Midterm I Review Solutions

MATH 100

[1] Show that $P \implies Q$ is equivalent to

(i) $(\sim P) \vee Q$
(ii) $(\sim Q) \implies (\sim P)$
(iii) $(P \wedge \sim Q) \implies (R \wedge \sim R)$

Proof.

(i) To prove the equivalence use a truth table:

P	Q	$\sim P$	$(\sim P) \vee Q$	$P \implies Q$
T	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**

(ii) Using the previous equivalence we can avoid using a truth table:

$$\begin{aligned}(\sim Q) \implies (\sim P) &\equiv Q \vee (\sim P)\\ &\equiv (\sim P) \vee Q\\ &\equiv P \implies Q\end{aligned}$$

(iii) Once again we avoid using a truth table:

$$\begin{aligned}(P \wedge \sim Q) \implies (R \wedge \sim R) &\equiv (P \wedge \sim Q) \implies \mathbf{F}\\ &\equiv \sim (P \wedge \sim Q) \vee \mathbf{F}\\ &\equiv (\sim P) \vee Q\\ &\equiv P \implies Q\end{aligned}$$

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[2] (1) Construct a truth table for $(P \implies Q) \vee (Q \implies P)$. Is this a tautology?
(2) Construct a truth table for $\{(\sim P) \implies [Q \wedge (\sim Q)]\} \implies P$.

Proof.

(1) The truth table takes the form:

P	Q	$P \implies Q$	$Q \implies P$	$(P \implies Q) \vee (Q \implies P)$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

Since all possible choices for P and Q produce the truth value **T** this implies that $(P \implies Q) \vee (Q \implies P)$ is indeed a tautology.

(2) The truth table takes the form:

P	Q	$\sim P$	$\sim Q$	$Q \wedge (\sim Q)$	$(\sim P) \implies [Q \wedge (\sim Q)]$	$\{(\sim P) \implies [Q \wedge (\sim Q)]\} \implies P$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**	**T**
F	**T**	**T**	**F**	**F**	**F**	**T**
F	**F**	**T**	**T**	**F**	**F**	**T**

Notice that we could have also arrived at the logical equivalence:

$$\begin{aligned}\{(\sim P) \implies [Q \wedge (\sim Q)]\} \implies P &\equiv \{(\sim P) \implies \mathbf{F}\} \implies P \\ &\equiv \{P \vee \mathbf{F}\} \implies P \\ &\equiv P \implies P \\ &\equiv (\sim P) \vee P \\ &\equiv \mathbf{T}\end{aligned}$$

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[3] Write the converse, contrapositive, and negation of the following conditional statements:

(1) If n is even, then n^2 is even.
(2) If $ab \neq 0$, then $a = 0$ or $b = 0$.
(3) If $a \neq 0$ or $b \neq 0$, then $ab \neq 0$.

Proof. Given $P \implies Q$ recall the following definitions:

- Converse: $Q \implies P$
- Contrapositive: $(\sim Q) \implies (\sim P)$
- Negation: $P \wedge (\sim Q)$

Now we proceed to identify P and Q for each part:

(1) Let $P := n$ is even and $Q := n^2$ is even.
   * Converse: If n^2 is even, then n is even.
   * Contrapositive: If n^2 is not even, then n is not even.
   * Negation: n is even and n^2 is not even.

(2) Let $P := (ab \neq 0)$ and $Q := (a = 0 \text{ or } b = 0)$.
   * Converse: If $a = 0$ or $b = 0$, then $ab \neq 0$.
   * Contrapositive: If $a \neq 0$ and $b \neq 0$, then $ab = 0$.
   * Negation: $ab \neq 0$, $a \neq 0$, and $b \neq 0$

(3) Let $P := (a \neq 0 \text{ or } b \neq 0)$ and $Q := (ab \neq 0)$.
   * Converse: If $ab \neq 0$, then $a \neq 0$ or $b \neq 0$.
   * Contrapositive: If $ab = 0$, then $a = 0$ and $b = 0$.
   * Negation: $a \neq 0$ or $b \neq 0$, and $ab = 0$.

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[4] A sequence $\{x_n\}$ is a Cauchy sequence provided that for each $\epsilon > 0$, there is a natural number N such that if $m, n > N$, then $|x_n - x_m| < \epsilon$. Without using any negative words, state what it means that $\{x_n\}$ is not a Cauchy sequence.

Proof. The given definition can be reformulated as:

A sequence $\{x_n\}$ is a Cauchy sequence provided that $\forall\ \epsilon > 0$,
$\exists\ N \in \mathbb{N}$ such that $\forall\ m, n > N$ we have $|x_n - x_m| < \epsilon$

Negation of this formulation leads to:

A sequence $\{x_n\}$ is not a Cauchy sequence provided that $\exists\ \epsilon > 0$,
$\forall\ N \in \mathbb{N}$ such that $\exists\ m, n > N$ satisfying $|x_n - x_m| \geq \epsilon$

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[5] An integer x has property P provided that for all integers a and b, whenever $x \mid ab$, $x \mid a$ or $x \mid b$. Explain what it means to say that x does not have property P.

