Demand Estimation in Models of Imperfect Competition*

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Abstract

We consider the identification and estimation of demand systems in models of imperfect competition. Under standard assumptions about demand and supply, the bias that arises from price endogeneity can be resolved without the use of instruments. We provide a constructive identification result where the causal price parameter can be expressed as a function of the covariance of unobserved shocks. The function is estimated efficiently by the output of ordinary least squares regression. Thus, with a covariance restriction on unobservable shocks, structural parameters can be point identified. Further, it can be possible to place bounds on the structural parameters without imposing a covariance restriction. We illustrate the methodology with applications to ready-to-eat cereal, cement, and airlines.

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1 Introduction

A central challenge of demand estimation is price endogeneity. If prices reflect demand shocks that are not observed by the econometrician, then ordinary least squares regression (OLS) does not recover the casual demand curve (Working, 1927). To correct for bias, researchers typically use estimation techniques that rely on instrumental variables. In this paper, we reconsider whether instruments are necessary to recover causal demand parameters.[1] We show that the supply-side assumptions already maintained in many models of imperfect competition dictate how prices respond to demand shocks. By leveraging these assumptions in estimation, it is possible to correct for endogeneity bias without isolating exogenous variation in prices.

Our methodology begins with an analysis of equilibrium variation in prices and quantities. We show that, with standard empirical models of imperfect competition, the bias in OLS estimates is a function of data and demand parameters. Thus, OLS estimates are informative, as they capture a blend of the demand curve and the endogenous response by firms. The supply-side assumptions may be used to construct bounds on the structural parameters and, with the addition of a covariance restriction that we introduce below, achieve point identification. Furthermore, accounting for supply-side behavior ensures that the estimated parameters align with economic intuition (e.g., that demand slopes downward). The methodology essentially uses economic theory to interpret and exploit all the relevant variation in the data.

Consider the class of models in which price is determined by additively separable markup and marginal cost terms, and demand takes a semi-linear form that nests the discrete-choice models common in empirical research (Berry, 1994). In this setting, the OLS bias can be decomposed into two components, each pertaining to: (i) the covariance between demand shocks and markups and (ii) the covariance between demand shocks and marginal costs. Using the supply-side model, the first component of bias is a function of the data. Therefore, a covariance restriction on demand shocks and marginal costs can be sufficient for identification. This intuition extends to a change-of-variables in prices, allowing the approach to be applied to models that have multiplicative markups, such as constant elasticity demand.

We first develop intuition using a model of a monopolist with constant marginal costs and linear demand (Section 2). Equilibrium variation in prices and quantities (p and q) is generated by uncorrelated demand and cost shocks (ξ and η) that are unobservable to the econometrician. We prove that consistent estimate of the price parameter, β, is given by

$$\hat{\beta} = -\sqrt{\left(\hat{\beta}^{OLS}\right)^2 + \frac{Cov(\hat{\xi}^{OLS}, q)}{Var(p)}}$$

where $\hat{\beta}^{OLS}$ is the price coefficient from an OLS regression of quantities on prices, and $\hat{\xi}^{OLS}$

[1] We consider instruments to be empirical objects that are an input in an estimation step (for example, two-stage least squares). Instruments may be constructed from other sources of data or estimated in a previous step.

is a vector of the OLS residuals. The information provided by OLS regression is sufficient for the consistent identification of the structural parameters. This holds whether variation arises predominately from demand shocks or from supply shocks—economic theory allows for identification of the structural parameter amidst a cloud of price-quantity pairs.

We obtain our baseline results (Section 3) under two common assumptions about demand and supply. We assume that demand is semi-linear in prices after a known transformation. This assumption nests many differentiated-products demand systems, including the random coefficients logit (e.g., Berry et al., 1995). On the supply side, we begin with firms that compete in prices à la Nash and have constant marginal costs. Our identification result is that the price parameter, β, solves a quadratic equation in which the coefficients are functions of the data and the covariance between unobserved demand shocks and marginal costs, $Cov(\xi, \eta)$. We provide a sufficient condition under which β is the lower root of the quadratic; if the condition holds then knowledge of this covariance point identifies β. We then derive a consistent three-stage estimator from the quadratic formula. The estimator is constructed from OLS coefficients and residuals, and Monte Carlo experiments indicate that it performs well in small samples.

Though economists have recognized that supply-side assumptions and covariance restrictions can be used together in estimation, identification results for general models of imperfect competition have not previously been formalized.[2] We show that these assumptions are often sufficient to point identify structural parameters. However, it is possible that a given covariance restriction can admit multiple solutions or even zero solutions; in the latter case the joint set of assumptions are rejected. We provide a means to calculate the full set of solutions, as well as an auxiliary condition that can be used to achieve point identification. To implement our methodology, we impose the *uncorrelatedness* assumption, $Cov(\xi, \eta) = 0$. This assumption could, alternatively, be used to construct a method-of-moments estimator. A method-of-moments estimator can obtain identical estimates with a greater computation burden, but it may converge to an incorrect solution if point identification has not been verified.

Even without exact knowledge of $Cov(\xi, \eta)$, supply-side restrictions can be used to place bounds on β. First, weaker assumptions about $Cov(\xi, \eta)$ that are motivated by the economic environment can be used to construct bounds on the causal parameters. For example, it may be reasonable to assume that there is positive correlation between unobserved shocks to supply and demand, in which case an upper bound on β is obtained. Second, certain values of β may be ruled out without any prior knowledge of $Cov(\xi, \eta)$. We show how to construct these *prior-free* bounds, which arise when the parameter values do not rationalize the data given the assumptions of the model.

[2]The international trade literature provides identification results for the special case where markups do not respond to demand shocks (e.g., Feenstra, 1994). We discuss this literature in more detail later. For applications in industrial organization, see Thomadsen (2005), Cho et al. (2018), and Li et al. (2018). Thomadsen (2005) assumes no unobserved demand shocks, and Cho et al. (2018) assume no unobserved cost shocks; both are special cases of a covariance restriction.

In Section 4, we consider generalizations of the assumptions used to develop our baseline results. First, we show additional covariance restrictions may be used to identify additional endogenous or nonlinear parameters. Next, we prove that identification of β is preserved with non-constant marginal costs if the non-constant portion can be brought into the model and estimated. We then consider multi-product firms, which is a straightforward extension of the single-product case used to develop notation earlier. Finally, we show that our approach is not dependent on the precise nature of the competitive game. Instead, it relies on the general property that prices can be structurally decomposed into additively separable marginal costs and markup terms. Our identification result and three-stage estimator are easily adapted to other models of competition, including Cournot. In Appendix B, we provide an extension to demand systems that are not semi-linear in prices. We focus on semi-linear demand in the main text for expositional clarity.

We provide three empirical applications in Section 5. The first application estimates demand in the ready-to-eat cereal industry using the model and data of Nevo (2000). Demand is specified as random coefficients logit. In this setting, there is no variation in product choice sets across markets, so the characteristic-based instruments used by Berry et al. (1995) are not available. We supplement the uncorrelatedness assumption with cross-product covariance restrictions in order to estimate the random coefficient parameters. We find that our estimates correspond closely to estimates that employ the instruments available in the dataset.

The second application examines the cement industry using the model and data of Fowlie et al. (2016) ["FRR"], extending the approach to Cournot competition. The setting allows for an assessment of the uncorrelatedness assumption, motivated by the same understanding of institutional details used to evaluate instruments. Unobserved demand shocks for cement reflect local construction activity, whereas marginal cost variation is primarily due to capacity utilization and coal prices. As coal prices are a good proxy for marginal cost shocks (after accounting for capacity constraints), uncorrelatedness is a reasonable assumption if coal prices are orthogonal to local construction activity. This orthogonality is the same assumption used to justify coal prices as a valid instrument in FRR. Not surprisingly, a three-stage estimator obtains results similar to two-stage least squares using the FRR instruments.

The third empirical application explores the airline industry using the model and data of Aguirregabiria and Ho (2012) ["AH"]. We focus on constructing bounds for β and σ, where σ is the coefficient on the conditional share in the nested logit demand system. We show that prior-free bounds rule out values of (β, σ), and we impose priors that are motivated by the economics of the market. Airlines face an opportunity cost for each sold seat because they cannot re-sell the seat later for a higher price. High demand increases this opportunity cost by increasing the probability that flights reach capacity. Therefore, we construct bounds under the assumption that demand shocks and marginal costs are positively correlated, within products and also within all nonstop or connecting flights in a market. These non-nested bounds provide the identified set for (β, σ), which include the unconditional bounds $\sigma > 0.599$ and $\beta <$ -0.067 for any value of the other parameter.

Section 6 provides two discussions that help frame the methodology we introduce. First, we argue that an understanding of institutional details can allow for an assessment of uncorrelatedness even though the structural error terms are (by definition) unobserved. Indeed, sometimes the institutional details will suggest that uncorrelatedness is unreasonable. Second, we relate uncorrelatedness to the instrumental variables approach. The most obvious similarity is that both approaches rely on orthogonality conditions that are not verifiable empirically but can be assessed with institutional details. This connection is especially clear with the so-called "Hausman" instruments—prices of the same good in other markets—for which consistency requires orthogonality among demand shocks across markets. However, the assumptions embedded by the two approaches are not generally nested: uncorrelatedness does not require any source of exogenous variation but does require a correctly-specified supply-side model.

Our research builds on several strands of literature in economics. Early research at the Cowles Foundation (Koopmans et al., 1950) examines the identifying power of covariance restrictions in linear systems of equations, and a number of articles pursued this agenda in subsequent years (e.g., Fisher, 1963, 1965; Wegge, 1965). Hausman and Taylor (1983) show that identification can be interpreted in terms of instrumental variables: a demand-side instrument is used to recover the (initially unobserved) supply-side shock, which is then a valid instrument in demand estimation under the covariance restriction. Matzkin (2016) and Chiappori et al. (2017) provide extensions to semi-parametric models. Estimators along these lines are consistent given sufficient variation in an excluded variable and the supply-side shock. By contrast, the three-stage estimator we introduce does not require either source of variation yet extends to imperfect competition nonetheless. In a more general sense, our approach is conceptually related to control function estimation procedures (e.g., Heckman, 1979).

A parallel literature explores the identification of supply and demand in models of international trade, building on the insight that a covariance restriction is sufficient to bound either the slope of supply or the slope of demand in linear models of perfect competition (Leamer, 1981). Feenstra (1994) considers monopolistic competition with constant elasticity demand. Identification is achieved if the econometrician observes at least two samples (e.g., from different countries) with equal slopes of demand and supply but unequal variances of the shocks. This approach is applied in Broda and Weinstein (2006, 2010), among other articles. With constant elasticity demand, firms do not adjust markups in response to demand shocks, and therefore these trade models do not allow for a primary source of endogeneity bias that arises in a more general model of firm behavior. Hottman et al. (2016) and Feenstra and Weinstein (2017) estimate models that allow for some variation in markups, but identification for these extensions is not established. Thus, to our knowledge, we are the first to prove that a covariance restriction can identify models where firms respond strategically to demand shocks.[3]

Another approach is to estimate supply and demand via maximum likelihood, under the assumption that the distributions of demand and cost shocks are known to the econometrician and independent. At least one seminal article in industrial organization, Bresnahan (1987), pursues this approach. In the marketing literature, Yang et al. (2003) estimate an oligopoly model of supply and demand using Bayesian techniques to lessen the computation burden. For discussions and extensions of this approach see Rossi et al. (2005), Dotson and Allenby (2010), and Otter et al. (2011). Published comments on Yang et al. (2003) point out that a coherent likelihood function may not exist due to multiple equilibria (Bajari, 2003; Berry, 2003). By contrast, our approach does not require a likelihood function and provides consistent estimates in the presence of multiple equilibria. Further, it does not require distributional assumptions. These advantages may make our approach relatively more palatable for oligopoly models.

Finally, price endogeneity has been a major focus of modern empirical and econometric research in industrial organization. Typically, the challenge is cast as a problem of finding valid instruments. Many possibilities have been developed, including the attributes of competing products (Berry et al., 1995; Gandhi and Houde, 2015), the prices of the same good in other markets (e.g., Hausman, 1996; Nevo, 2001; Crawford and Yurukoglu, 2012), or shifts in the equilibrium concept (e.g., Porter, 1983; Miller and Weinberg, 2017).[4] When valid instruments are available, the estimation techniques presented here may be used to construct overidentifying restrictions and test the model.

2 A Motivating Example: Monopoly Pricing

We introduce the supply-side identification approach with a motivating example of monopoly pricing, in the spirit of Rosse (1970). In each market $t = 1, \dots, T$, the monopolist faces a downward-sloping linear demand schedule, $q_t = \alpha + \beta p_t + \xi_t$, where q_t and p_t denote quantity and price, respectively, $\beta < 0$ is the price parameter, and ξ_t is mean-zero stochastic demand shock. Marginal cost is given by the function $c_t = \gamma + \eta_t$, where γ is some constant and η_t is a mean-zero stochastic cost shock. Prices are set to maximize profit. The econometrician observes vectors of prices, $p = [p_1, p_2, \dots, p_T]'$, and quantities, $q = [q_1, q_2, \dots, q_T]'$. The markets can be conceptualized as geographically or temporally distinct.

An OLS regression of q on p obtains a biased estimate of β if the monopolist's price reflects

[3]This trade literature has interesting historical antecedents. Leamer attributes an early version of his results to Schultz (1928). The identification argument of Feenstra (1994) is also proposed in Leontief (1929). Frisch (1933) provides an econometric critique of this argument.

[4]Byrne et al. (2016) proposes a novel set of instruments that leverage the structure of a discrete choice demand model with differentiated-products price competition. Nevo and Wolfram (2002) explore whether covariance restrictions can bound parameters (see footnote 41 of that article).

the unobservable demand shock, as is the case here given profit maximization. Formally,

$$\hat{\beta}^{OLS} = \frac{\hat{Cov}(p,q)}{\hat{Var}(p)} \quad \xrightarrow{p} \quad \beta + \frac{Cov(\xi,p)}{Var(p)} \tag{1}$$

The monopolist's profit-maximization conditions are such that price is equal to marginal cost plus a markup term: $p_t = \gamma + \eta_t - \left(\frac{dq}{dp}\right)^{-1} q_t$. Thus, the numerator of the OLS bias can be decomposed into the covariance between demand shocks and markups and the covariance between demand shocks and marginal cost shocks. This leads to our first theoretical result, which we obtain under the uncorrelatedness assumption that $Cov(\xi,\eta) = 0$:

Proposition 1. *Let the OLS estimates of (α,β) be $(\hat{\alpha}^{OLS},\hat{\beta}^{OLS})$ with probability limits $(\alpha^{OLS},\beta^{OLS})$, and denote the residuals at the limiting values as $\xi_t^{OLS} = q_t - \alpha^{OLS} - \beta^{OLS} p_t$. When demand shocks and cost shocks are uncorrelated, the probability limit of the OLS estimate can be expressed as a function of the true price parameter, the residuals from the OLS regression, prices, and quantities:*

$$\beta^{OLS} \equiv plim\left(\hat{\beta}^{OLS}\right) = \beta - \frac{1}{\beta + \frac{Cov(p,q)}{Var(p)}} \frac{Cov(\xi^{OLS},q)}{Var(p)} \tag{2}$$

Proof: We provide the proofs in this section for illustrative purposes; most subsequent proofs are confined to the appendix. Reformulate equation (1) as follows:

$$\beta^{OLS} = \beta + \frac{Cov(\xi, \eta - \frac{1}{\beta}q)}{Var(p)} = \beta - \frac{1}{\beta}\frac{Cov(\xi,q)}{Var(p)}$$

The first equality holds due to the first-order condition $p = \gamma + \eta_t - \frac{1}{\beta}q$. The second equality holds due to the uncorrelatedness assumption. As $\alpha + \beta p + \xi = \alpha^{OLS} + \beta^{OLS} p + \xi^{OLS}$, we have

$$\begin{aligned} Cov(\xi,q) &= Cov(\xi^{OLS} - (\beta - \beta^{OLS})p, q) \\ &= Cov(\xi^{OLS},q) - (\beta - \beta^{OLS})Cov(p,q) \\ &= Cov(\xi^{OLS},q) - \frac{1}{\beta}\frac{Cov(\xi,q)}{Var(p)}Cov(p,q) \end{aligned}$$

Collecting terms and rearranging implies

$$\frac{1}{\beta}Cov(\xi,q) = \frac{1}{\beta + \frac{Cov(p,q)}{Var(p)}} Cov(\xi^{OLS},q)$$

Plugging into the reformulation of equation (1) obtains the proposition. QED.

