Integration by Parts

MATH 19B
Nathan Marianovsky

Definition 1 (Integration by Parts)**.** The formula for integration by parts reads as:

$$\int_a^b u \, \mathrm{d}v = uv\Big|_a^b - \int_a^b v \, \mathrm{d}u \quad \text{and} \quad \int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

for definite integrals and indefinite integrals, respectively. The trouble with this method is knowing how to tell what u and $\mathrm{d}v$ should be. Look at the following:

L	ogarithmic
I	nverse
P	ower
E	xponential
T	rigonometric

Inverse refers to trigonometric functions such as $\arctan(x)$, $\arccos(x)$, and $\arcsin(x)$. Any problem involving integration by parts will usually have any combination of two from the above table. The one that is higher takes precedence on being u, the rest is usually bundled with the $\mathrm{d}v$.

Example 1. Evaluate $\int_{-1}^{2} xe^{6x} \, \mathrm{d}x$

Solution 1. Because the integrand is a polynomial and exponential multiplied, the polynomial takes precedence as u:

$$u = x \quad \mathrm{d}u = \mathrm{d}x$$
$$\mathrm{d}v = e^{6x} \, \mathrm{d}x \quad v = \frac{1}{6}e^{6x}$$

Evaluating the integral by parts gives:

$$\begin{aligned}
\int_{-1}^{2} xe^{6x} \, \mathrm{d}x &= \frac{x}{6}e^{6x}\Big|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2} e^{6x} \, \mathrm{d}x \\
&= \left(\frac{x}{6}e^{6x} - \frac{1}{36}e^{6x}\right)\Big|_{-1}^{2} \\
&= \left(\frac{1}{3}e^{12} - \frac{1}{36}e^{12}\right) - \left(-\frac{1}{6}e^{-6} - \frac{1}{36}e^{-6}\right) \\
&= \boxed{\frac{11}{36}e^{12} + \frac{7}{36}e^{-6}}
\end{aligned}$$

Example 2. Evaluate $\int x\sqrt{x+1}\ \mathrm{d}x$

Solution 2. This can actually be accomplished two ways:

(i) First consider integration by parts. The function inside is the product of a polynomial and a power function. Identify the parts as:

$$u = x \qquad \mathrm{d}u = \mathrm{d}x$$
$$\mathrm{d}v = (x+1)^{\frac{1}{2}}\,\mathrm{d}x \qquad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

The integral now becomes:

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\,\mathrm{d}x \\ &= \boxed{\frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C_1}\end{aligned}$$

(ii) Now by a simple substitution:

$$u = x+1$$
$$\mathrm{d}u = \mathrm{d}x$$

the integral becomes:

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int (u-1)\sqrt{u}\,\mathrm{d}u \\ &= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,\mathrm{d}u \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \\ &= \boxed{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C_2}\end{aligned}$$

These two results have shown something rather interesting. That even though both methods of integration produce solutions, they produce different solutions, or so you would think so. In reality these two answers are the same because there exists a family of antiderivatives for an indefinite integral. Thus these two answers are off of each other just by a constant. The easiest way to check this fact is to find the difference between the two solutions.

Example 3. Evaluate $\int \ln(x)\ \mathrm{d}x$

Solution 3. To approach this by parts consider that the function inside the is the multiplication of $\ln(x)$, a logarithmic function, and 1, a polynomial. The parts will take the form:

$$u = \ln(x) \qquad \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$
$$\mathrm{d}v = \mathrm{d}x \qquad v = x$$

The integral now becomes:

$$\begin{aligned}\int \ln(x)\,\mathrm{d}x &= x\ln(x) - \int \mathrm{d}x \\ &= \boxed{x\ln(x) - x + C}\end{aligned}$$

Example 4. Evaluate $\int x^2 e^x \, \mathrm{d}x$

Solution 4. Begin by identifying the parts:

$$\begin{aligned} u &= x^2 & \mathrm{d}u &= 2x \, \mathrm{d}x \\ \mathrm{d}v &= e^x \mathrm{d}x & v &= e^x \end{aligned}$$

