Improper Integrals & Comparison Test

MATH 19B
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IMPROPER INTEGRALS: TYPE I [1]

Definition 1 (Type I)**.** Improper integrals are nothing more than regular integrals that require a bit of fixing. The first type deals with the cases where the bounds go off to infinity. Since infinity is not defined as a number, it cannot represent the boundary of an interval for an integral. So formally:

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$
$$\int_{-\infty}^a f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_{-b}^a f(x)\,\mathrm{d}x$$
$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{m\to\infty}\int_{-m}^a f(x)\,\mathrm{d}x + \lim_{n\to\infty}\int_a^n f(x)\,\mathrm{d}x$$

If the limit(s) exists on the right hand side, the integral is said to converge. Otherwise the integral is divergent.

Example 1. Determine whether $\int_1^\infty \frac{\mathrm{d}x}{x}$ is convergent or divergent.

Solution 1. Rewrite using a limit and evaluate:

$$\begin{aligned}\int_1^\infty \frac{\mathrm{d}x}{x} &= \lim_{b\to\infty}\int_1^b \frac{\mathrm{d}x}{x}\\ &= \lim_{b\to\infty} \ln|x|\Big|_1^b\\ &= \lim_{b\to\infty}\Big(\ln|b| - \ln(0)\Big)\\ &= \infty\end{aligned}$$

Therefore, the integral is $\boxed{divergent}$.

Example 2. Find the intervals for p on which $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ converges and diverges.

Solution 2. Rewrite using a limit and evaluate:

$$\begin{aligned}\int_1^\infty \frac{\mathrm{d}x}{x^p} &= \lim_{b\to\infty}\int_1^b x^{-p}\,\mathrm{d}x\\ &= \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^b\\ &= \lim_{b\to\infty}\Big(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\Big)\end{aligned}$$

Now a little bit of analysis. If $p > 1$ then the limit will approach a finite value. If $p < 1$ the limit will go off to infinity. And lastly if $p = 1$ the above cannot say much since plugging in gives big issues. Looking back on Example 1 shows that at $p = 1$ this integral actually diverges. Therefore, this integral will behave as following depending on p:

$$\boxed{\text{Converges if } p > 1 \text{ and diverges if } p \le 1}$$

[1]If you would like to see why $\int_{-\infty}^\infty f(x)\,\mathrm{d}x \neq \lim_{b\to\infty}\int_{-b}^b f(x)\,\mathrm{d}x$, try evaluating $\int_{-\infty}^\infty x\,\mathrm{d}x$ using this definition and compare the result obtained when using the definition above.

Example 3. Determine whether $\int_{-\infty}^{0} \frac{\mathrm{d}x}{\sqrt{3-x}}$ is convergent or divergent.

Solution 3. Rewrite using a limit and evaluate:

$$\begin{aligned}
\int_{-\infty}^{0} \frac{\mathrm{d}x}{\sqrt{3-x}} &= \lim_{b\to\infty} \int_{-b}^{0} \frac{\mathrm{d}x}{\sqrt{3-x}} \\
&= \lim_{b\to\infty} \left(-\int_{3+b}^{3} u^{-\frac{1}{2}}\, \mathrm{d}u \right) \\
&= \lim_{b\to\infty} \int_{3}^{3+b} u^{-\frac{1}{2}}\, \mathrm{d}u \\
&= \lim_{b\to\infty} 2\sqrt{u}\Big|_{3}^{3+b} \\
&= \lim_{b\to\infty} \left(2\sqrt{3+b} - 2\sqrt{3} \right) \\
&= \infty
\end{aligned}$$

Therefore, the integral is $\boxed{\textit{divergent}}$.

Example 4. Determine whether $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$ is convergent or divergent.

Solution 4. Rewrite using a limit and evaluate:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} &= \lim_{m\to\infty} \int_{-m}^{a} \frac{\mathrm{d}x}{1+x^2} + \lim_{n\to\infty} \int_{a}^{n} \frac{\mathrm{d}x}{1+x^2} \\
&= \lim_{m\to\infty} \arctan(x)\Big|_{-m}^{a} + \lim_{n\to\infty} \arctan(x)\Big|_{a}^{n} \\
&= \lim_{m\to\infty} \Big(\arctan(a) - \arctan(-m) \Big) + \lim_{n\to\infty} \Big(\arctan(n) - \arctan(a) \Big) \\
&= -\left(-\frac{\pi}{2} \right) + \left(\frac{\pi}{2} \right) \\
&= \pi
\end{aligned}$$

Therefore, the integral is $\boxed{\textit{convergent}}$.

