Project Euler: Problem 31

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Problem (Coin Sums)**.** In England the currency is made of pound, £, and pence, p, and there are eight coins in general circulation:

$$1\text{p}, 2\text{p}, 5\text{p}, 10\text{p}, 20\text{p}, 50\text{p}, \pounds 1\ (100\text{p}), \pounds 2\ (200\text{p})$$

It is possible to make £2 in the following way:

$$(1 \times \pounds 1) + (1 \times 50\text{p}) + (2 \times 20\text{p}) + (1 \times 5\text{p}) + (1 \times 2\text{p}) + (3 \times 1\text{p})$$

How many different ways can £2 be made using any number of coins?

Solution. This problem is equivalent to finding the number of different solutions satisfying:

$$x_1 + 2x_2 + 5x_3 + 10x_4 + 20x_5 + 50x_6 + 100x_7 + 200x_8 = r \quad \text{where} \quad x_i \geq 0 \quad \forall i \in \{1, 2, \dots, 8\}$$

In this problem specifically we want $r = 200$, but the calculations can be generalized to any given sum r. Now in order to solve this problem, consider the method of generating functions. Each variable will have a corresponding generating function that shows its possibilities:

x_i	Generating Function	Simplified Form
x_1	$g_{x_1}(t) = 1 + t + t^2 + \cdots + t^r$	$g_{x_1}(t) = \frac{1-t^{r+1}}{1-t}$
x_2	$g_{x_2}(t) = 1 + t^2 + t^4 + \cdots + t^{\frac{r}{2}}$	$g_{x_2}(t) = \frac{1-t^{r+2}}{1-t^2}$
x_3	$g_{x_3}(t) = 1 + t^5 + t^{10} + \cdots + t^{\frac{r}{5}}$	$g_{x_3}(t) = \frac{1-t^{r+5}}{1-t^5}$
x_4	$g_{x_4}(t) = 1 + t^{10} + t^{20} + \cdots + t^{\frac{r}{10}}$	$g_{x_4}(t) = \frac{1-t^{r+10}}{1-t^{10}}$
x_5	$g_{x_5}(t) = 1 + t^{20} + t^{40} + \cdots + t^{\frac{r}{20}}$	$g_{x_5}(t) = \frac{1-t^{r+20}}{1-t^{20}}$
x_6	$g_{x_6}(t) = 1 + t^{50} + t^{100} + \cdots + t^{\frac{r}{50}}$	$g_{x_6}(t) = \frac{1-t^{r+50}}{1-t^{50}}$
x_7	$g_{x_7}(t) = 1 + t^{100} + t^{200} + \cdots + t^{\frac{r}{100}}$	$g_{x_7}(t) = \frac{1-t^{r+100}}{1-t^{100}}$
x_8	$g_{x_8}(t) = 1 + t^{200} + t^{400} + \cdots + t^{\frac{r}{200}}$	$g_{x_8}(t) = \frac{1-t^{r+200}}{1-t^{200}}$

Giving an overall generating function:

$$\begin{aligned} g(t) &= g_{x_1}(t) g_{x_2}(t) \dots g_{x_8}(t) \\ &= \frac{(1-t^{r+1})(1-t^{r+2})(1-t^{r+5})(1-t^{r+10})(1-t^{r+20})(1-t^{r+50})(1-t^{r+100})(1-t^{r+200})}{(1-t)(1-t^2)(1-t^5)(1-t^{10})(1-t^{20})(1-t^{50})(1-t^{100})(1-t^{200})} \end{aligned}$$

The solution to the problem now corresponds to finding the coefficient of t^r in the power series expansion of the above generating function. For this problem specifically we want the coefficient of t^{200} which happens to be 73,682 when calculated using Mathematica and can actually be found faster by replacing the numerator with just 1 since the lowest degree of t after 0 is going to be exactly 201 making it all extra junk.