Calculus 2
Recitation 9

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VOLUME BY INTEGRATION: CROSS SECTION METHOD

Definition 1 (Cross Section Method)**.** The volume of a cylinder is defined as the cross sectional area of the base multiplied by the height. Though why stop with cylinders? Extend this idea to any arbitrary shape and define volume as:

$$\textbf{Volume} = \int_a^b A(x)dx$$

where $A(x)$ represents the cross sectional area.

VOLUME BY INTEGRATION: WASHER METHOD

Definition 2 (Washer Method)**.** In order to find the volume formed by rotating a region in $\mathbb{R}^2$ about some axis, the washer method states that by taking perpendicular cross sections the volume is defined as:

$$\textbf{Volume} = \pi \int_a^b \Big((R(x))^2 - (r(x))^2 \Big) dx$$

where $R(x)$ and $r(x)$ represent the outer and inner radii of the perpendicular cross section. A common case is when the inner radius is just the x-axis or y-axis which implies that $r(x) = 0$ and the formula reduces to what is known as the disk method.

VOLUME BY INTEGRATION: CYLINDRICAL SHELL METHOD

Definition 3 (Cylindrical Shell Method)**.** Unlike the washer method, the shell method uses parallel cross sections and thus states that:

$$\textbf{Volume} = 2\pi \int_a^b r(x)h(x)dx$$

where $r(x)$ and $h(x)$ represent the radius and height of the parallel cross section respectively. Note that $h(x)$ can have cases where the cross sections are bound between two curves or cases when the bottom curve is the axis of rotation.

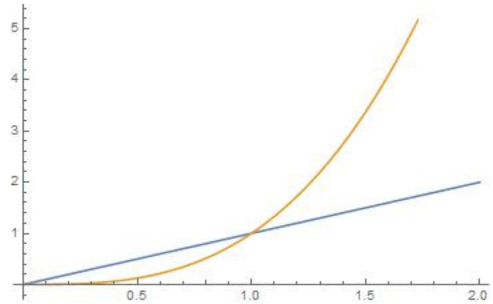
CHOOSING WHAT VARIABLE TO INTEGRATE OVER

Definition 4 (Integration Variable)**.** When setting up either of the methods, choosing which variable to integrate over can be difficult. To simplify the thinking process, here is a table describing all possible cases:

	Washer Method	**Cylindrical Shell Method**
Horizontal Axis of Rotation	x	y
Vertical Axis of Rotation	y	x

Example 1. *Find the volume generated by rotating the region enclosed by $y = x^3$ and $y = x$ about the x-axis.*

Solution 1. *First draw out the region:*



(i) *Washer Method:*
The perpendicular cross sections in this scenario will be vertical ones. The outer radius is given by $R(x) = x$ and the inner by $r(x) = x^3$. The interval is $[0, 1]$ for the enclosed region and the volume is given by:

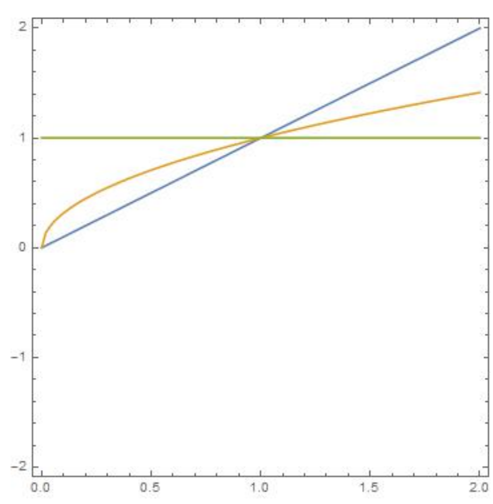
$$\textbf{\textit{Volume}} = \pi \int_0^1 \Big((x)^2 - (x^3)^2\Big)dx = \pi\Big(\frac{x^3}{3} - \frac{x^7}{7}\Big)\Big|_0^1 = \boxed{\frac{4\pi}{21}}$$

