Calculus 1 with Precalculus
Recitation 9

Nathan Marianovsky

EUCLIDEAN DISTANCE

Definition 1 (Distance Formula)**.** When given any pair of points (x_0, y_0), $(x_1, y_1) \in \mathbb{R}^2$, the distance between them is defined as:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

where d must be positive, and only zero when the two points are the same point.

Example 1. *Determine the distance between* $(3, -1)$ *and* $(7, 1)$.

Solution 1.

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \sqrt{(7-3)^2 + (1+1)^2} = \sqrt{20} = \boxed{2\sqrt{5}}$$

Example 2. *Determine the distance between* $(7, -3)$ *and* $(5, 3)$.

Solution 2.

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \sqrt{(7-5)^2 + (-3-3)^2} = \sqrt{40} = \boxed{2\sqrt{10}}$$

Example 3. *Determine the distance between* $(3, 0)$ *and* $(0, -2)$.

Solution 3.

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \sqrt{(3-0)^2 + (0+2)^2} = \boxed{\sqrt{13}}$$

MORE ON FUNCTIONS

Definition 2 (Classes of Common Functions)**.**

(a) A power function is a function in the form:

$$f(x) = x^n \quad \text{where} \quad n \in \mathbb{R}$$

(b) A polynomial is a function in the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \quad \text{where} \quad a_i \in \mathbb{R}\, \forall i \in \{0, 1, 2, 3, \ldots, n\} \quad \text{and} \quad n \in \mathbb{N}_0$$

If $a_n \neq 0$, then n is called the **degree** of the polynomial.

(c) A rational function is a function in the form:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are both polynomials.

Definition 3 (Vertical Line Test)**.** A function is, by definition, a mapping of an input to some output. A single input cannot map to two or more different output. On the other hand, different inputs that produce the same output are fine. To apply this, given the graph of some curve, the curve is a function iff by drawing a vertical line in any arbitrary position, the line will never intersect multiple points.

Definition 4 (Intercepts)**.** Given a function $y = f(x)$,

(a) the y-intercept is defined to be the point $(0, f(0))$.

(b) the x-intercept(s) is defined to be the point(s), $x_1, \ldots, x_n$, where $f(x_i) = 0\, \forall i \in \{1, 2, 3, \ldots, n\}$.

Example 4. *Determine the intercept(s) of* $f(x) = x^2 + 2x - 8$.

Solution 4.

(a) *x-intercepts:*

$$\boxed{\begin{aligned} &0 = x^2 + 2x - 8 \\ &0 = (x+4)(x-2) \\ &x = -4, 2 \end{aligned}}$$

(b) *y-intercept:*

$$\boxed{f(0) = (0)^2 + 2(0) - 8 = -8}$$

Example 5. *Determine the intercept(s) of* $f(x) = \sqrt{1-x}$.

Solution 5.

(a) *x-intercept:*

$$\boxed{\begin{aligned} &0 = \sqrt{1-x} \\ &0 = 1 - x \\ &x = 1 \end{aligned}}$$

(b) *y-intercept:*

$$\boxed{f(0) = (0)^2 + 2(0) - 8 = -8}$$

Example 6. *Determine the intercept(s) of*

$$f(x) = \begin{cases} 2x - 1 & x \leq 2 \\ 3 & x > 2 \end{cases}$$

Solution 6.

(a) *Only the left part of the function has an x-intercept:*

$$\boxed{\begin{aligned} &0 = 2x - 1 \\ &x = \frac{1}{2} \end{aligned}}$$

(b) *y-intercept:*

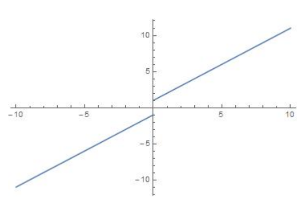
$$\boxed{f(0) = 2(0) - 1 = -1}$$

Example 7. *Determine the intercept(s) of*

$$f(x) = \begin{cases} x - 1 & x \leq 0 \\ x + 1 & x > 0 \end{cases}$$

Solution 7.

