Rocket Nozzle Design Experiment

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AEE 3162

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I. Introduction

This report details the Term Project for AEE 3162 Compressible Flow. This project aims to design a minimum length 2-dimensional Rocket Nozzle (also known as a Bell Nozzle) using the Method of Characteristics. The design is based on general specifications that were given and uses at least 35 characteristics to construct the rocket nozzle contour. A MATLAB code is implemented to calculate and draw the nozzle contour. The variation in Mach number, pressure, and temperature is also plotted. Table 1 illustrates the given data.

Table 1: Specifications for Rocket Nozzle Design

Specification	Value
Ambient Pressure	101 kPa
Specific heat ratio	1.4
R (gas constant)	287 J/kg*K
Stagnation temperature T_0 of propellant	1000 K
Stagnation Pressure p_0 of propellant	1500 kPa
Least number of Characteristic lines	35

II. Calculations

To determine the nozzle contours, the following calculations needed to be performed using the specified values.

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma - 1}{2} \times M_e^2\right)^{-\frac{\gamma}{\gamma-1}}$$

$$\frac{101 \times 10^\wedge 6}{1500 \times 10^\wedge 6} = \left(1 + \frac{1.4 - 1}{2} \times M_e^2\right)^{-\frac{1.4}{1.4-1}}$$

$$M_e = 2.41$$

Specific Heat:

$$C_p = \frac{\gamma R}{\gamma - 1}$$

$$C_p = \frac{1.4 \times 287}{1.4 - 1}$$

$$C_p = 1004.5\, J/kg * K$$

Exit Velocity:

$$V_e = \sqrt{2C_p(T_0 - T_e)}$$

$$V_e = \sqrt{2 * 1005 * (1000 - 288)}$$

$$V_e = 1196\, m/s$$

From Table A.5 in the textbook [1], For M = 2.4, ν = 36.75 degrees

$$\theta_{w,max} = \frac{\nu_{max}}{2}$$

$$\theta_{w,max} = \frac{36.75}{2}$$

$$\theta_{w,max} = 18.375°$$

Stagnation enthalpy:

$$h_0 = C_p T_0$$

$$h_0 = 1004.5 * 1000$$

$$h_0 = 1 \times 10^6 \, J$$

Specific kinetic energy:

$$k_e = \frac{1}{2} V_e^2$$

$$k_e = \frac{1}{2}(1196)\text{^}2$$

$$k_e = 715208 \, J$$

Percentage of specific stagnation enthalpy:

$$\% = \frac{k_e}{h_0} \times 100$$

$$\% = \frac{715208}{1 \times 10^6} \times 100$$

$$\% = 71.5\%$$

The best percentage kinetic energy relative to the specific stagnation enthalpy can occur when the gas expands to match the ambient pressure isentropically. At this point, the nozzle would have achieved maximum efficiency.

Ideally all the thermal energy would be converted to kinetic energy but the percentage can only approach 100% as some losses always occur due to boundary layer effects and incomplete expansion.

Number of Characteristic lines:

$$n_t = \frac{n(n+3)}{2}$$

Let nt = 35

$$n = 7$$

The first line compensates for the assumption of the straight sonic line:

$$\theta_1 = \nu_1 - \nu_{(M=1)}$$

Setting $\theta_1 = 0.375°$,

$$\nu_1 = 0.375°$$

$$(K_+)_1 = \theta_1 - \nu_1 = 0° \; along\ the\ C_+\ characteristic$$

$$(K_-)_1 = \theta_1 + \nu_1 = 0.75° \; along\ the\ C_+\ characteristic$$

$$(K_+)_i = \theta_i - \nu_i$$

$$(K_-)_i = \theta_i + \nu_i$$

From Table A.5 in the textbook [1], For $\nu_1 = 0.375°$, M = 1.04, $\mu = 74.06°$

