Calculus 2
Recitation 14

Nathan Marianovsky

REPRESENTING FUNCTIONS AS POWER SERIES

Definition 1 (Power Series Representation)**.** Given any arbitrary function, it can be represented as a power series so long as x is restricted to the interval of convergence of the power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^{kn+m}$$

Definition 2 (Differentiation and Integration of Power Series)**.** Given any arbitrary series, the derivative and integral of the series are defined as:

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$
$$\int \Big(\sum_{n=0}^{\infty} c_n(x-a)^n\Big)dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

so long as the differentiation and integration are bound to the same radius of convergence.

Definition 3 (Geometric Power Series)**.** A power series in the form:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{where} \quad |x| < 1$$

is nothing more than a geometric series with x as the common ratio.

Definition 4 (Binomial Series)**.** A power series in the form:

$$(x+y)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n y^{k-n} \quad \text{where} \quad \binom{k}{n} = \frac{k!}{n!(k-n)!}$$

is known as the binomial series and $\binom{k}{n}$ is the binomial coefficient.

Definition 5 (Properties of the Binomial Coefficient)**.** All binomial coefficients satisfy the following properties:

$$\binom{k}{n} = \binom{k}{k-n}$$
$$\binom{-k}{n} = (-1)^n \binom{k+(n-1)}{n}$$
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Example 1. *Find the power series representation of* $f(x) = \frac{1}{1+x^2}$.

Solution 1. *Use algebraic manipulation to make the function look like a geometric power series:*

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n x^{2n}}$$

Example 2. *Find the power series representation of* $f(x) = \frac{x^3}{2+x}$.

Solution 2. *Use algebraic manipulation to make the function look like a geometric power series:*

$$f(x) = \frac{x^3}{2+x} = \frac{x^3}{2}\frac{1}{1+\frac{x}{2}} = \frac{x^3}{2}\frac{1}{1-(-\frac{x}{2})} = \frac{x^3}{2}\sum_{n=0}^{\infty}\Big(-\frac{x}{2}\Big)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+3}}{2^{n+1}}}$$

Example 3. *Find the power series representation of* $f(x) = \frac{3}{1-x^4}$.

Solution 3. *Use algebraic manipulation to make the function look like a geometric power series:*

$$f(x) = \frac{3}{1-x^4} = 3\sum_{n=0}^{\infty}(x^4)^n = \boxed{\sum_{n=0}^{\infty}3x^{4n}}$$

Example 4. *Find the power series representation of* $f(x) = \frac{1}{(1-x)^2}$.

Solution 4. *Take the geometric power series and differentiate both sides:*

$$\frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \frac{d}{dx}\Big(\sum_{n=0}^{\infty}x^n\Big)$$
$$\frac{1}{(1-x)^2} = \boxed{\sum_{n=1}^{\infty}nx^{n-1}}$$

Example 5. *Find the power series representation of* $f(x) = \ln(1+x)$.

Solution 5. *Take the geometric series and integrate both sides:*

$$\int\frac{1}{1-x}dx = \int\sum_{n=0}^{\infty}x^n dx$$
$$-\ln|1-x| = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} + C$$
$$\ln(1-x) = \sum_{n=0}^{\infty}-\frac{x^{n+1}}{n+1} + C$$
$$\ln(1+x) = \sum_{n=0}^{\infty}-\frac{(-x)^{n+1}}{n+1} + C$$
$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} + C$$

Now to figure out C *use some value of* x. *Specifically use* $x=0$ *to get that* $C=0$ *and:*

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}}$$

Example 6. *Find the power series representation of* $f(x) = \frac{1}{(1-x)^k}$.

