# Calculus 1 with Precalculus
# Recitation 3

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## SYNTHETIC AND LONG DIVISION

**Definition 1** (Synthetic Division Setup)**.** Given any rational function in the form:

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{x-b}$$

To divide using synthetic division, on the left side write down $b$. On the right side place the constants in order from $a_n$ to $a_0$. To begin bring down the first term. Multiply the brought down term by $b$ and add it to the second term and bring down the result. Keep repeating this process until no terms are left to bring down. To read off the solution, the first term represents the coefficient of $x^{n-1}$, the second of $x^{n-2}$, and so on until the second last term. The last term represents the remainder. In the case where this factor divides the numerator perfectly, the remainder will be zero.

**Example 1.** *Perform long division or synthetic division to simplify:*

$$\frac{x^3 - 12x^2 - 42}{x-3}$$

**Solution 1.** *Using long division:*

$$\begin{array}{rl}
 & \phantom{-}x^2 \quad - 9x \quad - 27 \\ \cline{2-2}
x-3) & x^3 - 12x^2 \qquad\qquad - 42 \\
 & -x^3 + 3x^2 \\ \cline{2-2}
 & -9x^2 \\
 & 9x^2 - 27x \\ \cline{2-2}
 & -27x \quad - 42 \\
 & 27x \quad - 81 \\ \cline{2-2}
 & -123
\end{array}$$

*the result is:*

$$\frac{x^3 - 12x^2 - 42}{x-3} = \boxed{x^2 - 9x - 27 - \frac{123}{x-3}}$$

*Now using synthetic division place the denominator root outside on the left. Then place the variable constants inside and follow the procedure:*

$$\begin{array}{c|cccc}
 & 1 & -12 & 0 & -42 \\
3 & & 3 & -27 & -81 \\ \hline
 & 1 & -9 & -27 & -123
\end{array}$$

*Reading off the results gives:*

$$\frac{x^3 - 12x^2 - 42}{x-3} = \boxed{x^2 - 9x - 27 - \frac{123}{x-3}}$$

**Example 2.** *Perform long division or synthetic division to simplify:*

$$\frac{x^5 + x^3 - 8x^2 - 3x + 2}{x - 1}$$

**Solution 2.** *Using long division:*

$$
\begin{array}{rllllll}
 & x^4 & +x^3 & +2x^2 & -6x & -9 & \\
\cline{2-7}
x-1 \big) & x^5 & & +x^3 & -8x^2 & -3x & +22 \\
 & -x^5 & +x^4 & & & & \\
\cline{2-3}
 & & x^4 & +x^3 & & & \\
 & & -x^4 & +x^3 & & & \\
\cline{3-4}
 & & & 2x^3 & -8x^2 & & \\
 & & & -2x^3 & +2x^2 & & \\
\cline{4-5}
 & & & & -6x^2 & -3x & \\
 & & & & 6x^2 & -6x & \\
\cline{5-6}
 & & & & & -9x & +22 \\
 & & & & & 9x & -9 \\
\cline{6-7}
 & & & & & & 13
\end{array}
$$

*the result is:*

$$\frac{x^5 + x^3 - 8x^2 - 3x + 2}{x - 1} = \boxed{x^4 + x^3 + 2x^2 - 6x - 9 + \frac{13}{x - 1}}$$

*Now using synthetic division place the denominator root outside on the left. Then place the variable constants inside and follow the procedure:*

$$
\begin{array}{r|rrrrrr}
 & 1 & 0 & 1 & -8 & -3 & 22 \\
1 & & 1 & 1 & 2 & -6 & -9 \\
\hline
 & 1 & 1 & 2 & -6 & -9 & 13
\end{array}
$$

*Reading off the results gives:*

$$\frac{x^5 + x^3 - 8x^2 - 3x + 2}{x - 1} = \boxed{x^4 + x^3 + 2x^2 - 6x - 9 + \frac{13}{x - 1}}$$

