Alternating Series, Absolute Convergence, and Ratio Test

MATH 19B
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Alternating Series Test

Definition 1 (Alternating Series Test)**.** Given a series in the form:

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

The alternating series test states that the series will converge if the following conditions are met:

(i)

$$\lim_{n \to \infty} a_n = 0$$

(ii)

$$\left| a_{n+1} \right| \leq \left| a_n \right|$$

If one of the conditions is not met the series diverges.

Absolute and Conditional Convergence

Definition 2 (Absolute vs. Conditional)**.** Any arbitrary series, $\sum_{n=1}^{\infty} a_n$, that converges will satisfy one of the two following cases:

Case	Condition
Conditional Convergence	$\sum_{n=1}^{\infty} \left| a_n \right|$ diverges, but $\sum_{n=1}^{\infty} a_n$ converges
Absolute Convergence	$\sum_{n=1}^{\infty} \left| a_n \right|$ converges $\implies$ $\sum_{n=1}^{\infty} a_n$ converges

Ratio and Root Tests for Series

Definition 3 (Ratio and Root Tests)**.** For any arbitrary series, $\sum_{n=1}^{\infty} a_n$:

$$\lim_{n \to \infty} \left| a_n \right|^{\frac{1}{n}} = L \quad \text{for Root Test}$$
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \text{for Ratio Test}$$

Now depending on the value of L:

Condition	Result
$L = 1$	Ratio/Root test is inconclusive
$L > 1$	$\sum_{n=1}^{\infty} a_n$ diverges
$L < 1$	$\sum_{n=1}^{\infty} a_n$ converges

Example 1. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 1. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

The resulting sum is in the form of a p-series and is known to diverge. Now to check for conditional convergence, rewrite the series in the proper form:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \sum_{n=1}^{\infty} -\frac{(-1)^{n}}{\sqrt{n}}$$

Now using the alternating series test:

(i)

$$\lim_{n\to\infty} -\frac{1}{\sqrt{n}} = 0$$

(ii)

$$\begin{aligned} \left| -\frac{1}{\sqrt{n+1}} \right| &\le \left| -\frac{1}{\sqrt{n}} \right| \\ \frac{1}{\sqrt{n+1}} &\le \frac{1}{\sqrt{n}} \\ \sqrt{n} &\le \sqrt{n+1} \end{aligned}$$

Therefore, by the alternating series test the series $\boxed{converges\ conditionally}$.

Example 2. Determine whether $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 2. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{\ln(n)}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

The resulting sum diverges according to the integral test. Now to check for conditional convergence, rewrite the series in the proper form:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n} = \sum_{n=1}^{\infty} -(-1)^{n} \frac{\ln(n)}{n}$$

Now using the alternating series test:

(i)

$$\lim_{n\to\infty} -\frac{\ln(n)}{n} = 0$$

(ii)

$$\begin{aligned} \left| -\frac{\ln(n+1)}{n+1} \right| &\le \left| -\frac{\ln(n)}{n} \right| \\ \frac{\ln(n+1)}{n+1} &\le \frac{\ln(n)}{n} \\ n\ln(n+1) &\le (n+1)\ln(n) \end{aligned}$$

Therefore, by the alternating series test the series $\boxed{converges\ conditionally}$.

Example 3. Determine whether $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 3. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{(-3)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{3^n}{n^3}$$

The resulting sum diverges according to the divergence test. Now to check for conditional convergence, rewrite the series in the proper form:

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3} = \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n^3}$$

Now using the alternating series test:
(i)

$$\lim_{n \to \infty} \frac{3^n}{n^3} = \infty$$

Therefore, by the alternating series test the series $\boxed{diverges}$.

Example 4. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 4. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^4} \right| = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

The resulting sum is in the form of a p-series and is known to converge. Therefore, the series $\boxed{converges\ absolutely}$.

Example 5. Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{2n-1}{3n+1}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 5. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{2n-1}{3n+1} \right| = \sum_{n=1}^{\infty} \frac{2n-1}{3n+1}$$

The resulting sum diverges according to the divergence test. Now to check for conditional convergence use the alternating series test:
(i)

$$\lim_{n \to \infty} \frac{2n-1}{3n+1} = \frac{2}{3}$$

Therefore, by the alternating series test the series $\boxed{diverges}$.

