Properties of Integrals & Integration by Substitution

MATH 19B
Nathan Marianovsky

THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

Definition 1 (Derivative of an Integral Valued Function)**.** Given a function defined as:

$$k(x) = \int_{h(x)}^{g(x)} f(t)dt$$

the derivative of $k(x)$ can be calculated as:

$$\begin{aligned} k'(x) &= \frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt \\ &= \frac{d}{dx}\left[F(t)\Big|_{h(x)}^{g(x)}\right] && \text{where } F(t) \text{ is the antiderivative of } f(t) \\ &= \frac{d}{dx}\Big[F(g(x)) - F(h(x))\Big] \\ &= f(g(x))g'(x) - f(h(x))h'(x) && \text{where differentiation follows from the chain rule} \end{aligned}$$

Example 1. Find the derivative of $r(y)$ where:

$$r(y) = \int_3^y t^2 \sin(3t)\, \mathrm{d}t$$

Solution 1. To find the derivative, identify the components according to the formula:

$$\begin{aligned} g(y) &= y \\ h(y) &= 3 \\ f(t) &= t^2 \sin(3t) \end{aligned}$$

Therefore, the derivative of $r(y)$ is:

$$r'(y) = f(g(y))g'(y) - f(h(y))h'(y) = (y^2\sin(3y))(1) - (9\sin(9))(0) = \boxed{y^2\sin(3y)}$$

Example 2. Find the derivative of $w(x)$ where:

$$w(x) = \int_7^{\frac{1}{x}} \arctan(6t)\, \mathrm{d}t$$

Solution 2. Identify the components according to the formula:

$$\begin{aligned} g(x) &= \frac{1}{x} \\ h(x) &= 7 \\ f(t) &= \arctan(6t) \end{aligned}$$

Therefore, the derivative of $w(x)$ is:

$$w'(x) = f(g(x))g'(x) - f(h(x))h'(x) = \left(\arctan\left(\frac{6}{x}\right)\right)\left(-\frac{1}{x^2}\right) - (\arctan(42))(0) = \boxed{-\frac{\arctan\left(\frac{6}{x}\right)}{x^2}}$$

Example 3. Find the derivative of $p(x)$ where:

$$p(x) = \int_{5x}^{7x} \frac{u^2 - 4}{u^2 + 4}\, \mathrm{d}u$$

Solution 3. Identify the components according to the formula:

$$\begin{aligned} g(x) &= 7x \\ h(x) &= 5x \\ f(u) &= \frac{u^2 - 4}{u^2 + 4} \end{aligned}$$

Therefore, the derivative of $p(x)$ is:

$$p'(x) = f(g(x))g'(x) - f(h(x))h'(x) = \Big(\frac{49x^2 - 4}{49x^2 + 4}\Big)(7) - \Big(\frac{25x^2 - 4}{25x^2 + 4}\Big)(5) = \boxed{\frac{343x^2 - 28}{49x^2 + 4} - \frac{125x^2 - 20}{25x^2 + 4}}$$

Example 4. Find the interval on which the curve $y(x)$ is concave upward where:

$$y(x) = \int_0^x \frac{\mathrm{d}t}{2 + t + 2t^2}$$

Solution 4. To measure concavity the second derivative is needed. To find the first derivative consider the components according to the formula:

$$\begin{aligned} g(x) &= x \\ h(x) &= 0 \\ f(u) &= \frac{1}{2 + t + 2t^2} \end{aligned}$$

Therefore, the first derivative is:

$$y'(x) = f(g(x))g'(x) - f(h(x))h'(x) = \Big(\frac{1}{2 + x + 2x^2}\Big)(1) - \Big(\frac{1}{2}\Big)(0) = \frac{1}{2 + x + 2x^2}$$

By quotient rule, the second derivative is:

$$y''(x) = -\frac{1 + 4x}{(2 + x + 2x^2)^2}$$

Now to find the points of inflection:

$$\begin{aligned} 0 &= y''(x) \\ 0 &= -\frac{1 + 4x}{(2 + x + 2x^2)^2} \\ 0 &= 1 + 4x \\ x &= -\frac{1}{4} \end{aligned}$$

All that is left to check is the behavior to the left and right of the inflection point. Anything smaller when plugged into the second derivative will give a positive value, and anything bigger will give a negative value. Therefore the curve $y(x)$ is concave upward on the interval:

$$\boxed{\Big(-\infty, -\frac{1}{4}\Big)}$$

Properties of Integrals

Definition 2 (Properties)**.** Let $\alpha, \beta \in \mathbb{R}$ and $c \in (a, b)$. All integrals will satisfy the following properties:

$$\int_a^b [\alpha f(x) + \beta g(x)]\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x$$
$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

