Week 10 Attendance Solutions

MATH 23A

(1) (Q) Given the real-valued smooth function:

$$f : \mathbb{R}^3 \to \mathbb{R} \quad \text{given by} \quad f(x, y, z) = c$$

and smooth vector field:

$$V : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{given by} \quad V(x, y, z) = \begin{pmatrix} V_1(x, y, z) \\ V_2(x, y, z) \\ V_3(x, y, z) \end{pmatrix}$$

prove the following identities by direct calculation:

$$\nabla \times (\nabla f) = \overrightarrow{0} \quad \text{and} \quad \nabla \cdot (\nabla \times V) = 0$$

(A) By direct calculation we have:

* For the first identity:

$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

and as a result:

$$\nabla \times \nabla f = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{pmatrix} = \begin{pmatrix} f_{zy} - f_{yz} \\ -(f_{zx} - f_{xz}) \\ f_{yx} - f_{xy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where the last deduction follows from the fact that second partial derivatives commute.

* For the second identity we first have:

$$\nabla \times V = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \\ \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \\ \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{pmatrix}$$

and as a result:

$$\begin{aligned}
\nabla \cdot (\nabla \times V) &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \\ \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \\ \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{pmatrix} \\
&= \left(\frac{\partial V_3}{\partial x \partial y} - \frac{\partial V_2}{\partial x \partial z} \right) + \left(\frac{\partial V_1}{\partial y \partial z} - \frac{\partial V_3}{\partial y \partial x} \right) + \left(\frac{\partial V_2}{\partial z \partial x} - \frac{\partial V_1}{\partial z \partial y} \right) \\
&= \left(\frac{\partial V_3}{\partial x \partial y} - \frac{\partial V_3}{\partial y \partial x} \right) + \left(\frac{\partial V_2}{\partial z \partial x} - \frac{\partial V_2}{\partial x \partial z} \right) + \left(\frac{\partial V_1}{\partial y \partial z} - \frac{\partial V_1}{\partial z \partial y} \right) \\
&= 0
\end{aligned}$$

where the last deduction follows from the fact that second partial derivatives commute.

(2) (Q) Consider the vector field that corresponds to traveling along the latitude lines of the sphere:

$$\mathcal{V} : \mathbb{S}^2 \to \mathbb{R}^3 \quad \text{given by} \quad \mathcal{V}(x, y, z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

Show that the following path:

$$\overrightarrow{c}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}$$

is a flow of the vector field $\mathcal{V}$ (*Hint: Check that* $\overrightarrow{c}'(t) = \mathcal{V}(\overrightarrow{c}(t))$).

(A) By direct calculation:

$$\overrightarrow{c}'(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix}$$

and in comparison:

$$\mathcal{V}(\overrightarrow{c}(t)) = \begin{pmatrix} -(\sin(t)) \\ \cos(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix}$$