The integral now becomes:

$$\int x^2 e^x \, \mathrm{d}x = x^2 e^x - 2 \int x e^x \, \mathrm{d}x$$

Unfortunately this problem will require integration by parts once more. This time the parts are:

$$\begin{aligned} u &= x & \mathrm{d}u &= \mathrm{d}x \\ \mathrm{d}v &= e^x \, \mathrm{d}x & v &= e^x \end{aligned}$$

The solution take the form:

$$\begin{aligned} \int x^2 e^x \, \mathrm{d}x &= x^2 e^x - 2 \int x e^x \, \mathrm{d}x \\ &= x^2 e^x - 2\Big(x e^x - \int e^x \, \mathrm{d}x \Big) \\ &= x^2 e^x - 2x e^x - 2e^x + C \\ &= \boxed{(x^2 - 2x - 2)e^x + C} \end{aligned}$$

Example 5. Evaluate $\int_{-1}^{1} \arctan(x) \, \mathrm{d}x$

Solution 5. In this case the function inside is the multiplication of an inverse trigonometric function, $\arctan(x)$, and a polynomial, 1. Therefore, the parts are:

$$\begin{aligned} u &= \arctan(x) & \mathrm{d}u &= \frac{1}{1+x^2} \, \mathrm{d}x \\ \mathrm{d}v &= \mathrm{d}x & v &= x \end{aligned}$$

The integral now becomes:

$$\begin{aligned} \int_{-1}^{1} \arctan(x) \, \mathrm{d}x &= x \arctan(x) \Big|_{-1}^{1} - \int_{-1}^{1} \frac{x}{1+x^2} \, \mathrm{d}x \\ &= \arctan(1) + \arctan(-1) - \int_{-1}^{1} \frac{x}{1+x^2} \, \mathrm{d}x \\ &= - \int_{-1}^{1} \frac{x}{1+x^2} \, \mathrm{d}x \end{aligned}$$

To evaluate the remaining integral, make the substitution:

$$\begin{aligned} u &= 1 + x^2 \\ \mathrm{d}u &= 2x \, \mathrm{d}x \rightarrow \frac{\mathrm{d}u}{2} = \mathrm{d}x \end{aligned}$$

Reevaluating the bounds and plugging in gives:

$$\begin{aligned} - \int_{-1}^{1} \frac{x}{1+x^2} \, \mathrm{d}x &= -\frac{1}{2} \int_{2}^{2} \frac{\mathrm{d}u}{u} \\ &= \boxed{0} \end{aligned}$$

All of this work could have actually been avoided knowing that $\arctan(x)$ is an odd function and the integral of an odd function over a symmetrical region is always 0.

Example 6. Evaluate $\int \frac{\ln(x)}{x^2}\ \mathrm{d}x$

Solution 6. Begin by identifying the parts:

$$\begin{aligned} u &= \ln(x) \quad & \mathrm{d}u &= x^{-1}\ \mathrm{d}x \\ \mathrm{d}v &= x^{-2}\ \mathrm{d}x \quad & v &= -x^{-1} \end{aligned}$$

The integral now becomes:

$$\begin{aligned} \int \frac{\ln(x)}{x^2}\ \mathrm{d}x &= -\frac{\ln(x)}{x} + \int x^{-2}\ \mathrm{d}x \\ &= -\frac{\ln(x)}{x} - \frac{1}{x} + C \\ &= \boxed{-\frac{\ln(x)+1}{x} + C} \end{aligned}$$

Example 7. Evaluate $\int_1^2 (\ln(x))^2\ \mathrm{d}x$

Solution 7. Begin by identifying the parts:

$$\begin{aligned} u &= (\ln(x))^2 \quad & \mathrm{d}u &= \frac{2\ln(x)}{x}\ \mathrm{d}x \\ \mathrm{d}v &= \mathrm{d}x \quad & v &= x \end{aligned}$$

The integral now becomes:

$$\begin{aligned} \int_1^2 (\ln(x))^2\ \mathrm{d}x &= x(\ln(x))^2\Big|_1^2 - 2\int_1^2 \ln(x)\ \mathrm{d}x \\ &= \Big(x(\ln(x))^2 - 2x\ln(x) + 2x\Big)\Big|_1^2 \\ &= \boxed{2(\ln(2))^2 - 4\ln(2) + 2} \end{aligned}$$