The proposition makes clear that, among the objects that characterize β^{OLS}, only β does not have a well understood sample analog. Further, as β^{OLS} can be estimated consistently, the proposition suggests the possibility that β can be recovered from the data. Indeed, a closer inspection of equation (2) reveals that β solves a quadratic equation:

Proposition 2. *When demand shocks and marginal cost shocks are uncorrelated, β is point identified as the lower root of the quadratic equation*

$$\beta^2 + \beta\left(\frac{Cov(p,q)}{Var(p)} - \beta^{OLS}\right) + \left(-\frac{Cov(\xi^{OLS},q)}{Var(p)} - \frac{Cov(p,q)}{Var(p)}\beta^{OLS}\right) = 0 \tag{3}$$

and a consistent estimate of β is given by

$$\hat{\beta}^{\textit{3-Stage}} = -\sqrt{\left(\hat{\beta}^{OLS}\right)^2 + \frac{Cov(\hat{\xi}^{OLS},q)}{Var(p)}} \tag{4}$$

Proof: The quadratic equation is obtained as a re-expression of equation (2). An application of the quadratic formula provides the following roots:

$$\frac{-\left(\frac{Cov(p,q)}{Var(p)} - \beta^{OLS}\right) \pm \sqrt{\left(\frac{Cov(p,q)}{Var(p)} - \beta^{OLS}\right)^2 + 4\left(\frac{Cov(\xi^{OLS},q)}{Var(p)} + \frac{Cov(p,q)}{Var(p)}\beta^{OLS}\right)}}{2}.$$

In the univariate case, $\frac{Cov(p,q)}{Var(p)} = \beta^{OLS}$, which cancels out terms and obtains the probability limit analog of equation (4). It is easily verified that $\left(\beta^{OLS}\right)^2 + \frac{Cov(\xi^{OLS},q)}{Var(p)} > 0$ so both roots are real numbers. The upper root is positive, so β is point identified as the lower root. Equation (4) provides the empirical analog to the lower root. As the sample estimates of covariance terms are consistent for the limits, it provides a consistent estimate of β.

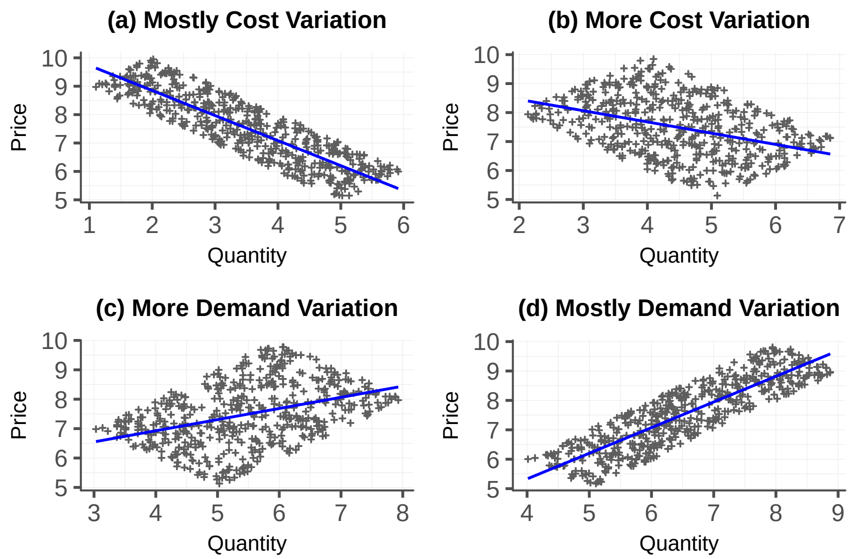
The first part of the proposition states that β solves a quadratic equation. There are two real roots, but only one is negative, so point identification is achieved. Further, an adjustment to the OLS estimator is sufficient to correct for bias. We label the estimator $\hat{\beta}^{\textit{3-Stage}}$ for reasons that become evident with the more general treatment later in the paper.

An additional simplification is available. Because $\xi_t^{OLS} = q_t - \alpha^{OLS} - \beta^{OLS}p_t$, we have $\frac{Cov(\xi^{OLS},q)}{Var(p)} = \frac{Var(q)}{Var(p)} - \beta^{OLS}\frac{Cov(p,q)}{Var(p)}$. Plugging into equation (4) obtains the following corollary:

Corollary 1. $\hat{\beta}^{\textit{3-Stage}} = -\sqrt{\frac{Var(q)}{Var(p)}}$.

In the monopoly model, the price parameter is identified from the relative variation in prices and quantities. To build additional intuition about this approach, we recast the monopoly problem in terms of supply and demand in Appendix A.1, providing a link to the Hayashi (2000) textbook treatment of bias with simultaneous equations.

Figure 1: Price and Quantity in the Monopoly Model



Notes: Figure displays equilibrium prices and quantities under four different specifications for the distribution of unobserved shocks to demand and marginal costs. The line in each figure indicates the slope obtained by OLS regression.

Consider the following numerical example. Let demand be given by $q_t = 10 - p_t + \xi_t$ and let marginal cost be $c_t = \eta_t$, so that $(\alpha, \beta, \gamma) = (10, -1, 0)$. Let the demand and cost shocks have independent uniform distributions. The monopolist sets price to maximize profit. As is well known, if both cost and demand variation is present then equilibrium outcomes provide a "cloud" of data points that do not necessarily correspond to the demand curve. To illustrate this, we consider four cases with varying degrees of cost and demand variation. In case (1), $\xi \sim U(0,2)$ and $\eta \sim U(0,8)$. In case (2), $\xi \sim U(0,4)$ and $\eta \sim U(0,6)$. In case (3), $\xi \sim U(0,6)$ and $\eta \sim U(0,4)$. In case (4), $\xi \sim U(0,8)$ and $\eta \sim U(0,2)$. We randomly take 500 draws for each case and calculate the equilibrium prices and quantities.

The data are plotted in Figure 1 along with OLS fits. The experiment represents the classic identification problem of demand estimation: the empirical relationship between equilibrium prices and quantities can be quite misleading about the slope of the demand function. However, Proposition 2 and Corollary 1 state that the structure of the model together with the OLS estimates allow for consistent estimates of the price parameter. Table 1 provides the required empirical objects. The OLS estimates, $\hat{\beta}^{OLS}$, are negative when there is greater variance in costs than demand shocks and positive when there is relatively more variation in demand shocks, as is also revealed in the scatter plots. By contrast, the adjustment term $\frac{Cov(\hat{\xi}^{OLS}, q)}{Var(p)}$ is larger if the cost and demand shocks have more similar support. Equation (4) yields estimates, $\hat{\beta}^{\textit{3-Stage}}$, that are nearly equal to the population value of -1.00. Note also that the variance of price and quantity are similar in each column, consistent with Corollary 1 and the data generating

Table 1: Numerical Illustration for the Monopoly Model

	(1) $Var(\eta) \gg Var(\xi)$	(2) $Var(\eta) > Var(\xi)$	(3) $Var(\eta) < Var(\xi)$	(4) $Var(\eta) \ll Var(\xi)$
$\hat{\beta}^{OLS}$	-0.87	-0.42	0.42	0.88
$Var(q)$	1.42	1.11	1.20	1.36
$Var(p)$	1.44	1.01	1.09	1.35
$Cov(\hat{\xi}^{OLS}, q)$	0.33	0.92	1.01	0.32
$\frac{Cov(\hat{\xi}^{OLS},q)}{Var(p)}$	0.23	0.91	0.93	0.24
$\hat{\beta}^{\text{3-Stage}}$	-0.995	-1.045	-1.051	-1.003

Notes: Based on numerically generated data that conform to the motivating example of monopoly pricing. The demand curve is $q_t = 10 - p_t + \xi_t$ and marginal costs are $c_t = \eta_t$, so that $(\beta_0, \beta, \gamma_0) = (10, -1, 0)$. In column (1), $\xi \sim U(0,2)$ and $\eta \sim U(0,8)$. In column (2), $\xi \sim U(0,4)$ and $\eta \sim U(0,6)$. In column (3), $\xi \sim U(0,6)$ and $\eta \sim U(0,4)$. In column (4), $\xi \sim U(0,8)$ and $\eta \sim U(0,2)$. Thus, the support of the cost shocks are largest (smallest) relative to the support of the demand shocks in the left-most (right-most) column.

process.[5]

3 Methodology: Bounds and Three-Stage Estimation

We present our main results through the lens of differentiated-products Bertrand competition. We provide identification conditions, show how bounds can be constructed without strong covariance restrictions, and then consider estimation via the three-stage approach and the method of moments. We illustrate small-sample properties with a numerical simulation. For extensions and a discussion, see Sections 4 and 6.

3.1 Data Generating Process

Let there be $j = 1, 2, \ldots, J$ products in each of $t = 1, 2, \ldots, T$ markets,[6] subject to downward-sloping demands. The econometrician observes vectors of prices, $p_t = [p_{1t}, p_{2t}, \ldots, p_{Jt}]'$, and quantities, $q = [q_{1t}, q_{2t}, \ldots, p_{Jt}]'$, corresponding to each market t, as well as a matrix of covariates $X_t = [x_{1t}\ x_{2t}\ \ldots\ x_{Jt}]$. The covariates are orthogonal to a pair of demand and marginal cost shocks (i.e,. $E[X\xi] = E[X\eta] = 0$) that are common knowledge among firms but unobserved by

[5]Inspection of Figure 1 further suggests that there is a connection between the magnitude of bias adjustment and the goodness-of-fit from the OLS regression of price on quantity. Starting with equation (4), a few lines of additional algebra obtains

$$\beta = -\frac{|\beta^{OLS}|}{\sqrt{R^2}}$$

where R^2 is from the OLS regression. This reformulation can fail if $R^2 = 0$ but numerical results indicate the formula is robust for values of R^2 that are approximately zero. We thank Peter Hull for this suggestion.

[6]In our empirical applications, we extend the notation to distinguish between geographic markets and time periods.

the econometrician.[7] We make the following assumptions on demand and supply:

Assumption 1 (Demand): *The demand schedule for each product is determined by the following semi-linear form:*

$$h_{jt} \equiv h(q_{jt}, w_{jt}; \sigma) = \beta p_{jt} + x_{jt}'\alpha + \xi_{jt} \tag{5}$$

where (i) $\frac{\partial h_{jt}}{\partial q_{jt}} > 0$, (ii) w_{jt} is a vector of observables and σ is a parameter vector, and (iii) the total derivatives of $h(\cdot)$ with respect to q exist as functions of the data and σ.

> *Example:* For the logit demand system, $h(q_{jt}; w_{jt}, \sigma) \equiv \ln(s_{jt}/w_{jt})$, where quantities are in shares ($q_{jt} = s_{jt}$) and w_{jt} is the share of the outside good (s_{0t}). There are no additional parameters in σ.

The demand assumption restricts attention to systems for which, after a transformation of quantities using observables (w_{jt}) and nonlinear parameters (σ), there is additive separability in prices, covariates, and the demand shock. The vector w_{jt} can be conceptualized as including the price and non-price characteristics of products, in particular those of other products that affect the demand of product j. Often, only a few observables are necessary to construct the transformation; in the logit discrete choice example above, the share of the outside good is a sufficient statistic to capture demand for other products.

Other typical demand systems also fit into this framework, including models with additional endogenous covariates. For the nested logit demand model, w_{jt} consists of two elements: the outside share (s_{0t}) and the within-group share ($\bar{s}_{j|g}$). One nonlinear parameter (σ) is needed for the transformation: $h(s_{jt}; w_{jt}, \sigma) \equiv \ln s_{jt} - \ln s_{0t} - \sigma \ln \bar{s}_{j|g,t}$. For the more flexible random coefficients logit demand system, $h(q_{jt}; w_{jt}, \sigma)$ can be defined as the mean utility and calculated using the contraction mapping of Berry et al. (1995) for any candidate σ vector. The demand assumption also nests monopolistic competition with linear demands (e.g., as in the motivating example). We derive these connections in some detail in Appendix B.

The third condition on the demand system allows us to complete an identification proof by constructing the first-order conditions implied by the demand system and the supply-side assumptions, which we introduce next. The total derivative is given by

$$\frac{dh_{jt}}{dq_{jt}} = \frac{\partial h_{jt}}{\partial q_{jt}} + \frac{\partial h_{jt}}{\partial w_{jt}} \cdot \frac{dw_{jt}}{dq_{jt}}$$

When an outcome variable is used in w_{jt} to construct the transformation, such as in the discrete choice demand systems mentioned above, it may be the case that $\frac{dw_{jt}}{dq_{jt}}$ depends upon what is held fixed under the competitive assumptions (e.g. the prices of other firms at the Bertrand-Nash equilibrium.). This should be accounted for when constructing the derivatives.

[7] We make the usual assumption that prices and covariates are linearly independent to allow for OLS estimation.

Example: For the logit demand system at a Bertrand-Nash equilibrium, $\frac{\partial h_{jt}}{\partial q_{jt}} = \frac{1}{s_{jt}}$, $\frac{\partial h_{jt}}{\partial w_{jt}} = -\frac{1}{s_{0t}}$, and $\frac{dw_{jt}}{dq_{jt}} = \frac{ds_{0t}/dp_{jt}}{ds_{jt}/dp_{jt}} = -\frac{s_{0t}}{1-s_{jt}}$. Thus we obtain $\frac{dh_{jt}}{dq_{jt}} = \frac{1}{s_{jt}(1-s_{jt})}$. These derivatives are calculated holding the prices of other products fixed.

Assumption 2 (Supply): *Each firm sells a single product and sets its price to maximize profit in each market. The firm takes the prices of other firms as given, knows the demand schedule in equation* (5)*, and has a linear constant marginal cost schedule given by*

$$c_{jt} = x'_{jt}\gamma + \eta_{jt}. \tag{6}$$

Under assumptions 1 and 2, there is a unique mapping from the data and parameters to the structural error terms (ξ, η). The supply-side assumption is strong but allows for a base set of identification results to be derived with minimal notation. In subsequent sections, we provide the additional notation necessary for models with multi-product firms, non-constant marginal costs, and Cournot-Nash competition. Note that supply and demand may depend on different covariates; this is captured when non-identical components of α and γ are equal to zero.

We further assume the existence of a Nash equilibrium in pure strategies, and that each firm satisfies the first-order condition

$$p_{jt} = c_{jt} - \frac{1}{\beta}\frac{dh_{jt}}{dq_{jt}}q_{jt}. \tag{7}$$

To obtain this expression, take the total derivative of h with respect to q_{jt}, re-arrange to obtain $\frac{dp_{jt}}{dq_{jt}} = \frac{1}{\beta}\frac{dh_{jt}}{dq_{jt}}$, and substitute into the more standard formulation of the first-order condition: $p = c - \frac{dp}{dq}q$. First-order conditions that admit multiple equilibria are unproblematic. It must be possible recover (ξ, η) from the data and parameters, but the mapping to prices from the parameters, exogenous covariates, and structural error terms need not be unique.

Our identification result relies on the markup being proportional to the reciprocal of the price parameter, which arises here due to the semi-linear demand system.[8] When this is the case, equilibrium prices respond to the demand shock through markup adjustments, which are fully determined by β, the structure of the model, and observables. Thus, first-order conditions may be useful in analyzing the covariance of demand shocks and prices, which determines the bias of the OLS estimate. As in the monopoly example of Section 2, this provides a basis to correct the bias from OLS estimation and solve for the true price parameter. We develop this identification argument below.

[8] The semi-linear structure is not strictly necessary. In practice, one could start with a known first-order condition and show that it takes the form $p_{jt} = c_{jt} - \frac{1}{\beta}f_{jt}$ for some function of the data f_{jt}. Further, we show how the results can be extended to demand systems that admit multiplicative markups in Appendix B.

3.2 Identification

We now formalize the identification argument for β, the price parameter. We assume the parameters in σ are known to the econometrician. The linear non-price parameters (α, γ) can be recovered trivially given β and σ.[9] We start by characterizing the OLS estimate of the price parameter, which is obtained from a regression of $h(\cdot)$ on p and x. The probability limit contains the standard bias term:

$$\beta^{OLS} \equiv \frac{Cov(p^*, h)}{Var(p^*)} = \beta + \frac{Cov(p^*, \xi)}{Var(p^*)} \tag{8}$$

where $p^* = [I - x(x'x)^{-1}x']p$ is a vector of residuals from a regression of p on x. Plugging in for price on the right-hand-side of equation (8) using the first-order conditions yields

$$\beta^{OLS} = \beta - \frac{1}{\beta}\frac{Cov(\frac{dh}{dq}q, \xi)}{Var(p^*)} + \frac{Cov(\eta, \xi)}{Var(p^*)}. \tag{9}$$

We express the unobserved demand shock ξ in terms of the OLS residuals and parameters to obtain our first general result: β solves a quadratic equation in which the coefficients are determined by $Cov(\xi, \eta)$ and objects with empirical analogs.

Proposition 3. *Under assumptions 1 and 2, the probability limit of the OLS estimate can be written as a function of the true price parameter, the residuals from the OLS regression, the covariance between demand and supply shocks, prices, and quantities:*

$$\beta^{OLS} = \beta - \frac{1}{\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}} \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} + \beta \frac{1}{\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}} \frac{Cov(\xi, \eta)}{Var(p^*)}. \tag{10}$$

The price parameter β solves the following quadratic equation:

$$\begin{aligned} 0 &= \beta^2 \\ &+ \left(\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \beta^{OLS} \right) \beta \\ &+ \left(-\beta^{OLS} \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} \right). \end{aligned} \tag{11}$$

Proof. See appendix.

