Applications of Divergence & Stokes' Theorem

MATH 23B

Physical Interpretation of the Curl

Let $\overrightarrow{V}$ represent a velocity vector field for a fluid and take some $p \in \mathbb{R}^3$. Associate to the point the closed disc:

$$\mathbb{D}_\rho(p) = \{w \in \mathbb{P} \mid \|p - w\| \leq \rho\} \text{ where } \mathbb{P} \text{ is some plane containing } p$$

Assuming the vector field is well-defined on the disc, we have by Stokes' Theorem:

$$\oiint_{\mathbb{D}_\rho(p)} (\nabla \times \overrightarrow{V}) \cdot \mathrm{d}\overrightarrow{S} = \oint_{\partial \mathbb{D}_\rho(p)} \overrightarrow{V} \cdot \mathrm{d}\overrightarrow{s}$$

By the Mean Value Theorem there exists some $q \in \mathbb{D}_\rho(p)$ such that:

$$\oiint_{\mathbb{D}_\rho(p)} (\nabla \times \overrightarrow{V}) \cdot \mathrm{d}\overrightarrow{S} = \left[(\nabla \times \overrightarrow{V})\Big|_q \cdot \overrightarrow{n} \right] \mathrm{Area}(\mathbb{D}_\rho(p))$$

Now by direct calculation:

$$\lim_{\rho \to 0} \frac{1}{\mathrm{Area}(\mathbb{D}_\rho(p))} \oint_{\partial \mathbb{D}_\rho(p)} \overrightarrow{V} \cdot \mathrm{d}\overrightarrow{s} = \lim_{\rho \to 0} \frac{1}{\mathrm{Area}(\mathbb{D}_\rho(p))} \oiint_{\mathbb{D}_\rho(p)} (\nabla \times \overrightarrow{V}) \cdot \mathrm{d}\overrightarrow{S} = \lim_{\rho \to 0} \frac{1}{\mathrm{Area}(\mathbb{D}_\rho(p))} \left[\left[(\nabla \times \overrightarrow{V})\Big|_q \cdot \overrightarrow{n} \right] \mathrm{Area}(\mathbb{D}_\rho(p)) \right]$$

$$= \lim_{\rho \to 0} \left[(\nabla \times \overrightarrow{V})\Big|_q \cdot \overrightarrow{n} \right] = \left[(\nabla \times \overrightarrow{V})\Big|_q \cdot \overrightarrow{n} \right]$$

From this we can justify the "rotation" description of the curl. Since the line integral represents the net velocity of the fluid around $\partial \mathbb{D}_\rho(p)$, then it must be that the dot product between the curl of the vector field and the normal corresponds to the turning/rotating effect of the fluid around the axis of the normal vector. It follows that if the curl of the vector field vanishes then the fluid is not rotating which is why we call such a vector field *irrotational*.

Physical Interpretation of the Divergence

Let $\overrightarrow{V}$ represent the velocity vector field for a fluid and take some $p \in \mathbb{R}^3$. Associate to the point the closed ball:

$$\mathbb{B}_\rho(p) = \{w \in \mathbb{R}^3 \mid \|p - w\| \leq \rho\}$$

Assuming the vector field is well-defined on the disc, we have by Divergence Theorem:

$$\oiint_{\partial \mathbb{B}_\rho(p)} \overrightarrow{V} \cdot \mathrm{d}\overrightarrow{S} = \iiint_{\mathbb{B}_\rho(p)} \nabla \cdot \overrightarrow{V} \, \mathrm{d}V$$

By the Mean Value Theorem there exists some $q \in \mathbb{B}_\rho(p)$ such that:

$$\iiint_{\mathbb{B}_\rho(p)} \nabla \cdot \overrightarrow{V} \, \mathrm{d}V = \left[(\nabla \cdot \overrightarrow{V})\Big|_q \right] \mathrm{Vol}(\mathbb{B}_\rho(p))$$

Now by direct calculation:

$$\lim_{\rho \to 0} \frac{1}{\mathrm{Vol}(\mathbb{B}_\rho(p))} \oiint_{\partial \mathbb{B}_\rho(p)} \overrightarrow{V} \cdot \mathrm{d}\overrightarrow{S} = \lim_{\rho \to 0} \frac{1}{\mathrm{Vol}(\mathbb{B}_\rho(p))} \iiint_{\mathbb{B}_\rho(p)} (\nabla \cdot \overrightarrow{V}) \, \mathrm{d}V = \lim_{\rho \to 0} \frac{1}{\mathrm{Vol}(\mathbb{B}_\rho(p))} \left[\left[(\nabla \cdot \overrightarrow{V})\Big|_q \right] \mathrm{Vol}(\mathbb{B}_\rho(p)) \right]$$

$$= \lim_{\rho \to 0} \left[(\nabla \cdot \overrightarrow{V})\Big|_q \right] = (\nabla \cdot \overrightarrow{V})\Big|_q$$

From this we can justify the "flow" description of the curl. Since the surface integral represents the net flux of the fluid through the surface $\partial \mathbb{B}_\rho(p)$, then it must be that the divergence of the vector field corresponds to the net flow of the fluid at a point. It follows that if the divergence of the vector field vanishes then the fluid is flowing out as much as it is flowing in to the surface which is why we call such a vector field *incompressible* or *divergence-free*.

