Geometric Series & Comparison Test

MATH 19B
Nathan Marianovsky

Geometric Series

Definition 1 (Geometric Series)**.** Given a series in the form:

$$\sum_{i=k}^{n} r^i$$

where r is the known as the *common ratio*, the sum of the series is defined as:

$$\sum_{i=k}^{n} r^i = \frac{r^k(1-r^{n-k+1})}{1-r}$$

where a is the first term of the series. Notice that the finite series is always convergent for any value of r. Now as $n \to \infty$:

$$\lim_{n\to\infty} \sum_{i=k}^{n} r^i = \lim_{n\to\infty} \frac{r^k(1-r^{n-k+1})}{1-r}$$

$$\sum_{i=k}^{\infty} r^i = \frac{r^k}{1-r} \quad \text{iff} \quad |r| < 1$$

Example 1. Determine whether $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Solution 1. First rewrite the sum in proper form:

$$\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{15}{2}\left(\frac{2}{3}\right)^{n}$$

Since the common ratio is less than 1, the series converges and the sum is given by:

$$\frac{15}{2}\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} = \frac{\left(\frac{15}{2}\right)\left(\frac{2}{3}\right)}{1-\frac{2}{3}} = \boxed{15}$$

Example 2. Determine whether $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$ converges or diverges. If it converges, find its sum.

Solution 2. First rewrite the sum in proper form:

$$\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{\pi}{3}\right)^{n}$$

Since the common ratio is greater than 1, the series $\boxed{diverges}$.

Example 3. Determine whether $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^n}$ converges or diverges. If it converges, find its sum.

Solution 3. First rewrite the sum in proper form:

$$\sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^n} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$$

Since the common ratio is less than 1, the series converges and the sum is given by:

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \boxed{\frac{1}{\sqrt{2}-1}}$$

Example 4. Determine whether $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$ converges or diverges. If it converges, find its sum.

Solution 4. First rewrite the sum in proper form:

$$\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}} = -\frac{5}{6}\sum_{n=1}^{\infty} \left(-\frac{6}{5}\right)^n$$

Since the common ratio is greater than 1, the series $\boxed{diverges}$.

Example 5. Determine whether $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$ converges or diverges. If it converges, find its sum.

Solution 5. First rewrite the sum in proper form:

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

Since the common ratio is less than 1 in both cases, the series converges and the sum is given by:

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} = \boxed{\frac{5}{2}}$$

Example 6. Determine whether $\sum_{n=1}^{\infty} (\cos(1))^n$ converges or diverges. If it converges, find its sum.

Solution 6. The sum is already in proper form with a common ratio that is less than 1. Therefore, the series converges and the sum is given by:

$$\sum_{n=1}^{\infty} (\cos(1))^n = \boxed{\frac{\cos(1)}{1-\cos(1)}}$$

Example 7. Determine whether $\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$ converges or diverges. If it converges, find its sum.

Solution 7. First rewrite the sum in proper form:

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = 5\sum_{n=0}^{\infty} \left(-\frac{64}{5}\right)^n$$

Since the common ratio is greater than 1, the series $\boxed{diverges}$.

Example 8. Determine whether $\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1}$ converges or diverges. If it converges, find its sum.

Solution 8. First rewrite the sum in proper form:

$$\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1} = 324\sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n$$

Since the common ratio is less than 1, the series converges and the sum is given by:

$$324\sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n = \frac{324}{1-\frac{4}{9}} = \boxed{\frac{2916}{5}}$$

General Divergence Test for Series

Definition 2 (Divergence Test)**.** Whenever dealing with any arbitrary series, this test provides a quick way of telling whether the series diverges. Given:

$$\sum_{n=k}^{\infty} a_n \quad \text{s.t.} \quad \lim_{n\to\infty} a_n \neq 0$$

the series diverges. Note that this is only a test for divergence, ergo if the limit does come out to be 0 the series does not necessarily converge.

Integral Test for Series

Definition 3 (Integral Test)**.** The integral test offers a way of determining the behavior of a series given that the sequence, a_n, is integrable. Now given a series, $\sum_{n=k}^{\infty} a_n$, below are the cases in which the integral test is useful:

Given	Condition
$\sum_{n=k}^{\infty} a_n$	If $\int_k^{\infty} a(n)\ \mathrm{d}n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges
$\sum_{n=k}^{\infty} a_n$	If $\int_k^{\infty} a(n)\ \mathrm{d}n$ diverges, then $\sum_{n=k}^{\infty} a_n$ diverges

Comparison Test for Series

Definition 4 (Comparison Test)**.** The comparison test offers a way of determining the behavior of a series without actually touching the original series. Now given a series, $\sum_{n=k}^{\infty} a_n$, below are the cases in which the comparison test is useful:

Scenario	Condition
$\sum_{n=k}^{\infty} a_n \leq \sum_{n=k}^{\infty} b_n$	If $\sum_{n=k}^{\infty} b_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges
$\sum_{n=k}^{\infty} c_n \leq \sum_{n=k}^{\infty} a_n$	If $\sum_{n=k}^{\infty} c_n$ diverges, then $\sum_{n=k}^{\infty} a_n$ diverges

Note that in the first scenario if $\sum_{n=k}^{\infty} b_n$ diverges, then $\sum_{n=k}^{\infty} a_n$ does not necessarily diverge. Similarly, in the second scenario if $\sum_{n=k}^{\infty} c_n$ converges, then $\sum_{n=k}^{\infty} a_n$ does not necessarily converge.

