Arc Length & Sequences

MATH 19B
Nathan Marianovsky

Arc Length

Definition 1 (Arc Length)**.** The length of any arbitrary curve, $f(x)$, on the interval $[a,b]$ is defined as:

$$\textbf{Arc Length} = \int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x$$

where $f'(x)$ represents the derivative.

Example 1. Find the length of the curve $y = 1 + 6x^{\frac{3}{2}}$ on the interval $[0,1]$.

Solution 1. According to the definition:

$$\textbf{Arc Length} = \int_0^1 \sqrt{1+(9x^{\frac{1}{2}})^2}\,\mathrm{d}x = \int_0^1 \sqrt{1+81x}\,\mathrm{d}x = \frac{2(1+81x)^{\frac{3}{2}}}{243}\bigg|_0^1 = \boxed{\frac{2}{243}\left[82^{\frac{3}{2}}-1\right]}$$

Example 2. Find the length of the curve $y = \frac{x^2}{2} - \frac{\ln(x)}{4}$ on the interval $[2,4]$.

Solution 2. According to the definition:

$$\begin{aligned}\textbf{Arc Length} &= \int_2^4 \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,\mathrm{d}x = \int_2^4 \sqrt{x^2-\frac{1}{16x^2}+\frac{1}{2}}\,\mathrm{d}x = \int_2^4 \sqrt{\frac{16x^4+8x^2+1}{16x^2}}\,\mathrm{d}x \\ &= \int_2^4 \sqrt{\frac{(4x^2+1)^2}{16x^2}}\,\mathrm{d}x = \int_2^4 \left(x+\frac{1}{4x}\right)\mathrm{d}x = \left(\frac{x^2}{2}+\frac{1}{4}\ln|x|\right)\bigg|_2^4 = \boxed{6+\frac{1}{4}\ln(2)}\end{aligned}$$

Example 3. Find the length of the curve $y = \ln(\cos(x))$ on the interval $\left[0,\frac{\pi}{3}\right]$.

Solution 3. According to the definition:

$$\begin{aligned}\textbf{Arc Length} &= \int_0^{\frac{\pi}{3}} \sqrt{1+\left(-\frac{\sin(x)}{\cos(x)}\right)^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2(x)}\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \sec(x)\,\mathrm{d}x \\ &= \ln|\sec(x)+\tan(x)|\bigg|_0^{\frac{\pi}{3}} = \boxed{\ln(2+\sqrt{3})}\end{aligned}$$

Example 4. Find the length of the curve $y = \cosh(x)$ on the interval $[0,1]$.

Solution 4. According to the definition:

$$\textbf{Arc Length} = \int_0^1 \sqrt{1+\sinh^2(x)}\,\mathrm{d}x = \int_0^1 \cosh(x)\,\mathrm{d}x = \sinh(x)\bigg|_0^1 = \boxed{\sinh(1)}$$

Notice that for this curve the arc length is exactly equal to the area under the curve on the interval.

Example 5. Find the length of the curve $y = \ln(\sec(x))$ on the interval $\left[0,\frac{\pi}{4}\right]$.

Solution 5. According to the definition:

$$\textbf{Arc Length} = \int_0^{\frac{\pi}{4}} \sqrt{1+\left(\frac{\sec(x)\tan(x)}{\sec(x)}\right)^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \sec(x)\,\mathrm{d}x = \ln|\sec(x)+\tan(x)|\bigg|_0^{\frac{\pi}{4}} = \boxed{\ln(1+\sqrt{2})}$$

Sequences

Definition 2 (Sequences)**.** A sequence, $\{a_n\}$, that satisfies:

$$\lim_{n\to\infty} a_n = L \quad \text{where } L \in \mathbb{R}$$

is said to converge. Otherwise the sequence diverges.

Special Limits

Definition 3 (Sequences)**.** These two limits come up often enough when dealing with sequences and series so much that it is worth mentioning:

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \qquad \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Example 6. Determine whether the sequence $a_n = \frac{3+5n^2}{n+n^2}$ converges or diverges.

