Quiz 1 Solutions

MATH 103A
July 31, 2018

(1) (Q) Compute $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$ for the following complex numbers:

a) $z = \frac{2+3i}{4-2i}$ b) $z = (1-i)^4$

c) $z = (1-i)^k$ d) $z = i^i$

(A)

(a) Multiply by the conjugate of the denominator to reconfigure the complex number:

$$z = \frac{2+3i}{4-2i} = \frac{2+3i}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{(2+3i)(4+2i)}{\|4+2i\|^2} = \frac{2+16i}{20} = \frac{1}{10} + \frac{4}{5}i$$

It follows that $\mathfrak{Re}(z) = \frac{1}{10}$ and $\mathfrak{Im}(z) = \frac{4}{5}$.

(b) The Binomial Theorem makes this complex number easy to expand:

$$z = (1-i)^4 = \sum_{k=0}^{4} \binom{4}{k} (-1)^k i^k = 1 - 4i - 6 + 4i + 1 = -4$$

It follows that $\mathfrak{Re}(z) = -4$ and $\mathfrak{Im}(z) = 0$.

(c) Using rectangular coordinates for an arbitrary k is not very efficient. Instead consider using Euler's formula to substitute $1 - i = e^{-\frac{\pi}{4}i}$:

$$z = (1-i)^k = \left(e^{-\frac{\pi}{4}i}\right)^k = e^{-\frac{k\pi}{4}i} = \cos\left(-\frac{k\pi}{4}\right) + i\sin\left(-\frac{k\pi}{4}\right) = \cos\left(\frac{k\pi}{4}\right) - i\sin\left(\frac{k\pi}{4}\right)$$

It follows that $\mathfrak{Re}(z) = \cos\left(\frac{k\pi}{4}\right)$ and $\mathfrak{Im}(z) = -\sin\left(\frac{k\pi}{4}\right)$.

(d) Using Euler's formula $i = e^{\frac{\pi}{2}i}$:

$$z = i^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{-\frac{\pi}{2}}$$

It follows that $\mathfrak{Re}(z) = e^{-\frac{\pi}{2}}$ and $\mathfrak{Im}(z) = 0$.

(2) (Q)

(a) For any $\alpha \in \mathbb{C}$ solve the equation $z^n = \alpha$. How many solutions are there?

(b) List out all of the solutions in rectangular coordinates if $n = 3$ and $\alpha = 1$.

(A)

(a) Using Euler's formula $\alpha = re^{i\theta}$:

$$\begin{aligned} z^n &= \alpha \\ z^n &= re^{i(\theta + 2\pi m)} \\ z &= r^{\frac{1}{n}} e^{\frac{\theta+2\pi m}{n} i} \\ z &= r^{\frac{1}{n}} \cos\left(\frac{\theta + 2\pi m}{n}\right) + ir^{\frac{1}{n}} \sin\left(\frac{\theta + 2\pi m}{n}\right) \end{aligned}$$

From here it is obvious that by allowing $m = 0, 1, 2, \ldots, n-1$ we achieve all unique values of z. The count shows that there are n solutions.

(b) For the case of where $\alpha = 1$ we refer to the solutions of the equation as the nth roots of unity. Using Euler's formula we identify $r = 1$ and $\theta = 0$ to obtain:

$$\begin{aligned}
z^3 &= 1 \\
z^3 &= e^{2\pi i m} \\
z &= e^{\frac{2\pi m}{3} i} \\
z &= \begin{cases} 1 & m = 0 \\ e^{\frac{2\pi}{3} i} & m = 1 \\ e^{\frac{4\pi}{3} i} & m = 2 \end{cases} \\
z &= \begin{cases} 1 & m = 0 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i & m = 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i & m = 2 \end{cases}
\end{aligned}$$