Example 6. Determine whether $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 6. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^3} \right| = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$

The sum on the right is in the form of a p-series and is known to converge. By comparison test the series $\boxed{converges\ absolutely}$.

Example 7. Determine whether $\sum_{n=1}^{\infty} \frac{(-22)^n}{n!}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 7. First to check for absolute convergence use the ratio test:

$$L = \lim_{n\to\infty} \left| \frac{(-22)^{n+1}}{(n+1)!} \times \frac{n!}{(-22)^n} \right| = 22 \lim_{n\to\infty} \frac{n!}{(n+1)n!} = 22 \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Since the value of L is less than 1, by the ratio test the series $\boxed{converges\ absolutely}$.

Example 8. Determine whether $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 8. First to check for absolute convergence use the root test:

$$L = \lim_{n\to\infty} \left| \left(\frac{n^2+1}{2n^2+1}\right)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2}$$

Since the value of L is less than 1, by the root test the series $\boxed{converges\ absolutely}$.

Example 9. Determine whether $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 9. First to check for absolute convergence use the root test:

$$L = \lim_{n\to\infty} \left| \frac{n^n}{3^{1+3n}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+3}} = \infty$$

Since the value of L is greater than 1, by the root test the series $\boxed{diverges}$.

Example 10. Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{n^2}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 10. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\arctan(n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\arctan(n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^2}$$

The sum on the right is in the form of a p-series and is known to converge. By comparison test the series $\boxed{converges\ absolutely}$.

Example 11. Determine whether $\sum_{n=1}^{\infty} \frac{n}{4^n}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 11. First to check for absolute convergence use the ratio test:

$$\lim_{n\to\infty} \left| \frac{n+1}{4^{n+1}} \times \frac{4^n}{n} \right| = \frac{1}{4} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{4}$$

Since the value of L is less than 1, by the ratio test the series $\boxed{converges\ absolutely}$.

Example 12. Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln^n(n)}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 12. First to check for absolute convergence use the root test:

$$L = \lim_{n\to\infty} \left| \frac{(-1)^n}{\ln^n(n)} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$$

Since the value of L is less than 1, by the root test the series $\boxed{converges\ absolutely}$.

Example 13. Determine whether $-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\frac{1}{25}+\dots$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 13. Rewrite the series in proper form:

$$-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\frac{1}{25}+\dots=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

Now to check for absolute convergence:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

The resulting sum is in the form of a p-series and is known to converge. Therefore, the series $\boxed{converges\ absolutely}$.

Example 14. Determine whether $-\frac{2}{5}+\frac{2}{6}-\frac{2}{7}+\frac{2}{8}-\frac{2}{9}+\dots$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 14. Rewrite the series in proper form:

$$-\frac{2}{5}+\frac{2}{6}-\frac{2}{7}+\frac{2}{8}-\frac{2}{9}+\dots=\sum_{n=5}^{\infty}(-1)^n\frac{2}{n}$$

Now to check for absolute convergence:

$$\sum_{n=5}^{\infty}\left|(-1)^n\frac{2}{n}\right|=\sum_{n=5}^{\infty}\frac{2}{n}$$

The resulting sum is in the form of a p-series and is known to diverge. Now to check for conditional convergence use the alternating series test:

(i)

$$\lim_{n\to\infty}\frac{2}{n}=0$$

(ii)

$$\begin{aligned}\left|\frac{2}{n+1}\right|\leq&\left|\frac{2}{n}\right|\\ n\leq&n+1\\ 0\leq&1\end{aligned}$$

Therefore, by the alternating series test the series $\boxed{converges\ conditionally}$.

Example 15. Determine whether $\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 15. First to check for absolute convergence use the ratio test:

$$\lim_{n\to\infty}\left|\frac{10^{n+1}}{(n+2)4^{2n+3}}\times\frac{(n+1)4^{2n+1}}{10^n}\right|=\frac{10}{4^2}\lim_{n\to\infty}\frac{n+1}{n+2}=\frac{5}{8}$$

Since the value of L is less than 1, by the ratio test the series $\boxed{converges\ absolutely}$.