Solution 6. *This can actually be accomplished two ways:*

(i) *Take the geometric power series and differentiate both sides a couple of times to notice the pattern:*

Relation	*Function*	*Power Series*
Original	$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$
First Derivative	$\frac{1}{(1-x)^2}$	$\sum_{n=1}^{\infty} nx^{n-1}$
Second Derivative	$\frac{2}{(1-x)^3}$	$\sum_{n=2}^{\infty} n(n-1)x^{n-2}$
Third Derivative	$\frac{6}{(1-x)^4}$	$\sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3}$
$\vdots$	$\vdots$	$\vdots$
$(K-1)^{th}$ *Derivative*	$\frac{(k-1)!}{(1-x)^k}$	$\sum_{n=k-1}^{\infty} n(n-1)(n-2)\ldots(n-(k-2))x^{n-(k-1)}$

Now use the $(K-1)^{th}$ *derivative to find the solution:*

$$\frac{(k-1)!}{(1-x)^k} = \sum_{n=k-1}^{\infty} n(n-1)(n-2)\ldots(n-(k-2))x^{n-(k-1)}$$

$$\frac{1}{(1-x)^k} = \sum_{n=k-1}^{\infty} \frac{n!}{(k-1)!(n-(k-1))!} x^{n-(k-1)}$$

$$\frac{1}{(1-x)^k} = \sum_{n=k-1}^{\infty} \binom{n}{k-1} x^{n-(k-1)}$$

$$\frac{1}{(1-x)^k} = \boxed{\sum_{m=0}^{\infty} \binom{m+(k-1)}{k-1} x^m}$$

(ii) *Use the binomial power series:*

$$\begin{aligned}
\frac{1}{(1-x)^k} &= (1-x)^{-k} \\
&= \sum_{m=0}^{\infty} \binom{-k}{m} (-x)^m \\
&= \sum_{m=0}^{\infty} \left((-1)^m \binom{m+(k-1)}{m}\right)\left((-1)^m x^m\right) \\
&= \sum_{m=0}^{\infty} \binom{m+(k-1)}{m} x^m \\
&= \boxed{\sum_{m=0}^{\infty} \binom{m+(k-1)}{k-1} x^m}
\end{aligned}$$

Example 7. *Find the power series representation of* $f(x) = \arctan(x)$.

Solution 7. *Use the geometric power series in the integral definition of* $\arctan(x)$*:*

$$\begin{aligned}
\arctan(x) &= \int_0^x \frac{dt}{1+t^2} \\
&= \int_0^x \frac{dt}{1-(-t^2)} \\
&= \int_0^x \sum_{n=0}^{\infty} (-t^2)^n dt \\
&= \sum_{n=0} (-1)^n \int_0^x t^{2n} dt \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\Big|_0^x \\
&= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}
\end{aligned}$$

Example 8. *Evaluate* $\int \ln(1+x^4)dx$ *as a power series.*

Solution 8. *Looking back at Example 5 shows how the logarithm can be rewritten:*

$$\begin{aligned}
\int \ln(1+x^4)dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{n+1}}{n+1} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int x^{4n+4} dx \\
&= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(n+1)(4n+5)} + C}
\end{aligned}$$

Example 9. *Evaluate* $\frac{d}{dx} J_0(x)$ *where* $J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}$*, which is the Bessel function of order 0.*

Solution 9.

$$\begin{aligned}
J_0'(x) &= \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \frac{d}{dx} x^{2n} \\
&= \sum_{n=1}^{\infty} (-1)^n \frac{2nx^{2n-1}}{2^{2n}(n!)^2} \\
&= \boxed{\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{2^{2n-1} n!(n-1)!}}
\end{aligned}$$

Example 10. *Evaluate* $\int \arctan(x^2)dx$ *as a power series and determine the radius and interval of convergence.*

Solution 10. *Looking back at Example 7 shows how arctan can be rewritten:*

$$\begin{aligned}\int \arctan(x^2)dx &= \int \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1}dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int x^{4n+2}dx \\ &= \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)} + C}\end{aligned}$$

Using the formula for the radius from Recitation 12:

$$R = \Big(\lim_{n\to\infty}\Big|\frac{(-1)^{n+1}}{(2n+3)(4n+7)} \times \frac{(2n+1)(4n+3)}{(-1)^n}\Big|\Big)^{-\frac{1}{4}} = \Big(\lim_{n\to\infty}\frac{(2n+1)(4n+3)}{(2n+3)(4n+7)}\Big)^{-\frac{1}{4}} = \Big(1\Big)^{-\frac{1}{4}} = \boxed{1}$$

