Sample Final Solutions

MATH 103A
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[1] Calculate:

(1)

$$\frac{(4+i)-(2-i)}{(1+i)^2} = \frac{2+2i}{(1+i)^2}\cdot\frac{(1-i)^2}{(1-i)^2} = \frac{2(1+i)(1-i)^2}{2^2} = \frac{2^2(1-i)}{2^2} = 1-i$$

(2)

$$\left(\frac{\sqrt{3}+i}{2-2i}\right)^{12} = \left(\frac{2e^{\frac{i\pi}{6}}}{2\sqrt{2}e^{-\frac{i\pi}{4}}}\right)^{12} = \left(\frac{e^{\frac{5i\pi}{12}}}{\sqrt{2}}\right)^{12} = \frac{e^{5i\pi}}{2^6} = -\frac{1}{64}$$

(3)

$$\left(\frac{2i}{1+i}\right)^{\frac{1}{5}} = \left(\frac{2e^{\frac{i\pi}{2}}}{\sqrt{2}e^{\frac{i\pi}{4}}}\right)^{\frac{1}{5}} = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{\frac{1}{5}} = 2^{\frac{1}{10}}e^{\frac{i\pi}{20}} = 2^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{20}\right)+i\sin\left(\frac{\pi}{20}\right)\right)$$

[2] (a) Compute the derivative of:

$$f'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\frac{(z+2)^3}{(z^2+iz+4)^2} = \frac{3(z+2)^2(z^2+iz+4)^2 - 2(z+2)^3(z^2+iz+4)(2z+i)}{(z^2+iz+4)^4}$$

(b) Check if the function:

$$f(z) = (x^3-3xy^2+y)+i(3x^2y-y^3-x)$$

satisfies the Cauchy-Riemann equations:

$$\begin{aligned}\frac{\partial f}{\partial x} &= (3x^2-3y^2)+i(6xy-1)\\ -i\frac{\partial f}{\partial y} &= -i\Big((-6xy+1)+i(3x^2-3y^2)\Big)\\ &= (3x^2-3y^2)+i(6xy-1)\end{aligned}$$

and find $f'(z)$:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = (3x^2-3y^2)+i(6xy-1)$$

(c) Check if:

$$u(x,y) = xy - x + y$$

is harmonic:

$$u_{xx}+u_{yy} = \frac{\partial^2}{\partial x^2}(xy-x+y) + \frac{\partial^2}{\partial y^2}(xy-x+y) = \frac{\partial}{\partial x}(y-1) + \frac{\partial}{\partial y}(x+1) = 0+0 = 0$$

and find its harmonic conjugate if it is harmonic. By use of the Cauchy-Riemann equations:

$$v(x,y) = \int -\frac{\partial u}{\partial y}\,\mathrm{d}x = \int -(x+1)\,\mathrm{d}x = -\frac{1}{2}x^2 - x + g(y)$$

and in order to calculate $g(y)$:

$$v_y = g'(y) = y - 1 = u_x$$

Thus, we are forced to have $g(y) = \frac{1}{2}y^2 - y + C$ showing that a complex conjugate will take the form:

$$v(x,y) = \frac{1}{2}(y^2-x^2) - (y+x) + C$$

[3] (a) Find the partial fraction decomposition of:

$$
\begin{aligned}
R(z) &= \frac{z}{(z^2+z+1)^2} = \frac{z}{\left(\frac{z^3-1}{z-1}\right)^2} = \frac{z}{\left(\frac{(z-e^{\frac{i\pi}{3}*0})(z-e^{\frac{i\pi}{3}*2})(z-e^{\frac{i\pi}{3}*4}))}{z-1}\right)^2} \\
&= \frac{z}{(z-e^{\frac{2i\pi}{3}})^2(z-e^{\frac{4i\pi}{3}})^2} \\
&= \frac{z}{\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^2\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^2} \\
&= \frac{A_0^1}{\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^2} + \frac{A_1^1}{z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} + \frac{A_0^2}{\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^2} + \frac{A_1^2}{z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}
\end{aligned}
$$

