

Midterm II Solutions

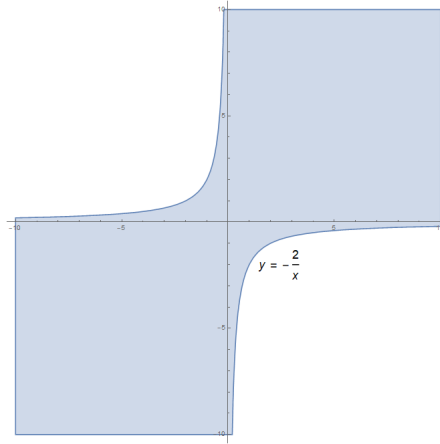
MATH 100

[1] Let R be a relation on the set of all real numbers \mathbb{R} so that $R \subseteq \mathbb{R} \times \mathbb{R}$. For each of the relation below, draw the picture of R in the plane, and determine if it is reflexive, symmetric, or transitive. Explain your answer.

- (1) $R = \{(x, y) \in \mathbb{R}^2 \mid xy \geq -2\}$
- (2) $R = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\}$
- (3) $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$

Proof.

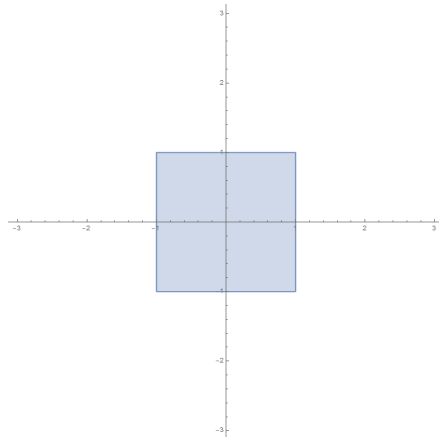
(1) When drawn out as a subset of \mathbb{R}^2 the relation takes the shape:



Now to check which properties the relation satisfies:

- * For any $x \in \mathbb{R}$ we have $x^2 \geq -2$ simply because x^2 is always non-negative. It follows that xR_x showing that R is reflexive.
- * For any $x, y \in \mathbb{R}$ we have $xy = yx \geq -2$. It follows that $xR_y \implies yR_x$ showing that R is symmetric.
- * Consider the setup where $-3R_0$ and $0R_1$. If the relation were transitive, then it would require that $-3R_1$ which is simply not true as $-3 \not\geq -2$. It follows that R is not transitive.

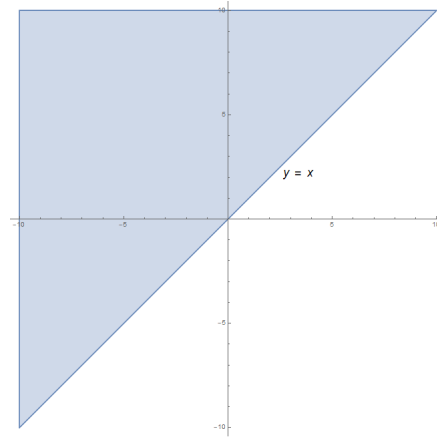
(2) When drawn out as a subset of \mathbb{R}^2 the relation takes the shape:



Now to check which properties the relation satisfies:

- * Consider the value $x = 3 \in \mathbb{R}$. By the condition on the relation, we see $|3| \not\leq 1$ implying that $(3, 3) \notin R$. It follows that R is not reflexive.
- * For any $x, y \in \mathbb{R}$ we see that if $|x| \leq 1$ and $|y| \leq 1$, then this is equivalent to saying $|y| \leq 1$ and $|x| \leq 1$. It follows that $xR_y \implies yR_x$ showing that R is symmetric.
- * For any $x, y, z \in \mathbb{R}$ such that xR_y and yR_z we have the setup $|x| \leq |y|$ and $|y| \leq |z|$. It directly follows that $|x| \leq |z|$ with the result $xR_y \& yR_z \implies xR_z$ showing that R is transitive.

(3) When drawn out as a subset of \mathbb{R}^2 the relation takes the shape:



Now to check which properties the relation satisfies:

- * For any $x \in \mathbb{R}$ we have $x \leq x$. It follows that xR_x showing that R is reflexive.
- * Consider the setup where $1R_2$. If the relation were symmetric, then it would require that $2R_1$ which is simply not true as $2 \not\leq 1$. It follows that R is not symmetric.
- * Consider the setup where $-3R_0$ and $0R_1$. If the relation were transitive, then it would require that $-3R_1$ which is simply not true as $-3 \not\leq -2$. It follows that R is not transitive.

■

[2] A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 3$, and

$$a_n = 3a_{n-1} - 2a_{n-2} \quad \text{for } n \geq 3$$

- (1) Compute a_3 and a_4 and conjecture a formula for a_n .
- (2) Prove the formula for a_n conjectured in (1) by (some version of) mathematical induction, or by other methods.

