Solving the Fibonacci Recurrence Relation

Solving Directly

Given the Fibonacci sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the recurrence relation $a_n = a_{n-1} + a_{n-2}$, we want to extract a closed form solution of a_n with the initial conditions $a_1 = a_2 = 1$. We make the *ansatz*[1] that $a_n = \alpha^n$ for some $\alpha \in \mathbb{R}$ and proceed directly to solve:

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \\ 0 &= a_n - a_{n-1} - a_{n-2} \\ 0 &= \alpha^n - \alpha^{n-1} - \alpha^{n-2} \\ 0 &= \alpha^{n-2}(\alpha^2 - \alpha - 1) \end{aligned}$$

Since an exponential will never vanish, we want to focus in on the *characteristic equation*:

$$\begin{aligned} 0 &= \alpha^2 - \alpha - 1 \\ \alpha &= \frac{1 \pm \sqrt{1+4}}{2} \\ \alpha &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

By convention we denote the solutions as $\varphi = \frac{1+\sqrt{5}}{2}$ (the *golden ratio*) and $\psi = \frac{1-\sqrt{5}}{2}$ satisfying the relationship:

$$-\psi^{-1} = -\frac{2}{1-\sqrt{5}} = -\frac{2}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = -\frac{2+2\sqrt{5}}{-4} = \frac{1+\sqrt{5}}{2} = \varphi$$

When solving *linear constant coefficient difference equations*, it can be proven that the most general solution will take the closed form:

$$a_n = A\varphi^n + B\psi^n$$

for some unknown $A, B \in \mathbb{R}$. Now we use the initial conditions to obtain the system of equations:

$$\begin{cases} a_1 = 1 = A\varphi + B\psi \\ a_2 = 1 = A\varphi^2 + B\psi^2 \end{cases}$$

There are many ways to solve the system for A and B, but using matrices seems to be the easiest:

$$\begin{aligned} \begin{pmatrix} \varphi & \psi \\ \varphi^2 & \psi^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} \varphi & \psi \\ \varphi^2 & \psi^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} A \\ B \end{pmatrix} &= \frac{1}{\varphi\psi^2 - \varphi^2\psi} \begin{pmatrix} \psi^2 & -\psi \\ -\varphi^2 & \varphi \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} A \\ B \end{pmatrix} &= \frac{1}{\varphi\psi(\psi - \varphi)} \begin{pmatrix} \psi^2 - \psi \\ \varphi - \varphi^2 \end{pmatrix} \\ \begin{pmatrix} A \\ B \end{pmatrix} &= \frac{1}{\varphi - \psi} \begin{pmatrix} (\psi+1) - \psi \\ \varphi - (\varphi+1) \end{pmatrix} \\ \begin{pmatrix} A \\ B \end{pmatrix} &= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

It directly follows that:

$$a_n = \frac{1}{\sqrt{5}}\Big(\varphi^n - \psi^n\Big)$$

[1]German for educated guess that is verified later by its results.

Explaining the Ansatz

Why did we know that $a_n = \alpha^n$ would work? To explain this we need to first define the *difference operator* (the discrete analogue of the derivative):

$$\Delta a_n = a_n - a_{n-1}$$

and consider the setup:

$$\Delta a_n = \lambda a_n$$

where $\lambda \in \mathbb{R}$ and is called the *eigenvalue* associated to the *eigenfunction* a_n. Supposing that a_n is an eigenfunction forces upon us:

$$\begin{aligned} \lambda a_n &= a_n - a_{n-1} \\ 0 &= (1-\lambda)a_n - a_{n-1} \end{aligned}$$

Now we approach via two separate cases:

- The trivial case is the one where $\lambda = 1$. In such a scenario we must have $a_{n-1} = 0$ for arbitrary n and this is obviously not the desired result.
- Assuming that $\lambda \neq 1$ we can isolate:

  $$a_n = \frac{1}{1-\lambda} a_{n-1}$$

  Repeated use of this identity yields:

  $$a_n = \left(\frac{1}{1-\lambda}\right)^2 a_{n-2} = \left(\frac{1}{1-\lambda}\right)^3 a_{n-3} = \cdots = \left(\frac{1}{1-\lambda}\right)^n a_0$$

