Week 7 Attendance Solutions

MATH 23A

(1) (Q) For a function $f(x,y)$, the general Taylor expansion centered about the point (a,b) is written down as:

$$f(x,y) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\frac{1}{n_1!n_2!}\frac{\partial^{n_1+n_2}f}{\partial x^{n_1}\partial y^{n_2}}\bigg|_{(a,b)}\right)(x-a)^{n_1}(y-b)^{n_2}$$

Let us consider the scenario in which our function can be decomposed as $f(x,y) = g(x) + h(y)$ for some single-variable functions $g(x)$ and $h(y)$. How is the Taylor expansion of $f(x,y)$ centered about the point (a,b) related to the Taylor expansions of $g(x)$ and $h(y)$ centered about the points $x = a$ and $y = b$ respectively? What happens to the coefficients that involve mixed partial derivatives?

(A) The answer is as simple as saying:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\frac{1}{(n_1+n_2)!}\frac{\partial^{n_1+n_2}f}{\partial x^{n_1}\partial y^{n_2}}\bigg|_{(a,b)}\right)(x-a)^{n_1}(y-b)^{n_2} = \sum_{n_1=0}^{\infty}\frac{g^{(n_1)}(a)}{n_1!}(x-a)^{n_1} + \sum_{n_2=0}^{\infty}\frac{h^{(n_2)}(b)}{n_2!}(y-b)^{n_2}$$

It may seem a little weird at first since the left-hand side has mixed partial derivatives, but a closer look reveals that:

$$\frac{\partial^{n_1+n_2}f}{\partial x^{n_1}\partial y^{n_2}}(g(x)+h(y)) = \frac{\partial^{n_1}f}{\partial x^{n_1}}h^{(n_2)}(y) = 0 = \frac{\partial^{n_2}f}{\partial y^{n_2}}g^{(n_1)}(x) = \frac{\partial^{n_1+n_2}f}{\partial y^{n_2}\partial x^{n_1}}(g(x)+h(y))$$

all mixed partial derivatives vanish as it holds for any choice of n_1, n_2 where $n_1, n_2 \neq 0$ simultaneously. The moral of the story is that if a function can be separated under addition into two single-variable functions, then the overall Taylor expansion is given by summing up the two Taylor expansions of the single-variable functions.

(2) (Q) A C^2 function (twice-differentiable function) always has the property that mixed partial derivatives commute, i.e.:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{where} \quad f : \mathbb{R}^n \to \mathbb{R} \quad \text{is given by} \quad f(x_1, \ldots, x_n) = y$$

for any $1 \leq i, j, \leq n$. Does there exist a C^2 function, g, such that it has the partial derivatives:

$$g_x(x,y,z) = x^2, \quad g_y(x,y,z) = 2xy, \quad \text{and} \quad g_z(x,y,z) = \sin(z)$$

(A) To ensure that such a g exists we have to check that partial derivatives commute, i.e.:

$$\begin{aligned} g_{xy}(x,y,z) &= 0 \neq 2y = g_{yx}(x,y,z) \\ g_{xz}(x,y,z) &= 0 = 0 = g_{zx}(x,y,z) \\ g_{yz}(x,y,z) &= 0 = 0 = g_{zy}(x,y,z) \end{aligned}$$

Since one of the pairs does not match up, it is enough to say that such a g does not exist.