Project Euler: Problem 2 Personal Comments

Nathan Marianovsky

Fibonacci Sequence in Closed Form

In the solution to this problem I only mentioned the closed form of the Fibonacci sequence and how to use it. Though this suffices for the problem, I believe it is important to know where results come from. So once more consider the Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

where this can be generated by the known recurrance relation:

$$F_n = F_{n-1} + F_{n-2} \quad \text{where} \quad F_0 = 0 \text{ and } F_1 = 1$$

Now to obtain the closed form we need to solve the above recurrence relation. Using similar techniques to the ones utilized in solving *linear-constant coeffiecient-homogeneous* ordinary differential equations, we guess that:

$$F_n = \alpha^n \quad \text{where} \quad \alpha \in \mathbb{R}$$

Plugging this into the recurrence relation and working out the algebra gives:

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ \alpha^n &= \alpha^{n-1} + \alpha^{n-2} \\ 0 &= \alpha^{n-2}\Big[\alpha^2 - \alpha - 1\Big] \end{aligned}$$

If $\alpha^{n-2} = 0 \implies \alpha = 0$ which is the trivial case. The remaining quadratic equation is what is known as the characteristic equation for the recurrence relation. Solving for α using the quadratic formula gives:

$$\alpha = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

Lets give each one a name:

$$\phi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1 - \sqrt{5}}{2}$$

where ϕ is known as the *golden ratio*. So now we have two linearly independent solutions. Using the superposition principle, the most general solution can be written as:

$$F_n = c_1\phi^n + c_2\psi^n \quad \text{where} \quad c_1, c_2 \in \mathbb{R}$$

The only thing left to do is to figure out the coefficients using the known intial conditions:

$$\begin{aligned} F_0 &= 0 = c_1 + c_2 \\ F_1 &= 1 = c_1\phi + c_2\psi \end{aligned}$$

Rewriting this system of equations as a matrix and solving gives:

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ \phi & \psi \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} \frac{\psi}{\psi-\phi} & -\frac{1}{\psi-\phi} \\ -\frac{\phi}{\psi-\phi} & \frac{1}{\psi-\phi} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\psi-\phi} \\ \frac{1}{\psi-\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \end{aligned}$$

Finally the closed form can be written as:

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n)$$

Past this point I am going to just use some algebraic manipulation to arrive at the form I used in the solution:

$$\psi = \frac{1-\sqrt{5}}{2} \times \frac{1+\sqrt{5}}{1+\sqrt{5}} = -\frac{4}{2(1+\sqrt{5})} = -\frac{1}{\frac{1+\sqrt{5}}{2}} = -\frac{1}{\phi} = -\phi^{-1}$$

Plugging this into the classed form for ψ finally gives the desired form:

$$F_n = \frac{1}{\sqrt{5}}\Big(\phi^n - (-\phi)^{-n}\Big)$$