta\eta + \xi. \end{aligned}$$

Similarly, the residual from a regression of h on x is:

$$h^* \approx \frac{1}{2}\left(\beta\eta + \xi - H^*\right). \tag{D.7}$$

Equations (D.6) and (D.7) provide an approximation for β.

$$\begin{aligned}
-\sqrt{\frac{Var(h^*)}{Var(p^*)}} &\approx -\sqrt{\frac{\frac{1}{4}Var\left(\beta\eta + \xi - H^*\right)}{\frac{1}{4}Var\left(\eta + \frac{1}{\beta}\xi - \frac{1}{\beta}H^*\right)}} \\
&\approx -\sqrt{\frac{\beta^2 Var\left(\eta + \frac{1}{\beta}\xi - \frac{1}{\beta}H^*\right)}{Var\left(\eta + \frac{1}{\beta}\xi - \frac{1}{\beta}H^*\right)}} \\
&\approx \beta
\end{aligned}$$

QED.

D.5 Proof of Lemma 1 (Monotonicity in $Cov(\xi, \eta)$)

We return to the quadratic formula for the proof. The lower root of a quadratic $ax^2 + bx + c$ is $L \equiv \frac{1}{2}\left(-b - \sqrt{b^2 - 4ac}\right)$. In our case, $a = 1$.

We wish to show that $\frac{\partial L}{\partial \gamma} < 0$, where $\gamma = Cov(\xi, \eta)$. We evaluate the derivative to obtain

$$\frac{\partial L}{\partial \gamma} = -\frac{1}{2}\left(1 + \frac{b}{(b^2 - 4c)^{\frac{1}{2}}}\right)\frac{\partial b}{\partial \gamma}.$$

We observe that, in our setting, $\frac{\partial b}{\partial \gamma} = \frac{1}{Var(p^*)}$ is always positive. Therefore, it suffices to show that

$$1 + \frac{b}{(b^2 - 4c)^{\frac{1}{2}}} > 0. \tag{D.8}$$

We have two cases. First, when $c < 0$, we know that $\left|\frac{b}{(b^2-4c)^{\frac{1}{2}}}\right| < 1$, which satisfies (D.8). Second, when $c > 0$, it must be the case that $b > 0$ also. Otherwise, both roots are positive, invalidating the model. When $b > 0$, it is evident that the left-hand side of (D.8) is positive. This demonstrates monotonicity.

Finally, we obtain the range of values for L by examining the limits as $\gamma \to \infty$ and $\gamma \to -\infty$. From the expression for L and the result that $\frac{\partial b}{\partial \gamma}$ is a constant, we obtain

$$\lim_{\gamma \to -\infty} L = 0$$

$$\lim_{\gamma \to \infty} L = -\infty$$

When $c < 0$, the domain of the quadratic function is $(-\infty, \infty)$, which, along with monotonicity, implies the range for L of $(0, -\infty)$. When $c > 0$, the domain is not defined on the interval $(-2\sqrt{c}, 2\sqrt{c})$, but L is equal in value at the boundaries of the domain. QED.

Additionally, we note that the upper root, $U \equiv \frac{1}{2}\left(-b+\sqrt{b^2-4ac}\right)$ is increasing in γ. When the upper root is a valid solution (i.e., negative), it must be the case that $c>0$ and $b>0$, and it is straightforward to follow the above arguments to show that $\frac{\partial U}{\partial \gamma}>0$ and that the range of the upper root is $[-\frac{1}{2}b, 0)$.

D.6 Proof of Proposition 4 (Covariance Bound)

The proof involves an application of the quadratic formula. Any generic quadratic, ax^2+bx+c, with roots $\frac{1}{2}\left(-b\pm\sqrt{b^2-4ac}\right)$, admits a real solution if and only if $b^2 \geq 4ac$. Given the formulation of (7), real solutions satisfy the condition:

$$\left(\frac{Cov(p^*,\lambda)}{Var(p^*)}+\frac{Cov(\xi,\eta)}{Var(p^*)}-\beta^{OLS}\right)^2 \geq 4\left(-\beta^{OLS}\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}\right).$$

As $a=1$, a solution is always possible if $c<0$. This is the sufficient condition for point identification from the text. If $c\geq 0$, it must be the case that $b\geq 0$; otherwise, both roots are positive. Therefore, a real solution is obtained if and only if $b\geq 2\sqrt{c}$, that is

$$\left(\frac{Cov(p^*,\lambda)}{Var(p^*)}+\frac{Cov(\xi,\eta)}{Var(p^*)}-\beta^{OLS}\right) \geq 2\sqrt{-\beta^{OLS}\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}}.$$

Solving for $Cov(\xi,\eta)$, we obtain the prior-free bound,

$$Cov(\xi,\eta) \geq Var(p^*)\beta^{OLS}-Cov(p^*,\lambda)+2Var(p^*)\sqrt{-\beta^{OLS}\frac{Cov(p^*,\lambda)}{Var(p^*)}-\frac{Cov\left(\xi^{OLS},\lambda\right)}{Var(p^*)}}.$$

This bound exists if the expression inside the radical is positive, which is the case if and only if the sufficient condition for point identification from Proposition 2 fails. QED.