(a) *Drawing out the graph*



shows that the piecewise function has a gap and never formally crosses the x-axis, thus there is $\boxed{\text{no } x - \text{intercept}}$.

(b) *y-intercept:*

$$\boxed{f(0) = (0) - 1 = -1}$$

Example 8. *Draw the graphs and find the points of intersection (if any) of* $f(x) = 3x+5$ *and* $g(x) = -x+3$.

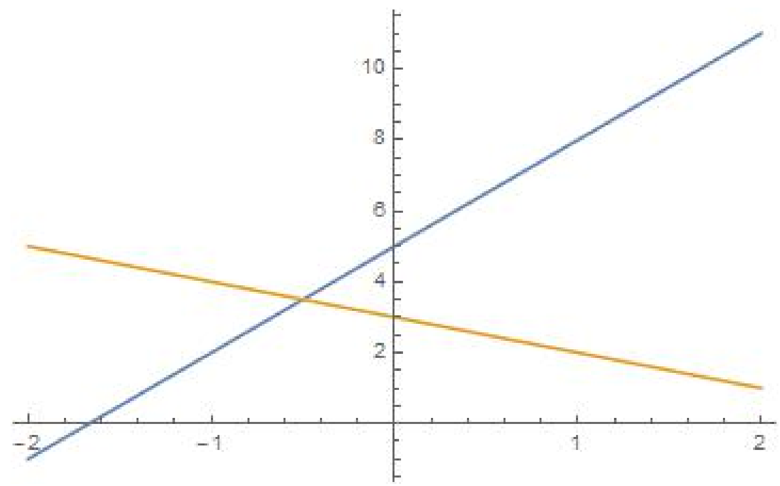
Solution 8. *For the points of intersection:*

$$\begin{aligned} f(x) &= g(x) \\ 3x+5 &= -x+3 \\ 4x &= -2 \\ x &= -\frac{1}{2} \end{aligned}$$

The corresponding output value is:

$$g(-.5) = -\Big(-\frac{1}{2}\Big) + 3 = \frac{7}{2}$$

The graph of the curves given is:



Example 9. *Draw the graphs and find the points of intersection (if any) of* $f(x) = x^2 - x$ *and* $g(x) = x - 1$.

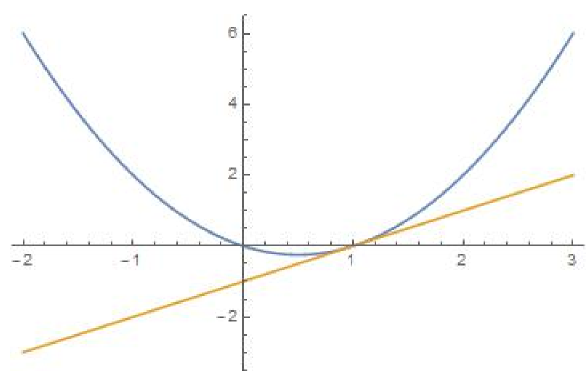
Solution 9. *For the points of intersection:*

$$\begin{aligned} f(x) &= g(x) \\ x^2 - x &= x - 1 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \\ x &= 1 \end{aligned}$$

The corresponding output value is:

$$g(1) = (1) - 1 = 0$$

The graph of the curves given is:



Example 10. *Draw the graphs and find the points of intersection (if any) of* $f(x) = x^3 + 1$ *and* $g(x) = x + 1$.

Solution 10. *For the points of intersection:*

$$\begin{aligned} f(x) &= g(x) \\ x^3 + 1 &= x + 1 \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x-1)(x+1) &= 0 \\ x &= -1, 0, 1 \end{aligned}$$

The corresponding output values are:

$$\begin{aligned} g(-1) &= (-1) + 1 = 0 \\ g(0) &= (0) + 1 = 1 \\ g(1) &= (1) + 1 = 2 \end{aligned}$$

The graph of the curves given is: