Answer Key to Group Exercises

MATH 11A - Discussion Sections C & F

(1) For each function transformation graph the result when acted on $f(x)$ and write down its corresponding formula given $f(x) = e^x$:
  (a) Shift down by 3 $\quad [g(x) = f(x) - 3 = e^x - 3]$
  (b) Shift right by 7 $\quad [g(x) = f(x-7) = e^{x-7}]$
  (c) Reflect about the x-axis $\quad [g(x) = -f(x) = -e^x]$
  (d) Reflect about the y-axis $\quad [g(x) = f(-x) = e^{-x}]$
  (e) Reflect about the origin $\quad [g(x) = -f(-x) = -e^{-x}]$

(2) Find the domain of the following functions:
  (a) $f(x) = \frac{7-x}{e^x - 1} \quad [x \in (-\infty, 0) \cup (0, \infty)]$
  (b) $g(x) = \frac{e^x}{x^2 - x - 1} \quad \left[x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right)\right]$
  (c) $h(x) = \frac{x^\pi - x + 1}{x^3 + x^2 - x} \quad \left[x \in \left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1-\sqrt{5}}{2}, 0\right) \cup \left(0, \frac{-1+\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, \infty\right)\right]$
  (d) $j(x) = \frac{1}{e^{x^2+2x-1}} \quad [x \in \mathbb{R}]$
  (e) $m(x) = \frac{x+\pi}{e^{\sin(x)}} \quad [x \in \mathbb{R}]$

(3) The half-life of Uranium-238 is 4.51×10^9 years. Given that we start with a sample of 500mg, answer the following:
  (a) Write down a formula that models how much of the sample is left after t years.
  (b) How much of the sample remains after $t = 10^{20}$ years?
  (c) At what value of t does the sample have 200mg left?

(4) The half-life of Polonium-210 is 138.376 days. Given that we start with a sample of 500mg, answer the following:
  (a) Write down a formula that models how much of the sample is left after t days. $\quad \left[P = 500\left(\frac{1}{2}\right)^{\frac{t}{138.376}}\right]$
  (b) How much of the sample remains after 2 years? $\quad \left[P(730) = 500\left(\frac{1}{2}\right)^{\frac{730}{138.376}}\right]$
  (c) At what value of t does the sample have 200mg left? $\quad \left[t = 138.376 \log_{\frac{1}{2}}\left(\frac{2}{5}\right)\right]$

(5) Determine whether the following functions are one-to-one (injective) on the provided domains:
  (a) $f(x) = \cos(x)$ where $x \in \mathbb{R}$ [No]
  (b) $g(x) = \cos(x)$ where $x \in [0, \pi]$ [Yes]
  (c) $h(x) = \frac{x^2 - x + 7}{x - 1}$ where $x \in (-\infty, 1) \cup (1, \infty)$ [No]
  (d) $j(x) = \ln(x)$ where $x \in (0, \infty)$ [Yes]
  (e) $m(x) = \ln|x|$ where $x \in \mathbb{R}$ [No]
  (f) $n(x) = \frac{x}{|x|}$ where $x \in (-\infty, 0) \cup (0, \infty)$ [No]

(6) Determine a formula for the inverse, if it exists:
  (a) $f(x) = \frac{2x-1}{5x+3}$ where $x \in \left(-\infty, -\frac{3}{5}\right) \cup \left(-\frac{3}{5}, \infty\right) \quad \left[f^{-1}(x) = -\frac{3x+1}{5x-2}\right]$
  (b) $g(x) = e^{x-3}$ where $x \in \mathbb{R} \quad [g^{-1}(x) = \ln(x) + 3]$
  (c) $h(x) = e^{(7-x)^3+10}$ where $x \in \mathbb{R} \quad \left[h^{-1}(x) = 7 - \sqrt[3]{\ln(x) - 10}\right]$
  (d) $j(x) = \ln(x^3 + 4)$ where $x \in \mathbb{R}$ [DNE]
  (e) $m(x) = \ln(x^2)$ where $x \in \mathbb{R}$ [DNE]
  (f) $n(x) = x^2$ where $x \in \mathbb{R}$ [DNE]
  (g) $p(x) = x^2$ where $x \in [0, \infty) \quad [p^{-1}(x) = \sqrt{x}]$

(7) Graph the following piecewise functions and determine if they are continuous:

a) $f(x) = \begin{cases} |x|, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ [Yes] b) $g(x) = \begin{cases} x+1, & x < 0 \\ x+10, & x \geq 0 \end{cases}$ [No] c) $h(x) = \begin{cases} x+1, & x < 0 \\ 0, & x = 0 \\ 1-x, & x > 0 \end{cases}$ [No]

