Rotations & Euler Angles

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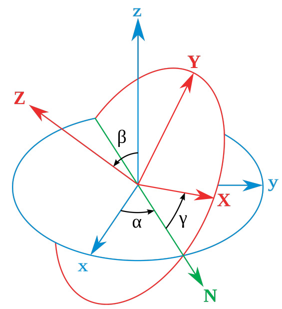
Euler Angles

Definition 1 (Proper Euler Angles)**.** Consider the following setup:



where:

- (x, y, z) and (X, Y, Z) represent the axes of the original and rotated frames
- N is the intersection of the xy and XY planes
- α is the angle between the x axis and the N axis
- β is the angle between the z axis and the Z axis
- γ is the angle between the N axis and the X axis

The angles defined above are known as *Proper Euler Angles* which provide us with:

- A rotation about the z axis that is determined by α
- A rotation about the N axis that is determined by β
- A rotation about the Z axis that is determined by γ

Rotations

Definition 2 (Intrinsic Rotation)**.** *Intrinsic rotations* are ones that occur about the axes of the rotating coordinate system XYZ, keeping xyz fixed. As a result the rotating coordinate system changes its orientation after each elemental rotation.

Example 1. *Begin by saying that the XYZ coordinate system coincides with the xyz system before we perform any rotation. Now let us say that we want to reach the final state above using intrinsic rotations. Follow these directions:*

- *Rotate the XYZ system by α about the Z axis, which is the same as the z axis.*
- *Rotate the new XYZ system by β about the new X axis, which is the same as the N axis above.*
- *Rotate the new XYZ system by γ about the new Z axis, which is the Z axis as in the image.*

where this whole rotation would be written down as $Z - X' - Z''$ about the angles α, β, and γ respectively.

Definition 3 (Intrinsic Rotation Matrix Form)**.** An intrinsic rotation given by $X - Y' - Z''$ about the angles α, β, and γ respectively is given by:

$$\mathcal{R} = X(\alpha)Y(\beta)Z(\gamma)$$

Definition 4 (Extrinsic Rotation)**.** *Extrinsic rotations* are ones that occur about the axes of the fixed coordinate system xyz, keeping xyz fixed.

Example 2. *Begin by saying that the XYZ coordinate system coincides with the xyz system before we perform any rotation. Now let us say that we want to reach the final state above using extrinsic rotations. Follow these directions:*

- *Rotate the XYZ system by α about the z axis.*
- *Rotate the new XYZ system by β about the x axis.*
- *Rotate the new XYZ system by γ about the z axis.*

where this whole rotation would be written down as $z - x - z$ about the angles α, β, and γ respectively.

Definition 5 (Extrinsic Rotation Matrix Form)**.** An extrinsic rotation given by $x - y - z$ about the angles α, β, and γ respectively is given by:

$$\mathcal{R} = Z(\gamma)Y(\beta)X(\alpha)$$

Definition 6 (Converting Between Extrinsic and Intrinsic)**.** If we are given an extrinsic rotation $x - y - z$ about the angles α, β, and γ respectively:

$$\mathcal{R} = Z(\gamma)Y(\beta)X(\alpha)$$

then this is equivalent to the intrinsic rotation $Z - Y' - X''$ about the angles γ, β, and α respectively.

Definition 7 (Gimbal Lock)**.** A *gimbal* is a ring that is suspended so it can rotate about an axis. Typically gimbals can be nested one within another to accommodate rotation about multiple axes such as:

Gimbal Lock references the loss of one degree of freedom in a three dimensional, three-gimbal mechanism that occurs when the axes of two of the three gimbals are driven into a parallel configuration.

Example 3. *Using Euler Angles, define the extrinsic rotation $z-y-x$ about the angles γ, β, and α respectively and observe what happens when $\beta = \frac{\pi}{2}$:*

$$\begin{aligned}
\mathcal{R} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 1 & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha)\cos(\gamma) + \cos(\alpha)\sin(\gamma) & -\sin(\alpha)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & 0 \\ -\cos(\alpha)\cos(\gamma) + \sin(\alpha)\sin(\gamma) & \cos(\alpha)\sin(\gamma) + \sin(\alpha)\cos(\gamma) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{pmatrix}
\end{aligned}$$

The final result ends up being a rotation that stays in the z direction no matter what α and γ are, thereby "locking" one of the directions.