Example 5. Determine whether $\int_{-\infty}^{0} xe^x \, \mathrm{d}x$ is convergent or divergent.

Solution 5. Rewrite using a limit:

$$\int_{-\infty}^{0} xe^x \, \mathrm{d}x = \lim_{b\to\infty} \int_{-b}^{0} xe^x \, \mathrm{d}x$$

Now identify the parts:

$$\begin{aligned}
u &= x & \mathrm{d}u &= \mathrm{d}x \\
\mathrm{d}v &= e^x \, \mathrm{d}x & v &= e^x
\end{aligned}$$

Plugging in gives:

$$\begin{aligned}
\lim_{b\to\infty} \int_{-b}^{0} xe^x \, \mathrm{d}x &= \lim_{b\to\infty} \left(xe^x\Big|_{-b}^{0} - \int_{-b}^{0} e^x \, \mathrm{d}x \right) \\
&= \lim_{b\to\infty} \left(xe^x - e^x \right)\Big|_{-b}^{0} \\
&= \lim_{b\to\infty} \Big((-1) - (-be^{-b} - e^{-b}) \Big) \\
&= -1
\end{aligned}$$

Therefore, the integral is $\boxed{\textit{convergent}}$.

Example 6. Determine whether $\int_{-\infty}^{\infty} xe^{-x^2}\ \mathrm{d}x$ is convergent or divergent.

Solution 6. Rewrite using a limit and evaluate:

$$\begin{aligned}\int_{-\infty}^{\infty} xe^{-x^2}\ \mathrm{d}x &= \lim_{m\to\infty}\int_{-m}^{a} xe^{-x^2}dx + \lim_{n\to\infty}\int_{a}^{n} xe^{-x^2}\ \mathrm{d}x \\ &= -\frac{1}{2}\left[\lim_{m\to\infty}\int_{-m^2}^{-a^2} e^u\ \mathrm{d}u + \lim_{n\to\infty}\int_{-a^2}^{-n^2} e^u\ \mathrm{d}u\right] \\ &= -\frac{1}{2}\left[\lim_{m\to\infty} e^u\Big|_{-m^2}^{-a^2} + \lim_{n\to\infty} e^u\Big|_{-a^2}^{-n^2}\right] \\ &= -\frac{1}{2}\left[\lim_{m\to\infty}\left(e^{-a^2}-e^{-m^2}\right) + \lim_{n\to\infty}\left(e^{-n^2}-e^{-a^2}\right)\right] \\ &= 0\end{aligned}$$

Therefore, the integral is $\boxed{convergent}$.

Example 7. Determine whether $\int_{0}^{\infty} \sin(x)\ \mathrm{d}x$ is convergent or divergent.

Solution 7. Rewrite using a limit and evaluate:

$$\int_0^{\infty}\sin(x)\ \mathrm{d}x = \lim_{b\to\infty}\int_0^b \sin(x)\ \mathrm{d}x = \lim_{b\to\infty} -\cos(x)\Big|_0^b = \lim_{b\to\infty}\Big(-\cos(b)+1\Big)$$

It is not clear as to where $\cos(b)$ will go as $b\to\infty$. It can be anywhere in the range $[-1,1]$, but since the limit does not approach a single value the integral is $\boxed{divergent}$.

Example 8. Determine whether $\int_{-\infty}^{\infty} \cos(\pi x)\ \mathrm{d}x$ is convergent or divergent.

Solution 8. Rewrite using a limit and evaluate:

$$\begin{aligned}\int_{-\infty}^{\infty}\cos(\pi x)\ \mathrm{d}x &= \lim_{m\to\infty}\int_{-m}^{a}\cos(\pi x)\ \mathrm{d}x + \lim_{n\to\infty}\int_{a}^{n}\cos(\pi x)\ \mathrm{d}x \\ &= \lim_{m\to\infty}\frac{\sin(\pi x)}{\pi}\Big|_{-n}^{a} + \lim_{n\to\infty}\frac{\sin(\pi x)}{\pi}\Big|_{a}^{n} \\ &= \frac{1}{\pi}\Big[\lim_{m\to\infty}(\sin(a\pi)-\sin(-m\pi)) + \lim_{n\to\infty}(\sin(n\pi)-\sin(a\pi))\Big] \\ &= \frac{1}{\pi}\Big[\lim_{m\to\infty}\sin(m\pi) + \lim_{n\to\infty}\sin(n\pi)\Big]\end{aligned}$$

It is not clear as to where either sine will go as $m,n\to\infty$. Both limits can be anywhere in the range $[-1,1]$, but since the limit does not approach a single value the integral is $\boxed{divergent}$.