(8) Simplify the following expressions in terms of a single logarithm:

(a) $7\log_{12}(x) + 21\log_{12}(y)$ $\left[\log_{12}\left(x^7 y^{21}\right)\right]$

(b) $3\log(x) - 6\log(y)$ $\left[\log\left(x^3 y^{-6}\right)\right]$

(c) $5\ln(x+y) - 21\ln(x) - 8\ln(y)$ $\left[\ln\left((x+y)^5 x^{-21} y^{-8}\right)\right]$

(d) $\log_7(x) + \ln(y)$ $\left[\ln\left(x^{\frac{1}{\ln(7)}} y\right)\right]$

(e) $\ln(y^2) - \ln(x) + 2\log(y)$ $\left[\ln\left(y^{\left(2+\frac{2}{\ln(10)}\right)} x^{-1}\right)\right]$

(9) Determine whether the following are true or false:

(a) If $x \in \mathbb{R}$, then $\sqrt{x^2} = x$. [False]

(b) If f and g are two continuous functions, then $f \circ g = g \circ f$. [False]

(c) For any given function f we have $f(x+y) = f(x) + f(y)$. [False]

(d) Given that f is a linear function, then we must have $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. [False]

(e) Given that f is a linear function that passes through the origin, then we must have $f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$. [True]

(f) If $x < y$ and f is a strictly decreasing function, then $f(x) > f(y)$. [True]

(g) If $\alpha, x \in \mathbb{R}$, then $\ln(x^\alpha) = \alpha\ln(x)$. [False]

(10) Determine a formula for the general term a_n given the following sequences:

(a) $\{1, -1, 1, -1, 1, -1, \dots\}$ $\left[a_n = (-1)^n \text{ where } n \geq 0\right]$

(b) $\{2, 3, 2, 3, 2, 3, \dots\}$ $\left[a_n = \frac{5}{2} + \frac{(-1)^n}{2} \text{ where } n \geq 1\right]$

(c) $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\}$ $\left[a_n = \frac{1}{2n+1} \text{ where } n \geq 0\right]$

(d) $\{1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots\}$ $\left[a_n = \left(-\frac{1}{3}\right)^n \text{ where } n \geq 0\right]$

(e) $\{1, 1.9, 1.99, 1.999, 1.9999, 1.99999, \dots\}$ $\left[a_n = 2 - \frac{1}{10^n} \text{ where } n \geq 0\right]$

(f) $\{1, r, r^2, r^3, r^4, \dots\}$ $[a_n = r^n \text{ where } n \geq 0]$

(g) $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\}$ $\left[a_n = 2^{\frac{2^n-1}{2^n}} \text{ where } n \geq 0\right]$

(11) Determine a formula for each situation:

(a) Linear function that passes through $(-2, 4)$ and $(2, 0)$. $[f(x) = -x + 2]$

(b) Exponential function that passes through $(-1, 10)$ and $(2, 30)$. $\left[f(x) = 30(3)^{\frac{x-2}{3}}\right]$

(c) Linear function that has slope $m = 5$ and passes through $(0, 10)$. $[f(x) = 5x + 10]$

(d) Quadratic function that has roots at $x = -3$ and $x = 3$ while passing through $(10, 10)$. $\left[f(x) = \frac{10}{91}(x-3)(x+3)\right]$

(e) Exponential function that has a horizontal asymptote of $y = -1$ and passes through $(1, 1)$ and $(5, 10)$. $\left[f(x) = 2\left(\frac{11}{2}\right)^{\frac{x-1}{4}} - 1\right]$

(12) Find the exact value of each expression without the use of a calculator:

(a) $\log_3\left(\frac{1}{27}\right)$ [-3]

(b) $\ln\left(\frac{1}{e^k}\right)$ where $k \in \mathbb{R}$ $[-k]$

(c) $e^{-2\ln(5)}$ $[5^{-2}]$

(d) $\ln\left(\ln\left(e^{e^{10}}\right)\right)$ [10]

(e) $2\log_2(6) - \log_2(15) + \log_2(20)$ $[\log_2(48)]$

(13) Solve the following for x exactly:

