Quiz 8 Solutions

MATH 100
December 3, 2018

(1) (Q) A relation R is defined on $\mathbb{R}$ by ${}_x\mathrm{R}_y$ if $x - y \in \mathbb{Z}$. Prove that R is an equivalence relation and determine the equivalence classes $\left[\frac{1}{2}\right]$ and $[\sqrt{2}]$.

(A) To check that R is an equivalence relation:

* For R to be reflexive means we want ${}_x\mathrm{R}_x$. It turns out this is trivially true as $x - x = 0 \in \mathbb{Z}$.
* For R to be symmetric means we want ${}_x\mathrm{R}_y$ to imply ${}_y\mathrm{R}_x$. By algebraic manipulation we have the following result $x - y = -(y - x)$, which states that if $x - y \in \mathbb{Z}$, then so is $y - x \in \mathbb{Z}$.
* For R to be transitive means we want ${}_x\mathrm{R}_y$ and ${}_y\mathrm{R}_z$ to imply ${}_x\mathrm{R}_z$. Observe that:

$$x - z = (x - y) + (y - z)$$

Since the right-hand side is the addition of two integers, it must be that $x - z \in \mathbb{Z}$.

Now to calculate the equivalence classes we want to take note of the fact that any $x \in \mathbb{R}$ can be decomposed as $x = x_0 + x_1$ where $x_0 \in \mathbb{Z}$ and $x_1 \in (0, 1)$. With this in mind, we can reformulate what it means for two elements to be equivalent. Specifically:

$$x - y = (x_0 + x_1) - (y_0 + y_1) = (x_0 - y_0) + (x_1 - y_1)$$

where $x_0 - y_0 \in \mathbb{Z}$ and $x_1 - y_1 \in (0, 1)$. For $x - y \in \mathbb{Z}$ we want $x_1 - y_1 = 0$. This means that two real numbers are equivalent if they have the same non-integer components.

* For the case of $\left[\frac{1}{2}\right]$ we want all real numbers who take the form _.5 where the underscore can be filled with any integer:

$$\left[\frac{1}{2}\right] = \left\{x \in \mathbb{R} \mid x - \frac{1}{2} \in \mathbb{Z}\right\}$$

* For the case of $[\sqrt{2}]$ we need to first determine the decomposition. It turns out that $1 < \sqrt{2} < 2$, which provides the intuition for $\sqrt{2} = 1 + (\sqrt{2} - 1)$:

$$[\sqrt{2}] = \left\{x \in \mathbb{R} \mid x - (\sqrt{2} - 1) \in \mathbb{Z}\right\}$$

(2) (Q) A relation R is defined on $\mathbb{R}\times\mathbb{R}$ by ${}_{(x_1,y_1)}\mathrm{R}_{(x_2,y_2)}$ if $x_1^2+y_1^2=x_2^2+y_2^2$. Prove that R is an equivalence relation and describe the equivalence classes geometrically.

(A) To check that R is an equivalence relation:

* For R to be reflexive means we want ${}_{(x,y)}\mathrm{R}_{(x,y)}$. It turns out this is trivially true as $x^2+y^2=x^2+y^2$ is a tautology.
* For R to be symmetric means we want ${}_{(x_1,y_1)}\mathrm{R}_{(x_2,y_2)}$ to imply ${}_{(x_2,y_2)}\mathrm{R}_{(x_1,y_1)}$. This is trivially true as writing $x_1^2+y_1^2=x_2^2+y_2^2$ is equivalent to $x_2^2+y_2^2=x_1^2+y_1^2$.
* For R to be reflexive means we want ${}_{(x_1,y_1)}\mathrm{R}_{(x_2,y_2)}$ and ${}_{(x_2,y_2)}\mathrm{R}_{(x_3,y_3)}$ to imply ${}_{(x_1,y_1)}\mathrm{R}_{(x_3,y_3)}$. The given information provides the system of equations:

$$\begin{aligned} x_1^2+y_1^2&=x_2^2+y_2^2\\ x_2^2+y_2^2&=x_3^2+y_3^2 \end{aligned}$$

It directly follows that $x_1^2+y_1^2=x_3^2+y_3^2$.

Now to calculate the equivalence classes we need to think about what it means to say that $x_1^2+y_1^2=x_2^2+y_2^2$. Recall that for any $p=(x,y)\in\mathbb{R}^2$ the Euclidean distance from the origin is calculated via $d^2(p)=x^2+y^2$. Thus, for points $p_1=(x_1,y_1)$ and $p_2=(x_2,y_2)$ we enforce $d^2(p_1)=d^2(p_2)$ which is equivalent to $d(p_1)=d(p_2)$. It follows that a single equivalence class consists of points in the plane who are the same radial distance away from the origin, i.e.:

$$\mathcal{S}_r=\{p\in\mathbb{R}^2\mid d(p)=r\}$$

represents all of the equivalence classes with $r\in[0,\infty)$. Drawing out a couple gives the picture: