Quiz 7 Solutions

MATH 103A
August 21, 2018

(1) (Q) For each of the following functions determine the type of singularity (*removable, pole of order* n *, or essential*) at $z = 0$:

a) $f(z) = \frac{\sinh(z)}{z}$ b) $f(z) = z^3 e^{\frac{1}{z}}$ c) $f(z) = \frac{\cos(z)\sin(z)}{z^4}$

(A)

(a) First we need the Laurent series:

$$f(z) = \frac{\sinh(z)}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n+1)!}$$

Since the expansion does not have any terms with z in the denominator, it follows that $z = 0$ is a removable singularity.

(b) First we need the Laurent series:

$$f(z) = z^3 e^{\frac{1}{z}} = z^3\sum_{n=0}^{\infty}\frac{1}{n!z^n} = \sum_{n=0}^{\infty}\frac{1}{n!z^{n-3}}$$

Since the expansion has infinitely many terms with z in the denominator, it follows that $z = 0$ is an essential singularity.

(c) First we need the Laurent series:

$$f(z) = \frac{\cos(z)\sin(z)}{z^4} = \frac{\sin(2z)}{2z^4} = \frac{1}{2z^4}\sum_{n=0}^{\infty}(-1)^n\frac{(2z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-4)^n}{(2n+1)!}z^{2n-3}$$

Since the expansion has finitely many terms with z in the denominator, it follows that $z = 0$ is a pole of order 3.

(2) (Q) Let $C = \{z \in \mathbb{C} \mid \|z\| = 2\}$ oriented counter-clockwise. Evaluate the following integral using the Residue Theorem:

$$\int_C \frac{\cosh(\pi z)}{(z^2+1)(z+1)}\,\mathrm{d}z$$

(A) The poles of the integrand correspond to $z = \pm i, -1$ whose residues are:

$$\begin{aligned}
\operatorname*{Res}_{z=i}\frac{\cosh(\pi z)}{(z^2+1)(z+1)} &= \lim_{z\to i}(z-i)\cdot\frac{\cosh(\pi z)}{(z^2+1)(z+1)} = \lim_{z\to i}\frac{\cosh(\pi z)}{(z+i)(z+1)}\\
&= \frac{\cosh(\pi i)}{2i(i+1)} = \frac{1}{2i-2}\cdot\frac{e^{\pi i}+e^{-\pi i}}{2} = -\frac{i+1}{4}\\
\operatorname*{Res}_{z=-i}\frac{\cosh(\pi z)}{(z^2+1)(z+1)} &= \lim_{z\to -i}(z+i)\cdot\frac{\cosh(\pi z)}{(z^2+1)(z+1)} = \lim_{z\to -i}\frac{\cosh(\pi z)}{(z-i)(z+1)}\\
&= \frac{\cosh(-\pi i)}{2i(i-1)} = -\frac{1}{2i+2}\cdot\frac{e^{\pi i}+e^{-\pi i}}{2} = \frac{1-i}{4}\\
\operatorname*{Res}_{z=-1}\frac{\cosh(\pi z)}{(z^2+1)(z+1)} &= \lim_{z\to -1}(z+1)\cdot\frac{\cosh(\pi z)}{(z^2+1)(z+1)} = \lim_{z\to -1}\frac{\cosh(\pi z)}{z^2+1}\\
&= \frac{\cosh(-\pi)}{2} = \frac{\cosh(\pi)}{2}
\end{aligned}$$

By the Residue Theorem it follows that:

$$\int\limits_C \frac{\cosh(\pi z)}{(z^2+1)(z+1)}\,\mathrm{d}z = 2\pi i \cdot \frac{\cosh(\pi) - i}{2} = \pi\Big(1 + i\cosh(\pi)\Big)$$