(a) $e^{7-4x} = 6$ $\left[x = \frac{7-\ln(6)}{4}\right]$

(b) $\ln(3x - 10) = 2$ $\left[x = \frac{e^2+10}{3}\right]$

(c) $\ln(x^2 - 1) = 3$ $\left[x = \pm\sqrt{e^3 + 1}\right]$

(d) $e^{2x} - 3e^x + 2 = 0$ $[x = 0, \ln(2)]$

(e) $\ln(\ln(x)) = 1$ $\left[x = e^e\right]$

(14) Solve the following inequalities for x:

(a) $\ln(x^2 - 1) > e$ $\left[|x| > \sqrt{e^e + 1}\right]$

(b) $e^{(x-1)^2} > 5$ $\left[x < 1 - \sqrt{\ln(5)} \text{ or } x > 1 + \sqrt{\ln(5)}\right]$

(c) $1 - 5\ln(x) < 7$ $\left[x > e^{-\frac{6}{5}}\right]$

(d) $1 < e^{3x-1} < 2$ $\left[\frac{1}{3} < x < \frac{1+\ln(x)}{3}\right]$

(e) $\ln|x - 3| \geq 5$ $[x \geq 3 + e^5 \text{ or } x \leq 3 - e^5]$

(15) Determine the inverse function of the following, if it exists:

(a) $f(x) = 1 + \sqrt{2 + 3x}$ where $x \in \left[-\frac{2}{3}, \infty\right)$ $\left[f^{-1}(x) = \frac{x^2-2x-1}{3}\right]$

(b) $g(x) = \frac{4x-1}{2x+3}$ where $x \in \left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$ $\left[g^{-1}(x) = -\frac{3x+1}{2x-4}\right]$

(c) $h(x) = e^{2x-1}$ where $x \in \mathbb{R}$ $\left[h^{-1}(x) = \frac{\ln(x)+1}{2}\right]$

(d) $j(x) = \frac{e^x}{1+2e^x}$ where $x \in \mathbb{R}$ $\left[j^{-1}(x) = \ln\left(\frac{x}{1-2x}\right)\right]$

(e) $m(x) = \begin{cases} \sqrt{|x|}, & x \leq 0 \\ -x, & x > 0 \end{cases}$ $\left[m^{-1}(x) = \begin{cases} -x^2, & x \leq 0 \\ -x, & x > 0 \end{cases}\right]$

(16) Find the domain and range of the following functions:

(a) $f(x) = \frac{2}{3x-1}$ $\left[\text{D:}\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right) \text{ and R:}(-\infty, 0) \cup (0, \infty)\right]$

(b) $g(x) = \sqrt{16 - x^4}$ $\left[\text{D:}[-2, 2] \text{ and R:}[0, 4]\right]$

(c) $h(x) = \ln(x^2 - 1)$ $\left[\text{D:}(-\infty, -1) \cup (1, \infty) \text{ and R:}\mathbb{R}\right]$

(d) $j(x) = 3 + \cos(2x)$ $\left[\text{D:}\mathbb{R} \text{ and R:}[2, 4]\right]$

(e) $m(x) = \left|\frac{x+3}{x^2-1}\right|$ $\left[\text{D:}(-\infty, -1) \cup (-1, 1) \cup (1, \infty) \text{ and R:}\mathbb{R}_{\geq 0}\right]$

(17) Determine whether the following sequences converge or diverge. If they converge determine the exact value they converge to.

(a) $a_n = \frac{1}{3n^4}$ [Convergent since $a_n \to 0$]

(b) $a_n = \frac{n^3-1}{n^3+1}$ [Convergent since $a_n \to 1$]

(c) $a_n = \frac{10^n}{1+9^n}$ [Divergent]

(d) $a_n = \ln(2n^2+1) - \ln(n^2+1)$ [Convergent since $a_n \to \ln(2)$]

(e) $a_n = \frac{\sqrt[3]{n}}{\sqrt[4]{n}+\sqrt{n}}$ [Convergent since $a_n \to 0$]

(18) For the following assume that $a_n \to \mathcal{L}$ and determine the value of $\mathcal{L}$ exactly:

(a) $a_{n+1} = \frac{1}{2}a_n + 1$ where $a_1 = 1$ [$\mathcal{L} = 2$]

(b) $a_{n+1} = 2a_n - 1$ where $a_1 = 2$ [$\mathcal{L} = 1$]

(c) $a_{n+1} = \sqrt{5a_n}$ where $a_1 = 1$ [$\mathcal{L} = 5$]

(d) $a_{n+1} = \frac{6}{1+a_n}$ where $a_1 = 1$ [$\mathcal{L} = 2$]

(e) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{25}{a_n}\right)$ where $a_1 = 100$ [$\mathcal{L} = 5$]

