Calculus 2
Recitation 1

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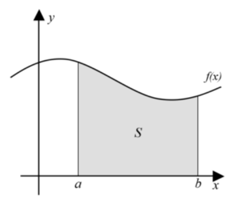
LEFT AND RIGHT APPROXIMATIONS BY RECTANGLES

Definition 1 (Area under a Curve)**.** Consider the area under an arbitrary curve $f(x)$ between $x = a$ and $x = b$:



In order to calculate the area under the curve it is natural to approach by defining upper and lower estimates. In this course, the functions will be monotonic, strictly increasing or decreasing, and therefore the upper and lower sums will be the left and right sums. Approximations can vary depending on the number of partitions taken, so to make it general, lets just say there are going to n partitions. Now define the width of each rectangle as:

$$\Delta x = \frac{b-a}{n}$$

Now that the width of each rectangle is defined, find the height of each rectangle:

x	a	$a+\Delta x$	$a+2\Delta x$	$\dots$	$b-\Delta x$	b
f(x)	$f(a)$	$f(a+\Delta x)$	$f(a+2\Delta x)$	$\dots$	$f(b-\Delta x)$	$f(b)$

To finish off, the left and right approximations are defined as:

$$\textbf{Area}_L = \sum_{i=0}^{n-1} f(x_i)\Delta x_i = \Delta x \sum_{i=1}^{n-1} f(x_i)$$
$$\textbf{Area}_R = \sum_{i=1}^{n} f(x_i)\Delta x_i = \Delta x \sum_{i=2}^{n} f(x_i)$$

where factoring out the Δx_i is justified since the width is going to be the same for all the rectangles. The left sum says to add up the values of $f(x)$ from the first values until the second to last one. The right sum says to add up the values from the second term until the last one. With this you can also define the difference between the estimates:

$$\left|\textbf{Area}_R - \textbf{Area}_L\right| = \left|\Delta x \sum_{i=1}^{n} f(x_i) - \Delta x \sum_{i=0}^{n-1} f(x_i)\right| = \Delta x \left|f(b) - f(a)\right|$$

where factoring out Δx outside of the absolute values is allowed since it is strictly positive. From this you can notice that the smaller the width gets the smaller the difference between these two estimates gets.

TRAPEZOIDAL AND MIDPOINT APPROXIMATIONS BY RECTANGLES

Definition 2 (Area under a Curve)**.** Once more consider the area under an arbitrary curve $f(x)$ between $x = a$ and $x = b$, but this time the goal is to refine the approximations so that the attained values are closer to the exact area. The first such approximation is the trapezoidal approximation. While this approximation also has its own definition as a sum, the preferable formula is:

$$\textbf{Area}_{TRAP} = \frac{\textbf{Area}_R + \textbf{Area}_L}{2}$$

so in other words the trapezoidal approximation is just the average of the left and right approximations. The midpoint approximation cannot be calculated as easily so much that it even requires a new table of values:

x	$\frac{(a)+(a+\Delta x)}{2}$	$\frac{(a+\Delta x)+(a+2\Delta x)}{2}$	$\frac{(a+2\Delta x)+(a+3\Delta x)}{2}$	$\ldots$	$\frac{(b-\Delta x)+(b)}{2}$
f*(x)	$f(\frac{(a)+(a+\Delta x)}{2})$	$f(\frac{(a+\Delta x)+(a+2\Delta x)}{2})$	$f(\frac{(a+2\Delta x)+(a+3\Delta x)}{2})$	$\ldots$	$f(\frac{(b-\Delta x)+(b)}{2})$

All that is left to do is to define the midpoint approximation:

$$\textbf{Area}_{MID} = \sum_{i=1}^{n} f^*(x_i)\Delta x_i = \Delta x \sum_{i=1}^{n} f^*(x_i)$$

where factoring out the Δx_i is justified since the width is going to be the same for all the rectangles and is exactly the same width as the one used in the left and right approximations.

