Week 5 Attendance Solutions

MATH 23A

(1) (Q) In the theory of classical mechanics everything is determined by Newton's three laws. Consider a scenario in which a particle is traveling on a circular path in $\mathbb{R}^2$ determined by the position vector:

$$\overrightarrow{r}(t) = \begin{pmatrix} k\cos(\omega t) \\ k\sin(\omega t) \end{pmatrix}$$

where $\omega \in \mathbb{R}_{\geq 0}$ is the angular velocity and $k \in \mathbb{R}_{\geq 0}$ is a constant. According to Newton's second law, the centripetal force the particle feels is going to be determined by:

$$\overrightarrow{F}(t) = m\overrightarrow{a}(t)$$

Calculate the acceleration from the given position vector, i.e. $\overrightarrow{a}(t) = \overrightarrow{r}''(t)$, and write down the centripetal force strictly in terms of m, ω, and $\overrightarrow{r}(t)$.

(A) By direct calculation the velocity and acceleration vectors are:

$$\overrightarrow{r}'(t) = \begin{pmatrix} -k\omega\sin(\omega t) \\ k\omega\cos(\omega t) \end{pmatrix} \quad \text{and} \quad \overrightarrow{r}''(t) = \begin{pmatrix} -k\omega^2\cos(\omega t) \\ -k\omega^2\sin(\omega t) \end{pmatrix}$$

Notice that in particular:

$$\overrightarrow{a}(t) = \begin{pmatrix} -k\omega^2\cos(\omega t) \\ -k\omega^2\sin(\omega t) \end{pmatrix} = -\omega^2\begin{pmatrix} k\cos(\omega t) \\ k\sin(\omega t) \end{pmatrix} = -\omega^2\overrightarrow{r}(t)$$

Therefore, the centripetal force is given by:

$$\overrightarrow{F}(t) = m\overrightarrow{a}(t) = -m\omega^2\overrightarrow{r}(t)$$

(2) (Q) In the study of complex analysis, any function that acts on the complex plane $\mathbb{C}$ can always be identified with acting on $\mathbb{R}^2$ via:

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{given by} \quad f(x,y) = u(x,y) + iv(x,y)$$

where $u(x,y)$ and $v(x,y)$ are the real and imaginary part of $f(x,y)$ respectively. If we were to say that this function is *holomorphic*, then it must satisfy the relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Prove that u is necessarily a *harmonic* function by showing it satisfies the equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(A) This follows by direct calculation:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}\Big) \\
&= \frac{\partial}{\partial x}\Big(\frac{\partial v}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(-\frac{\partial v}{\partial x}\Big) \\
&= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \\
&= \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial y \partial x} \\
&= 0
\end{aligned}$$