Quiz 5 Solutions

Section B

(1) (a) For any function that satisfies $f(x) \geq 0$ on $[a, b]$ then we must have $\int_a^b f(x)\, \mathrm{d}x \geq 0$. For the functions given in the problem equality to zero only occurs at $x = 0$, while the rest is strictly positive. Thus, the integrals will be strictly positive.

(b) Begin by expanding the numerator:

$$x^4(1-x)^4 = x^4 \cdot \sum_{i=0}^{4} \binom{4}{i}(-x)^i = x^4\left[\binom{4}{0} - \binom{4}{1}x + \binom{4}{2}x^2 - \binom{4}{3}x^3 + \binom{4}{4}x^4\right] = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$$

Now perform long division:

$$\begin{array}{r|rrrrrrr}
 & & & x^6 & -\,4x^5 & +\,5x^4 & -\,4x^2 & +\,4 \\ \hline
x^2+1 & x^8 & -\,4x^7 & +\,6x^6 & -\,4x^5 & +\,x^4 & & \\
 & -\,x^8 & & -\,x^6 & & & & \\ \cline{2-4}
 & & -\,4x^7 & +\,5x^6 & -\,4x^5 & & & \\
 & & 4x^7 & & +\,4x^5 & & & \\ \cline{3-5}
 & & & 5x^6 & & +\,x^4 & & \\
 & & & -\,5x^6 & & -\,5x^4 & & \\ \cline{4-6}
 & & & & & -\,4x^4 & & \\
 & & & & & 4x^4 & +\,4x^2 & \\ \cline{6-7}
 & & & & & & 4x^2 & \\
 & & & & & & -\,4x^2 & -\,4 \\ \cline{7-8}
 & & & & & & & -\,4
\end{array}$$

Using this the integration evaluates to:

$$\begin{aligned}
\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, \mathrm{d}x &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right) \mathrm{d}x \\
&= \frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\arctan(x)\Big|_0^1 \\
&= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\arctan(1) \\
&= \frac{22}{7} - \pi
\end{aligned}$$

(c) With the results of parts (a) and (b) we can say that $\frac{22}{7} - \pi > 0$ which implies $\frac{22}{7} > \pi$.

(2) Identify the trigonometric substitution:

$$x = b\tan(\theta) \quad \text{and} \quad \mathrm{d}x = b\sec^2(\theta)\, \mathrm{d}\theta$$

and plug in:

$$\int_{-a}^{\mathcal{L}-a} \frac{b\lambda(x)}{4\pi\epsilon_0(x^2+b^2)^{\frac{3}{2}}}\, \mathrm{d}x = \frac{b\lambda}{4\pi\epsilon_0} \int \frac{b\sec^2(\theta)}{(b^2(\tan^2(\theta)+1))^{\frac{3}{2}}}\, \mathrm{d}\theta = \frac{\lambda}{4\pi\epsilon_0 b} \int \frac{1}{\sec(\theta)}\, \mathrm{d}\theta$$

$$= \frac{\lambda}{4\pi\epsilon_0 b} \int \cos(\theta)\, \mathrm{d}\theta = \frac{\lambda}{4\pi\epsilon_0 b}\sin(\theta) = \frac{\lambda}{4\pi\epsilon_0 b}\frac{x}{\sqrt{x^2+b^2}}\Big|_{-a}^{\mathcal{L}-a} = \frac{\lambda}{4\pi\epsilon_0 b}\left[\frac{\mathcal{L}-a}{\sqrt{(\mathcal{L}-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right]$$

Section C

(1) Exactly the same as Section B.

(2) Identify the trigonometric substitution:

$$x = b\tan(\theta) \quad \text{and} \quad \mathrm{d}x = b\sec^2(\theta)\,\mathrm{d}\theta$$

and plug in:

$$\begin{aligned}
\int_{-a}^{\mathcal{L}-a} \frac{x\lambda(x)}{4\pi\epsilon_0(x^2+b^2)^{\frac{3}{2}}}\,\mathrm{d}x &= \frac{\lambda}{4\pi\epsilon_0}\int \frac{b^2\tan(\theta)\sec^2(\theta)}{(b^2(\tan^2(\theta)+1))^{\frac{3}{2}}}\,\mathrm{d}\theta \\
&= \frac{\lambda}{4\pi\epsilon_0 b}\int \frac{\tan(\theta)}{\sec(\theta)}\,\mathrm{d}\theta \\
&= \frac{\lambda}{4\pi\epsilon_0 b}\int \sin(\theta)\,\mathrm{d}\theta \\
&= -\frac{\lambda}{4\pi\epsilon_0 b}\cos(\theta) \\
&= -\frac{\lambda}{4\pi\epsilon_0 b}\frac{b}{\sqrt{x^2+b^2}}\bigg|_{-a}^{\mathcal{L}-a} \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{\sqrt{(\mathcal{L}-a)^2+b^2}}\right]
\end{aligned}$$

Note that this integral could have also been approached by a normal substitution (probably less work too!)