Midterm 1 - Review 1 Solutions

MATH 11A - Discussion Sections C & F

(1) To determine if a sequence is convergent or divergent we look for the limit:

(a)
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n^2} = 0+0=0 \implies \text{Convergent}$$

(b)
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{e^n+e^{-n}}{e^{2n}-1} = \lim_{n\to\infty}\frac{e^n+e^{-n}}{e^{2n}-1}\cdot\frac{e^{-2n}}{e^{-2n}} = \lim_{n\to\infty}\frac{e^{-n}+e^{-3n}}{1-e^{-2n}} = \frac{0+0}{1+0} = 0 \implies \text{Convergent}$$

(c)
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2-\left(\frac{1}{10}\right)^n\right) = 2-\lim_{n\to\infty}\left(\frac{1}{10}\right)^n = 2-0=2 \implies \text{Convergent}$$

(d)
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(3^{-n}-4^{-n}\right) = \lim_{n\to\infty}\left(\frac{1}{3}\right)^n - \lim_{n\to\infty}\left(\frac{1}{4}\right)^n = 0-0=0 \implies \text{Convergent}$$

(e)
$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{n^6+1}{n^7-n^5+3n^4+9} = \lim_{n\to\infty}\frac{n^6+1}{n^7-n^5+3n^4+9}\cdot\frac{n^{-7}}{n^{-7}}\\ &= \lim_{n\to\infty}\frac{n^{-1}+n^{-7}}{1-n^{-2}+3n^{-3}+9n^{-7}} = \frac{0+0}{1-0+0+0} = 0 \implies \text{Convergent}\end{aligned}$$

(2) For these limits it is a matter of algebraic manipulation:

(a)
$$\lim_{x\to1}\frac{5x^2-7x+1}{x^2-1} = \lim_{x\to1}\frac{(5x-2)(x-1)}{(x+1)(x-1)} = \lim_{x\to1}\frac{5x-2}{x+1} = \frac{3}{2}$$

(b)
$$\lim_{x\to0}\frac{\sqrt{x^2+9}-3}{x^2} = \lim_{x\to0}\frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \lim_{x\to0}\frac{(x^2+9)-(9)}{x^2(\sqrt{x^2+9}+3)} = \lim_{x\to0}\frac{1}{\sqrt{x^2+9}+3} = \frac{1}{6}$$

(c)
$$\lim_{x\to0}\frac{\frac{1}{x+3}-\frac{1}{3}}{x} = \lim_{x\to0}\frac{\frac{3-(x+3)}{3(x+3)}}{x} = \lim_{x\to0}\frac{-1}{3(x+3)} = -\frac{1}{9}$$

(d)
$$\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}\cdot\frac{e^{-3x}}{e^{-3x}} = \lim_{x\to\infty}\frac{1-e^{-6x}}{1+e^{-6x}} = \frac{1-0}{1+0} = 1$$

(e) Ignore

(3) Most limits can be evaluated using simple trickery and continuity properties:

(a)
$$\lim_{x\to1}\ln(\cos(x-1)) = \ln(\cos(0)) = \ln(1) = 0$$

(b) Let $u=2x$:
$$\lim_{x\to0}\frac{1-\cos(2x)}{3x} = \lim_{u\to0}\frac{1-\cos(u)}{\frac{3}{2}u} = \frac{2}{3}\lim_{u\to0}\frac{1-\cos(u)}{u} = 0$$

(c)
$$\lim_{x\to0}\frac{\csc(x)-\cot(x)}{x\csc(x)} = \lim_{x\to0}\frac{\csc(x)-\cot(x)}{x\csc(x)}\cdot\frac{\sin(x)}{\sin(x)} = \lim_{x\to0}\frac{1-\cos(x)}{x} = 0$$

(d)
$$\lim_{x\to\frac{\pi}{2}}\frac{\tan^2(x)+1}{\sec^2(x)} = \lim_{x\to\frac{\pi}{2}}\frac{\sec^2(x)}{\sec^2(x)} = 1$$

(e) As $x\to\infty$ we have the exponential decay to 0, but the cosine alternates. Thus the limit DNE.

(4) Use algebraic manipulation to rewrite some of the limits and evaluate others using the Squeeze Theorem:

(a)

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$
$$-x^4 \le x^4 \sin\left(\frac{1}{x}\right) \le x^4$$
$$\lim_{x\to 0} -x^4 \le \lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^4$$
$$0 \le \lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) \le 0 \implies \lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) = 0$$

(b)

$$-1 \le \cos(10x) \le 1$$
$$-e^{-x} \le e^{-x}\cos(10x) \le e^{-x}$$
$$\lim_{x\to\infty} -e^{-x} \le \lim_{x\to\infty} e^{-x}\cos(10x) \le \lim_{x\to\infty} e^{-x}$$
$$0 \le \lim_{x\to\infty} e^{-x}\cos(10x) \le 0 \implies \lim_{x\to\infty} e^{-x}\cos(10x) = 0$$

(c) Let $u = 3x$:

$$\lim_{x\to 0}\frac{\sin^2(3x)}{x^2} = \left(\lim_{x\to 0}\frac{\sin(3x)}{x}\right)^2 = \left(\lim_{u\to 0}\frac{\sin(u)}{\frac{u}{3}}\right)^2 = 9\left(\lim_{u\to 0}\frac{\sin(u)}{u}\right)^2 = 9(1^2) = 9$$

(d)

$$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4} = \lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}\cdot\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5} = \lim_{x\to -4}\frac{(x^2+9)-(25)}{(x+4)(\sqrt{x^2+9}+5)}$$
$$= \lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)} = \lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5} = -\frac{4}{5}$$

(e) Tangent has a vertical asymptote at $x = \frac{\pi}{2}$. If you observe the graph long enough you should be able to realize that from the left side the function approaches $+\infty$.

(5) Use algebraic manipulation to compute the limits:

(a)

$$\lim_{x\to -1}\frac{x^2-6x}{x^2-5x-6} = \lim_{x\to -1}\frac{x(x-6)}{(x+1)(x-6)} = \lim_{x\to -1}\frac{x}{x+1} = \text{DNE}$$

(b)

$$\lim_{h\to 0}\frac{(9+h)^3-729}{h} = \lim_{h\to 0}\frac{(h^3+27h^2+243h+729)-(729)}{h} = \lim_{h\to 0}(h^2+27h+243) = 243$$

(c)

$$\lim_{h\to 0}\frac{\sqrt{64+h}-8}{h}\lim_{h\to 0}\frac{\sqrt{64+h}-8}{h}\cdot\frac{\sqrt{64+h}+8}{\sqrt{64+h}+8} = \lim_{h\to 0}\frac{(64+h)-(64)}{h(\sqrt{64+h}+8)} = \lim_{h\to 0}\frac{1}{\sqrt{64+h}+8} = \frac{1}{16}$$

(d)

$$\lim_{x\to 0}\left(\frac{2}{x}-\frac{2}{x^2+x}\right) = \lim_{x\to 0}\frac{2(x+1)-2}{x(x+1)} = \lim_{x\to 0}\frac{2}{x+1} = 2$$

(e) First note that $\frac{10x+10}{|x+1|} = 10\cdot\frac{x+1}{|x+1|}$. We already know that the limit as $x \to 0$ of $\operatorname{sgn}(x) = \frac{x}{|x|}$ is undefined since the two sides approach different values. By the same reasoning the given function will approach an undefined value as $x \to -1$.

(6) Approach by using the definition:

(a)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2+1)-(x^2+1)}{h}$$
$$= \lim_{h\to 0} \frac{(x^2+2hx+h^2)-(x^2+1)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

(b)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^3-2(x+h))-(x^3-2x)}{h}$$
$$= \lim_{h\to 0} \frac{(x^3+3hx^2+3h^2x+h^3-2x-2h)-(x^3-2x)}{h} = \lim_{h\to 0}(3x^2+3hx+h^2-2) = 3x^2-2$$

(c)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \lim_{h\to 0} \frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(d)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{4(x+h)+2}{(x+h)+7} - \frac{4x+2}{x+7}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(4x+4h+2)(x+7)-(x+h+7)(4x+2)}{(x+h+7)(x+7)}$$
$$= \lim_{h\to 0} \frac{26h}{h(x+h+7)(x+7)} = \lim_{h\to 0} \frac{26}{(x+h+7)(x+7)} = \frac{26}{(x+7)^2}$$

(e)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{1-3(x+h)}-\sqrt{1-3x}}{h} \cdot \frac{\sqrt{1-3(x+h)}+\sqrt{1-3x}}{\sqrt{1-3(x+h)}+\sqrt{1-3x}}$$
$$= \lim_{h\to 0} \frac{(1-3x-3h)-(1-3x)}{h(\sqrt{1-3x-3h}+\sqrt{1-3x})} = \lim_{h\to 0} -\frac{3}{\sqrt{1-3x-3h}+\sqrt{1-3x}} = -\frac{3}{2\sqrt{1-3x}}$$

(7) First determine the slope (derivative at the point) and find the equation of the line. To determine the derivative I leave the work to you since it is the same process as (6).

(a) First the slope:

$$f'(x) = 3x^2-2 \implies f'(4) = 3(16)-2 = 46$$

Now the equation of the tangent line is given as:

$$y-57 = 46(x-4)$$

(b) First the slope:

$$f'(x) = \frac{1}{2\sqrt{x}} \implies f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

Now the equation of the tangent line is given as:

$$y-1 = \frac{1}{2}(x-1)$$

(c) First the slope:

$$f'(x) = -\frac{1}{x^2} \implies f'(2) = -\frac{1}{(2)^2} = -\frac{1}{4}$$

Now the equation of the tangent line is given as:

$$y-\frac{3}{2} = -\frac{1}{4}(x-2)$$

(8) Each component of the piecewise function is continuous, thus we only have to ensure that the function is continuous at the split. So at $x = a$ we must satisfy:

$$\begin{aligned} a^2 + 2a &= -1 \\ a^2 + 2a + 1 &= 0 \\ (a+1)^2 &= 0 \\ a &= -1 \end{aligned}$$

(9) First notice the following:

$$g(x) = \frac{x^2 + x - 6}{|x-2|} = \frac{(x+3)(x-2)}{|x-2|} = (x+3) \cdot \frac{x-2}{|x-2|}$$

(a) Approaching from the right side will have the fraction on the right evaluate to 1. Thus:

$$\lim_{x \to 2^+} g(x) = 5$$

(a) Approaching from the left side will have the fraction on the right evaluate to -1. Thus:

$$\lim_{x \to 2^-} g(x) = -5$$

(a) Since the right and left limits do not match we have

$$\lim_{x \to 2} g(x) = \text{DNE}$$

(10) (a) Since each component is continuous we just have to check the split:

$$\begin{aligned} (3)^3 + k &= k(3) - 5 \\ 27 + k &= 3k - 5 \\ 32 &= 2k \\ 16 &= k \end{aligned}$$

(b) Notice the following:

$$\frac{3x^2 + 2x - 8}{x+2} = \frac{(3x-4)(x+2)}{x+2} = 3x - 4$$

Of course the above only holds true if $x \neq -2$. Therefore, for the given piecewise function we would want $k = -4$.