Integration by Partial Fractions & Simpson's Rule

MATH 19B
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Integration by Partial Fractions

Definition 1 (Partial Fractions)**.** The idea behind partial fractions is to take a rational function and break it apart. Normally, this seems like going backwards because simplifying things usually means combining them, but breaking fractions apart makes integration easier. Now given a rational function, $f(x) = \frac{P(x)}{Q(x)}$, here are the cases and how to go about breaking them apart:

Case	Form	Breakup
Denominator is a product of distinct linear factors	$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)}$	$\frac{C_1}{a_1x+b_1} + \frac{C_2}{a_2x+b_2}$
Denominator is a product of a repeated linear factor	$\frac{P(x)}{(ax+b)^2}$	$\frac{C_1}{ax+b} + \frac{C_2}{(ax+b)^2}$
The denominator contains an irreducible term	$\frac{P(x)}{(ax+b)(k_0+\cdots+k_nx^n)}$	$\frac{C_1}{ax+b} + \frac{D_0+D_1x+\cdots+D_{n-1}x^{n-1}}{k_0+k_1x+\cdots+k_nx^n}$

In the case where an irreducible term is also repeated, follow the same pattern as the repeated linear factors. Also note that this method will only work if the denominator is a higher degree polynomial than the numerator. In the case where the degree of the numerator is equal to or bigger than the degree of the denominator, perform long division first.

Example 1. Evaluate $\int \frac{\mathrm{d}x}{(x+4)(x-1)}$

Solution 1. First break up the fraction inside according to the first case:

$$\begin{aligned}
\frac{1}{(x+4)(x-1)} &= \frac{C_1}{x+4} + \frac{C_2}{x-1} \\
\frac{1}{(x+4)(x-1)} &= \frac{C_1(x-1)}{(x+4)(x-1)} + \frac{C_2(x+4)}{(x+4)(x-1)} \\
1 &= C_1(x-1) + C_2(x+4) \\
\text{if } x = 1: \quad 1 &= 5C_2 \to C_2 = \frac{1}{5} \\
\text{if } x = -4: \quad 1 &= -5C_1 \to C_1 = -\frac{1}{5}
\end{aligned}$$

Now the integral becomes:

$$\begin{aligned}
\int \frac{\mathrm{d}x}{(x+4)(x-1)} &= \int \left(-\frac{\frac{1}{5}}{x+4} + \frac{\frac{1}{5}}{x-1} \right) \mathrm{d}x \\
&= \frac{1}{5}\Big(-\int \frac{\mathrm{d}x}{x+4} + \int \frac{\mathrm{d}x}{x-1} \Big) \\
&= \frac{1}{5}\Big(-\ln|x+4| + \ln|x-1| \Big) + C \\
&= \boxed{\frac{1}{5}\ln\left|\frac{x-1}{x+4}\right| + C}
\end{aligned}$$

Example 2. Evaluate $\int \frac{x^3-4x-10}{x^2-x-6}\,\mathrm{d}x$

Solution 2. Since the degree of the numerator is bigger than the degree of the denominator, long division must be used first:

$$\begin{array}{r|l} & x \quad +1 \\ \hline x^2-x-6 & x^3 \qquad -4x-10 \\ & -x^3+x^2+6x \\ \hline & \quad x^2+2x-10 \\ & \quad -x^2 \;+x \;+6 \\ \hline & \qquad 3x \; -4 \end{array}$$

Now the fraction becomes:

$$\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6} = x+1+\frac{3x-4}{(x-3)(x+2)}$$

Now reduce the last fraction according to the first case:

$$\begin{aligned} \frac{3x-4}{(x-3)(x+2)} &= \frac{C_1}{x-3}+\frac{C_2}{x+2} \\ \frac{3x-4}{(x-3)(x+2)} &= \frac{C_1(x+2)}{(x-3)(x+2)}+\frac{C_2(x-3)}{(x-3)(x+2)} \\ 3x-4 &= C_1(x+2)+C_2(x-3) \\ 3x-4 &= (C_1+C_2)x+(2C_1-3C_2) \end{aligned}$$

Now $C_1+C_2=3$ and $2C_1-3C_2=-4$. Solving this system gives $C_1=1$ and $C_2=2$. Now the integral becomes:

$$\begin{aligned} \int \frac{x^3-4x-10}{x^2-x-6}\,\mathrm{d}x &= \int \left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)\mathrm{d}x \\ &= \frac{x^2}{2}+x+\ln|x-3|+\ln|x+2|+C \\ &= \boxed{\frac{x^2}{2}+x+\ln|(x-3)(x+2)|+C} \end{aligned}$$