where the coefficients are given by:

$$
\begin{aligned}
A_0^1 &= \frac{1}{0!}\lim_{z\to -\frac{1}{2}+i\frac{\sqrt{3}}{2}} \frac{\mathrm{d}^0}{\mathrm{d}z^0}\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^2 R(z) \\
&= \lim_{z\to -\frac{1}{2}+i\frac{\sqrt{3}}{2}} \frac{z}{\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^2} \\
&= \frac{-\frac{1}{2}+i\frac{\sqrt{3}}{2}}{(i\sqrt{3})^2} = \frac{1}{6} - i\frac{1}{2\sqrt{3}} \\
A_0^2 &= \frac{1}{0!}\lim_{z\to -\frac{1}{2}-i\frac{\sqrt{3}}{2}} \frac{\mathrm{d}^0}{\mathrm{d}z^0}\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^2 R(z) \\
&= \lim_{z\to -\frac{1}{2}-i\frac{\sqrt{3}}{2}} \frac{z}{\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^2} \\
&= \frac{-\frac{1}{2}-i\frac{\sqrt{3}}{2}}{(-i\sqrt{3})^2} = \frac{1}{6} - i\frac{1}{2\sqrt{3}} \\
A_1^1 &= \frac{1}{1!}\lim_{z\to -\frac{1}{2}+i\frac{\sqrt{3}}{2}} \frac{\mathrm{d}^1}{\mathrm{d}z^1}\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^2 R(z) \\
&= \lim_{z\to -\frac{1}{2}+i\frac{\sqrt{3}}{2}} \frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^2} \\
&= \lim_{z\to -\frac{1}{2}+i\frac{\sqrt{3}}{2}} \frac{\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^2 - 2z\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)}{\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^4} \\
&= \frac{(i\sqrt{3})^2 - 2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(i\sqrt{3})}{(i\sqrt{3})^4} = \frac{1}{2^{\frac{3}{2}}} \\
A_1^2 &= \frac{1}{1!}\lim_{z\to -\frac{1}{2}-i\frac{\sqrt{3}}{2}} \frac{\mathrm{d}^1}{\mathrm{d}z^1}\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)^2 R(z) \\
&= \lim_{z\to -\frac{1}{2}-i\frac{\sqrt{3}}{2}} \frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^2} \\
&= \lim_{z\to -\frac{1}{2}-i\frac{\sqrt{3}}{2}} \frac{\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^2 - 2z\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)}{\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^4} \\
&= \frac{(-i\sqrt{3})^2 - 2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)(-i\sqrt{3})}{(-i\sqrt{3})^4} = -\frac{1}{2^{\frac{3}{2}}}
\end{aligned}
$$

(b) Find the principal value of [1]

$$
\begin{aligned}
\text{p.v.}(\text{Log}(1-i)) &= e^{\text{Log}(\text{Log}(1-i))} = e^{\text{Log}(\ln(|1-i|)+i\text{Arg}(1-i))} \\
&= e^{\text{Log}\left(\frac{1}{2}\ln(2)-\frac{i\pi}{4}\right)} \\
&= e^{\ln\left|\frac{1}{2}\ln(2)-\frac{i\pi}{4}\right|+i\text{Arg}\left(\frac{1}{2}\ln(2)-\frac{i\pi}{4}\right)} \\
&= e^{\ln\left(\sqrt{\frac{1}{4}\ln^2(2)+\frac{\pi^2}{16}}\right)+i\arctan\left(-\frac{\pi}{2\ln(2)}\right)} \\
&= \sqrt{\frac{1}{4}\ln^2(2)+\frac{\pi^2}{16}}\left[\cos\left(\arctan\left(-\frac{\pi}{2\ln(2)}\right)\right)+i\sin\left(\arctan\left(-\frac{\pi}{2\ln(2)}\right)\right)\right] \\
&= \sqrt{\frac{1}{4}\ln^2(2)+\frac{\pi^2}{16}}\left[\cos\left(\arctan\left(\frac{\pi}{2\ln(2)}\right)\right)-i\sin\left(\arctan\left(\frac{\pi}{2\ln(2)}\right)\right)\right] \\
&= \frac{\ln(2)}{2}\sqrt{1+\frac{\pi^2}{4\ln^2(2)}}\left[\frac{1}{\sqrt{1+\frac{\pi^2}{4\ln^2(2)}}}-i\frac{\pi}{2\ln(2)\sqrt{1+\frac{\pi^2}{4\ln^2(2)}}}\right] \\
&= \frac{\ln(2)}{2}-i\frac{\pi}{4}
\end{aligned}
$$

