Comments on the
Chebyshev Polynomials of the First Kind

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Definition 1 (Trigonometric Definition)**.** The *Chebyshev Polynomials of the First Kind* can be thought of as the unique polynomals satisfying:

$$T_n(x) = \cos(n \arccos(x))$$
$$T_n(\cos(x)) = \cos(nx)$$

where we restrict $x \in [-1, 1]$ so as to stay in the domain of the inverse trigonometric function.

Proposition 1 (Hyperbolic Definition)**.** One can substitute the cosine for hyperbolic cosine in the *Chebyshev Polynomials of the First Kind* to arrive at an equivalent definition:

$$T_n(x) = \cosh(n \operatorname{arccosh}(x))$$

Proof. Remember that:

$$\cosh(x) = \cos(ix) \quad \text{and} \quad \operatorname{arccosh}(x) = i \arccos(x)$$

Using this we have:

$$
\begin{aligned}
T_n &= \cos(n \arccos(x)) \\
&= \cos\left(\frac{n}{i} \operatorname{arccosh}(x)\right) \\
&= \cos(-in \operatorname{arccosh}(x)) \\
&= \cos(in \operatorname{arccosh}(x)) \\
&= \cosh(n \operatorname{arccosh}(x))
\end{aligned}
$$

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Proposition 2 (Reformulation)**.** The *Chebyshev Polynomials of the First Kind* can be rewritten as:

$$T_n(x) = \frac{1}{2}\left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\right)$$

Proof. We first note that:

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad \text{and} \quad \arccos(x) = -i \ln(x + \sqrt{x^2 - 1})$$

Using this we directly simplify the given definition:

$$
\begin{aligned}
T_n(x) &= \cos(n \arccos(x)) \\
&= \cos(-in \ln(x + \sqrt{x^2 - 1})) \\
&= \frac{1}{2}\left(e^{i(-in \ln(x+\sqrt{x^2-1}))} + e^{-i(-in \ln(x+\sqrt{x^2-1}))}\right) \\
&= \frac{1}{2}\left((x + \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^{-n}\right) \\
&= \frac{1}{2}\left((x + \sqrt{x^2 - 1})^n + \frac{1}{(x + \sqrt{x^2 - 1})^n} \cdot \frac{(x - \sqrt{x^2 - 1})^n}{(x - \sqrt{x^2 - 1})^n}\right) \\
&= \frac{1}{2}\left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\right)
\end{aligned}
$$

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Proposition 3 (Extending the Chebyshev Polynomials)**.** The definition of the *Chebyshev Polynomials of the First Kind* can be extended for all real values:

$$T_n(x) = \begin{cases} (-1)^n \cosh(n \operatorname{arccosh}(-x)) & x \leq -1 \\ \cos(n \arccos(x)) & |x| < 1 \\ \cosh(n \operatorname{arccosh}(x)) & x \geq 1 \end{cases}$$

Proof. The middle case is the one we already know. It has already been proven that hyperbolic cosine and its inverse can be used equivalently. Since the hyperbolic functions have a bigger domain, we let it define the region beyond $x = 1$. Now for all values below $x = -1$, consider the following:

$$T_n(x) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right)$$

$$T_n(-x) = \frac{1}{2} \left((-x + \sqrt{x^2 - 1})^n + (-x - \sqrt{x^2 - 1})^n \right)$$

$$= \frac{1}{2} \left((-1)^n (x - \sqrt{x^2 - 1})^n + (-1)^n (x + \sqrt{x^2 - 1})^n \right)$$

$$= (-1)^n T_n(x)$$

which tells us that the polynomials for negative values will either be exactly the same or off by a sign. With this we can write the extension below $x = -1$ the same as the other one with taking into account the sign:

$$T_n(x) = (-1)^n T_n(-x)$$

$$= (-1)^n \cosh(n \operatorname{arccosh}(-x))$$

for $x \leq -1$. ∎

Proposition 4 (Recurrence Definition)**.** The *Chebyshev Polynomials of the First Kind* can be calculated via the relations:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$

Proof. First check the two initial conditions:

$$T_0(x) = \cos(0 \cdot \arccos(x)) = 1$$

$$T_1(x) = \cos(1 \cdot \arccos(x)) = x$$

Now for the recurrence relation we have:

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$

$$2x T_n(x) = T_{n+1}(x) + T_{n-1}(x)$$

$$= \cos((n+1) \arccos(x)) + \cos((n-1) \arccos(x))$$

$$= 2 \cos \left(\frac{(n+1) + (n-1)}{2} \arccos(x) \right) \cos \left(\frac{(n+1) - (n-1)}{2} \arccos(x) \right)$$

$$= 2x \cos(n \arccos(x))$$

$$= 2x T_n(x)$$

∎

Proposition 5 (Orthogonality). The *Chebyshev Polynomials of the First Kind* form an orthogonal sequence of polynomials:

$$\int_{-1}^{1} T_n(x)T_m(x)w(x)dx = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases}$$

with respect to the weight:

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

Proof. To verify the orthogonality condition we make the substitution:

$$x = \cos(\theta) \text{ and } dx = -\sin(\theta)d\theta$$

and plug into the integral assuming that $n \neq m$:

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-\pi}^{0} T_n(\cos(\theta))T_m(\cos(\theta))\frac{-\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}d\theta$$

$$= \int_{-\pi}^{0} T_n(\cos(\theta))T_m(\cos(\theta))d\theta$$

$$= \int_{-\pi}^{0} \cos(n\theta)\cos(m\theta)d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{0} (\cos((n-m)\theta) + \cos((n+m)\theta))d\theta$$

$$= \frac{1}{2}\left[\frac{\sin((n-m)\theta)}{n-m} + \frac{\sin((n+m)\theta)}{n+m}\right]\Big|_{-\pi}^{0}$$

$$= \frac{1}{2}\left[\frac{\sin((n-m)\pi)}{n-m} + \frac{\sin((n+m)\pi)}{n+m}\right]$$

$$= 0$$

Now for the case where $n = m \neq 0$:

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-\pi}^{0} T_n^2(\cos(\theta))\frac{-\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}d\theta$$

$$= \int_{-\pi}^{0} T_n^2(\cos(\theta))d\theta$$

$$= \int_{-\pi}^{0} \cos^2(n\theta)d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{0} (\cos(2n\theta) + 1)d\theta$$

$$= \frac{1}{2}\left[\frac{\sin(2n\theta)}{2n} + \theta\right]\Big|_{-\pi}^{0}$$

$$= \frac{1}{2}\left[\frac{\sin(2n\pi)}{2n} + \pi\right]$$

$$= \frac{\pi}{2}$$

and finally for the case where $n = m = 0$:

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-\pi}^{0} T_0^2(\cos(\theta))\frac{-\sin(\theta)}{\sqrt{1-\cos^2(\theta)}}d\theta$$

$$= \int_{-\pi}^{0} T_0^2(\cos(\theta))d\theta$$

$$= \int_{-\pi}^{0} d\theta$$

$$= \theta\Big|_{-\pi}^{0}$$

$$= \pi$$

∎

Proposition 6 (Chebyshev Series)**.** Since the *Chebyshev Polynomials of the First Kind* form an orthogonal basis, they can be used to expand a function for $x \in [-1, 1]$ as:

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

where:

$$a_0 = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx$$

$$a_n = \frac{1}{\pi}\int_{-1}^{1}\frac{T_n(x)f(x)}{\sqrt{1-x^2}}dx$$

Proof. We determine the coefficients by abusing the orthogonality of the basis polynomials:

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

$$\frac{T_m(x)f(x)}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} a_n \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}$$

$$\int_{-1}^{1}\frac{T_m(x)f(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{1}\sum_{n=0}^{\infty} a_n \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx$$

where the integral and sum are allowed to switch order if we are staying inside the region of convergence. Now if $m \neq n$ the term goes to zero, thereby leaving only the terms where $n = m$. For this last piece there are two cases to consider. First if $m = n = 0$:

$$\frac{\pi}{2}a_0 = \int_{-1}^{1}\frac{T_0(x)f(x)}{\sqrt{1-x^2}}dx$$

$$a_0 = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx$$

and for the rest:

$$\pi a_m = \int_{-1}^{1}\frac{T_m(x)f(x)}{\sqrt{1-x^2}}dx$$

$$a_m = \frac{1}{\pi}\int_{-1}^{1}\frac{T_m(x)f(x)}{\sqrt{1-x^2}}dx$$

∎

Definition 2 (Chebyshev Equation). The *Chebyshev Equation* is a homogeneous second order linear differential equation in the form:

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

where $p \in \mathbb{R}$.

Proposition 7 (Solutions to the Chebyshev Equation). Assuming that the solution takes on a power series form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

we have two common situations:

(1) Setting $a_0 = 1$ and $a_1 = 0$:

$$y(x) = F(x) = 1 - \frac{p^2}{2!}x^2 + \frac{(p-2)p^2(p+2)}{4!}x^4 - \frac{(p-4)(p-2)p^2(p+2)(p+4)}{6!}x^6 + \dots$$

(2) Setting $a_0 = 0$ and $a_1 = 1$:

$$y(x) = G(x) = x - \frac{(p-1)(p+1)}{3!}x^3 + \frac{(p-3)(p-1)(p+1)(p+3)}{5!}x^5 - \dots$$

where the general solution is the linear combination of the two cases above.