Proposition 3 provides our core identification result. There are two main implications. First, the quadratic in equation (11) admits at most two solutions for a given value of $Cov(\xi, \eta)$. It

[9] An alternative interpretation is that the econometrician is considering a candidate σ and wishes to obtain corresponding estimates of (β, α, γ), as in the nested fixed-point estimation routine of Berry et al. (1995) and Nevo (2001) for the random coefficients logit demand system.

follows immediately that, with prior knowledge of $Cov(\xi, \eta)$, the price parameter β is set identified with a maximum of two elements (points). Indeed, as we show next, conditions exist that guarantee point identification. Second, if the econometrician does not have specific knowledge of $Cov(\xi, \eta)$, it nonetheless can be possible to bound β. We consider point identification first, as the intuition behind point identification maps neatly into how to construct bounds.

Assumption 3': *The econometrician has prior knowledge of $Cov(\xi, \eta)$.*

Proposition 4. ***(Point Identification)*** *Under assumptions 1 and 2, the price parameter β is the lower root of equation* (11) *if the following condition holds:*

$$0 \leq \beta^{OLS} \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)} \tag{12}$$

and, furthermore, β is the lower root of equation (11) *if and only if the following condition holds:*

$$-\frac{1}{\beta} \frac{Cov(\xi, \eta)}{Var(p^*)} \leq \frac{Cov\left(p^*, -\frac{1}{\beta}\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*, \eta\right)}{Var(p^*)} \tag{13}$$

Therefore, under assumptions 1, 2 and 3', β is point identified if either of these conditions holds.

Proof. See appendix.

The first (sufficient) condition is derived as a simple application of the quadratic formula: if the constant term in the quadratic of equation (11) is negative then the upper root of the quadratic is positive and β must be the lower root. For some model specifications, the condition can be proven analytically.[10] Otherwise it can be evaluated empirically using the data and assumptions 1 and 2. If the sufficient condition holds, then β is point identified with prior knowledge of $Cov(\xi, \eta)$ because all the terms in equation (11) are known or can be obtained from the data. If the condition fails, point identification of β is not guaranteed even with prior knowledge of $Cov(\xi, \eta)$, though the econometrician has reduced the identified set to two points.

The necessary and sufficient condition is more nuanced. Even with prior knowledge of $Cov(\xi, \eta)$, condition (13) contains elements that are not observed by the econometrician. Still, for some specifications, the condition can be verified analytically.[11] The condition holds under

[10] An example is a monopolist with a linear demand system. Following the logic of Corollary 1, we have

$$\beta^{OLS} \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)} \quad = \quad \frac{Var(q)}{Var(p)} > 0$$

[11] Consider again the example of a monopoly facing a linear demand system with $Cov(\xi, \eta) = 0$. In the proof of Proposition 4, we show that the necessary and sufficient condition is equivalent to $\beta < \frac{Cov\left(p^*, \xi\right)}{Var(p^*)} - \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)}$. With linear demand, we have that $\frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)} = \frac{Cov\left(p^*, q\right)}{Var(p^*)} = \beta^{OLS}$. Thus, the right-hand-side simplifies to $-\beta$ using equation (8). Because $\beta < 0$, $\beta < -\beta$ and the necessary and sufficient condition holds.

the standard intuition that prices increase both with demand and marginal cost shocks, provided that $Cov(\xi, \eta)$ is not too positive. To see this in the equation, note that the term $-\frac{1}{\beta}\xi$ is the shock to the inverse demand curve. The condition can fail if the empirical variation is driven predominately by demand shocks and the model dictates that prices decrease in the demand shock, which is possible with log-convex demand (Fabinger and Weyl, 2014).

3.3 Bounds

The model implies two complementary sets of bounds, neither of which requires exact knowledge of $Cov(\xi, \eta)$. We start by developing what we refer to as *bounds with priors*. If the econometrician has a prior over the plausible range of $Cov(\xi, \eta)$, along the lines of $m \leq Cov(\xi, \eta) \leq n$, then a posterior set for β can be constructed from the quadratic of equation (11). Each plausible $Cov(\xi, \eta)$ maps into one or two valid (i.e., negative) roots. Further, a monotonicity result that we formalize below establishes that, under either condition (12) or (13), there is a one-to-one mapping between the value of $Cov(\xi, \eta)$ and the lower root:

Lemma 1. ***(Monotonicity)*** *Under assumptions 1 and 2, a valid lower root of equation* (11) ***(i.e., one that is negative)*** *is decreasing in* $Cov(\xi, \eta)$*. The range of the function is* $(0, -\infty)$.

Proof. See appendix.

It follows immediately that a convex prior over $Cov(\xi, \eta)$ corresponds to convex posterior set. We suspect that, in practice, most priors will take the form $Cov(\xi, \eta) \geq 0$ or $Cov(\xi, \eta) \leq 0$. For example, an econometrician have reason to believe that higher quality products are more expensive to produce (yielding $Cov(\xi, \eta) \geq 0$) or that firms invest to lower the marginal costs of their best-selling products (yielding $Cov(\xi, \eta) \leq 0$). Priors of this firm generate one-sided bounds on β. Let $r(m)$ be the lower root of the quadratic evaluated at $Cov(\xi, \eta) = m$. Then under either condition (12) or (13), the prior $Cov(\xi, \eta) \geq m$ produces a posterior set of $(-\infty, r(m)]$, and the prior $Cov(\xi, \eta) \leq m$ produces a posterior set of $[r(m), 0)$.[12]

We now develop what we refer to as *prior-free bounds*. Even if the econometrician has no prior about $Cov(\xi, \eta)$, certain values may be possible to rule out because they imply that the observed data are incompatible with data generating process of the model. To see why, it is helpful to represent the quadratic of equation (11) as $az^2 + bz + c$, keeping in mind that one root is $z = \beta$ (< 0). Because $a = 1$, the quadratic forms a $\cup$-shaped parabola. If $c < 0$ then the existence of a negative root is guaranteed. However, if $c > 0$ then b must be positive and sufficiently large for a negative root to exist. By inspection of equation (11), this places restrictions on $Cov(\xi, \eta)$. We now state the result formally:

[12]Nevo and Rosen (2012) develop similar bounds for estimation with imperfect instruments, defined as instruments that are less correlated with the structural error term than the endogenous regressor.

Proposition 5. ***(Prior-Free Bound)*** *Under assumptions 1 and 2, the model and data may bound* $Cov(\xi, \eta)$ *from below. The bound is given by*

$$Cov(\xi,\eta) > Var(p^*)\beta^{OLS} - Cov\left(p^*, \frac{dh}{dq}q\right) + 2Var(p^*)\sqrt{\left(-\beta^{OLS}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}\right)}.$$

The bound exists if and only if the term inside the radical is non-negative. When β *is the lower root of equation* (11)*, this expression provides an upper bound on* β*.*

Proof. See appendix.

From the monotonicity result above, we can use the excluded values of $Cov(\xi, \eta)$ to rule out values of β as well. If point identification can be shown via the necessary and sufficient condition, then a prior-free upper bound for β is obtained by evaluating the lower root of equation (11) at the prior-free bound of $Cov(\xi, \eta)$.[13]

3.4 Estimation

The consistent estimation of β is possible if the conditions for point identification hold. The econometrician must have prior knowledge of $Cov(\xi, \eta)$. For the purposes of exposition, we proceed here under the uncorrelatedness assumption, $Cov(\xi, \eta) = 0$, though the mathematics extend to alternative restrictions.

Assumption 3 (Uncorrelatedness): $Cov(\xi, \eta) = 0$.

There are two natural approaches to estimation. The first is to apply the quadratic formula directly to equation (11). The second is to recast uncorrelatedness as a moment restriction of the form $E[\xi \cdot \eta] = 0$ and use the method of moments. Of these, the first is more novel, and so we open this section with the relevant theoretical result:

Corollary 2. ***(Three-Stage Estimator)*** *Under assumptions 1, 2, and 3, a consistent estimate of the price parameter* β *is given by*

$$\hat{\beta}^{\text{3-Stage}} = \frac{1}{2}\left(\hat{\beta}^{OLS} - \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \sqrt{\left(\hat{\beta}^{OLS} + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\hat{\xi}^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}}\right) \quad (14)$$

if either condition (12) *or condition* (13) *holds.*

The estimator is the empirical lower root of equation (11). It can be calculated in three stages: (i) regress $h(q)$ on p and x with OLS, (ii) regress p on x with OLS and obtain the

[13] Interestingly, prior-free bounds are available only if the sufficient condition for point identification (condition (12)) fails. When this occurs the term inside the radical is non-negative.

residuals p^*, and (iii) construct the estimator as shown. The computational burden of the estimator is trivial, which may be especially beneficial in practice if it is nested inside of a nonlinear routine for other parameters.[14]

We now develop a method-of-moments estimator that converges at the empirical root(s) of equation (11). Consider that the three-stage estimator rests on the moment condition $E[\xi \cdot \eta] = 0$. An alternative approach to estimation is to search numerically for a $\tilde{\beta}$ that satisfies the corresponding empirical moment, yielding

$$\hat{\beta}^{MM} = \arg\min_{\tilde{\beta}<0} \left[\frac{1}{T} \sum_t \frac{1}{|J_t|} \sum_{j \in J_t} \xi(\tilde{\beta}; w_{jt}, \sigma, X_{jt}) \cdot \eta(\tilde{\beta}; w_{jt}, \sigma, X_{jt}) \right]^2$$

where $\xi(\tilde{\beta}; w, \sigma, X)$ and $\eta(\tilde{\beta}; w, \sigma, X)$ are the estimated residuals given the data and the candidate parameter using equations (5)-(7), and the firms present in each market t are indexed by the set J_t. The linear parameters (α, γ) are concentrated out of the nonlinear optimization problem. We have confirmed in numerical experiments that $\hat{\beta}^{\textit{3-Stage}}$ and $\hat{\beta}^{MM}$ are equivalent to numerical precision. Some care must be taken with the method of moments: if condition (12) fails then the optimizer can reach the minimum (zero) at either the upper or lower root, and when the condition holds a local minimum may exist at the boundary value of the parameter space (as the optimizer attempts to reach the minimum for the positive root). Further, the three-stage estimator immediately returns a null value if no solution exists for a given $Cov(\xi, \eta)$, whereas the optimizer returns a solution.

There are three situations in which the method-of-moments approach may be preferred despite its greater computational burden. First, analytical solutions for $\frac{dh}{dq}$ may be unavailable with some specifications of the model, which diminishes the computation advantage of the three-stage estimator. Second, if valid instruments exist, then the additional moments can be incorporated in estimation, allowing for efficiency improvements and specification tests (e.g., Hausman, 1978; Hansen, 1982). Finally, one might consider adding additional covariates ($\tilde{X}$) to the supply equation that may be correlated with the demand shock. The three-stage estimator requires orthogonality between the unobserved demand shock and *all* the regressors (i.e., $E[\tilde{X}\xi] = 0$), whereas the method-of-moments approach can be pursued under a weaker assumption that allows for correlation between ξ and regressors that enter the cost function only.

[14] If condition (13) fails then the empirical analog to the upper root of equation (11) provides a consistent estimate. A more precise two-stage estimator is available for special cases in which the observed cost and demand shifters are uncorrelated. See Appendix C for details.

Table 2: Small-Sample Properties of Three-Stage and 2SLS Estimators

(a) Three-Stage: Mean Price Coefficient and Standard Errors

Observations	(1) $Var(\eta) \gg Var(\xi)$		(2) $Var(\eta) > Var(\xi)$		(3) $Var(\eta) < Var(\xi)$		(4) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.098)	-1.017	(0.201)	-1.018	(0.206)	-1.005	(0.099)
50	-1.001	(0.068)	-1.008	(0.136)	-1.007	(0.135)	-1.001	(0.068)
100	-1.001	(0.047)	-1.003	(0.094)	-1.004	(0.093)	-1.001	(0.047)
500	-1.000	(0.021)	-1.001	(0.041)	-1.001	(0.042)	-1.000	(0.021)

(b) Instrumental Variables: Mean Price Coefficient and Standard Errors

Observations	(1) $Var(\eta) \gg Var(\xi)$		(2) $Var(\eta) > Var(\xi)$		(3) $Var(\eta) < Var(\xi)$		(4) $Var(\eta) \ll Var(\xi)$	
25	-1.004	(0.105)	-1.039	(0.303)	-1.294	(3.721)	-5.481	(263.202)
50	-1.001	(0.072)	-1.018	(0.201)	-1.120	(1.829)	-2.285	(333.019)
100	-1.001	(0.050)	-1.008	(0.138)	-1.048	(0.332)	-1.473	(12.630)
500	-1.000	(0.022)	-1.001	(0.060)	-1.009	(0.138)	-1.061	(0.411)

Notes: Results are based on 10,000 simulations of data for each specification and number of observations. The demand curve is $q_t = 10 - p_t + \xi_t$ and marginal costs are $c_t = \eta_t$, so that $(\beta_0, \beta, \gamma_0) = (10, -1, 0)$. The two-stage least squares estimates are calculated using marginal costs (η) as an instrument in the demand equation. In specification (1), $\xi \sim U(0,2)$ and $\eta \sim U(0,8)$. In specification (2), $\xi \sim U(0,4)$ and $\eta \sim U(0,6)$. In specification (3), $\xi \sim U(0,6)$ and $\eta \sim U(0,4)$. In specification (4), $\xi \sim U(0,8)$ and $\eta \sim U(0,2)$.

3.5 Small-Sample Properties and Comparison to Instrumental Variables

To provide intuition about the small-sample properties of the estimator, we simulate data based on the motivating example of monopoly pricing in Section 2. Recall that $q_t = 10 - p_t + \xi_t$ and $c_t = \eta_t$, and that the error terms (ξ, η) have independent uniform distributions. We compare the three-stage estimator to a 2SLS estimator using the cost shock η_t as the excluded instrument. Thus, both the three-stage estimator and 2SLS rely on the orthogonality condition $Cov(\xi, \eta) = 0$, though only the latter approach requires the econometrician to observe η_t. We vary the support of η and ξ to compare environments in which empirical variation arises primarily from supply shocks or demand shocks. We consider sample sizes of 25, 50, 100, and 500 observations; for each specification and sample size, we randomly draw 10,000 datasets.

Table 2 summarizes the results for the three-stage estimator and 2SLS. Panel (a) shows that the three-stage estimator is accurate in every specification and sample size considered. Comparing specifications (1) and (2) (primarily cost variation) to specifications (3) and (4) (primarily demand variation) demonstrates that the precision of the estimator does not depend of the source of the variation. In particular, the standard errors are symmetric for specifications (1) and (4), as well as for specifications (2) and (3). Because the estimator exploits all of the variation in the data, its performance does not depend on whether empirical variation arises more from variation in supply conditions or demand conditions.

In contrast, 2SLS exploits only the price variation that can be attributed to costs. As shown in panel (b), it is less efficient than the three-stage estimator and has greater bias. The performance of 2SLS deteriorates as relatively more variation is caused by demand shocks, and this is exacerbated in smaller samples. The large bias in the upper right of panel (b) reflects a weak instrument problem, even though all the exogenous cost variation in observed. With specification (4), the mean F-statistics for the first-stage regression of p on η are 2.6, 4.2, 7.3, and 32.6 for markets with 25, 50, 100, and 500 observations, respectively. Thus, in settings where instrumental variables perform poorly, a three-stage estimator may still provide a precise estimate.[15]

4 Generalizations

The results developed thus far rely on an accurate model of the data generating process and some relatively strong (though common) restrictions on the form of demand and supply. In this section, we consider generalizations to non-constant marginal costs, multi-product firms, and non-Bertrand competition. First, we provide guidance on how additional restrictions may be imposed to estimate demand systems when σ is unknown. In Appendix B, we discuss how to generalize the demand assumption to incorporate, for example, constant elasticity demand.

4.1 Identification of Nonlinear Parameters

We now consider identification of the parameter vector, σ, which contains parameters that enter the demand system nonlinearly or load onto endogenous regressors other than price. Conditional on σ, knowledge of $Cov(\xi, \eta)$ can be sufficient to identify the linear parameters in the model, including β (Proposition 4). Thus, the single covariance restriction of Assumption 3 provides a function that maps σ to β and generates an identified set for (β, σ). To point identify (β, σ) when σ is unknown, supplemental moments must be employed. These moments may be generated from instruments or from covariance restrictions that extend the notion of uncorrelatedness. Covariance restrictions can be used to identify a large number of parameters without requiring the econometrician to isolate exogenous variation in each endogenous covariate.

Consider the relation between any demand shock and any marginal cost shock across products (j, k) and markets (t, s): (ξ_{jt}, η_{ks}). Assumption 3, which is sufficient to identify the price parameter, can be expressed as

$$E_{jt}[\xi_{jt} \cdot \eta_{jt}] = 0, \tag{15}$$

[15] Additionally, specifications (3) and (4) generate positive OLS coefficients. In small samples, a positive OLS coefficient might suggest that an identification strategy relying on cost shifters or other supply-side instruments will not be fruitful, because much of the variation in prices is determined by demand. Of course, for any particular instrument, the first-stage F-statistic can be used to assess relevance.

where the expectation is taken over products and markets. One could construct other first-order relations, such as

$$E_t[\xi_{jt} \cdot \eta_{kt}] = 0 \quad \forall j, k \tag{16}$$

where the expectation is taken over markets for each pair (j,k), providing $J \times J$ restrictions. We impose this set of moments in our first application, in which we estimate 13 nonlinear parameters in the random coefficients demand system of Nevo (2000). Other restrictions could be imposed on the time-series or cross-sectional correlation in shocks. We provide some intuition about the link between moments and parameters in each application. For a formal analysis of the sensitivity of parameters to estimation moments, see Andrews et al. (2017).

