Quiz 4 Solutions

MATH 103A
August 9, 2018

(1) (Q) Evaluate the following integrals where $\mathbb{S}^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$:

a) $\int\limits_{\mathbb{S}^1} \frac{\mathrm{d}z}{z}$ b) $\int\limits_{\mathbb{S}^1} z^i \, \mathrm{d}z$

(A)

(a) Use the parametrization $z = e^{i\theta}$ for $-\pi \leq \theta \leq \pi$:

$$\int\limits_{\mathbb{S}^1} \frac{\mathrm{d}z}{z} = \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{e^{i\theta}} \, \mathrm{d}\theta = i\theta\Big|_{-\pi}^{\pi} = 2\pi i$$

(b) Use the parametrization $z = e^{i\theta}$ for $-\pi \leq \theta \leq \pi$:

$$\begin{aligned}\int\limits_{\mathbb{S}^1} z^i \, \mathrm{d}z &= \int_{-\pi}^{\pi} e^{-\theta} \cdot ie^{i\theta} \, \mathrm{d}\theta = i\int_{-\pi}^{\pi} e^{(i-1)\theta} \, \mathrm{d}\theta = \frac{i}{i-1} e^{(i-1)\theta}\Big|_{-\pi}^{\pi} \\ &= \frac{i}{i-1}\Big(e^{(i-1)\pi} - e^{-(i-1)\pi}\Big) = (1-i)\frac{e^{\pi} - e^{-\pi}}{2} = (1-i)\sinh(\pi)\end{aligned}$$

(2) (Q) We aim to prove the Cauchy-Goursat Theorem. Given a holomorphic function $f(z)$ on the simply-connected region $\Omega \subset \mathbb{C}$, use Green's Theorem to deduce:

$$\int_{\partial\Omega} f(z) \, \mathrm{d}z = 0$$

(A) Use the fact that $f(z) = g(x, y) = u(x, y) + iv(x, y)$ and $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ to deduce:

$$\begin{aligned}\int_{\partial\Omega} f(z) \, \mathrm{d}z &= \int_{\partial\Omega} (u(x, y) + iv(x, y))(\mathrm{d}x + i\mathrm{d}y) \\ &= \int_{\partial\Omega} \bigg(\Big(u(x, y) + iv(x, y)\Big)\mathrm{d}x + \Big(-v(x, y) + iu(x, y)\Big)\mathrm{d}y\bigg) \\ &= \int_{\Omega} \Big(-v_x(x, y) + iu_x(x, y) - u_y(x, y) - iv_y(x, y)\Big)\mathrm{d}A \\ &= \int_{\Omega} \Big(-v_x(x, y) + iv_y(x, y) + v_x(x, y) - iv_y(x, y)\Big)\mathrm{d}A \\ &= \int_{\Omega} 0 \, \mathrm{d}A \\ &= 0\end{aligned}$$