Comments on Notation & Functions

Let me layout all of the notation I typically use, its corresponding definition, and some examples of usage:

I. Sets

In this course we strictly deal with the set of real numbers, $\mathbb{R}$. A subset, $\mathcal{S}$, of the real numbers is any collection of real numbers and is denoted by $\mathcal{S} \subseteq \mathbb{R}$ which is read as "$\mathcal{S}$ is a *subset* of $\mathbb{R}$". In some cases we want to make the relation stricter by writing $\mathcal{S} \subset \mathbb{R}$ which is read as "$\mathcal{S}$ is a *proper subset* of $\mathbb{R}$" which just implies that $\mathcal{S} \neq \mathbb{R}$. For examples of subsets consider the following:

$$\{1,2\} \subset \mathbb{R}, \quad [1,2) \subseteq \mathbb{R}, \quad \{0\} \cup (\pi, 100) \subset \mathbb{R}, \quad (-\infty, \infty) \subseteq \mathbb{R}, \quad \text{and} \quad (0, \infty) \subset \mathbb{R}$$

Of course there are a couple of subsets of the real numbers that have their own notation. So formally we have the following:

$$\begin{aligned}
\emptyset &= \{\} \quad \text{(The empty set)} \\
\mathbb{N} &= \{1,2,3,4,5,\dots\} \quad \text{(The set of natural numbers)} \\
\mathbb{Z} &= \{\dots,-3,-2,-1,0,1,2,3,\dots\} \quad \text{(The set of integers)} \\
\mathbb{Z}_{\geq 0} &= \{0,1,2,3,\dots\} \quad \text{(The set of non-negative integers)} \\
\mathbb{Q} &= \left\{ \frac{p}{q} \;\middle|\; p,q \in \mathbb{Z} \right\} \quad \text{(The set of rational numbers)} \\
\mathbb{R} &= \text{(The set of real numbers)} \\
\mathbb{R}_{>0} &= \{x \in \mathbb{R} \mid x > 0\} \quad \text{(The set of positive real numbers)} \\
\mathbb{R}_{\geq 0} &= \{x \in \mathbb{R} \mid x \geq 0\} \quad \text{(The set of non-negative real numbers)}
\end{aligned}$$

Whenever we want to talk about a specific element belonging to a set $\mathcal{S}$, we denote it by $x \in \mathcal{S}$ which is read as "x *belongs to* $\mathcal{S}$". For examples consider the following: $3 \in \mathbb{R}$, $1 \in (0,1)$, $0 \in \{0\}$, $7 \in \mathbb{N}$, and $\frac{9}{10} \in \mathbb{Q}$.

II. Functions

For *functions*, also known as *mappings*, we use the notation:

$$f : A \to B$$

which is read as "f is the function with *domain* A and *codomain* B". For the range of f we use the notation $f(A)$ which naturally satisfies the relation $f(A) \subseteq B$. With this structure you might note that the codomain of a function and its range need not be equal. As an example consider the mapping:

$$g : \mathbb{R} \to \mathbb{R} \quad \text{where} \quad g(x) = x^2$$

In this scenario we have $g(\mathbb{R}) = \mathbb{R}_{\geq 0}$. If it so happens that $f(A) = B$, then the function is said to be *onto*, or more formally *surjective*. In the example above if we restrict the codomain to be $\mathbb{R}_{\geq 0}$ then the function would be onto. Another important property of function is the idea of a *one-to-one* correspondence, or more formally function *injectivity*. A function is said to be one-to-one if its associated graph passes the *Horizontal Line Test*. In the case where the graph is not easy to draw we need a more formal approach to determining a one-to-one correspondence. So from a technical viewpoint, a function is said to be one-to-one if:

$$f(x) = f(y) \implies x = y$$

To prove such a condition, it is typically approached by utilizing a proof by contradiction. For example take the function g from above. We know this function to not be one-to-one, so here is the proof:

Proof: *Assume the opposite, so $g(x)$ is one-to-one. This means that given $g(x) = g(y)$ implies $x = y$. A simple counterexample is $f(1) = f(-1)$ because $1 \neq -1$. This contradiction implies that $g(x)$ is not one-to-one.* ■

Any function that is both onto, surjective, and one-to-one, injective, is said to be *bijective*. A bijection implies that there exists an *inverse function* denoted by $f^{-1} : B \to A$.