One could also construct restrictions on functions of these shocks, providing higher-order moments. In some cases, it may be reasonable to assume that the variance of the demand shock does not depend on the level of the cost shock, and vice versa, generating second-order moments of the form $E_{jt}[\xi_{jt}^2 \cdot \eta_{jt}]$ and $E_{jt}[\xi_{jt} \cdot \eta_{jt}^2]$. As another example, if product-level shocks are uncorrelated, it may be reasonable to assume that shocks are uncorrelated when aggregated by product group. These moments take the form

$$E_{gt}[\overline{\xi}_{gt} \cdot \overline{\eta}_{gt}] = 0, \tag{17}$$

where $\overline{\xi}_{gt} = \frac{1}{|g|}\sum_{j \in g} \xi_{jt}$ and $\overline{\eta}_{gt} = \frac{1}{|g|}\sum_{j \in g} \eta_{jt}$ are the mean demand and cost shocks within a group-market.[16] We impose related moment inequalities in our third application, in which we examine the demand system of Aguirregabiria and Ho (2012).

With these additional moments, the econometrician could pursue joint estimation via method-of-moments, searching over the parameter space for the pair (β,σ) that minimizes a weighted sum of the moments. Since the three-stage estimator is consistent for β conditional on σ, the econometrician could instead calculate $\hat{\beta}^{3\text{-}Stage}$ for each candidate parameter vector $\tilde{\sigma}$, selecting $(\hat{\beta}^{3\text{-}Stage}(\tilde{\sigma}), \tilde{\sigma})$ that minimize a weighted sum of the supplemental moments. The second approach can have an advantage in computational efficiency, especially if σ is low-dimensional.

4.2 Non-Constant Marginal Costs

If marginal costs are not constant in output, then unobserved demand shocks that change quantity also affect marginal cost. For example, consider a special case in which marginal costs take the form:

$$c_{jt} = x_{jt}'\gamma + g(q_{jt}; \lambda) + \eta_{jt} \tag{18}$$

Here $g(q_{jt}; \lambda)$ is some potentially nonlinear function that may (or may not) be known to the econometrician. Maintaining Bertrand competition and the baseline demand assumption, the

[16] Note that this assumption does not provide independent identifying power if (ξ, η) are jointly normal, because it would be implied by orthogonality.

first-order conditions of the firm are:

$$p_{jt} = \underbrace{x'_{jt}\gamma + g(q_{jt};\lambda) + \eta_{jt}}_{\text{Marginal Cost}} + \underbrace{\left(-\frac{1}{\beta}\frac{dh_{jt}}{dq_{jt}}q_{jt}\right)}_{\text{Markup}}.$$

Thus, provided $g'(\cdot;\lambda) \neq 0$, markup adjustments are no longer the only mechanism through with prices respond to demand shocks. This also can be seen from the OLS regression of $h(q_{jt}, w_{jt}; \sigma)$ on p and x, which yields a price coefficient with the following probability limit:

$$plim(\hat{\beta}^{OLS}) = \beta - \frac{1}{\beta}\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} + \frac{Cov(\xi, g(q;\lambda))}{Var(p^*)}$$

The third term on the right-hand-side shows that bias depends on how demand shocks affect the non-constant potion of marginal costs. Unless prior knowledge of $g(q_{jt};\lambda)$ can be brought to bear, additional restrictions are necessary to extend the identification results of the preceding section.

There are two ways to make progress. First, if $g'(\cdot;\lambda)$ can be signed, then it is possible to bound the price parameter, β. A leading example is that of capacity constraints, for which it might be reasonable to assume that $Cov(\xi,\eta) = 0$ and $g'(\cdot;\lambda) \geq 0$. Thus, bounds with priors can be constructed from $Cov(\xi, \eta^*) \geq 0$ where $\eta^*_{jt} \equiv \eta_{jt} + g(q_{jt};\lambda)$ is a composite error term. Prior knowledge of $Cov(\xi,\eta)$ is sufficient to at least set identify β in such a situation. Second, the econometrician may be able to estimate $g(q_{jt};\lambda)$, either in advance or simultaneously with the price coefficient. Supplemental covariance restrictions, such as those discussed in the previous section, can be used to identify λ.

Proposition 6. *Under assumptions 1 and 3 and a modified assumption 2 in which marginal costs take the semi-linear form of equation* (18)*, the price parameter β solves the following quadratic equation:*

$$\begin{aligned}
0 &= \left(1 - \frac{Cov(p^*, g(q;\lambda))}{Var(p^*)}\right)\beta^2 \\
&+ \left(\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, g(q;\lambda))}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\hat{\xi}^{OLS}, g(q;\lambda))}{Var(p^*)}\right)\beta \\
&+ \left(-\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov\left(\hat{\xi}^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}\right)
\end{aligned}$$

where $\hat{\beta}^{OLS}$ is the OLS estimate and $\hat{\xi}^{OLS}$ is a vector containing the OLS residuals.

Proof. See appendix.

With the above quadratic in hand, the remaining results of Section 3 extend naturally. Although the estimation of $g(q_{jt};\lambda)$ is not our focus, we note that a three-stage estimator of β could be obtained for any candidate parameters in λ, thereby facilitating computational efficiency.

4.3 Multi-Product Firms

We now provide the notation necessary to extend our results to the case of multi-product firms under our maintained assumptions. Let K^m denote the set of products owned by multi-product firm m. When the firm sets prices on each of its products to maximize joint profits, there are $|K^m|$ first-order conditions, which can be expressed as

$$\sum_{k\in K^m}(p_k-c_k)\frac{\partial q_k}{\partial p_j}=-q_j\ \ \forall j\in K^m.$$

The market subscript, t, is omitted to simplify notation. For demand systems satisfying Assumption 1,

$$\frac{\partial q_k}{\partial p_j}=\beta\frac{1}{\frac{dh_j}{dq_k}}.$$

where the derivative $\frac{dh_j}{dq_k}=\frac{\partial h_j}{\partial q_j}\frac{dq_j}{dq_k}+\frac{\partial h_j}{\partial w_j}\frac{dw_j}{dq_k}$ is calculated holding the prices of other products fixed. Therefore, the set of first-order conditions can be written as

$$\sum_{k\in K^m}(p_k-c_k)\frac{1}{dh_j/dq_k}=-\frac{1}{\beta}q_j\ \ \forall j\in K^m.$$

For each firm, stack the first-order conditions, writing the left-hand side as the product of a matrix A^m of loading components and a vector of markups, (p_j-c_j), for products owned by the firm. The loading components are given by $A^m_{i(j),i(k)}=\frac{1}{dh_j/dq_k}$, where $i(\cdot)$ indexes products within a firm. Next, invert the loading matrix to solve for markups as function of the loading components and $-\frac{1}{\beta}\boldsymbol{q}^m$, where $\boldsymbol{q}^m$ is a vector of the multi-product firm's quantities. Equilibrium prices equal marginal costs plus a markup, where the markup is determined by the inverse of A^m ($(A^m)^{-1}\equiv\Lambda^m$), quantities, and the price parameter:

$$p_j=c_j-\frac{1}{\beta}\left(\Lambda^m\boldsymbol{q}^m\right)_{i(j)}. \tag{19}$$

Here, $\left(\Lambda^m\boldsymbol{q}^m\right)_{i(j)}$ provides the entry corresponding to product j in the vector $\Lambda^m\boldsymbol{q}^m$. As the matrix Λ^m is not a function of the price parameter after conditioning on observables, this form of the first-order condition allows us to solve for β using a quadratic three-stage solution analogous to that in equation (14).[17] Letting $\tilde{h}\equiv\left(\Lambda^m\boldsymbol{q}^m\right)_{i(j)}$ be the multi-product analog for $\frac{dh}{dq}q$, we obtain a quadratic in β, and the remaining results of Section 3 then obtain easily:

[17] At this point, the reader may be wondering where the prices of other firms are captured under the adjusted first-order conditions for multi-product ownership. As is the case with single product firms, we expect prices of

Corollary 3. *Under assumptions 1 and 3, along with a modified assumption 2 that allows for multi-product firms, the price parameter β solves the following quadratic equation:*

$$\begin{aligned} 0 &= \beta^2 \\ &+ \left(\frac{Cov(p^*, \tilde{h})}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \hat{\beta}^{OLS} \right) \beta \\ &+ \left(-\frac{Cov(p^*, \tilde{h})}{Var(p^*)} \hat{\beta}^{OLS} - \frac{Cov(\hat{\xi}^{OLS}, \tilde{h})}{Var(p^*)} \right). \end{aligned}$$

where $\tilde{h}$ is constructed from the first-order conditions of multi-product firms.

4.4 Alternative Models of Competition

Though our main results are presented under Bertrand competition in prices, our method applies to a broader set of competitive assumptions. Consider, for example, Nash competition among profit-maximizing firms that have a single choice variable, a, and constant marginal costs. The individual firm's objective function is:

$$\max_{a_j | a_i, i \neq j} (p_j(a) - c_j) q_j(a).$$

This generalized model of Nash competition nests Bertrand ($a = p$) and Cournot ($a = q$). The first-order condition, holding fixed the actions of the other firms, is given by:

$$p_j(a) = c_j - \frac{{p_j}'(a)}{{q_j}'(a)} q_j(a).$$

In equilibrium, we obtain the structural decomposition $p = c + \mu$, where μ incorporates the structure of demand and its parameters. This decomposition provides a restriction on how prices move with demand shocks, aiding identification. It can be obtained in other contexts, including consistent conjectures and competition in quantities with increasing marginal costs. We provide one such extension in the empirical application to the cement industry.

The approach can be extended to demand systems that generate an alternative structure for equilibrium prices. For example, consider a monopolist facing a constant elasticity demand schedule. The optimal price is $p = c\frac{\epsilon}{1+\epsilon}$ where $\epsilon < 0$ is the elasticity of demand. In this model, the monopolist has a multiplicative markup that does not respond to demand shocks. As we show in Appendix B, it is straightforward to extend our identification results to this model and other demand systems that admit a more general class of multiplicative markups, which result when demand is semi-linear in log prices.[18]

other firm's products to be included in w_j, which is appropriate under Bertrand price competition.

[18]Our identification result relies on separability between marginal costs and markups, which may be obtained after a transformation of prices.

5 Empirical Applications

5.1 Ready-to-Eat Cereal

In our first application, we examine the pseudo-real cereals data of Nevo (2000).[19] The model features random coefficients logit demand and Bertrand competition among multi-product firms. We use the application to demonstrate that covariance restrictions can identify the price parameter, β, as well as the nonlinear parameters, σ. In the data, there is no variation in the product choice sets across markets, so the instruments defined in Berry et al. (1995) and Gandhi and Houde (2015) are unavailable. The instruments provided in the data set and employed in Nevo (2000) are constructed from the prices of the same product in other markets.

The indirect utility that consumer i receives from product j in market m and period t is given by

$$u_{ijmt} \quad = \quad x_j \alpha_i^* + \beta_i^* p_{jmt} + \zeta_j + \xi_{jmt} + \epsilon_{ijmt}$$

where ζ_j is a product fixed effect and ϵ_{ijmt} is a logit error term. The indirect utility provided by the outside good, $j = 0$, is given by $u_{i0mt} = \epsilon_{i0mt}$. The consumer-specific coefficients take the form

$$\begin{bmatrix} \alpha_i^* \\ \beta_i^* \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \Pi D_i + \Sigma \nu_i$$

where D_i is a vector of observed demographics and ν_i is vector of unobserved demographics that have independent standard normal distributions. Within the notation of Section 3, the non-linear parameters σ include the elements of Π and Σ, w_{jmt} is composed of the demographics, and $h(q_{jmt}, w_{jmt}; \tilde{\sigma})$ can be recovered using the contraction mapping of Berry et al. (1995) for any candidate parameters $\tilde{\sigma}$. The supply side of the model is simply the multiproduct version of our Assumption 2.

The data are a panel of 24 brands, 47 markets, and two quarters. We estimate the demand parameters, $\theta = (\beta, \alpha, \Pi, \Sigma)$, using the covariance restrictions $Cov(\xi_j, \eta_k) = 0$ for all j, k, as proposed in Section 4.1. The identifying assumption is that the demand shock of each product is orthogonal to its own marginal cost shock and those of all other products. The $J \times J$ (= 576) covariance restrictions are more than sufficient to identify the 13 nonlinear parameters in the Nevo (2000) specification. Roughly, cross-product covariance restrictions allow the empirical relationship between the shares of product j and the prices of product k to be interpreted as arising from model parameters (σ), rather than systematic correlation between demand shocks to one product and marginal cost shocks to its substitute.[20]

The point estimates are displayed in Table 3. Panel A shows results generated with the

[19] See also Dubé et al. (2012) and Knittel and Metaxoglou (2014).

[20] These restrictions still allow for correlation in demand shocks across products and marginal cost shocks across products.

Table 3: Point Estimates for Ready-to-Eat Cereal

Panel A: Available Instruments

Variable	Means	Standard Deviations	Interactions with Demographic Variables: Income	IncomeSq	Age	Child
Price	-62.714	3.313	588.463	-30.199	–	11.055
	(14.802)	(1.334)	(270.425)	(14.100)		(4.123)
Constant	–	0.558	2.292	–	1.285	–
		(0.163)	(1.208)		(0.631)	
Sugar	–	-0.006	-0.385	–	-0.052	–
		(0.014)	(0.121)		(0.026)	
Mushy	–	0.093	0.749	–	-1.353	–
		(0.185)	(0.802)		(0.667)	

Panel B: Covariance Restrictions

Variable	Means	Standard Deviations	Interactions with Demographic Variables: Income	IncomeSq	Age	Child
Price	-52.464	0.958	444.983	-23.337	–	-0.098
	(5.101)	(1.084)	(90.977)	(4.755)		(3.803)
Constant	–	0.170	1.811	–	0.357	–
		(0.167)	(0.680)		(0.493)	
Sugar	–	0.015	-0.195	–	0.022	–
		(0.017)	(0.045)		(0.026)	
Mushy	–	0.257	2.016	–	-0.853	–
		(0.169)	(0.537)		(0.357)	

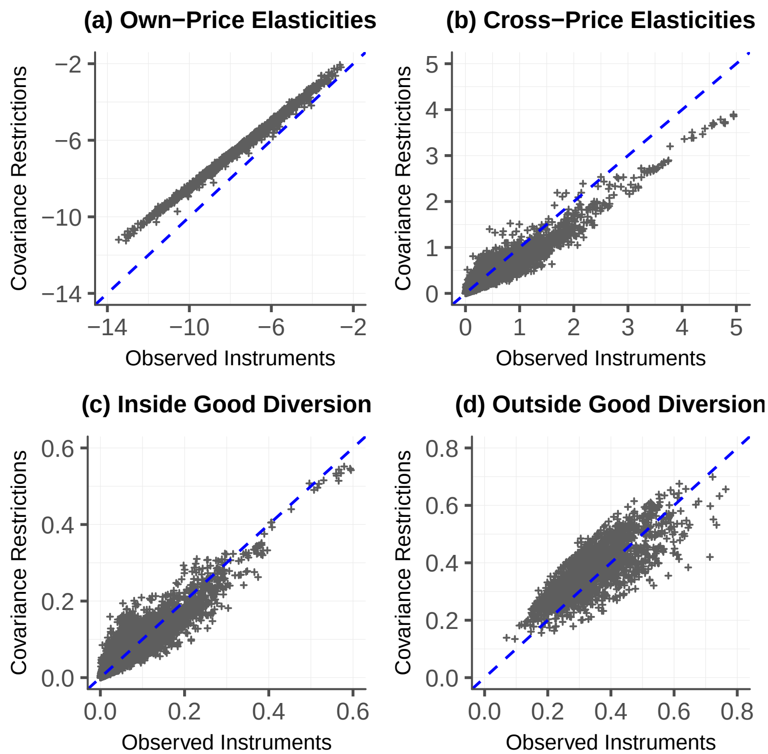
Notes: Table reports point estimates for the random-coefficients logit demand system estimated using the Nevo (2000) dataset. Panel A employs the available instruments in estimation and replicates the results of Knittel and Metaxoglou (2014). Panel B employs covariance restrictions described by the text.

instruments provided in the data set, and Panel B displays the estimates with covariance restrictions.[21] Overall, the different identification strategies produce similar patterns for the coefficients. A majority of the point estimates under covariance restrictions fall in the 95 percent confidence intervals implied by the specification with instruments, including that of the mean price parameter. Only one of the interaction terms that is statistically significant (Child$\times$Price) changes sign. The GMM standard errors are much smaller for the specification with covariance restrictions because we generated far more restrictions than the 40 instruments used by Nevo (2000). The estimated moments using covariance restrictions are small, with a mean absolute value of 0.000055 across the 576 sample moments. The correlation between own demand shocks and own marginal cost shocks is 0.015.

To more intuitively interpret the estimated parameters, Figure 2 compares the elasticities and diversion obtained with covariance restrictions (vertical axis) with those obtained with the

[21]We use a tight contraction mapping tolerance of 1e-14 and a modern optimization routine in each case. Our results with instruments replicate Knittel and Metaxoglou (2014).

