

Baby Rudin

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1 The Real and Complex Number Systems

1.1 Ordered Sets

We realize, considering the solutions to $x^2 = 2$, that the set of rational numbers \mathbb{Q} is incomplete. We start by defining some terms:

Definition 1. An **order** $<$ on S is a relation such that

1. If $x, y \in S$, then exactly one of $x < y$, $x = y$, or $x > y$ is true.
2. If $x, y, z \in S$ with $x < y$ and $y < z$, then $x < z$.

A set S is **ordered** if it admits an order. For example, \mathbb{Q} is an order if we take $x < y \iff y - x$ is positive.

If S is an ordered set and $E \subseteq S$, we call $\beta \in S$ an **upper bound** for E if $\beta \geq x$ for all $x \in E$. If such a β exists, we say E is **bounded above** (by β). We define lower bounds similarly.

If S is an ordered set and $E \subseteq S$ is bounded above, and there exists $\alpha \in S$ such that α is an upper bound for E and if $\gamma < \alpha$ then γ is not an upper bound for E , then we call α a **least upper bound** for E . We write $\alpha = \sup E$.

Similarly, we can define a **greatest lower bound** for E , which we write $\alpha = \inf E$.

An ordered set S is said to have the **least upper bound property** if every non-empty subset $E \subseteq S$ which is bounded above admits a least upper bound. We define the **greatest lower bound property** similarly. The following theorem demonstrates that these properties are in fact equivalent for all ordered sets.

Theorem 1. If S is an ordered set with the least upper bound property, and $\emptyset \neq B \subseteq S$ is bounded below, then $\inf B \in S$.

Proof. Let L be the set of lower bounds for B in S . Since B is bounded below, L is non-empty; and a subset of S by definition. Since S has the least upper bound property, there exists some $\alpha = \sup L \in S$. We claim that $\alpha = \inf B$ and in particular, $\inf B \in S$.

To show this, it suffices to show that α is a lower bound for B , and that if $\beta > \alpha$ in S , then β is not a lower bound for B . If α were not a lower bound for B , there must be some $x \in B$ with $x < \alpha$. However this would imply that x is an upper bound for L and thus $\alpha \geq x$, a contradiction.

If $\beta > \alpha$ and β were a lower bound for B (so $\beta \in L$), then since $\alpha = \sup L$, we would have $\beta \leq \alpha$, a contradiction.

Thus, $\alpha = \inf B \in S$. In particular, since this holds for all non-empty $B \subseteq S$ bounded below, S has the greatest lower bound property. ■

1.2 Fields

On to defining the notion of fields!

Definition 2. A **field** F is a set equipped with two operations, $+$ and \times with the following properties:

- F is a commutative group under addition, with additive identity 0. We denote the additive inverse of x to be $-x$.
- $F \setminus \{0\}$ is a commutative group under multiplication, with additive identity 1. We denote the multiplicative inverse of x to be $1/x$.
- $x \times (y + z) = x \times z + y \times z$ for all $x, y, z \in F$.

Proposition 1. If G is a group with operation \cdot , then for $x, y, z \in G$,

- (a) $x \cdot y = x \cdot z \rightarrow y = z$
- (b) $x \cdot y = x \rightarrow y = 1$
- (c) $x \cdot y = 1 \rightarrow y = x^{-1}$
- (d) $(x^{-1})^{-1} = x$

Proof. Statement (a) follows from multiplying x^{-1} to both sides. (b) follows by letting $z = 1$ in (a); (c) by taking $z = x^{-1}$; and (d) follows since $x^{-1} \cdot x = 1$ and by replacing x with x^{-1} in (c). ■

Notice that these properties can be applied both to the additive and multiplicative groups in a field F .

In addition to the above group properties, the distributive law leads to the following properties of fields:

Proposition 2. In a field F , for any $x, y \in F$, we have

- (a) $0 \times x = 0$.
- (b) If $x \neq 0$ and $y \neq 0$, then $x \times y \neq 0$.
- (c) $(-x) \times y = -(x \times y) = x \times (-y)$.
- (d) $(-x) \times (-y) = xy$.

Proof. For (a), notice that since $0 + 0 = 0$, we have $0 \times x = (0 + 0) \times x = 0 \times x + 0 \times x$, and the result follows from Proposition 1(b) on the additive group.

For (b), notice that if $x \times y = 0$ in this case, multiplying both sides of the equality by $y^{-1}x^{-1}$ would produce $1 = 0$, a contradiction.

For (c), notice that $0 = (x + (-x)) \times y = x \times y + (-x) \times y$, and the rest follows from Proposition 1(c). The other side follows similarly.

Statement (d) follows from (c) with $-y$ in place of y and using Proposition 1(d). ■

Now that we've sufficiently introduced ordered sets and fields, we wonder what would happen if we combined them...

Definition 3. An **ordered field** F is a field which is also an ordered set, with the following properties for all $x, y, z \in F$:

1. $x + y < x + z$ if $y < z$.
2. $x \times y > 0$ if $x > 0$ and $y > 0$

If $x > 0$, we say x is **positive**, and we say x is **negative** if $x < 0$. For example, \mathbb{Q} is an ordered field with the regular notions of $+$, \times and $<$.

The following properties hold for ordered fields:

Proposition 3. For $x, y, z \in F$ for some ordered field F , we have

1. If $x > 0$, then $-x < 0$.
2. If $x > 0$ and $y < z$ then $xy < xz$.
3. If $x < 0$ and $y < z$ then $xy > xz$.
4. If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
5. If $0 < x < y$ then $0 < y^{-1} < x^{-1}$.

Proof. For (a), if $x > 0$, then $0 = x + (-x) > 0 + (-x) = -x$. The other direction follows similarly.

For (b), notice that since $y < z$, we have $z - y > 0$ and thus $x(z - y) > 0$. Then, notice that $xz = x(z - y) + xy > xy$.

For (c), notice that $-x > 0$ by (a) so $-(xy) = (-x)y < (-x)z = -(xz)$. The result follows by adding $xy + xz$ to both sides.

For (d), if $x > 0$, then $x^2 = xx > x \times 0 > 0$, otherwise $-x > 0$ and the argument is analogous since $(-x)^2 = -(-x^2) = x^2$. In particular, $1 = 1 \times 1 = 1^2 > 0$.

For (e), notice that since $x > 0$, we must have $x^{-1} > 0$ otherwise $1 = x \times x^{-1} < 0$, contradicting (d). A similar argument holds for y^{-1} , and thus $x^{-1}y^{-1} > 0$. The result follows by multiplying the inequality by the positive value $x^{-1}y^{-1}$. ■

1.3 The Real Field

It is tempting to combine the definitions we have introduced so far: we can imagine some ordered field which has the least upper bound property, containing the rational numbers \mathbb{Q} . In fact, this is the familiar set of real numbers, \mathbb{R} , as shown in the following theorem:

Theorem 2 (Existence (and construction) of \mathbb{R}). *There exists an ordered field \mathbb{R} with the least upper bound property, and in fact, it contains \mathbb{Q} as a subfield. Specifically, this means that $\mathbb{Q} \subseteq \mathbb{R}$, and that the field operations in \mathbb{R} coincide with the field operations in \mathbb{Q} .*

Proof. This proof is presented in the Appendix. ■

Theorem 3. *From here, we can derive the following properties in \mathbb{R} :*

- (a) If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.
- (b) If $x < y$ in \mathbb{R} , there exists $q \in \mathbb{Q}$ with $x < q < y$.
- (c) For every $x > 0$ in \mathbb{R} , there exists one and only one positive $y \in \mathbb{R}$ such that $y^n = x$, for every positive integer n .
- (d) If $a, b > 0$ and $n > 0$, then $(ab)^{1/n} = a^{1/n}b^{1/n}$.

Proof. For (a), let $A = \{nx : n \in \mathbb{N}\}$. If (a) were true, A would be bounded above, and clearly non-empty, and thus admits a least upper bound $\alpha = \sup A$ in \mathbb{R} . However, $\alpha - x < \alpha$ is not an upper bound for A , so there exists some $\alpha - x < mx \leq \alpha$ in A . But then $\alpha < (m + 1)x \in A$, a contradiction.

For (b), notice that $y - x > 0$ so from (a), there exists $n \in \mathbb{N}$ such that $n(y - x) > 1$. Apply (a) again to find $m_1 > nx$ and $m_2 > -nx$ so that $-m_2 < nx < m_1$. Thus, there exists $m \in \mathbb{N}$ such that $m - 1 \leq nx < m$. Adding $n(y - x) > 1$ to both sides of $nx \geq m - 1$ yields $ny > m$ and so dividing both sides of $nx < m < ny$ by $n > 0$ yields the result.

For (c), notice first of all that the function $f(x) = x^n$ is monotonically increasing on the positive real numbers. To see this, factor

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + b^2a^{n-3} + ba^{n-2} + a^{n-1}) > n(b - a)a^{n-1} > 0$$

and similarly we can show $b^n - a^n < n(b - a)b^{n-1}$. These inequalities will also be helpful later. The fact that n -th positive real roots are unique follows immediately from the fact that x^n is strictly increasing.

Let $A = \{y > 0 : y^n < x\}$. Then, A is bounded above by $\max(1, x)$ and contains $\frac{x}{1+x}$. Since \mathbb{R} satisfies the least upper bound property, $\alpha = \sup A$ exists. We claim that $\alpha^n = x$.

Suppose otherwise. If $\alpha^n < x$, then for

$$0 < t < \min\left(1, \frac{x - \alpha^n}{n(\alpha + 1)^{n-1}}\right), \text{ we have } (\alpha + t)^n - \alpha^n < nt(\alpha + t)^{n-1} < nt(\alpha + 1)^{n-1} < x - \alpha^n$$

so $(\alpha + t)^n < x$, contradicting the fact that α is an upper bound for A .

If $\alpha^n > x$, then for

$$0 < t = \min\left(1, \frac{\alpha^n - x}{n\alpha^{n-1}}\right), \text{ we have } \alpha^n - (\alpha - t)^n < nt\alpha^{n-1} = \alpha^n - x$$

so $(\alpha - t)^n > x > z^n$ for each $z \in A$, so in particular $\alpha - t$ is a smaller upper bound than α for A , contradicting the fact that α is the **least** upper bound for A .

For (d), notice that $a^{1/n}b^{1/n} > 0$ since both terms are positive from (c), and that $(a^{1/n}b^{1/n})^n = ab$ since multiplication is commutative, so from (c), the n -th positive root is unique, $(ab)^{1/n} = a^{1/n}b^{1/n}$. ■

Definition 4. Let $x \in \mathbb{R}_{>0}$. Let n_0 be the largest integer such that $n_0 \leq x$. Then, given n_0, n_1, \dots, n_{k-1} , let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x.$$

Then, letting E be the set of the numbers

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}, k \in \mathbb{N},$$

we have $x = \sup E$. We define $n_0.n_1n_2n_3 \dots$ to be the **decimal expansion** of x .

1.4 The Extended Real Number System

Definition 5. The **extended real number system** consists of \mathbb{R} and two symbols $+\infty$ and $-\infty$, with the normal order and $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

It is customary to define the following results of operations:

- $x + \infty = +\infty, x - \infty = -\infty, \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$
- If $x > 0$, then $x \cdot (+\infty) = +\infty.$
- If $x < 0$, then $x \cdot (+\infty) = -\infty.$

1.5 The Complex Field

Definition 6. A **complex number** is an ordered pair (a, b) of real numbers. We denote \mathbb{C} the set of all complex numbers. Two ordered pairs are equal iff their components are equal. Also, we define

$$(a, b) + (c, d) = (a + c, b + d), (a, b) \times (c, d) = (ac - bd, ad + bc).$$

and note that the set of complex numbers forms a field, with $(0, 0)$ and $(1, 0)$ playing the roles of 0 and 1 respectively.

We define $i = (0, 1)$ and notice $i^2 = -1$. This notation is convenient as we can do away with the ordered pair notation and write $(a, b) = a + bi$ instead.

If $z = a + bi$ for $a, b \in \mathbb{R}$, then we define $\Re(z) = a$ and $\Im(z) = b$ to be the **real part** and **imaginary part** of z respectively. $\bar{z} = a - bi$ is called the **conjugate** of z .

The following properties of complex numbers will be helpful:

Theorem 4. If $z, w \in \mathbb{C}$, then

$$(a) \overline{z + w} = \bar{z} + \bar{w}.$$

$$(b) \overline{zw} = \bar{z} \cdot \bar{w}.$$

$$(c) z + \bar{z} = 2\Re(z), z - \bar{z} = 2i\Im(z).$$

$$(d) z\bar{z} \in \mathbb{R}_{\geq 0} \text{ when } z \neq 0.$$

Proof. For all of these, we will suppose $z = a + bi$ and $w = c + di$.

For (a), notice that $z + w = (a + c) + (b + d)i$ so

$$\overline{z + w} = (a + c) - (b + d)i = (a - bi) + (c - di) = \bar{z} + \bar{w}.$$

For (b), notice that $zw = (ac - bd) + (ad + bc)i$ and that

$$\bar{z} \cdot \bar{w} = (a - bi) \cdot (c - di) = (ac - bd) - (ad + bc)i = \overline{zw}.$$

For (c), notice that $z + \bar{z} = 2a = 2\Re(z)$ and $z - \bar{z} = 2bi = 2i\Im(z)$.

For (d), notice that $z\bar{z} = a^2 + b^2$, which is real and positive unless $a = b = 0$ (that is, $z = 0$). ■

Since the value $z\bar{z}$ is a non-negative real number, we can think of it as some measure of how ‘large’ z is: the following definition formalizes this:

Definition 7. We define $|z| = \sqrt{z\bar{z}}$, called the **magnitude** or **absolute value** of z , which exists and is unique from Theorem 3c. When $x \in \mathbb{R}$, $|x| = x$ when $x > 0$ and $-x$ otherwise. In particular, the fact that $x \leq |x|$ when $x \in \mathbb{R}$ is sometimes useful.

The following are helpful properties of the absolute value:

Theorem 5. If $z = a + bi, w = c + di \in \mathbb{C}$, then the following are true:

$$(a) |z| \geq 0, \text{ with equality iff } z = 0.$$

$$(b) |z| = |\bar{z}|.$$

$$(c) |zw| = |z| \cdot |w|.$$

$$(d) |\Re(z)| \leq |z|.$$

(e) $|z + w| \leq |z| + |w|$.

Proof. For (a), clearly $|0| = 0$. Otherwise, $z\bar{z}$ is a positive real number, and thus has a positive square root, which is precisely $|z|$ by definition.

For (b), notice that both sides evaluate to $\sqrt{a^2 + b^2}$.

For (c), if either z or w is 0, the result follows immediately. Otherwise, we can see

$$|zw| = \sqrt{zw\overline{zw}} = \sqrt{z\bar{z}w\bar{w}} = \sqrt{z\bar{z}}\sqrt{w\bar{w}} = |z||w|$$

where the third equality uses Theorem 3(d) since $z\bar{z}$ and $w\bar{w}$ are both positive from Theorem 4(d).

For (d), notice that

$$|\Re(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

where the third inequality follows since $f(x) = x^2$ is monotonically increasing as shown in Theorem 3(c). Notice that equality holds iff $b = 0$, that is, $z \in \mathbb{R}$.

For (e), notice that

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + w\bar{z} = |z|^2 + |w|^2 + (z\bar{w}) + (\overline{z\bar{w}}).$$

From Theorems 4(c) and 5(d), we have

$$(z\bar{w}) + (\overline{z\bar{w}}) = 2\Re(z\bar{w}) \leq |2\Re(z\bar{w})| \leq 2|z||w|.$$

Combining these yields

$$|z + w|^2 = |z|^2 + |w|^2 + (z\bar{w}) + (\overline{z\bar{w}}) \leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2.$$

The result follows by taking square roots, since both sides are non-negative real numbers. ■

Finally, we present a classic inequality, the Cauchy-Schwartz inequality:

Theorem 6. If $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}$, then $\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2$.

Proof. Let $A = \sum |a_j|^2, B = \sum |b_j|^2 \in \mathbb{R}$ and $C = \sum a_j \bar{b}_j$, so we claim $|C|^2 \leq AB$. If $B = 0$, then we must have $b_1 = b_2 = \dots = b_n = 0$ and the result follows immediately. Otherwise, if $B > 0$, then

$$\begin{aligned} 0 &\leq \sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)(\overline{Ba_j - Cb_j}) \\ &= B^2 \sum a_j \bar{a}_j - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + C\bar{C} \sum b_j \bar{b}_j \\ &= B^2 \sum |a_j|^2 - B\bar{C}C - BC\bar{C} + C\bar{C} \sum |b_j|^2 \\ &= B^2 A - B|C|^2 - B|C|^2 + |C|^2 B \\ &= B(AB - |C|^2) \end{aligned}$$

and since $B > 0$, we must have $AB - |C|^2 \geq 0$, as required. ■

1.6 Euclidean Spaces

Motivated by spaces like \mathbb{R}^2 and \mathbb{R}^3 , we generalize this notion.

Definition 8. For $k \in \mathbb{N}$, let \mathbb{R}^k be the set of all ordered k -tuples $\mathbf{x} = (x_1, x_2, \dots, x_k)$, where $x_i \in \mathbb{R}$ are called the **coordinates** of \mathbf{x} . We call these tuples **vectors**.

We define addition as component-wise addition, and scalar multiplication by a real number as component-wise multiplication, so that \mathbb{R}^k is closed under these two operations. The fact that \mathbb{R}^k satisfies these properties, as well as the operations satisfying the associative, commutative, and distributive laws make \mathbb{R}^k into a **vector space over \mathbb{R}** .

We define the **inner product** of \mathbf{x} and \mathbf{y} by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^k x_j y_j$$

and the **norm** of \mathbf{x} by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{j=1}^k x_j^2 \right)^{1/2}$$

Now, \mathbb{R}^k along with its associated inner product and norm is called **Euclidean k -space**.

We can now use this definition to prove some important properties of \mathbb{R}^k :

Theorem 7. Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$. Then, the following are true:

- (a) $|\mathbf{x}| \geq 0$.
- (b) $|\mathbf{x}| = 0$ iff $\mathbf{x} = \mathbf{0}$.
- (c) $|\alpha \mathbf{x}| = |\alpha| |\mathbf{x}|$.
- (d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$.
- (e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.
- (f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Proof. The facts (a), (b), and (c) follow directly from the definitions.

For (d), this is precisely the Cauchy-Schwartz inequality restricted to \mathbb{R} , applied to the components.

For (e), we write

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}| |\mathbf{y}| + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}| |\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2 \end{aligned}$$

and the result follows by taking the square root.

For (f), this is shown by replacing \mathbf{x} with $\mathbf{x} - \mathbf{y}$ and \mathbf{y} with $\mathbf{y} - \mathbf{z}$ in (e). ■

The fact that $\mathbf{x} \geq 0$ with equality iff $\mathbf{x} = \mathbf{0}$, and fact (f), called the Triangle Inequality, let us regard \mathbb{R}^k as a **metric space**.

$\mathbb{R}^1 = \mathbb{R}$ is called the (real) line, \mathbb{R}^2 is the plane. It is noteworthy that the norms in \mathbb{R}^2 and \mathbb{C} are consistent.

1.7 Appendix: Dedekind's construction of \mathbb{R}

Here we construct \mathbb{R} from \mathbb{Q} in 9 steps.

Step 1. Making \mathbb{R} a set.

The elements of \mathbb{R} will be certain subsets of \mathbb{Q} , called **cuts**. A cut is a set α with the following three properties:

- (I) $\emptyset \subset \alpha \subset \mathbb{Q}$.
- (II) If $p \in \alpha$ and $q \in \mathbb{Q}$, with $q < p$, then $q \in \alpha$.
- (III) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

We will use p, q, r, \dots for rational numbers, and $\alpha, \beta, \gamma, \dots$ for cuts.

Note that condition (II) implies the following useful statements:

- If $p \in \alpha$ and $q \notin \alpha$, then $p < q$.
- If $r \notin \alpha$ and $r < s$, then $s \notin \alpha$.

Step 2. Making \mathbb{R} an ordered set.

Define an order $<$ on the set of cuts such that $\alpha < \beta$ iff $\alpha \subset \beta$. This is clearly transitive, so it suffices to show that exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$ can be true at once. Notice that by the properties of sets, at most one of these can be true at once, so it suffices to show that at least one is.

To do this, suppose that the first two are false. Then, there exists some $q \in \alpha$ which is not in β . Then, if $p \in \beta$, then we must have $p < q$ from condition (II). However, since $q \in \alpha$ and $p < q$, this also means $p \in \alpha$, so we conclude $\beta \subset \alpha$ so $\alpha > \beta$ as required. Thus, \mathbb{R} is now an ordered set.

Step 3. Showing \mathbb{R} has the least-upper-bound property.

Let A be a non-empty subset of \mathbb{R} which is bounded above. Let $\gamma = \bigcup A$. We claim that $\gamma \in \mathbb{R}$ and $\gamma = \sup A$.

To show $\gamma \in \mathbb{R}$, we prove each condition individually. The first part of condition (I) follows directly from the fact that A is the non-empty union of non-empty subsets of \mathbb{Q} . The second follows from the fact that A is bounded above by some $\beta \neq \mathbb{Q}$. For condition (II), pick $p \in \gamma$ and $q \in \mathbb{Q}$. Since $p \in \gamma = \bigcup A$, there exists some $\alpha \in A$ for which $p \in \alpha$. Then, from condition (II) on α , we must have $q \in \alpha \subseteq \gamma$, as required. For condition (III), pick the same p, q, α . Then, from condition (III) on α , there must exist $r \in \alpha \subseteq \gamma$ with $p < r$, as required. Thus, $\gamma \in \mathbb{R}$.

To show that $\gamma = \sup A$, notice that by definition, γ is the superset of any $\alpha \in A$, so γ is an upper bound for A . Let $\beta < \gamma$. Then, there must be some p in γ but not in β . In particular, there must be some $\alpha \in A$ for which $p \in \alpha$ but not in β . However, this means $\alpha \not\subset \beta$ so β is not an upper bound for A , as required.

Thus, \mathbb{R} satisfies the least-upper-bound property.

Step 4. Defining addition on \mathbb{R} .

For $\alpha, \beta \in \mathbb{R}$, we define $\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\}$. For this, we let 0^* denote the set of all negative rational numbers. It's easily checked that $0^* \in \mathbb{R}$. We verify the 5 axioms which define \mathbb{R} as a commutative additive group with $+$.

Let $\alpha, \beta, \gamma \in \mathbb{R}$.

(A1) $\alpha + \beta \in \mathbb{R}$.

Notice that $\alpha + \beta$ is non-empty since α and β are non-empty. To show that $\alpha + \beta \neq \mathbb{Q}$, take elements $r' \notin \alpha, s' \notin \beta$. Then, $r' + s' > r + s$ for all $r \in \alpha, s \in \beta$, so in particular, $r' + s' \notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$, showing condition (I).

For condition (II) let $p = r + s \in \alpha + \beta$ for some $r \in \alpha, s \in \beta$, and $q < p$. Then, $q - s < p - s = r$ so $q - s \in \alpha$. Thus, $q = (q - s) + s \in \alpha + \beta$.

For condition (III), let $t > r$ in α , so that $t + s > r + s$ in $\alpha + \beta$.

(A2) $\alpha + \beta = \beta + \alpha$.

This follows from the definition and the commutativity of \mathbb{Q} :

$$\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\} = \{s + r : s \in \beta, r \in \alpha\} = \beta + \alpha$$

(A3) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

This follows as above from the definition and the associativity of \mathbb{Q} .

(A4) $\alpha + 0^* = \alpha$.

If $r \in \alpha$, and $s \in 0^*$, then $r + s < r \in \alpha$, so $\alpha + 0^* \subseteq \alpha$. For the other side, take $r \in \alpha$ and $t > r$ in α . Then, $r - t < 0$ so $r - t \in 0^*$ and thus $r = t + (r - t) \in \alpha + 0^*$ and thus $\alpha \subseteq \alpha + 0^*$, as required.

(A5) $-\alpha$ exists.

Let β be the set of p such that there exists $r > 0$ such that $-p - r \notin \alpha$. We claim that $\beta \in \mathbb{R}$ and $\alpha + \beta = 0^*$.

Let $s \notin \alpha$. Then, $s + 1 \in \beta$ since $(s + 1) - 1 \notin \alpha$, so β is non-empty. Also, if $r \in \alpha$, then $r \notin \beta$, so $\beta \neq \mathbb{Q}$, satisfying condition (I) for being a cut. For condition (II) let $p \in \beta$ and $q < p$ in \mathbb{Q} . Then, $-p - r \notin \alpha$ for some $r > 0$. Then, $-q - (-q + p + r) = -p - r \notin \alpha$, and $-q + p + r > r > 0$, so $q \in \beta$, proving condition (II). For condition (III), let $p \in \beta$. Then, there exists $r > 0$ so that $-p - r \notin \alpha$. Then, $p + r/2 > p$ is in β since $-(p + r/2) - (r/2) \notin \alpha$ and $r/2 > 0$, proving condition (III). So $\beta \in \mathbb{R}$.

Let $p \in \alpha$ and $q \in \beta$. Since $q \in \beta$, there exists $r > 0$ such that $-q - r \notin \alpha$. Thus, from condition (II) on α , $p < -q - r$ and thus $p + q < -r < 0$ so $p + q \in 0^*$. Since our choice of p and q was arbitrary, $\alpha + \beta \subseteq 0^*$.

Let $t < 0$. We want to find $p \in \alpha$ and $q \in \beta$ such that $p + q = t$. Let $s = -t/2$ so that $s > 0$. Then, by the Archimedean property, there exists an integer n such that $ns \in \alpha$ but $(n + 1)s \notin \alpha$. Take $p = ns$ and $q = -(n + 2)s = t - p$. Since $-(-(n + 2)s) - s \notin \alpha$ and $s > 0$, $q \in \beta$, so we are done!

Step 5. Towards an ordered field

Notice that $\alpha + \beta \subset \alpha + \gamma$ if $\beta \subset \gamma$, so the first condition for ordered fields is satisfied. It also follows that $\alpha > 0^*$ iff $-\alpha < 0^*$.

Step 6. Multiplication, but only somewhat

Let $\mathbb{R}^+ = \{\alpha \in \mathbb{R} : \alpha > 0^*\}$ be the positive real numbers. If $\alpha, \beta \in \mathbb{R}^+$, define $\alpha\beta$ to be the set of $p \leq rs$ in \mathbb{R}^+ , for some positive $r \in \alpha, s \in \beta$ with $r, s > 0$. We define 1^* to be the set of all rational numbers less than 1.