Example 3. Evaluate $\int_0^1 \frac{2x+3}{(x+1)^2}\,\mathrm{d}x$

Solution 3. First break up the fraction inside according to the second case:

$$\begin{aligned} \frac{2x+3}{(x+1)^2} &= \frac{C_1}{x+1}+\frac{C_2}{(x_1)^2} \\ \frac{2x+3}{(x+1)^2} &= \frac{C_1(x+1)}{(x+1)^2}+\frac{C_2}{(x+1)^2} \\ 2x+3 &= C_1(x+1)+C_2 \\ 2x+3 &= C_1x+(C_1+C_2) \end{aligned}$$

From this it is clear that $C_1=2$ and $C_2=1$. Now the integral becomes:

$$\begin{aligned} \int_0^1 \frac{2x+3}{(x+1)^2}\,\mathrm{d}x &= \int_0^1 \left(\frac{2}{x+1}+\frac{1}{(x+1)^2}\right)\mathrm{d}x = \left(2\ln|x+1|-\frac{1}{x+1}\right)\Big|_0^1 \\ &= \left(2\ln(2)-\frac{1}{2}\right)-\left(-1\right) = \boxed{2\ln(2)+\frac{1}{2}} \end{aligned}$$

Example 4. Evaluate $\int \frac{x^2+2x-1}{x^3+x}dx$

Solution 4. First break up the fraction inside according to the third case:

$$\begin{aligned}
\frac{x^2+2x-1}{x(x^2+1)} &= \frac{C_1}{x} + \frac{D_0 + D_1x}{x^2+1} \\
\frac{x^2+2x-1}{x(x^2+1)} &= \frac{C_1(x^2+1)}{x(x^2+1)} + \frac{(D_0 + D_1x)x}{x(x^2+1)} \\
x^2+2x-1 &= C_1(x^2+1) + (D_0 + D_1x)x \\
x^2+2x-1 &= (C_1 + D_1)x^2 + D_0x + C_1
\end{aligned}$$

From this it is clear that $C_1 = -1$, $D_0 = 2$, and $D_1 = 2$. Now the integral becomes:

$$\begin{aligned}
\int \frac{x^2+2x-1}{x^3+x}dx &= \int \Big(-\frac{1}{x} + \frac{2x+2}{x^2+1}\Big)dx \\
&= \int \Big(-\frac{1}{x} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\Big)dx \\
&= -\ln|x| + \ln|x^2+1| + 2\arctan(x) + C \\
&= \boxed{\ln\Big|x + \frac{1}{x}\Big| + 2\arctan(x) + C}
\end{aligned}$$

Example 5. Evaluate $\int_2^4 \frac{dx}{x^4-x^2}$

Solution 5. First break up the fraction inside:

$$\begin{aligned}
\frac{1}{x^2(x+1)(x-1)} &= \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x+1} + \frac{C_4}{x-1} \\
\frac{1}{x^2(x+1)(x-1)} &= \frac{C_1(x)(x+1)(x-1)}{x^2(x+1)(x-1)} + \frac{C_2(x+1)(x-1)}{x^2(x+1)(x-1)} + \frac{C_3(x^2)(x-1)}{x^2(x+1)(x-1)} + \frac{C_4(x^2)(x+1)}{x^2(x+1)(x-1)} \\
1 &= C_1(x^3-x) + C_2(x^2-1) + C_3(x^3-x^2) + C_4(x^3+x^2) \\
1 &= (C_1 + C_3 + C_4)x^3 + (C_2 - C_3 + C_4)x^2 - C_1x - C_2
\end{aligned}$$

From this it is clear that $C_1 = 0$, $C_2 = -1$, $C_3 = -\frac{1}{2}$, and $C_4 = \frac{1}{2}$. Now the integral becomes:

$$\begin{aligned}
\int_2^4 \frac{dx}{x^4-x^2} &= \int_2^4 \left(-\frac{1}{x^2} - \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\right)dx \\
&= \Big(\frac{1}{x} - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1|\Big)\Big|_2^4 \\
&= \Big(\frac{1}{4} - \frac{1}{2}\ln(5) + \frac{1}{2}\ln(3)\Big) - \Big(\frac{1}{2} - \frac{1}{2}\ln(3)\Big) \\
&= \boxed{-\frac{1}{4} + \frac{1}{2}\ln\Big(\frac{9}{5}\Big)}
\end{aligned}$$