and [2]:

$$
\text{p.v.}(i^i) = e^{\text{Log}(i^i)} = e^{\text{Log}\left((e^{\frac{i\pi}{2}})^i\right)} = e^{\text{Log}(e^{-\frac{\pi}{2}})} = e^{-\frac{\pi}{2}}
$$

(c) Find all the values of $\arcsin(2)$:

$$
\begin{aligned}
2 &= \sin(z) \\
2 &= \frac{e^{iz}-e^{-iz}}{2i} \\
4i &= e^{iz}-e^{-iz} \\
4ie^{iz} &= (e^{iz})^2-1 \\
0 &= (e^{iz})^2-4ie^{iz}-1 \\
e^{iz} &= \frac{4i\pm\sqrt{-16+4}}{2} \\
e^{iz} &= 2i\pm\sqrt{3}i \\
e^{iz} &= (2\pm\sqrt{3})i \\
iz &= \log((2+\sqrt{3})i), \log((2-\sqrt{3})i) \\
iz &= \ln(2+\sqrt{3})+i\left(\frac{\pi}{2}+2\pi n\right), \ln(2-\sqrt{3})+i\left(\frac{\pi}{2}+2\pi n\right) \\
z &= \left(\frac{\pi}{2}+2\pi n\right)-i\ln(2+\sqrt{3}), \left(\frac{\pi}{2}+2\pi n\right)-i\ln(2-\sqrt{3})
\end{aligned}
$$

[4] (a) Let Γ be the circle given by $|z-1|=3$ with positive orientation going around once. Evaluate:

$$
\oint_\Gamma \frac{\mathrm{d}z}{z^3}
$$

As $f(z)=\frac{1}{z^3}$ is analytic on Γ and $\text{Int}(\Gamma)$ it follows that the integral evaluates to 0 by the Cauchy-Goursat Theorem.

[1] The following approach is how the principal value of *any* complex number can be determined, namely $\text{p.v.}(z) = e^{\text{Log}(z)}$. However, a much shorter way to compute the value is to note that $\text{p.v.}(\text{Log}(z)) = \text{Log}(z)$, namely the principal value of the logarithm is the value of the logarithm itself (not surprisingly because the values already lie in the principal branch).

[2] To evaluate the cosine or sine of an arctangent draw a right triangle and label the sides accordingly by letting one of the angles satisfy $\tan(\theta) = \frac{\pi}{2\ln(2)}$.

(b) Let Γ be the circle given by $|z-1|=2$ with positive orientation going around once. Use the Cauchy Integral Formula to evaluate:

$$\oint_\Gamma \frac{ze^z}{(z+5)(z-1)^3}\,\mathrm{d}z$$

Recall that the Cauchy Integral Formula states:

$$f^{(n)}(z)=\frac{n!}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta$$

Recognizing that $z=1$ is the only singularity in the interior of Γ and matching the formula above to the given information we deduce that:

$$\oint_\Gamma \frac{ze^z}{(z+5)(z-1)^3}\,\mathrm{d}z=\oint_\Gamma \frac{\frac{ze^z}{z+5}}{(z-1)^3}\,\mathrm{d}z=\frac{2\pi i}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{ze^z}{z+5}\Big|_{z=1}=\pi i\frac{e^z(z^3+10z^2+35z+40)}{(z+5)^3}\Big|_{z=1}=\frac{43e\pi i}{108}$$

