Comments on Taylor Polynomials

I. Motivation

It might be a good idea to provide a reasoning why we bother with Taylor polynomials. Most of mathematics related to the study of functions typically deals with polynomials and their properties. Of these many properties, I would claim that *smoothness* is one of the most important since it tells use that polynomials can be differentiated infinitely many times. So now consider a scenario where we want to take any smooth function and define it as:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

This is known as defining the function as a *power series*. Next up we want to determine the coefficients that will satisfy the above relation in terms of derivatives. If you are willing to believe for now that you can differentiate an infinite sum without any issues, we have:

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

Now observe what happens when we evaluate the original function and first derivative at $x = a$:

$$f(a) = a_0 \quad \text{and} \quad f'(a) = (1!) \cdot a_1$$

This allows us to state:

$$a_0 = f(a) \quad \text{and} \quad a_1 = \frac{f'(a)}{1!}$$

In each evaluation the only term that remains is the first one in the sum and obviously leaving the $1!$ above is pointless, but I do it so that the pattern is apparent. In general we can compute the kth derivative:

$$f^{(k)}(x) = \frac{d^k}{dx^k}\sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} a_n \frac{d^k}{dx^k}(x-a)^n = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \cdot a_n (x-a)^{n-k}$$

Once again evaluation at $x = a$ provides:

$$f^{(k)}(a) = \frac{k!}{0!} \cdot a_k = k! a_k \implies a_k = \frac{f^{(k)}(a)}{k!}$$

This is exactly what we wanted. Now we can write the *Taylor Series* to f centered at $x = a$ as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For the purpose of this course all functions of interest will have a convergent taylor series, thus it makes sense to use the above. In fact, if we want to approximate f at $x = a$ we can define the *kth degree taylor polynomial* as:

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Naturally the only difference between the approximation and infinite series is that the approximation truncates at $n = k$. A special case usually considered is when $a = 0$and this is so much a special case that it even has its own corresponding name. So when $a = 0$ we refer to the Taylor Series as the *Maclaurin Series*.

II. Examples

Example 1 *Find the Maclaurin series for $f(x) = e^x$.*

Solution 1 *First we need all of the derivatives:*

$$e^x = f'(x) = f''(x) = \dots = f^{(n)}(x) = \dots$$

and evaluation at $x = 0$ provides:

$$1 = f(0) = f'(0) = f''(0) = \dots = f^{(n)}(0) = \dots$$

and as a result the Maclaurin series takes the form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Example 2 *Find the 4th degree Taylor polynomial for $g(x) = \sin(x)$ at $a = \frac{\pi}{2}$.*

Solution 2 *First we need all derivatives up til the 4th one:*

$$g'(x) = \cos(x), \quad g''(x) = -\sin(x), \quad g'''(x) = -\cos(x), \quad \textit{and} \quad g^{(4)}(x) = \sin(x)$$

and evaluation at $x = \frac{\pi}{2}$ provides:

$$g\left(\frac{\pi}{2}\right) = 1, \quad g'\left(\frac{\pi}{2}\right) = 0, \quad g''\left(\frac{\pi}{2}\right) = -1, \quad g'''\left(\frac{\pi}{2}\right) = 0, \quad \textit{and} \quad g^{(4)}\left(\frac{\pi}{2}\right) = 1$$

and as a result the 4th degree Taylor polynomial takes the form:

$$\begin{aligned} \mathcal{T}_4(x) &= g\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{g''\left(\frac{\pi}{2}\right)}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{g'''\left(\frac{\pi}{2}\right)}{3!}\left(x - \frac{\pi}{2}\right)^3 + \frac{g^{(4)}\left(\frac{\pi}{2}\right)}{4!}\left(x - \frac{\pi}{2}\right)^4 \\ &= 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} \end{aligned}$$

Example 3 *Find the 5th degree Taylor polynomial for $h(x) = \frac{1}{x+1}$ at $a = 1$.*

Solution 3 *First we need all derivatives up til the 5th one:*

$$h'(x) = -(x+1)^{-2}, \quad h''(x) = 2(x+1)^{-3}, \quad h'''(x) = -6(x+1)^{-4}, \quad h^{(4)}(x) = 24(x+1)^{-5} \textit{and} \quad h^{(5)}(x) = -120(x+1)^{-6}$$

and evaluation at $x = 1$ provides:

$$h(1) = \frac{1}{2}, \quad h'(1) = -\frac{1}{4}, \quad h''(1) = \frac{1}{4}, \quad h'''(1) = -\frac{3}{8}, \quad h^{(4)}(1) = \frac{3}{4}, \quad \textit{and} \quad h^{(5)}(1) = -\frac{15}{8}$$

and as a result the 4th degree Taylor polynomial takes the form:

$$\begin{aligned} \mathcal{T}_5(x) &= h(1) + h'(1)(x-1) + \frac{h''(1)}{2!}(x-1)^2 + \frac{h'''(1)}{3!}(x-1)^3 + \frac{h^{(4)}(1)}{4!}(x-1)^4 + \frac{h^{(5)}(1)}{5!}(x-1)^5 \\ &= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4 - \frac{1}{64}(x-1)^5 \end{aligned}$$