Example 6. Evaluate $\int_0^1 \frac{x^2}{(x-3)(x+2)^2}\ \mathrm{d}x$

Solution 6. First break up the fraction inside:

$$\begin{aligned}
\frac{x^2}{(x-3)(x+2)^2} &= \frac{C_1}{x-3} + \frac{C_2}{x+2} + \frac{C_3}{(x+2)^2} \\
\frac{x^2}{(x-3)(x+2)^2} &= \frac{C_1(x+2)^2}{(x-3)(x+2)^2} + \frac{C_2(x+2)(x-3)}{(x-3)(x+2)^2} + \frac{C_3(x-3)}{(x-3)(x+2)^2} \\
x^2 &= C_1(x^2+4x+4) + C_2(x^2-x-6) + C_3(x-3) \\
x^2 &= (C_1+C_2)x^2 + (4C_1 - C_2 + C_3)x + (4C_1 - 6C_2 - 3C_3)
\end{aligned}$$

Solving the above system system gives $C_1 = \frac{9}{25}$, $C_2 = \frac{16}{25}$, and $C_3 = -\frac{4}{5}$. Now the integral becomes:

$$\begin{aligned}
\int_0^1 \frac{x^2}{(x-3)(x+2)^2}\ \mathrm{d}x &= \int_0^1 \left(\frac{\frac{9}{25}}{x-3} + \frac{\frac{16}{25}}{x+2} - \frac{\frac{4}{5}}{(x+2)^2} \right) \mathrm{d}x \\
&= \left(\frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} \right) \Big|_0^1 \\
&= \left(\frac{9}{25}\ln(2) + \frac{16}{25}\ln(3) + \frac{4}{15} \right) - \left(\frac{9}{25}\ln(3) + \frac{16}{25}\ln(2) + \frac{4}{10} \right) \\
&= \boxed{\frac{7}{25}\ln\left(\frac{3}{2}\right) - \frac{2}{15}}
\end{aligned}$$

Example 7. Evaluate $\int \frac{e^{2x}}{e^{2x}+3e^x+2}\ \mathrm{d}x$

Solution 7. First a substitution:

$$\begin{aligned}
u &= e^x \\
\mathrm{d}u &= e^x\ \mathrm{d}x
\end{aligned}$$

Substituting in gives:

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}\ \mathrm{d}x = \int \frac{e^x e^x}{e^{2x}+3e^x+2}\ \mathrm{d}x = \int \frac{u}{u^2+3u+2}\ \mathrm{d}u = \int \frac{u}{(u+1)(u+2)}\ \mathrm{d}u$$

Now to break up the fraction inside:

$$\begin{aligned}
\frac{u}{(u+1)(u+2)} &= \frac{C_1}{u+1} + \frac{C_2}{u+2} \\
\frac{u}{(u+1)(u+2)} &= \frac{C_1(u+2)}{(u+1)(u+2)} + \frac{C_2(u+1)}{(u+1)(u+2)} \\
u &= C_1(u+2) + C_2(u+1) \\
u &= (C_1+C_2)u + (2C_1 + C_2)
\end{aligned}$$

Solving the above system gives $C_1 = -1$ and $C_2 = 2$. Now the integral becomes:

$$\begin{aligned}
\int \frac{u}{(u+1)(u+2)}\ \mathrm{d}u &= \int \left(-\frac{1}{u+1} + \frac{2}{u+2} \right) \mathrm{d}u \\
&= -\ln|u+1| + 2\ln|u+2| + C \\
&= \ln\left| \frac{(u+2)^2}{u+1} \right| + C \\
&= \boxed{\ln\left| \frac{(e^x+2)^2}{e^x+1} \right| + C}
\end{aligned}$$

Example 8. Evaluate $\int \sec(x)\ \mathrm{d}x$

Solution 8. This integral can easily be evaluated using a substitution after multiplying by a particular term. The only drawback is that in order to know what to multiply by the answer must be known. If the solution is not known, here is the intuitive approach:

$$\int \sec(x)\ \mathrm{d}x = \int \frac{\mathrm{d}x}{\cos(x)} = \int \frac{\cos(x)}{\cos^2(x)}\ \mathrm{d}x = \int \frac{\cos(x)}{1-\sin^2(x)}\ \mathrm{d}x = \int \frac{\mathrm{d}u}{1-u^2}$$

