Project Euler: Problem 45

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Problem (Triangular, Pentagonal, and Hexagonal)**.** Triangle, pentagonal, and hexagonal numbers are generated by the following formulae:

Triangle	$T_n = \frac{n(n+1)}{2}$	$1, 3, 6, 10, 15, \dots$
Pentagonal	$P_n = \frac{n(3n-1)}{2}$	$1, 5, 12, 22, 35, \dots$
Hexagonal	$H_n = n(2n-1)$	$1, 6, 15, 28, 45, \dots$

It can be verified that $T_{285} = P_{165} = H_{143} = 40755$. Find the next triangle number that is also pentagonal and hexagonal.

Solution. To approach this, perhaps it might be a good idea to have a way of telling whether a given number is triangular, hexagonal, or pentagonal. To do this, rewrite each of the generating formula into quadratic form and solve for n:

Sequence	Quadratic Form	Solution
Triangle	$0 = n^2 + n - 2T_n$	$n = \frac{-1 \pm \sqrt{1+8T_n}}{2}$
Pentagonal	$0 = 3n^2 - n - 2P_n$	$n = \frac{1 \pm \sqrt{1+24P_n}}{6}$
Hexagonal	$0 = 2n^2 - n - H_n$	$n = \frac{1 \pm \sqrt{1+8H_n}}{4}$

Since each of the sequences consists of strictly positive integers, we can drop the negative radical for all three solutions. Now this problem can be generalized to finding the n^{th} number that satisfies all three sequences. If a number does belong to all three sequences then the following conditions will be met:

Sequence	Condition
Triangle	$-1 + \sqrt{1+8T_n} \equiv 0 \pmod 2$
Pentagonal	$1 + \sqrt{1+24P_n} \equiv 0 \pmod 6$
Hexagonal	$1 + \sqrt{1+8H_n} \equiv 0 \pmod 4$

So keep iterating through the natural numbers until the n^{th} number that satisfies the above congruence relations is found.