Points 2 through 7 are along the first characteristic line

$$\Delta\theta = \frac{\theta_{w,max} - \theta_1}{n-1}$$

$$\Delta\theta = \frac{18.375 - 0.375}{7-1}$$

$$\Delta\theta = 3°$$

$$\theta_i = (i-1)\Delta\theta + \theta_1$$

$$\nu_i = \theta_i$$

Sample Calculation for θ_2

$$\theta_2 = (2-1)\times 3 + 0.375$$

$$\theta_2 = 3.375°$$

$$(K_-)_2 = 3.375 + 3.375$$

$$(K_-)_2 = 6.75°$$

Using Table A.5 for M_i and μ_i for v_i, when $v_2 = 3.375°$, $M_2 = 1.2$, $\mu_2 = 56.44°$

For the points on the centerline the K_- is the same as the K_- for that of the adjacent points

$$(i = 9, 16, 22, 27, 31, 34)$$

$$(K_-)_9 = (K_-)_2$$

$$(K_-)_9 = 6.75$$

$$\theta_i = 0$$

$$(K_-)_i = \nu_i$$

$$(K_+)_i = -\nu_i$$

$$(K_+)_9 = -6.75$$

$$v_i = \frac{1}{2}(K_- - K_+)$$

$$v_9 = \frac{1}{2}\left(6.75 - (-6.75)\right)`$$

$$v_9 = 6.75°$$

Using Table A.5 for M_i and μ_i for v_i, when $v_9 = 6.75$, $M_2 = 1.32$, $\mu_2 = 49.25°$

For points between the nozzle contour and center line, $(K_+)_i$ is the same along a given $C_+$and $(K_-)_i$ is the same, originating from point a.

$$(i = 7,14,20,25,29,32,34)$$

$$\theta_i = \frac{(K_+)_i + (K_-)_i}{2}$$

$$v_i = \frac{(K_-)_i - (K_+)_i}{2}$$

Sample Calculation for point 14:

$$(K_-)_7 = (K_-)_{14}$$

$$(K_-)_{14} = 36.75$$

$$(K_+)_9 = (K_+)_{14}$$

$$(K_+)_{14} = -6.75$$

$$\theta_{14} = \frac{-6.75 + 36.75}{2}$$

$$\theta_i = 15°$$

$$v_{14} = \frac{36.75 - (-6.75)}{2}$$

$$v_{14} = 21.75°$$

Using Table A.5 for M_i and μ_i for v_i, when $v_{14} = 21.75$, $M_2 = 1.84$, $\mu_2 = 32.92°$

For Points on the nozzle wall:

$$(i = 15, 21,26,30,33,35)$$

$$\theta_i = \theta_{i-1}$$

$$\nu_i = \nu_{i-1}$$

$$(K_+)_i = (K_+)_{i-1}$$

$$(K_-)_i = (K_-)_{i-1}$$

Sample Calculation for point 15:

$$\theta_{15} = \theta_{14}$$

$$\theta_{15} = 15°$$

$$\nu_{15} = \nu_{14}$$

$$\nu_{14} = 21.75°$$

$$(K_+)_{15} = (K_+)_{14}$$

$$(K_+)_{15} = -6.75$$

$$(K_-)_{15} = (K_-)_{14}$$

$$(K_-)_{15} = 36.75°$$

Using Table A.5 for M_i and μ_i for v_i, when $v_{15} = 21.75$, $M_2 = 1.84$, $\mu_2 = 32.92°$

Slope calculations:

Average Slope value from internal flow:

Along C_+:

$$\alpha_1 = \frac{(\theta_{i,1} + \theta_{i,2}) + (\mu_{i,1} + \mu_{i,2})}{2}$$

$$slope = \tan(\alpha_1)$$

Sample Calculation for line 1 -> 2:

$$\alpha_1 = \frac{(\theta_1 + \theta_2) + (\mu_1 + \mu_2)}{2}$$

$$\alpha_1 = \frac{(0.375 + 3.375) + (74.06 + 56.44)}{2}$$

$$\alpha_1 = 134.25°$$

$$slope = \tan(\alpha_1)$$

$$slope = \tan(134.25)$$

$$slope = -1.03$$

Along C_-:

$$\alpha_2 = \frac{(\theta_{i,1} + \theta_{i,2}) - (\mu_{i,1} + \mu_{i,2})}{2}$$

$$slope = \tan(\alpha_2)$$

Sample Calculation for line 2 ->9:

$$\alpha_2 = \frac{(\theta_2 + \theta_9) - (\mu_2 + \mu_9)}{2}$$

$$\alpha_2 = \frac{(3.375 + 0) - (56.44 + 49.25)}{2}$$

$$\alpha_2 = -51.16°$$

$$slope = \tan(\alpha_2)$$

$$slope = \tan(-51.16)$$

$$slope = -1.24$$

Average Slope angle from wall points:

$$\alpha_3 = \frac{(\theta_{i,1} + \theta_{i,2})}{2}$$

$$slope = \tan(\alpha_3)$$

Sample Calculation for 8 -> 15:

$$\alpha_3 = \frac{(\theta_8 + \theta_{15})}{2}$$

$$\alpha_3 = \frac{(18.375 + 15)}{2}$$

$$\alpha_3 = 16.69°$$

$$slope = \tan{(\alpha_3)}$$

$$slope = \tan{(16.69)}$$

$$slope = 0.3$$

Slope from throat:

$$\alpha = \frac{\left(\theta_{i,1} + \theta_{i,2}\right) - (\mu_{i,1} + \mu_{i,2})}{2}$$

For a -> 1 line along C_{-}:

$$\alpha = \frac{(\theta_a + \theta_1) - (\mu_a + \mu_1)}{2}$$

$$\alpha = \frac{(18.375 + 0.375) - (90 + 74.06)}{2}$$

$$\alpha = -72.66$$

$$slope = \tan{(\alpha_3)}$$

$$slope = \tan{(-72.66)}$$

$$slope = -3.2$$

Point 1 $(x_1, 0)$, Point a $(0, A_{throat})$:

$$Slope_{a\rightarrow 1} = \frac{0 - A_{throat}}{x_1 - 0}$$

III. Results

Table 2: Data for nozzle construction

Point #	K_-	K_+	$\theta_i(°)$	$\nu(°)$	$\mu(°)$	M
1	0.75	0	0.375	0.375	73.12	1.05
2	6.75	0	3.375	3.375	57.24	1.19
3	12.75	0	6.375	6.375	49.95	1.31
4	18.75	0	9.375	9.375	44.99	1.41
5	24.75	0	12.38	12.375	41.35	1.51
6	30.75	0	15.38	15.375	38.07	1.62
7	30.75	0	15.38	15.375	38.07	1.62
8	36.75	0	18.38	18.375	35.53	1.72
9	6.75	-6.75	0	6.75	49.03	1.32
10	12.75	-6.75	3	9.75	44.63	1.42
11	18.75	-6.75	6	12.75	40.76	1.53
12	24.75	-6.75	9	15.75	37.83	1.63
13	30.75	-6.75	12	18.75	35.32	1.73
14	30.75	-6.75	12	18.75	35.32	1.73
15	36.75	-6.75	15	21.75	32.96	1.84
16	12.75	-12.75	0	12.75	40.76	1.53
17	18.75	-12.75	3	15.75	37.83	1.63
18	24.75	-12.75	6	18.75	35.32	1.73
19	30.75	-12.75	9	21.75	32.96	1.84
20	30.75	-12.75	9	21.75	32.96	1.84
21	36.75	-12.75	12	24.75	31.08	1.94

22	18.75	-18.75	0	18.75	35.32	1.73
23	24.75	-18.75	3	21.75	32.96	1.84
24	30.75	-18.75	6	24.75	31.08	1.94
25	30.75	-18.75	6	24.75	31.08	1.94
26	36.75	-18.75	9	27.75	29.13	2.05
27	24.75	-24.75	0	24.75	31.08	1.94
28	30.75	-24.75	3	27.75	29.13	2.05
29	30.75	-24.75	3	27.75	29.13	2.05
30	36.75	-24.75	6	30.75	27.55	2.16
31	30.75	-30.75	0	30.75	27.55	2.16
32	36.75	-30.75	3	33.75	26.02	2.28
33	36.75	-30.75	3	33.75	26.02	2.28
34	36.75	-36.75	0	36.75	24.66	2.40
35	36.75	-36.75	0	36.75	24.66	2.40

