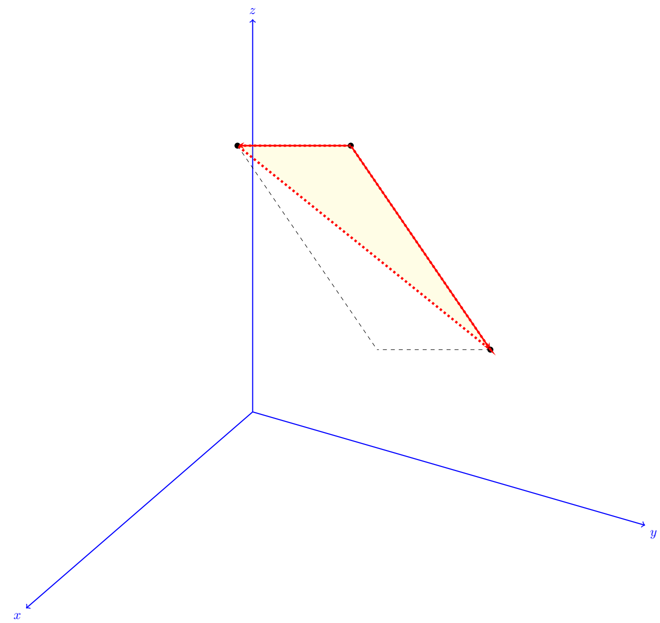
Week 2 Attendance Solutions

MATH 23A

(1) (Q) Let Δ represent the triangle formed by the three vertices $(0,1,3),(2,1,4),(1,3,2) \in \mathbb{R}^3$. Find the area of Δ.

(A) When drawn out, we have the following scenario:



where the two red vectors represent:

$$\vec{u} = \begin{pmatrix} 2-0 \\ 1-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 1-0 \\ 3-1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

The area of the parallelogram formed by the two vectors can be calculated via:

$$\text{Area}(\varparallelogram) = \|\vec{u} \times \vec{v}\|$$

which forces:

$$\text{Area}(\Delta) = \frac{\text{Area}(\varparallelogram)}{2}$$

Now by direct calculation:

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} (0)(-1)-(1)(2) \\ -[(2)(-1)-(1)(1)] \\ (2)(2)-(0)(1) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$$

Thus, the answer is:

$$\text{Area}(\Delta) = \frac{\|\vec{u} \times \vec{v}\|}{2} = \frac{\sqrt{(-2)^2+(3)^2+(4)^2}}{2} = \frac{\sqrt{29}}{2}$$

(2) (Q) Determine whether the following two vectors are orthogonal (perpendicular), parallel, or neither.

$$\overrightarrow{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{v} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

(A) There are actually a couple of approaches to solving this problem:

$*$ We can kill two birds with one stone by directly determining the angle from:

$$\overrightarrow{u} \cdot \overrightarrow{v} = \|\overrightarrow{u}\| \|\overrightarrow{v}\| \cos(\theta)$$

By direct calculation:

$$\cos(\theta) = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{\|\overrightarrow{u}\| \|\overrightarrow{v}\|} = \frac{(1)(-1) + (2)(2) + (3)(-3)}{\sqrt{(1)^2 + (2)^2 + (3)^2}\sqrt{(-1)^2 + (2)^2 + (-3)^2}} = -\frac{6}{14}$$

If the two vectors were perpendicular then $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, however both would force $\cos(\theta) = 0$ which is different from above telling us the vectors are not orthogonal. If the two vectors were parallel then $\theta = 0$ or $\theta = \pi$, however this would force $\cos(\theta) = 1$ or $\cos(\theta) = -1$ respectively. Once again this is different from above telling us the vectors are not parallel. Therefore, the vectors are neither orthogonal nor parallel.

$*$ This approach checks the orthogonal and parallel scenarios separately. If the two vectors were to be orthogonal, then they must satisfy $\overrightarrow{u} \cdot \overrightarrow{v} = 0$. By direct calculation we can see this to not be true:

$$\overrightarrow{u} \cdot \overrightarrow{v} = (1)(-1) + (2)(2) + (3)(-3) = -6$$

Therefore, they are not orthogonal. To see if the vectors are parallel there are two methods:

· If the two vectors were to be parallel then they must point in the same direction implying they should be scalar multiples of each other. Thus, we should check if there exists a $\alpha \in \mathbb{R}$ such that $\overrightarrow{u} = \alpha\overrightarrow{v}$:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\alpha \\ 2\alpha \\ -3\alpha \end{pmatrix}$$

The first and third coordinates force $\alpha = -1$, but the second coordinate says $\alpha = 1$. Since there is no one consistent α, they are not parallel.

· If the two vectors were to be parallel then they must satisfy $\overrightarrow{u} \times \overrightarrow{v} = \overrightarrow{0}$. By direct calculation:

$$\overrightarrow{u} \times \overrightarrow{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 2 & -3 \end{pmatrix} = \begin{pmatrix} (2)(-3) - (3)(2) \\ -[(1)(-3) - (3)(-1)] \\ (1)(2) - (2)(-1) \end{pmatrix} = \begin{pmatrix} -12 \\ 0 \\ 4 \end{pmatrix}$$

Since this is not the zero vectors it must be that the vectors are not parallel.

Therefore, the vectors are neither orthogonal nor parallel.