It can be shown, similarly to the proofs of (A1) to (A5), that the axioms of multiplication and distributivity hold for multiplication on \mathbb{R}^+ :

Let $\alpha, \beta > 0^*$ be arbitrary.

(M1) $\alpha\beta \in \mathbb{R}$.

We prove the three conditions separately. For condition (I): choose $r \in \alpha$ and $s \in \beta$ such that $r, s > 0$, which exist since $\alpha, \beta > 0^*$. Then, $rs \leq rs$ so $rs \in \alpha\beta$, so $\alpha\beta$ is non-empty. Now, let $p \notin \alpha$ and $q \notin \beta$. Then, for any $r \in \alpha$ and $s \in \beta$ with $r, s > 0$, then $r < p$ and $s < q$, so $rs < pq$, implying that $pq \notin \alpha\beta$, so $\alpha\beta \neq \mathbb{Q}$, proving condition (I).

For condition (II), let $q \in \alpha\beta$ and $p < q$ in \mathbb{Q} . Then, $q \leq rs$ for $r \in \alpha, s \in \beta$ for $r, s > 0$. Then, similarly $p < q \leq rs$ so $p \in \alpha\beta$. For condition (III), let $q \in \alpha\beta$ where $q \leq rs$ as above. Then, choose $t > r > 0$ and $u > s > 0$ in α and β respectively. Then, $tu \leq tu$ so $tu \in \alpha\beta$ with $tu > rs$. Thus, $\alpha\beta \in \mathbb{R}$.

(M2) $\alpha\beta = \beta\alpha$.

This follows directly from the definition, since

$$\{rs : r \in \alpha, s \in \beta, r, s > 0\} = \{sr : s \in \beta, r \in \alpha, r, s > 0\}.$$

(M3) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

We prove that both sets are the set A of all $p \leq rst$ for $r \in \alpha, s \in \beta, t \in \gamma$, where $r, s, t > 0$. Since the right side can be written as $(\beta\gamma)\alpha$ and is thus symmetric, it suffices to prove one side.

Let $p \in A$ so that $p \leq rst$, for some $r \in \alpha, s \in \beta, t \in \gamma$ all positive. Then, $rs \in \alpha\beta$, and $p \leq (rs)t$, so $p \in (\alpha\beta)\gamma$. So $A \subseteq (\alpha\beta)\gamma$. The proof of the other side is very similar, so we omit it.

(M4) $\alpha 1^* = \alpha$.

Let $p \in \alpha 1^*$, so that $p \leq rs$ for $r \in \alpha$ with $r > 0$ and $0 < s < 1$. Then, $p \leq rs < r$ so $p \in \alpha$, showing $\alpha 1^* \subseteq \alpha$. Now, let $t \in \alpha$. If $t < 0$, then $2t < t < 0$ so $2t \in \alpha$. Then, $t \leq (2t) \cdot (1/2) \in \alpha 1^*$. If $t > 0$, then let $r > t$ in α , from condition (III) of a cut. Then, $t/r < 1$, so $t \leq r \cdot (t/r) \in \alpha 1^*$, completing this proof.

(M5) α^{-1} exists.

For $\alpha > 0^*$, denote β to be the set of all $p > 0$ such that there exists $r > 0$ with $p^{-1} - r \notin \alpha$, along with all the non-positive rational numbers. We claim that $\beta \in \mathbb{R}$ and $\alpha\beta = 1^*$.

For condition (I), clearly $0 \in \beta$, so β is non-empty. Also, for any positive $q \in \alpha, p = \frac{1}{q+1} > 0$ and $p^{-1} - 1 = q \in \alpha$ so $p \notin \beta$, and thus $\beta \neq \mathbb{Q}$, proving condition (I) for cuts.

For condition (II), let $p \in \beta$ and $q < p$. If $q \leq 0$, then $q \in 0^* \cup \{0\} \subset \beta$ follows by definition. Otherwise, $p > 0$ so there exists some $r > 0$ such that $p^{-1} - r \notin \alpha$. Since $q < p$, we have $q^{-1} - r > p^{-1} - r \notin \alpha$ so $q^{-1} - r \notin \alpha$ and thus $q \in \beta$, as required.

Finally for condition (III), let $p \in \beta$. We want to find $q > p$ in β . If $p < 0$, this is easy since $0 \in \beta$. Otherwise, $p > 0$, so there exists $r > 0$ such that $p^{-1} - r \notin \alpha$. Then, let $q = (p^{-1} - r/2)^{-1} > p$ (if $p^{-1} = r/2$, choose $r/3$ instead). Then, $q^{-1} - r/2 = p^{-1} - r \notin \alpha$, so $q \in \beta$.

Now let $t \in \alpha\beta$, so that $t \leq rs$ for some positive $r \in \alpha, s \in \beta$. Then, $s^{-1} - u \notin \alpha$ for some $u > 0$. Also, $r - u \in \alpha$, so $r - u < s^{-1} - u$, implying $rs < 1$, so $t \leq rs < 1$ and thus $t \in 1^*$. This proves $\alpha\beta \subseteq 1^*$.

TODO Figure out how to get $1^* \subseteq \alpha\beta$, then show distributivity.

Step 7. Multiplication, for real

Now, define multiplication on all of \mathbb{R} by

$$\alpha\beta = \begin{cases} 0^* & \text{if } \alpha = 0^* \text{ or } \beta = 0^* \\ \alpha\beta & \text{if } \alpha > 0^*, \beta > 0^* \\ -(\alpha(-\beta)) & \text{if } \alpha > 0^*, \beta < 0^* \\ -((-\alpha)\beta) & \text{if } \alpha < 0^*, \beta > 0^* \\ (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \beta < 0^* \end{cases}$$

The proofs for the multiplication axioms and distributivity follow from Step 6, with repeated use of $\gamma = -(-\gamma)$. The proofs are omitted.

This proves that \mathbb{R} is an ordered field with the least-upper-bound property!

Step 8. Relation to \mathbb{Q}

Associate with each $q \in \mathbb{Q}$, a set $q^* = \{r \in \mathbb{Q} : r < q\}$. Notice that this definition is consistent with our definitions of 0^* and 1^* above (the asterisk was not a coincidence). These cuts satisfy the following relations:

- a) $r^* + s^* = (r + s)^*$.
- b) $r^* s^* = (rs)^*$.
- c) $r^* < s^*$ iff $r < s$.

This shows that \mathbb{Q} is isomorphic to \mathbb{Q}^* whose elements are the rational cuts. This relationship is the reason we can regard \mathbb{Q} as a subfield of \mathbb{R} .

1.8 Exercises

1. If r is rational and non-zero, and x is irrational, prove that $r + x$ and rx are irrational.

Proof. Suppose otherwise: then $(r + x) - r$ and $(rx)/r$ would be operations on rational numbers, and thus evaluate to rational numbers. However, these both evaluate to x , which is irrational, a contradiction. ■

2. Prove that there is no rational number whose square is 12.

Proof. We first prove that there is no rational number whose square is 3. Suppose that there existed $p/q \in \mathbb{Q}$ whose square is 3, and such that p and q share no factors, cancelling factors when appropriate. Then, multiplying this equality by q^2 , we have $p^2 = 3q^2$, showing that p is divisible by 3. But, this means $3p'^2 = 3q^2$, which in turn implies q is divisible by 3, contradicting the fact that p and q share no factors.

Now, if some rational number r had $r^2 = 12$, then $r/2$ would be a rational number such that $(r/2)^2 = 3$, contradicting our previous statement. ■

3. Prove that in a field F , the following statements follow from the axioms of multiplication:

- (a) If $x \neq 0$ and $xy = xz$ then $y = z$.
- (b) If $x \neq 0$ and $xy = x$ then $y = 1$.
- (c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proof. For (a), if $x \neq 0$, then $1/x \in F$, so we can multiply this to both sides, yielding the result. (b) follows by letting $z = 1$ in (a). (c) follows by letting $z = 1/x$ in (a). (d) follows by letting $x = 1/x$ and $y = 1/(1/x)$ in (c), noticing that $1 = 1/x \cdot x = 1/x \cdot 1/(1/x)$. ■

4. Let E be a non-empty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Prove that $\alpha \leq \beta$.

Proof. Since E is non-empty, let $x \in E$. Then, since α is a lower bound for E , in particular, $\alpha \leq x$. Similarly, $x \leq \beta$, so $\alpha \leq x \leq \beta$, as required. ■

5. Let A be a non-empty set of real numbers which is bounded below, and $-A$ be the set of all numbers $-x$ for $x \in A$. Prove that $\inf A = -\sup(-A)$.

Proof. Since A is non-empty and bounded below, and \mathbb{R} satisfies the least upper bound property, $\alpha = \inf A$ exists. We claim that $-\alpha$ is the least upper bound to $-A$.

First, notice that $-\alpha \geq -x$ for all $-x \in -A$, since $\alpha \leq x$ for all $x \in A$. Also, if $-\beta < -\alpha$ were a smaller upper bound for $-A$, then β would be a larger lower bound for A , contradicting the fact that α is the **greatest** lower bound for A . ■

6. Fix $b > 1$.

- (a) If $m, n, p, q \in \mathbb{Z}$ with $n, q > 0$ and $r = m/n = p/q$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$ so it makes sense to define $b^r = (b^m)^{1/n}$.

Proof. Let $x = (b^m)^{1/n}$ and $y = (b^p)^{1/q}$. Now, notice that $x^{nq} = (x^n)^q = (b^m)^q = b^{mq}$ and similarly $y^{nq} = b^{np}$. Since $m/n = p/q$, we have $qm = np$, so in fact $x^{nq} = y^{nq}$, so the result follows by the uniqueness of positive real nq -th roots. ■

- (b) Prove $b^{r+s} = b^r b^s$ if $r, s \in \mathbb{Q}$.

Proof. Write $r = m/n$ and $s = p/q$ for $m, n, p, q \in \mathbb{Z}$ with $n, q > 0$. Then, $r + s = (mq + np)/nq$. Now, notice that from properties of integer powers, $b^{mq+np} = b^{mq} b^{np}$. So,

$$(b^{r+s})^{nq} = (b^{(mq+np)/nq})^{nq} = b^{mq+np} = b^{mq} b^{np} = b^{r nq} b^{s nq} = (b^r b^s)^{nq}$$

and the result follows from the uniqueness of nq -th positive real roots. ■

- (c) If x is real, define $B(x)$ to be the set of all numbers b^t , where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence, we can define $b^x = \sup B(x)$ for every real x .

Proof. We first prove a helpful lemma: if $q > 0$ in \mathbb{Q} , then $b^q > 1$. Let $q = m/n$ for $m, n \in \mathbb{N}$ and $n \neq 0$. Since x^m and x^n are monotonically increasing, then if $b > 0$,

$$1 < b^{m/n} \iff 1^n = 1 < b^m \iff 1 < b$$

so $b^q > 1$ as required. It follows that if $r \leq t$, then $b^r \leq b^t$: simply consider $b^t - b^r = b^r(b^{t-r} - 1) \geq 0$, since both terms are non-negative.

In fact, the previous statement shows that b^r is indeed an upper bound for $B(r)$: if $t \leq r$ and thus $b^t \in B(r)$, then $b^t \leq b^r$. Since $r \leq r$, $b^r \in B(r)$, so any upper bound for $B(r)$ must be at least b^r . Thus b^r must be the least upper bound for $B(r)$, as required. ■

- (d) Prove $b^{x+y} = b^x b^y$ for all $x, y \in \mathbb{R}$.

Proof. First, we want to show that $b^x b^y$ is an upper bound for $B(x+y)$. Suppose that $q \leq x+y$ in \mathbb{Q} . Then, we can write $q = u+v$ for $u \leq x, v \leq y$ in \mathbb{Q} (from the Archimedean property, we can choose $u \in \mathbb{Q}$ in the non-empty interval $[q-y, x]$), so that $b^q = b^{u+v} = b^u b^v$. However, since $u \leq x$ and $v \leq y$, $b^u \leq b^x$ and $b^v \leq b^y$, so $b^q = b^u b^v \leq b^x b^y$ as required.

Now, suppose $r < b^x b^y$. We need to find some $q \leq x+y$ in \mathbb{Q} such that $b^q > r$, to show that r is not an upper bound for $B(x+y)$.

Dividing both sides by b^x yields $r/b^x < b^y = \sup B(y)$, so there exists some $v \leq y$ in \mathbb{Q} such that $b^v > r/b^x$ so $r < b^x b^v$. Dividing this new inequality by b^v yields $r/b^v < b^x = \sup B(x)$ so there exists $u \leq x$ in \mathbb{Q} such that $b^u > r/b^v$. Thus, $r < b^u b^v = b^{u+v}$ with $q = u+v \leq x+y$, as required. ■

7. Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the **logarithm of y to the base b** .)

- (a) For any positive integer n , we have $b^n - 1 \geq n(b - 1)$.

Proof. Factor $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \cdots + b + 1)$. Since $b^k > 1$ for every positive integer k (inductively), the above expression is greater than or equal to $(b - 1)(1 + 1 + \cdots + 1) = n(b - 1)$, where there are n 1's, as required. ■

- (b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

Proof. Since $f(x) = x^n$ is monotonically increasing, $b^{1/n} > 1$ since $b > 1$, so substituting $b^{1/n}$ for b in (a) yields the result. ■

- (c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

Proof. We have

$$n > \frac{b - 1}{t - 1} \geq \frac{n(b^{1/n} - 1)}{t - 1}$$

so cancelling the positive n from the outermost expressions, multiplying by the positive value $t - 1$, and adding 1 to both sides yields the result. ■

- (d) If w is such that $b^w < y$, then $b^{w+1/n} < y$ for sufficiently large n ; to see this, apply part (c) with $t = y \cdot b^{-w}$.

Proof. Following the prompt, we notice that letting $t = y \cdot b^{-w} > 1$ in (c), if $n > (b - 1)/(y \cdot b^{-w} - 1) \in \mathbb{R}$, then $b^{1/n} < y \cdot b^{-w}$, which after multiplying both sides by the positive value b^w , gives the result. ■

- (e) If $b^w > y$, then $b^{w-1/n} > y$ for sufficiently large n .

Proof. Similarly, letting $t = b^w/y$ in (c) yields $b^{1/n} < b^w/y$ for $n > (b - 1)/(b^w/y - 1) \in \mathbb{R}$, after which rearranging for y yields the result. ■

- (f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

Proof. Suppose otherwise. If $b^x < y$, then from (d), there exists some $n \in \mathbb{N}$ such that $b^{x+1/n} < y$, so $x + 1/n \in A$, contradicting the fact that x is an upper bound for A .

Similarly, if $b^x > y$, then from (e), there exists some $n \in \mathbb{N}$ such that $b^{x-1/n} > y$, and so for any $w \in A$, we have $b^w < y < b^{x-1/n}$ and thus $w < x - 1/n$ from the lemma in Exercise 6(c), making $x - 1/n$ a smaller upper bound for A , a contradiction. ■

- (g) Prove that this x is unique.

Proof. Suppose that $x \neq y$ in \mathbb{R} and $b^x = b^y$. Without loss of generality, suppose $x < y$. But then $b^x < b^y$ from the statement proved in Exercise 6(c), contradicting our assumption, so we are done. ■

8. Prove that no order can be defined in the complex field that turns it into an ordered field. Hint: -1 is a square.

Proof. Suppose otherwise. Then, notice that since $1 = 1^2$ and $-1 = i^2$ are both non-zero squares, they are both positive. Thus, their sum, 0, would have to be positive, a contradiction. ■

9. Suppose $z = a + bi$ and $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a **dictionary order** or **lexicographic order**, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Proof. First, we show that exactly one of $z < w$, $z = w$, or $z > w$ is true. Suppose that $z \neq w$, so that $a \neq c$ or $b \neq d$. If $a \neq c$, then either $a < c$, in which case $z < w$, or $a > c$, in which case $z > w$. Otherwise, either $b < d$, in which case $z < w$, or $b > d$, in which case $z > w$.

Next, we show that this order is transitive. Suppose $a + bi < c + di$ and $c + di < e + fi$. If $a < c$, then since $c \leq e$, $a < e$ and thus $a + bi < e + fi$. If $a = c$, then $b < d$. If further $c = e$, then $b < d < f$ and thus $a + bi < e + fi$. If further $c < e$, then $a = c < e$ so $a + bi < e + fi$. In all cases, $a + bi < e + fi$.

This ordered set does not have the least-upper-bound property: consider the set $A = i\mathbb{R} = \{ir : r \in \mathbb{R}\}$, which is bounded above by 1. We claim that though A is non-empty and bounded above, there does not exist a least upper bound. Let α be an upper bound for A . Notice that α must have a non-negative real component. If α has a strictly positive real component, say r , then $r/2$ will be smaller than r but still larger than all of A . Thus, a least upper bound will never have positive real component. If α had real component 0, then $\alpha + 1 \in A$ would be larger than it, a contradiction. Thus, no least upper bound for A exists. ■

10. Suppose $z = a + bi$, $w = u + vi$, and

$$a = \left(\frac{|w| + u}{2} \right)^{1/2}, \quad b = \left(\frac{|w| - u}{2} \right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Proof. Notice that $(a + bi)^2 = (a^2 - b^2) + i(2ab)$, so

$$z^2 = \left(\frac{|w| + u}{2} - \frac{|w| - u}{2} \right) + 2i \left(\sqrt{\frac{|w| + u}{2}} \cdot \sqrt{\frac{|w| - u}{2}} \right) = u + i\sqrt{|w|^2 - u^2} = u + i|v|$$

so $z^2 = u + iv$ when $v \geq 0$. Similarly,

$$\bar{z}^2 = (a^2 - b^2) - 2iab = u - i|v| = u + iv$$

if $v < 0$. Thus, if $v \geq 0$, then z and $-z$ are two complex square roots of w , and similarly $\pm \bar{z}$ are square roots of w if $v < 0$. The exception here is of course when $z = -z$, namely when $z = 0$ and thus $w = z^2 = 0$. ■

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z ?

Proof. Let $r = |z| \geq 0$ and $w = z/|z|$. Notice that $|w| = |z|/|z| = 1$, and $z = rw$ by definition. w and r are uniquely defined for all $z \in \mathbb{C}$ except 0, in which case $r = 0$ and w can be any complex number with unit absolute value.

Suppose $z \neq 0$ and $z = r_1 w_1 = r_2 w_2$ with the required conditions. Take absolute values to get $|z| = |r_1||w_1| = |r_2||w_2|$, which implies $|r_1| = |r_2|$. Since $r_1, r_2 \geq 0$, we must have $r_1 = r_2$, uniquely determining w_1 and w_2 , as required. ■

12. If z_1, \dots, z_n are complex, prove that

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|.$$

Proof. We proceed by induction on n . The base case is given by Theorem 7(e), as applied to complex numbers taken as ordered pairs of real numbers. Then, for $n > 2$, assuming the statement is true for $n - 1$ inductively, we have

$$\begin{aligned} |z_1 + z_2 + \dots + z_{n-1} + z_n| &\leq |z_1 + z_2 + \dots + z_{n-1}| + |z_n| \\ &\leq |z_1| + |z_2| + \dots + |z_{n-1}| + |z_n|, \end{aligned}$$

completing the induction. ■

13. If $x, y \in \mathbb{C}$, prove that

$$||x| - |y|| \leq |x - y|.$$

Proof. Notice $|x| = |(y - x) + x| \leq |y - x| + |x|$ so $|x - y| = |y - x| \geq |x| - |y|$.

Similarly, $|y| = |(y - x) + x| \leq |y - x| + |x|$ so $|x - y| \geq |y| - |x|$. Combining these two inequalities yields

$$-|x - y| \leq |x| - |y| \leq |x - y|$$

and so

$$||x| - |y|| \leq |x - y|,$$

as required. ■

14. If $z \in \mathbb{C}$ such that $|z| = 1$, compute

$$|1 + z|^2 + |1 - z|^2$$

Proof. Simply write

$$|1+z|^2 + |1-z|^2 = (1+z)(1+\bar{z}) + (1-z)(1-\bar{z}) = 1+z+\bar{z}+z\bar{z} + 1-z-\bar{z}+z\bar{z} = 4.$$

■

15. Under what conditions does equality hold in the Schwartz inequality?

Proof. Notice that in the proof, equality holds iff $\sum |Ba_j - Cb_j|^2 = 0$, and equivalently $a_j = Cb_j/B$ for each j . In particular, if the a_j are each the same multiple of b_j , then equality will hold. ■

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

- (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

Proof. Let $\mathbf{m} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$. Without loss of generality, suppose $\mathbf{y} = -\mathbf{x}$, translating when necessary. Also, let $t = \sqrt{r^2 - d^2/4}$, which is well defined since $2r > d$. Then, the set of vectors on the hyperplane orthogonal to the line crossing \mathbf{x} and \mathbf{y} with distance t to \mathbf{m} satisfy the required equation, and there are infinitely many. ■

- (b) If $2r = d$, there is exactly one such \mathbf{z} .

Proof. If such a \mathbf{z} existed, then $d = |\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}| = 2r = d$, so equality must hold in the triangle inequality. This happens when the three points are co-linear, and there is a unique point which satisfies this, namely $\mathbf{z} = \mathbf{m}$ as defined above. ■

- (c) If $2r < d$, there are no such \mathbf{z} .

Proof. If such a \mathbf{z} existed, then $d = |\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}| = 2r < d$, a contradiction. ■

How must these statements be modified if $k < 3$?

Proof. If $k = 2$, then (a) will have 2 solutions instead, and everything else remains the same. If $k = 1$, (a) will have no solutions. ■

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Proof. We expand:

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\ &= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 \end{aligned}$$

The interpretation of this is that in a parallelogram, the sums of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the four sides. ■

18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Proof. If $\mathbf{x} = 0$, then any non-zero vector will do. If \mathbf{x} has only one non-zero component, then choose i such that $x_i \neq 0$ and let \mathbf{y} be the vector with $y_i = 1$ and $y_j = 0$ if $j \neq i$. Otherwise, if $\mathbf{x} = (x_1, \dots, x_k)$, there exists $x_i \neq 0$. Then, let $y_j = x_j$ if $j \neq i$, and $y_i = \frac{x_1^2 + x_2^2 + \dots + x_k^2 - x_i^2}{x_i}$. Then,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^k x_j y_j = \left(\sum_{j=1}^k x_j x_j \right) - x_i^2 = x_1^2 + x_2^2 + \dots + x_k^2 - x_i^2 = 0$$

as required. Notice that \mathbf{y} is non-zero since there exists $y_k = x_k \neq 0$ for $k \neq i$ by presumption.

This statement is not true for $k = 1$: consider $\mathbf{x} = (1)$. ■

19. Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

iff $|\mathbf{x} - \mathbf{c}| = r$.

Proof. This is an example of **Circles of Apollonius**. $\mathbf{c} = \frac{1}{3}(4\mathbf{b} - \mathbf{a})$, $r = \frac{2}{3}|\mathbf{b} - \mathbf{a}|$. ■

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

Proof. TODO ■

2 Basic Topology

2.1 Finite, Countable and Uncountable Sets

Definition 9. Let A, B be two sets such that every element x of A is associated an element of B , denoted by $f(x)$. Then, f is a **function from A to B** or a **mapping from A into B** . The set A is called the **domain** of f , and the elements $f(x)$ are called the **values** of f . The set of all values of f is called the **range** of f .

Let A, B be two sets and f be a mapping of A into B . If $E \subseteq A$, then $f(E)$ is defined to be the set of all elements $f(x)$ for $x \in E$. We call $f(E)$ the **image** of E under f . In this notation, $f(A)$ is the range of f . It is clear that $f(A) \subseteq B$. If $f(A) = B$, we say that f maps A **onto** B , or that f is **surjective**.

It is important to note the difference between into and onto.

If $E \subseteq B$, then $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. $f^{-1}(E)$ is the **inverse image** of E under f . If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$.

If $f^{-1}(y)$ consists of at most one element of A for each $y \in B$, then f is a **one-to-one**, or **injective** mapping of A into B . A mapping f which is both injective and surjective is called **bijective**, or a **1-1 correspondence**.

If there exists a bijection between A and B , we say that A and B have the same **cardinal number**, or briefly that A and B are **equivalent**, and we write $A \sim B$. This relation has the reflexive, symmetric, and transitive properties, and is thus called an **equivalence relation**.

For any positive integer n , let J_n be the set $\{1, 2, 3, \dots, n\}$, and J be the set consisting of all positive integers. For any set A , we say

- (a) A is **finite** if $A \sim J_n$ for some n .
- (b) A is **infinite** if A is not finite.
- (c) A is **countable** if $A \sim J$.
- (d) A is **uncountable** if A is neither finite nor countable.
- (e) A is **at most countable** if A is finite or countable.

Countable sets are sometimes called **enumerable** or **denumerable**. Notice that if A, B are both finite, then $A \sim B$ iff they contain the same number of elements. The notion of bijection extends this idea to infinite sets.

For example, the set of all integers \mathbb{Z} is countable: consider the ordering

$$0, 1, -1, 2, -2, 3, -3, \dots$$

given by the bijection

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

An interesting property of \mathbb{Z} is thus that it is equivalent to one of its proper subsets. This is never true of finite sets, so in fact a set is infinite if it is equivalent to one of its proper subsets.

Definition 10. A **sequence** is a function f defined on the set J of positive integers. If $f(n) = x_n$ for $n \in J$, then we denote f by the symbol $\{x_n\}$ or sometimes x_1, x_2, \dots . The values x_n of f are called the **terms** of the sequence. If A is a set and $x_n \in A$ for each $n \in J$, then $\{x_n\}$ is a **sequence in A** , or a **sequence of elements in A** .

Theorem 8. Every infinite subset of a countable set A is countable.

Proof. Let $\{x_n\}$ be a sequence of distinct elements in A . Then, for any infinite subset E of A , we can take the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ corresponding to E , so E is countable. ■

The above theorem shows that countable sets are the ‘smallest infinity’ in some sense.

Definition 11. Let A and Ω be sets, and suppose that for each element $\alpha \in A$, there is associated a subset of Ω which we denote by E_α . We call the set whose elements are the sets E_α a **collection** or **family** of sets, denoted $\{E_\alpha\}$.

The **union** of the sets E_α is the set S such that $x \in S$ iff $x \in E_\alpha$ for at least one $\alpha \in A$, denoted

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

The **intersection** of the sts E_α is the set P such that $x \in P$ iff $x \in E_\alpha$ for every $\alpha \in A$, denoted

$$P = \bigcap_{\alpha \in A} E_\alpha.$$

If $A \cap B$ is not empty, we say that A and B **intersect**, otherwise they are **disjoint**.

