Calculus 2
Riemann Sum Examples

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Definition 1 (Exact Area)**.** When calculating the exact area under a curve, certain decisions have to be made. In a Riemann Sum, the width of each rectangle taken is going to be exactly the same. Specifically:

$$\Delta x = \frac{b-a}{n}$$

where a and b denote the beginning and end of the given interval. Now that the width is known, the x value of each partition is defined as:

$$x_i = a + i\Delta x$$

Now the exact area is defined as:

$$\textbf{Area} = \int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

Example 1. *Evaluate* $\int_0^2 (x^2+1)dx$ *using a Riemann Sum and compare to direct integration.*

Solution 1.

(i) *Under a Riemann Sum, the components needed are:*

$$\begin{aligned} a &= 0 \\ b &= 2 \\ \Delta x &= \frac{2}{n} \\ x_i &= \frac{2i}{n} \end{aligned}$$

Evaluation using a Riemann Sum gives:

$$\begin{aligned} \textbf{\textit{Area}} &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^2 + 1 \right) \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\frac{4}{n^2} \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1 \right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\frac{4}{n^2} \frac{n(2n+1)(n+1)}{6} + n \right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\frac{2}{3n}(2n^2+3n+1) + n \right] \\ &= \lim_{n\to\infty} \left[\frac{4}{n} + \frac{14}{3} + \frac{4}{3n^2} \right] \\ &= \boxed{\frac{14}{3}} \end{aligned}$$

(ii) *By direct evaluation:*

$$\int_0^2 (x^2+1)dx = \left(\frac{x^3}{3} + x\right)\Big|_0^2 = \boxed{\frac{14}{3}}$$

Example 2. *Evaluate $\int_3^5 (2x+1)dx$ using a Riemann Sum and compare to direct integration.*

Solution 2.

(i) *Under a Riemann Sum, the components needed are:*

$$\begin{aligned} a &= 3 \\ b &= 5 \\ \Delta x &= \frac{2}{n} \\ x_i &= 3 + \frac{2i}{n} \end{aligned}$$

Evaluation using a Riemann Sum gives:

$$\begin{aligned} \textbf{\textit{Area}} &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \Big(2\Big(3 + \frac{2i}{n}\Big) + 1\Big) \\ &= \lim_{n\to\infty} \frac{2}{n} \Big[\sum_{i=1}^{n} 7 + \frac{4}{n}\sum_{i=1}^{n} i\Big] \\ &= \lim_{n\to\infty} \frac{2}{n} \Big[7n + \frac{4}{n}\frac{n(n+1)}{2}\Big] \\ &= \lim_{n\to\infty} \Big[18 + \frac{8}{n^2} + \frac{4}{n}\Big] \\ &= \boxed{18} \end{aligned}$$

(ii) *By direct evaluation:*

$$\int_3^5 (2x+1)dx = (x^2 + x)\Big|_3^5 = (25+5) - (9+3) = \boxed{18}$$

Example 3. *Evaluate $\int_0^7 x^3 dx$ using a Riemann Sum and compare to direct integration.*

Solution 3.

(i) *Under a Riemann Sum, the components needed are:*

$$\begin{aligned} a &= 0 \\ b &= 7 \\ \Delta x &= \frac{7}{n} \\ x_i &= \frac{7i}{n} \end{aligned}$$

Evaluation using a Riemann Sum gives:

$$\begin{aligned} \textbf{\textit{Area}} &= \lim_{n\to\infty} \frac{7}{n} \sum_{i=1}^{n} \Big(\frac{7i}{n}\Big)^3 \\ &= \lim_{n\to\infty} \frac{2401}{n^4} \sum_{i=1}^{n} i^3 \\ &= \lim_{n\to\infty} \frac{2401}{n^4} \Big(\frac{n(n+1)}{2}\Big)^2 \\ &= \lim_{n\to\infty} \frac{2401}{4n^2}(n^2 + 2n + 1) \\ &= \lim_{n\to\infty} \Big(\frac{2401}{4} + \frac{2401}{2n} + \frac{2401}{4n^2}\Big) \\ &= \boxed{\frac{2401}{4}} \end{aligned}$$

(ii) *By direct evaluation:*

$$\int_0^7 x^3 dx = \frac{x^4}{4}\Big|_0^7 = \boxed{\frac{2401}{4}}$$

Example 4. *Evaluate* $\int_0^1 (x^2 - x)dx$ *using a Riemann Sum and compare to direct integration.*

Solution 4.

(i) *Under a Riemann Sum, the components needed are:*

$$\begin{aligned} a &= 0 \\ b &= 1 \\ \Delta x &= \frac{1}{n} \\ x_i &= \frac{i}{n} \end{aligned}$$

Evaluation using a Riemann Sum gives:

$$\begin{aligned} \textbf{\textit{Area}} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\left(\frac{i}{n}\right)^2 - \left(\frac{i}{n}\right) \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \Big[\frac{1}{n^2} \sum_{i=1}^{n} i^2 - \frac{1}{n} \sum_{i=1}^{n} i \Big] \\ &= \lim_{n\to\infty} \frac{1}{n} \Big[\frac{1}{n^2} \frac{n(2n+1)(n+1)}{6} - \frac{1}{n} \frac{n(n+1)}{2} \Big] \\ &= \lim_{n\to\infty} \Big[\frac{2n^2+3n+1}{6n^2} - \frac{n+1}{2n} \Big] \\ &= \lim_{n\to\infty} \Big[-\frac{1}{6} + \frac{1}{3n} - \frac{1}{2n} + \frac{1}{6n^2} \Big] \\ &= \boxed{-\frac{1}{6}} \end{aligned}$$

(ii) *By direct evaluation:*

$$\int_0^1 (x^2 - x)dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1 = \boxed{-\frac{1}{6}}$$