Proof. To say that x does not have property P is equivalent to stating that $\exists\, a, b \in \mathbb{Z}$ such that whenever $x \mid ab$, $x \nmid a$ and $x \nmid b$. ∎

[6] Prove that for sets A, B, and C: $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof. A direct proof can be easily obtained by starting on the right-hand side:

$$\begin{aligned} (A - B) \cap (B - C) &= (A \cap B^c) \cap (A \cap C^c) \\ &= A \cap B^c \cap C^c \\ &= A \cap (B^c \cap C^c) \\ &= A \cap (B \cup C)^c \\ &= A - (B \cup C) \end{aligned}$$

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[7] Prove that for any sets A and B: $\mathscr{P}(A) \subset \mathscr{P}(B) \implies A \subset B$.

Proof. Let us consider the contrapositive $A \not\subset B \implies \mathscr{P}(A) \not\subset \mathscr{P}(B)$ instead. Proving this is logically equivalent to the original statement and we proceed via proof by contradiction. Assuming that $A \not\subset B$ we aim to show that $\mathscr{P}(A) \subset \mathscr{P}(B)$. The initial assumption tells us that there exists some $x \in A$ such that $x \notin B$. By the definition of a power set, $\{x\} \in \mathscr{P}(A)$ yet $\{x\} \notin \mathscr{P}(B)$. Having found an element that belongs to one set but not the other provides us with the contradiction because $\mathscr{P}(A) \not\subset \mathscr{P}(B)$. ∎

[8] Let x be a positive real number. Prove that if $x - \frac{2}{x} > 1$, then $x > 2$ by the following methods:

(1) a direct proof.
(2) a proof by contrapositive.
(3) A proof by contradiction.

Proof.

(1) For a direct proof we can consider the following algebraic manipulation:

$$\begin{aligned} x - \frac{2}{x} &> 1 \\ x^2 - 2 &> x \\ x^2 - x - 2 &> 0 \\ (x-2)(x+1) &> 0 \end{aligned}$$

The polynomial on the left-hand has roots at $x = -1, 2$ giving us the two intervals of interest $(0, 2)$ and $(2, \infty)$. For $x \in (0, 2)$ we have that $x - 2 < 0$ and $x + 1 > 0$ implying the original inequality is false. For $x \in (2, \infty)$ we have that $x - 2 > 0$ and $x + 1 > 0$ implying the original inequality holds true. Therefore, it follows that $x > 2$.

(2) For a proof by contrapositive we consider the statement: If $x \leq 2$, then $x - \frac{2}{x} \leq 1$. From the algebraic manipulation from above we can see that if $x \leq 2$, then $(x-2)(x+1) \leq 0$ from which it follows that $x - \frac{2}{x} \leq 1$.

(3) For a proof by contradiction we assume that $x - \frac{2}{x} > 1$ and $x \leq 2$. Once again, using the algebraic manipulation from above we know that the first inequality is equivalent to $(x-2)(x+1) > 0$. The second inequality tells us $x - 2 < 0$ and $x + 1 > 0$ when used in combination with the fact that $x > 0$. Hence, the product is negative, which is a contradiction. Therefore, it must be that $x > 2$.

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[9] Let $n \in \mathbb{Z}$. Prove that $3 \mid (2n^2 + 1)$ iff $3 \nmid n$.

Proof.

$\Longrightarrow$ Begin by assuming that $3 \mid (2n^2 + 1)$. This is equivalent to saying $2n^2 + 1 \equiv 0 \pmod 3$ and can be reduced into:

$$\begin{aligned} 2n^2 + 1 &\equiv 0 \pmod 3 \\ 2n^2 &\equiv -1 \pmod 3 \\ 2n^2 &\equiv 2 \pmod 3 \\ 4n^2 &\equiv 4 \pmod 3 \\ n^2 &\equiv 1 \pmod 3 \\ n^2 - 1 &\equiv 0 \pmod 3 \\ (n-1)(n+2) &\equiv 0 \pmod 3 \end{aligned}$$

For the last line to be satisfied we need $n - 1 \equiv 0 \pmod 3$ or $n + 1 \equiv 0 \pmod 3$. These two equations are satisfied when $n = 1 + 3k$ and $n = 2 + 3k$ for some $k \in \mathbb{Z}$ respectively. More specifically we that when $n = 3k$ the equation becomes $-1 \equiv 0 \pmod 3$ which is of course a contradiction, thereby implying that $n \neq 3k$, or rather $3 \nmid n$.

$\Longleftarrow$ Assuming that $3 \nmid n$, it must be that $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. In both cases it follows that $n^2 \equiv 1 \pmod 3$ and so:

$$\begin{aligned} n^2 &\equiv 1 \pmod 3 \\ 2n^2 &\equiv 2 \pmod 3 \\ 2n^2 + 1 &\equiv 3 \pmod 3 \\ 2n^2 + 1 &\equiv 0 \pmod 3 \end{aligned}$$

Reading off the last line tells us that $3 \mid (2n^2 + 1)$.

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