Now break up the fraction inside:

$$\begin{aligned}
\frac{1}{(1-u)(1+u)} &= \frac{C_1}{1-u} + \frac{C_2}{1+u} \\
\frac{1}{(1-u)(1+u)} &= \frac{C_1(1+u)}{(1-u)(1+u)} + \frac{C_2(1-u)}{(1-u)(1+u)} \\
1 &= C_1(1+u) + C_2(1-u) \\
1 &= (C_1 - C_2)u + (C_1 + C_2)
\end{aligned}$$

Solving the above system gives $C_1 = \frac{1}{2}$ and $C_2 = \frac{1}{2}$. Now the integral becomes:

$$\begin{aligned}
\int \frac{\mathrm{d}u}{1-u^2} &= \frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right)\ \mathrm{d}u \\
&= \frac{1}{2}(-\ln|1-u| + \ln|1+u|) + C \\
&= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)} \cdot \frac{1+\sin(x)}{1+\sin(x)}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{1+2\sin(x)+\sin^2(x)}{1-\sin^2(x)}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{1+2\sin(x)+\sin^2(x)}{\cos^2(x)}\right| + C \\
&= \frac{1}{2}\ln|\sec^2(x) + 2\sec(x)\tan(x) + \tan^2(x)| + C \\
&= \frac{1}{2}\ln|(\sec(x)+\tan(x))^2| + C \\
&= \boxed{\ln|\sec(x)+\tan(x)| + C}
\end{aligned}$$

Example 9. Evaluate $\int \frac{\cos(x)}{\sin^2(x)+\sin(x)}\,\mathrm{d}x$

Solution 9. First a substitution:

$$u = \sin(x)$$
$$\mathrm{d}u = \cos(x)\,\mathrm{d}x$$

Substituting in gives:

$$\int \frac{\cos(x)}{\sin^2(x)+\sin(x)}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{u^2+u}$$

Now break up the fraction inside:

$$\begin{aligned}
\frac{1}{u(u+1)} &= \frac{C_1}{u} + \frac{C_2}{u+1} \\
\frac{1}{u(u+1)} &= \frac{C_1(u+1)}{u(u+1)} + \frac{C_2(u)}{u(u+1)} \\
1 &= C_1(u+1) + C_2 u \\
1 &= (C_1+C_2)u + C_1
\end{aligned}$$

From the above it is clear that $C_1 = 1$ and $C_2 = -1$. Now the integral becomes:

$$\begin{aligned}
\int \frac{\mathrm{d}u}{u^2+u} &= \int \left(\frac{1}{u} - \frac{1}{u+1}\right)\mathrm{d}u \\
&= \ln|u| - \ln|u+1| + C \\
&= \ln\left|\frac{u}{u+1}\right| + C \\
&= \boxed{\ln\left|\frac{\sin(x)}{\sin(x)+1}\right| + C}
\end{aligned}$$

Example 10. Evaluate $\int_9^{16} \frac{\sqrt{u}}{u-4}\,\mathrm{d}u$

Solution 10. First a substitution:

$$x = u^{\frac{1}{2}}$$
$$\mathrm{d}x = \frac{1}{2}u^{-\frac{1}{2}}\,\mathrm{d}u \to 2x\,\mathrm{d}x = \,\mathrm{d}u$$

Substituting in gives:

$$\int_9^{16} \frac{\sqrt{u}}{u-4}\,\mathrm{d}u = \int_3^4 \frac{2x^2}{x^2-4}\,\mathrm{d}x$$

Since the degrees are equal, long division is needed:

$$\begin{array}{r}
1 \\
x^2-4 \overline{)\; x^2 \qquad} \\
\underline{-x^2+4} \\
4
\end{array}$$

Now the integral becomes:

$$\begin{aligned}
\int_3^4 \frac{2x^2}{x^2-4}\,\mathrm{d}x &= 2\int_3^4 \left(1 + \frac{4}{x^2-4}\right)\mathrm{d}x = 2\int_3^4 \left(1 + \frac{1}{x-2} - \frac{1}{x+2}\right)\mathrm{d}x \\
&= 2\Big(x + \ln|x-2| - \ln|x+2|\Big)\Big|_3^4 = 2\Big(4 + \ln(2) - \ln(6)\Big) - 2\Big(3 - \ln(5)\Big) \\
&= \boxed{2 + 2\ln\left(\frac{5}{3}\right)}
\end{aligned}$$