D.7 Proof of Proposition 5 (Non-Constant Marginal Costs)

Under the semi-linear marginal cost schedule of equation (12), the plim of the OLS estimator is equal to

$$\text{plim}\hat{\beta}^{OLS} = \beta + \frac{Cov(\xi,g(q))}{Var(p^*)} - \frac{1}{\beta}\frac{Cov\left(\xi,\lambda\right)}{Var(p^*)}.$$

This is obtain directly by plugging in the first–order condition for p: $Cov(p^*,\xi) = Cov(g(q)+\eta-\frac{1}{\beta}\lambda-\hat{p},\xi) = Cov(\xi,g(q))-\frac{1}{\beta}Cov(\xi,\lambda)$ under the assumptions. Next, we re-express the terms including the unobserved demand shocks in in terms of OLS residuals. The unobserved demand shock may be written as $\xi = h(q)-x\beta_x-\beta p$. The estimated residuals are given by $\xi^{OLS} = \xi+\left(\beta-\beta^{OLS}\right)p^*$. As $\beta-\beta^{OLS} = \frac{1}{\beta}\frac{Cov(\xi,\lambda)}{Var(p^*)}-\frac{Cov(\xi,g(q))}{Var(p^*)}$, we obtain $\xi^{OLS} =$

$\xi + \left(\frac{1}{\beta}\frac{Cov(\xi,\lambda)}{Var(p^*)} - \frac{Cov(\xi,g(q))}{Var(p^*)}\right)p^*$. This implies

$$Cov\left(\xi^{OLS},\lambda\right) = \left(1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\xi,\lambda) - \frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\xi,g(q))$$
$$Cov\left(\xi^{OLS},g(q)\right) = \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov\left(\xi,\lambda\right) + \left(1 - \frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi,g(q))$$

We write the system of equations in matrix form and invert to solve for the covariance terms that include the unobserved demand shock:

$$\begin{bmatrix} Cov(\xi,\lambda) \\ Cov(\xi,g(q)) \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)} & -\frac{Cov(p^*,\lambda)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1} \begin{bmatrix} Cov(\xi^{OLS},\lambda) \\ Cov(\xi^{OLS},g(q)) \end{bmatrix}$$

where

$$\begin{bmatrix} 1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)} & -\frac{Cov(p^*,\lambda)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1} =$$
$$\frac{1}{1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}} \begin{bmatrix} 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} & \frac{Cov(p^*,\lambda)}{Var(p^*)} \\ -\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)} \end{bmatrix}.$$

Therefore, we obtain the relations

$$Cov(\xi,\lambda) = \frac{\left(1 - \frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi^{OLS},\lambda) + \frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\xi^{OLS},g(q))}{1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}}$$
$$Cov(\xi,g(q)) = \frac{-\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov(\xi^{OLS},\lambda) + \left(1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\xi^{OLS},g(q))}{1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}}.$$

In terms of observables, we can substitute in for $Cov(\xi,g(q)) - \frac{1}{\beta}Cov\left(\xi,\lambda\right)$ in the plim of the OLS estimator and simplify:

$$\begin{aligned}
&\left(1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}\right)\left(Cov(\xi,g(q)) - \frac{1}{\beta}Cov\left(\xi,\lambda\right)\right) \\
&= -\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov(\xi^{OLS},\lambda) + \left(1 + \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\xi^{OLS},g(q)) \\
&\quad - \frac{1}{\beta}\left(1 - \frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi^{OLS},\lambda) - \frac{1}{\beta}\frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\xi^{OLS},g(q)) \\
&= Cov(\xi^{OLS},g(q)) - \frac{1}{\beta}Cov(\xi^{OLS},\lambda).
\end{aligned}$$

Thus, we obtain an expression for the probability limit of the OLS estimator,

$$\text{plim}\hat{\beta}^{OLS} = \beta - \frac{\frac{Cov(\xi^{OLS},\lambda)}{Var(p^*)} - \beta\frac{Cov(\xi^{OLS},g(q))}{Var(p^*)}}{\beta + \frac{Cov(p^*,\lambda)}{Var(p^*)} - \beta\frac{Cov(p^*,g(q))}{Var(p^*)}},$$

and the following quadratic β.

$$\begin{aligned} 0 = &\left(1 - \frac{Cov(p^*, g(q))}{Var(p^*)}\right)\beta^2 \\ &+ \left(\frac{Cov(p^*, \lambda)}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, g(q))}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\xi^{OLS}, g(q))}{Var(p^*)}\right)\beta \\ &+ \left(-\frac{Cov(p^*, \lambda)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov(\xi^{OLS}, \lambda)}{Var(p^*)}\right). \end{aligned}$$

QED.