Quiz 8 Solutions

MATH 103A
August 23, 2018

(1) (Q) Use the method of residues over the upper half plane to evaluate:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+2x+2}$$

(A) Let $\gamma_1 = [-R, R]$ and $\gamma_2 = \{Re^{i\theta} \in \mathbb{C} \mid 0 \leq \theta \leq \pi\}$ with $\gamma = \gamma_1 \cup \gamma_2$. By definition we have the breakup:

$$\int\limits_{\gamma} \frac{\mathrm{d}z}{z^2+2z+2} = \int\limits_{\gamma_1} \frac{\mathrm{d}z}{z^2+2z+2} + \int\limits_{\gamma_2} \frac{\mathrm{d}z}{z^2+2z+2}$$

Now we evaluate two of these integrals to extract the value of the remaining one:

* By the quadratic formula we have the poles at $z = \frac{-2\pm 2i}{2} = -1 \pm i$. Furthermore, only one of these poles lies in the upper half plane, namely $z = -1+i$. Now by the Residue Theorem:

$$\begin{aligned}
\int\limits_{\gamma} \frac{\mathrm{d}z}{z^2+2z+2} &= 2\pi i \cdot \operatorname*{Res}_{z=-1+i} \frac{1}{z^2+2z+2} \\
&= 2\pi i \cdot \lim_{z\to -1+i} (z-(-1+i)) \cdot \frac{1}{z^2+2z+2} \\
&= 2\pi i \cdot \lim_{z\to -1+i} \frac{1}{z-(-1-i)} = 2\pi i \cdot \frac{1}{2i} \\
&= \pi
\end{aligned}$$

* For the right hand side, along γ_2 we want to use the fact that:

$$\left|\frac{1}{z^2+2z+2}\right| = \left|\frac{1}{R^2e^{2i\theta}+2Re^{i\theta}+2}\right| \leq \left|\frac{1}{Re^{i\theta}(Re^{i\theta}+2)}\right| \leq \frac{1}{R^2}$$

which forces:

$$\left|\int\limits_{\gamma_2} \frac{\mathrm{d}z}{z^2+2z+2}\right| \leq \int\limits_{\gamma_2} \left|\frac{1}{z^2+2z+2}\right| \mathrm{d}z \leq 2\pi R \cdot \frac{1}{R^2} = \frac{2\pi}{R}$$

With this we can conclude:

$$\lim_{R\to\infty} \int\limits_{\gamma_2} \frac{\mathrm{d}z}{z^2+2z+2} = 0$$

Combining the work above leads to:

$$\begin{aligned}
\lim_{R\to\infty} \int\limits_{\gamma} \frac{\mathrm{d}z}{z^2+2z+2} &= \lim_{R\to\infty} \int\limits_{\gamma_1} \frac{\mathrm{d}z}{z^2+2z+2} + \lim_{R\to\infty} \int\limits_{\gamma_2} \frac{\mathrm{d}z}{z^2+2z+2} \\
\pi &= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+2x+2}
\end{aligned}$$

(2) (Q) Use the method of residues over the upper half plane to evaluate (*Hint: To complexify the integral replace* $\sin(ax)$ *with* e^{iaz} *and only at the very end expand this exponential into real and imaginary components using Euler's formula*):

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}\,\mathrm{d}x \quad \text{where} \quad a \in \mathbb{R}_{\geq 0}$$

(A) Let $\gamma_1 = [-R, R]$ and $\gamma_2 = \{Re^{i\theta} \in \mathbb{C} \mid 0 \leq \theta \leq \pi\}$ with $\gamma = \gamma_1 \cup \gamma_2$. By definition we have the breakup:

$$\int\limits_{\gamma} \frac{ze^{iaz}}{z^4+4}\,\mathrm{d}z = \int\limits_{\gamma_1} \frac{ze^{iaz}}{z^4+4}\,\mathrm{d}z + \int\limits_{\gamma_2} \frac{ze^{iaz}}{z^4+4}\,\mathrm{d}z$$

Now we evaluate two of these integrals to extract the value of the remaining one:

* By factorization $z^4+4 = (z-(1-i))(z-(1+i))(z-(-1+i))(z-(-1-i))$, showing that two of the poles are in the upper half plane. Now by the Residue Theorem:

$$\int\limits_{\gamma} \frac{ze^{iaz}}{z^4+4}\,\mathrm{d}z = 2\pi i\left(\operatorname*{Res}_{z=-1+i} \frac{ze^{iaz}}{z^4+4} + \operatorname*{Res}_{z=1+i} \frac{ze^{iaz}}{z^4+4}\right)$$

$$= 2\pi i\left(\lim_{z\to -1+i} \frac{ze^{iaz}}{(z-(1-i))(z-(1+i))(z-(-1-i))} + \lim_{z\to 1+i} \frac{ze^{iaz}}{(z-(1-i))(z-(-1+i))(z-(-1-i))}\right)$$

$$= 2\pi i\left(\frac{(-1+i)e^{ia(-1+i)}}{8+8i} + \frac{(1+i)e^{ia(1+i)}}{-8+8i}\right) = \frac{\pi}{2}e^{-a}\sin(a)i$$

* For the right hand side, along γ_2 we want to use the fact that:

$$\left|\frac{ze^{iaz}}{z^4+1}\right| = \left|\frac{Re^{i\theta}e^{iaRe^{i\theta}}}{R^4e^{4i\theta}+1}\right| \leq \left|\frac{e^{iaR(\cos(\theta)+i\sin(\theta))}}{R^3}\right| = \left|\frac{e^{-aR\sin(\theta)}}{R^3}\right| \leq \frac{1}{R^3}$$

  which forces:

$$\left|\int\limits_{\gamma_2} \frac{ze^{iaz}}{z^4+1}\,\mathrm{d}z\right| \leq \int\limits_{\gamma_2} \left|\frac{ze^{iaz}}{z^4+1}\right|\,\mathrm{d}z \leq 2\pi R\cdot\frac{1}{R^3} = \frac{2\pi}{R^2}$$

  With this we can conclude:

$$\lim_{R\to\infty} \int\limits_{\gamma_2} \frac{z\sin(az)}{z^4+1}\,\mathrm{d}z = 0$$

Combining the work above leads to:

$$\lim_{R\to\infty} \int\limits_{\gamma} \frac{ze^{iaz}}{z^4+1}\,\mathrm{d}z = \lim_{R\to\infty} \int\limits_{\gamma_1} \frac{ze^{iaz}}{z^4+1}\,\mathrm{d}z + \lim_{R\to\infty} \int\limits_{\gamma_2} \frac{ze^{iaz}}{z^4+1}\,\mathrm{d}z$$

$$\frac{\pi}{2}e^{-a}\sin(a)i = \int_{-\infty}^{\infty} \frac{xe^{iax}}{x^4+1}\,\mathrm{d}x$$

$$\frac{\pi}{2}e^{-a}\sin(a)i = \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^4+1}\,\mathrm{d}x + i\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+1}\,\mathrm{d}x$$

Comparison of the real and imaginary parts provides:

$$\int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^4+1}\,\mathrm{d}z = 0$$

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+1}\,\mathrm{d}z = \frac{\pi}{2}e^{-a}\sin(a)$$