7. Activity for the Classroom. In this section, we consider the problem of estimating the position and velocity of a projectile, say, an artillery round, given a few noisy measurements of its position. We show through a series of exercises that the Kalman filter can average out the noise in the system and provide a relatively smooth profile of the projectile as it passes by a radar sensor. We then show that one can effectively predict the point of impact as well as the point of origin, so that troops on the ground can both duck for cover and return fire before the projectile lands. Although we computed the figures below in MATLAB, one could easily reproduce this work in another computing environment.

7.1. Problem Formulation. Assume that the state of the projectile is given by the vector $x = \begin{pmatrix} s_x & s_y & v_x & v_y \end{pmatrix}^T$, where s_x and s_y are the horizontal and vertical components of position, respectively, with corresponding velocity components v_x and v_y. We suppose that state evolves according to the discrete-time dynamical system

$$x_{k+1} = Fx_k + u + w_k,$$

where

$$F = \begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1-b & 0 \\ 0 & 0 & 0 & 1-b \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g\Delta t \end{pmatrix}.$$

In this model, Δt is the interval of time between measurements, $0 \leq b \ll 1$ is the drag coefficient, g is the gravitational constant, and the noise process w_k has zero mean with covariance $Q_k > 0$. Since the radar device is only able to measure the position of the projectile, we write the observation equation as

$$y_k = Hx_k + v_k, \quad \text{where} \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and the measurement noise v_k has zero mean with covariance $R_k > 0$.

7.2. Exercises. Throughout this classroom activity, we assume that $b = 10^{-4}$, $g = 9.8$, and $\Delta t = 10^{-1}$. For simplicity, we also assume $Q = 0.1 \cdot I_4$ and $R = 500 \cdot I_2$, though in practice these matrices would probably not be diagonal.

Exercise 1. Using an initial state $x_0 = \begin{pmatrix} 0 & 0 & 300 & 600 \end{pmatrix}^T$, evolve the system forward 1200 steps. Hint: The noise term w_k can be generated by using the transpose of the Cholesky factorization of Q; specifically, set `w = chol(Q)'*randn(4,1)`.

Exercise 2. Now assume that your radar system can only detect the projectile between the 400th and 600th time steps. Using the measurement equation, produce a plot of the projectile path and the noisy measurements. The noise term v_k can be generated similarly to the exercise above.

Exercise 3. Initialize the Kalman filter at $k = 400$. Use the position coordinates y_{400} to initialize the position and take the average velocity over 10 or so measurements to provide a rough velocity estimate. Use a large initial estimator covariance such as $P_{400} = 10^6 \cdot Q$. Then, using the Kalman filter, compute the next 200 state estimates using (3.9) and (3.12). Plot the estimated position over the graph of the previous exercise. Your image should be similar to Figures 7.1(a) and 7.1(b), the latter being a zoomed version of the former. In Figure 7.1(c), we see the errors generated by the measurement error as well as the estimation error. Note that the estimation error is much smaller than the measurement error.

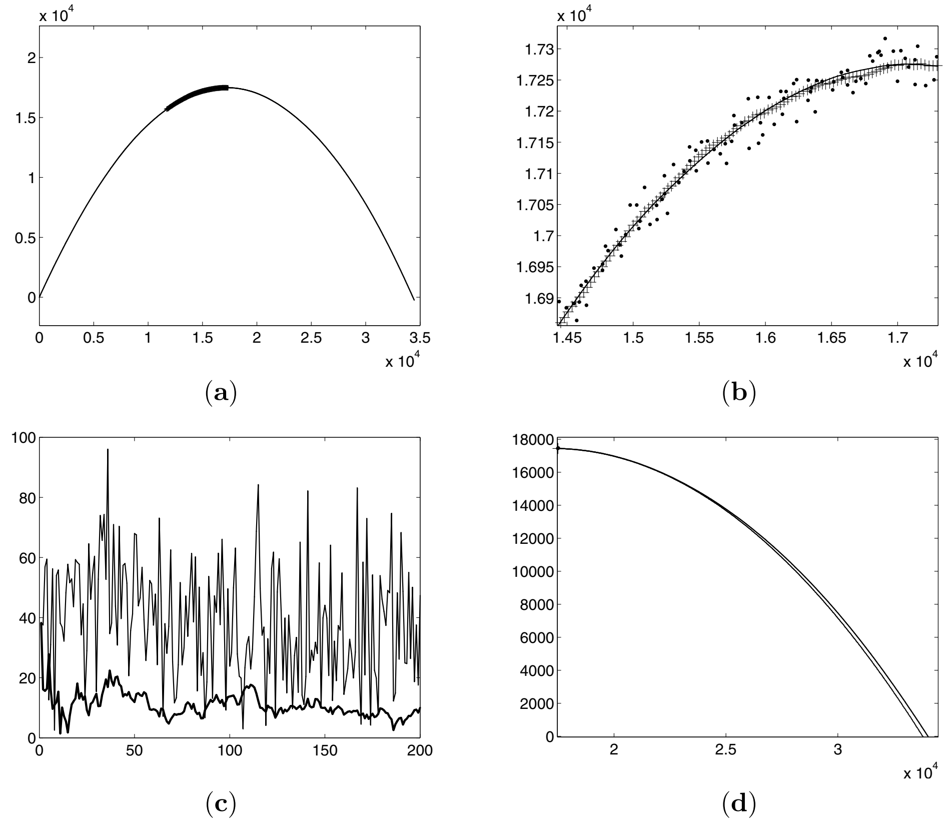

Fig. 7.1 *A plot of* (a) *the trajectory, the measurement points, and the estimates given by the Kalman filter. The dark region on the upper-left part of the trajectory corresponds to the part of the profile where measurement and state estimations are made. In* (b) *a zoom of the previous image is given. The measurements are peppered dots with* (+) *denoting the estimated points from the Kalman filter and the dotted line being the true profile. Note that the estimated and true positions are almost indistinguishable. In* (c) *a plot of errors produced by measurements and the Kalman filter is given. Notice that the estimation error is much smaller than the measurement error. Finally, in* (d) *a plot of the predicted trajectory is given. These lines are close, with the estimation error at the point of impact being roughly half a percent.*

Exercise 4. Using the last state estimate from the previous exercise $\hat{x}_{600}$, use the predictive estimation method described in (4.4) to trace out a projectile until the y-component of position crosses zero, that is, $s_y \approx 0$. Plot the path of the estimate against the true (noiseless) path and see how close the estimated point of impact is when compared with the true point of impact. Your graph should look like Figure 7.1(d).

Exercise 5 (bonus problem). Estimate the point of origin of the projectile by reversing the system and considering the problem

$$x_k = F^{-1}x_{k+1} - F^{-1}u - F^{-1}w.$$

This allows one to iterate backward in time.

Appendix A. Proof of the Gauss–Markov Theorem. In this section, we present a proof of Theorem 2.1, which states that among all linear unbiased estimators $\hat{x} =$