Quiz 4 Solutions

Section B

(1) (a) For any function that satisfies $f(x) \geq 0$ on $[a, b]$ then we must have $\int_a^b f(x)\, \mathrm{d}x \geq 0$. For the functions given in the problem equality to zero only occurs at $x = 0$, while the rest is strictly positive. Thus, the integrals will be strictly positive.

(b) Integration by parts is nice and all, but there is a shorter way to evaluate this integral. I make the *ansatz* that the antiderivatives will take the form:

$$\begin{aligned} f(x) &= (Ax^2 + Bx + C)e^x \\ g(x) &= (Mx^3 + Nx^2 + Kx + L)e^x \end{aligned}$$

Taking the derivatives provides:

$$\begin{aligned} f'(x) &= (Ax^2 + Bx + C)e^x + (2Ax + B)e^x \\ &= (Ax^2 + [2A + B]x + [B + C])e^x \\ g'(x) &= (Mx^3 + Nx^2 + Kx + L)e^x + (3Mx^2 + 2Nx + K)e^x \\ &= (Mx^3 + [3M + N]x^2 + [2N + K]x + [K + L])e^x \end{aligned}$$

Now matching these up against the integrands:

$$\begin{aligned} x^2e^x &= (Ax^2 + [2A + B]x + [B + C])e^x \\ x^3e^x &= (Mx^3 + [3M + N]x^2 + [2N + K]x + [K + L])e^x \end{aligned}$$

forces the following values on the coefficients:

$$A = 1, \quad B = -2, \quad C = 2, \quad M = 1, \quad N = -3, \quad K = 6, \quad \text{and} \quad L = -6$$

Using the above setup, the integrals become:

$$\begin{aligned} \int_0^1 x^2e^x\, \mathrm{d}x &= (x^2 - 2x + 2)e^x \Big|_0^1 = e - 2 \\ \int_0^1 x^3e^x\, \mathrm{d}x &= (x^3 - 3x^2 + 6x - 6)e^x \Big|_0^1 = -2e + 6 \end{aligned}$$

(c) With the results of parts (a) and (b) we can say that $e - 2 > 0$ which implies $e > 2$. Similarly $-2e + 6 > 0$ implies $3 > e$. Putting these together we have $2 < e < 3$ as desired.

(2) (a) Use the setup for integration by parts:

$$\begin{aligned} u = \ln(x) &\implies \mathrm{d}u = \frac{\mathrm{d}x}{x} \\ \mathrm{d}v = x^{-k}\, \mathrm{d}x &\implies v = \frac{x^{1-k}}{1-k} \end{aligned}$$

Plugging in gives:

$$\int \frac{\ln(x)}{x^k}\, \mathrm{d}x = \frac{x^{1-k}}{1-k}\ln(x) - \frac{1}{1-p}\int x^{-k}\, \mathrm{d}x = \frac{x^{1-k}}{1-k}\left[\ln(x) - \frac{1}{1-k}\right] + C$$

This corresponds to choice (a).

(b) The general case does not cover for $k = 1$, so to approach this we must go back to the integral. The integral can be done by substitution to get:

$$\int \frac{\ln(x)}{x} \, \mathrm{d}x = \frac{\ln^2(x)}{2} + C$$

SECTION C

(1) (a) Exactly the same as Section B.

(b) Exactly the same as Section B besides the evaluation: Using the above setup, the integrals become:

$$\int_{-1}^{0} x^2 e^x \, \mathrm{d}x = (x^2 - 2x + 2)e^x \Big|_{-1}^{0} = 2 - \frac{5}{e}$$
$$\int_{0}^{1} x^3 e^x \, \mathrm{d}x = (x^3 - 3x^2 + 6x - 6)e^x \Big|_{0}^{1} = -2e + 6$$

(c) With the results of parts (a) and (b) we can say that $2 - \frac{5}{e} > 0$ which implies $e > 2.5$. Similarly $-2e + 6 > 0$ implies $3 > e$. Putting these together we have $2.5 < e < 3$ as desired.

(2) Exactly the same as Section B.