Quiz 6 Solutions

Section B

(1) (a) By direct evaluation:

$$\Gamma(1) = \int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x = \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{b\to\infty}(1-e^{-b}) = 1$$

(b) Using the hint we have:

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,\mathrm{d}x \\ &= \lim_{b\to\infty} -x^n e^{-x}\Big|_0^b + n\int_0^\infty x^{n-1}e^{-x}\,\mathrm{d}x \\ &= \lim_{b\to\infty} -\frac{b^n}{e^b} + n\Gamma(n) \\ &= 0 + n(n-1)! \\ &= n!\end{aligned}$$

(2) (a) First identify that $\frac{1}{\phi} < 1$ implying that the series converges. Now using the formula for the infinite geometric series:

$$\sum_{n=1}^\infty \frac{1}{\phi^n} = \frac{\frac{1}{\phi}}{1-\frac{1}{\phi}} = \frac{1}{\phi-1} = \frac{1}{\frac{1+\sqrt{5}}{2}-1} = \frac{2}{-1+\sqrt{5}}\cdot\frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{2} = \phi$$

This corresponds to choice (c).

(b) First rewrite the series as:

$$\sum_{n=1}^\infty \frac{1}{\phi^{2n-1}} = \phi\sum_{n=1}^\infty\left(\frac{1}{\phi^2}\right)^n$$

Note that $\frac{1}{\phi^2} < 1$ implying that the series converges. Once again we use the formula for the infinite geometric series:

$$\phi\sum_{n=1}^\infty\left(\frac{1}{\phi^2}\right)^n = \phi\cdot\frac{\frac{1}{\phi^2}}{1-\frac{1}{\phi^2}} = \frac{\phi}{\phi^2-1} = \frac{\phi}{\frac{1+2\sqrt{5}+5}{4}-1} = \frac{\phi}{\phi} = 1$$

This corresponds to choice (a).

SECTION C

(1) Exactly the same as Section B.

(2) (a) Exactly the same as Section B.

(b) Note that $\frac{1}{\phi^2} < 1$ implying that the series converges. Now using the formula for the infinite geometric series:

$$\sum_{n=1}^{\infty} \left(\frac{1}{\phi^2}\right)^n = \frac{\frac{1}{\phi^2}}{1 - \frac{1}{\phi^2}} = \frac{1}{\phi^2 - 1} = \frac{1}{\frac{1+2\sqrt{5}+5}{4} - 1} = \frac{1}{\phi}$$

This corresponds to choice (b).