Week 9 Attendance Solutions

MATH 23A

(1) (Q) Determine three positive numbers whose sum is $\mathcal{S}$ and whose product is maximal.

(A) Let x, y, and z represent our three positive numbers. We want to optimize the function:

$$f(x,y,z) = xyz$$

subject to the constraint:

$$g(x,y,z) = x + y + z = \mathcal{S}$$

Now you have the choice of method:

* Using parametrization means we want to identify $x = 25 - y - z$ from the constraint which restricts the original function down to:

$$\tilde{f}(y,z) = f(\mathcal{S} - y - z, y, z) = (\mathcal{S} - y - z)yz = \mathcal{S}yz - y^2z - yz^2$$

Now to determine the critical points of $\tilde{f}$:

$$\nabla\tilde{f} = \begin{pmatrix} \mathcal{S}z - 2yz - z^2 \\ \mathcal{S}z - y^2 - 2yz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Subtracting the second component from the first lets us know:

$$0 = y^2 - z^2 = (y - z)(y + z)$$

Now since $y + z = 0$ forces one of the two variables to be negative we ignore this condition and focus only on $y - z = 0$ which is the same as $y = z$. Plugging this back gives into the first component forces:

$$0 = \mathcal{S}z - 2z^2 - z^2 = z(-3z + \mathcal{S}) \implies z = 0, \frac{\mathcal{S}}{3}$$

We throw away the possibility $z = 0$ since this would make the product exactly 0 which is definitely not maximal. It follows that $y = \frac{\mathcal{S}}{3}$ and consequently:

$$x = \mathcal{S} - y - z = \mathcal{S} - \frac{\mathcal{S}}{3} - \frac{\mathcal{S}}{3} = \frac{\mathcal{S}}{3}$$

Therefore, the maximal product is given by $f\left(\frac{\mathcal{S}}{3}, \frac{\mathcal{S}}{3}, \frac{\mathcal{S}}{3}\right) = \frac{\mathcal{S}^3}{27}$.

* Now using Lagrange multipliers we have the setup:

$$\nabla f = \lambda \nabla g$$
$$\begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This lets us know that all of the components must equate, i.e.:

$$yz = xz = xy$$

This only occurs when $x = y = z$ which plugging into the constraint forces:

$$\mathcal{S} = x + y + z = 3x$$

providing the solution triplet $\left(\frac{\mathcal{S}}{3}, \frac{\mathcal{S}}{3}, \frac{\mathcal{S}}{3}\right)$ with maximal product $f\left(\frac{\mathcal{S}}{3}, \frac{\mathcal{S}}{3}, \frac{\mathcal{S}}{3}\right) = \frac{\mathcal{S}^3}{27}$.

(2) (Q) Calculate the maximal volume a box can achieve when it has to be contained in the closed ball of radius r:

$$\mathcal{B}_r(0) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq r^2\}$$

(*Hint: If you center the box at the origin and let x, y, and z represent half of the length in each direction then the function to optimize takes the form $f(x, y, z) = (2x)(2y)(2z) = 8xyz$*).

(A) First to determine the normal critical points we must satisfy:

$$\nabla f = \vec{0}$$
$$\begin{pmatrix} 8yz \\ 8xz \\ 8xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For all three components to simultaneously vanish at least two of the variables have to be identically zero. Such points will not interest us as they force the volume to vanish which certainly doesn't correspond to a maximum. Now for this problem parametrization is not as natural, thus we will approach via Lagrange multipliers. We have the function:

$$f(x, y, z) = 8xyz$$

subject to the constraint:

$$g(x, y, z) = x^2 + y^2 + z^2 = r^2$$

The Lagrange multipliers setup takes the form:

$$\nabla f = \lambda \nabla g$$
$$\begin{pmatrix} 8yz \\ 8xz \\ 8xy \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Isolating λ in each component forces:

$$\lambda = \frac{4yz}{x} = \frac{4xz}{y} = \frac{4xy}{z}$$

Cross multiplication of all possible pairs of fractions provides:

$$y^2 z = x^2 z \implies y = x$$
$$yz^2 = x^2 y \implies x = z$$
$$xz^2 = xy^2 \implies y = z$$

where we have abandoned solutions involving a negative sign since the construction of f forced x, y, and z to be positive from the very beginning. These conditions plugged into the constraint informs us:

$$r^2 = x^2 + y^2 + z^2 = 3x^2$$

It follows that the parameters of our box are $\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$ with maximal volume:

$$f\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = 8\left(\frac{r}{\sqrt{3}}\right)^3 = \frac{8r^3}{3^{\frac{3}{2}}}$$

Non-surprisingly the maximal volume is achieved when our box is a cube.