Proof.

- (1) To compute the next two terms we have:

$$a_3 = 3a_2 - 2a_1 = 3(3) - 2(1) = 7$$

$$a_4 = 3a_3 - 2a_2 = 3(7) - 2(3) = 15$$

The sequence thus far takes the form $\{1, 3, 7, 15, \dots\}$. This motivates the conjecture $a_n = 2^n - 1$.

- (2) We proceed via proof by induction. For the base consider $n = 1$ which provides $a_1 = 2^1 - 1 = 1$. Now assuming $a_n = 2^n - 1$ we check that it satisfies the recurrence relation for $n + 1$:

$$\begin{aligned} 3a_n - 2a_{n-1} &= 3(2^n - 1) - 2(2^{n-1} - 1) \\ &= 3 \cdot 2^n - 3 - 2^n + 2 \\ &= 2 \cdot 2^n - 1 \\ &= 2^{n+1} - 1 \\ &= a_{n+1} \end{aligned}$$

■

[3] Show that for every positive integer n , $49 \mid (8^{n+1} - 7n - 8)$.

Proof. We proceed via proof by induction. For the base case consider $n = 1$ which reduces down to saying $49 \mid 49$, clearly a true statement. Now assuming that the statement holds true for n we can restate it as:

$$\begin{aligned} 8^{n+1} - 7n - 8 &\equiv 0 \pmod{49} \\ 8^{n+1} &\equiv 7n + 8 \pmod{49} \end{aligned}$$

Now to check that it holds true for $n + 1$:

$$\begin{aligned}
 8^{n+2} - 7(n+1) - 8 &\equiv 8 \cdot 8^{n+1} - 7n - 15 \pmod{49} \\
 &\equiv 8(7n+8) - 7n - 15 \pmod{49} \\
 &\equiv 56n + 64 - 7n - 15 \pmod{49} \\
 &\equiv 49(n+1) \pmod{49} \\
 &\equiv 0 \pmod{49}
 \end{aligned}$$

With $49 \mid (8^{n+2} - 7(n+1) - 8)$ we have completed the inductive step. ■

[4] For every positive integer n , prove the following formula:

$$\sum_{j=1}^n \left(\sum_{i=1}^j i \right) = \frac{n(n+1)(n+2)}{6}$$

Proof. We approach via direct proof. Recall the following formulas:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2} \quad \text{and} \quad \sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}$$

Plugging this in provides:

$$\begin{aligned}
 \sum_{j=1}^n \left(\sum_{i=1}^j i \right) &= \sum_{j=1}^n \left(\frac{j(j+1)}{2} \right) = \frac{1}{2} \left(\sum_{j=1}^n j^2 + \sum_{j=1}^n j \right) = \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right) \\
 &= \frac{n(n+1)}{4} \left(\frac{2n+1}{3} + 1 \right) = \frac{n(n+1)(n+2)}{6}
 \end{aligned}$$
■

[5] Prove or disprove:

- (1) There exist integers m and n such that $m^2 + m = n^2$.
- (2) There exist positive integers m and n such that $m^2 + m = n^2$.

Proof.

- (1) If we take $m = n = 0$, then the equation trivially holds true. Thus, the statement is proven.
- (2) Assuming the equation holds true, we can use it to deduce that:

$$\begin{aligned}
 m^2 + m &\equiv n^2 \pmod{n} \\
 m(m+1) &\equiv 0 \pmod{n}
 \end{aligned}$$

The last line only holds true if $m \equiv 0 \pmod{n}$ or $m \equiv n-1 \pmod{n}$. The first scenario has to be disregarded due to the fact that m and n have to be positive integers. In the second scenario we can deduce $m = (n-1) + kn$ for some $k \in \mathbb{Z}_{\geq 0}$. Plugging this in provides:

$$\begin{aligned}
 m^2 + m &= \left((k+1)n - 1 \right)^2 + \left((k+1)n - 1 \right) = (k+1)^2 n^2 - 2(k+1)n + 1 - (k+1)n - 1 \\
 &= (k+1)^2 n^2 - (k+1)n = n(k+1)((k+1)n - 1)
 \end{aligned}$$

For the last equality to equate to n^2 we must have $(k+1)((k+1)n - 1) = n$ for at least one value of k . Note that the factor of $k+1$ is strictly a natural number, which forces us to only consider $k = 0$:

$$m^2 + m = n(n-1) = n^2 - n$$

For $n^2 - n = n^2$ we conclude that $n = 0$, thereby forcing us to disregard the scenario because m and n have to be positive integers. Since none of the possible scenarios provide a solution, it follows that there is no solution to $m^2 + m = n^2$ for positive integers m and n . ■