Example 9. Determine whether $\int_{1}^{\infty} \frac{\ln(x)}{x}\ \mathrm{d}x$ is convergent or divergent.

Solution 9. Rewrite using a limit and evaluate:

$$\begin{aligned}\int_1^{\infty}\frac{\ln(x)}{x}\ \mathrm{d}x &= \lim_{b\to\infty}\int_1^b\frac{\ln(x)}{x}\ \mathrm{d}x \\ &= \lim_{b\to\infty}\int_0^{\ln(b)} u\ \mathrm{d}u \\ &= \lim_{b\to\infty}\frac{u^2}{2}\Big|_0^{\ln(b)} \\ &= \lim_{b\to\infty}\frac{\ln^2(b)}{2} \\ &= \infty\end{aligned}$$

Therefore, the integral is $\boxed{divergent}$.

IMPROPER INTEGRALS: TYPE II

Definition 2 (Type II)**.** The first type of improper integrals handled the case of infinite intervals. The second type of improper integrals handle discontinuities on an interval. Below are all the possible cases and how to handle them:

Case	Setup
Discontinuity at $x = a$	$\int_a^b f(x)\ \mathrm{d}x = \lim_{c\to a^+} \int_c^b f(x)\ \mathrm{d}x$
Discontinuity at $x = b$	$\int_a^b f(x)\ \mathrm{d}x = \lim_{c\to b^-} \int_a^c f(x)\ \mathrm{d}x$
Discontinuity at $x = c$ where $a \in (a, b)$	$\int_a^b f(x)\ \mathrm{d}x = \lim_{m\to c^-} \int_a^m f(x)\ \mathrm{d}x + \lim_{n\to c^+} \int_n^b f(x)\ \mathrm{d}x$

Just like the first type of improper integrals, if the limit(s) on the right hand side exists then the integral converges. Otherwise the integral is divergent.

Example 10. Determine whether $\int_0^3 \frac{\mathrm{d}x}{\sqrt{3-x}}$ is convergent or divergent.

Solution 10. There is a discontinuity at $x = 3$. Rewrite using a limit and evaluate:

$$\begin{aligned}
\int_0^3 \frac{\mathrm{d}x}{\sqrt{3-x}} &= \lim_{c\to 3^-} \int_0^c \frac{\mathrm{d}x}{\sqrt{3-x}} \\
&= \lim_{c\to 3^-} -\int_3^{3-c} u^{-\frac{1}{2}}\ \mathrm{d}u \\
&= \lim_{c\to 3^-} \int_{3-c}^{3} u^{-\frac{1}{2}}\ \mathrm{d}u \\
&= \lim_{c\to 3^-} 2\sqrt{u}\Big|_{3-c}^{3} \\
&= \lim_{c\to 3^-} \left(2\sqrt{3} - 2\sqrt{3-c}\right) \\
&= 2\sqrt{3}
\end{aligned}$$

Therefore, the integral is $\boxed{convergent}$.

Example 11. Determine whether $\int_0^1 \frac{\mathrm{d}x}{x}$ is convergent or divergent.

Solution 11. There is a discontinuity at $x = 0$. Rewrite using a limit and evaluate:

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}x}{x} &= \lim_{c\to 0^+} \int_c^1 \frac{\mathrm{d}x}{x} \\
&= \lim_{c\to 0^+} \ln|x|\Big|_c^1 \\
&= \lim_{c\to 0^+} \Big(-\ln|c|\Big) \\
&= \infty
\end{aligned}$$

Therefore, the integral is $\boxed{divergent}$.

Example 12. Find the intervals for p on which $\int_0^1 \frac{\mathrm{d}x}{x^p}$ converges and diverges.

Solution 12. There is a discontinuity at $x = 0$. Rewrite as a limit and evaluate:

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x^p} &= \lim_{c\to 0^+} \int_c^1 x^{-p}\,\mathrm{d}x \\ &= \lim_{c\to 0^+} \frac{x^{1-p}}{1-p}\bigg|_c^1 \\ &= \lim_{c\to 0^+} \left(\frac{1}{1-p} - \frac{c^{1-p}}{1-p}\right)\end{aligned}$$

Now a little bit of analysis. If $p > 1$ then the limit will go off to infinity. If $p < 1$ the limit will go to a finite value. And lastly if $p = 1$ the above cannot say much since plugging in gives big issues. Looking back on Example 11 shows that at $p = 1$ this integral actually diverges. Therefore, this integral will behave as following depending on p:

$$\boxed{\text{Converges if } p < 1 \text{ and diverges if } p \geq 1}$$

Example 13. Determine whether $\int_0^3 \frac{\mathrm{d}x}{x-1}$ is convergent or divergent.

