

# Calculus 2

## Double-Angle Formulas and Applications

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### DERIVATION OF DOUBLE-ANGLE FORMULAS

**Definition 1** (Double-Angle Formulas). Assuming that you do not have the double-angle formulas from trigonometry memorized, here is a simple approach to derive them. First assume that Euler's formula holds true:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

and exploit the properties of exponentials to arrive at the double-angle relationships:

$$(e^{i\theta})^2 = e^{i(2\theta)}$$

$$(\cos(\theta) + i \sin(\theta))^2 = \cos(2\theta) + i \sin(2\theta)$$

$$\cos^2(\theta) - \sin^2(\theta) + 2i \cos(\theta) \sin(\theta) = \cos(2\theta) + i \sin(2\theta)$$

Now if two complex numbers are equivalent, then their real components (*terms without an  $i$* ) and imaginary components (*terms with an  $i$* ) are equal to each other:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2 \cos^2(\theta) - 1 = 1 - 2 \sin^2(\theta)$$

$$\sin(2\theta) = 2 \cos(\theta) \sin(\theta)$$

**Example 1.** Evaluate  $\int_0^{\frac{\pi}{2}} \cos(\theta) \sin(\theta) d\theta$

**Solution 1.** This can actually be done two ways:

(i) Make the substitution:

$$u = \sin(\theta)$$

$$du = \cos(\theta) d\theta$$

Now reevaluation of the bounds and plugging in the substitution yields:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos(\theta) \sin(\theta) d\theta &= \int_0^1 u du \\ &= \frac{1}{2} u^2 \Big|_0^1 \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

(ii) Using the double angle formula for sine gives:

$$\int_0^{\frac{\pi}{2}} \cos(\theta) \sin(\theta) d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin(2\theta)}{2} d\theta$$

Now make the substitution:

$$u = 2\theta$$

$$du = 2d\theta \rightarrow \frac{du}{2} = d\theta$$

Now the integral becomes:

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2\theta)}{2} d\theta = \int_0^{\pi} \frac{\sin(u)}{4} du = -\frac{\cos(u)}{4} \Big|_0^{\pi} = \boxed{\frac{1}{2}}$$

**Example 2.** Evaluate  $\int \sin^2(x)dx$

**Solution 2.** Use the double-angle formula for cosine to redefine the  $\sin^2(x)$ :

$$\begin{aligned}\cos(2x) &= 1 - 2\sin^2(x) \\ 2\sin^2(x) &= 1 - \cos(2x) \\ \sin^2(x) &= \frac{1 - \cos(2x)}{2}\end{aligned}$$

Now the integral becomes:

$$\begin{aligned}\int \sin^2(x)dx &= \int \frac{1 - \cos(2x)}{2}dx \\ &= \frac{1}{2} \int dx - \frac{1}{2} \int \cos(2x)dx\end{aligned}$$

For the second integral above make the substitution:

$$\begin{aligned}u &= 2x \\ du &= 2dx \rightarrow \frac{du}{2} = dx\end{aligned}$$

Plugging in:

$$\begin{aligned}\frac{1}{2} \int dx - \frac{1}{2} \int \cos(2x)dx &= \frac{x}{2} - \frac{1}{2} \int \frac{\cos(u)}{2}du \\ &= \frac{x}{2} - \frac{\sin(u)}{4} + C \\ &= \boxed{\frac{x}{2} - \frac{\sin(2x)}{4} + C}\end{aligned}$$

**Example 3.** Evaluate  $\int \frac{\cos(x)}{\sin(2x)}dx$

**Solution 3.** Using the double angle formula for sine gives:

$$\begin{aligned}\int \frac{\cos(x)}{\sin(2x)}dx &= \int \frac{\cos(x)}{2\cos(x)\sin(x)}dx \\ &= \frac{1}{2} \int \frac{1}{\sin(x)}dx \\ &= \frac{1}{2} \int \csc(x)dx \\ &= \boxed{-\frac{\ln(\csc(x) + \cot(x))}{2} + C}\end{aligned}$$