Comments on the Chebyshev Polynomials of the First Kind

Nathan Marianovsky

Definition 1 (Trigonometric Definition)**.** The *Chebyshev Polynomials of the First Kind* can be thought of as the unique polynomials satisfying:

$$T_n(x) = \cos(n \arccos(x))$$
$$T_n(\cos(x)) = \cos(nx)$$

where we restrict $x \in [-1, 1]$ so as to stay in the domain of the inverse trigonometric function.

Proposition 1 (Hyperbolic Definition)**.** One can substitute the cosine for hyperbolic cosine in the *Chebyshev Polynomials of the First Kind* to arrive at an equivalent definition:

$$T_n(x) = \cosh(n \operatorname{arccosh}(x))$$

Proof. Remember that:

$$\cosh(x) = \cos(ix) \text{ and } \operatorname{arccosh}(x) = i \arccos(x)$$

Using this we have:

$$\begin{aligned} T_n &= \cos(n \arccos(x)) \\ &= \cos\left(\frac{n}{i} \operatorname{arccosh}(x)\right) \\ &= \cos(-in \operatorname{arccosh}(x)) \\ &= \cos(in \operatorname{arccosh}(x)) \\ &= \cosh(n \operatorname{arccosh}(x)) \end{aligned}$$

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Proposition 2 (Reformulation)**.** The *Chebyshev Polynomials of the First Kind* can be rewritten as:

$$T_n(x) = \frac{1}{2}\Big((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\Big)$$

Proof. We first note that:

$$\cos(x) = \frac{1}{2}\Big(e^{ix} + e^{-ix}\Big) \text{ and } \arccos(x) = -i \ln(x + \sqrt{x^2 - 1})$$

Using this we directly simplify the given definition:

$$\begin{aligned} T_n(x) &= \cos(n \arccos(x)) \\ &= \cos(-in \ln(x + \sqrt{x^2 - 1})) \\ &= \frac{1}{2}\Big(e^{i(-in \ln(x+\sqrt{x^2-1}))} + e^{-i(-in \ln(x+\sqrt{x^2-1}))}\Big) \\ &= \frac{1}{2}\Big((x + \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^{-n}\Big) \\ &= \frac{1}{2}\left((x + \sqrt{x^2 - 1})^n + \frac{1}{(x + \sqrt{x^2 - 1})^n} \cdot \frac{(x - \sqrt{x^2 - 1})^n}{(x - \sqrt{x^2 - 1})^n}\right) \\ &= \frac{1}{2}\Big((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\Big) \end{aligned}$$

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Proposition 3 (Recurrence Definition)**.** The *Chebyshev Polynomials of the First Kind* can be calculated via the relations:

$$\begin{aligned}T_0(x) &= 1\\ T_1(x) &= x\\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x)\end{aligned}$$

Proof. First check the two initial conditions:

$$\begin{aligned}T_0(x) &= \cos(0 \cdot \arccos(x)) = 1\\ T_1(x) &= \cos(1 \cdot \arccos(x)) = x\end{aligned}$$

Now for the recurrence relation we have:

$$\begin{aligned}T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x)\\ 2xT_n(x) &= T_{n+1}(x) + T_{n-1}(x)\\ &= \cos((n+1)\arccos(x)) + \cos((n-1)\arccos(x))\\ &= 2\cos\Big(\frac{(n+1)+(n-1)}{2}\arccos(x)\Big)\cos\Big(\frac{(n+1)-(n-1)}{2}\arccos(x)\Big)\\ &= 2x\cos(n\arccos(x))\\ &= 2xT_n(x)\end{aligned}$$

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Proposition 4 (Orthogonality)**.** The *Chebyshev Polynomials of the First Kind* form an orthogonal sequence of polynomials:

$$\int_{-1}^{1} T_n(x)T_m(x)w(x)dx = \begin{cases} 0 & n \neq m\\ \pi & n = m = 0\\ \frac{\pi}{2} & n = m \neq 0\end{cases}$$

with respect to the weight:

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

Proof. To verify the orthogonality condition we make the substitution:

$$x = \cos(\theta) \text{ and } dx = -\sin(\theta)d\theta$$

and plug into the integral assuming that $n \neq m$:

$$\begin{aligned}\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} &= \int_{-\pi}^{0} T_n(\cos(\theta))T_m(\cos(\theta))\frac{-\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}d\theta\\ &= \int_{-\pi}^{0} T_n(\cos(\theta))T_m(\cos(\theta))d\theta\\ &= \int_{-\pi}^{0} \cos(n\theta)\cos(m\theta)d\theta\\ &= \frac{1}{2}\int_{-\pi}^{0} (\cos((n-m)\theta) + \cos((n+m)\theta))d\theta\\ &= \frac{1}{2}\Big[\frac{\sin((n-m)\theta)}{n-m} + \frac{\sin((n+m)\theta)}{n+m}\Big]\Big|_{-\pi}^{0}\\ &= \frac{1}{2}\Big[\frac{\sin((n-m)\pi)}{n-m} + \frac{\sin((n+m)\pi)}{n+m}\Big]\\ &= 0\end{aligned}$$

Now for the case where $n = m \neq 0$:

$$\begin{aligned}
\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} &= \int_{-\pi}^{0} T_n^2(\cos(\theta))\frac{-\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}d\theta \\
&= \int_{-\pi}^{0} T_n^2(\cos(\theta))d\theta \\
&= \int_{-\pi}^{0} \cos^2(n\theta)d\theta \\
&= \frac{1}{2}\int_{-\pi}^{0} (\cos(2n\theta)+1)d\theta \\
&= \frac{1}{2}\left[\frac{\sin(2n\theta)}{2n}+\theta\right]\Big|_{-\pi}^{0} \\
&= \frac{1}{2}\left[\frac{\sin(2n\pi)}{2n}+\pi\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

and finally for the case where $n = m = 0$:

$$\begin{aligned}
\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} &= \int_{-\pi}^{0} T_0^2(\cos(\theta))\frac{-\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}d\theta \\
&= \int_{-\pi}^{0} T_0^2(\cos(\theta))d\theta \\
&= \int_{-\pi}^{0} d\theta \\
&= \theta\Big|_{-\pi}^{0} \\
&= \pi
\end{aligned}$$

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Proposition 5 (Chebyshev Series)**.** Since the *Chebyshev Polynomials of the First Kind* form an orthogonal basis, they can be used to expand a function for $x \in [-1, 1]$ as:

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

where:

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx \\
a_n &= \frac{1}{\pi}\int_{-1}^{1} \frac{T_n(x)f(x)}{\sqrt{1-x^2}}dx
\end{aligned}$$

Proof. We determine the coefficients by abusing the orthogonality of the basis polynomials:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n T_n(x) \\
\frac{T_m(x)f(x)}{\sqrt{1-x^2}} &= \sum_{n=0}^{\infty} a_n \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} \\
\int_{-1}^{1} \frac{T_m(x)f(x)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1} \sum_{n=0}^{\infty} a_n \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx
\end{aligned}$$

where the integral and sum are allowed to switch order if we are staying inside the region of convergence. Now if $m \neq n$ the term goes to zero, thereby leaving only the terms where $n = m$. For this last piece there are two cases to consider. First if $m = n = 0$:

$$\frac{\pi}{2} a_0 = \int_{-1}^{1} \frac{T_0(x) f(x)}{\sqrt{1-x^2}} dx$$
$$a_0 = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$$

and for the rest:

$$\pi a_m = \int_{-1}^{1} \frac{T_m(x) f(x)}{\sqrt{1-x^2}} dx$$
$$a_m = \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(x) f(x)}{\sqrt{1-x^2}} dx$$

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Definition 2 (Chebyshev Equation)**.** The *Chebyshev Equation* is a homogeneous second order linear differential equation in the form:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

where $p \in \mathbb{R}$.

Proposition 6 (Solutions to the Chebyshev Equation)**.** Assuming that the solution takes on a power series form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

we have two common situations:

(1) Setting $a_0 = 1$ and $a_1 = 0$:

$$y(x) = F(x) = 1 - \frac{p^2}{2!}x^2 + \frac{(p-2)p^2(p+2)}{4!}x^4 - \frac{(p-4)(p-2)p^2(p+2)(p+4)}{6!}x^6 + \dots$$

(2) Setting $a_0 = 0$ and $a_1 = 1$:

$$y(x) = G(x) = x - \frac{(p-1)(p+1)}{3!}x^3 + \frac{(p-3)(p-1)(p+1)(p+3)}{5!}x^5 - \dots$$

where the general solution is the linear combination of the two cases above.

Proof. To check these solutions first we assume that the solution takes on a power series form and define:

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Now plugging into the differential equation:

$$\begin{aligned}
0 &= (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y \\
&= (1-x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} a_n n x^{n-1} + p^2\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=1}^{\infty} a_n n x^n + p^2\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^n - a_1 x - \sum_{n=2}^{\infty} a_n n x^n + p^2 a_0 + p^2 a_1 x + p^2\sum_{n=2}^{\infty} a_n x^n \\
&= (2a_2 + p^2 a_0) + (6a_3 + (p^2-1)a_1)x + \sum_{n=2}^{\infty}\Big[a_{n+2}(n+2)(n+1) - a_n(n)(n-1) - a_n(n) + p^2 a_n\Big]x^n
\end{aligned}$$

Now if this polynomial is to equate to zero, each coefficient must simultaneously be zero giving:

$$\begin{aligned}
a_2 &= -\frac{p^2}{2}a_0 \\
a_3 &= \frac{1-p^2}{6}a_1 \\
a_{n+2} &= -\frac{(p-n)(p+n)}{(n+2)(n+1)}a_n
\end{aligned}$$

Now if we set $a_0 = 1$ and $a_1 = 0$ we get:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= -\frac{p^2}{2} \\
a_3 &= 0 \\
a_4 &= \frac{(p-2)p^2(p+2)}{4!} \\
&\vdots \\
a_{2n+1} &= 0 \\
a_{2n} &= \frac{(-1)^n p^2}{(2n)!}\prod_{i=1}^{n-1}(p-2(n-i))(p+2(n-i))
\end{aligned}$$

With this the solution takes on the form:

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_{2n}x^{2n} \\
&= 1 - \frac{p^2}{2}x^2 + \sum_{n=2}^{\infty}\frac{(-1)^n p^2}{(2n)!}\left[\prod_{i=1}^{n-1}(p-2(n-i))(p+2(n-i))\right]x^{2n} \\
&= F(x)
\end{aligned}$$

Now if we set $a_0 = 0$ and $a_1 = 1$ we get:

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 \\
a_2 &= 0 \\
a_3 &= -\frac{(p-1)(p+1)}{3!} \\
a_4 &= 0 \\
&\vdots \\
a_{2n+1} &= \frac{(-1)^n}{(2n+1)!}\prod_{i=1}^{n-1}(p-(2n+1-2i))(p+(2n+1-2i)) \\
a_{2n} &= 0
\end{aligned}$$

With this the solution takes on the form:

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1} \\
&= x - \frac{(p-1)(p+1)}{3!}x^3 + \sum_{n=2}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[\prod_{i=1}^{n-1}(p-(2n+1-2i))(p+(2n+1-2i))\right]x^{2n+1} \\
&= G(x)
\end{aligned}$$

Note that each of these solutions has the missing components of the other, therefore we form the general solution as the linear combination of the two. ■

Proposition 7 (Convergence of the Solutions)**.**

Proposition 8 (Chebyshev Polynomials from the Chebyshev Equation)**.** Notice that if from the above we have $p \in \mathbb{Z}$, the first and second case will terminate if p is even and odd respectively. Therefore, the two cases reduce down to a pth degree polynomial that is proportional to the pth Chebyshev polynomial by the relations:

$$T_p(x) = \begin{cases} (-1)^{\frac{p}{2}}F(x) & \exists k \in \mathbb{N} \text{ s.t. } p = 2k \\ (-1)^{\frac{p-1}{2}}pG(x) & \exists k \in \mathbb{N} \text{ s.t. } p = 2k+1 \end{cases}$$