Solution 6.

$$\lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = 5$$

The sequence $\boxed{converges}$.

Example 7. Determine whether the sequence $a_n = \cos\left(\frac{2}{n}\right)$ converges or diverges.

Solution 7.

$$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos(0) = 1$$

The sequence $\boxed{converges}$.

Example 8. Determine whether the sequence $a_n = \frac{e^n + e^{-n}}{e^{2n}-1}$ converges or diverges.

Solution 8.

$$\lim_{n\to\infty} \frac{e^n + e^{-n}}{e^{2n}-1} = 0$$

The sequence $\boxed{converges}$.

Example 9. Determine whether the sequence $a_n = \arctan(2n)$ converges or diverges.

Solution 9.

$$\lim_{n\to\infty} \arctan(2n) = \frac{\pi}{2}$$

The sequence $\boxed{converges}$.

Example 10. Determine whether the sequence $a_n = \frac{\sin(2n)}{1+\sqrt{n}}$ converges or diverges.

Solution 10. Use the Squeeze Theorem to evaluate the limit:

$$-\frac{1}{1+\sqrt{n}} \le \frac{\sin(2n)}{1+\sqrt{n}} \le \frac{1}{1+\sqrt{n}}$$

$$\lim_{n\to\infty}\left[-\frac{1}{1+\sqrt{n}} \le \frac{\sin(2n)}{1+\sqrt{n}} \le \frac{1}{1+\sqrt{n}}\right]$$

$$0 \le \lim_{n\to\infty} \frac{\sin(2n)}{1+\sqrt{n}} \le 0 \implies \lim_{n\to\infty} \frac{\sin(2n)}{1+\sqrt{n}} = 0$$

Therefore, the sequence $\boxed{converges}$.

Example 11. Determine whether the sequence $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$ converges or diverges.

Solution 11. Rewrite the sequence as:

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\} = \left\{2^{\frac{1}{2}}, 2^{\frac{3}{4}}, 2^{\frac{7}{8}}, \dots\right\}$$

From this it is clear that the sequence follows the formula:

$$a_n = 2^{\frac{2^n-1}{2^n}}$$

where the starting point is $n = 1$. Now to determine the convergence of this sequence:

$$\lim_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = 2$$

Therefore, the sequence $\boxed{converges}$.

Example 12. Determine whether the sequence $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$ converges or diverges.

Solution 12. Hopefully this sequence of numbers looks a little bit familiar since it is the Fibonacci sequence. The Fibonacci numbers are generated by the formula:

$$a_n = a_{n-1} + a_{n-2}$$

such that $a_1 = 1$ and $a_2 = 1$. Since the above is not in closed form, it is not possible to determine the limit. Although you could go through the trouble of finding the closed form for generating the Fibonacci numbers, to determine the convergence behavior of the sequence not that much work is required. Notice that this sequence is monotonically increasing, faster and faster. Therefore, as $n \to \infty$ the sequence will approach ∞. The sequence as a result $\boxed{diverges}$.

Example 13. Determine whether the sequence $\{5, 1, 5, 1, 5, 1, \dots\}$ converges or diverges.

Solution 13. Begin by defining the closed form of the sequence as:

$$a_n = 3 + 2(-1)^n$$

where the starting point is $n = 0$. Now to determine the convergence of this sequence:

$$\lim_{n\to\infty} 3 + 2(-1)^n = \{5, 1\}$$

As n approaches large values the term with the n can be either ± 1. Therefore, the limit can approach two values, but since it does not approach a singular value, the sequence $\boxed{diverges}$.

Example 14. Determine what values of r make the sequence $\{r, r^2, r^3, r^4, r^5, \dots\}$ converge.

Solution 14. Begin by defining the closed form of the sequence as:

$$a_n = r^n$$

where the starting point is $n = 1$. Now to determine the convergence of this sequence:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & |r| < 1 \\ 1 & r = 1 \\ \infty & |r| > 1 \end{cases}$$

Therefore, the sequence converges if $\boxed{|r| \leq 1}$.