Quiz 2 Solutions

MATH 103A
August 2, 2018

(1) (Q) Determine whether the following functions are entire (holomorphic on all finite points of $\mathbb{C}$):

a) $f(x,y) = y^2 - ix^2$

b) $f(x,y) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

(A)

(a) For the Cauchy-Riemann equations we identify $u(x,y) = y^2$ and $v(x,y) = -x^2$ to see that:

$$\begin{aligned} u_x &= 0 = 0 = v_y \\ u_y &= 2y \neq 2x = -v_x \end{aligned}$$

Since the equations are not satisfied for all of $\mathbb{C}$ it follows that $f(x,y)$ is not an entire function.

(b) For the Cauchy-Riemann equations we identify $u(x,y) = \sin(x)\cosh(y)$ and $v(x,y) = \cos(x)\sinh(y)$ to see that:

$$\begin{aligned} u_x &= \cos(x)\cosh(y) = v_y \\ u_y &= \sin(x)\sinh(y) = -v_x \end{aligned}$$

Since the equations are satisfied for all of $\mathbb{C}$ with continuous partial derivatives it follows that $f(x,y)$ is an entire function.

(2) (Q) Does the function:

$$f(z) = \begin{cases} \frac{\overline{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

satisfy the Cauchy-Riemann equations at $z = 0$? Is $f'(0)$ well-defined?

(A) It helps to rewrite:

$$\frac{\overline{z}^2}{z} = \frac{\overline{z}^3}{\|z\|^2} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3 - 3xy^2}{x^2+y^2} + \frac{y^3 - 3x^2y}{x^2+y^2}i = u(x,y) + iv(x,y)$$

For the Cauchy-Riemann equations we need the following calculations:

$$\begin{aligned} u_x(0) &= \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \\ u_y(0) &= \lim_{h\to 0} \frac{u(0,0+h) - u(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \\ v_x(0) &= \lim_{h\to 0} \frac{v(0+h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \\ v_y(0) &= \lim_{h\to 0} \frac{v(0,0+h) - v(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \end{aligned}$$

It follows that the Cauchy-Riemann equations are satisfied at $z = 0$. Now for the derivative behavior, we calculate directly:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{\overline{h}^2}{h}}{h} = \left(\lim_{h\to 0} \frac{h}{\overline{h}} \right)^{-2} = \text{DNE}$$