(ii) *Cylindrical Shell Method:*
The parallel cross sections in this scenario will be horizontal ones. The height of each cross section is going to be $h(y) = y^{\frac{1}{3}} - y$ and the radius $r(y) = y$ because each cross section is a distance y away from the axis of rotation. The interval is $[0, 1]$ for the enclosed region and the volume is given by:

$$\textbf{\textit{Volume}} = 2\pi \int_0^1 y(y^{\frac{1}{3}} - y)dy = 2\pi \int_0^1 (y^{\frac{4}{3}} - y^2)dy = 2\pi\Big(\frac{3}{7}x^{\frac{7}{3}} - \frac{1}{3}y^3\Big)\Big|_0^1 = \boxed{\frac{4\pi}{21}}$$

Example 2. *Find the volume generated by rotating the region enclosed by $y = x$ and $y = \sqrt{x}$ about $y = 1$.*

Solution 2. *First draw out the region:*



(i) *Washer Method:*
The perpendicular cross sections in this scenario will be vertical ones. The outer radius is given by $R(x) = 1 - x$ and the inner by $r(x) = 1 - \sqrt{x}$. The interval is $[0, 1]$ for the enclosed region and the volume is given by:

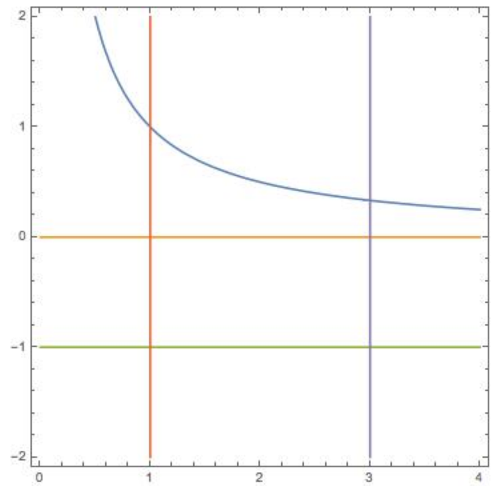
$$\textbf{\textit{Volume}} = \pi \int_0^1 \Big((1-x)^2 - (1-\sqrt{x})^2\Big)dx = \pi \int_0^1 \Big(x^2 - 3x + 2\sqrt{x}\Big)dx = \pi\Big(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{4x^{\frac{3}{2}}}{3}\Big)\Big|_0^1 = \boxed{\frac{\pi}{6}}$$

(ii) *Cylindrical Shell Method:*
The parallel cross sections in this scenario will be horizontal ones. The height of each cross section is going to be $h(y) = y - y^2$ and the radius $r(y) = 1 - y$ because each cross section is a distance $1 - y$ away from the axis of rotation. The interval is $[0, 1]$ for the enclosed region and the volume is given by:

$$\textbf{\textit{Volume}} = 2\pi \int_0^1 (1-y)(y-y^2)dy = 2\pi \int_0^1 (y^3 - 2y^2 + y)dy = 2\pi\Big(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\Big)\Big|_0^1 = \boxed{\frac{\pi}{6}}$$

Example 3. *Find the volume generated by rotating the region enclosed by $y = \frac{1}{x}$, $y = 0$, $x = 1$, and $x = 3$ about $y = -1$.*

Solution 3. *First draw out the region:*



(i) *Washer Method:*
The perpendicular cross sections in this scenario will be vertical ones. The outer radius is given by $R(x) = \frac{1}{x} + 1$ and the inner by $r(x) = 1$. The interval is $[1, 3]$ for the enclosed region and the volume is given by:

$$\textbf{\textit{Volume}} = \pi \int_1^3 \Big((x^{-1}+1)^2 - (1)^2\Big)dx = \pi \int_1^3 \Big(x^{-2} + 2x^{-1}\Big)dx = \pi\Big(-\frac{1}{x} + 2\ln|x|\Big)\Big|_1^3 = \boxed{\pi\Big[\frac{2}{3} + 2\ln(3)\Big]}$$

(ii) *Cylindrical Shell Method:*
The parallel cross sections in this scenario will be horizontal ones. The height of each cross section for the upper region is going to be $h(y) = \frac{1}{y} - 1$ and the radius $r(y) = y + 1$ because each cross section is a distance $y + 1$ away from the axis of rotation. The interval is $\Big[\frac{1}{3}, 1\Big]$ for the enclosed region and the volume is given by:

$$\textbf{\textit{Volume}}_1 = 2\pi \int_{\frac{1}{3}}^1 (1+y)(y^{-1} - 1)dy = 2\pi \int_{\frac{1}{3}}^1 \Big(y^{-1} - y\Big)dy = 2\pi\Big(\ln|y| - \frac{y^2}{2}\Big)\Big|_{\frac{1}{3}}^1 = 2\pi\Big[-\frac{4}{9} + \ln(3)\Big]$$