Example 16. Determine whether $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 16. First to check for absolute convergence use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^n2^{n+1}}{(n+1)^4}\times\frac{n^4}{(-1)^{n-1}2^n}\right|=2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^4=2$$

Since the value of L is greater than 1, by the ratio test the series $\boxed{diverges}$.

Example 17. Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 17. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n\ln(n)}\right| = \sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$$

The resulting sum diverges according to the integral test. Now to check for conditional convergence use the alternating series test:

(i)

$$\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$$

(ii)

$$\begin{aligned} \left|\frac{1}{(n+1)\ln(n+1)}\right| &\leq \left|\frac{1}{n\ln(n)}\right| \\ n\ln(n) &\leq (n+1)\ln(n+1) \end{aligned}$$

Therefore, by the alternating series test the series $\boxed{converges\ conditionally}$.

Example 18. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan(n))^n}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 18. First to check for absolute convergence use the root test:

$$L = \lim_{n\to\infty} \left|\frac{(-1)^n}{(\arctan(n))^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\arctan(n)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

Since the value of L is less than 1, by the root test the series $\boxed{converges\ absolutely}$.

Example 19. Determine whether $\sum_{n=1}^{\infty} \frac{\cos(\frac{n\pi}{3})}{n!}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 19. First to check for absolute convergence:

$$\sum_{n=1}^{\infty} \left|\frac{\cos(\frac{n\pi}{3})}{n!}\right| = \sum_{n=1}^{\infty} \frac{|\cos(\frac{n\pi}{3})|}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{n!}$$

Now use the ratio to determine the behavior of the right sum:

$$\lim_{n\to\infty} \left|\frac{1}{(n+1)!} \times n!\right| = \lim_{n\to\infty} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Since the value of L is less than 1 the sum on right will converge. By comparison test the series $\boxed{converges\ absolutely}$.

Example 20. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n}$ converges or diverges. If it converges, determine whether it is absolutely or conditionally convergent.

Solution 20. First to check for absolute convergence use the ratio test:

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)}{8^{n+1}} \times \frac{8^n}{(-1)^n n}\right| = \frac{1}{8} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{8}$$

Since the value of L is less than 1, by the ratio test the series $\boxed{converges\ absolutely}$.

Definition 4 (Interval and Radius of Convergence)**.** A series in the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^{kn+m} \quad \text{where} \quad k>0$$

is called a power series that is centered at $x=a$ with coefficients c_n. Given any arbitrary series it is common to wonder for what values of x the series will converge. To determine this use the ratio test:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{kn+k+m}}{c_n(x-a)^{kn+m}} \right| = |x-a|^k \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

By definition of the ratio test, the limit needs to be less than 1 in order to converge. Therefore, the radius of convergence is given by:

$$R = \left(\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \right)^{-\frac{1}{k}}$$

and the interval of convergence is defined as:

$$\Big(a-R, a+R\Big)$$

where the only uncertainties in the interval are the boundaries. To check for the convergence on the boundaries determine the behavior of:

$$\sum_{n=0}^{\infty} c_n (R)^n \quad \text{and} \quad \sum_{n=0}^{\infty} c_n (-R)^n$$

Example 21. For the power series $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^3}$, determine the radius and interval of convergence.

Solution 21. Notice that in this case:

$$a=0, \quad k=1, \quad \text{and} \quad c_n = \frac{1}{5^n n^3}$$

Therefore, the radius of convergence is:

$$R = \left(\lim_{n\to\infty} \left| \frac{1}{5^{n+1}(n+1)^3} \times 5^n n^3 \right| \right)^{-1} = \left(\frac{1}{5} \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^3 \right)^{-1} = \left(\frac{1}{5} \right)^{-1} = \boxed{5}$$

Now the interval without checking the boundaries is:

$$\Big(a-R, a+R\Big) = \Big(-5, 5\Big)$$

Now to check the boundaries plug in the values for x:

$$\sum_{n=1}^{\infty} \frac{5^n}{5^n n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$
$$\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

Notice that both of the series above converge. Therefore, the interval of convergence is:

$$\boxed{\Big[-5, 5\Big]}$$

Example 22. For the power series $\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n$ where $b > 0$, determine the radius and interval of convergence.