Integration by Completing the Square

Definition 2 (Completing the Square)**.** In some cases you could be given a rational function with an irreducible quadratic term in the denominator. In these types of cases completing the square helps make the integration easier. Given $ax^2 + bx + c$, to complete the square:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\ &= a\left(\left[x + \frac{b}{2a}\right]^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \end{aligned}$$

Example 11. Evaluate $\int \frac{\mathrm{d}x}{x^2+4x+5}$

Solution 11. Complete the square in the denominator:

$$\begin{aligned} \int \frac{\mathrm{d}x}{x^2 + 4x + 5} &= \int \frac{\mathrm{d}x}{x^2 + 4x + 4 - 4 + 5} \\ &= \int \frac{\mathrm{d}x}{(x+2)^2 + 1} \\ &= \int \frac{\mathrm{d}u}{u^2 + 1} \\ &= \arctan(u) + C \\ &= \boxed{\arctan(x+2) + C} \end{aligned}$$

Example 12. Evaluate $\int \frac{x}{x^2+6x+10}\, \mathrm{d}x$

Solution 12. Complete the square in the denominator and substitute:

$$\begin{aligned} \int \frac{x}{x^2 + 6x + 10}\, \mathrm{d}x &= \int \frac{x}{x^2 + 6x + 9 - 9 + 10}\, \mathrm{d}x \\ &= \int \frac{x}{(x+3)^2 + 1}\, \mathrm{d}x \\ &= \int \frac{u - 3}{u^2 + 1}\, \mathrm{d}u \\ &= \int \left(\frac{u}{u^2 + 1} - \frac{3}{u^2 + 1}\right) \mathrm{d}u \\ &= \frac{1}{2}\ln|u^2 + 1| - 3\arctan(u) + C \\ &= \boxed{\frac{1}{2}\ln|(x+3)^2 + 1| - 3\arctan(x+3) + C} \end{aligned}$$

INTEGRAL APPROXIMATION BY SIMPSON'S RULE

Definition 3 (Simpson's Rule)**.** While approximations such as the left, right, midpoint, and trapezoidal use straight line segments to work, there is yet another approximation that uses parabolas to approximate instead. This approximation is known as Simpson's Rule and is defined as:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

where:

$$\Delta x = \frac{b-a}{n}$$

Fortunately, there is a much shorter way of obtaining this approximation, rather than using the crude method above. Using the midpoint and trapezoidal approximations, Simpson's Rule is defined as:

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$$

Example 13. Approximate $\int_0^2 e^{x^2}\,\mathrm{d}x$ using Simpson's Rule with four subdivisions.

Solution 13. According to the shortcut method, if 4 subdivisions are wanted for Simpson's Rule, then 2 are needed for the midpoint and trapezoidal approximations since:

$$S_4 = \frac{1}{3}T_2 + \frac{2}{3}M_2$$

Now the width is defined as $\Delta x = 1$ with $n = 2$. The table of values for the left and right approximations is:

x	0	1	2
f(x)	1	e	e^4

The left and right approximations are:

$$\mathbf{L}_2 = \Delta x \sum_{i=1}^{n-1} f(x_i) = (1)\Big[1 + e\Big] \approx 3.72$$

$$\mathbf{R}_2 = \Delta x \sum_{i=2}^{n} f(x_i) = (1)\Big[e + e^4\Big] \approx 57.32$$

Therefore, the trapezoidal approximation is:

$$\mathbf{T}_2 = \frac{\mathbf{L}_2 + \mathbf{R}_2}{2} = \frac{3.72 + 57.32}{2} = 30.52$$

Now the table of values needed for the midpoint approximation is:

x	.5	1.5
f*(x)	$e^{\frac{1}{4}}$	$e^{\frac{9}{4}}$

Therefore, the midpoint approximation is:

$$\mathbf{M}_2 = \Delta x \sum_{i=1}^{n} f^*(x_i) = (1)\Big[e^{\frac{1}{4}} + e^{\frac{9}{4}}\Big] \approx 10.77$$

Now using the midpoint and trapezoidal approximations, Simpson's Rule gives:

$$S_4 = \frac{1}{3}T_2 + \frac{2}{3}M_2 = \frac{1}{3}(30.52) + \frac{2}{3}(10.77) = \boxed{17.35}$$