Integrals of Even and Odd Functions

Definition 3 (Even and Odd Properties)**.** Taking the integral of an even function, $f(x) = f(-x)$, over a symmetrical region can be simplified as:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x = 2\int_{-a}^0 f(x)\,\mathrm{d}x$$

Similarly given an odd function, $f(x) = -f(-x)$, the integral will obey:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

Average Value of a Function

Definition 4 (Average Value)**.** The average value of a function, $f(x)$, on an interval, $[a, b]$, is defined as:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Integration by Substitution

Definition 5 (The Substitution Rule)**.** Given an integral in the form:

$$\int f(g(x))g'(x)\,\mathrm{d}x$$

a substitution can be made:

$$u = g(x)$$
$$\mathrm{d}u = g'(x)\,\mathrm{d}x$$

such that the integral can be rewritten as:

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Example 5. Evaluate $\int \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$

Solution 5. Take notice that the top resembles the derivative of the inside of the bottom. Therefore let:

$$u = 1 - 4x^2$$
$$\mathrm{d}u = -8x\,\mathrm{d}x \to -\frac{\mathrm{d}u}{8} = x\,\mathrm{d}x$$

Plugging this into the integral gives:

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x &= -\frac{1}{8}\int \frac{\mathrm{d}u}{\sqrt{u}} \\ &= -\frac{1}{8}\int u^{-\frac{1}{2}}\,\mathrm{d}u \\ &= -\frac{1}{4}u^{\frac{1}{2}} + C \\ &= \boxed{-\frac{1}{4}\sqrt{1-4x^2} + C}\end{aligned}$$

Example 6. Evaluate $\int \cos(3z)\sin^{10}(3z)\,\mathrm{d}z$

Solution 6. Make the following substitution:

$$u = \sin(3z)$$
$$\mathrm{d}u = 3\cos(3z)\,\mathrm{d}z \to \frac{\mathrm{d}u}{3} = \cos(3z)\,\mathrm{d}z$$

Reevaluating the bounds using the substitution and plugging into the integral gives:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(3z)\sin^{10}(3z)\,\mathrm{d}z &= \frac{1}{3}\int_0^{-1} u^{10}\,\mathrm{d}u \\ &= -\frac{1}{3}\int_{-1}^{0} u^{10}\,\mathrm{d}u \\ &= -\frac{1}{33}u^{11}\Big|_{-1}^{0} \\ &= \boxed{-\frac{1}{33}}\end{aligned}$$

Example 7. Evaluate $\int_1^5 \left(1 - \frac{1}{w}\right)\cos(w - \ln(w))\,\mathrm{d}w$

Solution 7. Make the following substitution:

$$u = w - \ln(w)$$
$$\mathrm{d}u = \left(1 - \frac{1}{w}\right)\mathrm{d}w$$

Reevaluating the bounds using the substitution and plugging into the integral gives:

$$\begin{aligned}\int_1^5 \left(1 - \frac{1}{w}\right)\cos(w - \ln(w))\,\mathrm{d}w &= \int_1^{5-\ln(5)} \cos(u)\,\mathrm{d}u \\ &= \sin(u)\Big|_1^{5-\ln(5)} \\ &= \boxed{\sin(5 - \ln(5)) - \sin(1)}\end{aligned}$$

Example 8. Evaluate $\int \sec^2(4t)(3-\tan(4t))^3 \, \mathrm{d}t$

Solution 8. Make the following substitution:

$$u = 3 - \tan(4t)$$
$$\mathrm{d}u = -4\sec^2(4t)\,\mathrm{d}t \to -\frac{\mathrm{d}u}{4} = \sec^2(4t)\,\mathrm{d}t$$

Plugging this into the integral gives:

$$\begin{aligned}\int \sec^2(4t)(3-\tan(4t))^3\,\mathrm{d}t &= -\frac{1}{4}\int u^3\,\mathrm{d}u \\ &= -\frac{1}{16}u^4 + C \\ &= \boxed{-\frac{(3-\tan(4t))^4}{16} + C}\end{aligned}$$

Example 9. Evaluate $\int_0^1 3(8y-1)e^{4y^2-y}\,\mathrm{d}y$

Solution 9. Make the following substitution:

$$u = 4y^2 - y$$
$$\mathrm{d}u = (8y-1)\,\mathrm{d}y$$

Reevaluating the bounds using the substitution and plugging into the integral gives:

$$\begin{aligned}\int_0^1 3(8y-1)e^{4y^2-y}\,\mathrm{d}y &= 3\int_0^1 (8y-1)e^{4y^2-y}\,\mathrm{d}y \\ &= 3\int_0^3 e^u\,\mathrm{d}u \\ &= 3e^u\Big|_0^3 \\ &= \boxed{3e^3 - 3}\end{aligned}$$