Figure 2: Estimated Elasticities and Diversion Ratios for Ready-to-Eat Cereal



Notes: The figure compares results of estimation with covariance restrictions and the observed instruments. Panel A focuses on own-price elasticities, Panel B on cross-price elasticities, Panel C on diversion among the inside goods, and Panel D on diversion to the outside good. Each dot in Panels A and D represents a product-market-period observation. Each dot in Panels B and C represents a product×product combination in a given market-period. 45-degree lines are provided to facilitate comparison. The pseudo-real data are a balanced panel of 24 products, 47 markets, and two time periods.

instruments provided in the pseudo-real data (horizontal axis).[22] The top panels show that the own-price and cross-price elasticities are quite similar on an observation-by-observation basis; the bivariate correlation coefficients are 0.997 and 0.960, respectively. Covariance restrictions produce slightly smaller elasticities in magnitude. For example, the mean own-price elasticity is -6.27 with covariance restrictions and -7.38 with instruments. This is due to a statistically insignificant difference in the estimate of β. The bottom panels show that the diversion ratios, which do not depend on β, fall along the $45°$-degree line.

[22]See Nevo (2000) for elasticities formulas for the random coefficients logit model and Conlon and Mortimer (2018) for a discussion of diversion rations.

5.2 The Portland Cement Industry

Our second empirical application uses the setting and data of Fowlie et al. (2016) ["FRR"], which examines market power in the cement industry and its effects on the efficacy of environmental regulation. The model features Cournot competition among undifferentiated cement plants facing capacity constraints.[23] We use the application to demonstrate that knowledge of institutional details can help evaluate the uncorrelatedness assumption.

We begin by extending our results to Cournot competition with non-constant marginal costs. Let $j = 1, \ldots, J$ firms produce a homogeneous product demanded by consumers according to $h(Q; w) = \beta p + x'\gamma + \xi$, where $Q = \sum_j q_j$, and p represents a price common to all firms in the market. Marginal costs are semi-linear, as in equation (18), possibly reflecting capacity constraints. Working with aggregated first-order conditions, it is possible to show that the OLS regression of $h(Q; w_{jt})$ on price and covariates yields:

$$plim\left(\hat{\beta}^{OLS}\right) = \beta - \frac{1}{\beta}\frac{1}{J}\frac{Cov\left(\xi, \frac{dh}{dq}Q\right)}{Var(p^*)} + \frac{Cov(\xi, \overline{g})}{Var(p^*)}$$

where J is the number of firms in the market and $\overline{g} = \frac{1}{J}\sum_{j=1}^{J} g(q_j; \lambda)$ is the average contribution of $g(q, \lambda)$ to marginal costs. Bias arises due to markup adjustments and the correlation between unobserved demand and marginal costs generated through $g(q; \lambda)$.[24] The identification result provided in Section 4.2 for models with non-constant marginal costs extends.

Corollary 4. *In the Cournot model, the price parameter β solves the following quadratic equation:*

$$\begin{aligned} 0 \;=\; & \left(1 - \frac{Cov(p^*, \overline{g})}{Var(p^*)}\right)\beta^2 \\ + \; & \left(\frac{1}{J}\frac{Cov\left(p^*, \frac{dh}{dq}Q\right)}{Var(p^*)} + \frac{Cov(\xi, \overline{\eta})}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, \overline{g})}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\hat{\xi}^{OLS}, \overline{g})}{Var(p^*)}\right)\beta \\ + \; & \left(-\frac{1}{J}\frac{Cov\left(p^*, \frac{dh}{dq}Q\right)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{1}{J}\frac{Cov\left(\hat{\xi}^{OLS}, \frac{dh}{dq}Q\right)}{Var(p^*)}\right) \end{aligned}$$

The derivation tracks exactly the proof of Proposition 6. For the purposes of the empirical exercise, we compute the three-stage estimator as the empirical analog to the lower root.

Turning to the application, FRR examine 20 distinct geographic markets in the United States

[23] A published report of the Environment Protection Agency (EPA) states that "consumers are likely to view cement produced by different firms as very good substitutes.... there is little or no brand loyalty that allows firms to differentiate their product" EPA (2009).

[24] Bias due to markup adjustments dissipates as the number of firms grows large. Thus, if marginal costs are constant then the OLS estimate is likely to be close to the population parameter in competitive markets. In Monte Carlo experiments, we have found similar results for Bertrand competition and logit demand.

annually over 1984-2009. Let the demand curve in market m and year t have a logit form:

$$h(Q_{mt}; w) \equiv \ln(Q_{mt}) - \ln(M_m - Q_{mt}) = \alpha_r + \beta p_{mt} + \xi_{mt}$$

where M_m is the "market size" of the market. We assume $M_r = 2 \times \max_t\{Q_{mt}\}$ for simplicity.[25] Further, let marginal costs take the "hockey stick" form of FRR:

$$\begin{aligned} c_{jmt} &= \gamma + g(q_{jmt}) + \eta_{jmt} \\ g(q_{jmt}) &= 2\lambda_2 1\{q_{jmt}/k_{jm} > \lambda_1\}(q_{jmt}/k_{jm} - \lambda_1) \end{aligned}$$

where k_{jm} and q_{jmt}/k_{jm} are capacity and utilization, respectively. Marginal costs are constant if utilization is less than the threshold $\lambda_1 \in [0, 1]$, and increasing linearly at rate determined by $\lambda_2 \geq 0$ otherwise. The two unobservables, (ξ, η), capture demand shifts and shifts in the constant portion of marginal costs.

The institutional details of the industry suggest that uncorrelatedness may be reasonable. Demand is procyclical because cement is used in construction projects; given the demand specification this cyclicality enters through the unobserved demand shock. On the supply side, the two largest cost components are "materials, parts, and packaging" and "fuels and electricity" (EPA, 2009). Both depend on the price of coal. With regard to "fuels and electricity," most cement plants during the sample period rely on coal as their primary fuel, and electricity prices are known to correlate with coal prices. With regard to "material, parts, and packaging," the main input in cement manufacture is limestone, which requires significant amounts of electricity to extract (National Stone Council, 2008). Thus, an assessment of uncorrelatedness hinges largely on the relationship between construction activity and coal prices.

In this context, there is a theoretical basis for orthogonality: if coal suppliers have limited market power and roughly constant (realized) marginal costs, then coal prices should not respond much to demand. Indeed, this is precisely the identification argument of FRR, as both coal and electricity prices are included in the set of excluded instruments.

Table 4 summarizes the results of demand estimation. The 3-Stage estimator is implemented taking as given the nonlinear cost parameters obtained in FRR: $\lambda_1 = 0.869$ and $\lambda_2 = 803.65$. In principle, these could be estimated simultaneously via the method of moments, provided some demand shifters can be excluded from marginal costs, but estimation of these parameters is not our focus. As shown, the mean price elasticity of demand obtained with the 3-Stage estimator under uncorrelatedness is -1.15. This is statistically indistinguishable from the 2SLS elasticity estimate of -1.07, which is obtained using the FRR instruments: coal prices, natural gas prices, electricity prices, and wage rates. The closeness of the 3-Stage and 2SLS is not coincidental and

[25]We use logit demand rather than the constant elasticity demand of FRR because it fits easily into our framework. The 2SLS results are unaffected by the choice. Similarly, the 3-Stage estimator with logit obtains virtually identical results as a method-of-moments estimator with constant elasticity demand that imposes uncorrelatedness.

Table 4: Point Estimates for Cement

Estimator:	3-Stage	2SLS	OLS
Elasticity	-1.15	-1.07	-0.47
of Demand	(0.18)	(0.19)	(0.14)

Notes: The sample includes 520 market-year observations over 1984-2009. Bootstrapped standard errors are based on 200 random samples constructed by drawing markets with replacement.

instead reflects that the identifying assumptions are quite similar. Indeed, the main difference is whether the cost shifters are treated as observed (FRR) or unobserved (3-Stage).

If the econometrician does not know (and cannot identify) the nonlinear parameters in the cost function, then consistent estimates cannot be obtained with our methodology. Further, prior-free bounds are unavailable as the empirical upper root of the quadratic in Corollary 4 is positive. Nonetheless, some progress can be made using posterior bounds. Define the composite marginal cost shock, $\eta^*_{jmt} = g(q_{jmt}) + \eta_{jmt}$, as inclusive of the capacity effects. Given the upward-sloping marginal costs, we have $Cov(\xi, \overline{\eta}^*) \geq 0$ if $Cov(\xi, \overline{\eta}) = 0$. This restriction generates an upper bound on the demand elasticity of -0.69, ruling out the OLS point estimate.

5.3 The Airline Industry

In our third empirical application, we examine demand for airline travel using the setting and data of Aguirregabiria and Ho (2012) ["AH"].[26] AH explores why airlines form hub-and-spoke networks; here, we focus on demand estimation only. The model features differentiated-products Bertrand competition among multi-product firms facing a nested logit demand system. We use the application to demonstrate how bounds can narrow the identified set for (β, σ).

The nested logit demand system can be expressed as

$$h(s_{jmt}, w_{jmt}; \sigma) \equiv \ln s_{jmt} - \ln s_{0mt} - \sigma \ln \overline{s}_{jmt|g} = \beta p_{jmt} + x'_{jmt}\alpha + \xi_{jmt} \tag{20}$$

where s_{jmt} is the market share of product j in market m in period t. The conditional market share, $\overline{s}_{j|g} = s_j / \sum_{k \in g} s_k$, is the the choice probability of product j given that its "group" of products, g, is selected. The outside good is indexed as $j = 0$. Higher values of σ increase within-group consumer substitution relative to across-group substitution. In contrast to the typical expression for the demand system, we place $\sigma \ln \overline{s}_{jmt|g}$ on the left-hand side so that the right-hand side contains a single endogenous regressor: price.

The data are drawn from the *Airline Origin and Destination Survey* (DB1B) survey, a ten percent sample of airline itineraries, for the four quarters of 2004. Markets are defined as di-

[26]We thank Victor Aguirregabiria for providing the data. Replication is not exact because the sample differs somewhat from what is used in the AH publication and because we employ a different set of fixed effects in estimation.

rectional round trips between origin and destination cities. Consumers within a market choose among airlines and whether to take a nonstop or one-stop itinerary. Thus, each airline offers zero, one, or two products per market. The nesting parameter, σ, governs consumer substitution within each product group: nonstop flights, one-stop flights, and the outside good. The supply side of the model is the multiproduct version of Assumption 2.[27]

The institutional details of the airline industry suggest that the covariance between demand shocks and marginal cost shocks is positive. The marginal production cost to carry an additional passenger is small and roughly constant because each additional passenger has little impact on the inputs needed to fly the plane from one airport to another. However, the airline bears an opportunity cost for each sold seat, as they can no longer sell the seat at a higher price to another passenger. When the airline expects the flight to be at capacity, this opportunity cost may become large (Williams, 2017). Thus, because positive shocks to demand produce more full flights, it is reasonable to assume $Cov(\xi, \eta) \geq 0$.

Under that assumption, it is possible to reject values (β, σ) that produce negative correlation in product-specific shocks. The econometrician can combine this prior with the prior-free bounds developed earlier. Finally, one can consider reasonable extensions of the priors over the correlation between demand and supply shocks. Following the logic from Section 4.1, we impose the group-level moment

$$E_{gmt}[\overline{\xi}_{gmt} \cdot \overline{\eta}_{gmt}] \geq 0, \tag{21}$$

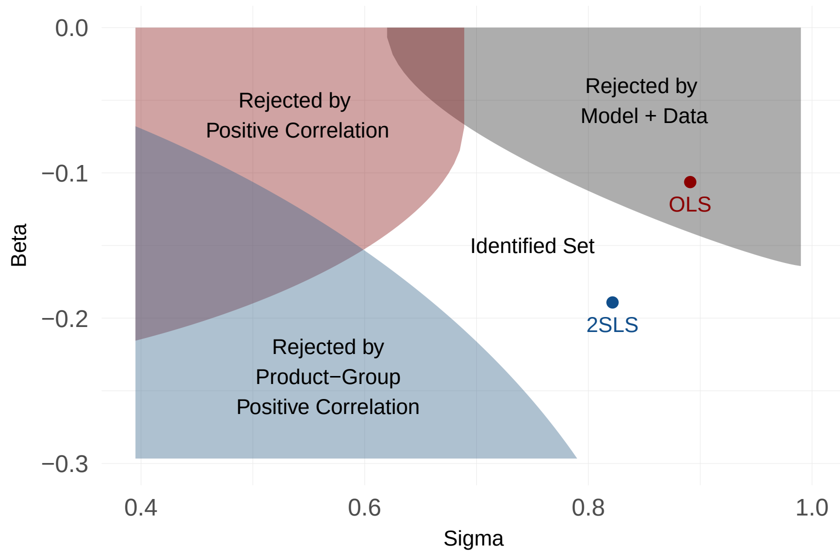
where $\overline{\xi}_{gmt} = \frac{1}{|g|}\sum_{j\in g}\xi_{jmt}$ and $\overline{\eta}_{gmt} = \frac{1}{|g|}\sum_{j\in g}\eta_{jmt}$ are the mean demand and cost shocks within a group-market-period. If the correlation in product-level shocks is weakly positive, it is reasonable to assume that the group-level shocks are also weakly positive, through a similar deduction. By rejecting parameter values that fail to generate the data or that deliver negative correlation, the econometrician can narrow the identified set.

Figure 3 displays the rejected regions based on the model and above priors on unobserved shocks. The gray region corresponds to the parameter values rejected by the prior-free bounds. Based on these bounds only, some values of β can be rejected if $\sigma \geq 0.62$. As σ becomes larger, a more negative β is required to rationalize the data within the context of the model. If $\sigma = 0.80$ then it must be that $\beta \leq -0.11$. The dark red region corresponds to parameter values that generate negative correlation between demand and supply shocks. These can be rejected under the prior that $Cov(\xi, \eta) \geq 0$. The dark blue region provides the corresponding set for the prior $Cov(\overline{\xi}, \overline{\eta}) \geq 0$.

The three regions overlap, but no region is a strict subset of the other. The remaining non-rejected values provide the identified set. In our application, we are able to rule out values of σ

[27]The covariates include an indicator for nonstop itineraries, the distance between the origin and destination cities, and a measure of the airline's "hub sizes" at the origin and destination cities. We also include airline fixed effects and route$\times$quarter fixed effects. The latter expands on the city$\times$quarter fixed effects described by AH. Market size, which determines the market share of the outside good, is equal to the total population in the origin and destination cities.

Figure 3: Identified Parameter Set Under Priors



Notes: Figure displays candidate parameter values for (σ, β). The gray region indicates the set of parameters that cannot generate the observed data from the assumptions of the model. The red region indicates the set of parameters that generate $Cov(\xi, \eta) < 0$, and the blue region indicates parameters that generate $Cov(\overline{\xi}, \overline{\eta}) < 0$. The identified set is obtained by rejecting values in the above regions under the assumption of (weakly) positive correlation. For context, the OLS and the 2SLS estimates are plotted, along with 3-Stage estimates I-IV. The parameter σ can only take values on $[0, 1)$.

less than 0.599 for any value of β, as these lower values cannot generate positive correlation in both product-level and product-group-level shocks. Similarly, we obtain an upper bound on β of -0.067 across all values of σ. For context, we plot the OLS and the 2SLS estimates in Figure 3. The OLS estimate falls in a rejected region and can be ruled out by the structure of the model alone. The 2SLS estimate, in contrast, falls within the identified set. This result is not mechanical, as these point estimates are generated with non-nested assumptions.

6 Discussion

6.1 Assessing Covariance Restrictions

In many applications, econometricians may have detailed knowledge of the determinants of demand and marginal cost, even if these determinants are unobserved in the data. Such knowledge allows the econometrician to assess whether covariance restrictions along the lines of $Cov(\xi, \eta) = 0$ are reasonable. Covariance restrictions need not (and should not) be a “black box” that provides identification. The distinction between *understood* and *observed* is important, as the econometrician may have reasonable priors about the relationship between structural error terms even though they are (by definition) unobserved. For a constructive example, see the discussion about the cement industry in Section 5.2.

Sometimes knowledge of institutional details suggests that uncorrelatedness may be *unreasonable* as an identifying assumption. Products with greater unobserved quality might be more expensive to produce, demand shocks could raise or lower marginal costs (e.g., due to capacity constraints), or firms might invest to lower the costs of their best-selling products. In these cases, three-stage estimates under uncorrelatedness would be inconsistent unless the confounding variation can be absorbed by control variables or fixed effects. Rich panel data provides the econometrician with the means to correct for several first-order determinants of correlation, as we show below. Even without these controls, it may be possible to sign $Cov(\xi, \eta)$, allowing the econometrician to set identify parameters using bounds with priors. As with any identification strategy, careful attention must be paid to the institutional details.

