# Comments on Norms

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**Definition 1** (Norms)**.** Given a vector space $V$ over some field $F$, the *norm* is defined to be the function:

$$\rho : V \to \mathbb{R}$$

s.t.:

(a) Given $\alpha \in F$ and $v \in V$:

$$\rho(\alpha v) = |\alpha|\rho(v)$$

(b) Given $v_1, v_2 \in V$ the triangle inequality holds:

$$\rho(v_1 + v_2) \leq \rho(v_1) + \rho(v_2)$$

(c) Given $v \in V$ the norm is non-negative:

$$\rho(v) \geq 0$$

where equality holds iff $v = 0$.

Note that typically fields of interest include the real or complex numbers.

**Proposition 1** (Euclidean Norm)**.** Given the vector space $V = \mathbb{R}^n$ over the field $F = \mathbb{R}$, the Euclidean norm is defined as:

$$\|v\| = \sqrt{v^T v} = \sqrt{\sum v_i^2}$$

where $v \in V$.

*Proof.* We just have to ensure that the three conditions for a norm hold true:

(a)

$$\begin{aligned}\|v\| &= \sqrt{(\alpha v^T)(\alpha v)} \\ &= \sqrt{\alpha^2}\sqrt{v^T v} \\ &= |\alpha|\sqrt{v^T v}\end{aligned}$$

(b) Using the Cauchy-Schwarz Inequality:

$$\begin{aligned}\|v_1 + v_2\|^2 &= \sum (v_{1i} + v_{2i})^2 \\ \|v_1 + v_2\|^2 &= \sum v_{1i}^2 + \sum v_{2i}^2 + 2\sum v_{1i}v_{2i} \\ \|v_1 + v_2\|^2 &= \|v_1\|^2 + \|v_2\|^2 + 2\sum v_{1i}v_{2i} \\ \|v_1 + v_2\|^2 &\leq \|v_1\|^2 + \|v_2\|^2 + 2\sqrt{\sum v_{1i}^2}\sqrt{\sum v_{2i}^2} \\ \|v_1 + v_2\|^2 &\leq \|v_1\|^2 + \|v_2\|^2 + 2\|v_1\|\|v_2\| \\ \|v_1 + v_2\|^2 &\leq (\|v_1\| + \|v_2\|)^2 \\ \|v_1 + v_2\| &\leq \|v_1\| + \|v_2\|\end{aligned}$$

(c) For this consider the fact that the inside of the norm is a sum of strictly positive values. That sum will only be zero is each component is zero. As for the positive behavior consider the fact that:

$$\sqrt{x} \geq 0$$

which confirms that norm is strictly positive and zero only if the sum is zero which implies all of the components are zero.

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**Proposition 2** (Energy Norm)**.** Given the vector space $V = \mathbb{R}^n$ over the field $F = \mathbb{R}$, the energy norm is defined as:

$$\|v\|_A = \sqrt{v^T A v}$$

where $v \in V$ and $A \in \mathbb{R}^{n^2}$ is known to be positive-definite and symmetric.

*Proof.* We just have to ensure that the three conditions for a norm hold true:

(a)

$$\begin{aligned}
\|v\|_A &= \sqrt{(\alpha v^T)A(\alpha v)} \\
&= \sqrt{\alpha^2}\sqrt{v^T A v} \\
&= |\alpha|\sqrt{v^T A v}
\end{aligned}$$

(b) Using the Cauchy-Schwarz Inequality and the fact that $A$ is a symmetric matrix:

$$\begin{aligned}
&\|v_1 + v_2\|_A^2 = (v_1 + v_2)^T A(v_1 + v_2) \\
&\|v_1 + v_2\|_A^2 = (v_1^T + v_2^T)(Av_1 + Av_2) \\
&\|v_1 + v_2\|_A^2 = v_1^T A v_1 + v_2^T A v_2 + v_1^T A v_2 + v_2^T A v_1 \\
&\|v_1 + v_2\|_A^2 = \|v_1\|_A^2 + \|v_2\|_A^2 + 2v_2^T A v_1 \\
&\|v_1 + v_2\|_A^2 = \|v_1\|_A^2 + \|v_2\|_A^2 + 2v_2^T A^{\frac{1}{2}} A^{\frac{1}{2}} v_1 \\
&\|v_1 + v_2\|_A^2 = \|v_1\|_A^2 + \|v_2\|_A^2 + 2((A^{\frac{1}{2}})^T v_2)^T (A^{\frac{1}{2}} v_1) \\
&\|v_1 + v_2\|_A^2 \leq \|v_1\|_A^2 + \|v_2\|_A^2 + \|(A^{\frac{1}{2}})^T v_2\| \|A^{\frac{1}{2}} v_1\| \\
&\|v_1 + v_2\|_A^2 \leq \|v_1\|_A^2 + \|v_2\|_A^2 + \|A^{\frac{1}{2}} v_2\| \|A^{\frac{1}{2}} v_1\| \\
&\|v_1 + v_2\|_A^2 \leq \|v_1\|_A^2 + \|v_2\|_A^2 + \|v_2\|_A \|v_1\|_A \\
&\|v_1 + v_2\|_A^2 \leq (\|v_1\|_A + \|v_2\|_A)^2 \\
&\|v_1 + v_2\|_A \leq \|v_1\|_A + \|v_2\|_A
\end{aligned}$$

(c) For this consider that if $A$ is positive-definite, then:

$$v^T A v \geq 0$$

with equality only when $v = 0$. So the square guarantees that we only take on positive values where zero is achieved iff the vector is identically the zero vector.

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