Week 3 Attendance Solutions

MATH 23A

(1) (Q) Given two planes $\mathcal{P}$ and $\mathcal{Q}$, we want to discuss the notion of a *distance* between them. Note that by intuition this can only work when the planes are parallel as otherwise there would not be a consistent value for the distance. Assuming the planes are parallel, it must be that the normal vectors are parallel. Thus, we can take the normal vector and some point on $\mathcal{P}$ denoted by $\overrightarrow{n}$ and $\overrightarrow{p_0}$ respectively to form the line:

$$\overrightarrow{l}(t) = \overrightarrow{p_0} + t\,\overrightarrow{n}$$

For some value of $t \in \mathbb{R}$, say $t = t_0$, we will reach a point on $\mathcal{Q}$ that is going to be denoted by $\overrightarrow{l}(t_0)$, the point of intersection between $\mathcal{Q}$ and $\overrightarrow{l}(t)$. We now have enough information to write down the vector $\overrightarrow{r} = \overrightarrow{l}(t_0) - \overrightarrow{p_0}$ and define our distance as:

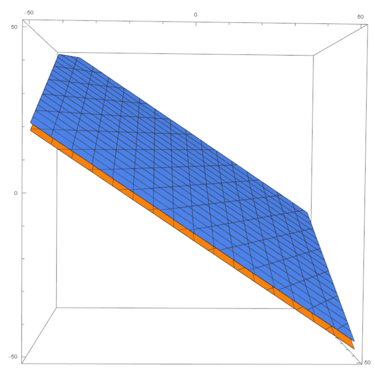
$$\text{dist}(\mathcal{P}, \mathcal{Q}) = \|\overrightarrow{r}\|$$

As a computational exercise we take our planes to be:

$$\mathcal{P} = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 14\} \quad \text{and} \quad \mathcal{Q} = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 7\}$$

Calculate $\text{dist}(\mathcal{P}, \mathcal{Q})$ (*Hint: A good starting point on $\mathcal{P}$ would be* $(14, 0, 0)$).

(A) When drawn out our scenario looks like:



As recommended, we take the starting point and direction vector to be:

$$\overrightarrow{p_0} = \begin{pmatrix} 14 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

which provides:

$$\overrightarrow{l}(t) = \begin{pmatrix} 14 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 + t \\ 2t \\ 3t \end{pmatrix}$$

Now to find t_0 we must satisfy:

$$\begin{aligned} 7 &= (14 + t_0) + 2(2t_0) + 3(3t_0) \\ 7 &= 14 + 14t_0 \\ t_0 &= -\frac{1}{2} \end{aligned}$$

Finally the distance is:

$$\text{dist}(\mathcal{P}, \mathcal{Q}) = \left\| \begin{pmatrix} \frac{27}{2} \\ -1 \\ -\frac{3}{2} \end{pmatrix} - \begin{pmatrix} 14 \\ 0 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{3}{2} \end{pmatrix} \right\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-1\right)^2 + \left(-\frac{3}{2}\right)^2} = \frac{\sqrt{14}}{2}$$

(2) (Q) Show that the following identity is true (one way is to compute both sides and show that they match up):

$$\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) = (\overrightarrow{u} \cdot \overrightarrow{w})\overrightarrow{v} - (\overrightarrow{u} \cdot \overrightarrow{v})\overrightarrow{w}$$

(A) We compute each side separately:

* First:

$$\overrightarrow{v} \times \overrightarrow{w} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$$

and now:

$$\begin{aligned} \overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{pmatrix} \\ &= \begin{pmatrix} u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\ u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1) \\ u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2) \end{pmatrix} \end{aligned}$$

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$$\begin{aligned} (\overrightarrow{u} \cdot \overrightarrow{w})\overrightarrow{v} - (\overrightarrow{u} \cdot \overrightarrow{v})\overrightarrow{w} &= \left[\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right] \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - \left[\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right] \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ &= (u_1w_1 + u_2w_2 + u_3w_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - (u_1v_1 + u_2v_2 + u_3v_3) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ &= \begin{pmatrix} u_1v_1w_1 + u_2v_1w_2 + u_3v_1w_3 \\ u_1v_2w_1 + u_2v_2w_2 + u_3v_2w_3 \\ u_1v_3w_1 + u_2v_3w_2 + u_3v_3w_3 \end{pmatrix} - \begin{pmatrix} u_1v_1w_1 + u_2v_2w_1 + u_3v_3w_1 \\ u_1v_1w_2 + u_2v_2w_2 + u_3v_3w_2 \\ u_1v_1w_3 + u_2v_2w_3 + u_3v_3w_3 \end{pmatrix} \\ &= \begin{pmatrix} u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\ u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1) \\ u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2) \end{pmatrix} \end{aligned}$$