Limit Comparison Test for Series

Definition 5 (Limit Comparison Test)**.** The limit comparison test offers a way of determining the behavior of a series in relation to a similar series. Now given the original series, $\sum_{n=k}^{\infty} a_n$, and a series to compare to, $\sum_{n=k}^{\infty} b_n$, below are the cases in which the limit comparison test is useful:

Condition
If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ such that $L \in (0, \infty)$, then either both series diverge or both converge
If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then the original series converges iff the comparison series converges
If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then the original series diverges iff the comparison series diverges

Note that in the second scenario if the comparison series diverges, the original series can still converge. Similarly, in the third scenario if the comparison series converges, the original series can still diverge.

Example 9. Determine whether $\sum_{n=1}^{\infty} \arctan(n)$ converges or diverges.

Solution 9. Using the divergence test:

$$\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2}$$

Since the limit does not approach 0, by the divergence test the series $\boxed{diverges}$.

Example 10. Determine whether $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$ converges or diverges.

Solution 10. Using the divergence test:

$$\lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = 1$$

Since the limit does not approach 0, by the divergence test the series $\boxed{diverges}$.

Example 11. Determine whether $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ converges or diverges.

Solution 11. Using the divergence test:

$$\lim_{n\to\infty} \frac{n+1}{2n-3} = \frac{1}{2}$$

Since the limit does not approach 0, by the divergence test the series $\boxed{diverges}$.

Example 12. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

Solution 12. Using the integral test:

$$\int_1^{\infty} \frac{dn}{n} = \lim_{a\to\infty} \int_1^a \frac{dn}{n} = \lim_{a\to\infty} \ln|n|\Big|_1^a = \lim_{a\to\infty} \ln(a) = \infty$$

Therefore, by the integral test the series $\boxed{diverges}$.

Example 13. Determine the values of p that make $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge and diverge.

Solution 13. Using the integral test:

$$\int_1^{\infty} \frac{dn}{n^p} = \lim_{a\to\infty} \int_1^a \frac{dn}{n^p} = \lim_{a\to\infty} \frac{n^{1-p}}{1-p}\Big|_1^a = \lim_{a\to\infty} \left(\frac{a^{1-p}}{1-p} - \frac{1}{1-p}\right)$$

Now a little bit of analysis. If $p > 1$ then the limit will approach a finite value. If $p < 1$ the limit will go off to infinity. And lastly if $p = 1$ the above cannot say much since plugging in gives big issues. Looking back on Example 12 shows that at $p = 1$ this series actually diverges. Therefore, this series will behave as following depending on p:

$$\boxed{\text{Converges if } \ p > 1 \ \text{ and diverges if } \ p \leq 1}$$

Example 14. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges.

Solution 14. Using the integral test:

$$\int_1^{\infty} \frac{dn}{n^2+1} = \lim_{a\to\infty} \int_1^a \frac{dn}{n^2+1} = \lim_{a\to\infty} \arctan(n)\Big|_1^a = \lim_{a\to\infty} \Big(\arctan(a) - \arctan(1)\Big) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Therefore, by the integral test the series $\boxed{converges}$.

Example 15. Determine whether $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ converges or diverges.

Solution 15. Using the integral test:

$$\int_1^{\infty} \frac{dn}{2n+3} = \lim_{a\to\infty} \int_1^a \frac{dn}{2n+3} = \lim_{a\to\infty} \frac{1}{2} \ln|2n+3|\Big|_1^a = \frac{1}{2} \lim_{a\to\infty} \Big(\ln|2a+3| - \ln(5)\Big) = \infty$$

Therefore, by the integral test the series $\boxed{diverges}$.

Example 16. Determine whether $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converges or diverges.

Solution 16. A good series to compare to is given by:

$$\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1} \leq \sum_{n=1}^{\infty} \frac{n^2-1}{3n^4} = \sum_{n=1}^{\infty} \frac{1}{3n^2} - \sum_{n=1}^{\infty} \frac{1}{3n^4}$$

According to Example 13 both of the series on the right will converge. Therefore, by the comparison test the original series $\boxed{converges}$.

Example 17. Determine whether $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$ converges or diverges.

Solution 17. A good series to compare to is given by:

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{3}{n^{\frac{3}{2}}}$$

According to Example 13 the series on the right will converge. Therefore, by the comparison test the original series $\boxed{converges}$.

Example 18. Determine whether $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$ converges or diverges.

Solution 18. A good series to compare to is given by:

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

According to Example 14 the series on the right will converge. Therefore, by the comparison test the original series $\boxed{converges}$.

Example 19. Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges or diverges.