The operations \cup and \cap are commutative and associative, but they also distribute:

Theorem 9. For sets A, B, C , we have

$$E = A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = F.$$

Suppose that $x \in E$. Then, $x \in A$ and at least one of $x \in B$ or $x \in C$. Thus at least one of $x \in (A \cap B)$ or $x \in (A \cap C)$, so $x \in F$.

Similarly, if $x \in F$, then at least x is in at least one of $A \cap B$ or $A \cap C$, so $x \in A$, and at least one of B or C , so $x \in E$, completing the proof.

Theorem 10. Let $\{E_n\}$ where $n \in J$ be a sequence of countable sets, and let S be their countable union. Then S is countable.

Proof. Let every E_n be arranged in a sequence $\{x_{nk}\}_{k \in J}$. Then, consider the sequence

$$\underbrace{x_{11}}_{}, \underbrace{x_{21}, x_{12}}_{}, \underbrace{x_{31}, x_{22}, x_{13}}_{}, \underbrace{x_{41}, x_{32}, x_{23}, x_{14}}_{}, \dots$$

which enumerates S . If any two of the sets E_n have elements in common, these will appear more than once in S . So the sequence is at most countable, but since $E_1 \subseteq S$, it is at least countable, so it is countable. ■

Corollary 1. Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable. Then, $T = \bigcup_{\alpha \in A} B_\alpha$ is at most countable.

This follows since T is equivalent to a subset of S in the above theorem.

Theorem 11. Let A be a countable set, and B_n be the set of all n -tuples (a_1, \dots, a_n) in A , where the a_k need not be distinct. Then B_n is countable.

Proof. $B_1 = A$ is countable. Suppose that B_{n-1} is countable for $n > 1$. Then, the elements of B_n are of the form (b, a) for $b \in B_{n-1}$ and $a \in A$, so the set of all pairs $(b, a) \sim A$ and is thus countable. Then, B_n is the countable union of countable sets, and is thus countable. The result follows by induction on n . ■

Corollary 2. The set of all rational numbers is countable.

Proof. Associate with each $r = m/n \in \mathbb{Q}$ the pair (m, n) of integers. The set of pairs (m, n) is countable, and thus \mathbb{Q} is countable. ■

But not all infinite sets are countable:

Theorem 12. The set A of all sequences in $\{0, 1\}$ is uncountable.

Proof. Suppose otherwise, so that A is countable, and thus admits a sequence $\{s_n\}$. Then, construct a sequence $t \in A$ such that $t_j = 1 - s_{jj}$ so $t_j \neq s_{jj}$ for all $j \in J$. Then, $t \notin \{s_n\}$ since it differs from each sequence at at least one point. Thus, $t \notin A$, a contradiction. ■

Notice the above theorem can be combined with the binary representation of real numbers to show that the real numbers are uncountable.

2.2 Metric Spaces

Definition 12. A set X , whose elements are **points**, is a **metric space** if for any two points $p, q \in X$ there is a real number $d(p, q)$ called the **distance** from p to q , such that

- (a) $d(p, q) \geq 0$, with equality iff $p = q$.
- (b) $d(p, q) = d(q, p)$.
- (c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

Any function with these three properties is called a **distance function**, or a **metric**.

We can define metrics in the Euclidean spaces \mathbb{R}^k , with the distance $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. It is important to note that every subset of a metric space is still a metric space under the same distance function.

Definition 13. We denote the **segment** (a, b) to be the set of real numbers x such that $a < x < b$. Similarly, the **interval** $[a, b]$ is the set of all real numbers x such that $a \leq x \leq b$.

If $a_i < b_i$ for all $i \in J_k$, then the set of all points $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ such that $a_i \leq x_i \leq b_i$ is called a **k -cell**.

If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the **open** (or **closed** respectively) **ball** B is the set of all $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $\leq r$ respectively).

A set $E \subseteq \mathbb{R}^k$ is **convex** iff $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ whenever $\mathbf{x}, \mathbf{y} \in E$ and $\lambda \in (0, 1)$.

Balls and k -cells are convex.

Definition 14. Let X be a metric space. Then,

- (a) A **neighbourhood** of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$ for some $r > 0$, where r is the **radius** of $N_r(p)$.

- (b) A point p is a **limit point** of the set E if every neighbourhood of p contains a point $q \neq p$ such that $q \in E$. (That is, points in E get arbitrarily close to p .)
- (c) If $p \in E$ is not a limit point, it is called an **isolated point**.
- (d) A set E is **closed** iff every limit point of E is in E .
- (e) A point p is an **interior point** of E if there is a neighbourhood N of p such that $N \subseteq E$.
- (f) E is **open** if every point in E is an interior point of E .
- (g) The **complement** of E , denoted E^c , is the set of all points in X but not in E .
- (h) E is **perfect** if E is closed and every point of E is a limit point of E .
- (i) E is **bounded** if there exists a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
- (j) E is **dense** in X if every point in X is a limit point of E , or a point of E (or both). (So points in E get arbitrarily close to any point in X).

Theorem 13. Every neighbourhood is an open set.

Proof. Let $E = N_r(p)$ be a neighbourhood, and q be any point in E . Then, $d(p, q) < r$ so $d(p, q) = r - h$ for some positive h . Then for any $s \in N_h(q)$, we have

$$d(s, p) \leq d(s, q) + d(q, p) < h + (r - h) = r$$

so $s \in E$ and thus $N_h(q) \subseteq E$. Thus E is open. ■

Theorem 14. If p is a limit point of E , then every neighbourhood of p contains infinitely many points of E .

Proof. Suppose not, so that some neighbourhood $N_r(p)$ contains finitely many points in E . Then, there exists some closest point $q \neq p$ in that neighbourhood with $d(q, p) = h > 0$. Then, the neighbourhood $N_{h/2}(p)$ contains no points in E which are not p , contradicting the definition of a limit point. ■

Corollary 3. A finite point set has no limit points.

Proof. If p were a limit point in a finite point set E , then some neighbourhood must have infinitely many points in E , a contradiction. ■

Theorem 15. Let $\{E_\alpha\}$ be a possibly infinite collection of sets. Then

$$A = \left(\bigcup_{\alpha} E_{\alpha} \right)^c = \bigcap_{\alpha} E_{\alpha}^c = B.$$

Proof. Let $x \in A$. Since it is not in $\bigcup_{\alpha} E_{\alpha}$, x is not in any of the E_{α} . Thus, it is in each of the E_{α}^c , so $x \in B$.

Let $y \in B$. By definition, y is in E_{α}^c , so it is not in any of the E_{α} . Thus, $y \notin \bigcup_{\alpha} E_{\alpha}$ so $y \in A$, completing the proof. ■

Theorem 16. A set E is open iff its complement is closed.

Proof. Suppose E^c is not closed. Then, there exists a limit point of E^c , say x which is not in E^c (and thus in E). However this means that there are points in E^c in neighbourhoods arbitrarily close to x : so no neighbourhood around x will be a subset of E , so E is not open, proving the contrapositive of the forward statement.

Now suppose E is not open, so there exists some $x \in E$ for which no neighbourhood is entirely contained within E . Thus, every neighbourhood of x contains some point in E^c , so x is a limit point of E^c not in E^c , so E^c is not closed, proving the contrapositive of the backwards statement, completing the proof. ■

Corollary 4. *A set F is closed iff its complement is open.* ■

Theorem 17. *With the previous theorems, we can prove the following:*

- (a) *For any possibly infinite collection $\{G_\alpha\}$ of open sets, $A = \bigcup_\alpha G_\alpha$ is open.*
- (b) *For any possibly infinite collection $\{F_\alpha\}$ of closed sets, $B = \bigcap_\alpha F_\alpha$ is closed.*
- (c) *For any finite collection G_1, \dots, G_n of open sets, $C = \bigcap_{i=1}^n G_i$ is open.*
- (d) *For any finite collection F_1, \dots, F_n of closed sets, $D = \bigcup_{i=1}^n F_i$ is closed.*

Proof. For (a), let $x \in A$ be arbitrary. Then, $x \in G_\alpha$ for some open set G_α , so $N_r(x) \subseteq G_\alpha \subseteq A$. Thus, x is interior in A and A is open. For (b), notice that $\{F_\alpha^c\}$ is a collection of open sets, so by (a), $\bigcup_\alpha F_\alpha^c = (\bigcap_\alpha F_\alpha)^c$ is open, so $B = \bigcap_\alpha F_\alpha$ is closed.

For (c), let $x \in C$. Then, there are neighbourhoods $N_{r_i}(x) \subseteq G_i$ for some positive r_i . Then, taking $r = \min_{i=1}^n r_i$, we have $N_r(x) \subseteq N_{r_i}(x) \subseteq G_i$ for each $i \in J_n$, so $N_r(x) \subseteq C$ so x is interior to C , and thus C is open. Statement (d) reduces to (c) with reasoning similar to (b). ■

Notice that in parts (c) and (d) in the previous theorem, finiteness of the collections is essential. Otherwise, the minimum of the radii of your neighbourhoods could be 0, and no neighbourhood would exist. For instance, take $G_n = (-1/n, 1/n)$.

Definition 15. *If X is a metric space, and $E \subseteq X$ and E' is the set of all limit points of E in X , then the **closure** of E is the set $\bar{E} = E \cup E'$.*

Theorem 18. *If X is a metric space with $E \subseteq X$, then*

- (a) *\bar{E} is closed.*
- (b) *$E = \bar{E}$ iff E is closed.*
- (c) *$\bar{E} \subseteq F$ for every closed set $F \subseteq X$ with $E \subseteq F$.*

Notice that (a) and (c) imply \bar{E} is the smallest closed subset of X containing E , so the name ‘closure’ is appropriate.

Proof. For (a), if $p \in X$ and $p \notin \bar{E}$, then p is neither a point of E nor a limit point of E . Hence, p has a neighbourhood which does not intersect E , so \bar{E}^c is open and \bar{E} is closed.

For (b), if E is closed, then $E' \subseteq E$ so $\bar{E} = E$. If $\bar{E} = E$, then (a) implies E is closed.

For (c), let $F \subseteq X$ be a closed set such that $E \subseteq F$. Let $x \in \bar{E}$. If $x \in E$, then clearly $x \in F$. If $x \in E'$ but not in F , then x would be a limit point of $F \supset E$ not in F , contradicting the fact that F is closed. ■

Theorem 19. *Let E be a non-empty set of real numbers which is bounded above. Let $y = \sup E$, then $y \in \bar{E}$, hence $y \in E$ if E is closed.*

Proof. Suppose that y were not in \overline{E} , so it is neither a point in E nor a limit point of E , so there exists a neighbourhood $(y - \epsilon, y + \epsilon)$ which does not contain any elements in E . But then $y - \epsilon/2$ would be a smaller upper bound for E than y , contradicting the minimality of y , proving the first statement.

Then, if E is closed, $y = \sup E \in \overline{E} = E$. ■

Remark. Suppose $E \subseteq Y \subseteq X$ where X is a metric space. We say E is an open subset of X when every point $p \in E$ has associated to it a positive real number r such that $d(p, q) < r$ implies $q \in E$. But we know Y is also a metric space under the same distance function, so we can extend our definitions to Y .

We say E is **open relative to** Y when to each $p \in E$, there exists a positive r such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.

Theorem 20. Suppose $Y \subseteq X$. Then a subset $E \subseteq Y$ is open relative to Y iff $E = Y \cap G$ for some open subset G of X .

Proof. Suppose $E \subseteq Y$ is open relative to Y . Then, for every $p \in E$, we associate some positive r_p such that $N_{r_p}(p) \cap Y \subseteq E$. Let $G = \bigcup_{p \in E} N_{r_p}(p) \subseteq X$. Since each $N_{r_p}(p)$ is open, the infinite union G is open. Then, clearly $E \subseteq Y \cap G$. By our choice of $N_{r_p}(p)$, $N_{r_p}(p) \cap Y \subseteq E$ for each $p \in E$, so $G \cap Y \subseteq E$. Thus, $E = G \cap Y$.

Conversely, if G is open in X and $E = G \cap Y$, then every $p \in E$ has a neighbourhood $N_{r_p}(p) \subseteq G$, so $N_{r_p}(p) \cap Y \subseteq E$ and thus E is open relative to Y . ■

2.3 Compact Sets

Definition 16. Let X be a metric space and $E \subseteq X$. Then, an **open cover** of E is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \bigcup_\alpha G_\alpha$.

Definition 17. A subset K of a metric space X is said to be **compact** if every open cover of K contains a finite subcover.

Formally, if $\{G_\alpha\}$ is an open cover of K , then there exist indices $\alpha_1, \dots, \alpha_n$ such that $K \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$.

Clearly every finite set is compact.

Theorem 21. Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to X iff K is compact relative to Y .

Proof. Suppose K is compact relative to Y , and we have some open cover $\{G_\alpha\}$ in X . Then, letting $H_\alpha = Y \cap G_\alpha$, $\{H_\alpha\}$ is still an open cover of K since $K \subseteq Y$. In fact, each H_α is open relative to Y from Theorem 20, and a subset of Y , so $\{H_\alpha\}$ is an open cover of K in Y . Since K is compact in Y , there exists some finite subcover $\{H_{\alpha_k}\}_{k=1}^n$ of K . Then, $G \subseteq \bigcup_{k=1}^n H_{\alpha_k} \subseteq \bigcup_{k=1}^n G_{\alpha_k}$ so $\{G_{\alpha_k}\}_{k=1}^n$ is a finite subcover of K .

Now, suppose K is compact relative to X , and we have some open cover $\{H_\alpha\}$ in Y . Since each H_α is open relative to Y , we can write $H_\alpha = G_\alpha \cap Y$ for some open subset G_α of X . Then, $\{G_\alpha\}$ is an open cover of K in X and thus a finite subcover $\{G_{\alpha_k}\}_{k=1}^n$ exists. Then $\{H_{\alpha_k}\}_{k=1}^n$ is a finite subcover of K in Y , so K is compact relative to Y , completing the proof. ■

Theorem 22. Compact subsets of metric spaces are closed.

Proof. Let K be compact in X . Let $x \in E^c$. For each $y \in E$, let G_y be some neighbourhood of y with radius less than $\frac{1}{2}d(x, y)$. Then, $\{G_y\}$ forms an open cover of K . Since K is compact, there exist y_1, \dots, y_n such that G_{y_1}, \dots, G_{y_n} forms a finite subcover of K .

Then, there exists a closest point y^* within y_1, \dots, y_n to x , with distance r from x . Then, there are no points in E closer than $r/2$ to x , so E^c contains a neighbourhood of x completely contained in E^c so E^c is open and thus E is closed. ■

Theorem 23. *Closed subsets of compact sets are compact.*

Proof. Let $F \subseteq K \subseteq X$ be closed relative to X and K compact. Then, let $\{G_\alpha\}$ be an open cover of F , and append the open set F^c to make it an open cover of $G \subseteq X$. Then, since K is compact, there exists some finite subcover $\{G_{\alpha_k}\}_{k=1}^n$ of K . Then the finite subcover without F^c is also a finite subcover of F , so F is compact. ■

Corollary 5. *If F is closed and K is compact, then $F \cap K$ is compact.*

Proof. $F \cap K \subseteq K$ is closed as the finite intersection of closed sets, and thus compact by the previous theorem. ■

Theorem 24. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\bigcap K_\alpha$ is non-empty.*

Proof. Let K be some element of $\{K_\alpha\}$. Suppose, for the sake of contradiction, that there are no points in K which are in every K_α . That is, for every $x \in K$, there is some K_α^c such that $x \in K_\alpha^c$. So $K \subseteq \bigcup_\alpha K_\alpha^c$, so $\{K_\alpha^c\}$ forms an open cover of K , so since K is compact, there exists a finite subcover $\{K_{\alpha_k}^c\}_{k=1}^n$ of K . But then $K \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$, contrary to our assumption. ■

Corollary 6 (Cantor's intersection theorem). *If $\{K_n\}$ is a sequence of non-empty compact sets such that $K_n \supseteq K_{n+1}$, then $\bigcap_{n=1}^\infty K_n$ is non-empty.*

Proof. Notice that $K_{k_1} \cap \dots \cap K_{k_n} = K_k$ where $k = \max(k_1, \dots, k_n)$, so any finite intersection is non-empty. Thus, the result follows by applying the above theorem. ■

Theorem 25. *If E is an infinite subset of a compact set K , then E has a limit point in K .*

Proof. Suppose that E has no limit points in K . Then, every $x \in K$ will have some sufficiently small neighbourhood N_x which contains at most one point in E . However, no finite subcollection of $\{N_x\}$ can cover E and thus K , contradicting the fact that K is compact. ■

Theorem 26 (Nested intervals). *If $\{I_n\}$ is a sequence of intervals in \mathbb{R} such that $I_n \supseteq I_{n+1}$, then $\bigcap_{n=1}^\infty I_n$ is not empty.*

Proof. Let $I_n = [a_n, b_n]$, and A be the set of all a_n which is non-empty and bounded above, say by b_1 . Then, let $\alpha = \sup A$. For any m, n , we have $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$, so $\alpha \leq b_m$ for any b_m . Thus, $\alpha \in I_m$ for all m , and thus $\bigcap_{n=1}^\infty I_n$ is non-empty. ■

Theorem 27. *Let k be a positive integer. If $\{I_n\}$ is a sequence of k -cells such that $I_n \supseteq I_{n+1}$ for all n , then $\bigcap_{n=1}^\infty I_n$ is non-empty.*

Proof. Taking the intervals corresponding to each dimension, we can invoke the previous theorem to find values x_i in the intersections of each of the intervals, so $\mathbf{x} = (x_1, \dots, x_k) \in \bigcap_{n=1}^\infty I_n$. ■

Theorem 28. *Every k -cell is compact.*

Proof. Suppose $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$. Then, let $\delta = |\mathbf{b} - \mathbf{a}|$, where $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{b} = (b_1, \dots, b_k)$. Then, notice that $|\mathbf{x} - \mathbf{y}| \leq \delta$, whenever $\mathbf{x}, \mathbf{y} \in I$.

Now suppose for the sake of contradiction that some open cover $\{G_\alpha\}$ admits no finite subcover of I . Subdividing each interval in half determines 2^k smaller k -cells whose union is I . At least one of these cells, I_1 cannot be covered by any finite subcollection of $\{G_\alpha\}$. Continue this indefinitely to get a sequence of nested intervals $\{I_n\}$, all of which cannot be covered by a finite subcollection of $\{G_\alpha\}$.

By the previous theorem, there exists some \mathbf{x} in every one of these intervals, so $\mathbf{x} \in G_\alpha$ for some α . Since G_α is open, there must exist some neighbourhood $N_r(\mathbf{x})$ completely contained in G_α . However, since the diameter of the intervals decreases by a factor of 2 each time, for sufficiently large n , $I_n \subseteq N_r(\mathbf{x}) \subseteq G_\alpha$, contradicting the fact that I_n cannot be covered by any finite subcollection of $\{G_\alpha\}$, completing the proof. ■

Theorem 29 (Heine-Borel). *The following are equivalent for sets $E \subseteq \mathbb{R}^k$:*

- (a) *E is closed and bounded.*
- (b) *E is compact.*
- (c) *Every infinite subset of E has a limit point in E .*

Proof. The fact that (b) implies (c) is precisely Theorem 25.

To show that (c) implies (a), suppose that every infinite subset of E has a limit point in E , but E is not closed and bounded.

- If E is not closed, there exists a limit point x of E which is not contained in E . We construct an infinite subset of E which has only x as a limit point. Let x_1 be an arbitrary point in $N_1(x) \cap E$ and $r_1 = d(x, x_1)$. Now, let x_n be a point in E which is closer than $\min(1/n, r_{n-1}/2)$ to x . This must exist since x is a limit point. This defines a sequence $\{x_n\}$. Let X be the infinite set of these points. Then clearly x is a limit point of X . However, we still have to show that it is the only limit point of X . Let $y \neq x$ be another limit point. Since $d(x, y) > 0$ and the r_n are monotonically decreasing, there exists some n such that $r_n \leq d(x, y) \leq r_{n+1}$. taking $r_0 = d(x, y)$ and $r < \min(r_0 - r_n, r_0 - r_{n+1})$, the neighbourhood with radius r around y has no points in X and is thus not a limit point of X .
- Similarly, if E is not bounded, let $w \in E$ be arbitrary. Then we can create a sequence $\{x_n\}$ such that $d(w, x_n) > n$ and $x_n \in E$. Then, if $y \in E$, then there are only finitely many x_n such that $d(w, x_n) < d(w, y) + 1$, so there are not infinitely many x_n in $N_1(y) \cap E$, so y is not a limit point. Thus, X has no limit points in E .

Finally, to show (a) implies (b), suppose E is closed and bounded. Since E is bounded, we can surround it with a sufficiently large k -cell I . That is, $E \cap I = E$ is compact, as the intersection of a closed set and a compact set. ■

Remark. Notice that the above theorem was specific to \mathbb{R}^k . One may ask how much of the theorem extends to general metric spaces. In general, (b) and (c) are equivalent in any metric space, but (a) does not imply (b) and (c).

Theorem 30 ((Weierstrass)). *Every bounded infinite subset of \mathbb{R}^k has a limit point in \mathbb{R}^k .*

Proof. Let $E \subseteq \mathbb{R}^k$ be bounded. Then, $E \subseteq I$ for some k -cell I . Since I is compact, E is an infinite subset of I and thus has a limit point in $I \subseteq \mathbb{R}^k$ from the above theorem. ■

2.4 Perfect Sets

Theorem 31. *Let P be a non-empty perfect set in \mathbb{R}^k . Then, P is uncountable.*

Proof. Suppose P is non-empty but not uncountable. Since P has limit points, it must be infinite, and thus countable. Enumerate the points of P as x_1, x_2, \dots

Let V_1 be any neighbourhood of x_1 . Suppose we have V_n such that $V_n \cap P$ is non-empty. Since every point of P is a limit point of P , there exists some neighbourhood V_{n+1} such that $\overline{V_{n+1}} \subseteq V_n$, $x_n \notin \overline{V_{n+1}}$, and $V_{n+1} \cap P$ is non-empty.

Put $K_n = \overline{V_n} \cap P$. Notice that $\overline{V_n}$ is closed and bounded and thus compact. The K_n are thus compact as closed subsets of $\overline{V_n}$. Notice K_n form a decreasing sequence of compact sets, each of which are non-empty by construction, so $\bigcap_{n=1}^{\infty} K_n$ must be non-empty. In particular, since $K_1 \subseteq P$, there must be some x_k in this intersection. However, the construction forbids this. ■

Corollary 7. *Every interval $[a, b]$ with $a < b$ is uncountable. In particular, the set of all real numbers is uncountable.*

Proof. Every interval is perfect and thus uncountable. The set of real numbers is larger. ■

Are combinations of intervals the only perfect sets? No: we construct a perfect set in \mathbb{R} with no segment, the **Cantor set**.

Let $E_0 = [0, 1]$. Remove the segment $(1/3, 2/3)$ so that $E_1 = [0, 1/3] \cup [2/3, 1]$. Removing the middle thirds indefinitely yields a decreasing sequence of compact sets, whose infinite intersection is non-empty. Call this set the Cantor set.

The Cantor set contains no segment, since every segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right)$$

has a point in common with P . It can be shown that every segment (α, β) shares a segment of the above form, so the Cantor set contains no segment.

To show that P is perfect, let $x \in P$ and let S be any segment containing x . Let I_n be an interval chosen such that n is so large that $I_n \subseteq S$. Then, let either endpoint is in P , so x is a limit point and P is perfect.

2.5 Connected Sets

Definition 18. *Two subsets A, B of a metric space X are said to be **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.*

*A set $E \subseteq X$ is said to be **connected** if it is not a union of two non-empty separated sets.*

Theorem 32. *A subset E of the real line \mathbb{R} is connected iff it has the property that if $x, y \in E$ and $x < z < y$, then $z \in E$.*

Proof. Suppose otherwise, so that for some $x, y \in E$, there exists some $z \in \mathbb{R}$ such that $x < z < y$ but $z \notin E$. Then we claim $A = E \cap (-\infty, z)$ and $B = E \cap (z, \infty)$ form a separating pair for E .

Clearly $A \cup B = E$. Also, clearly A and B are non-empty, since they contain x and y respectively. Now, notice that everything in A is $\leq z$, and everything in B is $> z$. Thus, they are disjoint. A similar argument holds for the other pair. Thus, A and B form a separating pair for E , contradicting the fact that E is connected.

Conversely, suppose that E is not connected, and in particular admits the separation (A, B) . Then, let $x \in A$ and $y \in B$, and suppose without loss of generality that $x < y$. Let $z = \sup(A \cap [x, y]) \in \bar{A}$. Since A and B are separated, $z \notin B$.

If $z \notin A$, then $x < z < y$ but $z \notin A \cup B = E$. Otherwise, if $z \in A$, then it mustn't be in \bar{B} , so there exists $z_1 \in (z, y)$ such that $z_1 \notin B$. Then $x < z_1 < y$ with $z_1 \notin E$. In either case, the contrapositive holds, completing the proof. ■

2.6 Exercises

1. Prove that the empty set is a subset of every set.

Proof. The definition of set inclusion is vacuously true here. ■

2. A complex number z is said to be **algebraic** if there are integers a_0, \dots, a_n not all zero such that

$$a_0 z^n + \dots + a_n z^0 = 0.$$

Prove that the set of all algebraic numbers is countable. Hint: For every positive integer N , there are only finitely many equations with $n + |a_0| + \dots + |a_n| = N$.

Proof. For any n , let S_n be the set of tuples $(k, a_0, a_1, \dots, a_k)$ such that $k + |a_0| + \dots + |a_k| = n$. Notice that this set is finite, and that each corresponding equation can only have finitely many roots. Thus, the set of algebraic numbers A_n whose polynomials map to a given S_n is finite. Then, $A = \bigcup_{i=1}^{\infty} A_i$ is countable. ■

3. Prove that there exist real numbers which are not algebraic.

Proof. If every real number were algebraic, there would be countably many of them, contradicting the uncountability of the reals. ■

4. Is the set of all irrational real numbers countable?

Proof. If the set of irrational real numbers were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be countable, contradicting the uncountability of the reals. ■

5. Construct a bounded set of real numbers with exactly three limit points.

Proof. Let $A = \{k + 1/n : n \in \mathbb{N}, k \in \{0, 1, 2\}\}$. We claim that the limit points of A are precisely 0, 1, and 2. Clearly, all three are limit points (for any $r > 0$, take $1/n < r$ by the Archimedean property of \mathbb{R} ; then $k + 1/n$ is within r of k).