Proof. To check these solutions first we assume that the solution takes on a power series form and define:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Now plugging into the differential equation:

$$0 = (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y$$

$$= (1 - x^2)\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - x\sum_{n=1}^{\infty} a_n n x^{n-1} + p^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1) x^{n} - \sum_{n=1}^{\infty} a_n n x^{n} + p^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^{n} - \sum_{n=2}^{\infty} a_n n(n-1) x^{n} - a_1 x - \sum_{n=2}^{\infty} a_n n x^{n} + p^2 a_0 + p^2 a_1 x + p^2 \sum_{n=2}^{\infty} a_n x^n$$

$$= (2a_2 + p^2 a_0) + (6a_3 + (p^2 - 1)a_1)x + \sum_{n=2}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n(n)(n-1) - a_n(n) + p^2 a_n \right] x^n$$

Now if this polynomial is to equate to zero, each coefficient must simultaneously be zero giving:

$$a_2 = -\frac{p^2}{2}a_0$$

$$a_3 = \frac{1 - p^2}{6}a_1$$

$$a_{n+2} = -\frac{(p-n)(p+n)}{(n+2)(n+1)}a_n$$

Now if we set $a_0 = 1$ and $a_1 = 0$ we get:

$$a_0 = 1$$
$$a_1 = 0$$
$$a_2 = -\frac{p^2}{2}$$
$$a_3 = 0$$
$$a_4 = \frac{(p-2)p^2(p+2)}{4!}$$
$$\vdots$$
$$a_{2n+1} = 0$$
$$a_{2n} = \frac{(-1)^n p^2}{(2n)!} \prod_{i=1}^{n-1} (p - 2(n-i))(p + 2(n-i))$$

With this the solution takes on the form:

$$y(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$$
$$= 1 - \frac{p^2}{2} x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n p^2}{(2n)!} \left[\prod_{i=1}^{n-1} (p - 2(n-i))(p + 2(n-i)) \right] x^{2n}$$
$$= F(x)$$

Now if we set $a_0 = 0$ and $a_1 = 1$ we get:

$$a_0 = 0$$
$$a_1 = 1$$
$$a_2 = 0$$
$$a_3 = -\frac{(p-1)(p+1)}{3!}$$
$$a_4 = 0$$
$$\vdots$$
$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} \prod_{i=1}^{n-1} (p - (2n+1-2i))(p + (2n+1-2i))$$
$$a_{2n} = 0$$

With this the solution takes on the form:

$$y(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$
$$= x - \frac{(p-1)(p+1)}{3!} x^3 + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\prod_{i=1}^{n-1} (p - (2n+1-2i))(p + (2n+1-2i)) \right] x^{2n+1}$$
$$= G(x)$$

Note that each of these solutions has the missing components of the other, therefore we form the general solution as the linear combination of the two. ∎

Proposition 8 (Convergence of the Solutions). The power series solution to the Chebyshev equation converges iff $|x| \leq 1$.

Proof. We begin by determining the limit:

$$\lim_{n \to \infty} \left| \frac{a_{n+2}}{a_n} \right| = |x^2| \lim_{n \to \infty} \left| \frac{(p-n)(p+n)}{(n+2)(n+1)} \right| = |x|^2$$

and now by the ratio test:

$$|x|^2 < 1 \implies |x| < 1$$

With this we have the region and have to check the boundaries independently. So for $x = -1$ we have:

$$y = \frac{(p-1)(p+1)}{3!} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \left[\prod_{i=1}^{n-1} (p - (2n+1-2i))(p + (2n+1-2i)) \right]$$

$$- \frac{p^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n p^2}{(2n)!} \left[\prod_{i=1}^{n-1} (p - 2(n-i))(p + 2(n-i)) \right]$$

where both infinite series converge according to the alternating series test. Lastly for $x = 1$:

$$y = 1 - \frac{(p-1)(p+1)}{3!} + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\prod_{i=1}^{n-1} (p - (2n+1-2i))(p + (2n+1-2i)) \right]$$

$$+ 1 - \frac{p^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n p^2}{(2n)!} \left[\prod_{i=1}^{n-1} (p - 2(n-i))(p + 2(n-i)) \right]$$

and once again by the alternating series test both infinite series converge above. Therefore, the final region of convergence included the boundaries and is given by $[-1, 1]$. ∎

Proposition 9 (Chebyshev Polynomials from the Chebyshev Equation). Notice that if from the above we have $p \in \mathbb{Z}$, the first and second case will terminate if p is even and odd respectively. Therefore, the two cases reduce down to a pth degree polynomial that is proportional to the pth Chebyshev polynomial by the relations:

$$T_p(x) = \begin{cases} (-1)^{\frac{p}{2}} F(x) & \exists k \in \mathbb{N} \text{ s.t. } p = 2k \\ (-1)^{\frac{p-1}{2}} p G(x) & \exists k \in \mathbb{N} \text{ s.t. } p = 2k+1 \end{cases}$$