Midterm II Solutions

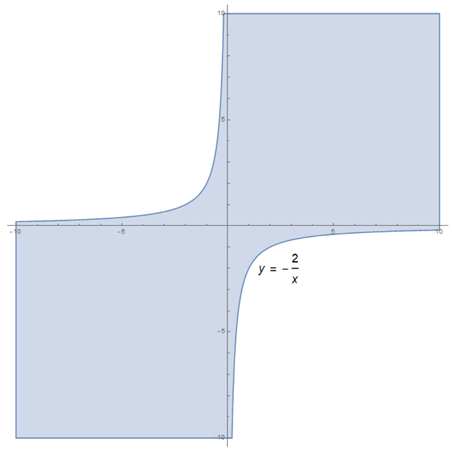
MATH 100

[1] Let R be a relation on the set of all real numbers $\mathbb{R}$ so that $\mathrm{R} \subseteq \mathbb{R} \times \mathbb{R}$. For each of the relation below, draw the picture of R in the plane, and determine if it is reflexive, symmetric, or transitive. Explain your answer.

(1) $\mathrm{R} = \{(x, y) \in \mathbb{R}^2 \mid xy \geq -2\}$
(2) $\mathrm{R} = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\}$
(3) $\mathrm{R} = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$

Proof.

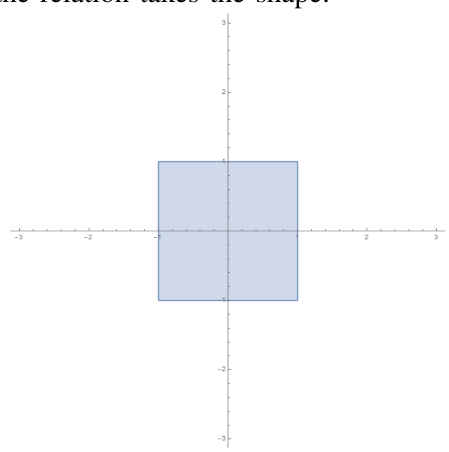
(1) When drawn out as a subset of $\mathbb{R}^2$ the relation takes the shape:



Now to check which properties the relation satisfies:

* For any $x \in \mathbb{R}$ we have $x^2 \geq -2$ simply because x^2 is always non-negative. It follows that ${}_x\mathrm{R}_x$ showing that R is reflexive.
* For any $x, y \in \mathbb{R}$ we have $xy = yx \geq -2$. It follows that ${}_x\mathrm{R}_y \implies {}_y\mathrm{R}_x$ showing that R is symmetric.
* Consider the setup where ${}_{-3}\mathrm{R}_0$ and ${}_0\mathrm{R}_1$. If the relation were transitive, then it would require that ${}_{-3}\mathrm{R}_1$ which is simply not true as $-3 \not\geq -2$. It follows that R is not transitive.

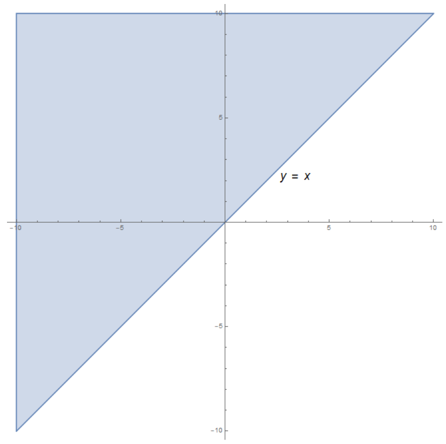
(2) When drawn out as a subset of $\mathbb{R}^2$ the relation takes the shape:



Now to check which properties the relation satisfies:

* Consider the value $x = 3 \in \mathbb{R}$. By the condition on the relation, we see $|3| \not\leq 1$ implying that $(3, 3) \notin \mathrm{R}$. It follows that R is not reflexive.
* For any $x, y \in \mathbb{R}$ we see that if $|x| \leq 1$ and $|y| \leq 1$, then this is equivalent to saying $|y| \leq 1$ and $|x| \leq 1$. It follows that ${}_x\mathrm{R}_y \implies {}_y\mathrm{R}_x$ showing that R is symmetric.
* For any $x, y, z \in \mathbb{R}$ such that ${}_x\mathrm{R}_y$ and ${}_y\mathbb{R}_z$ we have the setup $|x| \leq 1, |y| \leq 1$ and $|y| \leq 1, |z| \leq 1$. It directly follows that $|x| \leq 1, |z| \leq 1$ with the result ${}_x\mathrm{R}_y$ & ${}_y\mathrm{R}_z \implies {}_x\mathrm{R}_z$ showing that R is transitive.

(3) When drawn out as a subset of $\mathbb{R}^2$ the relation takes the shape:



Now to check which properties the relation satisfies:

* For any $x \in \mathbb{R}$ we have $x \leq x$. It follows that ${}_x\mathrm{R}_x$ showing that R is reflexive.
* Consider the setup where ${}_1\mathrm{R}_2$. If the relation were symmetric, then it would require that ${}_2\mathrm{R}_1$ which is simply not true as $2 \not\leq 1$. It follows that R is not symmetric.
* For any $x, y, z \in \mathbb{R}$ we have ${}_x\mathrm{R}_y$ and ${}_y\mathrm{R}_z$ telling us $x \leq y$ and $y \leq z$ respectively. It directly follows that $x \leq z$ and so ${}_x\mathrm{R}_z$ which implies that R is transitive.