Let $y \notin \{0, 1, 2\}$. Consider $k \in \{0, 1, 2\}$. If $y < k$ or $y > k + 1$, then taking some radius smaller than the distance to $[k, k + 1]$ fails. Otherwise, we have $y - k \in [1/(n + 1), 1/(n - 1)]$ for some n . Taking a radius smaller than the distance to the nearest of the three points $k + 1/(n + 1), k + 1/n, k + 1/(n - 1)$ fails. ■

6. Let E' be the set of all limit points of a set E . Prove that E' is closed. Prove that E and \bar{E} have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

7. Let A_1, A_2, \dots be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$.

Proof. Let $C = \bigcup_{i=1}^n \overline{A_i}$. Let $x \in \overline{B_n} = B_n \cup B'_n$. If $x \in B_n$, then $x \in A_k \subseteq \overline{A_k} \subseteq C$. If $x \in B'_n$, then construct a sequence $\{x_k\}$ such that $d(x_k, x) < 1/k$ and $x_k \in B_n$. Since $\{x_n\}$ is countable, there exists at least one set A_j for which $x_k \in A_j$ infinitely many times. Thus, $x \in A'_j \subseteq \overline{A_j} \subseteq C$. Putting this together, $\overline{B_n} \subseteq C$.

Now, let $y \in C$. If $y \in A_j$ for some j , then $y \in A_j \subseteq B_n \subseteq \overline{B_n}$. If $y \in A'_j$ for some j , then every neighbourhood of y contains an element $y_k \in A_j \subseteq B_n$, so $y \in B'_n \subseteq \overline{B_n}$. The result follows. ■

- (b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$. Show, by an example, that this inclusion can be proper.

Proof. Let $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$, then $x \in \overline{A_j}$ for some j . Either $x \in A_j \subseteq B \subseteq \overline{B}$, or $x \in A'_j$, so there are elements $x_k \in A_j \subseteq B$ in every neighbourhood of x , ie. $x \in \overline{B}$.

The inclusion can be proper: let A_n be the set of rationals such that $q = m/n$ in lowest form. Then, $B = \mathbb{Q}$, whose closure is \mathbb{R} , whereas the right side is still \mathbb{Q} , as $\overline{A_i} = A_i$ for all i . ■

8. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E ? Answer the same question for closed sets in \mathbb{R}^2 .

Proof. For open sets, yes. Every interior point has a neighbourhood completely contained within E , so every smaller neighbourhood will contain (infinitely many) points in E . The same is not true in general for closed sets. For example, consider $\{(0, 1/n) : n \in \mathbb{N}\} \cup \{(0, 0)\}$, in which 1 is not a limit point. ■

9. Let E° denote the set of all interior points of a set E , called the **interior** of E .

- (a) Prove that E° is always open.

Proof. This is immediate. ■

- (b) Prove that E is open iff $E^\circ = E$.

Proof. If E is open, then every point in E is interior, so $E^\circ = E$. Similarly, if $E = E^\circ$, then E is open from (a). ■

- (c) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.

Proof. Let $x \in G$. Then x is an interior point of $G \subseteq E$, and thus an interior point of E . ■

- (d) Prove that the complement of E° is the closure of the complement of E .

Proof. Let $x \notin E^\circ$, so x is not an interior point of E . So every neighbourhood of x is not completely contained in E , so there exists a point in E^c in every neighbourhood of x . In other words, $x \in E^{c'} \subseteq \overline{E^c}$.

Now let $y \in \overline{E^c}$. If $y \in E^c$, then $y \notin E^\circ \subseteq E$. If $y \in E^{c'}$, then every neighbourhood of y contains an element of E^c , and thus no neighbourhood of y is completely contained in E , so $y \notin E^\circ$. ■

- (e) Do E and \overline{E} always have the same interiors?

Proof. No: let $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ be a subset of \mathbb{R} and let $E = A^c$. Then, $\bar{E} = \mathbb{R}$ so its interior is \mathbb{R} . However, 0 is not an interior point of E : every neighbourhood contains at least one point $1/n$ not in E . ■

(f) Do E and E° always have the same closures?

Proof. No: consider $E = \mathbb{Q}$. Then $\bar{E} = \mathbb{R}$ and $E^\circ = \emptyset$, whose closure is also empty. ■

10. Let X be an infinite set. For $p, q \in X$, define $d(p, q) = 1 - \delta_{p,q}$. Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. By definition, $d(p, q) \geq 0$, with equality iff $p = q$. Now let $p, q, r \in X$. Then if $p \neq r$, then $d(p, r) = 1$, while $d(p, q) + d(q, r)$ is at least 1, since at least one of the terms must be non-zero. If $p = r$, then either $q = r = p$ in which case equality holds, or $q \neq r$, in which case $0 \leq 2$. Thus d is a metric.

Notice that for any $E \subseteq X$, a neighbourhood of size $1/2$ around any point $x \in E$ consists only of the point x (any other point has $d(x, y) = 1 > 1/2$). Thus every point in any set E is an interior point, so every set is open. Similarly, considering any set $F \subseteq X$, F^c is open, so F is closed.

Now, notice that finite sets are always compact. Infinite sets can be covered by singletons, but thus admit no finite subcover. Thus, a set in X is compact iff it is finite. ■

11. For $x, y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof. We prove them individually:

$d_1(x, y)$: This is not a metric: it violates the Triangle inequality with $(x, y, z) = (0, 1, 2)$.

$d_2(x, y)$: This is a metric: this is clearly non-negative, and zero iff $|x - y| = 0$ and thus $x = y$. Also, the Triangle inequality holds.

$d_3(x, y)$: This is not a metric since $d(-1, 1) = 0$ while $1 \neq -1$.

$d_4(x, y)$: This is not a metric since $d(2, 1) = 0$ while $2 \neq 1$.

$d_5(x, y)$: Maybe. ■

12. Let $K = \{1/n : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$. Prove that K is compact without invoking Heine-Borel.

Proof. Let $\{G_\alpha\}$ be any open cover of K . Then, $0 \in K$ is in some G_α , and since G_α is open, there must exist some neighbourhood $N_r(0) \subseteq G_\alpha$. Thus, if $n > 1/r$, $1/n \in G_\alpha$. Taking this, and one open set for each of the remaining $1/n$ yields a finite cover for K . ■

13. Construct a compact set of real numbers whose limits form a countable set.

Proof. Let $A = \{(1 + 2^{-m})/n : m, n \in \mathbb{N}\}$. Notice that $\{1/n : n \in \mathbb{N}\} \cup \{0\} \subseteq A'$. We claim these are the only limit points. Let $y \in [0, 2]$ not in the above set of limit points. Then, $\frac{1}{n+1} < y < \frac{1}{n}$ for some n . Now for every $k > n$ in \mathbb{N} , $\frac{1}{m_k+1} < y - \frac{1}{k} \leq \frac{1}{m_k}$ for some $m_k \in \mathbb{N}$. Notice that as k increases, m_k decreases, so the interval $\left[\frac{1}{m_k+1}, \frac{1}{m_k}\right]$ only gets larger. Thus, there are no elements in

$$A \cap \left(\frac{1}{n+1} + \frac{1}{m_{n+1}+1}, \frac{1}{n+1} + \frac{1}{m_{n+1}} \right),$$

a neighbourhood of y by construction. Thus y is not a limit point, and thus A has countably many limit points. ■

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Proof. Let $G_n = (1/(n+2), 1/n)$ for $n \in \mathbb{N}$. This is a cover since for any $r \in (0, 1)$, there exists n such that $1/(n+2) < r < 1/n$ so $r \in G_n$. However, notice that $1/(n+1)$ is only in G_n , so every subcover of $(0, 1)$ would have to contain G_n for every $n \in \mathbb{N}$, forcing it to be countable, as required. ■

15. Show that Cantor's intersection theorem becomes false if the condition of compactness is weakened to either solely closed or bounded.

Proof. Suppose the restriction is weakened to closed sets. Then let $K_r = \{x \in \mathbb{R} : x \leq r\}$, each of which is closed. Also,

$$K_{a_1} \cap \dots \cap K_{a_n} = K_{\min(a_1, \dots, a_n)} \neq \emptyset$$

but $\bigcap_{r \in \mathbb{R}} K_r = \emptyset$.

Now let $F_k = (0, 1/k)$ be bounded sets. Then

$$F_{k_1} \cap \dots \cap F_{k_n} = F_{\max(k_1, \dots, k_n)} \neq \emptyset$$

but $\bigcap_{k \in \mathbb{N}} F_k = \emptyset$. ■

16. Regard \mathbb{Q} as a metric space with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ with $2 < p^2 < 3$. Show that E is closed and bounded in \mathbb{Q} but not compact. Is E open in \mathbb{Q} ?

Proof. E is bounded below and above by 1 and 2 respectively. Also, E is closed since \mathbb{Q} is dense in \mathbb{R} . To show E is not compact, we employ a similar construction as in Exercise 14. E is open in \mathbb{Q} , since a neighbourhood sufficiently small can be chosen around every point, completely contained in E . ■

17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Perfect?

Proof. E is uncountable: there is a bijection between E and the real numbers between 0 and 1, simply by replacing 4's with 0's and 7's with 1's and interpreting the resulting string as a binary expansion. E is not dense in $[0, 1]$ since there are no elements of E in $(0.5, 0.6)$, as the first digit will always be 5. E is clearly bounded.

Suppose that x contained a digit other than 4 or 7. That is, suppose $x = 0.x_1x_2x_3 \dots x_k \dots$ with $x_k \notin \{4, 7\}$. Then, choosing a neighbourhood of size $2 \cdot 10^{-(k+1)}$ assures that either x_k or x_{k+1}

will not be either a 4 or a 7. Thus, the neighbourhood will contain no elements in E . Thus, the contrapositive states that every limit point of E must be in E , so E is closed and thus compact.

Let $x \in E'$. We can find elements x_n in E arbitrarily close to x by switching the n -th digit between a 4 and a 7, acquiring elements in E which are a distance $3 \cdot 10^{-n}$ from x . Thus, E is also perfect. ■

18. Is there a non-empty perfect set in \mathbb{R} which contains no rational number?

Proof. Yes: enumerate the rational numbers between 0 and 1 as q_n . Then, let A be the set of real numbers between 0 and 1 such that the n -th ternary digit of any $x \in A$ is chosen to be different from the n -th ternary digit of q_n . In this way, A contains no rational numbers.

A is non-empty: this must be the case if $[0, 1]$ is to be uncountable. A is closed: suppose x is a limit point of A so that we can find elements of A arbitrarily close to x . In particular, we can find elements of A whose first n ternary digits match. If x were not in A , then there would be some n for which $x_n \neq (q_n)_n$, but we can find some element of A which matches x to more than n digits, a contradiction. Finally, A is perfect since we can always generate close elements in A to any x by choosing the other possibility for any one of the digits. ■

19. Prove the following:

- a) If A, B are closed and disjoint in some metric space X , they are separate.

Proof. Since A and B are closed, $A = \bar{A}$ and $B = \bar{B}$, so (A, \bar{B}) and (\bar{A}, B) are both disjoint sets (they are both (A, B)). ■

- b) Prove the same for disjoint open sets.

Proof. Suppose otherwise, and without loss of generality, suppose $x \in \bar{A} \cap B$. Then, some neighbourhood of x is completely contained in B . Also, there exists some element of A in this neighbourhood, so there exists $y \in A \cap B$, contradicting the disjointness of A and B . ■

- c) Fix $p \in X$, $\delta > 0$, and define $A = \{q \in X : d(p, q) < \delta\}$. Define B similarly, with $>$ instead of $<$. Show A and B are separated.

Proof. If B is empty, we are done. (Notice A is never empty: $p \in A$).

Let $\alpha = \sup_{q \in A} d(p, q)$, which exists since the set is bounded above by δ . Similarly, let $\beta = \inf_{q \in B} d(p, q)$, since the set is bounded below by δ and non-empty by assumption. We have $\alpha \leq \delta \leq \beta$. Now $\bar{A} = \{q \in X : d(p, q) \leq \alpha\}$ and $\bar{B} = \{q \in X : d(p, q) \geq \beta\}$, so the result follows since $\alpha \leq \beta$ so the required sets are disjoint. ■

- d) Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

Proof. Suppose a, b are distinct elements of a connected metric space X . Let $\delta = d(a, b) > 0$. Then, for every $r \in (0, \delta)$, let A_r and B_r be the sets of elements of X with distance less than, or greater than respectively, r . Then, there must be an element z_r in X with $d(z_r, a) = r$, otherwise A_r and B_r form a separation of X , as in (c). Thus, there exists an injection from the uncountable set $(0, \delta)$ to X so X is uncountable. ■

20. Are closures and interiors of connected sets always connected? (Look at subsets of \mathbb{R}^2 .)

Proof. Closures of connected sets are connected. Suppose E is connected but \bar{E} is not. Then, \bar{E} admits a separation (A, B) . Then, $(A \cap E, B \cap E)$ forms a separation of E , a contradiction. However, this is not the same for interiors: take $A = N_1(-2) \cup N_1(2) \cup L(-2, 2) \subseteq \mathbb{C}$ where $L(a, b)$ is the line segment between a and b . Then $A^\circ = N_1(-2) \cup N_1(2)$ which is disconnected. ■

21. Let A, B be separated subsets of \mathbb{R}^k . Suppose $a \in A, b \in B$, and define $p(t) = (1 - t)a + tb$ for $t \in \mathbb{R}$. Put $A_0 = p^{-1}(A)$ and $B_0 = p^{-1}(B)$.

a) Prove that A_0 and B_0 are separated subsets of \mathbb{R}^1 .

Proof. Notice firstly that both sets are non-empty as they contain 0 and 1 respectively. Now suppose, for the sake of contradiction that $t \in \bar{A}_0 \cap B_0$, so that there are t_0 arbitrarily close to t in A_0 , and also $t \in B_0$. This implies there are $p(t_0)$ arbitrarily close to $p(t)$ in A and also $p(t) \in B$, so $p(t) \in \bar{A} \cap B$, contradicting the fact that A, B are separated. ■

b) Prove that there exists $t_0 \in (0, 1)$ such that $p(t_0) \notin A \cup B$.

Proof. Suppose otherwise, so that $A_0 \cup B_0 = [0, 1]$. Let $\beta = \inf B_0 \in \bar{B}_0$. We know $\beta \notin A_0$ since $A_0 \cap \bar{B}_0 = \emptyset$. However, since $A_0 \cup B_0 = [0, 1]$, there are points in A_0 arbitrarily close to β . This implies $\beta \in \bar{A}_0$ so $\beta \notin B_0$. This implies $\beta \notin A_0 \cup B_0 = [0, 1]$ which is impossible, forming our desired contradiction. ■

c) Prove that every convex subset of \mathbb{R}^k is connected.

Proof. If a subset E were not connected, then the above construction shows that there exists an incomplete line in E , so it fails to be convex. ■

22. A metric space is called **separable** if it contains a countable dense subset. Show that \mathbb{R}^k is separable. Hint: Consider the set of points which have only rational coordinates.

Proof. Let \mathbb{Q}^k be the set of k -tuples of rational numbers. Notice that in every open ball, we can completely enclose a k -cell. This k -cell is determined by k intervals in \mathbb{R} , each of which contain a rational number since \mathbb{Q} is dense. Taking these as coordinates yields an element of \mathbb{Q}^k in the original neighbourhood. Now \mathbb{Q}^k is countable by an earlier theorem, so \mathbb{R}^k is separable. ■

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a **base** for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ containing x , we have $x \in V_\alpha \subseteq G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable base. Hint: take all neighbourhoods with rational radius and center in some countable dense subset of X .

Proof. As the hint suggests, take $\{V_\alpha\} = \{N_r(x) : x \in D, r \in \mathbb{Q}\}$, for some countable dense subset D of X . Then, let G be an open set. Then, G is the union of all the elements of $\{V_\alpha\}$ which are subsets of G . This works since \mathbb{R} satisfies the least-upper-bound property, so a sequence of rationals can approximate any radius arbitrarily closely. Also, $\{V_\alpha\}$ is countable since there exists a surjection from it to $D \times \mathbb{Q}$ which is countable. ■

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Proof. Fix $\delta > 0$ and pick $x_1 \in X$. Then, choose $x_{j+1} \in X$ inductively such that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$, if possible. This process must terminate after a finite number of steps, otherwise the set $A = \{x_n : n \in \mathbb{N}\}$ is an infinite set without a limit point. Then, X can be covered by finitely many neighbourhoods of size δ .

Take $\delta = 1/n, n \in \mathbb{N}$. Then, let K be the union of all the sets of centers (x_i) in the above construction, for each $\delta = 1/n$. We claim K is dense. Let $x \in X$ and $r > 0$. We aim to find some $y \in N_r(x) \cap Y$. Let $n \in \mathbb{N}$ such that $1/n < r$. If no elements of K are in $N_{1/n}(x) \subseteq N_r(x)$, then the above construction for x_i could have proceeded one more step, contradicting our construction. Thus K is a countable dense set in X , so X is separable. ■

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. Hint: For every positive integer n , there are finitely many neighbourhoods of radius $1/n$ whose union covers K .

Proof. For $n \in \mathbb{N}$, consider the cover $\{N_{1/n}(x) : x \in K\}$. Since K is compact, some finite subset of these neighbourhoods must cover K . From the same conclusion as in Exercise 24, the set G of all the centres of the neighbourhoods for $n \in \mathbb{N}$ forms a countable dense set in K , so K is separable. ■

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact.

Proof. By Exercise 24, X is separable. By Exercise 23, X has a countable base, say $\{V_n\}$. Then every open cover of X has a countable subcover $\{G_n\} \subseteq \{V_n\}$. Suppose that there exists no finite subcover of $\{G_n\}$ which covers X . Then, the complement $F_n = (G_1 \cup \dots \cup G_n)^c$ is non-empty for each n but $\bigcap F_n$ is empty. Let E be a set which contains a point from each F_n , and let $x \in X$ be a limit point of E , which exists by presumption. But then $x \notin G_n$ for any G_n , a contradiction. ■

27. Define a point p in a metric space X to be a **condensation point** of a set $E \subseteq X$ if every neighbourhood of p contains uncountably many points of E .

Suppose $E \subseteq \mathbb{R}^k$ is uncountable, and let P be the set of all condensation points of E . Prove that P is perfect and that at most countably many points of E are not in P . In other words, show that $P^c \cap E$ is at most countable. Hint: Let $\{V_n\}$ be a countable base of \mathbb{R}^k , let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Proof. We follow the construction of W outlined in the hint. Let $x \in W^c$. Then x is not in any V_n for which $E \cap V_n$ is at most countable. In fact, this means that if $x \in V_n$, then $E \cap V_n$ must be uncountable. Now consider $N_r(x)$. It is an open set, so it can be written as the union of some subcollection $V_{n_1} \cup \dots \cup V_{n_k}$ of $\{V_n\}$, of which each $V_{n_i} \cap E$ is uncountable. So $N_r(x) \cap E \supseteq V_{n_1} \cap E$ is an uncountable set, so $x \in P$.

Now, let $y \in W$, so $y \in V_n$ for some n such that $E \cap V_n$ is at most countable. Then, V_n contains some neighbourhood of y which contains at most countably many points in E , so $y \notin P$. Thus $W \subseteq P^c$ and so $P \subseteq W^c$, so we conclude $P = W^c$.

The fact that $P^c \cap E = W \cap E$ is at most countable follows by construction. Now we claim W^c is perfect. The fact that W^c is closed follows directly from the fact that it is the complement of a countable union of open sets. It suffices to show that each point of W^c is a limit point of W^c . Let $x \in W^c$. If some neighbourhood of x did not contain any points in W^c , then there must be only points in $W \cap E$. However, there are at most countably many of these points, and any neighbourhood of x is uncountable. Thus the result follows. ■

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is most countable. (Corollary: Every countable closed set in \mathbb{R}^k has isolated points.) Hint: Exercise 27.

Proof. If F is at most countable, we are done. Otherwise, if F is uncountable, the proof from Exercise 27 extends to $F \subseteq X$, so if P is the set of condensation points of F , then $F = P \cup (F \setminus P)$ with $F \setminus P$ at most countable, as required.

For the corollary, let E be a countable closed set in the separable metric space \mathbb{R}^k . If it had no isolated points, it would be perfect and thus uncountable, a contradiction. ■

29. Prove that every open set in \mathbb{R} is the union of an at most countable collection of disjoint segments. Hint: Exercise 22.

Proof. Let A be open. For every x , let $I_x = (a, b)$ be the largest segment in A containing x . That is, let $a = \inf\{z : (z, x) \in A\}$, $b = \sup\{y : (x, y) \subseteq A\}$ in the extended reals $\overline{\mathbb{R}}$. Then, $a, b \notin A$ otherwise a larger interval exists (since A is open). Also, every $c \in (a, b)$ must be in A as otherwise if $x \leq c$, there would exist some $x \leq c < y < b$ for which $(x, y) \not\subseteq A$ and similarly for the other case.

Now, if (a, b) and (c, d) share at least one point, then $c < b$ and $d > a$. Since $a, c \notin A$, we must have $a \geq c$ and $c \geq a$, so $a = c$. Similarly $b = d$ so the intervals are the same. That is, if two intervals share any point, they must be the same interval. Now, every interval I_x contains at least one rational since \mathbb{Q} is dense, so there exists an injection from our set of open sets to the countable set of rational numbers. Thus, our set is countable, as required. ■

30. Imitate the proof of Theorem 31 to obtain the following result:

If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$ where F_n is a closed subset of \mathbb{R}^k , then at least one F_n has a non-empty interior.

Equivalently: If G_n are dense open subsets of \mathbb{R}^k , then $\bigcap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in \mathbb{R}^k).

(This is a special case of Baire's theorem; see Exercise 3.22 for the general case).

Proof. Let $x_1 \in G_1$, so it has a neighbourhood V_1 whose closure is in G_1 . Since G_2 is dense, V_1 contains a point x_2 in G_2 for which there exists a neighbourhood whose closure is completely in $V_1 \cap G_2$. Repeat this process indefinitely, choosing x_n in $V_{n-1} \cap G_n$ and some neighbourhood V_n around it whose closure is completely contained in $V_{n-1} \cap G_n$. This yields a nested sequence of non-empty closed and bounded sets $\{\overline{V_n}\}$, so $\bigcap_{n=1}^{\infty} \overline{V_n}$ is non-empty. But inductively

$$\overline{V_n} \subseteq V_{n-1} \cap G_n \subseteq \overline{V_{n-1}} \cap G_n \subseteq \left[\bigcap_{k=1}^{n-1} G_k \right] \cap G_n = \bigcap_{k=1}^n G_k$$

so any element in $\bigcap_{n=1}^{\infty} \overline{V_n}$ is also in $\bigcap_{k=1}^{\infty} G_k$, so the latter is non-empty, proving the given equivalent statement. ■

3 Numerical Sequences and Series

3.1 Convergent Sequences

Definition 19. A sequence $\{p_n\}$ in a metric space X **converges** if there exists $p \in X$ such that for every $\epsilon > 0$, there exists an integer N such that $n \geq N$ implies $d(p_n, p) < \epsilon$. In this case, we say p is the limit of $\{p_n\}$, or $p_n \rightarrow p$, or $\lim_{n \rightarrow \infty} p_n = p$. A sequence which does not converge **diverges**. A sequence is **bounded** if its range is bounded.

We prove some important properties of convergent sequences in metric spaces.

Theorem 33. Let $\{p_n\}$ be a sequence in a metric space X .

- (a) $\{p_n\}$ converges to $p \in X$ iff every neighbourhood of p contains all but finitely many n .
- (b) If $p, p' \in X$ and $\{p_n\}$ converges to both p and p' , then $p = p'$.
- (c) If $\{p_n\}$ converges, then it is bounded.
- (d) If $E \subseteq X$ and p is a limit point of E , then there is a sequence $\{p_n\}$ in E such that $p_n \rightarrow p$.

Proof. For (a), suppose that $p_n \rightarrow p$. Then every neighbourhood of size $\epsilon > 0$ must contain all p_n for some $n \geq N$ by definition. Now suppose every neighbourhood of p contains all but finitely many p_n . Then for any $\epsilon > 0$, the neighbourhood of size ϵ contains all but finitely many p_n , so there exists N such that $n \geq N$ implies $p_n \in N_\epsilon(x)$.

For (b), if $p \neq p'$, then $d(p, p') = r > 0$. Selecting $\epsilon < r/2$ will force all but finitely many p_n lie in $N_{r/2}(p) \cap N_{r/2}(p') = \emptyset$, a contradiction.

For (c), let N such that $n \geq N$ implies $d(p, p_n) < 1$. Then, $d(p, p_n) \leq \max\{d(p, p_1), \dots, d(p, p_{N-1}), 1\}$.

For (d), take p_n to be some element of E in $N_{1/n}(p)$, which exists since p is a limit point of E . This converges to p , given $N > 1/\epsilon$. ■

The following theorem states we can combine convergent sequences the way we expect.

Theorem 34. Suppose $\{s_n\}, \{t_n\}$ are complex sequences and $s_n \rightarrow s, t_n \rightarrow t$.

- (a) $(s_n + t_n) \rightarrow s + t$.
- (b) $cs_n \rightarrow cs, c + s_n \rightarrow c + s$.
- (c) $s_n t_n \rightarrow st$.
- (d) $1/s_n \rightarrow 1/s$ given $s_n, s \neq 0$.

Proof. For (a), let $\epsilon > 0$ be arbitrary and choose N_s, N_t such that $n \geq N_s$ ($n \geq N_t$) implies $|s_n - s| < \epsilon/2$ ($|t_n - t| < \epsilon/2$). Then, if $n \geq \max(N_s, N_t)$,

$$|s_n + t_n - (s + t)| \stackrel{\Delta}{\leq} |s_n - s| + |t_n - t| < \epsilon$$

and the result follows.

For the first part of (b), if $c = 0$, the result is immediate. Otherwise, let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies $|s - s_n| < \epsilon/|c|$. Then $n \geq N$ implies $|cs - cs_n| < \frac{|c|\epsilon}{|c|} = \epsilon$, completing the proof. The second part is similarly easy.