USING PROPERTIES OF A FUNCTION TO TELL THE BEHAVIOR OF THE APPROXIMATION

Definition 3 (Overestimate vs. Underestimate)**.** Depending on the given function $f(x)$, the four different approximations may be either underestimates or overestimates. The following tables characterize the approximations based on the monotonic behavior and concavity of $f(x)$:

Area$_L$

	Concave Up	Concave Down
Increasing	Underestimate	Underestimate
Decreasing	Overestimate	Overestimate

Area$_R$

	Concave Up	Concave Down
Increasing	Overestimate	Overestimate
Decreasing	Underestimate	Underestimate

Area$_{TRAP}$

	Concave Up	Concave Down
Increasing	Overestimate	Underestimate
Decreasing	Overestimate	Underestimate

Area$_{MID}$

	Concave Up	Concave Down
Increasing	Underestimate	Overestimate
Decreasing	Underestimate	Overestimate

Example 1. *Approximate the area under $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using five partitions for all the approximations.*

Solution 1. *First calculate the width of each rectangle:*

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{5} = 1$$

and now the table of values that are going to be used for the left and right approximations:

x	*0*	*1*	*2*	*3*	*4*	*5*
f(x)	*25*	*24*	*21*	*16*	*9*	*0*

Using the above, the left and right approximations are:

$$\boxed{\begin{aligned} \textbf{\textit{Area}}_L &= \Delta x \sum_{i=0}^{n-1} f(x_i) = (1)\Big[25 + 24 + 21 + 16 + 9\Big] = 95 \\ \textbf{\textit{Area}}_R &= \Delta x \sum_{i=1}^{n} f(x_i) = (1)\Big[24 + 21 + 16 + 9 + 0\Big] = 70 \end{aligned}}$$

Now the trapezoidal approximation is:

$$\boxed{\textbf{\textit{Area}}_{TRAP} = \frac{\textbf{\textit{Area}}_R + \textbf{\textit{Area}}_L}{2} = \frac{70+95}{2} = \frac{165}{2}}$$

To calculate the midpoint approximation, a new table of values is going to be needed:

x	*.5*	*1.5*	*2.5*	*3.5*	*4.5*
f*(x)	*24.75*	*22.75*	*18.75*	*12.75*	*4.75*

Using the above the midpoint approximation is:

$$\boxed{\textbf{\textit{Area}}_{MID} = \Delta x \sum_{i=1}^{n} f^*(x_i) = (1)\Big[24.75 + 22.75 + 18.75 + 12.75 + 4.75\Big] = 83.75}$$

Now just to compare, lets find the exact area as an integral:

$$\boxed{\textbf{\textit{Area}} = \int_0^5 (25 - x^2)dx = \left(25x - \frac{x^3}{3}\right)\Bigg|_0^5 = \left(125 - \frac{125}{3}\right) - \Big(0\Big) = \frac{250}{3} \approx 83.33}$$

Example 2. *Approximate the area under $f(x) = e^{-x^2}$ from $x = -2$ to $x = 2$ using four partitions for all the approximations.*

Solution 2. *First calculate the width of each rectangle:*

$$\Delta x = \frac{b-a}{n} = \frac{2-(-2)}{4} = 1$$

and now the table of values that are going to be used for the left and right approximations:

x	*-2*	*-1*	*0*	*1*	*2*
f(x)	e^{-4}	e^{-1}	*1*	e^{-1}	e^{-4}

Using the above, the left and right approximations are:

$$\boxed{\begin{aligned} \textbf{\textit{Area}}_L &= \Delta x \sum_{i=1}^{n-1} f(x_i) = (1)\left[e^{-4} + e^{-1} + 1 + e^{-1}\right] \approx 1.75 \\ \textbf{\textit{Area}}_R &= \Delta x \sum_{i=2}^{n} f(x_i) = (1)\left[e^{-1} + 1 + e^{-1} + e^{-4}\right] \approx 1.75 \end{aligned}}$$

Now the trapezoidal approximation is:

$$\boxed{\textbf{\textit{Area}}_{TRAP} = \frac{\textbf{\textit{Area}}_R + \textbf{\textit{Area}}_L}{2} = \frac{1.75 + 1.75}{2} = 1.75}$$

To calculate the midpoint approximation, a new table of values is going to be needed:

x	*-1.5*	*-.5*	*.5*	*1.5*
f*(x)	$e^{-2.25}$	$e^{-.25}$	$e^{-.25}$	$e^{-2.25}$

Using the above the midpoint approximation is:

$$\boxed{\textbf{\textit{Area}}_{MID} = \Delta x \sum_{i=1}^{n} f^*(x_i) = (1)\left[e^{-2.25} + e^{-.25} + e^{-.25} + e^{-2.25}\right] \approx 1.77}$$

Now just to compare, the exact area as computed by wolfram-alpha is approximately:

$$\boxed{\textbf{\textit{Area}} = \int_{-2}^{2} e^{-x^2} dx \approx 1.76}$$

Common Sums

Definition 4 (Useful Formulas)**.** Before moving into Riemann Sums, it is useful to know how to evaluate a couple of common sums:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Properties of Sums

Definition 5 (Properties)**.** All sums will satisfy the following properties:

$$\sum_{i=1}^{n}[f(x_i) + g(x_i)] = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} g(x_i)$$
$$\sum_{i=1}^{n} cf(x_i) = c\sum_{i=1}^{n} f(x_i)$$
$$\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{m} f(x_i) + \sum_{i=m+1}^{n} f(x_i)$$

Definite Integral and Riemann Sum

Definition 6 (Exact Area)**.** When thinking back to the approximations, the decision has to always be made on how many partitions are taken. Not surprisingly, the more partitions taken, the more accurate the answer. Using this, what happens if $n \to \infty$? Obviously, in the real world, this is not possible, but theoretically there are no issues. As an infinite number of partitions are taken, the exact value of the area under the curve $f(x)$ between $x = a$ and $x = b$ is computed as:

$$\textbf{Area} = \int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

Now similar to the approximations, under the Riemann sum the values needed are:

$$\Delta x = \frac{b-a}{n}$$
$$x_i = a + i\Delta x$$

The First Fundamental Theorem of Calculus

Definition 7 (Direct Evaluation of Definite Integral)**.** Given a continuous and bounded function $f(x)$, the definite integral can be evaluated as:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $x = a$ and $x = b$ represent the bounds of the region and $F(x)$ is the antiderivative of $f(x)$.

Example 3. *Evaluate* $\int_0^3 (x^3 - 6x)dx$ *using a Riemann Sum and by direct evaluation.*

Solution 3.

(i) *From the given integral it is clear that* $a = 0$ *and* $b = 3$*. Using this gives:*

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$
$$x_i = a + i\Delta x = \frac{3i}{n}$$
$$f(x_i) = \Big(\frac{3i}{n}\Big)^3 - 6\Big(\frac{3i}{n}\Big) = \frac{27i^3}{n^3} - \frac{18i}{n}$$

Now evaluation of the Riemann Sum gives:

$$\begin{aligned}
\int_0^3 (x^3 - 6x)dx &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{27i^3}{n^3} - \frac{18i}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \Big[\frac{27}{n^3} \sum_{i=1}^{n} i^3 - \frac{18}{n} \sum_{i=1}^{n} i\Big] \\
&= \lim_{n\to\infty} \frac{3}{n} \Big[\frac{27}{n^3} \Big(\frac{n(n+1)}{2}\Big)^2 - \frac{18}{n}\Big(\frac{n(n+1)}{2}\Big)\Big] \\
&= \lim_{n\to\infty} \frac{3}{n} \Big[\frac{27}{4n}(n^2 + 2n + 1) - 9(n+1)\Big] \\
&= \lim_{n\to\infty} \frac{3}{n} \Big[-\frac{9}{4}n + \frac{27}{4n} + \frac{9}{2}\Big] \\
&= \lim_{n\to\infty} \Big(-\frac{27}{4} + \frac{81}{4n^2} + \frac{27}{2n}\Big) \\
&= \boxed{-\frac{27}{4}}
\end{aligned}$$

(ii) *By direct evaluation:*

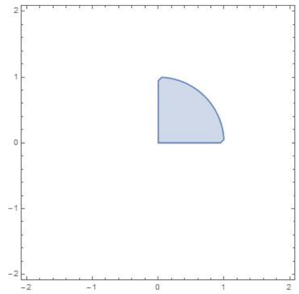
$$\begin{aligned}
\int_0^3 (x^3 - 6x)dx &= \Big(\frac{x^4}{4} - 3x^2\Big)\Big|_0^3 \\
&= \Big(\frac{81}{4} - 27\Big) - \Big(0\Big) \\
&= \boxed{-\frac{27}{4}}
\end{aligned}$$

Geometric Approach to Evaluation of Definite Integrals

Definition 8 (Exact Area)**.** The definite integral is defined as the area under a curve. In certain cases the region can be recognized as a familiar shape with a known formula. This method saves plenty of time avoiding calculations, but can only be applied in certain scenarios.

Example 4. *Evaluate* $\int_0^1 \sqrt{1-x^2}dx$

Solution 4. *The curve in this case is* $f(x) = \sqrt{1-x^2}$ *bounded by* $x = 0$ *and* $x = 1$*. Plotting this* $f(x)$*:*

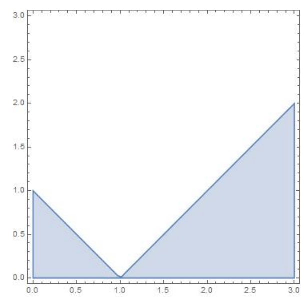


This is just a quarter of a circle with $r = 1$*. The integral will be positive in this scenario, same as the area:*

$$\textbf{\textit{Area}} = \frac{\pi r^2}{4} = \boxed{\frac{\pi}{4}}$$

Example 5. *Evaluate* $\int_0^3 |x-1|dx$

Solution 5. *The curve in this case is* $f(x) = |x-1|$*; bounded by* $x = 0$ *and* $x = 3$*. Plotting this* $f(x)$*:*



The region takes the form of just two right triangles. The integral will be positive in this region, same as the area:

$$\textbf{\textit{Area}} = \frac{b_1 h_1}{2} + \frac{b_2 h_2}{2} = \frac{1}{2} + \frac{4}{2} = \boxed{\frac{5}{2}}$$

Common Antiderivatives

Definition 9 (Antiderivatives)**.**

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int e^x dx = e^x + C$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$	$\int a^x dx = \frac{a^x}{\ln(a)} + C$
$\int \cos(x)dx = \sin(x) + C$	$\int \sin(x)dx = -\cos(x) + C$
$\int \sec^2(x)dx = \tan(x) + C$	$\int \csc^2(x)dx = -\cot(x) + C$
$\int \tan(x)\sec(x)dx = \sec(x) + C$	$\int \cot(x)\csc(x)dx = -\csc(x) + C$
$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$	$\int \frac{dx}{1+x^2} = \arctan(x) + C$

Example 6. *Evaluate* $\int_{-1}^{3} x^5 dx$ *directly.*

Solution 6. *Using the power rule for integration:*

$$\int_{-1}^{3} x^5 dx = \frac{x^6}{6}\Big|_{-1}^{3} = \frac{729}{6} - \frac{1}{6} = \boxed{\frac{364}{3}}$$

Example 7. *Evaluate* $\int_{\pi}^{2\pi} \cos(\theta) d\theta$ *directly.*

Solution 7. *From the table:*

$$\int_{\pi}^{2\pi} \cos(\theta) d\theta = \sin(\theta)\Big|_{\pi}^{2\pi} = \sin(2\pi) - \sin(\pi) = \boxed{0}$$

Example 8. *Evaluate* $\int_{1}^{9} \frac{1}{2x} dx$ *directly.*

Solution 8. *From the table:*

$$\int_{1}^{9} \frac{1}{2x} dx = \frac{1}{2} \ln|x|\Big|_{1}^{9} = \frac{1}{2}\ln(9) - \frac{1}{2}\ln(1) = \boxed{\frac{\ln(9)}{2}}$$

Example 9. *Evaluate* $\int_{-1}^{1} e^{u+1} du$ *directly.*

Solution 9. *From the table:*

$$\int_{-1}^{1} e^{u+1} du = e\int_{-1}^{1} e^u du = e\Big(e^u\Big|_{-1}^{1}\Big) = e\Big(e - e^{-1}\Big) = \boxed{e^2 - 1}$$

Example 10. *Evaluate* $\int_{0}^{\frac{\pi}{4}} \frac{1+\cos(\,x)}{\cos^2(x)} dx$ *directly.*

Solution 10. *From the table:*

$$\begin{aligned}
\int_{0}^{\frac{\pi}{4}} \frac{1+\cos^2(x)}{\cos^2(x)} dx &= \int_{0}^{\frac{\pi}{4}} \Big(\frac{1}{\cos^2(x)} + 1\Big) dx \\
&= \int_{0}^{\frac{\pi}{4}} \sec^2(x) dx + \int_{0}^{\frac{\pi}{4}} dx \\
&= \Big(\tan(x) + x\Big)\Big|_{0}^{\frac{\pi}{4}} \\
&= \Big(\tan\Big(\frac{\pi}{4}\Big) + \frac{\pi}{4}\Big) - \Big(\tan(0) + 0\Big) \\
&= \boxed{1 + \frac{\pi}{4}}
\end{aligned}$$

Example 11. *Evaluate* $\int \frac{\sin(2x)}{\sin(x)} dx$

Solution 11. *By simplifying and using the table:*

$$\begin{aligned}
\int \frac{\sin(2x)}{\sin(x)} dx &= \int \frac{2\sin(x)\cos(x)}{\sin(x)} dx \\
&= 2\int \cos(x) dx \\
&= 2(\sin(x) + C) \\
&= \boxed{2\sin(x) + D}
\end{aligned}$$