The interval without checking the boundaries is:

$$\Big(a - R, a + R\Big) = \Big(-1, 1\Big)$$

Check the boundaries by plugging in the values for x*:*

$$\sum_{n=0}^{\infty}(-1)^n \frac{(1)^{4n+3}}{(2n+1)(4n+3)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)}$$
$$\sum_{n=0}^{\infty}(-1)^n \frac{(-1)^{4n+3}}{(2n+1)(4n+3)} = \sum_{n=0}^{\infty} \frac{(-1)^{5n+3}}{(2n+1)(4n+3)}$$

Notice that both of the series above will converge absolutely. Therefore, the interval of convergence is:

$$\boxed{\Big[-1, 1\Big]}$$

Example 11. *Evaluate* $\int \frac{\ln(1-t)}{t}dt$ *as a power series.*

Solution 11. *Looking back at Example 5 shows how the logarithm can be written:*

$$\int \frac{\ln(1-t)}{t}dt = \int \frac{\sum_{n=0}^{\infty}(-1)^n \frac{(-t)^{n+1}}{n+1}}{t}dt = \sum_{n=0}^{\infty} -\frac{1}{n+1}\int t^n dt = \boxed{\sum_{n=0}^{\infty} -\frac{t^{n+1}}{(n+1)^2} + C}$$

Example 12. *Use the power series expansion of* $\arctan(x)$ *to define an expression for* π *as the sum of an infinite series.* [1]

Solution 12. *Begin with Example 7 and choose* $x = 1$*:*

$$\begin{aligned}\arctan(1) &= \sum_{n=0}^{\infty}(-1)^n \frac{(1)^{2n+1}}{2n+1} \\ \frac{\pi}{4} &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1} \\ \pi &= \boxed{4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}}\end{aligned}$$

[1] In this example $x = 1$, but it is not the only "good" option for x. For example, $x = \frac{1}{\sqrt{3}}$ or $x = \sqrt{3}$ are also valid choices that can be used to approximate π.

Example 13. *Express* $f(x) = \frac{3}{x^2+x-2}$ *as a power series.*

Solution 13. *Use the method of partial fractions to break up the rational function and apply the geometric power series:*

$$\begin{aligned}
\frac{3}{x^2+x-2} &= \frac{1}{x-1} - \frac{1}{x+2} \\
&= -\frac{1}{1-x} - \frac{1}{2}\frac{1}{1-(-\frac{x}{2})} \\
&= -\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\Big(-\frac{x}{2}\Big)^n \\
&= \boxed{\sum_{n=0}^{\infty}\Big(\frac{(-1)^{n+1}}{2^{n+1}} - 1\Big)x^n}
\end{aligned}$$

Example 14. *Evaluate* $\frac{d}{dx}f(x)$ *where* $f(x) = \arctan(3x)$. *Express the solution as a power series.*

Solution 14. *This can be accomplished two ways:*

(i) *Find the derivative and express the solution as a power series:*

$$\begin{aligned}
\frac{d}{dx}\arctan(3x) &= \frac{3}{1+9x^2} \\
&= \frac{3}{1-(-9x^2)} \\
&= 3\sum_{n=0}^{\infty}(-9x^2)^n \\
&= \boxed{3\sum_{n=0}^{\infty}(-1)^n 9^n x^{2n}}
\end{aligned}$$

(ii) *Express the original function as a power series and differentiate:*

$$\begin{aligned}
\frac{d}{dx}\arctan(3x) &= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{(3x)^{2n+1}}{2n+1} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}}{2n+1}\frac{d}{dx}x^{2n+1} \\
&= \boxed{3\sum_{n=0}^{\infty}(-1)^n 9^n x^{2n}}
\end{aligned}$$

Example 15. *Express* $\ln(2)$ *as a sum of an infinite series.*

Solution 15. *Use the power series expansion for the logarithm from Example 5 when* $x = 1$*:*

$$\boxed{\ln(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}}$$

Example 16. *Express* 125 *as a sum of an infinite series.*

Solution 16. *Use the result of Example 6 for* $k = 3$*:*

$$125 = \frac{1}{(1-(\frac{4}{5}))^3} = \sum_{m=0}^{\infty}\binom{m+(3-1)}{3-1}\Big(\frac{4}{5}\Big)^m = \boxed{\sum_{m=0}^{\infty}\binom{m+2}{2}\frac{4^m}{5^m}}$$