  Letting $\alpha = \frac{1}{1-\lambda}$ we obtain the closed form:

  $$a_n = a_0 \alpha^n$$

  where a_0 represents the initial condition for a single primitive. Thus, any primitive for a *difference equation* given in the form:

  $$a_n + \xi_{n-1}a_{n-1} + \xi_{n-2}a_{n-2} + \cdots + \xi_{n-k}a_{n-k} = 0 \quad \text{where} \quad \xi_{n-k}, \ldots, \xi_{n-1} \in \mathbb{R}$$

  is always an exponential function, thereby motivating the ansatz.

Solving via Linear Algebra

Another method to solve the Fibonacci recurrence relation uses the matrix setup:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$$

- We now turn our attention towards the matrix:

  $$\Omega = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

  and want to find its eigenvalues with the associated eigenvectors. To accomplish this we compute the characteristic equation:

  $$p(\lambda) = \det(\Omega - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

  Non-surprisingly this characteristic equation is exactly the same one found when we solved directly. The eigenvalues are the values for which $p(\lambda) = 0$ giving $\lambda = \varphi, \psi$. To find the eigenvectors we consider each case separately:

  - For $\lambda = \varphi$ we want to solve the system:

    $$\left(\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 1 & -\varphi & 0 \end{array}\right) \implies \left(\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 1+\varphi-\varphi^2 & \varphi-\varphi & 0 \end{array}\right) \implies \left(\begin{array}{cc|c} 1-\varphi & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

    Reading off the last augmented matrix tells us that the eigenvector takes the form:

    $$v_\varphi = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$$

- For $\lambda = \psi$ we want to solve the system:

$$\left(\begin{array}{cc|c} 1-\psi & 1 & 0 \\ 1 & -\psi & 0 \end{array}\right) \implies \left(\begin{array}{cc|c} 1-\psi & 1 & 0 \\ 1+\psi-\psi^2 & \psi-\psi & 0 \end{array}\right) \implies \left(\begin{array}{cc|c} 1-\psi & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

  Reading off the last augmented matrix tells us that the eigenvector takes the form:

$$v_\psi = \begin{pmatrix} \psi \\ 1 \end{pmatrix}$$

- Having found the eigenvalues and eigenvectors we note that the algebraic and geometric multiplicities of each eigenvalue are equal[2], thereby implying that Ω is diagonalizable, i.e. $\Omega = P\Lambda P^{-1}$ for particular matrices P and Λ. More specifically:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \uparrow & \uparrow \\ v_\varphi & v_\psi \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix}$$

  Putting this together yields:

$$\Omega = P\Lambda P^{-1} = \frac{1}{\varphi-\psi}\begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}\begin{pmatrix} 1 & -\psi \\ -1 & \varphi \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}\begin{pmatrix} 1 & -\psi \\ -1 & \varphi \end{pmatrix}$$

- Now you should be asking yourself...why did we bother diagonalizing the matrix? Precisely due to this property:

$$\begin{aligned} \Omega^n = (P\Lambda P^{-1})^n &= \underbrace{(P\Lambda P^{-1})(P\Lambda P^{-1})\dots(P\Lambda P^{-1})}_{n \text{ copies}} \\ &= (P\Lambda^2 P^{-1})\underbrace{(P\Lambda P^{-1})(P\Lambda P^{-1})\dots(P\Lambda P^{-1})}_{n-2 \text{ copies}} \\ &= (P\Lambda^3 P^{-1})\underbrace{(P\Lambda P^{-1})(P\Lambda P^{-1})\dots(P\Lambda P^{-1})}_{n-3 \text{ copies}} \\ &\vdots \\ &= P\Lambda^n P^{-1} \end{aligned}$$