We highlight that econometricians with panel data may be able to incorporate fixed effects that absorb otherwise confounding correlations. To illustrate, consider the following generalized demand and cost functions:

$$\begin{aligned} h(q_{jt}, w_{jt}; \sigma) &= \beta p_{jt} + x_{jt}' \alpha + D_j + F_t + E_{jt} \\ c_{jt} &= g(q_{jt}; \lambda) + x_{jt}' \gamma + U_j + V_t + W_{jt} \end{aligned}$$

with $Cov(E_{jt}, W_{jt}) = 0$. Let the unobserved shocks be $\xi_{jt} = D_j + F_t + E_{jt}$ and $\eta_{jt} = U_j + V_t + W_{jt}$. Further, let $h(\cdot)$ and $g(\cdot)$ be known up to parameters. If products with higher quality have higher marginal costs then $Cov(U_j, D_j) > 0$. The econometrician can account for the relationship by estimating D_j for each firm; the residual $\xi_{jt}^* = \xi_{jt} - D_j$ is uncorrelated with U_j. Similarly, if costs are higher (or lower) in markets with high demand then market fixed effects can be incorporated. In this manner, panel data reduce the remaining unobserved correlation to product-specific deviations within a market, E_{jt} and W_{jt}, allowing the econometrician to proceed with the three-stage approach. Of course, the econometrician must assess whether the restriction $Cov(E_{jt}, W_{jt}) = 0$ is appropriate in the empirical setting.

6.2 Relation to Instruments

To further build intuition on covariance restrictions, we draw some connections between estimation under uncorrelatedness and the instrumental variation approach. In our view, the most obvious similarity is that both approaches rely on orthogonality conditions—$E[\xi \cdot \eta] = 0$ or $E[\xi Z] = 0$ for instruments Z—that are not verifiable empirically but can be assessed with knowledge of institutional details. A stylized model makes this connection clear: Suppose marginal costs are determined by a single variable w that is orthogonal to the demand-side structural error term. If the variable is observed, then IV estimation can proceed under $E[\xi Z] = 0$ with $Z = w$ and if it is unobserved than $E[\xi \cdot \eta] = 0$ with $\eta = w$ allows for estimation via uncorrelatedness. Indeed, some existing articles on covariance restrictions refer to the supply-side structural error term as providing an "unobserved instrument" that identifies demand (e.g., Hausman and Taylor, 1983; Matzkin, 2016).[28]

In general, the assumptions embedded by the two approaches are not nested. Consider the case where marginal costs are the sum of an observed and unobserved variable: $c = Z + \eta$. When $Cov(\eta, \xi) \neq 0$, the IV conditions may still be satisfied, whereas three-stage estimation requires both Z and η to be orthogonal to ξ. On the other hand, the conditions needed for consistent three-stage estimation are not sufficient for IV, as IV requires that *Z does not enter the demand equation*.[29] Satisfying this assumption is one of the key challenges in finding a plausible instrument and is not necessary for estimation under uncorrelatedness. In addition to this theoretical distinction, the IV approach has an additional empirical requirement related to the weak instruments problem: the instrumental variables approach requires sufficient variation in the observed instrument Z, whereas estimation under uncorrelatedness can proceed even if the cost determinants (Z and η) exhibit no variation.

Finally, estimation with uncorrelatedness requires a correctly-specified supply side, whereas IV requires a less formal theory of supply. Of course, many research articles that estimate demand maintain supply-side specifications for counterfactual experiments, and some articles also use supply-side moments to improve efficiency in demand estimation (e.g., Berry et al., 1995). Nonetheless, researchers sometimes express a preference for demand to estimated separately, which ensures that at least the demand estimates are not influenced by misspecification on the supply side (e.g., Dubé and Chintagunta, 2003). Econometricians relying on uncorrelatedness for identification do not have that option—demand and supply must be estimated jointly, increasing the importance of efforts to validate the supply-side assumptions.

7 Conclusion

Our objective has been to evaluate the identifying power of typical supply-side restrictions in models of imperfect competition. Our main result is that price endogeneity can be resolved by interpreting an OLS estimate through the lens of a theoretical model. With a covariance restriction, the demand system is point identified, and weaker assumptions generate bounds on the structural parameters. Thus, causal demand parameters can be recovered without the availability of exogenous price variation. Though we focus our results on specific, widely-used assumptions about demand and supply, we view our method as not particular to these

[28] Hausman and Taylor (1983) proposes a two-stage approach for the estimation of supply and demand models of perfect competition: First, the supply equation is estimated with 2SLS using an instrument taken from the demand-side of the model. Second, the supply-side error term is recovered and, under the assumption of uncorrelatedness ($Cov(\xi, \eta) = 0$), it serves as a valid instrument for the estimation of demand. Interestingly, the first known application of instrumental variables also uses this approach (Wright (1928)).

[29] If it does, it would violate either the relevance condition or the exclusion restriction, depending on interpretation.

assumptions. Rather, the main insight is that information about supply-side behavior can be modeled to adjust the observed relationships between quantity and price. When price can be structurally decomposed into marginal cost and a markup, our method provides a direct way to correct for endogeneity arising from the latter component. We hope that the methods developed help facilitate research in areas for which strong instruments are unavailable or difficult to find.

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A Linear Models of Supply and Demand

In this appendix we recast the monopoly model of Section 2 in terms of supply and demand, providing an alternative proof for Proposition 2 that builds explicitly on the canonical textbook treatment of simultaneous equation bias in chapter 3 of Hayashi (2000).

We then develop the case of perfect competition with linear demand and marginal costs, which has many similarities to monopoly and one critical difference. The model was a primary focus of previous articles addressing demand identification using covariance restrictions (e.g., Koopmans et al., 1950; Hausman and Taylor, 1983; Matzkin, 2016).

A.1 Intuition from Simultaneous Equations: A Link to Hayashi

To start, given the first-order conditions of the monopolist, $p_t + (\frac{dq}{dp})^{-1} q_t = \gamma + \eta_t$ for $\frac{dq}{dp} = \beta$, equilibrium in the model can be characterized as follows:

$$\begin{aligned} q_t^d &= \alpha + \beta p_t + \xi_t && \text{(demand)} \\ q_t^s &= \beta\gamma - \beta p_t + \nu_t && \text{(supply)} \\ q_t^d &= q_t^s && \text{(equilibrium)} \end{aligned} \tag{A.1}$$

where $\nu_t \equiv \beta\eta_t$. The only distinction between this model and that of Hayashi is that slope of the supply schedule is determined (solely) by the price parameter of the demand equation, rather than by the increasing marginal cost schedules of perfect competitors.[30]

If market power is the reason that the supply schedule slopes upwards, as it is with our monopoly example, then uncorrelatedness suffices for identification because the model fully pins down how firms adjust prices with demand shocks. Repeating the steps of Hayashi, we have:

$$\beta^{OLS} \equiv plim\left(\hat{\beta}^{OLS}\right) = \beta\left(\frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}\right). \tag{A.2}$$

If variation in the data arises solely due to cost shocks (i.e., $Var(\xi) = 0$) then the OLS estimator is consistent for β. If instead variation arises solely due to demand shocks (i.e., $Var(\nu) = 0$) then the OLS estimator is consistent for $-\beta$. A third special case arises if the demand and cost shocks have equal variance (i.e., $Var(\nu) = Var(\xi)$). Then $\beta^{OLS} = 0$, exactly halfway between the demand slope (β) and the supply slope ($-\beta$). Thus the adjustment required to bring the OLS coefficient in line with either the demand or supply slope is maximized, in terms of absolute value.

It is when variation in the data arises due both cost and demand shocks that the OLS estimate is difficult to interpret. With uncorrelatedness, however, the OLS residuals provide the information required to correct bias. A few lines of algebra obtain:

Lemma A.1. *Under uncorrelatedness, we have*

$$\beta^2 = \left(\beta^{OLS}\right)^2 + \frac{Cov(q, \xi^{OLS})}{Var(p)}. \tag{A.3}$$

[30] A implication of equation (A.1) is that it can be possible to estimate demand parameters by estimating the *supply side* of the model, taking as given the demand system and the nature of competition. We are aware of precisely one article that employs such a method: Thomadsen (2005) estimates a model of price competition among spatially-differentiated duopolists with (importantly) constant marginal costs.

and

$$Cov(q, \xi^{OLS}) = \frac{Var(\nu)Var(\xi)}{Var(\nu) + Var(\xi)}. \quad \text{(A.4)}$$

Proof: See appendix D.

The first equation is a restatement of Proposition 2. The second equation expresses the correction term as function of $Var(\nu)$ and $Var(\xi)$. Notice that the correction term equals zero if variation in the data arises solely due to either cost or demand shocks—precisely the cases for which OLS estimator obtains β and $-\beta$, respectively. Further, the correction term is maximized if $Var(\nu) = Var(\xi)$ which, as developed above, is when the largest adjustment is required because $\beta^{OLS} = 0$.

A.2 Perfect Competition

As a point of comparison, consider perfect competition with linear demand and supply curves. Let marginal costs be given by $c = x'\gamma + \lambda q + \eta$. Firms are price-takers and each has a first-order condition given by $p = x'\gamma + \lambda q + \eta$. The firm-specific supply curve is $q^s = -\frac{1}{\lambda}x'\gamma + \frac{1}{\lambda}p - \frac{\eta}{\lambda}$. Aggregating across firms and assuming with linearity in demand, we have the following market-level system of equations:

$$\begin{aligned} Q^D &= \beta p + x'\alpha + \xi && \text{(Demand)} \\ Q^S &= \frac{J}{\lambda}p - \frac{J}{\lambda}x'\gamma - \frac{J}{\lambda}\eta && \text{(Supply)} \\ Q^D &= Q^S && \text{(Equilibrium)} \end{aligned} \quad \text{(A.5)}$$

where Q^D and Q^S represent market quantity demanded and supplied, respectively. Though similar in structure to the monopoly problem described by (A.1), the supply slope in this case depends on the number of firms and the slope of marginal costs. In the monopoly problem above, the supply slope is fully determined by the demand parameter.

In this setting, uncorrelatedness allows for the consistent estimation of the price coefficient, but only if the supply slope $\frac{J}{\lambda}$ can be pinned down with additional moments. Hausman and Taylor (1983) propose the following methodology: (i) under the exclusion restriction $\gamma_k = 0$ for some k, estimate the supply-schedule using x_k as an instrument for p; (ii) recover estimates of the supply-side shock; (iii) use these estimated supply-side errors as instruments in demand estimation. Under uncorrelatedness, these supply-side errors are orthogonal to demand shocks. Matzkin (2016) proposes a similar procedure but relaxes the assumption of linearity.

B Demand System Applications and Extensions

The demand system of equation (5) is sufficiently flexible to nest monopolistic competition with linear demands (e.g., as in the motivating example) and the discrete choice demand models that support much of the empirical research in industrial organization. We illustrate with some typical examples:

1. *Nested logit demand:* Following the exposition of Cardell (1997), let the firms be grouped into $g = 0, 1, \ldots, G$ mutually exclusive and exhaustive sets, and denote the set of firms in

group g as $\mathscr{J}_g$. An outside good, indexed by $j = 0$, is the only member of group 0. Then the left-hand-side of equation (5) takes the form

$$h(s_j, w_j; \sigma) \equiv \ln(s_j) - \ln(s_0) - \sigma \ln(\bar{s}_{j|g})$$

where $\bar{s}_{j|g} = \sum_{j \in \mathscr{J}_g} \frac{s_j}{\sum_{j \in \mathscr{J}_g} s_j}$ is the market share of firm j within its group. The parameter $\sigma \in [0, 1)$ determines the extent to which consumers substitute disproportionately among firms within the same group. If $\sigma = 0$ then the logit model obtains. We can construct the markup by calculating the total derivative of h with respect to s. At the Bertrand-Nash equilibrium,

$$\frac{dh_j}{ds_j} = \frac{1}{s_j \left(\frac{1}{1-\sigma} - s_j + \frac{\sigma}{1-\sigma} \bar{s}_{j|g} \right)}.$$

Thus, we verify that the derivatives can be expressed as a function of data and the non-linear parameters, allowing for three-stage estimation. In our third application, we use the nested logit model to estimate bounds on the structural parameters (Section 5.3).

2. *Random coefficients logit demand:* Modifying slightly the notation of Berry (1994), let the indirect utility that consumer $i = 1, \ldots, I$ receives from product j be

$$u_{ij} = \beta p_j + x_j' \alpha + \xi_j + \left[\sum_k x_{jk} \sigma_k \zeta_{ik} \right] + \epsilon_{ij}$$

where x_{jk} is the kth element of x_j, ζ_{ik} is a mean-zero consumer-specific demographic characteristic, and ϵ_{ij} is a logit error. We have suppressed market subscripts for notational simplicity. Decomposing the right-hand side of the indirect utility equation into $\delta_j = \beta p_j + x_j' \alpha + \xi_j$ and $\mu_{ij} = \sum_k x_{jk} \sigma_k \zeta_{ik}$, the probability that consumer i selects product j is given by the standard logit formula

$$s_{ij} = \frac{\exp(\delta_j + \mu_{ij})}{\sum_k \exp(\delta_k + \mu_{ik})}.$$

Integrating yields the market shares: $s_j = \frac{1}{I} \sum_i s_{ij}$. Berry et al. (1995) prove that a contraction mapping recovers, for any candidate parameter vector $\tilde{\sigma}$, the vector $\delta(s, \tilde{\sigma})$ that equates these market shares to those observed in the data. This "mean valuation" is $h(s_j, w_j; \tilde{\sigma})$ in our notation. The three-stage estimator can be applied to recover the price coefficient, again taking some $\tilde{\sigma}$ as given. At the Bertrand-Nash equilibrium, dh_j/ds_j takes the form

$$\frac{dh_j}{ds_j} = \frac{1}{\frac{1}{I} \sum_i s_{ij}(1 - s_{ij})}.$$

Thus, with the uncorrelatedness assumption the linear parameters can be recovered given the candidate parameter vector $\tilde{\sigma}$. The identification of σ is a distinct issue that has received a great deal of attention from theoretical and applied research (e.g., Waldfogel, 2003; Romeo, 2010; Berry and Haile, 2014; Gandhi and Houde, 2015; Miller and Weinberg, 2017). We demonstrate how to estimate these parameters using additional covariance restrictions in our first application (Section 5.1).

The semi-linear demand assumption (Assumption 1) can be modified to allow for semi-linearity in a transformation of prices, $f(p_{jt})$:

$$h_{jt} \equiv h(q_{jt}, w_{jt}; \sigma) = \beta f(p_{jt}) + x'_{jt}\alpha + \xi_{jt} \tag{B.1}$$

Under this modification assumptions, it is possible to employ a method-of-moments approach to estimate the structural parameters. When $f(p_{jt}) = \ln p_{jt}$, the three-stage estimator and identification results are applicable, under the modified assumptions that ξ is orthogonal to $\ln X$ and that $\ln \eta$ and ξ are uncorrelated.

The optimal price for these demand systems takes the form $p_{jt} = \mu_{jt} c_{jt}$, where μ_{jt} is a markup that reflects demand parameters and (in general) demand shocks. It follows that the probability limit of an OLS regression of h on $\ln p$ is given by:

$$\beta^{OLS} = \beta - \frac{1}{\beta}\frac{Cov(\ln \mu, \xi)}{Var(\ln p^*)} + \frac{Cov(\ln \eta, \xi)}{Var(\ln p^*)}. \tag{B.2}$$

Therefore, the results developed in this paper are extend in a straightforward manner. We opt to focus on semi-linear demand throughout this paper for clarity.

A special case that is often estimated in empirical work is when h and $f(p)$ are logarithms:

> *Constant elasticity demand:* With the modified demand assumption of equation (B.1), the constant elasticity of substitution (CES) demand model of Dixit and Stiglitz (1977) can be incorporated:
>
> $$\ln(q_{jt}/q_t) = \alpha + \beta \ln\left(\frac{p_{jt}}{\Pi_t}\right) + \xi_{jt}$$
>
> where q_t is an observed demand shifter, Π_t is a price index, and β provides the constant elasticity of demand. This model is often used in empirical research on international trade and firm productivity (e.g., De Loecker, 2011; Doraszelski and Jaumandreeu, 2013). Due to the constant elasticity, profit-maximization and uncorrelatedness imply $Cov(p, \xi) = 0$, and OLS produces unbiased estimates of the demand parameters.[31] Indeed, this is an excellent illustration of our basic argument: so long as the data generating process is sufficiently well understood, it is possible to characterize the bias of OLS estimates.

The demand assumption in Equation (5) accommodates many rich demand systems. Consider the linear demand system, $q_{jt} = \alpha_j + \sum_k \beta_{jk} p_k + \xi_{jt}$, which sometimes appears in identification proofs (e.g., Nevo, 1998) but is seldom applied empirically due to the large number of price coefficients. In principle, the system could be formulated such that $h(q_{jt}, w_{jt}; \sigma) \equiv q_{jt} - \sum_{k \neq j} \beta_{jk} p_k$. In addition to the own-product uncorrelatedness restrictions that could identify β_{jj}, one could impose cross-product covariance restrictions to identify β_{jk} $(j \neq k)$. We discuss these cross-product covariance restrictions in Section 4.1. A similar approach could be used with the almost ideal demand system of Deaton and Muellbauer (1980).

[31] The international trade literature following Feenstra (1994) consider non-constant marginal costs, which requires an additional restriction. See section 4.2 for an extension of our methodology to non-constant marginal costs.