Solution 22. Notice that in this case:

$$k = 1 \quad \text{and} \quad c_n = \frac{n}{b^n}$$

Therefore, the radius of convergence is:

$$R = \Big(\lim_{n\to\infty} \Big|\frac{n+1}{b^{n+1}} \times \frac{b^n}{n}\Big|\Big)^{-1} = \Big(\frac{1}{b}\lim_{n\to\infty} \frac{n}{n+1}\Big)^{-1} = \Big(\frac{1}{b}\Big)^{-1} = \boxed{b}$$

Now the interval without checking the boundaries is:

$$\Big(a - R, a + R\Big) = \Big(a - b, a + b\Big)$$

Now to check the boundaries plug in the values for x:

$$\sum_{n=1}^{\infty} \frac{n}{b^n}(-b)^n = \sum_{n=1}^{\infty}(-1)^n n$$

$$\sum_{n=1}^{\infty} \frac{n}{b^n}(b)^n = \sum_{n=1}^{\infty} n$$

Notice that both of the series above diverge. Therefore, the interval of convergence is:

$$\boxed{\Big(a - b, a + b\Big)}$$

Example 23. For the power series $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{3^n \ln(n)}$, determine the radius and interval of convergence.

Solution 23. Notice that in this case:

$$a = 0, \quad k = 1, \quad \text{and} \quad c_n = \frac{(-1)^n}{3^n \ln(n)}$$

Therefore, the radius of convergence is:

$$R = \Big(\lim_{n\to\infty} \Big|\frac{(-1)^{n+1}}{3^{n+1}\ln(n+1)} \times \frac{3^n \ln(n)}{(-1)^n}\Big|\Big)^{-1} = \Big(\frac{1}{3}\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}\Big)^{-1} = \Big(\frac{1}{3}\Big)^{-1} = \boxed{3}$$

Now the interval without checking the boundaries is:

$$\Big(a - R, a + R\Big) = \Big(-3, 3\Big)$$

Now to check the boundaries plug in the values for x:

$$\sum_{n=2}^{\infty}(-1)^n \frac{3^n}{3^n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

$$\sum_{n=2}^{\infty}(-1)^n \frac{(-3)^n}{3^n \ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

Notice that the first one converges, but the second one diverges. Therefore, the interval of convergence is:

$$\boxed{\Big(-3, 3\Big]}$$

Example 24. For the power series $\sum_{n=1}^{\infty} n!(2x-1)^n$, determine the radius and interval of convergence.

Solution 24. First rewrite the sum in proper form:

$$\sum_{n=1}^{\infty} n!(2x-1)^n = \sum_{n=1}^{\infty} 2^n n!\Big(x-\frac{1}{2}\Big)^n$$

Notice that in this case:

$$a=\frac{1}{2}, \quad k=1, \quad \text{and} \quad c_n = 2^n n!$$

Therefore, the radius of convergence is:

$$R=\Big(\lim_{n\to\infty}\Big|\frac{2^{n+1}(n+1)!}{2^n n!}\Big|\Big)^{-1} = \Big(2\lim_{n\to\infty}\frac{(n+1)n!}{n!}\Big)^{-1} = \Big(2\lim_{n\to\infty}(n+1)\Big)^{-1} = \Big(\infty\Big)^{-1} = \boxed{0}$$

Since the radius of convergence is 0, the interval of convergence is just the center:

$$\boxed{\Big\{\frac{1}{2}\Big\}}$$

Example 25. For the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, determine the radius and interval of convergence. [1]

Solution 25. Notice that in this case:

$$a=0, \quad k=2, \quad \text{and} \quad c_n=\frac{(-1)^n}{(2n)!}$$

Therefore, the radius of convergence is:

$$R=\Big(\lim_{n\to\infty}\Big|\frac{(-1)^{n+1}}{(2n+2)!}\times\frac{(2n)!}{(-1)^n}\Big|\Big)^{-\frac{1}{2}} = \Big(\lim_{n\to\infty}\frac{(2n)!}{(2n+1)(2n+1)(2n)!}\Big)^{-\frac{1}{2}} = \Big(0\Big)^{-\frac{1}{2}} = \boxed{\infty}$$

Since the radius of convergence is ∞, the interval of convergence is all real numbers:

$$\boxed{\Big(-\infty,\infty\Big)}$$

Example 26. For the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^{2n}(n!)^2}$, determine the radius and interval of convergence. [2]