**Example 3.** *Solve for $x$ in $x^3 - x^2 - x + 1 = 0$.*

**Solution 3.** *When solving polynomials of degree greater than two, as human beings, there is only one thing we can try. If the polynomial has integer coefficients, it may happen to be that one of the factors of the constant may be a solution. In this case, the factors of $1$ are $-1$ and $1$. Using $x = 1$ satisfies the given equation, showing is a root. Now that one of the roots has been found, by factoring:*

$$x^3 - x^2 - x + 1 = (x - 1) \cdot \frac{x^3 - x^2 - x + 1}{x - 1}$$

*Now using synthetic division:*

$$
\begin{array}{r|rrrr}
 & 1 & -1 & -1 & 1 \\
1 & & 1 & 0 & -1 \\
\hline
 & 1 & 0 & -1 & 0
\end{array}
$$

*and reading off the result is:*

$$x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1) = (x - 1)^2(x + 1)$$

*With the polynomial fully factored, the roots are clearly* $\boxed{x = -1, 1}$.

**Example 4.** *Solve for $x$ in $x^3 + x + 2 = 0$.*

**Solution 4.** *Like before, if the polynomial has integer coefficients, one of the factors of the constant may be a solution. In this case the factors of $2$ are $-1$, $1$, $-2$, and $2$. So if $x = -1$:*

$$x^3 + x + 2 = (-1)^3 + (-1) + 2 = 0$$

*Now that one of the roots has been found, by factoring:*

$$x^3 + x + 2 = (x + 1) \cdot \frac{x^3 + x + 2}{x + 1}$$

*Now using synthetic division:*

$$\begin{array}{r|rrrr} & 1 & 0 & 1 & 2 \\ -1 & & -1 & 1 & -2 \\ \hline & 1 & -1 & 2 & 0 \end{array}$$

*and reading off the result is:*

$$x^3 + x + 2 = (x + 1)(x^2 - x + 2)$$

*With the polynomial factored enough, the first root is clearly $\boxed{x = -1}$. For the remaining use the quadratic formula:*

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{1}{2} \pm \frac{\sqrt{-7}}{2} = \boxed{\frac{1}{2}\left(1 \pm \sqrt{7}i\right)}$$

**Example 5.** *Solve for $x$ in $5x^3 + 4x^2 - 8x = -8$.*

**Solution 5.** *Like before, if the polynomial has integer coefficients, one of the factors of the constant may be a solution. In this case the factors of $8$ are $-1$, $1$, $-2$, $2$, $-4$, $4$, $-8$, and $8$. So if $x = -2$:*

$$5x^3 + 4x^2 - 8x + 8 = 5(-8) + 4(4) + 8(2) + 8 = 0$$

*Now that one of the roots has been found, by factoring:*

$$5x^3 + 4x^2 - 8x + 8 = (x + 2) \cdot \frac{5x^3 + 4x^2 - 8x + 8}{x + 2}$$

*Now using synthetic division:*

$$\begin{array}{r|rrrr} & 5 & 4 & -8 & 8 \\ -2 & & -10 & 12 & -8 \\ \hline & 5 & -6 & 4 & 0 \end{array}$$

*and reading off the result is:*

$$5x^3 + 4x^2 - 8x + 8 = (x + 2)(5x^2 - 6x + 4)$$

*With the polynomial factored enough, the first root is clearly $\boxed{x = -2}$. For the remaining use the quadratic formula:*

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{3}{5} \pm \frac{\sqrt{-44}}{10} = \boxed{\frac{3}{5} \pm \frac{\sqrt{11}}{5}i}$$

# Summations

**Definition 2.** Given any sequence of $n$ numbers $\{a_1, a_2, a_3, \dots, a_n\}$, the total sum is denoted as:

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

**Definition 3** (Properties)**.** Sums are known as linear operators, thus they obey the following rules:

(a) Given $k \in \mathbb{R}$:

$$\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$$

(b)

$$\sum_{i=1}^{n} \Big(a_i + b_i\Big) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

**Definition 4** (Common Sums)**.** These few sums may come in handy:

(a) Given $k \in \mathbb{R}$:

$$\sum_{i=1}^{n} k = kn$$

(b)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

**Example 6.** *Rewrite the following in summation notation and determine the sum:*

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64$$

**Solution 6.**

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = \sum_{i=1}^{8} i^2 = \frac{8(8+1)(2(8)+1)}{6} = \boxed{204}$$

**Example 7.** *Rewrite the following in summation notation and determine the sum:*

$$3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30$$

**Solution 7.**

$$3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30 = \sum_{i=1}^{10} 3i = 3\sum_{i=1}^{10} i = 3 \cdot \frac{10(10+1)}{2} = \boxed{165}$$

**Example 8.** *Determine the sum of:*

$$\sum_{i=1}^{6} \frac{1}{i}$$

**Solution 8.**

$$\sum_{i=1}^{6} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \boxed{\frac{49}{20}}$$

**Example 9.** *Determine the sum of:*

$$\sum_{j=1}^{4}(3j+1)$$

**Solution 9.**

$$\sum_{j=1}^{4}(3j+1) = 3\sum_{j=1}^{4} j + \sum_{j=1}^{4} 1 = 3\cdot\frac{4(4+1)}{2} + 4 = \boxed{34}$$

**Example 10.** *Determine the sum of:*

$$\sum_{j=1}^{7}(j^2+2)$$

**Solution 10.**

$$\sum_{j=1}^{7}(j^2+2) = \sum_{j=1}^{7} j^2 + \sum_{j=1}^{7} 2 = \frac{7(7+1)(2(7)+1)}{6} + 14 = \boxed{154}$$

**Example 11.** *Determine the sum of:*

$$\sum_{j=1}^{n}(a_j - a_{j-1})$$

**Solution 11.**

$$\begin{aligned}\sum_{j=1}^{n}(a_j - a_{j-1}) &= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) \\ &= (\cancel{a_1} - a_0) + (\cancel{a_2} - \cancel{a_1}) + (\cancel{a_3} - \cancel{a_2}) + \cdots + (\cancel{a_{n-1}} - \cancel{a_{n-2}}) + (a_n - \cancel{a_{n-1}}) \\ &= \boxed{a_n - a_0}\end{aligned}$$

*This is known as a telescoping series.*

**Example 12.** *Determine the sum of:*

$$\sum_{j=1}^{7}(3^j - 3^{j-1})$$

**Solution 12.** *Identify this as a telescoping series and evaluate:*

$$\sum_{j=1}^{7}(3^j - 3^{j-1}) = 3^7 - 3^0 = \boxed{2186}$$

**Example 13.** *Determine the sum of:*

$$\sum_{j=1}^{9} 2(\ln(j+1) - \ln(j))$$

**Solution 13.** *Identify this as a telescoping series and evaluate:*

$$\sum_{j=1}^{9} 2(\ln(j+1) - \ln(j)) = 2\Big(\ln(10) - \ln(1)\Big) = \boxed{2\ln(10)}$$

**Example 14.** *Determine the sum of:*

$$\sum_{j=1}^{15}(-1)^j$$

**Solution 14.**

$$\sum_{j=1}^{15}(-1)^j = \cancel{(-1)} + \cancel{(1)} + \cancel{(-1)} + \cancel{(1)} + \cdots + \cancel{(-1)} + \cancel{(1)} + (-1) = \boxed{-1}$$

**Example 15.** *Determine the sum of:*

$$\sum_{j=1}^{k}(-1)^j$$

**Solution 15.** *Using the previous example shows that whenever $k$ is odd, the sum is $-1$. On the other hand if the previous example only had 14 entries, then everything would have canceled out, thus:*

$$\boxed{\sum_{j=1}^{k}(-1)^j = \begin{cases} -1 & \exists\ n\ \textit{s.t.}\ k = 2n+1 \\ 0 & \exists\ n\ \textit{s.t.}\ k = 2n \end{cases}}$$