Solution 19. A good series to compare to is given by:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

According to Example 13 the series on the right will converge. Therefore, by the comparison test the original series $\boxed{converges}$.

Example 20. Determine whether $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$ converges or diverges.

Solution 20. A good series to compare to is given by:

$$\sum_{n=1}^{\infty} \frac{n-1}{n4^n} \leq \sum_{n=1}^{\infty} \frac{n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

The series on the right is a geometric series with a common ratio less than 1 making the series convergent. Therefore, by the comparison test the original series $\boxed{converges}$.

Example 21. Determine whether $\sum_{n=1}^{\infty} \frac{1}{3^n-n}$ converges or diverges.

Solution 21. To compare use $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a convergent series:

$$\lim_{n\to\infty} \frac{\frac{1}{3^n-n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n-n} = 1$$

Using the limit comparison test, the original series $\boxed{converges}$.

Example 22. Determine whether $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Solution 22. To compare use $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent series:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{\cos\left(\frac{1}{n}\right)}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

Using the limit comparison test, the original series $\boxed{diverges}$.

Example 23. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges or diverges.

Solution 23. To compare use $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent series:

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1$$

Using the limit comparison test, the original series $\boxed{converges}$.

Example 24. Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ converges or diverges.

Solution 24. To compare use $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent series:

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$$

Using the limit comparison test, the original series $\boxed{diverges}$.

Example 25. Determine whether $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$ converges or diverges.

Solution 25. To compare use $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent series:

$$\lim_{n\to\infty} \frac{\frac{n^2+2}{n^4+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+2n^2}{n^4+5} = 1$$

Using the limit comparison test, the original series $\boxed{converges}$.

Example 26. Determine whether $\sum_{n=1}^{\infty} \frac{4n^2+n+9}{7n^3+13}$ converges or diverges.

Solution 26. To compare use $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent series:

$$\lim_{n\to\infty} \frac{\frac{4n^2+n+9}{7n^3+13}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4n^3+n^2+9n}{7n^3+13} = \frac{4}{7}$$

Using the limit comparison test, the original series $\boxed{diverges}$.

Example 27. Determine whether $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges or diverges.

Solution 27. To compare use $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent series:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-\frac{2\cos\left(\frac{1}{n^2}\right)}{n^3}}{-\frac{2}{n^3}} = \cos(0) = 1$$

Using the limit comparison test, the original series $\boxed{converges}$.

TELESCOPING SERIES

Definition 6 (Telescoping Series)**.** A series in the form:

$$\sum_{i=1}^{n}(a_i - a_{i-1})$$

when expanded has many cancellations such that:

$$\begin{aligned}\sum_{i=1}^{n}(a_i - a_{i-1}) &= (\cancel{a_1} - a_0) + (\cancel{a_2} - \cancel{a_1}) + (\cancel{a_3} - \cancel{a_2}) + \cdots + (\cancel{a_{n-1}} - \cancel{a_{n-2}}) + (a_n - \cancel{a_{n-1}})\\ &= a_n - a_0\end{aligned}$$

Now that the finite case has been covered, what exactly happens at infinity? First remember that if a given series, $\sum_{i=1}^{\infty} a_n$, converges, then $\lim_{n\to\infty} a_n = 0$:

$$\lim_{n\to\infty}\sum_{i=1}^{n}(a_i - a_{i-1}) = \lim_{n\to\infty}(a_n - a_0)$$

$$\sum_{i=1}^{\infty}(a_i - a_{i-1}) = -a_0$$

Example 28. Determine whether $\sum_{n=0}^{\infty}\frac{1}{n^2+3n+2}$ converges or diverges. If it converges, find its sum.

Solution 28. First workout the finite series using the method of partial fractions:

$$\sum_{i=0}^{n}\frac{1}{i^2+3i+2} = \sum_{i=0}^{n}\frac{1}{(i+2)(i+1)} = \sum_{i=0}^{n}\left(\frac{1}{i+1} - \frac{1}{i+2}\right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right) = 1 - \frac{1}{n+2}$$

Now for the infinite case:

$$\sum_{i=0}^{\infty}\frac{1}{i^2+3i+2} = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{i^2+3i+2} = \lim_{n\to\infty}\left(1 - \frac{1}{n+2}\right) = \boxed{1}$$

Example 29. Determine whether $\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3}$ converges or diverges. If it converges, find its sum.

Solution 29. First workout the finite series using the method of partial fractions:

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{i^2+4i+3} = \sum_{i=1}^{n}\frac{1}{(i+1)(i+3)} = \frac{1}{2}\sum_{i=0}^{n}\left(\frac{1}{i+1} - \frac{1}{i+3}\right)\\ &= \frac{1}{2}\left[\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right)\right]\\ &= \frac{5}{12} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}\end{aligned}$$

Now for the infinite case:

$$\sum_{i=0}^{\infty}\frac{1}{i^2+4i+3} = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{i^2+4i+3} = \lim_{n\to\infty}\left(\frac{5}{12} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}\right) = \boxed{\frac{5}{12}}$$