Project Euler: Problem 137 Personal Comments

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Closed Form of Fibonacci Generating Function

In the solution I provided, I made a big jump by saying that:

$$A_F(x) = xF_1 + x^2F_2 + x^3F_3 + \cdots = \frac{x}{1-x-x^2}$$

which turns an infinite series into a finite closed form. But so where did this closed form come from? Before moving on lets define the generating function in a proper form:

$$A_F(x) = \sum_{i=1}^{\infty} x^i F_i$$

Now observe what happens when we use the known recurrence relation for the Fibonacci sequence:

$$\begin{aligned}
A_F(x) &= \sum_{i=1}^{\infty} x^i F_i \\
&= F_0 + xF_1 + \sum_{i=2}^{\infty} x^i F_i \\
&= F_0 + xF_1 + \sum_{i=2}^{\infty} x^i (F_{i-1} + F_{i-2}) \\
&= F_0 + xF_1 + \sum_{i=2}^{\infty} x^i F_{i-1} + \sum_{i=2}^{\infty} x^i F_{i-2} \\
&= F_0 + xF_1 + \sum_{k=1}^{\infty} x^{k+1} F_k + \sum_{k=0}^{\infty} x^{k+2} F_k \\
&= F_0 + x^2F_0 + xF_1 + x\sum_{k=1}^{\infty} x^k F_k + x^2 \sum_{k=1}^{\infty} x^k F_k \\
&= x + xA_F(x) + x^2 A_F(x)
\end{aligned}$$

where I exploited the Fibonacci numbers because placing the F_0 in the above calculations is meaningless since it is 0. Now use the above to solve for $A_F(x)$:

$$\begin{aligned}
A_F(x) &= x + xA_F(x) + x^2A_F(x) \\
A_F(x)(1-x-x^2) &= x \\
A_F(x) &= \frac{x}{1-x-x^2}
\end{aligned}$$