(c) Let Γ be the circle given by $|z|=3$ with positive orientation going around once. Use the Residue Theorem to evaluate:

$$\oint_\Gamma\left(z^3e^{\frac{1}{z}}+\frac{z^2+z+1}{(z-1)^2(z-2)}\right)\mathrm{d}z=\oint_\Gamma z^3e^{\frac{1}{z}}\,\mathrm{d}z+\oint_\Gamma \frac{z^2+z+1}{(z-1)^2(z-2)}\,\mathrm{d}z$$

In order to evaluate the first integral we use a series expansion:

$$\oint_\Gamma z^3e^{\frac{1}{z}}\,\mathrm{d}z=\oint_\Gamma z^3\left(\sum_{n=0}^{\infty}\frac{1}{n!z^n}\right)\mathrm{d}z=\sum_{n=0}^{\infty}\frac{1}{n!}\oint_\Gamma\frac{\mathrm{d}z}{z^{n-3}}=\frac{2\pi i}{4!}=\frac{\pi i}{12}$$

For the second integral we recognize that the integrand has a simple pole at $z=2$ and a pole of order 2 at $z=1$. Thus, we compute the residues to be:

$$\begin{aligned}
\mathrm{Res}_{z=2}\frac{z^2+z+1}{(z-1)^2(z-2)}&=\lim_{z\to2}(z-2)\cdot\frac{z^2+z+1}{(z-1)^2(z-2)}=\lim_{z\to2}\frac{z^2+z+1}{(z-1)^2}=7\\
\mathrm{Res}_{z=1}\frac{z^2+z+1}{(z-1)^2(z-2)}&=\frac{1}{1!}\lim_{z\to1}\frac{\mathrm{d}^1}{\mathrm{d}z^1}\left((z-1)^2\frac{z^2+z+1}{(z-1)^2(z-2)}\right)\\
&=\lim_{z\to1}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2+z+1}{z-2}\\
&=\lim_{z\to1}\frac{(2z+1)(z-2)-(z^2+z+1)}{(z-2)^2}\\
&=-6
\end{aligned}$$

allowing us to compute the second integral as:

$$\oint_\Gamma\frac{z^2+z+1}{(z-1)^2(z-2)}\,\mathrm{d}z=2\pi i\left(\mathrm{Res}_{z=1}\frac{z^2+z+1}{(z-1)^2(z-2)}+\mathrm{Res}_{z=2}\frac{z^2+z+1}{(z-1)^2(z-2)}\right)=2\pi i(-6+7)=2\pi i$$

Putting everything together provides:

$$\oint_\Gamma\left(z^3e^{\frac{1}{z}}+\frac{z^2+z+1}{(z-1)^2(z-2)}\right)\mathrm{d}z=\oint_\Gamma z^3e^{\frac{1}{z}}\,\mathrm{d}z+\oint_\Gamma\frac{z^2+z+1}{(z-1)^2(z-2)}\,\mathrm{d}z=\frac{\pi i}{12}+2\pi i=\frac{25\pi i}{12}$$

[5] (a) Find the Laurent series expansion centered at $z=0$ of:

$$f(z)=\frac{1}{(z-1)(z-3)}$$

in the annulus region given by $1<|z|<3$. We begin by computing the partial fraction decomposition to be:

$$\frac{1}{(z-1)(z-3)}=\frac{\frac{1}{2}}{z-3}-\frac{\frac{1}{2}}{z-1}$$

and are interested in the expansion:

$$\frac{1}{(z-1)(z-3)}=\frac{\frac{1}{2}}{1-z}-\frac{\frac{1}{2}}{3-z}=-\frac{1}{2z}\frac{1}{1-\frac{1}{z}}-\frac{1}{6}\frac{1}{1-\frac{z}{3}}=-\frac{1}{2z}\sum_{n=0}^{\infty}\frac{1}{z^n}-\frac{1}{6}\sum_{n=0}^{\infty}\frac{z^n}{3^n}=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}}+\frac{z^n}{3^{n+1}}\right)$$