Table 3: X and Y coordinates for Contour plot

x	y
0.3107	0.0000
0.4647	0.3637
0.5399	0.4863
0.5986	0.5712
0.6489	0.6406
0.6985	0.7078
0.6985	0.7078
0.7444	0.7702
0.7545	0.0000
0.8959	0.1589
1.0094	0.2815
1.1085	0.3869
1.2064	0.4922
1.2064	0.4922
1.2996	0.5945
1.0774	0.0000
1.2277	0.1298

1.3614	0.2463
1.4956	0.3656
1.4956	0.3656
1.6260	0.4851
1.4126	0.0000
1.5814	0.1211
1.7530	0.2481
1.7530	0.2481
1.9232	0.3792
1.7868	0.0000
1.9986	0.1304
1.9986	0.1304
2.2132	0.2689
2.2563	0.0000
2.5213	0.1426
2.5213	0.1426
2.8439	0.0000
2.8439	0.0000

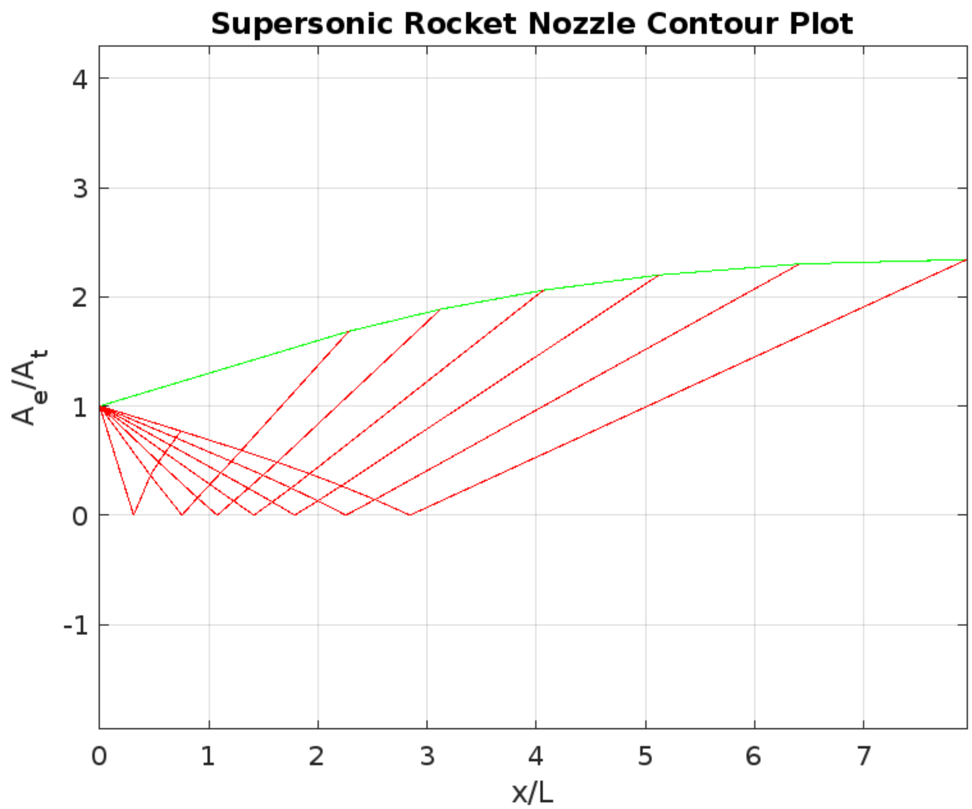

Figure 1: Plot of Rocket Nozzle Contour

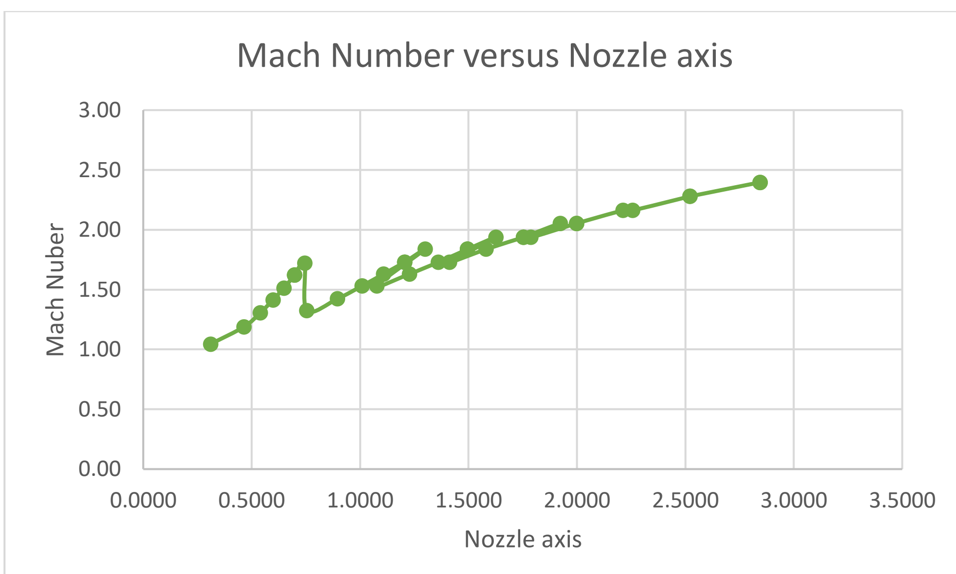

Figure 2: Plot of Mach number vs nozzle axis

IV. Code

```
Me = 2.4;
n = 7;
gamma = 1.4;
p = n;
Gp = gamma + 1;
Gm = gamma - 1;
T_0 = 1000
p_0 = 1500000000
k_min = zeros(p, p);
k_plus = zeros(p, p);
Theta = zeros(p, p);
nu = zeros(p, p);
mu = zeros(p, p);
Mach = zeros(p, p);
x = zeros(p, p);
y = zeros(p, p);


nu_max = 36.75;
Theta_max = nu_max / 2;
y_0 = 1;
x_0 = 0;
dt = Theta_max - 18;
Theta(:,1) = (dt:3:Theta_max);
nu(:, 1) = Theta(:, 1);
k_min(:, 1) = Theta(:, 1) + nu(:, 1);
k_plus(:, 1) = Theta(:, 1) - nu(:, 1);


for i = 1:p
    Mach(i, 1) = find_mach(nu(i, 1), gamma);
    mu(i, 1) = asind(1 / Mach(i, 1));
end


x(1, 1) = x_0-y_0 / tand(Theta(1, 1) - mu(1, 1));
y(1, 1) = 0;


for i = 2:p
    s_char = tand(Theta(i, 1) - mu(i, 1))
    s_avg = tand((Theta(i - 1, 1) + mu(i - 1, 1) + Theta(i, 1) + mu(i, 1)) / 2)
```

```
    x(i, 1) = ((y(i - 1, 1) - x(i - 1, 1) * s_avg) - (y_0 -x_0 * s_char)) /
(s_char - s_avg);
    y(i, 1) = y(i - 1, 1) + (x(i, 1) - x(i - 1, 1)) * s_avg;
end


for j = 2:p
    for i = 1:n + 1 - j
        k_min(i, j) = k_min(i + 1, j - 1);

        if i == 1
            Theta(i, j) = 0;
            nu(i,j) = k_min(i,j);
            k_plus (i,j) = -k_min(i,j);

        else
            k_plus(i, j) = k_plus(i - 1, j);
            Theta(i, j) = (k_min(i, j) + k_plus(i, j)) / 2;
            nu(i, j) = (k_min(i, j) - k_plus(i, j)) / 2;
        end

Mach(i, j) = find_mach(nu(i, j), gamma);
mu(i, j) = asind(1 / Mach(i, j));


        if i == 1
            s_char = tand((Theta(i + 1, j - 1) - mu(i + 1, j - 1) + Theta(i, j) -
mu(i, j)) / 2)
            x(i, j) = x(i + 1, j - 1) - y(i + 1, j - 1) / s_char;
            y(i, j) = 0;


        else
            s_char = tand((Theta(i + 1, j - 1) - mu(i + 1, j - 1) + Theta(i, j) -
mu(i, j)) / 2)
            s_avg = tand((Theta(i - 1, j) + mu(i - 1, j) + Theta(i, j) + mu(i,
j)) / 2)
            x(i, j) = ((y(i - 1, j) - x(i - 1, j) * s_avg) - (y(i + 1, j - 1) -
x(i + 1, j - 1) * s_char)) / (s_char - s_avg);
            y(i, j) = y(i - 1, j) + (x(i, j) - x(i - 1, j)) * s_avg;
        end
    end
end


xwall = zeros(1, n + 1);
ywall = zeros(1, n + 1);
xwall(1) = x_0;
```

```
ywall(1) = y_0;


for j = 2:n + 1
    if j == 2
        wall_slope = tand(Theta_max)
        interior_slope = tand(Theta(p, 1) - mu(p, 1))
    else
        wall_slope = tand((Theta(n - j + 3, j - 2) + Theta(n - j + 2, j - 1)) /
2)
        interior_slope = tand(Theta(n - j + 2, j - 1) + mu(n - j + 2, j - 1))
    end
    xwall(j) = ((y(n - j + 2, j - 1) - x(n - j + 2, j - 1) * interior_slope) -
(ywall(j - 1) - xwall(j - 1) * wall_slope)) / (wall_slope - interior_slope);
    ywall(j) = ywall(j - 1) + (xwall(j) - xwall(j - 1)) * wall_slope;
end


plot(xwall, ywall);
hold on;
plot(xwall, ywall, 'g');
hold on;


for i = 1:n
    plot([0, x(i, 1)], [y_0, y(i, 1)],'r');
    plot([x(n + 1 - i, i), xwall(i + 1)], [y(n + 1 - i, i), ywall(i + 1)],'r');
end


for i = 1:n - 1
    plot(x(1:n + 1 - i, i), y(1:n + 1 - i, i),'r');
end
 for t=1:n
        for r=2:n+1-t
            plot([x(t,r) x(t+1,r-1)],[y(t,r) y(t+1,r-1)],'r')
        end
 end


xlabel('x/L');
ylabel('A_e/A_t');
title('Supersonic Rocket Nozzle Contour Plot');
axis equal;
grid on;


disp(mu)
```

```
disp(x)
disp(y)
disp(Mach)
disp(Theta)
disp(nu)
disp(k_min)
disp(k_plus)
disp(s_avg)
disp(s_char)






grid on;
```

```
function Mach = find_mach(nu, gamma)


Gp = gamma + 1;
Gm = gamma - 1;
Mach_range = linspace(1, 10, 1000);
nu_vals = zeros(size(Mach_range));
for i = 1:length(Mach_range)
        Mach_squared = Mach_range(i)^2;
        nu_vals(i) = sqrt(Gp / Gm) * atand(sqrt(Gm * (Mach_squared - 1) / Gp))
...
                     - atand(sqrt(Mach_squared - 1));
end
abs_diff = zeros(size(nu_vals));
for i = 1:length(nu_vals)
        abs_diff(i) = abs(nu_vals(i) - nu);
end
[~, idx] = min(abs_diff);
Mach = Mach_range(idx);
end
```

References

1] J.D. Anderson, “Modern Compressible Flow,” 4 th ed., McGraw-Hill.

[2] C.S. Subramanian, Lecture Notes.

[3] J.E. John and T.G. Keith, Gas Dynamics 3 rd Ed., Pearson.

[4] "Rocket Thrust equations," NASA Available: https://www.grc.nasa.gov/www/k-12/BGP/rktthsum.html.