  Now we also need to check that for any diagonal matrix:

$$\Lambda^n = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^n & 0 & \dots & 0 \\ 0 & 0 & \lambda_3^n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k^n \end{pmatrix}$$

  Proof. A proof by induction should suffice:
  - For the base case take $n = 1$ and notice that it obviously holds true.
  - Assuming the above statement as the inductive hypothesis we check the behavior for $n + 1$:

$$\begin{aligned} \Lambda^{n+1} = \Lambda\Lambda^n &= \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k \end{pmatrix}\begin{pmatrix} \lambda_1^n & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^n & 0 & \dots & 0 \\ 0 & 0 & \lambda_3^n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k^n \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1^{n+1} & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^{n+1} & 0 & \dots & 0 \\ 0 & 0 & \lambda_3^{n+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k^{n+1} \end{pmatrix} \end{aligned}$$

$\blacksquare$

[2] This just tells us that the number of associated eigenvectors each eigenvalue has is equal to the multiplicity of the root in the characteristic equation.

- Now we put everything together! Knowing that:

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a_n\\a_{n-1}\end{pmatrix}=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}$$

We can first write:

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a_2\\a_1\end{pmatrix}=\begin{pmatrix}a_3\\a_2\end{pmatrix}$$

and use this to deduce:

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a_3\\a_2\end{pmatrix}=\begin{pmatrix}a_4\\a_3\end{pmatrix}$$
$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^2\begin{pmatrix}a_2\\a_1\end{pmatrix}=\begin{pmatrix}a_4\\a_3\end{pmatrix}$$

Once more:

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a_4\\a_3\end{pmatrix}=\begin{pmatrix}a_5\\a_4\end{pmatrix}$$
$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^3\begin{pmatrix}a_2\\a_1\end{pmatrix}=\begin{pmatrix}a_5\\a_4\end{pmatrix}$$

Generalizing the pattern above provides:

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{n-1}\begin{pmatrix}a_2\\a_1\end{pmatrix}=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}$$

Using the work from above we can avoid multiplying matrices needlessly:

$$\begin{aligned}
\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{n-1}\begin{pmatrix}a_2\\a_1\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & \psi\\1 & 1\end{pmatrix}\begin{pmatrix}\varphi & 0\\0 & \psi\end{pmatrix}^{n-1}\begin{pmatrix}1 & -\psi\\-1 & \varphi\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & \psi\\1 & 1\end{pmatrix}\begin{pmatrix}\varphi^{n-1} & 0\\0 & \psi^{n-1}\end{pmatrix}\begin{pmatrix}1 & -\psi\\-1 & \varphi\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n} & \psi^{n}\\\varphi^{n-1} & \psi^{n-1}\end{pmatrix}\begin{pmatrix}1 & -\psi\\-1 & \varphi\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n}-\psi^{n} & -\varphi\psi(\varphi^{n-1}-\psi^{n-1})\\\varphi^{n-1}-\psi^{n-1} & -\varphi\psi(\varphi^{n-2}-\psi^{n-2})\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n}-\psi^{n} & \varphi^{n-1}-\psi^{n-1}\\\varphi^{n-1}-\psi^{n-1} & \varphi^{n-2}-\psi^{n-2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n-1}(\varphi+1)-\psi^{n-1}(\psi+1)\\\varphi^{n-2}(\varphi+1)-\psi^{n-2}(\psi+1)\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n-1}(\varphi^2)-\psi^{n-1}(\psi^2)\\\varphi^{n-2}(\varphi^2)-\psi^{n-2}(\psi^2)\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}\\
\frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n+1}-\psi^{n+1}\\\varphi^{n}-\psi^{n}\end{pmatrix}&=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}
\end{aligned}$$

Reading off of the bottom row we finally arrive at the same formula as before:

$$a_n=\frac{1}{\sqrt{5}}\Big(\varphi^n-\psi^n\Big)$$