Week 4 Attendance Solutions

MATH 23A

(1) (Q) Prove the following limit exists (*Hint: Convert the limit into a one-dimensional limit using polar coordinates*):

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

(A) As recommended we approach evaluation of the limit by converting to polar coordinates:

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0} r^2\ln(r^2) = \lim_{r\to 0} 2r^2\ln(r) = 2\lim_{r\to 0}\frac{\ln(r)}{\frac{1}{r^2}}$$

The right hand side results in an indeterminate form. Thus, by L'Hôpital's rule:

$$2\lim_{r\to 0}\frac{\ln(r)}{\frac{1}{r^2}} = 2\lim_{r\to 0}\frac{\frac{1}{r}}{-\frac{2}{r^3}} = -\lim_{r\to 0} r^2 = 0$$

(2) (Q) In the theory of quantum mechanics, a non-relativistic particle's information is encoded into what is known as the wave function. We want to consider the scenario of the "Particle in a Box" where the time-dependent Schrödinger's equation reads off as:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

where $i=\sqrt{-1}$ is the imaginary number and $\hbar, m$ are positive constants. Show that the following wave function satisfies the above equation by calculating the partial derivatives and plugging it in:

$$\psi(x,t) = A\sin(k(x+1))e^{-\frac{iEt}{\hbar}}$$

where k, A are constants. The identity $E=\frac{\hbar^2k^2}{2m}$ might come in handy.

(A) The necessary partial derivatives are:

$$\begin{aligned}
\frac{\partial}{\partial t}\psi(x,t) &= \frac{\partial}{\partial t}\Big(A\sin(k(x+1))e^{-\frac{iEt}{\hbar}}\Big) = -\frac{iAE}{\hbar}\sin(k(x+1))e^{-\frac{iEt}{\hbar}}\\
\frac{\partial}{\partial x}\psi(x,t) &= \frac{\partial}{\partial x}\Big(A\sin(k(x+1))e^{-\frac{iEt}{\hbar}}\Big) = Ak\cos(k(x+1))e^{-\frac{iEt}{\hbar}}\\
\frac{\partial^2}{\partial x^2}\psi(x,t) &= \frac{\partial}{\partial x}\Big(Ak\cos(k(x+1))e^{-\frac{iEt}{\hbar}}\Big) = -Ak^2\sin(k(x+1))e^{-\frac{iEt}{\hbar}}
\end{aligned}$$

Plugging this into both left and right hand sides shows that this wave function satisfies the given equation:

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\psi(x,t) &= i\hbar\Big(-\frac{iAE}{\hbar}\sin(k(x+1))e^{-\frac{iEt}{\hbar}}\Big) = AE\sin(k(x+1))e^{-\frac{iEt}{\hbar}}\\
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) &= -\frac{\hbar^2}{2m}\Big(-Ak^2\sin(k(x+1))e^{-\frac{iEt}{\hbar}}\Big) = AE\sin(k(x+1))e^{-\frac{iEt}{\hbar}}
\end{aligned}$$