Examples of Tensor Products

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To follow up on some of the material covered during section I figure it would be helpful to provide more examples where the best way of describing the elements of any vector space is to write down a basis. Letting for all of the following examples V and W be finite-dimensional vector spaces over the field $\mathbb{F}$ with bases $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ and $\mathcal{B}_W = \{w_1, \ldots, w_m\}$, the basis for $V \otimes_{\mathbb{F}} W$ is given by $\mathcal{B}_{V\otimes_{\mathbb{F}} W} = \{v_i \otimes_{\mathbb{F}} w_j\}_{1\leq i\leq n, 1\leq j\leq m}$.

(1) Let us consider the case of $V = W = \mathbb{R}$ as real vector spaces ($\mathbb{F} = \mathbb{R}$). In this scenario we can write down $\mathcal{B}_V = \{a\}$ and $\mathcal{B}_W = \{b\}$ for $a, b \in \mathbb{R}$ from which we obtain $\mathcal{B}_{V\otimes_{\mathbb{R}} W} = \{a \otimes_{\mathbb{R}} b\}$, but from the properties of a tensor:

$$a \otimes_{\mathbb{R}} b = a(1 \otimes_{\mathbb{R}} b) = 1 \otimes_{\mathbb{R}} ab$$

In other words, we can write down an even simpler basis for the tensor product:

$$\mathcal{B}'_{V\otimes_{\mathbb{R}} W} = \{1 \otimes_{\mathbb{R}} c\} \quad \text{for} \quad c \in \mathbb{R}$$

Since there is a single degree of freedom in the basis we can identify $\dim(V \otimes_{\mathbb{R}} W) = 1$. Furthermore, having the identity element of the field in the tensor does not add any new information, hence justifying the identification $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}$ as real vector spaces.

(2) Let us consider the case of $V = \mathbb{C}$ and $W = \mathbb{R}$ as real vector spaces ($\mathbb{F} = \mathbb{R}$). In this scenario we can write $\mathcal{B}_V = \{a, bi\}$ and $\mathcal{B}_W = \{c\}$ for $a, b, c \in \mathbb{R}$ from which we obtain $\mathcal{B}_{V\otimes_{\mathbb{R}} W} = \{a \otimes_{\mathbb{R}} c, bi \otimes_{\mathbb{R}} c\}$, but from the properties of a tensor:

$$a \otimes_{\mathbb{R}} c = c(a \otimes_{\mathbb{R}} 1) = ac \otimes_{\mathbb{R}} 1 \quad \text{and} \quad bi \otimes_{\mathbb{R}} c = c(bi \otimes_{\mathbb{R}} 1) = bci \otimes_{\mathbb{R}} 1$$

In other words, we can write down an even simpler basis for the tensor product:

$$\mathcal{B}'_{V\otimes_{\mathbb{R}} W} = \{m \otimes_{\mathbb{R}} 1, ni \otimes_{\mathbb{R}} 1\} \quad \text{for} \quad m, n \in \mathbb{R}$$

Since there are two degrees of freedom in the basis we can identify $\dim(V \otimes_{\mathbb{R}} W) = 2$. Furthermore, having the identity element of the field in the tensor does not add any new information, hence justifying the identification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{C}$ as real vector spaces.

(3) Let us consider the case of $V = W = \mathbb{C}$ as real vector spaces ($\mathbb{F} = \mathbb{R}$). In this scenario we can write $\mathcal{B}_V = \{a, bi\}$ and $\mathcal{B}_W = \{c, di\}$ for $a, b, c, d \in \mathbb{R}$ from which we obtain $\mathcal{B}_{V\otimes_{\mathbb{R}} W} = \{a \otimes_{\mathbb{R}} c, a \otimes_{\mathbb{R}} di, bi \otimes_{\mathbb{R}} c, bi \otimes_{\mathbb{R}} di\}$, but from the properties of a tensor:

$$\begin{aligned} a \otimes_{\mathbb{R}} c &= c(a \otimes_{\mathbb{R}} 1) = ac(1 \otimes_{\mathbb{R}} 1) \\ a \otimes_{\mathbb{R}} di &= a(1 \otimes_{\mathbb{R}} di) = ad(1 \otimes_{\mathbb{R}} i) \\ bi \otimes_{\mathbb{R}} c &= c(bi \otimes_{\mathbb{R}} 1) = bc(i \otimes_{\mathbb{R}} 1) \\ bi \otimes_{\mathbb{R}} di &= b(i \otimes_{\mathbb{R}} di) = bd(i \otimes_{\mathbb{R}} i) \end{aligned}$$

In other words, we can write down an even simpler basis for the tensor product:

$$\mathcal{B}'_{V\otimes_{\mathbb{R}} W} = \{1 \otimes_{\mathbb{R}} 1, 1 \otimes_{\mathbb{R}} i, i \otimes_{\mathbb{R}} 1, i \otimes_{\mathbb{R}} i\}$$

Since there are four independent elements in the basis we can identify $\dim(V \otimes_{\mathbb{R}} W) = 4$.

(4) Let us consider the case of $V = W = \mathbb{C}$ as complex vector spaces ($\mathbb{F} = \mathbb{C}$). In this scenario we can write $\mathcal{B}_V = \{z_1\}$ and $\mathcal{B}_W = \{z_2\}$ for $z_1, z_2 \in \mathbb{C}$ from which we obtain $\mathcal{B}_{V\otimes_{\mathbb{C}} W} = \{z_1 \otimes_{\mathbb{C}} z_2\}$, but from the properties of a tensor:

$$z_1 \otimes_{\mathbb{C}} z_2 = z_1(1 \otimes_{\mathbb{C}} z_2) = 1 \otimes_{\mathbb{C}} z_1 z_2$$

In other words, we can write down an even simpler basis for the tensor product:

$$\mathcal{B}'_{V\otimes_{\mathbb{R}} W} = \{1 \otimes_{\mathbb{C}} z_3\} \quad \text{for} \quad z_3 \in \mathbb{C}$$

Since there is a single degree of freedom in the basis we can identify $\dim(V \otimes_{\mathbb{C}} W) = 1$. Furthermore, having the identity element of the field in the tensor does not add any new information, hence justifying the identification $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ as complex vector spaces.

(5) Let us consider the case of $V = \mathbb{C}$ and $W = \mathbb{R}^3$ as real vector spaces ($\mathbb{F} = \mathbb{R}$). In this scenario we can write $\mathcal{B}_V = \{1, i\}$ and $\mathcal{B}_W = \{(1,0,0),(0,1,0),(0,0,1)\}$ from which we obtain:

$$\mathcal{B}_{V\otimes_{\mathbb{R}}W} = \{1 \otimes_{\mathbb{R}} (1,0,0), 1 \otimes_{\mathbb{R}} (0,1,0), 1 \otimes_{\mathbb{R}} (0,0,1), i \otimes_{\mathbb{R}} (1,0,0), i \otimes_{\mathbb{R}} (0,1,0), i \otimes_{\mathbb{R}} (0,0,1)\}$$

Since there are six independent basis elements we can identify $\dim(V \otimes_{\mathbb{C}} W) = 6$.

From all of the above examples it should become apparent that $\dim(V \otimes_{\mathbb{F}} W) = \dim(V)\dim(W)$. To see why this is so, consider the fact that the basis elements of the tensor product have to choose an $1 \leq i \leq n$ and $1 \leq j \leq m$ so as to form $v_i \otimes_{\mathbb{F}} w_j$. Hence, there are n and m choices, respectively, providing a total of mn basis elements for the tensor product.