(b) Determine the annulus of convergence of the Laurent series:

$$\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}}$$

To accomplish the task we split up the given series into two parts:

$$\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}} = \sum_{j=-\infty}^{-1} \frac{z^j}{2^{-j}} + \sum_{j=0}^{\infty} \frac{z^j}{2^j} = \sum_{j=1}^{\infty} \frac{z^{-j}}{2^j} + \sum_{j=0}^{\infty} \frac{z^j}{2^j} = \sum_{j=1}^{\infty} \frac{1}{2^j z^j} + \sum_{j=0}^{\infty} \frac{z^j}{2^j}$$

Now to determine the region of convergence for the first component we use the Ratio Test:

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \left| \frac{2^j z^j}{2^{j+1} z^{j+1}} \right| = \frac{1}{2|z|} \lim_{j\to\infty} 1 = \frac{1}{2|z|}$$

For convergence we require $\frac{1}{2|z|} < 1$, or equivalently $\frac{1}{2} < |z|$. Similarly, for the second component we use the Ratio Test:

$$\lim_{j\to\infty} \left| \frac{b_{j+1}}{b_j} \right| = \lim_{j\to\infty} \left| \frac{z^{j+1}}{2^{j+1}} \cdot \frac{2^j}{z^j} \right| = \frac{|z|}{2} \lim_{j\to\infty} 1 = \frac{|z|}{2}$$

For convergence we require $\frac{|z|}{2} < 1$, or equivalently $|z| < 2$. The only region in which both components of the original Laurent series converge is given by $\frac{1}{2} < |z| < 2$.

(c) Classify the zeros and singularities of the function $\sin\left(1 - \frac{1}{z}\right)$.

∗ In order to classify the zeroes we are interested in solving:

$$\begin{aligned}
0 &= \sin\left(1 - \frac{1}{z}\right) \\
0 &= \sin\left(\frac{z-1}{z}\right) \\
0 &= \frac{e^{i\frac{z-1}{z}} - e^{-i\frac{z-1}{z}}}{2i} \\
0 &= e^{i\frac{z-1}{z}} - e^{-i\frac{z-1}{z}} \\
0 &= \left(e^{i\frac{z-1}{z}}\right)^2 - 1 \\
e^{2\pi i k} &= e^{2i\frac{z-1}{z}} \\
\pi k &= \frac{z-1}{z} \\
\pi k z &= z - 1 \\
1 &= (1 - \pi k) z \\
z &= \frac{1}{1 - \pi k}
\end{aligned}$$

where $k \in \mathbb{Z}$. To determine the order of each zero we would ideally need to write out a Laurent series centered at each zero, but this is far too tedious. Instead we observe the nature of the zeroth and first degree terms. The constant term will obviously vanish as the given function vanishes at the zero, however the coefficient tied to the first degree term will not vanish as:

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin\left(\frac{z-1}{z}\right)\Bigg|_{z=\frac{1}{1-\pi k}} = \cos\left(\frac{z-1}{z}\right) \cdot \frac{1}{z^2}\Bigg|_{z=\frac{1}{1-\pi k}} = \cos(\pi k) \cdot (1-\pi k)^2 = (-1)^k (1-\pi k)^2$$

Thus, such an expansion would be able to only factor out a single copy of $\left(z - \frac{1}{1-\pi k}\right)$. Therefore, the order of each zero is one.