Example 8. Evaluate $\int \cos(x)e^x\ \mathrm{d}x$

Solution 8. The inside composes of trigonometric and exponential functions. Therefore, the parts are:

$$\begin{aligned} u &= e^x \quad & \mathrm{d}u &= e^x\ \mathrm{d}x \\ \mathrm{d}v &= \cos(x)\ \mathrm{d}x \quad & v &= \sin(x) \end{aligned}$$

The integral now becomes:

$$\int \cos(x)e^x\ \mathrm{d}x = \sin(x)e^x - \int \sin(x)e^x\ \mathrm{d}x$$

Integration by parts is needed once more, this time the parts are:

$$\begin{aligned} u &= e^x \quad & \mathrm{d}u &= e^x\ \mathrm{d}x \\ \mathrm{d}v &= \sin(x)\ \mathrm{d}x \quad & v &= -\cos(x) \end{aligned}$$

The integral becomes:

$$\begin{aligned} \int \cos(x)e^x\ \mathrm{d}x &= \sin(x)e^x - \int \sin(x)e^x\ \mathrm{d}x \\ &= \sin(x)e^x - \Big(-\cos(x)e^x + \int \cos(x)e^x\ \mathrm{d}x\Big) = \sin(x)e^x + \cos(x)e^x - \int \cos(x)e^x\ \mathrm{d}x \end{aligned}$$

Solving for the integral gives:

$$\int \cos(x)e^x\ \mathrm{d}x = \boxed{\frac{1}{2}(\sin(x)+\cos(x))e^x + C}$$

Example 9. Evaluate $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) \, \mathrm{d}x$

Solution 9. The above has a combination of a trigonometric function, $\arctan$, and a polynomial, 1. Therefore, the parts are:

$$\begin{aligned} u &= \arctan\left(\frac{1}{x}\right) & \mathrm{d}u &= -\frac{1}{1+x^2} \, \mathrm{d}x \\ \mathrm{d}v &= \mathrm{d}x & v &= x \end{aligned}$$

The integral now becomes:

$$\begin{aligned} \int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) \mathrm{d}x &= x \arctan\left(\frac{1}{x}\right)\bigg|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{1+x^2} \, \mathrm{d}x \\ &= \left[x \arctan\left(\frac{1}{x}\right) + \frac{1}{2} \ln|1+x^2|\right]\bigg|_1^{\sqrt{3}} \\ &= \left(\sqrt{3} \arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2}\ln(4)\right) - \left(\arctan(1) + \frac{1}{2}\ln(2)\right) \\ &= \boxed{\left(\frac{1}{2\sqrt{3}} - \frac{1}{4}\right)\pi + \frac{1}{2}\ln(2)} \end{aligned}$$

Example 10. Evaluate $\int_1^4 e^{\sqrt{x}} \, \mathrm{d}x$

Solution 10. First a substitution:

$$\begin{aligned} w &= x^{\frac{1}{2}} \\ \mathrm{d}w &= \frac{1}{2} x^{-\frac{1}{2}} \, \mathrm{d}x \rightarrow 2x^{\frac{1}{2}} \, \mathrm{d}w = \mathrm{d}x \rightarrow 2w \, \mathrm{d}w = \mathrm{d}x \end{aligned}$$

Reevaluating the bounds and plugging in gives:

$$\int_1^4 e^{\sqrt{x}} \, \mathrm{d}x = 2 \int_1^2 w e^w \, \mathrm{d}w$$

which is a familiar integration by parts problem. Identify the parts as:

$$\begin{aligned} u &= w & \mathrm{d}u &= \mathrm{d}w \\ \mathrm{d}v &= e^w \, \mathrm{d}w & v &= e^w \end{aligned}$$

Now the integral becomes:

$$\begin{aligned} 2\int_1^2 w e^w \, \mathrm{d}w &= 2\left[w e^w \Big|_1^2 - \int_1^2 e^w \, \mathrm{d}w\right] \\ &= 2\left[w e^w - e^w\right]\Big|_1^2 \\ &= 2\left[\left(2e^2 - e^2\right) - \left(e - e\right)\right] \\ &= \boxed{2e^2} \end{aligned}$$