Fibonacci Comments

Let $\{a_n\}_{n=1}^{\infty}$ denote the Fibonacci sequence satisfying $a_n = a_{n-1} + a_{n-2}$ with the initial conditions $a_1 = a_2 = 1$.

Lemma 1. Let φ denote the golden ratio, i.e. $\varphi = \frac{1+\sqrt{5}}{2}$, satisfying $\varphi^2 = \varphi + 1$. The Fibonacci sequence will satisfy the following identity:

$$\varphi^n = a_n\varphi + a_{n-1}$$

Proof. We proceed via proof by induction:

- For the base case consider $n = 2$ giving:

$$\begin{aligned} \varphi^2 &= a_2\varphi + a_1 \\ \varphi^2 &= \varphi + 1 \end{aligned}$$

- Now we take the given identity as the inductive hypothesis and show that it holds true for $n + 1$:

$$\begin{aligned} \varphi^{n+1} &= \varphi \cdot \varphi^n = \varphi \cdot (a_n\varphi + a_{n-1}) \\ &= a_n\varphi^2 + a_{n-1}\varphi = a_n(\varphi + 1) + (a_{n+1} - a_n)\varphi \\ &= a_{n+1}\varphi + a_n \end{aligned}$$

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Corollary 1. Let ψ denote the other primitive of the Fibonacci sequence, i.e. $\psi = \frac{1-\sqrt{5}}{2}$, satisfying $\psi^n = \psi + 1$. The Fibonacci sequence will satisfy the following identity:

$$\psi^n = a_n\psi + a_{n-1}$$

Proposition 1. The Fibonacci sequence will satisfy the following identity:

$$a_{n+2}^2 - a_n^2 = a_{2n+2}$$

Proof. We proceed via a direct proof. By definition:

$$a_{n+1} = a_n + a_{n-1} \quad \text{and} \quad a_{n+1}^2 = a_n^2 + a_{n-1}^2 + 2a_na_{n-1}$$

and:

$$a_{n+2} = a_{n+1} + a_n \quad \text{and} \quad a_{n+2}^2 = a_{n+1}^2 + a_n^2 + 2a_{n+1}a_n$$

Now using the previous lemma in combination with the above:

$$\begin{aligned} a_{2n+2} &= \frac{1}{\sqrt{5}}\left(\varphi^{2n+2} - \psi^{2n+2}\right) \\ &= \frac{1}{\sqrt{5}}\left[\varphi^2\left(a_n^2\varphi^2 + a_{n-1}^2 + 2a_na_{n-1}\varphi\right) - \phi^2\left(a_n^2\phi^2 + a_{n-1}^2 + 2a_na_{n-1}\phi\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[a_n^2(\varphi^4 - \psi^4) + a_{n-1}^2(\varphi^2 - \psi^2) + 2a_na_{n-1}(\varphi^3 - \psi^3)\right] \\ &= a_n^2 \cdot a_4 + a_{n-1}^2 \cdot a_2 + 2a_na_{n-1} \cdot a_3 \\ &= 3a_n^2 + a_{n-1}^2 + 4a_na_{n-1} \\ &= (a_n^2 + a_{n-1}^2 + 2a_na_{n-1}) + (2a_n^2 + 2a_na_{n-1}) \\ &= a_{n+1}^2 + (2a_n^2 + 2a_na_{n-1}) \\ &= (a_{n+1}^2 + a_n^2) + (a_n^2 + 2a_na_{n-1}) \\ &= (a_{n+2}^2 - 2a_{n+1}a_n) + (a_n^2 + 2a_na_{n-1}) \\ &= (a_{n+2}^2 + a_n^2) + 2a_n(a_{n-1} - a_{n+1}) \\ &= a_{n+2}^2 + a_n^2 - 2a_n^2 \\ &= a_{n+2}^2 - a_n^2 \end{aligned}$$

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