Double-Angle Formulas and Applications

MATH 19B
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Definition 1 (Derivation)**.** Assuming that you do not have the double-angle formulas from trigonometry memorized, here is a simple approach to derive them. First assume that Euler's formula holds true:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

and exploit the properties of exponentials to arrive at the double-angle relationships:

$$\begin{aligned}(e^{i\theta})^2 &= e^{i(2\theta)} \\ (\cos(\theta) + i\sin(\theta))^2 &= \cos(2\theta) + i\sin(2\theta) \\ \cos^2(\theta) - \sin^2(\theta) + 2i\cos(\theta)\sin(\theta) &= \cos(2\theta) + i\sin(2\theta)\end{aligned}$$

Now if two complex numbers are equivalent, then their real components (*terms without an* i) and imaginary components (*terms with an* i) are equal to each other:

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta) \\ \sin(2\theta) &= 2\cos(\theta)\sin(\theta)\end{aligned}$$

Example 1. Evaluate $\int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\ \mathrm{d}\theta$

Solution 1. This can actually be done two ways:

(i) Make the substitution:

$$\begin{aligned}u &= \sin(\theta) \\ \mathrm{d}u &= \cos(\theta)\ \mathrm{d}\theta\end{aligned}$$

Now reevaluation of the bounds and plugging in the substitution yields:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\ \mathrm{d}\theta &= \int_0^1 u\ \mathrm{d}u \\ &= \frac{1}{2}u^2\Big|_0^1 \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

(ii) Using the double angle formula for sine gives:

$$\int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \frac{\sin(2\theta)}{2}\ \mathrm{d}\theta$$

Now make the substitution:

$$\begin{aligned}u &= 2\theta \\ \mathrm{d}u &= 2\ \mathrm{d}\theta \to \frac{\mathrm{d}u}{2} = \mathrm{d}\theta\end{aligned}$$

Now the integral becomes:

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2\theta)}{2}\ \mathrm{d}\theta = \int_0^{\pi} \frac{\sin(u)}{4}\ \mathrm{d}u = -\frac{\cos(u)}{4}\Big|_0^{\pi} = \boxed{\frac{1}{2}}$$

Example 2. Evaluate $\int \sin^2(x)\ \mathrm{d}x$

Solution 2. Use the double-angle formula for cosine to redefine the $sin^2(x)$:

$$\begin{aligned}
\cos(2x) &= 1 - 2\sin^2(x) \\
2\sin^2(x) &= 1 - \cos(2x) \\
\sin^2(x) &= \frac{1-\cos(2x)}{2}
\end{aligned}$$

Now the integral becomes:

$$\begin{aligned}
\int \sin^2(x)\ \mathrm{d}x &= \int \frac{1-\cos(2x)}{2}\ \mathrm{d}x \\
&= \frac{1}{2}\int\ \mathrm{d}x - \frac{1}{2}\int \cos(2x)\ \mathrm{d}x
\end{aligned}$$

For the second integral above make the substitution:

$$\begin{aligned}
u &= 2x \\
\mathrm{d}u &= 2\ \mathrm{d}x \rightarrow \frac{\mathrm{d}u}{2} = \mathrm{d}x
\end{aligned}$$

Plugging in:

$$\begin{aligned}
\frac{1}{2}\int\ \mathrm{d}x - \frac{1}{2}\int \cos(2x)\ \mathrm{d}x &= \frac{x}{2} - \frac{1}{2}\int \frac{\cos(u)}{2}\ \mathrm{d}u \\
&= \frac{x}{2} - \frac{\sin(u)}{4} + C \\
&= \boxed{\frac{x}{2} - \frac{\sin(2x)}{4} + C}
\end{aligned}$$

Example 3. Evaluate $\int \frac{\cos(x)}{\sin(2x)}\ \mathrm{d}x$

Solution 3. Using the double angle formula for sine gives:

$$\begin{aligned}
\int \frac{\cos(x)}{\sin(2x)}\ \mathrm{d}x &= \int \frac{\cos(x)}{2\cos(x)\sin(x)}\ \mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{\sin(x)}\ \mathrm{d}x \\
&= \frac{1}{2}\int \csc(x)\ \mathrm{d}x \\
&= \boxed{-\frac{\ln(\csc(x)+\cot(x))}{2} + C}
\end{aligned}$$