Solution 13. There is a discontinuity at $x = 1$. Rewrite using two limits and evaluate:

$$\begin{aligned}\int_0^3 \frac{\mathrm{d}x}{x-1} &= \lim_{m\to 1^-} \int_0^m \frac{\mathrm{d}x}{x-1} + \lim_{n\to 1^+} \int_n^3 \frac{\mathrm{d}x}{x-1} \\ &= \lim_{m\to 1^-} \ln|x-1|\bigg|_0^m + \lim_{n\to 1^+} \ln|x-1|\bigg|_n^3 \\ &= \lim_{m\to 1^-} \Big(\ln|m-1| - 0\Big) + \lim_{n\to 1^+} \Big(\ln(2) - \ln|n-1|\Big) \\ &= -\infty\end{aligned}$$

Therefore, the integral is $\boxed{divergent}$.

Example 14. Determine whether $\int_0^{\frac{\pi}{2}} \sec(x)\,\mathrm{d}x$ is convergent or divergent.

Solution 14. There is a discontinuity at $x = \frac{\pi}{2}$. Rewrite as a limit and evaluate:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sec(x)\,\mathrm{d}x &= \lim_{c\to \frac{\pi}{2}^-} \int_0^c \sec(x)\,\mathrm{d}x \\ &= \lim_{c\to \frac{\pi}{2}^-} \ln|\sec(x) + \tan(x)|\bigg|_0^c \\ &= \lim_{c\to \frac{\pi}{2}^-} \ln|\sec(c) + \tan(c)| \\ &= \infty\end{aligned}$$

Therefore, the integral is $\boxed{divergent}$.

Example 15. Determine whether $\int_1^2 \ln(x-1)\,\mathrm{d}x$ is convergent or divergent.

Solution 15. There is a discontinuity at $x = 1$. Rewrite as limit and evaluate:

$$\begin{aligned}\int_1^2 \ln(x-1)\,\mathrm{d}x &= \lim_{c\to 1^+} \int_c^2 \ln(x-1)\,\mathrm{d}x \\ &= \lim_{c\to 1^+} \Big((x-1)\ln(x-1) - (x-1)\Big)\bigg|_c^2 \\ &= \lim_{c\to 1^+} \Big(-1 - (c-1)\ln(c-1) + (c-1)\Big) = -1\end{aligned}$$

Therefore, the integral is $\boxed{convergent}$.

Comparison Test for Improper Integrals

Definition 3 (Comparison Test)**.** In all the previous examples the integration was possible, but unfortunately this is not always the case. How then can you determine whether an integral converges or diverges without integrating? The comparison test offers a way of determining the behavior of an integral without actually touching the integral. Now given an integral, $\int_a^b f(x)\ \mathrm{d}x$, below are the cases in which the comparison test is useful:

Scenario	Condition
$\int_a^b f(x)\ \mathrm{d}x \le \int_a^b g(x)\ \mathrm{d}x$	If $\int_a^b g(x)\ \mathrm{d}x$ converges, then $\int_a^b f(x)\ \mathrm{d}x$ converges
$\int_a^b h(x)\ \mathrm{d}x \le \int_a^b f(x)\ \mathrm{d}x$	If $\int_a^b h(x)\ \mathrm{d}x$ diverges, then $\int_a^b f(x)\ \mathrm{d}x$ diverges

Note that in the first scenario if $\int_a^b g(x)\ \mathrm{d}x$ diverges, then $\int_a^b f(x)\ \mathrm{d}x$ does not necessarily diverge. Similarly, in the second scenario if $\int_a^b h(x)\ \mathrm{d}x$ converges, then $\int_a^b f(x)\ \mathrm{d}x$ does not necessarily converge.

Example 16. Determine whether $\int_1^\infty \frac{\cos^2(x)}{1+x^2}\ \mathrm{d}x$ converges or diverges by comparison test.

