Notes on Curvature & Acceleration Components

MATH 23A

Derivation

We are given a curve $\mathcal{C} \subset \mathbb{R}^3$, which is explicitly parametrized via:

$$\vec{r} : \mathbb{R} \to \mathcal{C} \quad \text{given by} \quad \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

For such a parametrization we can always compute the velocity $\vec{v}(t) = \vec{r}\,'(t)$ and associated speed $v(t) = \|\vec{v}(t)\|$. Now define the *unit velocity vector*:

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)}$$

and note that by definition $\vec{T}(t) \cdot \vec{T}(t) = 1$. Since the dot product respects the product rule we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\vec{T}(t) \cdot \vec{T}(t)\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(1\Big) \\ \frac{\mathrm{d}\vec{T}}{\mathrm{d}t} \cdot \vec{T} + \vec{T} \cdot \frac{\mathrm{d}\vec{T}}{\mathrm{d}t} &= 0 \\ 2\vec{T} \cdot \frac{\mathrm{d}\vec{T}}{\mathrm{d}t} &= 0 \end{aligned}$$

Interpreting this result states that $\vec{T} \perp \frac{\mathrm{d}\vec{T}}{\mathrm{d}t}$. This perpendicular vector allows us to write down the *unit normal vector*:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

We are now ready to define the *curvature* of the curve. If the curve were to be parametrized by *arclength*, represented by s, then the curvature is calculated via:

$$\kappa = \left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right\|$$

Since we are not typically parametrized by arclength, we instead obtain via the chain rule:

$$\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\| = \left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\right\| = \left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right\|\left\|\frac{\mathrm{d}s}{\mathrm{d}t}\right\| = \kappa v$$

where v represents the same speed from above. Plugging this information into the definition of the unit normal vector provides:

$$\vec{T}\,' = \kappa v \vec{N}$$

Next up we want to use the setup $\vec{v}(t) = v(t)\vec{T}(t)$ and differentiate it to obtain:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(v(t)\vec{T}(t)\Big) \\ \vec{a} &= \frac{\mathrm{d}v}{\mathrm{d}t}\vec{T} + v\vec{T}\,' \\ \vec{a} &= \frac{\mathrm{d}v}{\mathrm{d}t}\vec{T} + \kappa v^2 \vec{N} \end{aligned}$$

where $\vec{a}$ is the acceleration vector. From here we identify the *tangential* and *normal components of acceleration* as:

$$a_T = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \text{and} \quad a_N = \kappa v^2$$

Rewriting the Formulas

The above might have you a bit scared as the calculations are not exactly in a simple form. To rewrite the formulas above into a simpler form we want to take advantage of the fact that $\overrightarrow{T} \perp \overrightarrow{N}$ so that:

$$\begin{aligned} \overrightarrow{a} \cdot \overrightarrow{T} &= \left(a_T \overrightarrow{T} + a_N \overrightarrow{N}\right) \cdot \overrightarrow{T} \\ \overrightarrow{a} \cdot \overrightarrow{T} &= a_T \overrightarrow{T} \cdot \overrightarrow{T} + a_N \overrightarrow{N} \cdot \overrightarrow{T} \\ \overrightarrow{a} \cdot \overrightarrow{T} &= a_T \end{aligned}$$

The last line is close to want we truly want. To identify a_T strictly in terms of $\overrightarrow{v}$, v, and $\overrightarrow{a}$ we have:

$$\boxed{a_T = \overrightarrow{a} \cdot \overrightarrow{T} = \overrightarrow{a} \cdot \frac{\overrightarrow{v}}{v} = \frac{\overrightarrow{a} \cdot \overrightarrow{v}}{v}}$$

Once again playing on the orthonormality of $\overrightarrow{T}$ and $\overrightarrow{N}$:

$$\begin{aligned} \overrightarrow{a} \times \overrightarrow{T} &= \left(a_T \overrightarrow{T} + a_N \overrightarrow{N}\right) \times \overrightarrow{T} \\ \overrightarrow{a} \times \overrightarrow{T} &= a_T \overrightarrow{T} \times \overrightarrow{T} + a_N \overrightarrow{N} \times \overrightarrow{T} \\ \overrightarrow{a} \times \overrightarrow{T} &= a_N \overrightarrow{N} \times \overrightarrow{T} \\ \|\overrightarrow{a} \times \overrightarrow{T}\| &= a_N \|\overrightarrow{N} \times \overrightarrow{T}\| \\ \|\overrightarrow{a} \times \overrightarrow{T}\| &= a_N \|\overrightarrow{N}\| \|\overrightarrow{T}\| \\ \|\overrightarrow{a} \times \overrightarrow{T}\| &= a_N \end{aligned}$$

The last line can be rewritten as:

$$\boxed{a_N = \|\overrightarrow{a} \times \overrightarrow{T}\| = \left\|\overrightarrow{a} \times \frac{\overrightarrow{v}}{v}\right\| = \frac{\|\overrightarrow{a} \times \overrightarrow{v}\|}{v}}$$

