Comments on Some Problems

I want to clarify some problems that I covered during section because I feel I may not have covered them in full detail. Specifically consider the following:

(1) Evaluate $\lim\limits_{x\to 0^+} \sin(x)\ln(4x)$.

- First notice that $\sin(x)\ln(4x) = \frac{\ln(4x)}{(\sin(x))^{-1}}$. Using this setup we must have:

$$\lim_{x\to 0^+} \frac{\ln(4x)}{(\sin(x))^{-1}} \to \frac{-\infty}{+\infty}$$

  This justifies using L'Hospital's rule so that:

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln(4x)}{(\sin(x))^{-1}} &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-(\sin(x))^{-2}\cos(x)} = -\lim_{x\to 0^+} \frac{\sin(x)}{x}\cdot\frac{\sin(x)}{\cos(x)} \\ &= -\lim_{x\to 0^+} \frac{\sin(x)}{x}\cdot \lim_{x\to 0^+} \frac{\sin(x)}{\cos(x)} = -1\cdot 0 = 0\end{aligned}$$

- Another way to approach this problem is to use a Taylor series. Since we are approaching zero we can use a Maclaurin series to rewrite one of the functions. Note that:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

  I leave this to you to check (just a matter of finding the Taylor series for $\sin(x)$ at $a = 0$). Plugging this into the limit provides:

$$\begin{aligned}\lim_{x\to 0^+} \sin(x)\ln(4x) &= \lim_{x\to 0^+} \ln(4x)\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right] \\ &= \lim_{x\to 0^+} x\ln(4x) - \frac{1}{3!}\lim_{x\to 0^+} x^3\ln(4x) + \frac{1}{5!}\lim_{x\to 0^+} x^5\ln(4x) - \frac{1}{7!}\lim_{x\to 0^+} x^7\ln(4x) + \dots\end{aligned}$$

  Now it is a matter of determining what happens to each limit above. Consider evaluating the limit:

$$\lim_{x\to 0^+} x^k\ln(4x) = \lim_{x\to 0^+} \frac{\ln(4x)}{x^{-k}} \to \frac{-\infty}{+\infty}$$

  where $k > 1$. Using L'Hospital's rule we have:

$$\lim_{x\to 0^+} \frac{\ln(4x)}{x^{-k}} = \lim_{x\to 0^+} \frac{x^{-1}}{-kx^{-k-1}} = -\frac{1}{k}\lim_{x\to 0^+} x^k = 0$$

  Thus, each limit in the infinite sum above will go to zero implying the overall limit approaches zero.

(2) Find the minimal cost needed to make an open top rectangular box where the length is twice the width and volume is 10m^3. Assume that the cost for materials is $\frac{\$10}{\text{m}^2}$ for the base and $\frac{\$6}{\text{m}^2}$ for the sides.

- The volume is given as $V = lwh$ for a rectangular box and for our situation we actually have $10 = 2w^2h$. A little reorganization provides:

$$h = \frac{5}{w^2}$$

For the cost function we must have:

$$\begin{aligned}\text{Cost} &= (\text{Cost of materials for base} \times \text{Area of base}) + (\text{Cost of materials for side} \times \text{Area of sides})\\ C &= (10lw) + 6(2lh + 2wh)\\ &= 20w^2 + 24wh + 12wh\\ &= 20w^2 + 36wh\end{aligned}$$

Now using the relationship between the height and width from above the cost function takes the form:

$$C = 20w^2 + \frac{180}{w}$$

Since the cost function is now in terms of only one variable we can take the derivative:

$$C' = 40w - \frac{180}{w^2}$$

and identify the critical points:

$$0 = \frac{40w^3 - 180}{w^2} \implies 40w^3 - 180 = 0 \implies w = \left(\frac{9}{2}\right)^{\frac{1}{3}}$$

Using the First Derivative Test you can also show that this point is exactly the minimum of our function. Take note that $w = 0$ is also a critical point since it makes the derivative undefined, but definitely not of interest to us since none of the dimensions can be zero to have a volume of 10m^3. Thus, the minimal cost is:

$$C\left(\left(\frac{9}{2}\right)^{\frac{1}{3}}\right) = \$20\left(\frac{9}{2}\right)^{\frac{2}{3}} + \$180\left(\frac{2}{9}\right)^{\frac{1}{3}} \approx \$163.541$$