(19) Calculate the following:

(a) $\lim\limits_{n\to\infty} \frac{\sqrt{n}+n^2}{2n-n^2}$ [0]

(b) $\lim\limits_{x\to\infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$ [4]

(c) $\lim\limits_{t\to\infty} (\sqrt{9t^2+t} - 3t)$ $\left[\frac{1}{6}\right]$

(d) $\lim\limits_{x\to\infty} [\ln(x^2) - \ln(x^2+1)]$ [0]

(e) $\lim\limits_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$ [1]

(20) Evaluate the following limits, if they exist:

(a) $\lim\limits_{x\to 5} \frac{x^2-6x+5}{x-5}$ [4]

(b) $\lim\limits_{h\to 0} \frac{(4+h)^2-16}{h}$ [8]

(c) $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right)$ [1]

(d) $\lim\limits_{x\to -2} \frac{x+2}{x^3+8}$ $\left[\frac{1}{12}\right]$

(e) $\lim\limits_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ $\left[\frac{1}{2}\right]$

(21) In the context of *Classical Mechanics* as first described by Sir Isaac Newton, *momentum* is defined as $p = mv$ where m and v are the mass and velocity respectively of the object of interest. In the context of *Special Relativity*, momentum is defined as:

$$p^* = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c is the speed of light. What happens as $v \to c^-$? [$p \to mc$ but $p^* \to \infty$]

(22) Evaluate the following limits:

(a) $\lim_{n\to\infty}\left(2^{-n}+3^{-n}\right)$ [0]

(b) $\lim_{n\to\infty}\frac{6n+5}{n-7}$ [6]

(c) $\lim_{n\to\infty}\frac{\sqrt{n}+n^2}{100n-3n^2}$ $\left[-\frac{1}{3}\right]$

(d) $\lim_{n\to\infty}\frac{e^{3n}-e^{-3n}}{10e^{3n}+e^{-2n}}$ [1] $\left[\frac{1}{10}\right]$

(e) $\lim_{n\to\infty}\frac{n^4-8n^2+n}{n^3-n+90}$ [∞]

(23) Determine whether the following approach negative or positive infinity:

(a) $\lim_{n\to 3^+}\frac{e^n}{(n-3)^5}$ [+]

(b) $\lim_{n\to -4^-}\frac{n+3}{n+4}$ [+]

(c) $\lim_{n\to 5^+}\frac{n^2-5n}{n^2-10n+25}$ [+]

(d) $\lim_{n\to 3^+}\ln(n^2-9)$ [-]

(e) $\lim_{x\to(2\pi)^-}x\csc(x)$ [-]

(24) Evaluate the following limits:

(a) $\lim_{h\to 0}\frac{(4+h)^2-16}{h}$ [8]

(b) $\lim_{h\to 0}\frac{(2+h)^3-8}{h}$ [12]

(c) $\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$ $\left[\frac{1}{2}\right]$

(d) $\lim_{h\to 0}\left(\frac{1}{h}-\frac{1}{h^2+h}\right)$ [1]

(e) $\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}$ $\left[\frac{1}{128}\right]$

(25) Evaluate the following limits:

(a) $\lim_{x\to 3}(2x-|x-3|)$ [6]

(b) $\lim_{x\to -6}\frac{2x+12}{|x+6|}$ [DNE]

(c) $\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ [$-\infty$]

(d) $\lim_{x\to -2}\frac{2-|x|}{2+x}$ [1]

(e) $\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}$ [-4]

(26) Is it true that $\frac{x^2+x-6}{x-2}=x+3$ for all $x\in\mathbb{R}$? If not, explain why. Next determine whether $\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}(x+3)$ is true or not. [False and True respectively]

(27) Assuming $a_i, b_j\in\mathbb{R}$ for $i\leq n$ and $j\leq m$, prove the following:

$$\lim_{x\to\infty}\frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}=\begin{cases}0 & n<m\\ \frac{a_n}{b_n} & n=m\\ \infty & n>m\end{cases}$$

[For each case divide by the highest power of x and compute the limit accordingly]

(28) Evaluate the following using an analytical approach:

$$\lim_{n\to\infty}n^{\frac{1}{n}}=1$$

(29) Find numbers $a, b \in \mathbb{R}$ s.t.:

$$\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1 \quad [a = b = 4]$$

(30) Evaluate the following using an analytical approach:

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$$

(31) The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is given by:

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & r < R \\ \dfrac{GM}{r^2} & r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r? [Yes]