* In order to classify the singularities we first note that $z = 0$ is the only singularity and to determine the nature of $z = 0$ we would ideally need to write out a Laurent expansion, but this is far too tedious. We can instead observe that with $z_k = \frac{1}{1-\pi k}$ we have $z_k \to 0$ as $k \to \infty$ while $f(z_k) = 0$. As this sequence causes the function to vanish we instead consider:

$$u_k = \frac{1}{1 - 2\pi k - \frac{\pi}{2}}$$

which still satisfies $u_k \to 0$ when $k \to \infty$, but now:

$$f(u_k) = \sin\left(1 - \frac{1}{\frac{1}{1-2\pi k-\frac{\pi}{2}}}\right) = \sin\left(2\pi k + \frac{\pi}{2}\right) = 1$$

In other words, $z = 0$ has to be an essential singularity.

[6] (a) In order to find a branch of $\log(z)$ that is analytic at $z = -i$ observe that the principal branch will suffice as this branch is analytic everywhere but the negative real axis. Thus, choosing $f(z) = \text{Log}(z)$ we find that $f'(z) = \frac{1}{z}$.

(b) * For any contour Γ connecting $-i$ to i in the right-half plane we can deform it into the portion of the unit circle that is in the right-half plane, denoted by Γ'. It directly follows that:

$$\int_\Gamma \frac{\mathrm{d}z}{z} = \int_{\Gamma'} \frac{\mathrm{d}z}{z} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ie^{i\theta}}{e^{i\theta}}\,\mathrm{d}\theta = i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta = \pi i$$

* The approach above requires the fact that the integrand is analytic in the right-half plane, but for exactly the same reasoning we can just use the Fundamental Theorem of Calculus:

$$\int_\Gamma \frac{\mathrm{d}z}{z} = \text{Log}(i) - \text{Log}(-i) = \frac{\pi i}{2} - \left(-\frac{\pi i}{2}\right) = \pi i$$

(c) We are given $p(z) = a_0 + a_1 z + a_2 z^2$ with $M = \max_{z\in\mathbb{S}^1} |p(z)|$ and by using the Cauchy Estimate on the unit circle we have:

$$|p^{(n)}(0)| \leq Mn!$$

Thus, plugging in $n = 0, 1, 2$ we arrive correspondingly at $|a_0| \leq M$, $|a_1| \leq M$, and $|a_2| \leq M$.

(d) In order to view the order of the zero at ∞ for $f(z)$ we instead observe the behavior of the singularity at $z = 0$ for:

$$g(z) = f\left(\frac{1}{z}\right) = \frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)}$$

Now taking $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ and $q(z) = q_0 + q_1 z + \cdots + q_m z^m$ where $p_n, q_m \neq 0$ and $n < m$, the above becomes:

$$g(z) = \frac{p_0 + \frac{p_1}{z} + \cdots + \frac{p_n}{z^n}}{q_0 + \frac{q_1}{z} + \cdots + \frac{q_m}{z^m}} = \frac{p_0 z^m + p_1 z^{m-1} + \cdots + p_n z^{m-n}}{q_0 z^m + q_1 z^{m-1} + \cdots + q_m}$$

It directly follows that $g(z)$ has no singularity at $z = 0$ as $q_m \neq 0$ by assumption, directly showing that $f(z)$ has no zero at $z = \infty$.

(e) Let Γ be any loop encircling the origin with positive orientation going around once. We aim to utilize the standard Laurent series expansions in order to evaluate:

$$\begin{aligned}
\oint_\Gamma e^{\frac{1}{z}} \sin\left(\frac{1}{z}\right) \mathrm{d}z &= \oint_\Gamma \left(\sum_{n=0}^{\infty} \frac{1}{n! z^n}\right)\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)! z^{2k+1}}\right) \mathrm{d}z \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{n!(2k+1)!} \oint_\Gamma \frac{\mathrm{d}z}{z^{n+2k+1}} \\
&= \frac{1}{0!(2(0)+1)!} \cdot 2\pi i \\
&= 2\pi i
\end{aligned}$$