C Two-Stage Estimation

In the presence of an additional restriction, we can produce a more precise estimator that can be calculated with one fewer stage. When the observed cost and demand shifters are uncorrelated, there is no need to project the price on demand covariates when constructing a consistent estimate, and one can proceed immediately using the OLS regression. We formalize the additional restriction and the estimator below.

Assumption 5: *Let the parameters $\alpha^{(k)}$ and $\gamma^{(k)}$ correspond to the demand and supply coefficients for covariate k in X. For any two covariates k and l, $Cov(\alpha^{(k)}x^{(k)}, \gamma^{(l)}x^{(l)}) = 0$.*

Proposition C.1. *Under assumptions 1-3 and 5, a consistent estimate of the price parameter β is given by*

$$\hat{\beta}^{\text{2-Stage}} = \frac{1}{2}\left(\hat{\beta}^{OLS} - \frac{\hat{Cov}\left(p, \frac{dh}{dq}q\right)}{\hat{Var}(p)} - \sqrt{\left(\hat{\beta}^{OLS} + \frac{\hat{Cov}\left(p, \frac{dh}{dq}q\right)}{\hat{Var}(p)}\right)^2 + 4\frac{\hat{Cov}\left(\hat{\xi}^{OLS}, \frac{dh}{dq}q\right)}{\hat{Var}(p)}}\right) \tag{C.1}$$

when the auxiliary condition, $\beta < \frac{Cov(p^*,\xi)}{Var(p^*)}\frac{Var(p)}{Var(p^*)} - \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}$, holds.

The estimator can be expressed entirely in terms of the data, the OLS coefficient, and the OLS residuals. The first stage is an OLS regression of $h(q;\cdot)$ on p and x, and the second stage is the construction of the estimator as in equation (C.1). Thus, we eliminate the step of projecting p on x. This estimator is consistent under the assumption that any covariate affecting demand does not covary with marginal cost. The auxiliary condition parallels that of the three-stage estimator, and we expect that it holds in typical cases.

D Proofs

Lemma: A Consistent and Unbiased Estimate for ξ

The following proof shows a consistent and unbiased estimate for the unobserved term in a linear regression when one of the covariates is endogenous. Though demonstrated in the context of semi-linear demand, the proof also applies for any endogenous covariate, including when (transformed) quantity depends on a known transformation of price, as no supply-side assumptions are required. For example, we may replace p with $\ln p$ everywhere and obtain the same results.

Lemma D.1. *A consistent and unbiased estimate of ξ is given by $\xi_1 = \xi^{OLS} + \left(\beta^{OLS} - \beta\right)p^*$*

We can construct both the true demand shock and the OLS residuals as:

$$\xi = h(q) - \beta p - x'\alpha$$
$$\xi^{OLS} = h(q) - \beta^{OLS}p - x'\alpha^{OLS}$$

where this holds even in small samples. Without loss of generality, we assume $E[\xi] = 0$. The true demand shock is given by $\xi_0 = \xi^{OLS} + (\beta^{OLS} - \beta)p + x'(\alpha^{OLS} - \alpha)$. We desire to show that an alternative estimate of the demand shock, $\xi_1 = \xi^{OLS} + (\beta^{OLS} - \beta)p^*$, is consistent and unbiased. (This eliminates the need to estimate the true α parameters). It suffices to show that $(\beta^{OLS}-\beta)p^* \to (\beta^{OLS}-\beta)p + x'(\alpha^{OLS}-\alpha)$. Consider the projection matrices

$$\begin{aligned} Q &= I - P(P'P)^{-1}P' \\ M &= I - X(X'X)^{-1}X', \end{aligned}$$

where P is an $N \times 1$ matrix of prices and X is the $N \times k$ matrix of covariates x. Denote $Y \equiv h(q) = P\beta + X\alpha + \xi$. Our OLS estimators can be constructed by a residualized regression

$$\begin{aligned} \alpha^{OLS} &= \left((XQ)'QX\right)^{-1}(XQ)'Y \\ \beta^{OLS} &= \left((PM)'MP\right)^{-1}(PM)'Y. \end{aligned}$$

Therefore

$$\begin{aligned} \alpha^{OLS} =& \left(X'QX\right)^{-1}\left(X'QP\beta + X'QX\alpha + X'Q\xi\right) \\ =&\alpha + \left(X'QX\right)^{-1}X'Q\xi. \end{aligned}$$

Similarly,

$$\begin{aligned} \beta^{OLS} =& \left(P'MP\right)^{-1}\left(P'MP\beta + P'MX\alpha + P'M\xi\right) \\ =&\beta + \left(P'MP\right)^{-1}P'M\xi. \end{aligned}$$

We desire to show

$$P^*(\beta^{OLS}-\beta) \to P(\beta^{OLS}-\beta) + X(\alpha^{OLS}-\alpha).$$

Note that $P^* = MP$. Then

$$\begin{aligned} P^*(\beta^{OLS}-\beta) \to& P(\beta^{OLS}-\beta) + X(\alpha^{OLS}-\alpha) \\ MP\left(P'MP\right)^{-1}P'M\xi \to& P\left(P'MP\right)^{-1}P'M\xi + X\left(X'QX\right)^{-1}X'Q\xi \\ -X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'M\xi \to& X\left(X'QX\right)^{-1}X'Q\xi \\ -X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'\left[I - X(X'X)^{-1}X'\right]\xi \to& X\left(X'QX\right)^{-1}X'\left[I - P(P'P)^{-1}P'\right]\xi \\ -X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'\xi \to& X\left(X'QX\right)^{-1}X'\xi \\ +X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'X(X'X)^{-1}X'\xi \quad & - X\left(X'QX\right)^{-1}X'P(P'P)^{-1}P'\xi. \end{aligned}$$

We will show that the following two relations hold, which proves consistency and completes the proof.

1. $X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'\xi = X\left(X'QX\right)^{-1}X'P(P'P)^{-1}P'\xi$
2. $X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'X(XX\left(X'QX\right)^{-1}X'X)^{-1}X'\xi \to X\left(X'QX\right)^{-1}X'\xi$

Part 1: Equivalence

It suffices to show that $X(X'X)^{-1}X'P\,(P'MP)^{-1} = X\,(X'QX)^{-1}\,X'P(P'P)^{-1}$.

$$
\begin{aligned}
X(X'X)^{-1}X'P\left(P'MP\right)^{-1} =& X\left(X'QX\right)^{-1}X'P(P'P)^{-1} \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1}X'P(P'P)^{-1}\left(P'MP\right) \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1}X'P(P'P)^{-1}\left(P'P\right) \\
&- X\left(X'QX\right)^{-1}X'P(P'P)^{-1}\left(P'X(X'X)^{-1}X'P\right) \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1}X'P \\
&- X\left(X'QX\right)^{-1}X'\left[I-Q\right]X(X'X)^{-1}X'P \\
X(X'X)^{-1}X'P =& X\left(X'QX\right)^{-1}X'P \\
&- X\left(X'QX\right)^{-1}X'X(X'X)^{-1}X'P \\
&+ X\left(X'QX\right)^{-1}X'QX(X'X)^{-1}X'P \\
X(X'X)^{-1}X'P =& X(X'X)^{-1}X'P
\end{aligned}
$$

QED.

Part 2: Consistency (and Unbiasedness)

Because $X(X'X)^{-1}X'P = X\,(X'QX)^{-1}\,X'P(P'P)^{-1}\,(P'MP)$, as shown above:

$$
\begin{aligned}
X(X'X)^{-1}X'P\left(P'MP\right)^{-1}P'X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X\left(X'QX\right)^{-1}X'P(P'P)^{-1}P'X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X\left(X'QX\right)^{-1}X'\left[I-Q\right]X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X\left(X'QX\right)^{-1}X'X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
-X(X'X)^{-1}X'\xi & \\
X\left(X'QX\right)^{-1}X'\xi - X(X'X)^{-1}X'\xi &\to X\left(X'QX\right)^{-1}X'\xi \\
X(X'X)^{-1}X'\xi &\to 0.
\end{aligned}
$$

The last line, where the projection of ξ onto the exogenous covariates X converges to zero, holds by assumption. We say that two vectors converge if the mean absolute deviation goes to zero as the sample size gets large. Note that also $E[X(X'X)^{-1}X'\xi] = 0$, so ξ_1 is both a consistent and unbiased estimate for ξ_0. QED.

Proof of Proposition 3 (Set Identification)

From the text, we have $\hat{\beta}^{OLS} \xrightarrow{p} \beta + \frac{Cov(p^*,\xi)}{Var(p^*)}$. The general form for a firm's first-order condition is $p = c + \mu$, where c is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = X\gamma + \eta$. If we substitute the first-order condition $p^* = X\gamma + \eta + \mu - \hat{p}$ into the bias term from the OLS regression, we obtain

$$\begin{aligned}\frac{Cov(p^*,\xi)}{Var(p^*)} &= \frac{Cov(\xi, X\gamma+\eta+\mu-\hat{p})}{Var(p^*)}\\ &= \frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{Cov(\xi,\mu)}{Var(p^*)}\end{aligned}$$

where the second line follows from the exogeneity assumption ($E[X\xi]=0$). Under our demand assumption, the unobserved demand shock may be written as $\xi = h(q) - x\alpha - \beta p$. At the probability limit of the OLS estimator, we can construct a consistent estimate of the unobserved demand shock as $\xi = \xi^{OLS} + \left(\beta^{OLS}-\beta\right)p^*$ (see Lemma D.1 above). From the prior step in this proof, $\beta^{OLS}-\beta = \frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{Cov(\xi,\mu)}{Var(p^*)}$. Therefore, $\xi = \xi^{OLS} + \left(\frac{Cov(\eta,\xi)}{Var(p^*)}+\frac{Cov(\mu,\xi)}{Var(p^*)}\right)p^*$. This implies

$$\begin{aligned}\frac{Cov\left(\xi,\mu\right)}{Var(p^*)} =& \frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\left(\frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\right)\frac{Cov(p^*,\mu)}{Var(p^*)}\\ \frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\left(1-\frac{Cov(p^*,\mu)}{Var(p^*)}\right) =& \frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)}\\ \frac{Cov\left(\xi,\mu\right)}{Var(p^*)} =& \frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)}\end{aligned}$$

When we substitute this expression in for β^{OLS}, we obtain

$$\begin{aligned}\beta^{OLS} &= \beta+\frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{\frac{Cov(p^*,\mu)}{Var(p^*)}}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\\ \beta^{OLS} &= \beta+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)}+\frac{1}{1-\frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\end{aligned}$$

Thus, we obtain an expression for the OLS estimator in terms of the OLS residuals, the residualized prices, the markup, and the correlation between unobserved demand and cost shocks. If the markup can be parameterized in terms of observables and the correlation in unobserved shocks can be calibrated, we have a method to estimate β from the OLS regression. Under our supply and demand assumptions, $\mu = -\frac{1}{\beta}\frac{dh}{dq}q$, and plugging in obtains the first equation of the proposition:

$$\beta^{OLS} = \beta - \frac{1}{\beta+\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)}+\beta\frac{1}{\beta+\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}.$$

The second equation in the proposition is obtained by rearranging terms. QED.

Proof of Proposition 4 (Point Identification)

Part (1). We first prove the sufficient condition, i.e., that under assumptions 1 and 2, β is the lower root of equation (11) if the following condition holds:

$$0 \leq \beta^{OLS} \frac{Cov(p^*, \frac{dh}{dq} q)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)} \tag{D.1}$$

Consider a generic quadratic, $ax^2 + bx + c$. The roots of the quadratic are $\frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$. Thus, if $4ac < 0$ and $a > 0$ then the upper root is positive and the lower root is negative. In equation (11), $a = 1$, and $4ac < 0$ if and only if equation (D.1) holds. Because the upper root is positive, $\beta < 0$ must be the lower root, and point identification is achieved given knowledge of $Cov(\xi, \eta)$. QED.

Part (2). In order to prove the necessary and sufficient condition for point identification, we first state and prove a lemma:

Lemma D.2. *The roots of equation* (11) *are* β *and* $\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*, \frac{dh}{dq} q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}$.

Proof of Lemma D.2. We first provide equation (11) for reference:

$$\begin{aligned}
0 &= \beta^2 \\
&+ \left(\frac{Cov(p^*, \frac{dh}{dq} q)}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \beta^{OLS}\right)\beta \\
&+ \left(-\beta^{OLS} \frac{Cov(p^*, \frac{dh}{dq} q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)}\right)
\end{aligned}$$

To find the roots, begin by applying the quadratic formula

$$\begin{aligned}
(r_1, r_2) &= \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4AC}\right) \\
&= \frac{1}{2}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi, \eta)}{Var(p^*)}\right. \\
&\quad \left. \pm\sqrt{\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 + 4\beta^{OLS}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}}\right) \\
&= \frac{1}{2}\left[\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right. \\
&\quad \pm\left(\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)\right. \\
&\quad \left.\left. +4\beta^{OLS}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right. \\
&\quad \left.\pm\sqrt{\left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)}\right)
\end{aligned} \tag{D.2}$$

Looking inside the radical, consider the first part: $\left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}$

$$\begin{aligned}
&\left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi - p^*(\beta^{OLS} - \beta), \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} - 4\frac{Cov(p^*, \xi)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} - 4\left(\frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{Cov(\xi, -\frac{1}{\beta}\frac{dh}{dq}q)}{Var(p^*)}\right)\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&= \left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)}\left(1 + \frac{1}{\beta}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - 4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}
\end{aligned} \tag{D.3}$$

To simplify this expression, it is helpful to use the general form for a firm's first-order condition, $p = c + \mu$, where c is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = X\gamma + \eta$. It follows that

$$\begin{aligned}
p^* &= X\gamma + \eta + \mu - \hat{p} \\
&= X\gamma + \eta - \frac{1}{\beta}\frac{dh}{dq}q - \hat{p}
\end{aligned}$$

Therefore

$$Cov(p^*,\xi) = Cov(\xi,\eta) - \frac{1}{\beta}Cov(\xi, \frac{dh}{dq}q)$$

and

$$\begin{aligned} Cov(\xi, \frac{dh}{dq}q) &= -\beta\left(Cov(p^*,\xi) - Cov(\xi,\eta)\right) \\ \frac{Cov(\xi, \frac{dh}{dq}q)}{Var(p^*)} &= -\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right) \end{aligned} \tag{D.4}$$

Returning to equation (D.3), we can substitute using equation (D.4) and simplify:

$$\begin{aligned}
&\left(\beta^{OLS} + \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)}\left(1 + \frac{1}{\beta}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - 4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
=&\left(\beta^{OLS}\right)^2 + \left(\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^2 + 2\beta^{OLS}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - 4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&+ 4\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)} + 4\frac{1}{\beta}\frac{Cov\left(\xi, \frac{dh}{dq}q\right)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
=&\left(\beta + \frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2 + \left(\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^2 + 2\left(\beta + \frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - 4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&- 4\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right) - 4\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right)\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
=&\left(\beta + \frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2 + \left(\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^2 + 2\left(\beta + \frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&- 4\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)}\right) - 4\left(\frac{Cov(p^*,\xi)}{Var(p^*)}\right)\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)} \\
=&\beta^2 + \left(\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2 + \left(\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right)^2 + 2\beta\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} \\
&- 2\beta\frac{Cov\left(p^*,\xi\right)}{Var(p^*)} - 2\frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)} \\
=&\left(\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2 + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)}
\end{aligned}$$

Now, consider the second part inside of the radical in equation (D.2):

$$\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)$$
$$= \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta + \frac{Cov(\xi,\eta)}{Var(p^*)} - \frac{1}{\beta}\frac{Cov(\xi, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\right)$$
$$= \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 + 2\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(\xi, \frac{dh}{dq}q)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}$$
$$= -\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right) + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}$$
$$= \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}$$

Combining yields a simpler expression for the terms inside the radical of equation (D.2):

$$\left(\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2 + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)}$$
$$+ \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}$$
$$= \left(\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right) - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)}\right)^2 + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2$$
$$+ 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}$$
$$= \left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2$$

Plugging this back into equation (D.2), we have:

$$\begin{aligned}
(r_1, r_2) &= \frac{1}{2}\Bigg(\beta^{OLS} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
&\qquad \pm\sqrt{\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\Bigg) \\
&= \frac{1}{2}\Bigg(\beta + \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} - \frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
&\qquad \pm\sqrt{\left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\Bigg)
\end{aligned}$$

The roots are given by

$$\frac{1}{2}\left(\beta+\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}+\beta+\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}-\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}+\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)$$
$$=\beta$$

and

$$\frac{1}{2}\left(\beta+\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}-\beta-\frac{Cov\left(p^{*},\frac{dh}{dq}q\right)}{Var(p^{*})}+\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}\right)$$
$$=\frac{Cov\left(p^{*},\xi\right)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}$$

which completes the proof of the intermediate result. QED.

Part (3). Consider the roots of equation (11), β and $\frac{Cov(p^{*},\xi)}{Var(p^{*})}-\frac{Cov(p^{*},\frac{dh}{dq}q)}{Var(p^{*})}-\frac{Cov(\xi,\eta)}{Var(p^{*})}$. The price parameter β may or may not be the lower root.[32] However, β is the lower root iff

$$\begin{aligned}
\beta &< \frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
\beta &< -\beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} + \beta\frac{Cov(p^*,-\frac{1}{\beta}\frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
\beta &< -\beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} + \beta\frac{Cov(p^*,p^*-c)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
\beta &< \beta\frac{Var(p^*)}{Var(p^*)} - \beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} - \beta\frac{Cov(p^*,\eta)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
0 &< -\beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} - \beta\frac{Cov(p^*,\eta)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
0 &< \frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} + \frac{Cov(p^*,\eta)}{Var(p^*)} + \frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)}
\end{aligned}$$

The third line relies on the expression for the markup, $p-c=-\frac{1}{\beta}\frac{dh}{dq}q$. The final line holds because $\beta<0$ so $-\beta>0$. It follows that β is the lower root of (11) iff

$$-\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)} \leq \frac{Cov\left(p^*,-\frac{1}{\beta}\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*,\eta\right)}{Var(p^*)}$$

in which case β is point identified given knowledge of $Cov(\xi,\eta)$. QED.

Proof of Lemma 1 (Monotonicity in $Cov(\xi,\eta)$)

We return to the quadratic formula for the proof. The lower root of a quadratic ax^2+bx+c is $L \equiv \frac{1}{2}\left(-b-\sqrt{b^2-4ac}\right)$. In our case, $a=1$.

[32]Consider that the first root is the upper root if

$$\beta + \frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} > 0$$

because, in that case,

$$\sqrt{\left(\beta + \frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2} = \beta + \frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}$$

When $\beta + \frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} < 0$, then $\sqrt{\left(\beta + \frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2} = -\left(\beta + \frac{Cov\left(p^*,\frac{dh}{dq}q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)$, and the first root is then the lower root (i.e., minus the negative value).

We wish to show that $\frac{\partial L}{\partial \gamma} < 0$, where $\gamma = Cov(\xi, \eta)$. We evaluate the derivative to obtain

$$\frac{\partial L}{\partial \gamma} = -\frac{1}{2}\left(1 + \frac{b}{(b^2 - 4c)^{\frac{1}{2}}}\right)\frac{\partial b}{\partial \gamma}.$$

We observe that, in our setting, $\frac{\partial b}{\partial \gamma} = \frac{1}{Var(p^*)}$ is always positive. Therefore, it suffices to show that

$$1 + \frac{b}{(b^2 - 4c)^{\frac{1}{2}}} > 0. \tag{D.5}$$

We have two cases. First, when $c < 0$, we know that $\left|\frac{b}{(b^2-4c)^{\frac{1}{2}}}\right| < 1$, which satisfies (D.5). Second, when $c > 0$, it must be the case that $b > 0$ also. Otherwise, both roots are positive, invalidating the model. When $b > 0$, it is evident that the left-hand side of (D.5) is positive. This demonstrates monotonicity.

Finally, we obtain the range of values for L by examining the limits as $\gamma \to \infty$ and $\gamma \to -\infty$. From the expression for L and the result that $\frac{\partial b}{\partial \gamma}$ is a constant, we obtain

$$\lim_{\gamma \to -\infty} L = 0$$
$$\lim_{\gamma \to \infty} L = -\infty$$

When $c < 0$, the domain of the quadratic function is $(-\infty, \infty)$, which, along with monotonicity, implies the range for L of $(0, -\infty)$. When $c > 0$, the domain is not defined on the interval $(-2\sqrt{c}, 2\sqrt{c})$, but L is equal in value at the boundaries of the domain. QED.

Additionally, we note that the upper root, $U \equiv \frac{1}{2}\left(-b + \sqrt{b^2 - 4ac}\right)$ is increasing in γ. When the upper root is a valid solution (i.e., negative), it must be the case that $c > 0$ and $b > 0$, and it is straightforward to follow the above arguments to show that $\frac{\partial U}{\partial \gamma} > 0$ and that the range of the upper root is $[-\frac{1}{2}b, 0)$.

Proof of Proposition 5 (Prior-Free Bounds)

The proof is again an application of the quadratic formula. Any generic quadratic, $ax^2 + bx + c$, with roots $\frac{1}{2}\left(-b \pm \sqrt{b^2 - 4ac}\right)$, admits a real solution if and only if $b^2 > 4ac$. Given the formulation of (11), real solutions satisfy the condition:

$$\left(\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \beta^{OLS}\right)^2 > 4\left(-\beta^{OLS}\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)}\right).$$

As $a = 1$, a solution is always possible if $c < 0$. This is the sufficient condition for point identification from the text. If $c > 0$, it must be the case that $b > 0$; otherwise, both roots are positive. Therefore, a real solution is obtained if and only if $b > 2\sqrt{c}$, that is

$$\left(\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}+\frac{Cov(\xi,\eta)}{Var(p^*)}-\beta^{OLS}\right)>2\sqrt{\left(-\beta^{OLS}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}-\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)}\right)}.$$

Solving for $Cov(\xi,\eta)$, we obtain the prior-free bound,

$$Cov(\xi,\eta)>Var(p^*)\beta^{OLS}-Cov(p^*,\frac{dh}{dq}q)+2Var(p^*)\sqrt{\left(-\beta^{OLS}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}-\frac{Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)}{Var(p^*)}\right)}.$$

This bound exists if the expression inside the radicals is positive, which is the case if and only if the sufficient condition for point identification from Proposition 4 fails. QED.

Proof of Proposition 6 (Non-Constant Marginal Costs)

Under the semi-linear marginal cost schedule of equation (18), the plim of the OLS estimator is equal to

$$\text{plim}\hat{\beta}^{OLS}=\beta+\frac{Cov(\xi,g(q))}{Var(p^*)}-\frac{1}{\beta}\frac{Cov\left(\xi,\frac{dh}{dq}q\right)}{Var(p^*)}.$$

This is obtain directly by plugging in the first–order condition for p: $Cov(p^*,\xi)=Cov(g(q)+\eta-\frac{1}{\beta}\frac{dh}{dq}q-\hat{p},\xi)=Cov(\xi,g(q))-\frac{1}{\beta}Cov(\xi,\frac{dh}{dq}q)$under the assumptions. Next, we re-express the terms including the unobserved demand shocks in in terms of OLS residuals. The unobserved demand shock may be written as $\xi=h(q)-x\beta_x-\beta p$. The estimated residuals are given by $\xi^{OLS}=\xi+\left(\beta-\beta^{OLS}\right)p^*$. As $\beta-\beta^{OLS}=\frac{1}{\beta}\frac{Cov\left(\xi,\frac{dh}{dq}q\right)}{Var(p^*)}-\frac{Cov(\xi,g(q))}{Var(p^*)}$, we obtain $\xi^{OLS}=\xi+\left(\frac{1}{\beta}\frac{Cov\left(\xi,\frac{dh}{dq}q\right)}{Var(p^*)}-\frac{Cov(\xi,g(q))}{Var(p^*)}\right)p^*$. This implies

$$Cov\left(\xi^{OLS},\frac{dh}{dq}q\right)=\left(1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\right)Cov(\xi,\frac{dh}{dq}q)-\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}Cov(\xi,g(q))$$
$$Cov\left(\xi^{OLS},g(q)\right)=\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov\left(\xi,\frac{dh}{dq}q\right)+\left(1-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi,g(q))$$

We write the system of equations in matrix form and invert to solve for the covariance terms that include the unobserved demand shock:

$$\begin{bmatrix} Cov(\xi,\frac{dh}{dq}q) \\ Cov(\xi,g(q)) \end{bmatrix}=\begin{bmatrix} 1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} & -\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1-\frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1}\begin{bmatrix} Cov(\xi^{OLS},\frac{dh}{dq}q) \\ Cov(\xi^{OLS},g(q)) \end{bmatrix}$$

where

$$\begin{bmatrix} 1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} & -\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1-\frac{Cov(p^*,g(q))}{Var(p^*)} \end{bmatrix}^{-1} =$$

$$\frac{1}{1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}-\frac{Cov(p^*,g(q))}{Var(p^*)}}\begin{bmatrix} 1-\frac{Cov(p^*,g(q))}{Var(p^*)} & \frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} \\ -\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)} \end{bmatrix}.$$

Therefore, we obtain the relations

$$Cov(\xi,\frac{dh}{dq}q)=\frac{\left(1-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi^{OLS},\frac{dh}{dq}q)+\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}Cov(\xi^{OLS},g(q))}{1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}-\frac{Cov(p^*,g(q))}{Var(p^*)}}$$

$$Cov(\xi,g(q))=\frac{-\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov(\xi^{OLS},\frac{dh}{dq}q)+\left(1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\right)Cov(\xi^{OLS},g(q))}{1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}-\frac{Cov(p^*,g(q))}{Var(p^*)}}.$$

In terms of observables, we can substitute in for $Cov(\xi,g(q))-\frac{1}{\beta}Cov\left(\xi,\frac{dh}{dq}q\right)$ in the plim of the OLS estimator and simplify:

$$\begin{aligned}
&\left(1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)\left(Cov(\xi,g(q))-\frac{1}{\beta}Cov\left(\xi,\frac{dh}{dq}q\right)\right)\\
=&-\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov(\xi^{OLS},\frac{dh}{dq}q)+\left(1+\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}\right)Cov(\xi^{OLS},g(q))\\
&-\frac{1}{\beta}\left(1-\frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi^{OLS},\frac{dh}{dq}q)-\frac{1}{\beta}\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}Cov(\xi^{OLS},g(q))\\
=&Cov(\xi^{OLS},g(q))-\frac{1}{\beta}Cov(\xi^{OLS},\frac{dh}{dq}q).
\end{aligned}$$

Thus, we obtain an expression for the probability limit of the OLS estimator,

$$\text{plim}\hat{\beta}^{OLS}=\beta-\frac{\frac{Cov(\xi^{OLS},\frac{dh}{dq}q)}{Var(p^*)}-\beta\frac{Cov(\xi^{OLS},g(q))}{Var(p^*)}}{\beta+\frac{Cov(p^*,\frac{dh}{dq}q)}{Var(p^*)}-\beta\frac{Cov(p^*,g(q))}{Var(p^*)}},$$

and the following quadratic β.

$$
\begin{aligned}
0 = &\left(1 - \frac{Cov(p^*, g(q))}{Var(p^*)}\right)\beta^2 \\
&+ \left(\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, g(q))}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\xi^{OLS}, g(q))}{Var(p^*)}\right)\beta \\
&+ \left(-\frac{Cov(p^*, \frac{dh}{dq}q)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov(\xi^{OLS}, \frac{dh}{dq}q)}{Var(p^*)}\right).
\end{aligned}
$$

QED.

Proof of Lemma A.1

The proof is by construction. Note that model has the solutions $p_t^* = \frac{1}{2}\left(-\frac{\alpha}{\beta} - \frac{\xi_t}{\beta} + \gamma + \frac{\nu_t}{\beta}\right)$ and $q_t^* = \frac{1}{2}(\alpha + \xi_t + \beta\gamma + \nu_t)$, where again $\nu_t \equiv \beta\eta_t$. The following objects are easily derived:

$$
\begin{aligned}
Cov(p, \xi) &= -\frac{1}{2\beta}Var(\xi) \qquad & Cov(p, \nu) &= \frac{1}{2\beta}Var(\nu)2 \\
Var(p) &= \frac{Var(\nu) + Var(\xi)}{(2\beta)^2} \qquad & Var(q) &= \frac{1}{4}(Var(\xi) + Var(\nu))
\end{aligned}
$$

Using the above, we have

$$
\begin{aligned}
Cov(p, q) &= Cov(p, \alpha + \beta p + \xi) = \beta Var(p) + Cov(p, \xi) = \beta\frac{Var(\nu) + Var(\xi)}{(2\beta)^2} - \frac{2\beta}{(2\beta)^2}Var(\xi) \\
&= \frac{\beta Var(\nu) + \beta Var(\xi) - 2\beta Var(\xi)}{(2\beta)^2} = \beta\frac{Var(\nu) - Var(\xi)}{(2\beta)^2}
\end{aligned}
$$

And that obtains equation (A.2):

$$
plim\left(\hat{\beta}^{OLS}\right) \equiv \beta^{OLS} = \frac{Cov(p, q)}{Var(p)} = \beta\frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}
$$

Equation (A.4) requires an expression for $Cov(q, \xi^{OLS})$. Define

$$
plim(\hat{\xi}^{OLS}) \equiv \xi^{OLS} = q - \alpha^{OLS} - \beta^{OLS}p
$$

Then, plugging into $Cov(q, \xi^{OLS})$ using the objects derived above, we have

$$\begin{aligned}
Cov(q, \xi^{OLS}) &= Cov(q, q - \beta^{OLS} p) \\
&= Var(q) - \beta^{OLS} Cov(p, q) \\
&= \frac{1}{4}(Var(\xi) + Var(\nu)) - \left(\beta \frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}\right)\left(\beta \frac{(Var(\nu) - Var(\xi))}{(2\beta)^2}\right) \\
&= \frac{1}{4}\left(\frac{[Var(\xi) + Var(\nu)]^2 - [Var(\nu) - Var(\xi)]^2}{Var(\nu) + Var(\xi)}\right) \\
&= \frac{Var(\xi)Var(\nu)}{Var(\nu) + Var(\xi)}
\end{aligned}$$

We turn now to equation (A.3). Based on the above, we have that

$$\frac{Cov(q, \xi^{OLS})}{Var(p)} = \left(\frac{Var(\xi)Var(\nu)}{Var(\nu) + Var(\xi)}\right) \frac{(2\beta)^2}{Var(\nu) + Var(\xi)} = (2\beta)^2 \frac{Var(\xi)Var(\nu)}{[Var(\nu) + Var(\xi)]^2}$$

and now only few more lines of algebra are required:

$$\begin{aligned}
(\beta^{OLS})^2 + \frac{Cov(q, \xi^{OLS})}{Var(p)} &= \beta^2 \left[\frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}\right]^2 + (2\beta)^2 \frac{Var(\xi)Var(\nu)}{[Var(\nu) + Var(\xi)]^2} \\
&= \frac{\beta^2[Var^2(\nu) + Var^2(\xi) - 2Var(\nu)Var(\xi)] + 4\beta^2 Var(\nu)Var(\xi)}{[Var(\nu) + Var(\xi)]^2} \\
&= \frac{\beta^2[Var^2(\nu) + Var^2(\xi) + 2Var(\nu)Var(\xi)]}{[Var(\nu) + Var(\xi)]^2} \\
&= \beta^2 \frac{[Var(\nu) + Var(\xi)]^2}{[Var(\nu) + Var(\xi)]^2} = \beta^2
\end{aligned}$$

QED.

Proof of Proposition C.1 (Two-Stage Estimator)

Suppose that, in addition to assumptions 1-3, that marginal costs are uncorrelated with the exogenous demand factors (Assumption 5). Then, the expression $\frac{1}{\beta + \frac{Cov\left(p, \frac{dh}{dq} q\right)}{Var(p)}} \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p)}$ is equal to $\frac{1}{\beta + \frac{Cov\left(p^*, \frac{dh}{dq} q\right)}{Var(p^*)}} \frac{Cov\left(\xi^{OLS}, \frac{dh}{dq} q\right)}{Var(p^*)}$.

Assumption 4 implies $Cov(\hat{p}, c) = 0$, allowing us to obtain

$$\begin{aligned}
Cov(\hat{p}, \beta(\hat{p} + p^* - c)) &= \beta Var(\hat{p}) \\
Cov(p - p^*, \beta(\hat{p} + p^* - c)) &= \beta Var(p) - \beta Var(p^*) \\
Var(p)\beta + Cov\left(p, \frac{dh}{dq}q\right) &= Var(p^*)\beta + Cov\left(p^*, \frac{dh}{dq}q\right) \\
\left(\beta + \frac{Cov\left(p, \frac{dh}{dq}q\right)}{Var(p)}\right)\frac{1}{Var(p^*)} &= \left(\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}\right)\frac{1}{Var(p)} \\
\frac{1}{\beta + \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p^*)} &= \frac{1}{\beta + \frac{Cov\left(p, \frac{dh}{dq}q\right)}{Var(p)}}\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p)}.
\end{aligned}$$

Therefore, the probability limit of the OLS estimator can be written as:

$$\text{plim}\hat{\beta}^{OLS} = \beta - \frac{1}{\beta + \frac{Cov\left(p, \frac{dh}{dq}q\right)}{Var(p)}}\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p)}.$$

The roots of the implied quadratic are:

$$\frac{1}{2}\left(\beta^{OLS} - \frac{Cov\left(p, \frac{dh}{dq}q\right)}{Var(p)} \pm \sqrt{\left(\beta^{OLS} + \frac{Cov\left(p, \frac{dh}{dq}q\right)}{Var(p)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, \frac{dh}{dq}q\right)}{Var(p)}}\right)$$

which are equivalent to the pair $\left(\beta, \beta\left(1 - \frac{Var(p^*)}{Var(p)}\right) + \frac{Cov(p^*, \xi)}{Var(p^*)} - \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p)}\right)$. Therefore, with the auxiliary condition $\beta < \frac{Cov(p^*, \xi)}{Var(p^*)}\frac{Var(p)}{Var(p^*)} - \frac{Cov\left(p^*, \frac{dh}{dq}q\right)}{Var(p^*)}$, the lower root is consistent for β. QED.