Solution 16. Since $0 \le \cos^2(x) \le 1$:

$$\int_1^\infty \frac{\cos^2(x)}{1+x^2}\ \mathrm{d}x \le \int_1^\infty \frac{\mathrm{d}x}{1+x^2}$$

Now evaluating the right hand side gives:

$$\begin{aligned}\int_1^\infty \frac{\mathrm{d}x}{1+x^2} &= \lim_{b\to\infty}\int_1^b \frac{\mathrm{d}x}{1+x^2}\\ &= \lim_{b\to\infty} \arctan(x)\Big|_1^b\\ &= \lim_{b\to\infty}\Big(\arctan(b)-\arctan(1)\Big)\\ &= \frac{\pi}{2}-\frac{\pi}{4}\\ &= \frac{\pi}{4}\end{aligned}$$

Since the bigger integral converges, by comparison test the original integral is $\boxed{convergent}$.

Example 17. Determine whether $\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{x\sin(x)}$ converges or diverges by comparison test.

Solution 17. Since $0 \le \sin(x) \le 1$ means that ignoring the sine term in the denominator would make the function smaller:

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{x} \le \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{x\sin(x)}$$

Now evaluating the left hand side gives:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{x} &= \lim_{c\to0^+}\int_c^{\frac{\pi}{2}} \frac{\mathrm{d}x}{x}\\ &= \lim_{c\to0^+} \ln|x|\Big|_c^{\frac{\pi}{2}}\\ &= \lim_{c\to0^+}\left(\ln\left(\frac{\pi}{2}\right)-\ln|c|\right)\\ &= \infty\end{aligned}$$

Since the smaller integral diverges, by comparison test the original integral is $\boxed{divergent}$.

Example 18. Determine whether $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\ \mathrm{d}x$ converges or diverges by comparison test.

Solution 18. At very large values of x the 1 in the denominator becomes insignificant. So ignoring the 1 gives:

$$\int_1^\infty \frac{x}{\sqrt{1+x^6}}\ \mathrm{d}x \leq \int_1^\infty \frac{x}{x^3}\ \mathrm{d}x = \int_1^\infty \frac{1}{x^2}\ \mathrm{d}x$$

Now referring to Example 2 shows that the integral on the right will converge. Now by comparison test the original integral is $\boxed{convergent}$.

Example 19. Determine whether $\int_1^\infty \frac{2+e^{-x}}{x}\ \mathrm{d}x$ converges or diverges by comparison test.

Solution 19. At very large values of x the exponential term in the numerator becomes insignificant. So ignoring this term gives:

$$\int_1^\infty \frac{2+e^{-x}}{x}\ \mathrm{d}x \geq \int_1^\infty \frac{2}{x}\ \mathrm{d}x = 2\int_1^\infty \frac{1}{x}\ \mathrm{d}x$$

Now referring to Example 1 shows that the integral on the right will diverge. Now by comparison test the original integral is $\boxed{divergent}$.

Example 20. Determine whether $\int_1^\infty \frac{\mathrm{d}x}{x+e^{2x}}$ converges or diverges by comparison test.

Solution 20. At very large values of x the linear term in the denominator becomes insignificant. So by ignoring this term gives:

$$\int_1^\infty \frac{\mathrm{d}x}{x+e^{2x}} \leq \int_1^\infty e^{-2x}\ \mathrm{d}x$$

Now evaluating the integral on the right:

$$\begin{aligned}\int_1^\infty e^{-2x}\ \mathrm{d}x &= \lim_{b\to\infty}\int_1^b e^{-2x}\ \mathrm{d}x \\ &= \lim_{b\to\infty} -\frac{1}{2}e^{-2x}\Big|_1^b \\ &= -\frac{1}{2}\lim_{b\to\infty}\left(e^{-2b}-e^{-2}\right) \\ &= \frac{1}{2}e^{-2}\end{aligned}$$

Since the bigger integral converges, by comparison test the original integral is $\boxed{convergent}$.

Example 21. Determine whether $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\ \mathrm{d}x$ converges or diverges by comparison test.

Solution 21. Ignoring the exponential term in the numerator will make the whole function bigger giving:

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\ \mathrm{d}x \leq \int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = \int_0^1 \frac{\mathrm{d}x}{x^{\frac{1}{2}}}$$

Now referring to Example 12 shows that the integral on the right will converge. Now by comparison test the original integral is $\boxed{convergent}$.

Example 22. Determine whether $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}+x^3}$ converges or diverges by comparison test.

Solution 22. On this interval the cubic term can be ignored such that:

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}+x^3} \leq \int_0^1 \frac{\mathrm{d}x}{x^{\frac{1}{2}}}$$

Now referring to Example 12 shows that the integral on the right will converge. Now by comparison test the original integral is $\boxed{convergent}$.