# Week 8 Attendance Solutions

MATH 23A

(1) (Q) Assuming $|x| < 1$ and $|y| < 1$ write down the Taylor series centered at $(0,0)$ for the function:

$$f(x,y) = \frac{1}{1-x-y+xy}$$

(*Hint: Think about the Geometric series.*)

(A) First notice the following:

$$f(x,y) = \frac{1}{1-x-y-xy} = \frac{1}{(1-x)(1-y)} = \frac{1}{1-x}\frac{1}{1-y}$$

In this scenario our function is decomposed as $f(x,y) = g(x)h(y)$ where:

$$g(x) = \frac{1}{1-x} \quad \text{and} \quad h(y) = \frac{1}{1-y}$$

The Taylor expansions of $g$ and $h$ at $x = 0$ and $y = 0$ respectively take the form:

$$g(x) = \sum_{n_1=0}^{\infty} x^{n_1} \quad \text{and} \quad h(y) = \sum_{n_2=0}^{\infty} y^{n_2}$$

providing us with the Taylor expansion of $f$:

$$f(x,y) = g(x)h(y) = \sum_{n_1=0}^{\infty} x^{n_1} \cdot \sum_{n_2=0}^{\infty} y^{n_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} x^{n_1} y^{n_2}$$

(2) (Q) Find the absolute maximum and minimum of the function:

$$g(x,y) = x^2 - y^2$$

on the domain:

$$\mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

(*Hint: Find all normal critical points that belong to the interior first. Then parametrize the boundary, restrict $g$ to the boundary, and find all critical points on the boundary. Finally compute the value of $g$ at all these critical points and determine the smallest and biggest values.*)

(A) For the normal critical points we want to satisfy:

$$\nabla g = \begin{pmatrix} 2x \\ -2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \overrightarrow{0}$$

Clearly the only solution occurs when $x = y = 0$. Now we restrict our attention to the boundary which can be parametrized via $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t < 2\pi$. The restriction of the function to the boundary takes the form:

$$\tilde{g}(t) = g(\cos(t), \sin(t)) = \cos^2(t) - \sin^2(t) = \cos(2t)$$

To find the critical points of $\tilde{g}$ we must satisfy:

$$\tilde{g}'(t) = -2\sin(2t) = 0$$

This occurs when $2t = k\pi$ for $k \in \mathbb{Z}$. This is equivalent to $t = \frac{\pi}{2}k$. On our domain the acceptable values are $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Among the points $(0,0), (1,0), (0,1), (-1,0), (0,-1)$ we have $g$ obtaining $1$ and $-1$ as the absolute maximum and minimum on $\mathbb{D}$.