For (c), we use the following manipulation:

$$s_n t_n - st = s_n t_n - s_n t + s_n t - st = s_n(t_n - t) + (s_n - s)t$$

Choose N_s such that $n \geq N_s$ implies $|s_n - s| < \epsilon/2|t|$ and N_t such that $n \geq N_t$ implies $|t_n - t| < \epsilon/2S$, where $S = \max_{n \in \mathbb{N}} |s_n|$. Then, if $n \geq \max(N_s, N_t)$,

$$|s_n t_n - st| = |s_n(t_n - t) + (s_n - s)t| \leq |s_n(t_n - t)| + |(s_n - s)t| = |s_n||t_n - t| + |s_n - s||t| \leq S \cdot \frac{\epsilon}{2S} + \frac{\epsilon}{2|t|} \cdot |t| = \epsilon$$

as required. If $|t| = 0$, then choose N_s such that $n \geq N_s$ implies $|t_n| < \epsilon/S$, and the result follows.

For (d), we can use (c) to conclude $(1/s_n)s_n \rightarrow 1$, and since $s_n \rightarrow s$, we must have $1/s_n \rightarrow 1/s$. ■

Theorem 35. *The following statements are true:*

1. Suppose $\mathbf{x}_n \in \mathbb{R}^k$ and $\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n})$. Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \dots, \alpha_k)$ iff each $\{\alpha_{i,n}\}$ converges to α_i .
2. Suppose $\{\mathbf{x}_n\}, \{\mathbf{y}_n\}$ are sequences in \mathbb{R}^k , and $\{c_n\}$ is a sequence of real numbers, and $\mathbf{x}_n \rightarrow \mathbf{x}$, $\mathbf{y}_n \rightarrow \mathbf{y}$, and $c_n \rightarrow c$. Suppose further that $\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n})$ and $\mathbf{y}_n = (\beta_{1,n}, \dots, \beta_{k,n})$.

Then,

$$\mathbf{x}_n + \mathbf{y}_n \rightarrow \mathbf{x} + \mathbf{y}, \quad \mathbf{x}_n \cdot \mathbf{y}_n \rightarrow \mathbf{x} \cdot \mathbf{y}, \quad c_n \mathbf{x}_n \rightarrow c\mathbf{x}$$

Proof. For (a), suppose $\{\mathbf{x}_n\}$ converges to \mathbf{x} . So for $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|\mathbf{x}_n - \mathbf{x}| < \epsilon$. Then for any i , $|\alpha_{i,n} - \alpha_i| < \epsilon$ and the result follows.

Suppose each $\{\alpha_{i,n}\}$ converges to α_i . Let $\epsilon > 0$. For each i , we can choose $N_i \in \mathbb{N}$ such that $|\alpha_{i,n} - \alpha_i| < \epsilon/\sqrt{k}$ so that if $n \geq \max(N_1, \dots, N_k)$, then

$$|\mathbf{x}_n - \mathbf{x}| = \sqrt{\sum_{i=1}^k |\alpha_{i,n} - \alpha_i|^2} \leq \sqrt{k \cdot \frac{\epsilon^2}{k}} = \epsilon,$$

completing the proof.

For the first part of (b),

$$(\mathbf{x}_n + \mathbf{y}_n)_i = \alpha_{i,n} + \beta_{i,n} \rightarrow \alpha_i + \beta_i = (\mathbf{x} + \mathbf{y})_i$$

so the result follows from part (a).

For the second part,

$$\mathbf{x}_n \cdot \mathbf{y}_n = \sum_{i=1}^k \alpha_{i,n} \beta_{i,n} \rightarrow \sum_{i=1}^k \alpha_i \beta_i = \mathbf{x} \cdot \mathbf{y}.$$

For the final part,

$$(\beta_n \mathbf{x}_n)_i = \beta_n \alpha_{i,n} \rightarrow \beta \alpha_i = (\beta \mathbf{x})_i$$

so the result follows from part (a). ■

Definition 20. Given a sequence $\{p_n\}$, consider an increasing sequence of positive integers $\{n_k\}$. Then the sequence $\{p_{n_k}\}$ is a **subsequence** of $\{p_n\}$. If $\{p_{n_k}\}$ converges, its limit is called a **subsequential limit** of $\{p_n\}$.

Theorem 36. $\{p_n\}$ converges to p iff every subsequence of $\{p_n\}$ converges to p .

Proof. Suppose $p_n \rightarrow p$. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|p_n - p| < \epsilon$. Then if $\{p_{n_k}\}$ is a subsequence of p_n , then there exists some $K \in \mathbb{N}$ such that $k \geq K$ implies $n_k \geq N$. So for $k \geq K$, we have $|p_{n_k} - p| < \epsilon$.

If every subsequence of p_n converges to p , then in particular, the subsequence with $n_k = k$ (the entire sequence) converges, so we are done! ■

Theorem 37. The following are true:

1. If $\{p_n\}$ is a sequence in a compact metric space X , then some subsequence of $\{p_n\}$ converges to a point of X .
2. Every bounded sequence in \mathbb{R}^k contains a convergent subsequence.

Proof. For (a), let K be the image of $\{p_n\}$. Then if K is finite, there exists some $p \in K$ which is attained infinitely often. The subsequence of these points converges to p . Otherwise, K is an infinite subset of a compact metric space, and thus admits a limit point in X . The result follows from Theorem 3.2(d).

The result in (b) follows from (a) since bounded subsets of \mathbb{R}^k lie inside a compact subset of \mathbb{R}^k (namely the closure of the image of the sequence). ■

Theorem 38. The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X .

Proof. Let E^* be the set of all subsequential limits of $\{p_n\}$ and let q be a limit point of E^* . Choose n_1 such that $p_{n_1} \neq q$. If this doesn't exist, E^* has one point and we are done. Put $\delta = d(q, p_{n_1})$. Then build n_k inductively, so that given n_1, \dots, n_{i-1} , we choose $x \in E^*$ such that $d(x, q) < 2^{-i}\delta$. Since $x \in E^*$, we can choose $n_i > n_{i-1}$ such that $d(x, p_{n_i}) < 2^{-i}\delta$. Then $d(q, p_{n_i}) < 2^{1-i}\delta$ so $p_{n_i} \rightarrow q$ so $q \in E^*$. ■

3.2 Cauchy Sequences

Definition 21. A sequence $\{p_n\}$ in a metric space X is said to be a **Cauchy sequence** if for every $\epsilon > 0$, there exists an integer N such that $d(p_n, p_m) < \epsilon$ if $n, m \geq N$.

Let E be a non-empty subset of a metric space X and let S be the set of all real numbers of the form $d(p, q)$ with $p, q \in E$. Then $\text{diam } E = \sup S$ is the **diameter** of E .

Clearly a sequence $\{p_n\}$ is Cauchy iff $\text{diam } E_N \rightarrow 0$, where $E_n = \{p_n, p_{n+1}, \dots\}$.

Theorem 39. The following are true:

1. If E is a subset of a metric space X , then $\text{diam } E = \text{diam } \bar{E}$.
2. If K_n is a sequence of compact sets in X such that $K_n \supseteq K_{n+1}$ and $\text{diam } K_n \rightarrow 0$, then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

Proof. For (a), notice that $E \subseteq \bar{E}$ so $\text{diam } E \leq \text{diam } \bar{E}$. If $\text{diam } \bar{E} > \text{diam } E$, there exists a pair of elements $x, y \in \bar{E}$ with $|x - y| > \text{diam } E$. If one of x, y is not in E , then a neighbourhood of size less than $(|x - y| - \text{diam } E)/2$ around the element in \bar{E} has no elements in E , a contradiction. A similar argument can be made if both are not in E .

For (b), we know from Nested interval theorem that $S = \bigcap_{n=1}^{\infty} K_n$ is non-empty. If S had more than one point, say a and b , then $\text{diam } S_n \geq d(a, b) > 0$ all n and thus $\text{diam } S_n \not\rightarrow 0$, a contradiction. ■

Theorem 40. *The following are true:*

1. *In any metric space X , every convergent sequence is a Cauchy sequence.*
2. *If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X , then $\{p_n\}$ converges to some point of X .*
3. *In \mathbb{R}^k , every Cauchy sequence converges.*

Proof. For (a), let $\epsilon > 0$. Then there exists N such that $n \geq N$ implies $d(p_n, p) \leq \epsilon/2$. Then for any $m, n \geq N$, we have $d(p_n, p_m) \leq d(p_n, p) + d(p_m, p) < \epsilon$.

For (b), let $E_n = \{p_n, p_{n+1}, \dots\}$. Then since $\{p_n\}$ is Cauchy, $\text{diam } \overline{E_n} \rightarrow 0$. Also, $\overline{E_n}$ are subsets of X and thus compact. Finally, $\overline{E_n} \supseteq \overline{E_{n+1}}$ by definition. By Theorem 3.10, $\bigcap_{n=1}^{\infty} \overline{E_n}$ consists of one point. Let this point be p . Since $\text{diam } \overline{E_n} \rightarrow 0$, points in $\{p_n\}$ get arbitrarily close to p so $\{p_n\}$ converges to $p \in X$.

For (c), Cauchy sequences are bounded, and thus their image is contained in a compact subset of \mathbb{R}^k . The result follows from (b). ■

Definition 22. *A metric space in which every Cauchy sequence converges is **complete**.*

Theorem 3.11 says that all compact metric spaces and all Euclidean spaces are complete.

Also, every closed subset of a complete metric space is complete.

Definition 23. *A sequence $\{s_n\}$ of real numbers is said to be*

- **monotonically increasing** if $s_n \leq s_{n+1}$ for all n ,
- **monotonically decreasing** if $s_n \geq s_{n+1}$ for all n

The class of monotonic sequences consists of the increasing and decreasing sequences.

Theorem 41. *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges iff it is bounded.*

Proof. Suppose $\{s_n\}$ is increasing and bounded. Then its image admits a least upper bound α . For every $\epsilon > 0$, there exists N such that $s - \epsilon < s_N \leq s$, otherwise $s - \epsilon$ would be a smaller upper bound. For $n \geq N$, $s - \epsilon < s_N \leq s_n \leq s$, as required.

The converge follows from Theorem 3.2(c). ■

3.3 Upper and Lower Limits

Definition 24. *If $\{s_n\}$ is a sequence of real numbers such that for any $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n \geq M$, then we say $s_n \rightarrow +\infty$.*

Similarly, if for any $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n \leq M$, then we say $s_n \rightarrow -\infty$.

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of possibly infinite subsequential limits. Then, let $s^ = \sup E$ and $s_* = \inf E$, called the **upper** and **lower limits** of $\{s_n\}$ respectively.*

We say

$$\limsup_{n \rightarrow \infty} s_n = s^* \text{ and } \liminf_{n \rightarrow \infty} s_n = s_*.$$

Theorem 42. *Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of possibly infinite subsequential limits of $\{s_n\}$, and $s^* = \limsup s_n$. Then the following are true:*

1. $s^* \in E$.

2. If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with these properties.

Proof. For (a), if $s_n \rightarrow \pm\infty$, then every subsequence goes to $\pm\infty$ so $s^* \in E$. Otherwise, it is bounded and thus E is closed by Theorem 3.7, so $s^* = \sup E \in \bar{E} = E$.

For (b), notice that if this weren't the case, then there are $s_n \geq x$ for n arbitrarily large. Taking these s_n as a subsequence yields a subsequence with limit $\geq x > s^*$, contradicting the maximality of s^* in E .

Finally, let $a, b \in E \subseteq \mathbb{R}$ both satisfy these above properties. Without loss of generality, $a > b$. Let $x \in \mathbb{R}$ such that $b < x < a$. Then statement (b) holds for b so $s_n < x < a$ for $n \geq N$. Then a cannot satisfy (a). ■

For example, the sequence containing all rational numbers $\{s_n\}$ has every real number as a subsequential limit. Also, its \limsup is $+\infty$ and \liminf is $-\infty$.

Also, it is noteworthy that a sequence converges iff its \limsup is equal to its \liminf .

Theorem 43. If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

$$\liminf_{n \rightarrow \infty} s_n \leq \liminf_{n \rightarrow \infty} t_n \text{ and } \limsup_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} t_n$$

Proof. The cases with infinities are simple casework. Thus, suppose all values involved are finite real numbers. Suppose, for the sake of contradiction that $\liminf_{n \rightarrow \infty} s_n > \liminf_{n \rightarrow \infty} t_n$. Then, there exists some $y \in E_t$ strictly less than all the $x \in E_s$, where E_s and E_t are the sets of subsequential limits of $\{s_n\}$ and $\{t_n\}$ respectively.

Consider the subsequence $\{t_{n_k}\}$ which converges to this y . Suppose without loss of generality that each $n_k \geq N$. Then each term in $\{s_{n_k}\}$ is at most the corresponding term in $\{t_{n_k}\}$. In particular, some subsequence $\{s_{n_{k_i}}\}$ converges to a value at most y , a contradiction, since some $x \in E_s$ is at most y .

The corresponding proof for \limsup follows similarly. ■

3.4 Some Special Sequences

Theorem 44. We compute the limits of some helpful sequences, using the helpful fact that if $0 \leq x_n \leq s_n$ for $n \geq N$, then $s_n \rightarrow 0$ implies $x_n \rightarrow 0$.

1. If $p > 0$, then $n^{-p} \rightarrow 0$.
2. If $p > 0$, then $p^{1/n} \rightarrow 1$.
3. $n^{-n} \rightarrow 1$.
4. If $p > 0$ and $\alpha \in \mathbb{R}$, then $\frac{n^\alpha}{(1+p)^n} \rightarrow 0$.
5. If $|x| < 1$, then $x^n \rightarrow 0$.

Proof. For (a), take $n > (1/\epsilon)^{1/p}$.

For (b), $p = 1$ is obvious. If $p > 1$, let $x_n = p^{1/n} - 1 > 0$. Then, by the binomial theorem, $1 + nx_n \leq (1 + x_n)^n = p$ so $0 < x_n \leq \frac{p-1}{n}$ and the rightmost terms go to 0. $p < 1$ follows by taking reciprocals.

For (c), let $x_n = n^{1/n} - 1 > 0$. Then $n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$. Then $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$. and the result follows.

For (d), let $k > \max(0, \alpha)$ be an integer. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1) \cdots (n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$$

which approaches zero since $\alpha - k < 0$ and by (a).

Statement (e) follows by taking $\alpha = 0$ in (d). ■

3.5 Series

Definition 25. Given $\{a_n\}$ we associate a sequence $\{s_n\}$ where

$$s_n = \sum_{k=1}^n a_k = a_1 + a_2 + \cdots + a_n,$$

where s_n are called the **partial sums** of $\{a_n\}$.

We also write

$$\{s_n\} = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n.$$

If $s_n \rightarrow s$, we say the series **converges**, and that

$$\sum_{n=1}^{\infty} a_n = s.$$

The notion of Cauchy sequences on the partial sums can be extended to series:

Theorem 45. $\sum a_n$ converges iff for every $\epsilon > 0$, there exists an integer N such that

$$\left| \sum_{k=m}^n a_k \right| < \epsilon$$

for all $n \geq m \geq N$.

Proof. Notice that $\sum_{k=m}^n a_k = s_n - s_m$, so the condition is equivalent to the sequence $\{s_n\}$ being Cauchy. The result follows from the fact that Cauchy sequences and convergent sequences are equivalent in \mathbb{R}^k . ■

In particular, taking $m = n$ we have $|a_n| < \epsilon$ for $n \geq N$. More formally,

Theorem 46. If $\sum a_n$ converges, then $a_n \rightarrow 0$.

Theorem 47. A series of non-negative terms converges if and only if its partial sums form a bounded sequence.

Proof. The partial sums form a monotonically increasing sequence, so the result follows as in Theorem 3.14. ■

We now prove a helpful theorem called the ‘Comparison test’.

Theorem 48. *The following are true:*

- (a) *If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.*
- (b) *If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.*

Proof. For (a), let $\{s_n\}$ be the sequence of partial sums of a_n and $C = \sum c_n$. Also, notice that $c_n \geq 0$. Let $\epsilon > 0$. Then since $\sum c_n$ converges, there exists an integer N such that for $n \geq m \geq N$,

$$\left| \sum_{k=m}^n c_k \right| < \epsilon.$$

Then, for any $n \geq m \geq N$,

$$\left| \sum_{k=m}^n a_k \right| \leq \sum_{k=m}^n |a_k| \leq \sum_{k=m}^n c_k \leq \left| \sum_{k=m}^n c_k \right| < \epsilon$$

so $\sum a_n$ converges.

For (b), suppose without loss of generality that $\sum d_n$ diverges to $+\infty$. Then let M be a real number. Then for $n \geq N$,

$$\sum_{k=1}^n a_k \geq \sum_{k=1}^n d_k > M$$

so $\sum a_n$ diverges too. ■

3.6 Series of Non-negative Terms

Theorem 49. *If $0 \leq x < 1$, then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \geq 1$, the series diverges.

Proof. If $0 \leq x < 1$, let $s_n = \sum_{k=0}^n x^k$. It can be shown by induction that $s_n = \frac{1-x^{n+1}}{1-x}$. Then,

$$\left| s_n - \frac{1}{1-x} \right| = \left| \frac{1}{1-x} - \frac{x^{n+1}}{1-x} - \frac{1}{1-x} \right| = x^{n+1} \cdot \frac{1}{1-x} \rightarrow 0$$

so the result follows.

If $x \geq 1$, then $s_n \geq n$. The result follows immediately. ■

The following theorem states that a rather ‘thin’ subsequence of decreasing sequences determines the convergence of $\sum a_n$.

Theorem 50 (Cauchy). *Suppose $a_1 \geq a_2 \geq \dots \geq 0$. Then the series $\sum a_n$ converges if and only if the series*

$$S := \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \dots$$

converges.

Proof. Let $\{b_n\}$ be a sequence related to a_n such that $b_n = a_{f(n)}$ where $f(n)$ is the largest power of two at most n . That is, the sequence $\{b_n\}$ starts $(a_1, a_2, a_2, a_4, a_4, a_4, a_4, \dots)$. Since a_n is decreasing, $\{b_n\}$ is a strictly decreasing sequence where each term is at least the corresponding a_n . Also, notice S is $\sum b_n$, so the fact that $\sum a_n$ converges if S converges follows by the comparison test.

Now suppose $\sum a_n$ converges. Then,

$$\sum_{i=1}^{2^k} = a_1 + a_2 + (a_3 + a_4) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \geq \frac{1}{2}a_1 + a_2 + 2a_4 + \dots + 2^{k-1}a_{2^k} = \frac{1}{2} \sum_{i=1}^{2^k} b_i$$

so the result follows again by the comparison test. ■

Theorem 51. *The series $\sum n^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Proof. We can write $2^k a_{2^k} = \frac{2^k}{2^{kp}} = 2^{(1-p)k}$ so $\sum_{k=0}^{\infty} (2^{(1-p)})^k$ converges iff $0 \leq 2^{1-p} < 1$ from Theorem 3.26, which occurs when $p > 1$. The result follows by the previous theorem, since the original sequence was decreasing. ■

Theorem 52. *If $p > 1$, then*

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges. Otherwise, the series diverges.

Proof. Write

$$2^k a_{2^k} = \frac{2^k}{2^k (\log 2^k)^p} = \frac{1}{(k \log 2)^p} = \frac{c}{k^p}$$

with $c = \frac{1}{(\log 2)^p}$ so by Theorem 3.27, $\sum a_n$ converges iff

$$c \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges, but that happens iff $p > 1$ by the previous theorem. ■

3.7 The Number e

Definition 26. *We define*

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This is well-defined since

$$s_n = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots + \frac{1}{1 \cdot 2 \cdot \dots \cdot n} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} < 3$$

There is an equivalent definition:

Theorem 53. *We have*

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Proof. Let $s_n = \sum_{k=0}^n \frac{1}{k!}$ and $t_n = \left(1 + \frac{1}{n}\right)^n$. By the binomial theorem,

$$t_n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \cdot \frac{n!}{(n-k)!n^k} = \sum_{k=0}^n \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right).$$

Clearly $t_n \leq s_n$ termwise, so $\limsup_{n \rightarrow \infty} t_n \leq e$.

Also, if $m \leq n$, then

$$t_n \geq \sum_{k=0}^m \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right)$$

which, as n increases, becomes

$$\liminf_{n \rightarrow \infty} t_n \geq \sum_{k=0}^m \frac{1}{k!}.$$

Letting m go to infinity yields $e \leq \liminf_{n \rightarrow \infty} t_n$ so the result follows. ■

As a sidenote, the following bound on the error of the partial sums proves helpful:

Remark. Notice that

$$0 < e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n!n}$$

which shows that this infinite sum representation converges extremely quickly.

Theorem 54. e is irrational.

Proof. Suppose otherwise, so that $e = p/q$ for positive integers p, q . By the above remark, $0 < q!(e - s_q) < \frac{1}{q}$. Then both $q!s_q$ and $q!e$ are integers, so the above inequality implies the existence of an integer between 0 and $1/q$, a contradiction. ■

3.8 The Root and Ratio Tests

Theorem 55 (Root Test). Given $\sum a_n$, put $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$. Then,

- (a) If $\alpha < 1$, then $\sum a_n$ converges.
- (b) If $\alpha > 1$, then $\sum a_n$ diverges.
- (c) If $\alpha = 1$, the test gives no information.

Proof. For (a), we know then that for large enough n , $\sqrt[n]{|a_n|} \leq \alpha$ so $|a_n| \leq \alpha^n$. The result follows from the comparison test. For (b), notice that for large enough n , $|a_n| \geq 1$, so the series diverges by the contrapositive of Theorem 3.23.

For (c), the series $\sum 1/n$ and $\sum 1/n^2$ work. ■

Theorem 56 (Ratio Test). The series $\sum a_n$ converges if $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ and diverges if $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for all $n \geq n_0$ where n_0 is some fixed integer.

Proof. For the first part, for $n \geq N$, we will have $\left| \frac{a_{n+1}}{a_n} \right| = \beta < 1$. Then $|a_n| \leq |a_N| \beta^{n-N}$ so the result follows by comparing with a geometric series. For the second part, the fact that $|a_{n+1}| \geq |a_n|$ eventually, means that $a_n \not\rightarrow 0$, so the series cannot converge. ■

The previous two tests are very useful for determining convergence. In most cases, the ratio test is easier to use, but we show in the following theorem that the root test is stronger.

Theorem 57. For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \limsup_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$$

Proof. Let $\alpha = \limsup_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$. If $\alpha = +\infty$, we are done. Otherwise, suppose some subsequence $\{c_{n_k}\}$ had the property that $\lim_{k \rightarrow \infty} \sqrt[n_k]{c_{n_k}} = \beta > \alpha$. Then, there must exist infinitely many terms $\{c_n\}$ with $c_n^n > \alpha^n$. But this cannot occur if the ratios of consecutive terms does not exceed α infinitely often, a contradiction. The left inequality is similar. ■

3.9 Power Series

Definition 27. Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a **power series**. The numbers c_n are called the **coefficients** of the series, and z is a complex number.

Theorem 58. Given a power series as above, let

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}, R = \frac{1}{\alpha}.$$

Then $\sum c_n z^n$ converges if $|z| < R$ and diverges when $|z| > R$.

Proof. The value for α is derived from the result of the root test, so this proof is just like unwrapping a present you wrapped for yourself. ■

3.10 Summation by Parts

Theorem 59. Given two sequences $\{a_n\}, \{b_n\}$, put $A_n = \sum_{k=0}^n a_k$ if $n \geq 0$ and $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^q a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Proof. Expand the sum and telescope differently:

$$\begin{aligned} \sum_{n=p}^q a_n b_n &= \sum_{n=p}^q (A_n - A_{n-1}) b_n \\ &= A_q b_q - A_{q-1} b_q + A_{q-1} b_{q-1} - A_{q-2} b_{q-1} + \cdots + A_{p+1} b_{p+1} - A_p b_{p+1} + A_p b_p - A_{p-1} b_p \\ &= A_q b_q + (A_{q-1} (b_{q-1} - A_{q-1} b_q) + \cdots + A_p (b_p - A_p b_{p+1}) - A_{p-1} b_p \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \end{aligned}$$

as required. ■

Theorem 60. Let $\{a_n\}$ be a sequence such that

- (a) The partial sums A_n of $\sum a_n$ form a bounded sequence.
- (b) $b_0 \geq b_1 \geq \dots$.
- (c) $b_n \rightarrow 0$.

Then $\sum a_n b_n$ converges.

Proof. The above theorem suggests using the Cauchy criterion. Let $\epsilon > 0$ and suppose the A_n are bounded by M . We first manipulate:

$$\begin{aligned} \left| \sum_{n=p}^q a_n b_n \right| &= \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right| \\ &\leq M \left[\left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) \right| + |b_q| + |b_p| \right] \\ &= 2M b_q \end{aligned}$$

so choosing N such that $q \geq p \geq N$ implies $b_q < \epsilon/(2M)$, completing the proof. ■

Theorem 61. Suppose $\{c_n\}$ is a sequence such that

- (a) $|c_1| \geq |c_2| \geq \dots$.
- (b) $c_{2m-1} \geq 0, c_{2m} \leq 0$.
- (c) $c_n \rightarrow 0$.

Then $\sum c_n$ converges.

Proof. This follows from the previous theorem, letting $a_n = (-1)^{n+1}$ and $b_n = |c_n|$. ■

Theorem 62. Suppose the radius of convergence of $\sum c_n z^n$ is 1, and that $|c_0| \geq |c_1| \geq \dots$ and $c_n \rightarrow 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$ except possibly at $z = 1$.

Proof. Let $a_n = z^n$ and $b_n = |c_n|$. Then the A_n are bounded by $\frac{2}{|1-z|}$, so Theorem 3.42 applies. ■

3.11 Absolute Convergence

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

Theorem 63. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof. Let $\epsilon > 0$ be arbitrary and N such that the Cauchy criterion holds for $n \geq m \geq N$. Then, for these same n, m , we have

$$\left| \sum_{k=m}^n a_k \right| \leq \sum_{k=m}^n |a_k| < \epsilon$$

so $\sum a_n$ converges by the Cauchy criterion. ■

Notice that absolute convergence is convergence if your terms are positive. Series which converge but not absolutely converge **non-absolutely**.

We see that we can manipulate absolutely convergent sums much like finite sums. Let's explore the types of operations we can perform on general series.

Theorem 64. *If $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$ and $\sum ca_n = cA$ for any c .*

Proof. The first statement follows after noticing that the partial sums of $\sum(a_n + b_n)$ are simply the sums of the partial sums of $\sum a_n$ and $\sum b_n$. The second statement follows similarly. ■

Can we multiply infinite series? The answer is slightly more complicated.

Definition 28. *Given $\sum a_n$ and $\sum b_n$, we put*

$$c_n = \sum_{k=0}^n a_k b_{n-k}$$

*and call $\sum c_n$ the **product** of the two series. This is motivated by considering the coefficients of the related generating series $\sum a_n z^n$ and $\sum b_n z^n$.*

The following theorem, due to Mertens, characterizes a large set of situations where this notion of product is consistent.

Theorem 65 (Mertens). *Suppose that*

(a) $\sum a_n$ converges absolutely.

(b) $\sum a_n = A$.

(c) $\sum b_n = B$.

(d) $c_n = \sum_{k=0}^n a_k b_{n-k}$ as above.

Then $\sum c_n = AB$.

Proof. Let A_n, B_n and C_n be the n -th partial sums of a_n, b_n, c_n respectively. Also, let $\beta = B_n - B$. We manipulate C_n , essentially switching the order of summations:

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_n (B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0. \end{aligned}$$

Thus if we let

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0,$$

it suffices to prove that $\gamma_n \rightarrow 0$. Let $\epsilon > 0$ be arbitrary and choose N such that $|\beta_n| < \epsilon$ for $n \geq N$. Also, let $\alpha = \sum |a_n|$. Then

$$\begin{aligned} |\gamma_n| &= |a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0| \\ &\leq |a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_N \beta_{n-N}| + |a_{N+1} \beta_{n-N-1} + \cdots + a_n \beta_0| \\ &\leq |a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_N \beta_{n-N}| + \epsilon \alpha \end{aligned}$$

so if we fix N and let $n \rightarrow \infty$, then we have

$$\limsup_{n \rightarrow \infty} |\gamma_n| \leq \epsilon \alpha$$

so the result follows since ϵ was arbitrary. ■

We now wonder whether this definition of product makes sense in the sense that if $\sum c_n$ converges, whether it will have the sum AB . Abel proved that this is true.

Theorem 66. *If $\sum a_n$, $\sum b_n$, $\sum c_n$ converge to A, B, C and $c_n = a_0b_n + \cdots + a_nb_0$, then $C = AB$.*

Proof. Note: This is an incomplete proof, as it assumes that power series are continuous without justification. We will fill in this gap in Theorem 8.2. This proof will be replaced later, since it is most likely circular or wrong.

Let $a(z) = \sum a_n z^n$, $b(z) = \sum b_n z^n$, and $c(z) = \sum c_n z^n$. Then we (will) have seen that $c(z) = a(z)b(z)$ if $c_n = a_0b_n + \cdots + a_nb_0$, if we consider partial sums of $c(z)$. Then the result follows by substituting $z = 1$. ■

3.12 Rearrangements

Definition 29. *Let $\{k_n\}$ be a bijection from \mathbb{N} to itself. Then letting $a'_n = a_{k_n}$, we say that $\sum a'_n$ is a **rearrangement** of $\sum a_n$.*

Notice that the partial sums of these have no reason to be the same, so the question arises: should they converge? Will their sums be the same if so?

We prove Riemann's rearrangement theorem, which says that all hell breaks loose if our sequence doesn't converge absolutely:

Theorem 67 (Riemann Rearrangement). *Let $\sum a_n$ which converges non-absolutely, and $-\infty \leq \alpha \leq \beta \leq +\infty$. Then, there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that*

$$\liminf_{n \rightarrow \infty} s'_n = \alpha \text{ and } \limsup_{n \rightarrow \infty} s'_n = \beta.$$

Proof. This proof is well-studied. We present an informal proof hopefully sufficient to provide intuition. The actual proof is long-winded.

Let a_n^+ and a_n^- denote the subsequences of positive and negative terms of a_n . If either subsequence were finite, then the other would overpower it and eventually either diverge or converge (after which a_n would converge absolutely since the subsequence has all the same sign). Thus both must be infinite.

If $\alpha \geq 0$, take positive terms until our partial sums go above α . We must be able to do this, since both subseries diverge. Similarly, if $\alpha < 0$, take negative terms. Then, repeat the following process indefinitely. Take positive terms until our partial sums go above β , then negative terms until our partial sums return below α . This is again made possible since both subsequences diverge.

Taking the terms which are 'turning points' on both sides, we have sequences which converge to α and β , so the result follows. ■

Now, the following theorem states that absolutely convergent series do indeed behave nicely:

Theorem 68. *If $\sum a_n$ converges absolutely, then every rearrangement of $\sum a_n$ converges, and further they converge to the same sum.*

Proof. Let $\sum a'_n = \sum a_{k_n}$ be a rearrangement of $\sum a_n$. Let $\epsilon > 0$. Take $N \in \mathbb{N}$ such that if $n \geq m \geq N$, then

$$\left| \sum_{k=m}^n |a_k| \right| < \epsilon.$$

Then, take $N' = \max(k_1, \dots, k_N) + 1$, so that no $n' \geq m' \geq N'$ will be in $\{1, 2, \dots, N\}$. Then

$$\left| \sum_{k=m'}^{n'} |a'_k| \right| \leq \limsup_{n \rightarrow \infty} \left| \sum_{k=N+1}^n |a'_k| \right| \leq \epsilon$$

so the result follows by the Cauchy criterion. ■

3.13 Exercises

1. Prove that convergence of $\{s_n\}$ implies convergence of $|s_n|$. Is the converse true?

Proof. If $s_n \rightarrow 0$, then the result follows by definition. Otherwise, suppose $s_n \rightarrow s$ and $s > 0$. We can find N such that $n \geq N$ implies $s/2 < s_n < 3s/2$, so that $||s_n| - |s|| = |s_n - s| < \epsilon$. The proof when $s < 0$ is similar, except we take $\epsilon = -s/2$.

The converse is not true: consider the sequence $s_n = (-1)^n$ for which $|s_n| = 1$ converges trivially, but for which no N exists for $\epsilon < 2$. ■

2. Calculate $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$.

Proof. We manipulate:

$$\sqrt{n^2 + n} - n = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + 1/n} + 1}.$$

We prove that $\sqrt{1 + 1/n} \rightarrow 1$: notice that

$$|\sqrt{1 + 1/n} - 1| = \left| \frac{1 + 1/n - 1}{\sqrt{1 + 1/n} + 1} \right| \leq \frac{1}{2n} \rightarrow 0$$

so

$$\sqrt{n^2 + n} - n = \frac{1}{\sqrt{1 + 1/n} + 1} \rightarrow \frac{1}{1 + 1} = \frac{1}{2}.$$

■

3. If $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$, prove that $\{s_n\}$ converges and that $s_n < 2$.

Proof. Notice that the function $f(x) = \sqrt{2 + \sqrt{x}}$ is monotonic in the sense that if $a < b$, then $f(a) < f(b)$. Also, $s_2 = \sqrt{2 + \sqrt{s_1}} \geq \sqrt{2} = s_1$. Also, if $s_{n-1} \leq s_n$, then

$$s_n = \sqrt{2 + \sqrt{s_{n-1}}} \leq \sqrt{2 + \sqrt{s_n}} = s_{n+1}.$$

so it follows by induction that s_n is an monotonically increasing sequence.

We claim $s_n < 2$ inductively. The base case is true, and if $n \geq 1$,

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + \sqrt{2}} < 2.$$

Thus $\{s_n\}$ converges by Theorem 3.14. ■

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Proof. We claim that $s_* = 0$ and $s^* = 1$. First, the fact that $0 \leq s_n \leq 1$ with $s_{2m} \leq 1/2$ can be shown inductively. The base case follows immediately. Otherwise, dividing by 2 preserves the conditions, and adding $1/2$ to s_{2m} gives a number at most 1. It suffices now to find two sequences converging to 0 and 1.

the sequence $\{s_{2^k}\}$ is identically 0. The sequence $\{s_{2^{k+1}-1}\} = \left(0, \frac{1}{2}, \frac{3}{4}, \dots, \frac{2^k-1}{2^k}, \dots\right)$ converges to 1. ■

5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n,$$

provided the sum on the right is not of the form $+\infty - \infty$.

Proof. If both α and β are $+\infty$, the result proves itself. Likewise if α and β are both $-\infty$, so it suffices to consider the finite cases.

Let $\alpha = \limsup_{n \rightarrow \infty} a_n$ and $\beta = \limsup_{n \rightarrow \infty} b_n$. Suppose that $\limsup_{n \rightarrow \infty} (a_n + b_n) = \gamma > \alpha + \beta$. Then, some sequence $\{s_n\}$ with $s_n = a_n + b_n$ converges to γ . For each s_n , either $a_n > \alpha$ or $b_n > \beta$. Thus, either $a_n > \alpha$ infinitely often, or $b_n > \beta$ infinitely often. In either of these cases, either $\limsup_{n \rightarrow \infty} a_n > \alpha$, or $\limsup_{n \rightarrow \infty} b_n > \beta$, a contradiction. ■

6. Investigate the behaviour of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$.

Proof. The partial sums $s_n = \sqrt{n+1} - 1$ diverge to $+\infty$, so the sum diverges. ■

(b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$.

Proof. Manipulate:

$$0 \leq a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{(n+1) - n}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{n^{3/2}}$$

so since $\sum n^{-3/2}$ converges, $\sum a_n$ converges by the comparison test. ■

(c) $a_n = (\sqrt[n]{n} - 1)^n$.

Proof. Since we have

$$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = \limsup_{n \rightarrow \infty} (\sqrt[n]{n} - 1) = 0 < 1,$$

the series $\sum a_n$ converges by the root test. ■

(d) $a_n = \frac{1}{1+z^n}$ for complex z .

Proof. Notice that if $|z| < 1$, then $a_n \rightarrow 1 \neq 0$, so the sum does not converge. If $|z| > 1$, notice that the ratios

$$\left| \frac{a_n}{a_{n-1}} \right| = \left| \frac{1+z^{n-1}}{1+z^n} \right| \rightarrow \left| \frac{1}{z} \right| < 1$$

so $\sum a_n$ converges by the ratio test. If $|z| = 1$, the sum diverges. ■

7. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$, if $a_n \geq 0$.

Proof. We consider the ratio:

$$\left| \frac{b_n}{b_{n-1}} \right| = \left| \frac{n-1}{n} \cdot \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}} \right| < \left| \frac{\sqrt{a_n}}{\sqrt{a_{n-1}}} \right|$$

so

$$\limsup_{n \rightarrow \infty} \left| \frac{b_n}{b_{n-1}} \right| \leq \limsup_{n \rightarrow \infty} \sqrt{\frac{a_n}{a_{n-1}}} < 1$$

since $\sum a_n$ converges by the ratio test. ■

8. If $\sum a_n$ converges and $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof. Let $B = \sum b_n$. We can write

$$\sum a_n b_n = \sum a_n (b_n - B) + B \sum a_n$$

for which the first sum converges by Theorem 3.42, and the second sum converges as a scalar multiple of a convergent series. ■

9. Find the radius of convergence of the following power series:

- (a) $\sum n^3 z^n$.

Proof. In order for $\sum n^3 z^n$ to converge, we must have

$$\limsup_{n \rightarrow \infty} \left| \frac{n^3 z^n}{(n-1)^3 z^{n-1}} \right| = |z| < 1$$

so the radius of convergence is 1. ■

- (b) $\sum \frac{2^n}{n!} z^n$.

Proof. Using the ratio test, we must have

$$\limsup_{n \rightarrow \infty} \left| \frac{2^n (n-1)! z^n}{2^{n-1} n! z^{n-1}} \right| = \limsup_{n \rightarrow \infty} \frac{2}{n} |z| = 0 < 1$$

which holds regardless of z , so the radius of convergence is $+\infty$. ■

- (c) $\sum \frac{2^n}{n^2} z^n$.

Proof. Using the ratio test, we must have

$$\limsup_{n \rightarrow \infty} \left| \frac{2^n (n-1)^2 z^n}{2^{n-1} n^2 z^{n-1}} \right| = \limsup_{n \rightarrow \infty} 2 \left| \frac{(n-1)^2}{n^2} \right| |z| = 2|z| < 1$$

so the radius of convergence is $1/2$. ■

- (d) $\sum \frac{n^3}{3^n} z^n$.

Proof. By the ratio test, we must have

$$\limsup_{n \rightarrow \infty} \left| \frac{n^3 3^{n-1} z^n}{(n-1)^3 3^n z^{n-1}} \right| = \limsup_{n \rightarrow \infty} \frac{1}{3} \left| \frac{n^3}{(n-1)^3} \right| |z| = \frac{1}{3} |z| < 1$$

so the radius of convergence is 3. ■

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are non-zero. Prove that the radius of convergence is at most 1.

Proof. Let $\{b_n\}$ be the sequence of non-zero coefficients. Then $|b_n| \geq 1$ so $\sqrt[n]{|b_n|} \geq 1$. Thus any subsequential limit of the $\sqrt[n]{|b_n|}$ will converge to a value at least 1, so

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|} \geq 1$$

and thus the radius of convergence is at most 1. ■

11. Suppose $a_n > 0$ and $s_n = a_1 + \cdots + a_n$ and $\sum a_n$ diverges.

- (a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

Proof. We prove the contrapositive. Let $b_n = \frac{a_n}{1+a_n}$ and suppose $\sum b_n$ converges. Then we know $b_n \rightarrow 0$ so $1 - b_n = \frac{1}{1+a_n} \rightarrow 1$. Thus for $0 < \epsilon < \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that if $n \geq N$,

$$\left| \frac{1}{1+a_n} - 1 \right| = \left| \frac{b_n}{a_n} - 1 \right| < \frac{1}{2}$$

so $a_n < 2b_n$ if $n \geq N$. This implies that

$$\sum_{n=N}^m a_n < 2 \sum_{n=N}^m b_n < M$$

where $m \geq N$ and M is a bound on the partial sums of $\sum b_n$ which exists since $\sum b_n$ converges. So the partial sums of $\sum a_n$ monotonically increasing and bounded above, so $\sum a_n$ converges. ■

- (b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Proof. Notice that s_n is a monotonically increasing sequence since $a_n > 0$, so

$$\begin{aligned} \frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+2}}{s_{N+2}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + \frac{a_{N+2}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\ &= \frac{a_{N+1} + a_{N+2} + \cdots + a_{N+k}}{s_{N+k}} \\ &= 1 - \frac{s_N}{s_{N+k}} \end{aligned}$$

Then, letting $\epsilon = 1/2 > 0$, we can fix N and choose $k \geq N$ such that $s_{N+k} > 2s_N$ since the s_n are unbounded, so that

$$\frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+2}}{s_{N+2}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}} > \frac{1}{2}$$

so $\sum \frac{a_n}{s_n}$ is not Cauchy and thus diverges. ■

- (c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

Proof. We have $s_n > s_{n-1}$ so

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

so

$$\sum_{n=2}^m \frac{a_n}{s_n^2} \leq \sum_{n=2}^m \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{s_1} - \frac{1}{s_m} \rightarrow \frac{1}{a_1}$$

and thus $\sum \frac{a_n}{s_n^2}$ converges by the comparison test. ■

(d) What can be said about

$$\sum \frac{a_n}{1 + na_n} \text{ and } \sum \frac{a_n}{1 + n^2 a_n}?$$

Proof. The first series converges for some a_n and diverges for some other a_n . But $\frac{a_n}{1+n^2 a_n} < \frac{a_n}{n^2 a_n} = \frac{1}{n^2}$ so the second series converges by the comparison test and the p -test. ■

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Proof. Since $a_n > 0$, the sequence r_n is decreasing, so

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} > \frac{a_m + \cdots + a_{n-1}}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}.$$

Now for any $\epsilon > 0$, suppose $\epsilon < 1/2$ without loss of generality. Then, for any $n \in \mathbb{N}$, we can find $m \geq n$ such that $\frac{r_n}{r_m} < 1 - \epsilon$ since $r_n \rightarrow 0$, so $\sum_{k=m}^n \frac{a_k}{r_k} = 1 - \frac{r_n}{r_m} > \epsilon$. Then $\sum \frac{a_k}{r_k}$ is not Cauchy and thus diverges. ■

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Proof. Since $r_{n+1} < r_n$, we have $\frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2$. Reorganizing yields

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

since $a_n = r_n - r_{n+1}$. Now,

$$\left| \sum_{k=m}^n \frac{a_k}{\sqrt{r_k}} \right| < \left| 2 \sum_{k=m}^n (\sqrt{r_k} - \sqrt{r_{k+1}}) \right| = |2\sqrt{r_m} - 2\sqrt{r_n}|$$

which can be made arbitrarily small since $r_n \rightarrow 0$ so $\sqrt{r_n} \rightarrow 0$. Thus $\sum \frac{a_n}{\sqrt{r_n}}$ converges. ■

13. Let $\sum a_n$ and $\sum b_n$ converge absolutely. Then, letting $c_n = \sum_{k=1}^n a_k b_{n-k}$, prove that $\sum c_n$ converges absolutely.

Proof. Notice that

$$|c_n| = \left| \sum_{k=1}^n a_k b_{n-k} \right| \leq \sum_{k=1}^n |a_k| |b_{n-k}|.$$

and that letting d_n be the positive terms on the right side, $\sum d_n$ converges by Merten's theorem. Thus, $\sum c_n$ converges absolutely by the comparison test. ■

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means $\sigma_n = \frac{1}{n+1}(s_0 + \dots + s_n)$.

(a) If $s_n \rightarrow s$, prove that $\sigma_n \rightarrow s$.

Proof. Let $\epsilon > 0$ be arbitrary and $N_0 \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2$ for $n \geq N_0$. Notice then that

$$|\sigma_n - s| \leq \sum_{k=0}^{N_0} \frac{|s_k - s|}{n+1} + \sum_{k=N_0}^n \frac{|s_k - s|}{n+1} < \frac{S}{n+1} + \frac{\epsilon}{2}$$

where $S = \sum_{k=0}^{N_0} |s_k - s|$ is finite. Then, letting $N = \max(N_0, 2S/\epsilon)$, we notice that

$$|\sigma_n - s| \leq \frac{S}{n+1} + \frac{\epsilon}{2} \leq \epsilon$$

if $n \geq N$, so $\sigma_n \rightarrow s$. ■

(b) Construct a sequence $\{s_n\}$ which does not converge, but σ_n does.

Proof. Consider $s_n = (-1)^n$. Then clearly s_n does not converge as the partial sums alternate between 0 and 1, but

$$\sigma_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{1}{n+1} & \text{if } n \text{ odd} \end{cases}$$

does. ■

(c) Can it happen that $s_n > 0$ for all n and that $\limsup_{n \rightarrow \infty} s_n = +\infty$, but $\sigma_n \rightarrow 0$?

Proof. Yes, let $s_0 = 1$, then $s_n = \sqrt[3]{n} > 0$ if $\sqrt[3]{n}$ is an integer, otherwise let $s_n = 1/n^2 > 0$. Then,

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k < \frac{1}{n+1} \left(1 + 2n^{2/3} + \sum_{i=0}^n 1/i^2 \right) < 2n^{-1/3} + \frac{\pi^2}{6} n^{-1} \rightarrow 0$$

but for any $M \in \mathbb{R}$, there exists some integer $k > M$, and thus $s_{(k-1)^2} > M$, so $\limsup_{n \rightarrow \infty} s_n = +\infty$, as required. ■

(d) Put $a_n = s_n - s_{n-1}$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^n k a_k.$$

Assume that $na_n \rightarrow 0$ and $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. (Notice this provides a converse of (a) given $na_n \rightarrow 0$.)

Proof. We begin manipulating:

$$\begin{aligned}
 s_n - \sigma_n &= s_n - \frac{1}{n+1} \sum_{k=0}^n s_k \\
 &= \frac{n}{n+1} s_n - \frac{1}{n+1} \sum_{k=0}^{n-1} s_k \\
 &= \frac{1}{n+1} \left[- \sum_{k=1}^{n-1} s_k + n s_n - s_0 \right] \\
 &= \frac{1}{n+1} \left[\sum_{k=1}^n k s_k - \sum_{k=1}^n k s_{k-1} \right] \\
 &= \frac{1}{n+1} \sum_{k=1}^n k a_k.
 \end{aligned}$$

so clearly if $na_n \rightarrow 0$ and $\sigma_n \rightarrow \sigma$, then eventually $|\sigma_n - \sigma| < \epsilon/2$ and $\sum_{k=1}^n k a_k < (n+1)/2 \cdot \epsilon$, so $|s_n - \sigma| < \epsilon$ for any $\epsilon > 0$, and thus $\{s_n\}$ converges. ■

- (e) Prove (d) with the weaker hypothesis that na_n is bounded by $M < \infty$ and $\sigma_n \rightarrow \sigma$. Show that $s_n \rightarrow \sigma$, by completing the following outline:

- (i) If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^n (s_n - s_i).$$

Proof. We manipulate:

$$\begin{aligned}
 s_n &= \frac{1}{n-m} \sum_{k=0}^n s_k - \frac{1}{n-m} \sum_{k=0}^m s_k + \frac{n-m}{n-m} \cdot s_n - \frac{1}{n-m} \sum_{k=0}^n s_k + \frac{1}{n-m} \sum_{k=0}^m s_k \\
 &= \frac{1}{n-m} \left[\sum_{k=0}^n s_k - \sum_{k=0}^m s_k \right] + \frac{1}{n-m} \left[\sum_{k=0}^n (s_n - s_k) - \sum_{k=0}^m (s_n - s_k) \right] \\
 &= \frac{1}{n-m} [(n+1)\sigma_n - (m+1)\sigma_m] + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k) \\
 &= \frac{n+1}{n-m} \sigma_n - \frac{m+1}{n-m} \sigma_m + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k)
 \end{aligned}$$

so

$$\begin{aligned}
 s_n - \sigma_n &= \left(\frac{n+1}{n-m} - 1 \right) \sigma_n - \frac{m+1}{n-m} \sigma_m + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k) \\
 &= \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^n (s_n - s_i),
 \end{aligned}$$

as required. ■

- (ii) For these i ,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Proof. We can write $s_n - s_i$ as a telescoping sum of a_n :

$$|s_n - s_i| = \left| \sum_{k=i+1}^n (s_k - s_{k-1}) \right| \stackrel{\Delta}{\leq} \sum_{k=i+1}^n |a_k| \leq \sum_{k=i+1}^n \frac{M}{i+1} = \frac{(n-i)M}{i+1},$$

since we have $k \geq i+1$ and thus

$$|a_k| \leq \frac{M}{k} \leq \frac{M}{i+1}.$$

and the latter inequality follows immediately from the fact that $i \geq m+1$ in the previous part of the proof. ■

(iii) Fix $\epsilon > 0$ and associate to every n the integer m such that

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1.$$

Then $(m+1)/(n-m) \leq 1/\epsilon$ and $|s_n - s_i| < M\epsilon$ so $\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq M\epsilon$ so the result follows.

Proof. We have $m \leq \frac{n-\epsilon}{1+\epsilon}$, so $(m+1)/(n-m) \leq 1/\epsilon$ follows from manipulation. Similarly, we have $\frac{n-m-1}{m+2} < \epsilon$. Thus,

$$|s_n - s_i| \leq \frac{(n-m-1)M}{m+2} < M\epsilon$$

and thus

$$|s_n - \sigma_n| \leq \frac{m+1}{n-m} |\sigma_n - \sigma_m| + \frac{1}{n-m} |s_n - s_i| < M\epsilon$$

for sufficiently large m, n such that the first term vanishes (since $\sigma_n \rightarrow \sigma$). Thus

$$\limsup_{n \rightarrow \infty} |s_n - \sigma_n| \leq M\epsilon,$$

so s_n converges to σ since ϵ was arbitrary. ■

15. Extend Definition 3.21 to points in \mathbb{R}^k . Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, 3.55 are true in this more general setting.

Proof. Each proof is presented in a sufficiently general fashion such that replacing $|x - y|$ with the respective metric in \mathbb{R}^k suffices to prove each theorem for infinite sums in \mathbb{R}^k . ■

16. Fix $\alpha > 0$. Choose $x_1 > \sqrt{\alpha}$ and let

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $x_n \rightarrow \sqrt{\alpha}$.

Proof. Notice that $x_n > \sqrt{\alpha}$ for each n from the AM-GM inequality, and thus $\alpha/x_n < \sqrt{\alpha}$. Thus, $x_{n+1} < x_n$ and the sequence is monotonically decreasing and bounded below. Thus, it converges to some $L \geq \sqrt{\alpha}$. Then, notice that

$$L = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right) = \frac{1}{2}(L + \alpha/L)$$

which has solutions $\pm\sqrt{\alpha}$, so $\lim_{n \rightarrow \infty} x_n = \sqrt{\alpha}$, as required. ■

- (b) Put $\epsilon = x_n - \sqrt{\alpha}$ and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that if $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^n}.$$

Proof. We have

$$\frac{\epsilon_n^2}{2x_n} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{1}{2} \left(x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n} \right) = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right) - \sqrt{\alpha} = \epsilon_{n+1}.$$

and the second part of the inequality follows from the fact proven in part (a) that $x_n > \sqrt{\alpha}$ for all n . The second fact follows inductively by the above statement. ■

- (c) This presents a great algorithm for computing square roots. Show that if $\alpha = 3$ and $x_1 = 2$, that $\epsilon_1/\beta < 0.1$ so $\epsilon_5 < 4 \cdot 10^{-16}$ and $\epsilon_6 < 4 \cdot 10^{-32}$.

Proof. We have $\beta = 2\sqrt{3}$ and $\epsilon_1 = 2 - \sqrt{3}$ so

$$\frac{\epsilon_1}{\beta} = \frac{2 - \sqrt{3}}{2\sqrt{3}} = \frac{4 - 3}{2\sqrt{3}(2 + \sqrt{3})} < \frac{1}{10.5} < 0.1$$

since $\sqrt{3} < 1.5$. Then $\epsilon_5 < \beta \cdot 10^{-2^4} < 4 \cdot 10^{-16}$ and similarly $\epsilon_6 < \beta \cdot 10^{-2^5} < 4 \cdot 10^{-32}$ as required. ■

17. Fix $\alpha > 1$ and take $x_1 > \sqrt{\alpha}$ and

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

- (a) Prove that $x_1 > x_3 > x_5 > \dots$.
 (b) Prove that $x_2 < x_4 < x_6 < \dots$.

Proof. Manipulating, we have

$$x_{n+2} = \frac{\alpha + x_{n+1}}{1 + x_{n+1}} = \frac{(1 + x_n)(\alpha + x_{n+1})}{(1 + x_n)(1 + x_{n+1})} = \frac{\alpha + \alpha x_n + \alpha + x_n}{1 + x_n + \alpha + x_n} < x_n$$

if and only if

$$\alpha + \alpha x_n + \alpha + x_n < x_n + x_n^2 + \alpha x_n + x_n^2 \iff 0 < \alpha < x_n^2$$

so it suffices to show that $x_{2k+1} > \sqrt{\alpha}$ and $x_{2k} < \sqrt{\alpha}$ inductively to conclude both results.

Now, if $x_n > \sqrt{\alpha}$, then

$$x_{n+1} - x_n = \frac{\alpha - x_n^2}{1 + x_n} = (x_n - \sqrt{\alpha}) \frac{\sqrt{\alpha} + x_n}{1 + x_n} > x_n - \sqrt{\alpha}$$

since $\sqrt{\alpha} > 1$. Then $x_{n+1} < \sqrt{\alpha}$. The argument is symmetric for the other condition, completing the proof. ■

- (c) Prove that $x_n \rightarrow \sqrt{\alpha}$.

Proof. The subsequence $\{x_{2k+1}\}$ is decreasing and bounded below by $\sqrt{\alpha}$, and is thus convergent. It must converge to a value $L \geq \sqrt{\alpha}$ such that

$$L = \lim_{n \rightarrow \infty} x_{n+2} = \lim_{n \rightarrow \infty} \frac{\alpha + x_n}{1 + x_n} = \frac{\alpha + L}{1 + L},$$

whose solutions are $\pm\sqrt{\alpha}$. Thus, $L = \sqrt{\alpha}$. ■

- (d) Compare the rapidity of convergence of this process with the previous one.

Proof. TODO: It converges slower, but I'm not able to show this... ■

18. Replace the recursive definition in Exercise 16 by

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{1}{p} \cdot \frac{\alpha}{x_n^{p-1}}$$

where p is a fixed positive integer. Describe the behaviour of the resulting sequences $\{x_n\}$.

Proof. Notice that according to the definition, we must have $x_{n+1} \geq \sqrt[p]{\alpha}$ by the AM-GM inequality. Then let $x_1 > \sqrt[p]{\alpha}$ without loss of generality, then by a similar argument as above, $x_{n+1} < x_n$ so the sequence converges since it is monotonically decreasing and bounded below. It must converge to its unique fixed point at least $\sqrt[p]{\alpha}$, namely $x_n \rightarrow \sqrt[p]{\alpha}$. ■

19. Associate to each sequence $a = \{\alpha_n\} \in \{0, 2\}^{\mathbb{N}}$ the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is the Cantor set.

Proof. First, notice that for any real number $r \in [0, 1]$, we can associate a ternary representation $\{a_n\} \in \{0, 1, 2\}^{\mathbb{N}}$ such that $r = \sum_{n=0}^{\infty} a_n 3^{-n}$, so this question reduces to showing that r is in the Cantor set iff it contains no 1s in its ternary representation.

Notice that by construction, the middle thirds of each segment that we remove in the i -th iteration have a 1 as the i -th trit. So $\{0, 2\}^{\mathbb{N}} \supseteq C$. The converse follows again by noticing that any $r \in \{0, 2\}^{\mathbb{N}}$ is in each closed partial C_n , and thus in its infinite intersection C . ■

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p .

Proof. Let $\epsilon > 0$ be arbitrary and let $I \in \mathbb{N}$ such that $d(p_{n_i}, p) < \epsilon/2$ for all $i \geq I$. Also, let $N \in \mathbb{N}$ such that for all $n \geq m \geq N$, we have $d(p_n, p_m) < \epsilon/2$. Let N_1 be some n_i for $i \geq I$ such that $n_i \geq N$. Then for all $n \geq N_1$, we have $d(p_n, p) \leq d(p_n, p_{N_1}) + d(p_{N_1}, p) \leq \epsilon$ so $p_n \rightarrow p$ as required. ■

21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed non-empty and bounded sets in a complete metric space X , if $E_n \supseteq E_{n+1}$, and if $\text{diam } E_n \rightarrow 0$, then $\bigcap_{n=1}^{\infty} E_n$ contains a single point.

Proof. Let $p_n \in E_n$. Then $\{p_n, p_{n+1}, \dots\} \subseteq E_n$ so $\text{diam}\{p_n, p_{n+1}, \dots\} \rightarrow 0$, ie. $\{p_n\}$ is Cauchy. Since X is complete, $p_n \rightarrow p$ for some $p \in X$. But now for any n , the sequence $\{p_n, p_{n+1}, \dots\}$ converges to $p \in E_n$ since E_n is closed. Thus, $p \in \bigcap_{n=1}^{\infty} E_n$. Now p must be the unique point in $\bigcap_{n=1}^{\infty} E_n$ otherwise the diameters of E_n would not fall below some positive real value. ■

22. Suppose X is a non-empty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X . Prove Baire's theorem, namely that $\bigcap_{n=1}^{\infty} G_n$ is non-empty, and in fact dense in X .

Proof. Let S be a neighbourhood in X . We show that $S \cap \bigcap_{n=1}^{\infty} G_n$ is non-empty. We define a sequence $\{F_n\}$ of closed and bounded subsets of X : let $E_0 = S$, and construct E_n inductively as follows: take x_n from the dense open set $E_{n-1} \cap G_n$, and $\overline{E_n}$ some neighbourhood of x_n with $\overline{E_n} \subseteq G_n \cap E_{n-1}$. This defines a sequence $\{E_n\}$ such that $\overline{E_n} \subseteq S \cap \bigcap_{i=1}^n G_i$ inductively. Then, $\{F_n = \overline{E_n}\}$ forms a sequence of closed and bounded sets in the complete metric space X , so by the previous theorem, $\bigcap_{n=1}^{\infty} F_n \subseteq S \cap \bigcap_{n=1}^{\infty} G_n$ is non-empty, as required. ■

23. Suppose $\{p_n\}, \{q_n\}$ are Cauchy sequences in a metric space X . Show that the sequence $\{d(p_n, q_n)\}$ converges.

Proof. Let $\epsilon > 0$ be arbitrary. Let $N_p \in \mathbb{N}$ such that if $n \geq m \geq N_p$, then $d(p_n, p_m) < \epsilon/2$. Define N_q similarly. Then, for any $n \geq m \geq \max(N_p, N_q)$, we have

$$d(p_n, q_n) \leq d(p_m, q_m) + d(p_n, p_m) + d(q_n, q_m) < d(p_m, q_m) + \epsilon,$$

and similarly $d(p_m, q_m) \leq d(p_n, q_n) + \epsilon$, so $|d(p_n, q_n) - d(p_m, q_m)| < \epsilon$, so the sequence $\{d(p_n, q_n)\}$ is Cauchy in \mathbb{R} and thus convergent since \mathbb{R} is complete. ■

24. Let X be a metric space.

- (a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ equivalent if $d(p_n, q_n) \rightarrow 0$. Show that this is an equivalence relation.

Proof. Clearly $d(p_n, p_n) \rightarrow 0$ so $\{p_n\} \sim \{p_n\}$. Also, since $d(p_n, q_n) = d(q_n, p_n)$, the relation is symmetric. Finally, if $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$, then $0 \leq d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n) \rightarrow 0$ converges to 0 by the comparison test. Thus the relation is transitive. These properties together make \sim an equivalence relation on the Cauchy sequences in X . ■

- (b) Let X^* be the set of equivalence classes under the above relation. If $P, Q \in X^*$ and $\{p_n\} \in P, \{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n \rightarrow \infty} d(p_n, q_n),$$

which exists from Exercise 23. Show that the number $\Delta(P, Q)$ is unchanged if we replace $\{p_n\}$ or $\{q_n\}$ by equivalent sequences, so that Δ is a distance function in X^* .

Proof. Suppose $D = \Delta(P, Q) = \lim_{n \rightarrow \infty} d(p_n, q_n)$ for Cauchy sequences $\{p_n\} \in P, \{q_n\} \in Q$. Suppose that $\{r_n\} \in P$. Then, notice that

$$d(p_n, q_n) - d(p_n, r_n) \stackrel{\Delta}{\leq} d(r_n, q_n) \stackrel{\Delta}{\leq} d(p_n, r_n) + d(p_n, q_n)$$

so $\lim_{n \rightarrow \infty} d(r_n, q_n) = D$ by the Squeeze theorem so we are done. ■

- (c) Prove that the resulting metric space X^* is complete.

Proof. It is always true that convergent sequences are Cauchy. It suffices to show that Cauchy sequences in X^* converge in X^* . Suppose that $\{P_n\}$ is Cauchy with respect to the distance metric Δ . Then, for each n , there exists a Cauchy sequence in Q_{kn} such that $\lim_{k \rightarrow \infty} \Delta(P_n, Q_{kn}) = 0$. Then, for each n , we extract a subsequence Q'_n of Q_{kn} such that $\Delta(P_n, Q'_n) < 1/n$ for all n . Then,

$$\Delta(Q'_n, Q'_m) \stackrel{\Delta}{\leq} \Delta(P_n, Q'_n) + \Delta(P_n, P_m) + \Delta(P_m, Q'_m) < 1/n + 1/m + \epsilon < 3\epsilon$$

for sufficiently large m, n , where $\epsilon > 0$ can be made arbitrarily small. Thus $\{Q'_n\}$ is Cauchy. Also, if $\{Q'_n\} \in P$, then

$$\Delta(P, P_n) \leq \Delta(P, Q'_n) + \Delta(Q'_n, P_n) < 2\epsilon$$

so $P_n \rightarrow P$ is convergent, as required. ■

- (d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p . Let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$, so that $\phi : p \mapsto P_p$ is an **isometry** (a distance-preserving mapping) of X onto X^* .

Proof. This is trivial if we take the representative sequence to be the constant sequence of p and q : the limit is that of a constant sequence. ■

- (e) Prove that $\phi(X)$ is dense in X^* , and that $\phi(X) = X^*$ if X is complete. By (d), we may identify X and $\phi(X)$ and thus regard X as embedded in the complete metric space X^* . We call X^* the **completion** of X .

Proof. Take $P \in X^*$ and $\{p_n\} \in P$ with $p_n \in X$. Then $P_{p_n} \rightarrow P$ since $\{p_n\}$ is Cauchy, so $\phi(X)$ is dense in X^* . Similarly, if X were complete, then every element $P \in X^*$ admits a constant sequence, so $X^* \subseteq \phi(X)$ completing the proof. ■

25. Let X be a metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24).

Proof. The completion is \mathbb{R} ! Clearly $\mathbb{Q}^* \subseteq \mathbb{R}$ by definition, and every element $r \in \mathbb{R}$ has some Cauchy sequence in \mathbb{Q} which converges to it, namely the one generated by its decimal expansion. ■

4 Continuity

4.1 Limits of Functions

Definition 30. Let X, Y be metric spaces and $E \subseteq X$. Let f map E into Y , and p be a limit point of E . We write $f(x) \rightarrow q$ as $x \rightarrow p$, or equivalently

$$\lim_{x \rightarrow p} f(x) = q$$

iff there is a point $q \in Y$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ such that $0 < d_X(x, p) < \delta$.

Note that $p \in X$, but that p need not be in E . Also, even if $p \in E$, we do not necessarily have $f(p) = \lim_{x \rightarrow p} f(x)$. This property is interesting and we will formalize it later.

Theorem 69. Let X, Y, E, f, p as in Definition 4.1. Then, $f(x) \rightarrow q$ when $x \rightarrow p$ iff $f(p_n) \rightarrow q$ for every sequence $\{p_n\}$ in $E \setminus \{p\}$ which converges to p .

Proof. Suppose that $f(x) \rightarrow q$ as in Definition 4.1. Then, let $\epsilon > 0$ be arbitrary, and let δ such that Definition 4.1 holds. Then, if $\{p_n\}$ is some sequence converging to p , then eventually $d_X(p_n, p) < \delta$, so $d_Y(f(p_n), q) < \epsilon$ eventually. Thus $f(p_n) \rightarrow q$.

Now suppose that $f(x) \nrightarrow q$ as in Definition 4.1. Then, there exists some $\epsilon > 0$ such that for every $\delta > 0$, we can find some $x \in E$ such that $0 < d_X(x, p) < \delta$ but $d_Y(f(x), q) \geq \epsilon$. Let $\delta = 1/n$ for every n , giving us some sequence x_n such that $d_Y(f(x_n), q) \geq \epsilon$. Thus $f(x_n) \nrightarrow q$, proving the contrapositive. ■

Corollary 8. If f has a limit at p , then this limit is unique.

Definition 31. If f, g are complex functions, then we denote $f + g$ to be the function such that $(f + g)(x) = f(x) + g(x)$. We define $f - g$, fg and f/g similarly. If $f(x) = c$ for all x , then we say f is **constant**, and write $f = c$. If $f(x) \geq g(x)$ for all x , we sometimes write $f \geq g$.

Theorem 70. Let $E \subseteq X$ a metric space, p a limit point of E , and f and g complex functions on E . Furthermore, let $A = \lim_{x \rightarrow p} f(x)$ and $B = \lim_{x \rightarrow p} g(x)$. Then,

1. $\lim_{x \rightarrow p} (f + g)(x) = A + B$.
2. $\lim_{x \rightarrow p} (fg)(x) = AB$.
3. $\lim_{x \rightarrow p} (f/g)(x) = A/B$ if $B \neq 0$.
4. If f and g map into \mathbb{R}^k , then additionally $\lim_{x \rightarrow p} (f \cdot g)(x) = A \cdot B$.

Proof. This follows immediately from the analogous properties of sequences. ■

4.2 Continuous Functions

Definition 32. Suppose X, Y are metric spaces, $E \subseteq X$, $p \in E$, and $f : E \rightarrow Y$. Then, f is **continuous** at p if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \epsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$. If f is continuous at every point in E , then f is **continuous on E** .

Notice that every function f is continuous at isolated points of p , since we can always choose $\delta > 0$ sufficiently small such that $E \cap N_\delta(x) = \{p\}$ so $d_Y(f(x), f(p)) = 0 < \epsilon$.

Theorem 71. Let everything as Definition 4.5. If p is a limit point of E , then f is continuous at p iff $f(x) \rightarrow f(p)$ as $x \rightarrow p$.

Theorem 72. Suppose X, Y, Z are metric spaces, $f : (E \subseteq X) \rightarrow Y$ and $g : f(E) \rightarrow Z$, and $h = g \circ f$. Then, if f is continuous at $p \in E$ and g is continuous at $f(p)$, then h is continuous at p .

Proof. Let $\epsilon > 0$. Since g is continuous at $f(p)$, there exists some $\delta > 0$ such that $d_Z(g(f(x)), g(f(p))) < \epsilon$ for all $f(x) \in f(E)$ such that $0 < d_Y(f(x), f(p)) < \delta$. However, since f is continuous, we can find $\delta' > 0$ such that $d_Y(f(x), f(p)) < \delta$ whenever $0 < d_X(x, p) < \delta'$. In other words, we've found $\delta' > 0$ such that $d_Z(h(x), h(p)) < \epsilon$ whenever $0 < d_X(x, p) < \delta'$, so h is continuous at p as required. ■

Theorem 73. A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y .

Proof. Suppose f is continuous on X . Then, if $y \in V$ for some open V , then there exists some neighbourhood $N_\epsilon(y)$ around y . By the definition of continuity, this admits $N_\delta(f^{-1}(y))$ in $f^{-1}(V)$ for positive δ , so $f^{-1}(V)$ is open in X , so the forward direction is done since V was chosen arbitrarily.

Now for the contrapositive, suppose $f^{-1}(V)$ is open for every open subset V of Y . Then, for any $x \in X$, for any $\epsilon > 0$, the set $V = N_\epsilon(f(x))$ is an open set in Y , so $f^{-1}(V)$ is open by assumption. Therefore, $f^{-1}(f(x)) = x$ is an interior point of $f^{-1}(V)$, so in particular, this gives us the $\delta > 0$ we need for continuity. ■

Corollary 9. A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y .

Theorem 74. If f, g are continuous complex functions on X , then $f + g, fg$ and f/g are continuous on X .

Proof. These follow directly from the corresponding theorems on sequences, and the equivalence of the sequential definition of continuity. ■

Theorem 75. Let f_1, f_2, \dots, f_k be real functions on X and \mathbf{f} be the mapping of X into \mathbb{R}^k defined by

$$\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_k(x)).$$

then \mathbf{f} is continuous iff each of the functions f_1, \dots, f_k is continuous.

Furthermore, if $\mathbf{f}, \mathbf{g} : X \rightarrow \mathbb{R}^k$ are continuous, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous.

Proof. Part (a) follows from the fact that

$$|f_i(x) - f_i(y)| \leq |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{j=1}^k |f_j(x) - f_j(y)|^2}, \forall i \in \{1, 2, \dots, k\},$$

and part (b) follows from (a) and Theorem 4.9. ■

Now, we show that a very large class of functions is continuous. Firstly, the component function $f((x_1, \dots, x_k)) = x_i$ is continuous for any $i \in \{1, 2, \dots, k\}$. This shows that polynomials in the components are continuous, and thus rational functions on (x_i) are continuous. Finally, it can be seen that $f(\mathbf{x}) = |\mathbf{x}|$ is continuous.

4.3 Continuity and Compactness

Definition 33. A mapping $f : E \rightarrow \mathbb{R}^k$ is **bounded** if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

Theorem 76. Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then $f(X)$ is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of $f(X)$. Then, since f is continuous, each $f^{-1}(G_\alpha)$ is open, forming an open cover for X . Since X is compact,

$$X \subseteq \bigcup_{i=1}^n f^{-1}(G_i)$$

for some finite subset $\{G_i\}$ of $\{G_\alpha\}$. Then, since $f(f^{-1}(E)) \subseteq E$ for all $E \subseteq X$, we have $f(X) = \bigcup_{i=1}^n G_i$, so $f(X)$ is compact. ■

Theorem 77. If $f : X \rightarrow \mathbb{R}^k$, then $f(X)$ is closed and bounded. Thus, f is bounded.

Proof. By Theorem 4.14, $f(X)$ is compact. Thus it is closed and bounded. In particular it is bounded. ■

Theorem 78 (Extreme Value Theorem). Suppose f is a continuous real function on a compact metric space X , and

$$M = \sup_{p \in X} f(p), \quad m = \inf_{p \in X} f(p),$$

then f attains M and m .

Proof. From Theorem 4.14, $f(X)$ is compact and thus closed and bounded. Hence $f(X)$ contains M and m , proving the result. ■

Theorem 79. Suppose f is a continuous bijection from a compact metric space X onto a metric space Y . Then, f^{-1} is a continuous mapping of Y onto X .

Proof. To use Theorem 4.8, it suffices to show that $f(V)$ is open for every open set V in X . But V^c is closed and thus compact in X , and thus $f(V^c)$ is compact and thus closed in Y . Since f is bijective, $f(V)$ is the complement of $f(V^c)$, thus $f(V)$ is open. ■

Definition 34. A mapping $f : X \rightarrow Y$ is **uniformly continuous** on X if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \epsilon, \forall p, q \in X : d_X(p, q) < \delta.$$

Theorem 80. If f is a continuous mapping from a compact metric space X into a metric space Y , then f is uniformly continuous on X .

Proof. We prove the contrapositive: suppose f is not uniformly continuous on X . Then, for some $\epsilon > 0$, we can find $p_n, q_n \in X$ such that $d_X(p_n, q_n) < 1/n$ but $d_Y(f(p_n), f(q_n)) > \epsilon$. Then, since X is compact, we can find a subsequence (p_{n_k}) which converges, then take a subsequence $(q_{n_{k_m}})$ of (q_{n_k}) which converges, again since X is compact. We relabel these subsequences (r_n) and (s_n) . By the Squeeze Theorem, $r_n, s_n \rightarrow x_0$ for some $x_0 \in X$. However, $d_Y(f(r_n), f(s_n)) > \epsilon$, so f cannot be continuous at x_0 by the sequential definition of continuity, completing the proof. ■

Theorem 81. Let E be a non-compact set in \mathbb{R} . Then:

1. There exists a continuous unbounded function on E .
2. There exists a continuous and bounded function on E which does not attain its maximum.
3. If E is bounded, there exists a continuous function on E which is not uniformly continuous.

Proof. For (a), if E is unbounded, then $f(x) = x$ is continuous but unbounded. Otherwise, E is not closed, so there exists a limit point x_0 of E which is not in E . Then $f(x) = 1/(x - x_0)$ is continuous but unbounded (consider the sequence converging to x_0).

For (b), if $f(x)$ is the function constructed in part (a), then $g(x) = -1/|f(x)|$ is bounded above by 0 and continuous, but never attains its maximum of 0.

For (c), notice E is thus not closed. Then $f(x)$ from part (a) is not uniformly continuous. ■

4.4 Continuity and Connectedness

Theorem 82. If $f : X \rightarrow Y$ and E is a connected subset of X , then $f(E)$ is connected.

Proof. We prove the contrapositive: suppose $f(E) = A \cup B$ and A, B are non-empty separated sets. Then, letting $A' = E \cap f^{-1}(A)$ and $B' = E \cap f^{-1}(B)$, we see that $E = A' \cup B'$.

Now

$$\overline{A'} \cap B' \subseteq E \cap f^{-1}(\overline{A}) \cap f^{-1}(B) = \emptyset$$

since $\overline{A} \cap B = \emptyset$, and repeating the argument for A' and $\overline{B'}$ shows that (A', B') is a separation of E , proving the contrapositive. ■

Theorem 83 (Intermediate Value Theorem). Let f be a continuous real function on the interval $[a, b]$. If $f(a) < c < f(b)$, then $f(x) = c$ for some $x \in (a, b)$.

Proof. If not, then $f([a, b]) \cap (f(a), c)$ and $f([a, b]) \cap (c, f(b))$ form a disconnection of $f([a, b])$, contradicting Theorem 4.22. ■

4.5 Discontinuities

Definition 35. If $f : X \rightarrow Y$ and $x \in X$ is such that f is not continuous at x , we say f is **discontinuous** at x , or that f **has a discontinuity** at x .

Let $f : (a, b) \rightarrow Y$. Let $x \in [a, b)$. We define $f(x+) = q$ if $f(t_n) \rightarrow q$ for any sequence $\{t_n\}$ in (x, b) with $t_n \rightarrow x$. We define $f(x-)$ symmetrically. Clearly $\lim_{t \rightarrow x} f(t)$ exists iff $f(x-) = f(x+) = \lim_{t \rightarrow x} f(t)$.

Definition 36. Let $f : (a, b) \rightarrow Y$. If f is discontinuous at x , but $f(x-)$ and $f(x+)$ both exist, f is said to have a **discontinuity of the first kind**, or a **simple discontinuity** at x . Otherwise, the discontinuity is said to be of the **second kind**.

If f has a simple discontinuity at x , either $f(x+) \neq f(x-)$, or $f(x+) = f(x-) \neq f(x)$.

4.6 Monotonic functions

Definition 37. Let $f : (a, b) \rightarrow \mathbb{R}$. Then f is **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If instead $f(x) \geq f(y)$ with the same conditions, then f is **monotonically decreasing**.

Theorem 84. Let f be monotonically increasing on (a, b) . Then $f(x+)$ and $f(x-)$ exist at every point $x \in (a, b)$.

Proof. Every sequence converging to x from the left has a monotonically increasing subsequence. Then $f(x_n)$ is monotonically increasing and bounded above, say by $f(x + \delta)$. Thus $f(x_n)$ converges by the Monotone Convergence Theorem. The argument for $f(x+)$ is analogous. ■

Corollary 10. Monotonic functions have no discontinuities of the second kind.

Theorem 85. Let f be monotonic on (a, b) . Then the set of discontinuities of f in (a, b) is at most countable.

Proof. We associate each discontinuity x in (a, b) with a rational number r such that $f(x-) < r < f(x+)$, forming an injection from the set of discontinuities to \mathbb{Q} , proving the result. ■

4.7 Infinite Limits and Limits at Infinity

Definition 38. The set $(c, +\infty)$ is considered a neighbourhood of $+\infty$. Similarly for $(-\infty, c)$ being neighbourhoods of $-\infty$.

Definition 39. Let $f : (E \subseteq \mathbb{R}) \rightarrow Y$. We say $f(t) \rightarrow y$ as $t \rightarrow x$, where $A, x \in \overline{\mathbb{R}}$, if for every neighbourhood U of y , there is a neighbourhood V of x such that $V \cap E$ is non-empty, and $f(t) \in U$ for all $t \in V \cap E \setminus \{x\}$. Limits of functions still combine in the same ways.

4.8 Exercises

1. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ with $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$ for all $x \in \mathbb{R}$. Is f continuous?

Proof. No, consider the function f which is 1 at 0 and 0 everywhere else. Then f satisfies the condition everywhere but is discontinuous at $x = 0$. ■

2. Let $f : X \rightarrow Y$ be continuous. Prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every $E \subseteq X$. Show that this inequality can be proper.

Proof. Let E be arbitrary and $x \in \overline{E}$. If $x \in E$, then clearly $f(x) \in f(E) \subseteq \overline{f(E)}$. Otherwise, x is a limit point of E , so some sequence (x_n) in E converges to x , so since f is continuous, the sequence $f(x_n) \rightarrow f(x)$, and thus $f(x) \in \overline{f(E)}$, as required.

Let $f(x) = 1$ for all $x < 1$ and $f(x) = 1/x$ if $x > 1$. Then f is continuous on \mathbb{R} , but $0 \in \overline{f(\mathbb{R})}$ but is not in $f(\mathbb{R})$, so the inclusion can be proper. ■

3. Let $f : X \rightarrow \mathbb{R}$ be continuous. Let $Z(f)$ be the kernel of f . Prove that $Z(f)$ is closed.

Proof. Let $x_n \rightarrow x$ be a convergent sequence in $Z(f)$. Then $f(x_n) = 0$ for all n , and thus $f(x_n) \rightarrow 0$. Since f is continuous, this implies that $f(x) = 0$, and thus that $x \in Z(f)$, completing the proof. ■

4. Let $f, g : X \rightarrow Y$ be continuous, and E be dense in X . Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

4 Continuity

Proof. Let $V \subseteq f(X)$ be open. Since f is continuous, $f^{-1}(V)$ is open, and since E is dense, we can find $p \in E \cap f^{-1}(V)$, which corresponds to $p \in E$ such that $f(p) \in V$, so in particular, $f(p)$ is an element of $f(E)$ in V . Since V was arbitrary, $f(E)$ is dense in $f(X)$.

Notice that E is a dense subset of $Z = Z(g - f)$. Thus, $\bar{E} = X \subseteq \bar{Z} \subseteq X$, so in particular, $X = Z$ since Z is closed from the previous Exercise, completing the proof. ■

5. Let $f : (E \subseteq \mathbb{R}) \rightarrow \mathbb{R}$ be continuous, and E be closed. Prove that there exist continuous functions $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g = f$ on E . These are called **continuous extensions** of f from E to \mathbb{R} . Show that the result becomes false if E is not closed, and extend the result to vector-valued functions.

Proof. We know that E^c is open and thus a countable union of disjoint open segments of \mathbb{R} . For each of these segments (a, b) , assign $g(at + b(1 - t)) = tf(a) + (1 - t)f(b)$, and $g(x) = f(x)$ on E . Then $g = f$ on E , and g is continuous at the endpoints a and b , as required.

Now, if $E = \{\pm 1/n : n \in \mathbb{N}^+\}$. Then let $f(1/n) = 1 + 1/n$ and $f(-1/n) = -1 + 1/n$. Then the sequences $f(1/n) \rightarrow 1$ and $f(-1/n) \rightarrow -1$ while both $\pm 1/n \rightarrow 0$. Thus any g which agrees with f on E must be discontinuous at 0, and thus no such continuous extension of f exists. ■

6. If f is defined on E , the **graph** of f is the set of points $(x, f(x))$ for $x \in E$. Suppose E is compact. Prove that f is continuous on E iff its graph is compact.

Proof. First, suppose f is continuous. Let $\{G_\alpha\}$ be an open cover of $(E, f(E))$. Then since f is continuous, E and $f(E)$ are both compact, so we can choose elements $\{G_n\}$ such that the restrictions of the G_n to X cover E , and similarly G_m to cover $f(E)$. These two sets combined give a finite cover for the graph of f .

Now suppose the graph of f is compact. Then [**TODO**] ■

7. If $E \subseteq \mathbb{R}^2$ and f is defined on E , the **restriction** of f to E is the function whose domain is E , such that $g(p) = f(p)$ for $p \in E$. Define $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f(0, 0) = g(0, 0) = 0$, $f(x, y) = xy^2/(x^2 + y^4)$, $g(x, y) = xy^2/(x^2 + y^6)$ otherwise. Prove that f is bounded on \mathbb{R}^2 , and that g is unbounded in every neighbourhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; but nevertheless that the restrictions of both f and g to every straight line in \mathbb{R}^2 are continuous!

Proof. Notice that $\frac{ab}{a^2 + b^2} \leq \frac{1}{2}$ since $(a - b)^2 = a^2 - 2ab + b^2 \geq 0$ for any a, b . Thus f is bounded. Now, we have $g(1/n^3, 1/n) = n/2$ so g is unbounded in any neighbourhood of $(0, 0)$. Finally, $f(1/n^2, 1/n) = 1/2$ for any n , so f is discontinuous at $(0, 0)$.

However, if $x = cy$, $f(cy, y) = \frac{cy^3}{c^2y^2 + y^4} = \frac{cy}{c^2 + y^2} \rightarrow 0$ as $y \rightarrow 0$, so f is continuous on the line $x = cy$. On any other line not passing through $(0, 0)$, the restriction is purely within the domain of the continuous definition. Similarly, $g(cy, y) = \frac{cy}{c^2 + y^4} \rightarrow 0$ as $y \rightarrow 0$, so g is continuous on any straight line. ■

8. Let $f : E \rightarrow \mathbb{R}$ be uniformly continuous on a bounded set $E \subseteq \mathbb{R}$. Prove that f is bounded on \mathbb{R} .

Proof. Suppose not, and let x_n be a sequence in E such that $f(x_n) \geq n$ for all $n \in \mathbb{N}$. Then, (x_n) admits a convergent subsequence (x_{n_k}) , which converges to x_0 , which is not necessarily in E . Then any sufficiently small neighbourhood of x_0 contains elements $f(x_{n_k})$ which are arbitrarily far apart, contradicting the fact that f is uniformly continuous, proving the result. ■

9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\epsilon > 0$, there exists $\delta > 0$ such that $\text{diam } f(E) < \epsilon$ for all $E \subseteq X$ with $\text{diam } E < \delta$.

Proof. Suppose the new definition holds. Then, let $\epsilon > 0$ be arbitrary and δ as in the definition. Then, for any two $x, y \in E$ such that $d_X(x, y) < \delta$, the set $E = \{x, y\}$ has diameter strictly less than δ , and thus $\text{diam } f(\{x, y\}) < \epsilon$, corresponding to the fact that $d_Y(f(x), f(y)) < \epsilon$.

Now, suppose the old definition holds, and $\epsilon > 0$ be arbitrary. Let $\delta > 0$ as in the old definition with $\epsilon/2 > 0$. Furthermore, let E be a set such that $\text{diam } E < \delta$. Now, for any pair $x, y \in E$, we have $d_X(x, y) \leq \text{diam } E < \delta$, so $d_Y(f(x), f(y)) < \epsilon$. Thus, $\text{diam } f(E) = \sup\{d_Y(f(x), f(y)) : x, y \in E\} \leq \epsilon/2 < \epsilon$, as required. ■

10. Complete the details of the following alternative proof of Theorem 4.19.

Proof. Plot twist: I accidentally wrote this proof as I went along proving while copying over the notes. ■

11. Suppose $f : X \rightarrow Y$ is uniformly continuous, and let $\{x_n\}$ be Cauchy in X . Prove that $\{f(x_n)\}$ is Cauchy in Y . Use this result to give an alternative proof of the theorem stated in Exercise 13.

Proof. Let $E_n = \{x_n, x_{n+1}, \dots\}$. Let $\epsilon > 0$. We want to show that $\text{diam } f(E_n) < \epsilon$ for sufficiently large N . Choose $\delta > 0$ such that the definition from Exercise 4.9 holds. Then choose $N \in \mathbb{N}$ such that $\text{diam } E_n < \delta$ for all $n \geq N$, which exists since $\{x_n\}$ is Cauchy. Then, for any $n \geq N$, we have $\text{diam } E_n < \delta$ and thus $\text{diam } f(E_n) < \epsilon$ as required. ■

12. A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.

Proof. Let X, Y, Z be metric spaces and $f : (E \subseteq X) \rightarrow Y$ and $g : Y \rightarrow Z$ be uniformly continuous. Let $h = g \circ f$. Then h is uniformly continuous on E .

Let $\epsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $\text{diam } E < \delta$ implies $\text{diam } g(E) < \epsilon$. Furthermore, there exists $\nu > 0$ such that $\text{diam } F < \nu$ implies $\text{diam } f(F) < \delta$. Then if $\text{diam } F < \nu$, then $\text{diam } f(F) < \delta$ and thus $\text{diam } h(F) = \text{diam}(g \circ f)(F) < \epsilon$, as required. ■

13. Let E be a dense subset of a metric space X , and let $f : E \rightarrow \mathbb{R}$ be uniformly continuous. Prove that f has a continuous extension from E to X . Could the range space \mathbb{R} be replaced by \mathbb{R}^k ? By any compact metric space? By any complete metric space? By any metric space?

Proof. We define $g : X \rightarrow Y$ such that $g = f$ on E , and $g(x) = \lim_{n \rightarrow \infty} f(x_n)$ for any sequence $x_n \rightarrow x$. We must show that this definition is well defined: specifically that the right hand side does not depend on our choice of x_n .

Let $\{x_n\}$ and $\{y_n\}$ be two sequences which converge to $x \in X$, and let $\epsilon > 0$ be arbitrary. We wish to show that $\lim_{n \rightarrow \infty} d_Y(f(x_n), f(y_n)) < \epsilon$. Now since f is uniformly continuous, there exists some $\delta > 0$ such that if $d_X(x_n, y_n) < \delta$, then $d_Y(f(x_n), f(y_n)) < \epsilon$. Since $x_n, y_n \rightarrow x$, we can choose $N \in \mathbb{N}$ sufficiently large so that $d_X(x_n, x), d_X(y_n, x) < \delta/2$, such that $d_X(x_n, y_n) \overset{\Delta}{<} \delta$, and thus $d_Y(f(x_n), f(y_n)) < \epsilon$. The result follows. ■

14. Let I be the closed unit interval, and $f : I \rightarrow I$ be continuous. Prove that $f(x) = x$ for at least one $x \in I$.

Proof. If $f(0) = 0$ or $f(1) = 1$, we are done. Otherwise, $f(0) > 0$ and $f(1) < 1$, so the function $g(x) = f(x) - x$ is positive at 0 and negative at 1. By the Intermediate Value Theorem, $g(x) = f(x) - x = 0$ so $f(x) = x$ for some $x \in (0, 1) \subseteq I$, as required. ■

15. A mapping $f : X \rightarrow Y$ is **open** if $f(V)$ is open in Y whenever V is open in X . Prove that every continuous open mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ is monotonic.

Proof. Suppose the contrary, specifically that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a non-monotonic continuous open mapping. Then there exists a segment (a, b) such that either the supremum or the infimum of f on (a, b) is not achieved at either endpoint. Then, by the Extreme Value Theorem, f attains this extremum within the segment, say at $c \in (a, b)$. Then, $f((a, b))$ is not open, as it contains no neighbourhoods of $f(c)$, contradicting the fact that f is open. ■

16. Let $[x]$ denote the floor of x , and $\{x\} = x - [x]$. What discontinuities do $[x]$ and $\{x\}$ have?

Proof. They are both discontinuous at the integers. ■

17. Let $f : (a, b) \rightarrow \mathbb{R}$. Prove that the set of points at which f has a simple discontinuity is at most countable.

Proof. Following the hint, let $E_<$ be the set of all simple discontinuities x of f such that $f(x-) < f(x+)$. Then, associate to each $x \in E_<$ a triple of rational numbers (p, q, r) such that $f(x-) < p < f(x+)$, $a < q < t < x$ implies $f(t) < p$, and $x < t < r < b$ implies $f(t) > p$. We claim that every triple represents a unique simple discontinuity of f :

Suppose that x and x' are simple discontinuities and $x < x'$ WLOG, both represented by the triples (p, q, r) . Then in the interval (x, x') lie in both the intervals (x, r) and (q, x') , and thus $f(t)$ must be both less than, and greater than, p at the same time, a contradiction. Thus, we must have $x = x'$, as required.

We can define $E_>$ and associate triples a similar way. For $E_>$, we associate to each discontinuity (p, q, r) where p is between $f(x)$ and $f(x-) = f(x+)$, q such that $f(t) < p$ in (q, x) and (x, r) . Again, these triples map uniquely from the discontinuities in $E_>$. Since each of these sets must be at most countable, their union is also at most countable. ■

18. Every rational x can be written as $x = m/n$ with $n > 0$ and $(m, n) = 1$. When $x = 0$, take $n = 1$. Then, consider $f(x) = 1/n$ if x is rational, and 0 otherwise. Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Proof. Let $x \in \mathbb{R}$. For sufficiently small neighbourhoods of x , $N_\delta(x)$ contains no fractions of the form m/n for some $n < N$. Thus, $f(x_k) = 1/n < 1/N$ for x_k sufficiently close to x . Thus, every sequence converging to x has $f(x_k) \rightarrow 0$. Thus, f is continuous at x iff $f(x) = 0$, iff x is irrational. Otherwise, sequences from both sides converge to 0, so f has simple discontinuities at the rationals. ■

19. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ has the intermediate value property, and that $f^{-1}(\{r\})$ is closed for all $r \in \mathbb{Q}$. Prove that f is continuous.

Proof. First, we show that there cannot be any sequence $\{x_n\}$ converging to any x such that $f(x_n) > r > f(x)$ for some $r \in \mathbb{Q}$ and all $n \in \mathbb{N}$, otherwise there exist t_n between x_n and x for each n such that $f(t_n) = r$, and thus $r > f(x) = r$ since $f^{-1}(\{r\})$ is closed, a contradiction. Thus, if $r < f(x) < s$ for $r, s \in \mathbb{Q}$ and $\{x_n\}$ is a sequence converging to x , there must only be finitely many terms x_{n_k} not in (r, s) , and thus $f(x_{n_k}) \rightarrow f(x)$, so f is continuous. ■

20. If E is a non-empty subset of a metric space X , define the distance from $x \in X$ to E by $\rho_E(x) = \inf_{z \in E} d(x, z)$.

a) Prove that $\rho_E(x) = 0$ iff $x \in \bar{E}$.

Proof. If $x \in \bar{E}$, then either $x \in E$ and the result is trivial, or x is a limit point of E and thus there is a sequence $\{x_n\}$ which gets arbitrarily close to x . Conversely, if $\rho_E(x) = 0$, then there exists some sequence $\{x_n\}$ in E such that $d(x, x_n) < 1/n$ for every n , so $x_n \rightarrow x$ and thus $x \in \bar{E}$. ■

b) Prove that ρ_E is a uniformly continuous function on X by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y), \forall x, y \in X$$

Proof. Suppose without loss of generality that $\rho_E(x) \geq \rho_E(y)$. Then, let $\{z_n\} \in E^{\mathbb{N}}$ such that $d(y, z_n) - \rho_E(y) < 1/n$ for all n . Then,

$$\rho_E(x) \leq d(x, z_n) \stackrel{\Delta}{\leq} d(x, y) + d(y, z_n) < d(x, y) + \rho_E(y) + 1/n$$

for all n . Then, taking the limit as $n \rightarrow \infty$, we have $|\rho_E(x) - \rho_E(y)| = \rho_E(x) - \rho_E(y) \leq d(x, y)$ as required. Then ρ_E is 1-Lipschitz and thus uniformly continuous. ■

21. Suppose K, F are disjoint sets in X , where K is compact and F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$ and $q \in F$. Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Proof. For every $x \in K$, let G_x be the open ball around x with radius $\rho_F(x)/2$, so that each G_x is disjoint from F . Then $\{G_x : x \in K\}$ forms an open cover of K , so since K is compact, there exists a finite set x_1, \dots, x_n such that $G_{x_1} \cup \dots \cup G_{x_n} \supseteq K$. Then taking $\delta = \min(\rho_F(x_1), \dots, \rho_F(x_n))/2 > 0$, the result follows.

To show that compactness is required for at least one of the sets, let $X = \{(x, y) : y \leq -1/x, x \geq 1\}$ and $Y = \{(x, y) : y \geq 1/x, x \geq 1\}$ in \mathbb{R}^2 . Clearly both sets are closed and disjoint, but the points $(n, -1/n) \in X$ and $(n, 1/n) \in Y$ get arbitrarily close as $n \rightarrow \infty$. ■

22. Let A, B be disjoint non-empty closed sets in a metric space X , and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}.$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, and that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B . This establishes a converse of Exercise 3: Every closed set $A \subseteq X$ is $Z(f)$ for some continuous real f on X . Setting

$$V = f^{-1}([0, 1/2)) \text{ and } W = f^{-1}((1/2, 1]),$$

show that V, W are open and disjoint, and that $A \subseteq V, B \subseteq W$, and thus that pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called **normality**.

Proof. The fact that f is continuous stems from the fact that ρ_E is uniformly continuous and thus continuous, for any E , and the rules about combining continuous functions. It is also clear that the range of f lies in $[0, 1]$, and that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B .

4 Continuity

Now define V and W as in the statement. Clearly V and W are disjoint, $A \subseteq V$ and $B \subseteq W$. The fact that V and W are open comes from the fact that f is continuous, and that the definitions can be rewritten $V = f^{-1}((-1/2, 1/2))$ and $W = f^{-1}((1/2, 3/2))$ respectively as the inverse image of an open set. Thus, metric spaces are **normal**. ■

23. A function $f : (a, b) \rightarrow \mathbb{R}$ is **convex** iff

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for $x, y \in (a, b)$ and $\lambda \in (0, 1)$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex.

If f is convex in (a, b) and $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof. We begin by proving the inequality. Notice that we can write $t = \lambda s + (1 - \lambda)u$ for some $\lambda \in (0, 1)$. Then,

$$\frac{f(t) - f(s)}{t - s} = \frac{f(\lambda s + (1 - \lambda)u) - f(s)}{\lambda s + (1 - \lambda)u - s} \stackrel{\text{conv.}}{\leq} \frac{\lambda f(s) + (1 - \lambda)f(u) - f(s)}{(1 - \lambda)(u - s)} = \frac{f(u) - f(s)}{u - s}.$$

A similar argument can be used to show the second inequality. [**TODO**] ■

24. Let $f : (a, b) \rightarrow \mathbb{R}$ be continuous such that

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}, \forall x, y \in (a, b).$$

Show that f is convex.

Proof. Let $x, y \in (a, b)$ and $\lambda \in (0, 1)$. Let

$$(a_0, z_0, b_0) = \left(x, \frac{x + y}{2}, y\right).$$

Then, define (a_n, z_n, b_n) recursively: if $\lambda \geq z_{n-1}$, take

$$(a_n, z_n, b_n) = \left(z_{n-1}, \frac{z_{n-1} + b_{n-1}}{2}, b_{n-1}\right),$$

otherwise, take

$$(a_n, z_n, b_n) = \left(a_{n-1}, \frac{a_{n-1} + z_{n-1}}{2}, z_{n-1}\right).$$

Then, $z_n \rightarrow \lambda x + (1 - \lambda)y$. Also, it can be shown inductively that if $z_n = \lambda_n x + (1 - \lambda_n)y$, then $f(z_n) \leq \lambda_n f(x) + (1 - \lambda_n)f(y)$, so the result follows since f is continuous. ■

25. If $A, B \subseteq \mathbb{R}^k$, define $A + B = \{x + y : x \in A, y \in B\}$.

- a) If K is compact and C is closed in \mathbb{R}^k , prove that $K + C$ is closed.

Proof. Let $z_n = x_n + y_n \rightarrow z$ in $K + C$, with $x_n \in K$ and $y_n \in C$. By the Bolzano-Weierstrass Theorem, (x_n) admits a convergent subsequence $(x_{n_k}) \rightarrow x \in K$, so $y_{n_k} = z_{n_k} - x_{n_k}$ is a convergent sequence in C . Say $y_{n_k} \rightarrow y \in C$. Then $z = x + y \in K + C$ so $K + C$ is closed. ■

- b) Let α be irrational. Let $C_1 = \mathbb{Z}$ and $C_2 = \alpha\mathbb{Z}$. Show that C_1, C_2 are closed subsets of \mathbb{R} whose sum $C_1 + C_2$ is **not** closed, by showing that $C_1 + C_2$ is a countable dense subset of \mathbb{R} .

Proof. C_1 and C_2 are countable sets of isolated points and are thus closed sets in \mathbb{R} . Also, $C_1 + C_2$ is countable, as every element can be associated to a pair of integers $(x, y) \mapsto x + \alpha y$.

Notice that $S = \{x_n := n\alpha - \lfloor n\alpha \rfloor : n \in \mathbb{Z}\}$ is an infinite subset of the compact set $[0, 1]$. Thus, Bolzano-Weierstrass shows the existence of a convergent sequence in S , and thus we can find pairs (x_i, x_j) arbitrarily close together, and thus $|x_i - x_j| = |x_{i-j}| < 1/n$ for any n . Then, given any segment (a, b) with length at least $1/n$, at least one multiple of x_{i-j} must exist inside the segment, so $S = (C_1 + C_2)/\mathbb{Z}$ is dense in $(0, 1)$ and thus the result. ■

26. Suppose X, Y, Z are metric spaces such that Y is compact. Let $f : X \rightarrow Y$, and $g : Y \rightarrow Z$ be a continuous bijection, and let $h = g \circ f$.

Prove that f is uniformly continuous if h is uniformly continuous. Prove also that f is continuous if h is continuous. Show, that the compactness of Y cannot be omitted from the hypotheses, even when both X and Z are compact.

Proof. From Theorem 4.17, g^{-1} is a continuous function, and thus $g(Y)$ is compact. Then, g^{-1} is continuous on a compact domain, and thus uniformly continuous. Then, $f = g^{-1} \circ h$ is uniformly continuous by Exercise 4.12. Similarly, g^{-1} is continuous so $f = g^{-1} \circ h$ is continuous by Theorem 4.7.

Let $X = [0, 2\pi]$, $Y = [0, 2\pi)$, and $Z = \{x \in \mathbb{R}^2 : |x| = 1\}$, and notice that X and Z are compact. Also, let $f : X \rightarrow Y$ with $f(x) = \pi - x$ for $x \in [0, \pi]$ and $f(x) = 3\pi - x$ on $(\pi, 2\pi)$. Clearly f is discontinuous at π . Also, let $g(t) = \{\cos t, \sin t\}$ be a continuous bijection from Y to Z . Then, $h = g \circ f$ is a clockwise path from $(0, 1)$ to $(0, -1)$ along the unit circle, which is uniformly continuous. However, f is discontinuous, as required. ■

5 Differentiation

5.1 The Derivative of a Real Function

Definition 40. Let $f : [a, b] \rightarrow \mathbb{R}$. Then for any $x \in [a, b]$, define

$$\phi(t) = \frac{f(t) - f(x)}{t - x}$$

and

$$f'(x) = \lim_{t \rightarrow x} \phi(t) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}.$$

given the limit exists. This f' , whose domain is exactly the set of points x where the above limit exists, is called the **derivative** of f .

If f' is defined at x , we say f is **differentiable** at x . If f is differentiable at every point of a set $E \subseteq [a, b]$, we say f is differentiable on E . Extending the above limit definition with left-hand and right-hand limits, we get the left-hand and right-hand derivatives, respectively. Notice thus that f' is not defined on the endpoints a nor b .

Theorem 86. Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable at $x \in (a, b)$. Then f is continuous at x .

Proof. As $t \rightarrow x$, we have

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \rightarrow f'(x) \cdot 0 = 0.$$

■

Notice that the converse is not true: consider $f(x) = |x|$ on $[-1, 1]$.

Theorem 87. Suppose f, g are defined on $[a, b]$ and differentiable at $x \in [a, b]$. Then $f + g, fg, f/g$ are differentiable at x , and

$$(a) (f + g)'(x) = f'(x) + g'(x);$$

$$(b) (fg)'(x) = f'(x)g(x) + f(x)g'(x);$$

$$(c) (f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Proof. (a) follows from properties of limits. (b) follows from writing

$$\frac{(fg)(t) - (fg)(x)}{t - x} = \frac{f(t)[g(t) - g(x)] + [f(t) - f(x)]g(x)}{t - x} \rightarrow f(x)g'(x) + f'(x)g(x).$$

Similarly, (c) follows from writing

$$\frac{(f/g)(t) - (f/g)(x)}{t - x} = \frac{1}{g(t)g(x)} \left[g(x) \frac{f(t) - f(x)}{t - x} - f(x) \frac{g(t) - g(x)}{t - x} \right] \rightarrow \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

■

It can be shown inductively that polynomials are differentiable, and that $(x^n)' = nx^{n-1}$. Also, 5.3(c) shows that rational functions are differentiable, except at points where the denominator is zero.

Theorem 88. Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable at $x \in [a, b]$. Also, suppose g is defined on an interval I containing the range of f and differentiable at $f(x)$. Then, $(g \circ f)'(x) = g'(f(x))f'(x)$.

Proof. We have

$$\frac{h(t) - h(x)}{t - x} = \frac{g(f(t)) - g(f(x))}{t - x} = \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \cdot \frac{f(t) - f(x)}{t - x} \rightarrow g'(f(x))f'(x)$$

where the first term converges since $f(t) \rightarrow f(x)$ as f is continuous. ■

5.2 Mean Value Theorems

Definition 41. Let f be a real-valued function on a metric space X . We say f has a **local maximum** at a point $p \in X$ when there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in N_\delta(p)$. A **local minimum** is defined similarly.

Theorem 89. Let $f : [a, b] \rightarrow \mathbb{R}$ with a local maximum at $x \in (a, b)$. If $f'(x)$ exists, then $f'(x) = 0$.

Proof. Define f and x as above, and suppose $f'(x)$ exists. If $f'(x) = c > 0$, then there must exist some $y > x$ sufficiently close (say, at most δ away) from x such that $\phi(y) > 0$ but this implies $f(y) > f(x)$, a contradiction. The argument against $f'(x) < 0$ is symmetric. ■

Theorem 90 (Cauchy's MVT). If f, g are continuous real functions on $[a, b]$ which are differentiable on (a, b) , then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Notice that differentiability is not required at the endpoints.

Proof. Let $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. It suffices to show that $h'(x) = 0$ for some $x \in (a, b)$. Notice that $h(a) = h(b)$. If h is constant on $[a, b]$, then $h'(x) = 0$ everywhere. Otherwise, h admits a local extremum, and thus has derivative 0 at that extremum, completing the proof. ■

Corollary 11 (Mean Value Theorem). If $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on (a, b) , then

$$f(b) - f(a) = (b - a)f'(x)$$

for some $x \in (a, b)$.

Proof. Take $g(x) = x$. ■

Theorem 91. Suppose f is differentiable in (a, b) .

- (a) If $f' \geq 0$ on (a, b) , then f is monotonically non-decreasing.
- (b) If $f' = 0$ on (a, b) , then f is constant.
- (c) If $f' \leq 0$ on (a, b) , then f is monotonically non-increasing.

n

Proof. For any pair $a \leq x < y \leq b$, there exists some $z \in (x, y)$ such that $f'(z)(y - x) = f(y) - f(x)$, so each result follows trivially. ■

5.3 The Continuity of Derivatives

We have seen that derivatives need not be continuous, but the following theorem states they still have the intermediate value property:

Theorem 92 (Darboux property of derivatives). *Suppose $f : [a, b] \rightarrow \mathbb{R}$ is differentiable and $f'(a) < \lambda < f'(b)$. Then, there exists a point $x \in (a, b)$ such that $f'(x) = \lambda$.*

Proof. Let $h(t) = f(t) - \lambda t$, and notice that $h'(a) = f'(a) - \lambda < 0$ and similarly $h'(b) > 0$. Then there exist $a < s < t < b$ such that $h(s) < h(a)$ and $h(t) < h(b)$. Then h is continuous on the compact interval $[s, t]$ and thus attains a minimum at $u \in (s, t)$. Then $h'(u) = 0$ so $f'(u) = \lambda$ as required. ■

Corollary 12. *If f is differentiable on $[a, b]$, then f' has no simple discontinuities on $[a, b]$. (Discontinuities of the second kind as still possible)*

5.4 L'Hopital's Rule

Theorem 93 (L'Hopital's Rule).

6 TODOs

1. Finish the proof for multiplicative inverses in Appendix 1
2. Exercise 1.20
3. Exercise 2.11
4. Exercise 3.7 has a weird \leq vs. $<$ problem. See if you can spot it.
5. Exercise 3.17d
6. Fix the theorem/example/etc. numbers so they match with Rudin
7. Exercise 4.6
8. Exercise 4.23