Calculus 2
Recitation 2

Nathan Marianovsky

THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

Definition 1 (Derivative of an Integral Valued Function)**.** Given a function defined as:

$$k(x) = \int_{h(x)}^{g(x)} f(t)dt$$

the derivative of $k(x)$ can be calculated as:

$$
\begin{aligned}
k'(x) &= \frac{d}{dx} \int_{h(x)}^{g(x)} f(t)dt \\
&= \frac{d}{dx} \left[F(t) \Big|_{h(x)}^{g(x)} \right] \qquad \text{where } F(t) \text{ is the antiderivative of } f(t) \\
&= \frac{d}{dx} \left[F(g(x)) - F(h(x)) \right] \\
&= f(g(x))g'(x) - f(h(x))h'(x) \quad \text{where differentiation follows from the chain rule}
\end{aligned}
$$

Example 1. *Find the derivative of $r(y)$ where:*

$$r(y) = \int_{3}^{y} t^2 \sin(3t)dt$$

Solution 1. *To find the derivative, identify the components according to the formula:*

$$
\begin{aligned}
g(y) &= y \\
h(y) &= 3 \\
f(t) &= t^2 \sin(3t)
\end{aligned}
$$

Therefore, the derivative of $r(y)$ is:

$$r'(y) = f(g(y))g'(y) - f(h(y))h'(y) = (y^2 \sin(3y))(1) - (9\sin(9))(0) = \boxed{y^2 \sin(3y)}$$

Example 2. *Find the derivative of $w(x)$ where:*

$$w(x) = \int_{7}^{\frac{1}{x}} \arctan(6t)dt$$

Solution 2. *Identify the components according to the formula:*

$$
\begin{aligned}
g(x) &= \frac{1}{x} \\
h(x) &= 7 \\
f(t) &= \arctan(6t)
\end{aligned}
$$

Therefore, the derivative of $w(x)$ is:

$$w'(x) = f(g(x))g'(x) - f(h(x))h'(x) = \left(\arctan\left(\frac{6}{x}\right) \right)\left(-\frac{1}{x^2} \right) - (\arctan(42))(0) = \boxed{-\frac{\arctan\left(\frac{6}{x}\right)}{x^2}}$$

Example 3. *Find the derivative of $p(x)$ where:*

$$p(x) = \int_{5x}^{7x} \frac{u^2 - 4}{u^2 + 4} du$$

Solution 3. *Identify the components according to the formula:*

$$g(x) = 7x$$
$$h(x) = 5x$$
$$f(u) = \frac{u^2 - 4}{u^2 + 4}$$

Therefore, the derivative of $p(x)$ is:

$$p'(x) = f(g(x))g'(x) - f(h(x))h'(x) = \left(\frac{49x^2 - 4}{49x^2 + 4}\right)(7) - \left(\frac{25x^2 - 4}{25x^2 + 4}\right)(5) = \boxed{\frac{343x^2 - 28}{49x^2 + 4} - \frac{125x^2 - 20}{25x^2 + 4}}$$

Example 4. *Find the interval on which the curve $y(x)$ is concave upward where:*

$$y(x) = \int_0^x \frac{dt}{2 + t + 2t^2}$$

Solution 4. *To measure concavity the second derivative is needed. To find the first derivative consider the components according to the formula:*

$$g(x) = x$$
$$h(x) = 0$$
$$f(u) = \frac{1}{2 + t + 2t^2}$$

Therefore, the first derivative is:

$$y'(x) = f(g(x))g'(x) - f(h(x))h'(x) = \left(\frac{1}{2 + x + 2x^2}\right)(1) - \left(\frac{1}{2}\right)(0) = \frac{1}{2 + x + 2x^2}$$

By quotient rule, the second derivative is:

$$y''(x) = -\frac{1 + 4x}{(2 + x + 2x^2)^2}$$

Now to find the points of inflection:

$$0 = y''(x)$$
$$0 = -\frac{1 + 4x}{(2 + x + 2x^2)^2}$$
$$0 = 1 + 4x$$
$$x = -\frac{1}{4}$$

All that is left to check is the behavior to the left and right of the inflection point. Anything smaller when plugged into the second derivative will give a positive value, and anything bigger will give a negative value. Therefore the curve $y(x)$ is concave upward on the interval:

$$\boxed{\left(-\infty, -\frac{1}{4}\right)}$$

PROPERTIES OF INTEGRALS

Definition 2 (Properties)**.** All integrals will satisfy the following properties:

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c \int_a^b f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

INTEGRALS OF EVEN AND ODD FUNCTIONS

Definition 3 (Even and Odd Properties)**.** Taking the integral of an even function, $f(x) = f(-x)$, over a symmetrical region can be simplified as:

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$$

Similarly given an odd function, $f(x) = -f(-x)$, the integral will obey:

$$\int_{-a}^a f(x)dx = 0$$

AVERAGE VALUE OF A FUNCTION

Definition 4 (Average Value)**.** The average value of a function, $f(x)$, on an interval, $[a, b]$, is defined as:

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)dx$$

INTEGRATION BY SUBSTITUTION

Definition 5 (The Substitution Rule)**.** Given an integral in the form:

$$\int f(g(x))g'(x)dx$$

a substitution can be made:

$$u = g(x)$$
$$du = g'(x)dx$$

such that the integral can be rewritten as:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 5. *Evaluate* $\int \frac{x}{\sqrt{1-4x^2}}dx$

Solution 5. *Take notice that the top resembles the derivative of the inside of the bottom. Therefore let:*

$$u = 1 - 4x^2$$

$$du = -8xdx \rightarrow -\frac{du}{8} = xdx$$

Plugging this into the integral gives:

$$\int \frac{x}{\sqrt{1-4x^2}}dx = -\frac{1}{8}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{8}\int u^{-\frac{1}{2}}du$$

$$= -\frac{1}{4}u^{\frac{1}{2}} + C$$

$$= \boxed{-\frac{1}{4}\sqrt{1-4x^2} + C}$$

Example 6. *Evaluate* $\int \cos(3z)\sin^{10}(3z)dz$

Solution 6. *Make the following substitution:*

$$u = \sin(3z)$$

$$du = 3\cos(3z)dz \rightarrow \frac{du}{3} = \cos(3z)dz$$

Reevaluating the bounds using the substitution and plugging into the integral gives:

$$\int_0^{\frac{\pi}{2}} \cos(3z)\sin^{10}(3z)dz = \frac{1}{3}\int_0^{-1} u^{10}du$$

$$= -\frac{1}{3}\int_{-1}^0 u^{10}du$$

$$= -\frac{1}{33}u^{11}\Big|_{-1}^0$$

$$= \boxed{-\frac{1}{33}}$$

Example 7. *Evaluate* $\int_1^5 \left(1 - \frac{1}{w}\right)\cos(w - \ln(w))dw$

Solution 7. *Make the following substitution:*

$$u = w - \ln(w)$$

$$du = \left(1 - \frac{1}{w}\right)dw$$

Reevaluating the bounds using the substitution and plugging into the integral gives:

$$\int_1^5 \left(1 - \frac{1}{w}\right)\cos(w - \ln(w))dw = \int_1^{5-\ln(5)} \cos(u)du$$

$$= \sin(u)\Big|_1^{5-\ln(5)}$$

$$= \boxed{\sin(5 - \ln(5)) - \sin(1)}$$

Example 8. *Evaluate* $\int \sec^2(4t)(3 - \tan(4t))^3 dt$

Solution 8. *Make the following substitution:*

$$u = 3 - \tan(4t)$$

$$du = -4\sec^2(4t)dt \rightarrow -\frac{du}{4} = \sec^2(4t)dt$$

Plugging this into the integral gives:

$$\int \sec^2(4t)(3 - \tan(4t))^3 dt = -\frac{1}{4}\int u^3 du$$

$$= -\frac{1}{16}u^4 + C$$

$$= \boxed{-\frac{(3 - \tan(4t))^4}{16} + C}$$

Example 9. *Evaluate* $\int_0^1 3(8y - 1)e^{4y^2 - y}dy$

Solution 9. *Make the following substitution:*

$$u = 4y^2 - y$$
$$du = (8y - 1)dy$$

Reevaluating the bounds using the substitution and plugging into the integral gives:

$$\int_0^1 3(8y - 1)e^{4y^2 - y}dy = 3\int_0^1 (8y - 1)e^{4y^2 - y}dy$$

$$= 3\int_0^3 e^u du$$

$$= 3e^u \Big|_0^3$$

$$= \boxed{3e^3 - 3}$$