■

[2] A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 3$, and

$$a_n = 3a_{n-1} - 2a_{n-2} \quad \text{for} \quad n \geq 3$$

(1) Compute a_3 and a_4 and conjecture a formula for a_n.
(2) Prove the formula for a_n conjectured in (1) by (some version of) mathematical induction, or by other methods.

Proof.

(1) To compute the next two terms we have:

$$\begin{aligned} a_3 &= 3a_2 - 2a_1 = 3(3) - 2(1) = 7 \\ a_4 &= 3a_3 - 2a_2 = 3(7) - 2(3) = 15 \end{aligned}$$

The sequence thus far takes the form $\{1, 3, 7, 15, \ldots\}$. This motivates the conjecture $a_n = 2^n - 1$.

(2) We proceed via proof by induction. For the base consider $n = 1$ which provides $a_1 = 2^1 - 1 = 1$. Now assuming $a_n = 2^n - 1$ we check that it satisfies the recurrence relation for $n + 1$:

$$\begin{aligned} 3a_n - 2a_{n-1} &= 3(2^n - 1) - 2(2^{n-1} - 1) \\ &= 3 \cdot 2^n - 3 - 2^n + 2 \\ &= 2 \cdot 2^n - 1 \\ &= 2^{n+1} - 1 \\ &= a_{n+1} \end{aligned}$$

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[3] Show that for every positive integer n, $49 \mid (8^{n+1} - 7n - 8)$.

Proof. We proceed via proof by induction. For the base case consider $n = 1$ which reduces down to saying $49 \mid 49$, clearly a true statement. Now assuming that the statement holds true for n we can restate it as:

$$\begin{aligned} 8^{n+1} - 7n - 8 &\equiv 0 \pmod{49} \\ 8^{n+1} &\equiv 7n + 8 \pmod{49} \end{aligned}$$

Now to check that it holds true for $n+1$:

$$\begin{aligned}
8^{n+2} - 7(n+1) - 8 &\equiv 8 \cdot 8^{n+1} - 7n - 15 \pmod{49} \\
&\equiv 8(7n+8) - 7n - 15 \pmod{49} \\
&\equiv 56n + 64 - 7n - 15 \mod 49 \\
&\equiv 49(n+1) \pmod{49} \\
&\equiv 0 \pmod{49}
\end{aligned}$$

With $49 \mid (8^{n+2} - 7(n+1) - 8)$ we have completed the inductive step. ■

[4] For every positive integer n, prove the following formula:

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{j} i \right) = \frac{n(n+1)(n+2)}{6}$$

Proof. We approach via direct proof. Recall the following formulas:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \quad \text{and} \quad \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Plugging this in provides:

$$\begin{aligned}
\sum_{j=1}^{n} \left(\sum_{i=1}^{j} i \right) &= \sum_{j=1}^{n} \left(\frac{j(j+1)}{2} \right) = \frac{1}{2} \left(\sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j \right) = \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right) \\
&= \frac{n(n+1)}{4} \left(\frac{2n+1}{3} + 1 \right) = \frac{n(n+1)(n+2)}{6}
\end{aligned}$$

■

[5] Prove or disprove:

(1) There exist integers m and n such that $m^2 + m = n^2$.
(2) There exist positive integers m and n such that $m^2 + m = n^2$.

Proof.

(1) If we take $m = n = 0$, then the equation trivially holds true. Thus, the statement is proven.
(2) Assuming the equation holds true, we can use it to deduce that:

$$\begin{aligned}
m^2 + m &\equiv n^2 \pmod{n} \\
m(m+1) &\equiv 0 \pmod{n}
\end{aligned}$$

The last line only holds true if $m \equiv 0 \pmod{n}$ or $m \equiv n - 1 \pmod{n}$.

* In the first scenario we can deduce $m = kn$ for some $k \in \mathbb{Z}_{>0}$. Plugging this in provides:

$$m^2 + m = k^2 n^2 + kn$$

Equating this to the right-hand side provides $k^2n^2 + kn = n^2$ which reduces down to $n\Big((k^2-1)n + k\Big) = 0$. The only possible solutions occur when $n = 0$ (disregard this possibility) and:

$$n = -\frac{k}{k^2 - 1} = \frac{k}{1 - k^2}$$

For all $k \neq 1$ it follows that the above states $n < 0$ which is to be disregarded. For the exception $k = 1$, from above it reduces down to saying $n = 0$ which is also to be disregarded.

* In the second scenario we can deduce $m = (n-1) + kn$ for some $k \in \mathbb{Z}_{\geq 0}$. Plugging this in provides:

$$\begin{aligned}
m^2 + m &= \Big((k+1)n - 1\Big)^2 + \Big((k+1)n - 1\Big) = (k+1)^2n^2 - 2(k+1)n + 1 - (k+1)n - 1 \\
&= (k+1)^2n^2 - (k+1)n = n(k+1)((k+1)n - 1)
\end{aligned}$$

For the last equality to equate to n^2 we must have $(k+1)((k+1)n-1) = n$ for at least one value of k. Note that the factor of $k+1$ is strictly a natural number, which forces us to only consider $k = 0$:

$$m^2 + m = n(n-1) = n^2 - n$$

For $n^2 - n = n^2$ we conclude that $n = 0$, thereby forcing us to disregard the scenario because m and n have to be positive integers.

Since none of the possible scenarios provide a solution, it follows that there is no solution to $m^2 + m = n^2$ for positive integers m and n.

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