Project Euler: Problem 6 Personal Comments

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Sum of First n Natural Numbers

To derive the formula for finding the sum of the first n natural numbers consider the following:

$$s_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$
$$s_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

The second line is just the first rewritten backwards. Now add the two equations:

$$2s_n = n(n+1)$$

which yields the final formula:

$$s_n = \frac{n(n+1)}{2} = \sum_{i=1}^{n} i$$

Sum of Squares of First n Natural Numbers

Consider the following telescoping series:

$$\sum_{i=1}^{n}((1+i)^3 - i^3) = (2^{\cancel{3}} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + (\cancel{n^3} - \cancel{(n-1)^3}) + ((n+1)^3 - \cancel{n^3}) = (n+1)^3 - 1$$

On the other hand this summation can also be written as:

$$\sum_{i=1}^{n}((1+i)^3 - i^3) = \sum_{i=1}^{n}(i^3 + 3i^2 + 3i + 1 - i^3) = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + n$$

Now comparing both sides and solving for the sum of the squares gives:

$$3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + n = (n+1)^3 - 1$$
$$\sum_{i=1}^{n} i^2 = \frac{(n+1)^3 - 1 - n}{3} - \sum_{i=1}^{n} i$$
$$\sum_{i=1}^{n} i^2 = \frac{(n+1)^3 - 1 - n}{3} - \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n^3 + 3n^2 + 2n}{3} - \frac{n^2 + n}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$