Finally to compute the curvature we take $\kappa v^2 = a_N$ and rewrite it as:

$$\boxed{\kappa = \frac{\|\overrightarrow{a} \times \overrightarrow{v}\|}{v^3}}$$

Examples

- Consider the curve parametrized by:

$$\overrightarrow{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}$$

The associated velocity and acceleration vectors are given by:

$$\overrightarrow{v}(t) = \overrightarrow{r}'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{a}(t) = \overrightarrow{r}''(t) = \begin{pmatrix} -\cos(t) \\ -\sin(t) \\ 0 \end{pmatrix}$$

The speed is calculated to be $v(t) = \|\overrightarrow{v}(t)\| = \sqrt{\cos^2(t) + \sin^2(t) + 1} = \sqrt{2}$. It follows that the curvature is:

$$\kappa(t) = \frac{\|\overrightarrow{a}(t) \times \overrightarrow{v}(t)\|}{v^3(t)} = \frac{1}{(\sqrt{2})^3}\left\| \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos(t) & -\sin(t) & 0 \\ -\sin(t) & \cos(t) & 1 \end{pmatrix} \right\| = \frac{1}{(\sqrt{2})^3}\left\| \begin{pmatrix} -\sin(t) \\ \cos(t) \\ -1 \end{pmatrix} \right\| = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}$$

Turns out that the curvature of the *helix* is constant and the acceleration components are given as:

$$a_T(t) = \frac{\overrightarrow{a}(t) \cdot \overrightarrow{v}(t)}{v(t)} = \frac{0}{\sqrt{2}} = 0 \quad \text{and} \quad a_N(t) = \kappa(t) v^2(t) = \frac{1}{2} \cdot (\sqrt{2})^2 = 1$$

- Consider the curve parametrized by:

$$\overrightarrow{r}(t) = \begin{pmatrix} \cos^3(t) \\ \sin^3(t) \\ 0 \end{pmatrix}$$

The associated velocity and acceleration vectors are given by:

$$\overrightarrow{v}(t) = \overrightarrow{r}'(t) = \begin{pmatrix} -3\cos^2(t)\sin(t) \\ 3\sin^2(t)\cos(t) \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{a}(t) = \overrightarrow{r}''(t) = \begin{pmatrix} -3(-2\cos(t)\sin^2(t) + \cos^3(t)) \\ 3(2\sin(t)\cos^2(t) - \sin^3(t)) \\ 0 \end{pmatrix}$$

The speed is calculated to be $v(t) = \|\overrightarrow{v}(t)\| = \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} = 3|\sin(t)\cos(t)|$. For the sake of simplicity we will compute everything at $t = \frac{\pi}{4}$. The above reduces down to:

$$\overrightarrow{v}\left(\frac{\pi}{4}\right) = \begin{pmatrix} -\frac{3}{(\sqrt{2})^3} \\ \frac{3}{(\sqrt{2})^3} \\ 0 \end{pmatrix}, \quad \overrightarrow{a}\left(\frac{\pi}{4}\right) = \begin{pmatrix} \frac{3}{(\sqrt{2})^3} \\ \frac{3}{(\sqrt{2})^3} \\ 0 \end{pmatrix}, \quad \text{and} \quad v\left(\frac{\pi}{4}\right) = \frac{3}{2}$$

It follows that the curvature is:

$$\kappa\left(\frac{\pi}{4}\right) = \frac{\left\|\overrightarrow{a}\left(\frac{\pi}{4}\right) \times \overrightarrow{v}\left(\frac{\pi}{4}\right)\right\|}{v^3\left(\frac{\pi}{4}\right)} = \left(\frac{2}{3}\right)^3 \left\| \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{(\sqrt{2})^3} & \frac{3}{(\sqrt{2})^3} & 0 \\ -\frac{3}{(\sqrt{2})^3} & \frac{3}{(\sqrt{2})^3} & 0 \end{pmatrix} \right\| = \frac{8}{9} \left\| \begin{pmatrix} 0 \\ 0 \\ \frac{18}{(\sqrt{2})^6} \end{pmatrix} \right\| = \frac{8}{9} \cdot \frac{9}{4} = 2$$

The acceleration components are given as:

$$a_T\left(\frac{\pi}{4}\right) = \frac{\overrightarrow{a}\left(\frac{\pi}{4}\right) \cdot \overrightarrow{v}\left(\frac{\pi}{4}\right)}{v\left(\frac{\pi}{4}\right)} = \frac{0}{\frac{3}{2}} = 0 \quad \text{and} \quad a_N\left(\frac{\pi}{4}\right) = \kappa\left(\frac{\pi}{4}\right) v^2\left(\frac{\pi}{4}\right) = 2 \cdot \frac{9}{4} = \frac{9}{2}$$

The two curves can be visualized as the standard helix and astroid shapes:

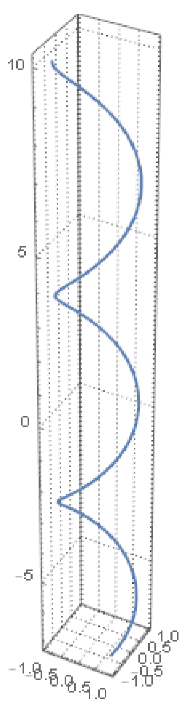

Fig. 1: First Example

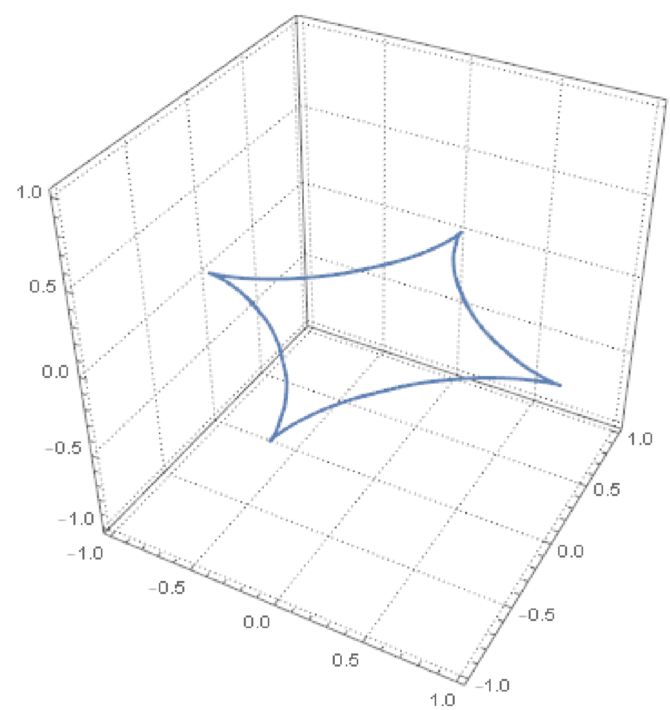

Fig. 2: Second Example

A Look Beyond Just Curvature (Frenet-Serret Frame)

The setup on the first two pages almost provides the setup for what is known as the *Frenet-Serret Frame*, an orthonormal basis that changes along the curve as the parameter t changes. Let us define the *unit binormal vector* as:

$$\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$$

By the same argument as on the first page we must have $\overrightarrow{B} \perp \frac{\mathrm{d}\overrightarrow{B}}{\mathrm{d}t}$ where:

$$\overrightarrow{B}' = \overrightarrow{T}' \times \overrightarrow{N} + \overrightarrow{T} \times \overrightarrow{N}' = \overrightarrow{T} \times \overrightarrow{N}'$$

by using the fact that $\overrightarrow{T}' \parallel \overrightarrow{N}$. Comparing $\overrightarrow{B}$ and $\overrightarrow{B}'$ forces $\overrightarrow{B}' \parallel \overrightarrow{N}$ and so we identify:

$$\overrightarrow{B}' = -\tau v \overrightarrow{N}$$

where τ represents the *torsion*. One last thing that we need to observe is:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{N}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{B}(t) \times \overrightarrow{T}(t) \\
\overrightarrow{N}' &= \overrightarrow{B}' \times \overrightarrow{T} + \overrightarrow{B} \times \overrightarrow{T}' \\
&= -\tau v \overrightarrow{N} \times \overrightarrow{T} + (\overrightarrow{T} \times \overrightarrow{N}) \times \overrightarrow{T}' \\
&= \tau v \overrightarrow{B} + \left(\overrightarrow{T} \times (\overrightarrow{N} \times \overrightarrow{T}') + \overrightarrow{N} \times (\overrightarrow{T}' \times \overrightarrow{T})\right) \\
&= \tau v \overrightarrow{B} + \left(\kappa v \overrightarrow{N} \times (\overrightarrow{N} \times \overrightarrow{T})\right) \\
&= \tau v \overrightarrow{B} - \kappa v \overrightarrow{N} \times \overrightarrow{B} \\
&= \tau v \overrightarrow{B} - \kappa v \overrightarrow{T}
\end{aligned}$$

where we used the Jacobi Identity:

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$$

on the 4th line. Putting everything together provides the Frenet-Serret Frame:

$$\begin{aligned}
\overrightarrow{T}' &= \kappa v \overrightarrow{N} \\
\overrightarrow{N}' &= -\kappa v \overrightarrow{T} + \tau v \overrightarrow{B} \\
\overrightarrow{B}' &= -\tau v \overrightarrow{N}
\end{aligned}$$

which looks a little nicer in matrix form:

$$\begin{pmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa v & 0 \\ -\kappa v & 0 & \tau v \\ 0 & -\tau v & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{pmatrix}$$

To see what this frame looks like geometrically I suggest you visit this page.