The horizontal cross sections ranging from $\Big[0, \frac{1}{3}\Big]$ have $h(x) = 2$ and $r(x) = 1 + y$. The volume for the lower region is given by:

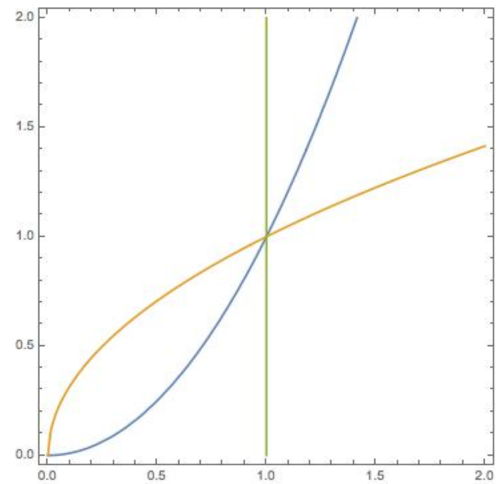
$$\textbf{\textit{Volume}}_2 = 2\pi \int_0^{\frac{1}{3}} (1+y)(2)dy = 4\pi\Big(y + \frac{y^2}{2}\Big)\Big|_0^{\frac{1}{3}} = \frac{14\pi}{9}$$

Now the total volume is:

$$\textbf{\textit{Volume}} = \textbf{\textit{Volume}}_1 + \textbf{\textit{Volume}}_2 = -\frac{8\pi}{9} + 2\pi\ln(3) + \frac{14\pi}{9} = \boxed{\frac{2\pi}{3} + 2\pi\ln(3)}$$

Example 4. *Find the volume generated by rotating the region enclosed by $y = x^2$ and $y = \sqrt{x}$ about $x = 1$.*

Solution 4. *First draw out the region:*



(i) *Washer Method:*
The perpendicular cross sections in this scenario will be horizontal ones. The outer radius is given by $R(y) = 1 - y^2$ and the inner by $r(y) = 1 - \sqrt{y}$. The interval is $[0, 1]$ for the enclosed region and the volume is given by:

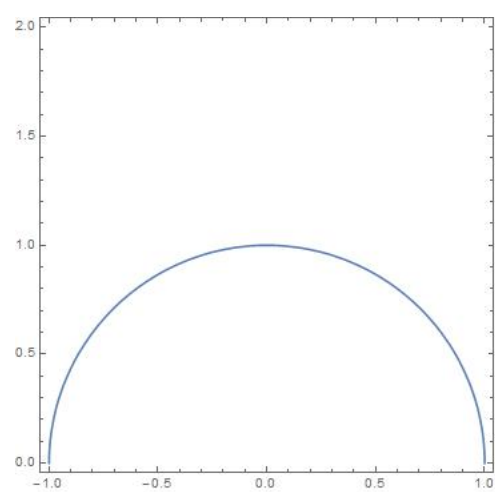
$$\textbf{\textit{Volume}} = \pi \int_0^1 \Big((1-y^2)^2 - (1-\sqrt{y})^2\Big)dy = \pi \int_0^1 \Big(y^4 - 2y^2 - y + 2\sqrt{y}\Big)dy$$
$$= \pi\Big(\frac{y^5}{5} - \frac{2y^3}{3} - \frac{y^2}{2} + \frac{4y^{\frac{3}{2}}}{3}\Big)\Big|_0^1 = \boxed{\frac{11\pi}{30}}$$

(ii) *Cylindrical Shell Method:*
The parallel cross sections in this scenario will be vertical ones. The height of each cross section is going to be $h(x) = \sqrt{x} - x^2$ and the radius $r(x) = 1 - x$ because each cross section is a distance $1 - x$ away from the axis of rotation. The interval is $[0, 1]$ for the enclosed region and the volume is given by:

$$\textbf{\textit{Volume}} = 2\pi \int_0^1 (1-x)(\sqrt{x}-x^2)dx = 2\pi \int_0^1 \Big(x^3-x^2-x^{\frac{3}{2}}+\sqrt{x}\Big)dx = 2\pi\Big(\frac{x^4}{4}-\frac{x^3}{3}-\frac{2x^{\frac{5}{2}}}{5}+\frac{2x^{\frac{3}{2}}}{3}\Big)\Big|_0^1 = \boxed{\frac{11\pi}{30}}$$

Example 5. *Find the volume of a sphere with radius r through the washer and cylindrical shell methods.*

Solution 5. *First draw out the curve, $y = \sqrt{r^2 - x^2}$, that is going to be rotated around the x-axis:*



(i) *Washer Method:*
The perpendicular cross sections in this scenario will be vertical ones. The outer radius is given by $R(x) = \sqrt{r^2 - x^2}$ *and the inner by* $r(x) = 0$*. The interval is* $[-r, r]$ *for the enclosed region and the volume is given by:*

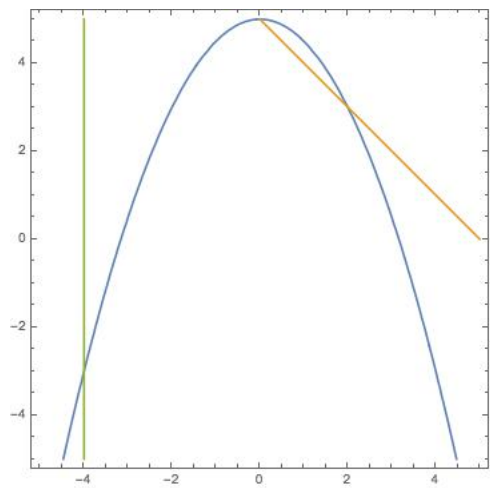
$$\textbf{Volume} = \pi \int_{-r}^{r} \left((\sqrt{r^2 - x^2})^2 \right) dx = \pi \int_{-r}^{r} (r^2 - x^2) dx = 2\pi \int_{0}^{r} (r^2 - x^2) dx = 2\pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_0^r = \boxed{\frac{4}{3}\pi r^3}$$

(ii) *Cylindrical Shell Method:*
The parallel cross sections in this scenario will be horizontal ones. The height of each cross section is going to be $h(y) = 2\sqrt{r^2 - y^2}$ *and the radius* $r(y) = y$ *because each cross section is a distance* y *away from the axis of rotation. The interval is* $[0, r]$ *for the enclosed region and the volume is given by:*

$$\textbf{Volume} = 2\pi \int_{0}^{r} y(2\sqrt{r^2 - y^2}) dy = 4\pi \int_{0}^{r} y\sqrt{r^2 - y^2} dy = 2\pi \int_{0}^{r^2} u^{\frac{1}{2}} du = \frac{4}{3}\pi u^{\frac{3}{2}} \Big|_0^{r^2} = \boxed{\frac{4}{3}\pi r^3}$$

Example 6. *Find the volume generated by rotating the region enclosed by* $y = 5 - \frac{x^2}{2}$ *and* $x + y = 5$ *about* $x = -4$.

Solution 6. *First draw out the region:*



(i) *Washer Method:*
The perpendicular cross sections in this scenario will be horizontal ones. The outer radius is given by $R(y) = 4 + \sqrt{10 - 2y}$ *and the inner by* $r(y) = 9 - y$*. The interval is* $[3, 5]$ *for the enclosed region and the volume is given by:*

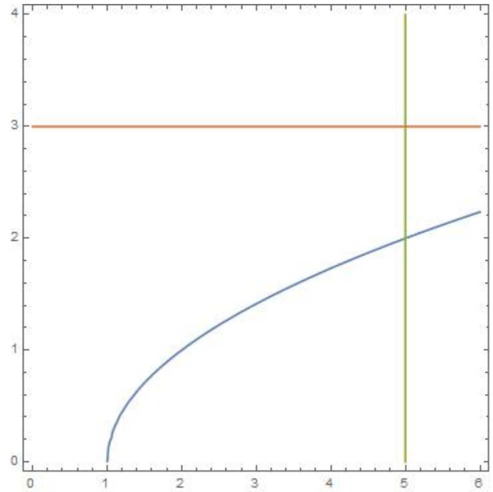
$$\begin{aligned}
\textbf{Volume} &= \pi \int_{3}^{5} \left((4 + \sqrt{10 - 2y})^2 - (9 - y)^2 \right) dy = \pi \int_{3}^{5} \left(26 + 8\sqrt{10 - 2y} - 2y - (9 - y)^2 \right) dy \\
&= \pi \left(26y - \frac{8(10 - 2y)^{\frac{3}{2}}}{3} - y^2 + \frac{(9 - y)^3}{3} \right) \Big|_3^5 \\
&= \pi \left[\left(105 + \frac{64}{3} \right) - \left(141 - \frac{64}{3} \right) = \boxed{\frac{20\pi}{3}} \right.
\end{aligned}$$

(ii) *Cylindrical Shell Method:*
The parallel cross sections in this scenario will be vertical ones. The height of each cross section is going to be $h(x) = -\frac{x^2}{2} + x$ *and the radius* $r(x) = 4 + x$ *because each cross section is a distance* $4 + x$ *away from the axis of rotation. The interval is* $[0, 2]$ *for the enclosed region and the volume is given by:*

$$\begin{aligned}
\textbf{Volume} &= 2\pi \int_{0}^{2} (4 + x)\left(-\frac{x^2}{2} + x \right) dx = 2\pi \int_{0}^{2} \left(-x^2 + 4x - \frac{x^3}{2} \right) dx = 2\pi \left(-\frac{x^3}{3} + 2x^2 - \frac{x^4}{8} \right) \Big|_0^2 \\
&= 2\pi \left(-\frac{8}{3} + 8 - 2 \right) = \boxed{\frac{20\pi}{3}}
\end{aligned}$$

Example 7. *Find the volume generated by rotating the region enclosed by* $y = \sqrt{x-1}$*,* $y = 0$*, and* $x = 5$ *about* $y = 3$*.*

Solution 7. *First draw out the region:*



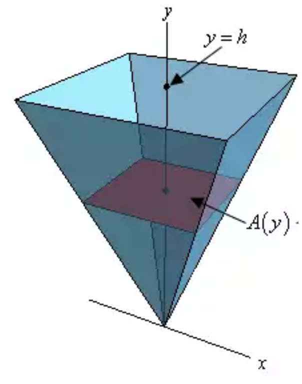
(i) *Washer Method:*
The perpendicular cross sections in this scenario will be vertical ones. The outer radius is given by $R(x) = 3$ *and the inner by* $r(x) = 3 - \sqrt{x-1}$*. The interval is* $[1, 5]$ *for the enclosed region and the volume is given by:*

$$\textbf{\textit{Volume}} = \pi \int_1^5 \left((3)^2 - (3 - \sqrt{x-1})^2\right) dx = \pi \int_1^5 \left(6\sqrt{x-1} - x + 1\right) dx$$
$$= \pi\left(4(x+1)^{\frac{3}{2}} - \frac{x^2}{2} + x\right)\Big|_1^5 = \boxed{24\pi}$$

(ii) *Cylindrical Shell Method:*
The parallel cross sections in this scenario will be horizontal ones. The height of each cross section is going to be $h(y) = 4 - y^2$ *and the radius* $r(y) = 3 - y$ *because each cross section is a distance* $3 - y$ *away from the axis of rotation. The interval is* $[0, 2]$ *for the enclosed region and the volume is given by:*

$$\textbf{\textit{Volume}} = 2\pi \int_0^2 (3-y)(4-y^2)dy = 2\pi \int_0^2 \left(y^3 - 3y^2 - 4y + 12\right) dy = 2\pi\left(\frac{y^4}{4} - y^3 - 2y^2 + 12y\right)\Big|_0^2 = \boxed{24\pi}$$

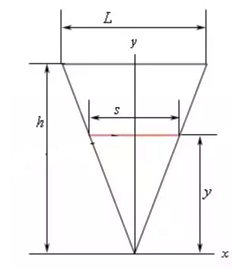
Example 8. *Find the volume of the square pyramid whose base is h and side length L:*



Solution 8. *Take horizontal cross sections and sum them up to find the formula. The area of a small cross section will be:*

$$A(y) = s^2(y)$$

where s represents the side length of the cross section. Now look at a vertical cross section:



Using similar triangles:

$$\frac{s}{y} = \frac{L}{h} \implies s = \frac{L}{h}y$$

Now the volume will be:

$$\textbf{\textit{Volume}} = \int_a^b A(y)dy = \int_0^h \frac{L^2}{h^2}y^2 dy = \frac{L^2}{3h^2}y^3\Big|_0^h = \boxed{\frac{L^2h}{3}}$$