Comments on Hypergeometric Differential Equation

Winter 2019

INTRODUCTION

For a homogeneous linear second-order differential equation of the form:

$$y'' + p(x)y' + q(x)y = 0$$

we say that x_0 is a regular singular point if $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at x_0. For the setup above the functions p and q are restricted to meromorphic functions (functions that have at most countably many isolated singularities).

HYPERGEOMETRIC EQUATION SINGULARITIES

We are interested in looking at an equation of the form:

$$x(1-x)y'' + (\gamma - (1+\alpha+\beta)x)y' - \alpha\beta y = 0$$

for $\alpha, \beta, \gamma \in \mathbb{R}$. In order to identify the singular points we begin by rewriting the above into:

$$y'' + \frac{\gamma - (1+\alpha+\beta)x}{x(1-x)}y' - \frac{\alpha\beta}{x(1-x)}y = 0$$

It is clear from here that $x_0 = 0, 1$ because the functions have singularities there. In order to determine if these are regular singular points we identify $p(x) = \frac{\gamma-(1+\alpha+\beta)x}{x(1-x)}$ and $q(x) = -\frac{\alpha\beta}{x(1-x)}$ to see that:

$$\begin{aligned} x_0 = 0: \quad xp(x) &= \frac{\gamma - (1+\alpha+\beta)x}{1-x} \quad \text{and} \quad x^2p(x) = -\frac{\alpha\beta x}{1-x} \\ x_0 = 1: \quad (x-1)p(x) &= \frac{\gamma - (1+\alpha+\beta)x}{x} \quad \text{and} \quad (x-1)^2p(x) = \frac{\alpha\beta(x-1)}{x} \end{aligned}$$

In both cases we see that $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic.

FIRST SINGULAR POINT: $x_0 = 0$

In order to achieve a power series solution here we use the Frobenius Method to write down:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

and directly plug in to obtain:

$$\begin{aligned} 0 &= (x-x^2)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (\gamma - (1+\alpha+\beta)x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+r} \\ 0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}\gamma(n+r)a_n x^{n+r-1} \\ &\quad - \sum_{n=0}^{\infty}(1+\alpha+\beta)(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}\alpha\beta a_n x^{n+r} \\ 0 &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(n+r-1)(n+r-2)a_{n-1}x^{n+r-1} + \sum_{n=0}^{\infty}\gamma(n+r)a_n x^{n+r-1} \\ &\quad - \sum_{n=1}^{\infty}(1+\alpha+\beta)(n+r-1)a_{n-1}x^{n+r-1} - \sum_{n=1}^{\infty}\alpha\beta a_{n-1}x^{n+r-1} \\ 0 &= \sum_{n=1}^{\infty}\left[\Big[(n+r)(n+r-1) + \gamma(n+r)\Big]a_n - \Big[(n+r-1)(n+r-2) + (1+\alpha+\beta)(n+r-1) + \alpha\beta\Big]a_{n-1}\right]x^{n+r-1} \\ &\quad + \Big(r(r-1) + \gamma r\Big)a_0 x^{r-1} \end{aligned}$$

To determine the valid values of r we solve the indicial equation:

$$\begin{aligned}
0 &= r(r-1)+\gamma r \\
0 &= r(r-(1-\gamma)) \\
r &= 0, 1-\gamma
\end{aligned}$$

<u>SECOND SINGULAR POINT: $x_0 = 1$</u>

In order to achieve a power series solution here we use the Frobenius Method to write down:

$$y=\sum_{n=0}^{\infty} a_n (x-1)^{n+r}, \quad y'=\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1}, \quad y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-2}$$

and directly plug in to obtain:

$$\begin{aligned}
0 &= (x-x^2)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-2} \\
&\quad + (\gamma-(1+\alpha+\beta)x)\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1} - \alpha\beta\sum_{n=0}^{\infty}a_n(x-1)^{n+r} \\
0 &= -\Big((x-1)^2+(x-1)\Big)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-2} \\
&\quad + \Big((\gamma-(1+\alpha+\beta)) - (1+\alpha+\beta)(x-1)\Big)\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1} - \alpha\beta\sum_{n=0}^{\infty}a_n(x-1)^{n+r} \\
0 &= -\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-1} \\
&\quad + (\gamma-(1+\alpha+\beta))\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1} - (1+\alpha+\beta)\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r} \\
&\quad - \alpha\beta\sum_{n=0}^{\infty}a_n(x-1)^{n+r} \\
0 &= -\sum_{n=1}^{\infty}(n+r-1)(n+r-2)a_{n-1}(x-1)^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-1} \\
&\quad + (\gamma-(1+\alpha+\beta))\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1} - (1+\alpha+\beta)\sum_{n=1}^{\infty}(n+r-1)a_{n-1}(x-1)^{n+r-1} \\
&\quad - \alpha\beta\sum_{n=1}^{\infty}a_{n-1}(x-1)^{n+r-1} \\
0 &= \sum_{n=1}^{\infty}\left[(n+r)(\gamma-n-r-\alpha-\beta)a_n - \Big[(n+r-1)(n+r+\alpha+\beta-1)-\alpha\beta\Big]a_{n-1}\right]x^{n+r-1} \\
&\quad + \Big(-r(r-1)+(\gamma-(1+\alpha+\beta))r\Big)a_0x^{r-1}
\end{aligned}$$

To determine the valid values of r we solve the indicial equation:

$$\begin{aligned}
0 &= -r(r-1)+(\gamma-(1+\alpha+\beta))r \\
0 &= -r^2+(\gamma-(\alpha+\beta))r \\
0 &= -r(r-(\gamma-(\alpha+\beta))) \\
r &= 0, \gamma-\alpha-\beta
\end{aligned}$$

THIRD SINGULAR POINT: $x_0 = \infty$

To determine if the given differential equation has a singular at ∞ we aim to use the substitution $w = \frac{1}{x}$ and see if $w = 0$ is a singularity. To rewrite the differential equation using this substitution we need the following:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}x} = -w^2 \frac{\mathrm{d}y}{\mathrm{d}w}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\Big(- w^2 \cdot \frac{\mathrm{d}y}{\mathrm{d}w}\Big) = w^4 \frac{\mathrm{d}^2y}{\mathrm{d}w^2} + 2w^3 \frac{\mathrm{d}y}{\mathrm{d}w}$$

Plugging this into the given differential equation yields:

$$\begin{aligned}
0 &= (x - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (\gamma - (1 + \alpha + \beta)x)\frac{\mathrm{d}y}{\mathrm{d}x} - \alpha\beta y \\
0 &= \Big(\frac{1}{w} - \frac{1}{w^2}\Big)\Big(w^4 \frac{\mathrm{d}^2y}{\mathrm{d}w^2} + 2w^3 \frac{\mathrm{d}y}{\mathrm{d}w}\Big) + \Big(\gamma - (1 + \alpha + \beta)\frac{1}{w}\Big)\Big(- w^2 \frac{\mathrm{d}y}{\mathrm{d}w}\Big) - \alpha\beta y \\
0 &= (w^2 - w)(y''w + 2y') + ((1 + \alpha + \beta)w - \gamma w^2)y' - \alpha\beta y \\
0 &= w^2(w - 1)y'' + ((\alpha + \beta - 1)w + (2 - \gamma)w^2)y' - \alpha\beta y \\
0 &= y'' + \frac{(\alpha + \beta - 1) + (2 - \gamma)w}{w(w - 1)}y' - \frac{\alpha\beta}{w^2(w - 1)}y
\end{aligned}$$

From the above we see that $w_0 = 0$ is a regular singular point as $wp(w) = \frac{(\alpha+\beta-1)+(2-\gamma)w}{w-1}$ and $w^2q(w) = -\frac{\alpha\beta}{w-1}$ are both analytic at the origin. In order to achieve a power series solution here we use the Frobenius Method to write down:

$$y = \sum_{n=0}^{\infty} a_n w^{n+r}, \quad y' = \sum_{n=0}^{\infty}(n + r)a_n w^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n w^{n+r-2}$$

and directly plug in to obtain:

$$\begin{aligned}
0 &= (w^2 - w)\sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n w^{n+r-2} + ((\alpha + \beta - 1)w + (2 - \gamma)w^2)\sum_{n=0}^{\infty}(n + r)a_n w^{n+r-1} - \alpha\beta\sum_{n=0}^{\infty} a_n w^{n+r} \\
0 &= \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n w^{n+r} - \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n w^{n+r-1} + \sum_{n=0}^{\infty}(\alpha + \beta - 1)\gamma(n + r)a_n w^{n+r} \\
&\quad + \sum_{n=0}^{\infty}(2 - \gamma)(n + r)a_n w^{n+r+1} - \sum_{n=0}^{\infty}\alpha\beta a_n w^{n+r} \\
0 &= \sum_{n=1}^{\infty}(n + r - 1)(n + r - 2)a_{n-1} w^{n+r-1} - \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n w^{n+r-1} + \sum_{n=1}^{\infty}(\alpha + \beta - 1)\gamma(n + r - 1)a_{n-1} w^{n+r-1} \\
&\quad + \sum_{n=2}^{\infty}(2 - \gamma)(n + r - 2)a_{n-2} w^{n+r-1} - \sum_{n=1}^{\infty}\alpha\beta a_{n-1} w^{n+r-1} \\
0 &= \Big(- r(r - 1)\Big)a_0 w^{r-1} + \Bigg(\Big[r(r - 1) + (\alpha + \beta - 1)\gamma r - \alpha\beta\Big]a_0 - (r + 1)ra_1\Bigg)w^r \\
&\quad + \sum_{n=2}^{\infty}\Bigg[(2 - \gamma)(n + r - 2)a_{n-2} + \Big[(n + r - 1)((n + r - 2) + (\alpha + \beta - 1)\gamma) - \alpha\beta\Big]a_{n-1} - (n + r)(n + r - 1)a_n\Bigg]w^{n+r-1}
\end{aligned}$$

To determine the valid values of r we solve the indicial equation:

$$\begin{aligned}
&0 - r(r - 1) \\
&r = 0, 1
\end{aligned}$$

Radius of Convergence for $x_0 = 0, 1$

In order to determine the radius of convergence for the first two regular singular points we observe the following setups:

- For $x_0 = 0$:

$$0 = \Big[(n+r)(n+r-1) + \gamma(n+r)\Big]a_n - \Big[(n+r-1)(n+r-2) + (1+\alpha+\beta)(n+r-1) + \alpha\beta\Big]a_{n-1}$$

$$\frac{a_n}{a_{n-1}} = \frac{(n+r-1)(n+r-2) + (1+\alpha+\beta)(n+r-1) + \alpha\beta}{(n+r)(n+r-1) + \gamma(n+r)}$$

- For $x_0 = 1$:

$$0 = (n+r)(\gamma - n - r - \alpha - \beta)a_n - \Big[(n+r-1)(n+r+\alpha+\beta-1) - \alpha\beta\Big]a_{n-1}$$

$$\frac{a_n}{a_{n-1}} = \frac{(n+r-1)(n+r+\alpha+\beta-1) - \alpha\beta}{(n+r)(\gamma - n - r - \alpha - \beta)}$$

In both cases we achieve $\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| = 1$. Recall that the Ratio Test states that given a series $\sum b_n$ it will converge absolutely if $\lim_{n\to\infty}\left|\frac{b_n}{b_{n-1}}\right| < 1$. For both of the cases above we take $b_n = a_n x^{n+r-1}$ so that:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{b_n}{b_{n-1}}\right| &= \lim_{n\to\infty}\left|\frac{a_n x^{n+r}}{a_{n-1}x^{n+r-1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}x\right| \\
&= |x| \lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right| \\
&= |x|
\end{aligned}$$

Thus, to guarantee convergence we must have $|x| < 1$, i.e. the radius of convergence is 1.