Proposition 1 (Useful Identity)**.** Given any two vectors $u, v \in \mathbb{R}^3$, the following holds true:

$$u \times v = u_\times v$$

where:

$$u_\times = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Proof. This can be shown by direct computation of the determinant using the cofactor expansion and by reorganizing the linear result as the multiplication of a matrix and the vector on the right side:

$$\begin{aligned} u \times v &= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \\ &= \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \end{aligned}$$

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Proposition 2 (Rotation Matrix for Rotation About Any Axis)**.** A rotation about a given a unit vector $\hat{k} \in \mathbb{R}^3$ with angle θ is determined by the rotation matrix:

$$\mathcal{R} = \begin{pmatrix} \cos(\theta)(1-k_1^2) + k_1^2 & k_1 k_1(1-\cos(\theta)) - k_3 \sin(\theta) & k_1 k_3(1-\cos(\theta)) + k_2 \sin(\theta) \\ k_1 k_2(1-\cos(\theta)) + k_3 \sin(\theta) & \cos(\theta)(1-k_2^2) + k_2^2 & k_2 k_3(1-\cos(\theta)) - k_1 \sin(\theta) \\ k_1 k_3(1-\cos(\theta)) - k_2 \sin(\theta) & k_2 k_3(1-\cos(\theta)) + k_1 \sin(\theta) & \cos(\theta)(1-k_3^2) + k_3^2 \end{pmatrix}$$

Proof. If we accept Rodrigues' Rotation Formula:

$$v' = \cos(\theta) v + \sin(\theta)(\hat{k} \times v) + (1-\cos(\theta))\hat{k}(\hat{k} \cdot v)$$

then if we simply pull out the vector on the right side and combine we get:

$$\begin{aligned} v' &= \cos(\theta) v + \sin(\theta)\hat{k}_\times v + (1-\cos(\theta))\hat{k}\hat{k}^T v \\ &= \Big[\cos(\theta) I_3 + \sin(\theta)\hat{k}_\times + (1-\cos(\theta))\hat{k} \otimes \hat{k}\Big] v \\ &= \left[\begin{pmatrix} \cos(\theta) & 0 & 0 \\ 0 & \cos(\theta) & 0 \\ 0 & 0 & \cos(\theta) \end{pmatrix} + \begin{pmatrix} 0 & -k_3 \sin(\theta) & k_2 \sin(\theta) \\ k_3 \sin(\theta) & 0 & -k_1 \sin(\theta) \\ -k_2 \sin(\theta) & k_1 \sin(\theta) & 0 \end{pmatrix} \right. \\ &\qquad \left. + (1-\cos(\theta)) \begin{pmatrix} k_1^2 & k_1 k_2 & k_1 k_3 \\ k_1 k_2 & k_2^2 & k_2 k_3 \\ k_1 k_3 & k_2 k_3 & k_3^2 \end{pmatrix} \right] v \\ &= \begin{pmatrix} \cos(\theta)(1-k_1^2) + k_1^2 & k_1 k_1(1-\cos(\theta)) - k_3 \sin(\theta) & k_1 k_3(1-\cos(\theta)) + k_2 \sin(\theta) \\ k_1 k_2(1-\cos(\theta)) + k_3 \sin(\theta) & \cos(\theta)(1-k_2^2) + k_2^2 & k_2 k_3(1-\cos(\theta)) - k_1 \sin(\theta) \\ k_1 k_3(1-\cos(\theta)) - k_2 \sin(\theta) & k_2 k_3(1-\cos(\theta)) + k_1 \sin(\theta) & \cos(\theta)(1-k_3^2) + k_3^2 \end{pmatrix} v \end{aligned}$$

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