Solution 26. Notice that in this case:

$$a=0, \quad k=2, \quad \text{and} \quad c_n=\frac{(-1)^n}{2^{2n}(n!)^2}$$

Therefore, the radius of convergence is:

$$\begin{aligned}
R &= \Big(\lim_{n\to\infty}\Big|\frac{(-1)^{n+1}}{2^{2n+2}((n+1)!)^2}\times\frac{2^{2n}(n!)^2}{(-1)^n}\Big|\Big)^{-\frac{1}{2}} \\
&= \Big(\frac{1}{2^2}\lim_{n\to\infty}\frac{n!n!}{((n+1)n!)((n+1)n!)}\Big)^{-\frac{1}{2}} \\
&= \Big(\frac{1}{4}\lim_{n\to\infty}\frac{1}{(n+1)^2}\Big)^{-\frac{1}{2}} \\
&= \Big(0\Big)^{-\frac{1}{2}} \\
&= \boxed{\infty}
\end{aligned}$$

Since the radius of convergence is ∞, the interval of convergence is all real numbers:

$$\boxed{\Big(-\infty,\infty\Big)}$$

[1] This is the power series expansion of $\cos(x)$

[2] In case you are curious, this specific power series is known as Bessel's function of order 0. The Bessel function shows up in many places such as in the solutions to the radial Schrödinger equation for a free particle, the dynamics of floating bodies, the heat conduction in cylindrical objects, etc.

Example 27. For the power series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}(x-4)^n$, determine the radius and interval of convergence.

Solution 27. Notice that in this case:

$$a = 4, \quad k = 1, \quad \text{and} \quad c_n = \frac{n}{n^3+1}$$

Therefore, the radius of convergence is:

$$R = \Big(\lim_{n\to\infty} \Big|\frac{n+1}{(n+1)^3+1} \times \frac{n^3+1}{n}\Big|\Big)^{-1} = \Big(\lim_{n\to\infty} \frac{n+1}{n} \lim_{n\to\infty} \frac{n^3+1}{(n+1)^3+1}\Big)^{-1} = \Big(1\Big)^{-1} = \boxed{1}$$

Now the interval without checking the boundaries is:

$$\Big(a-R, a+R\Big) = \Big(3, 5\Big)$$

Now to check the boundaries plug in the values for x:

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}(5-4)^n = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$$
$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}(3-4)^n = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+1}$$

Notice that both of the series above converge. Therefore, the interval of convergence is:

$$\boxed{\Big[3, 5\Big]}$$

Example 28. For the power series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, determine the radius and interval of convergence. [3]

Solution 28. Notice that in this case:

$$a = 0, \quad k = 1, \quad \text{and} \quad c_n = \frac{1}{n!}$$

Therefore, the radius of convergence is:

$$R = \Big(\lim_{n\to\infty} \Big|\frac{1}{(n+1)!} \times n!\Big|\Big)^{-1} = \Big(\lim_{n\to\infty} \frac{n!}{(n+1)n!}\Big)^{-1} = \Big(\lim_{n\to\infty} \frac{1}{n+1}\Big)^{-1} = \Big(0\Big)^{-1} = \boxed{\infty}$$

Since the radius of convergence is ∞, the interval of convergence is all real numbers:

$$\boxed{\Big(-\infty, \infty\Big)}$$

Example 29. For the power series $\sum_{n=1}^{\infty} n! x^{3n}$, determine the radius and interval of convergence.

Solution 29. Notice that in this case:

$$a = 0, \quad k = 3, \quad \text{and} \quad c_n = n!$$

Therefore, the radius of convergence is:

$$R = \Big(\lim_{n\to\infty} \Big|\frac{(n+1)!}{n!}\Big|\Big)^{-\frac{1}{3}} = \Big(\lim_{n\to\infty} \frac{(n+1)n!}{n!}\Big)^{-\frac{1}{3}} = \Big(\lim_{n\to\infty} (n+1)\Big)^{-\frac{1}{3}} = \Big(\infty\Big)^{-\frac{1}{3}} = \boxed{0}$$

Since the radius of convergence is 0, the interval of convergence is just the center:

$$\boxed{\Big\{0\Big\}}$$

[3] This is the power series expansion of e^x