COULOMB'S LAW $\implies$ GAUSS' LAW

The electric field, $\overrightarrow{E}(\overrightarrow{r})$, of a single stationary point charge is given by:

$$\overrightarrow{E}(\overrightarrow{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

where $\overrightarrow{r}$ is the position vector, $r = \|\overrightarrow{r}\|$, $r\hat{r} = \overrightarrow{r}$, q is the charge, and ε_0 is the permittivity of free space. Now for a given charge distribution $\rho(\overrightarrow{s})$ the above can be generalized to a continuous spectrum:

$$\overrightarrow{E}(\overrightarrow{r}) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_W \frac{\rho(\overrightarrow{s})(\overrightarrow{r}-\overrightarrow{s})}{\|\overrightarrow{r}-\overrightarrow{s}\|^3}\,\mathrm{d}^3V$$

where the d^3V denotes that this integral is applied to all components of the vector. Taking the divergence we arrive at:

$$\nabla\cdot\overrightarrow{E}(\overrightarrow{r}) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_W \left(\nabla\cdot\frac{\rho(\overrightarrow{s})(\overrightarrow{r}-\overrightarrow{s})}{\|\overrightarrow{r}-\overrightarrow{s}\|^3}\right)\mathrm{d}^3V$$

Notice that the divergence is taken with respect to $\overrightarrow{r}$, implying $\overrightarrow{s}$ acts as a constant. Therefore, it suffices to just compute:

$$\nabla\cdot\frac{\overrightarrow{r}}{r^3} = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\cdot\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \frac{3}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

The above is certainly true if $\overrightarrow{r}\neq 0$. To handle the exception it takes a bit of extra work to show:

$$\iiint\limits_W \left(\nabla\cdot\frac{\overrightarrow{r}}{r^3}\right)\mathrm{d}V = \oiint\limits_{\partial W}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} = 4\pi$$

if $\overrightarrow{0}\in W$. To accomplish this we want to delete a small neighborhood around the origin and use the Divergence Theorem:

$$\begin{aligned}
0 &= \iiint\limits_{W\setminus\mathbb{B}_\rho(\overrightarrow{0})}\left(\nabla\cdot\frac{\overrightarrow{r}}{r^3}\right)\mathrm{d}V = \oiint\limits_{\partial(W\setminus\mathbb{B}_\rho(\overrightarrow{0}))}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} = \oiint\limits_{\partial W}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} - \oiint\limits_{\partial\mathbb{B}_\rho(\overrightarrow{0})}\frac{\overrightarrow{r}}{r^3}\cdot\frac{\overrightarrow{r}}{r}\,\mathrm{d}S\\
&= \oiint\limits_{\partial W}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} - \oiint\limits_{\partial\mathbb{B}_\rho(\overrightarrow{0})}\frac{\rho^2}{\rho^4}\,\mathrm{d}S = \oiint\limits_{\partial W}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} - \frac{1}{\rho^2}\oiint\limits_{\partial\mathbb{B}_\rho(\overrightarrow{0})}\mathrm{d}S\\
&= \oiint\limits_{\partial W}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} - \frac{1}{\rho^2}\mathrm{Area}(\partial\mathbb{B}_\rho(\overrightarrow{0})) = \oiint\limits_{\partial W}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} - \frac{1}{\rho^2}\cdot 4\pi\rho^2\\
&= \oiint\limits_{\partial W}\frac{\overrightarrow{r}}{r^3}\cdot\mathrm{d}\overrightarrow{S} - 4\pi
\end{aligned}$$

If familiar with the Dirac delta "function" then we can use this to also formulate:

$$\nabla\cdot\frac{\overrightarrow{r}}{r^3} = 4\pi\delta(\overrightarrow{r})$$

While the Dirac delta may not be a standard function, it is actually still integrable when viewed as a distribution:

$$\nabla\cdot\overrightarrow{E}(\overrightarrow{r}) = \frac{1}{\varepsilon_0}\iiint\limits_W \rho(\overrightarrow{s})\delta(\overrightarrow{r}-\overrightarrow{s})\,\mathrm{d}^3V = \frac{\rho(\overrightarrow{r})}{\varepsilon_0}$$

where we used the property:

$$\int_{-\infty}^{\infty} f(x)\delta(x_0 - x)\,\mathrm{d}x = f(x_0)$$

Consequently the above can be reworked so as to derive Coulomb's Law from Gauss' Law implying the two are equivalent statements.

GAUSS' LAW: DIFFERENTIAL FORM $\implies$ INTEGRAL FORM

Assuming the differential form of Gauss' Law, namely:

$$\nabla \cdot \overrightarrow{E}(\overrightarrow{r}) = \frac{\rho(\overrightarrow{r})}{\varepsilon_0}$$

for an electric field arising from a continuous charge distribution, we have by the Divergence Theorem:

$$\Phi_{\overrightarrow{E}} = \oiint\limits_{\partial\Sigma} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{S} = \iiint\limits_{\Sigma} \nabla \cdot \overrightarrow{E}\, \mathrm{d}V = \frac{1}{\varepsilon_0} \iiint\limits_{\Sigma} \rho(\overrightarrow{r})\, \mathrm{d}V = \frac{Q(\Sigma)}{\varepsilon_0}$$

where $\partial\Sigma$ is any closed surface that encloses the volume Σ and $Q(\Sigma)$ represents the charge inside the volume. This conclusion can be stated as saying that *the flux of the electric field through a closed surface is directly proportional to the charge stored inside the region bound by the surface*. Of course you can also start with the integral form and derive the differential form implying that the two are equivalent.

FARADAY'S LAW: DIFFERENTIAL FORM $\implies$ INTEGRAL FORM

For time-dependent electric, $\overrightarrow{E}$, and magnetic, $\overrightarrow{B}$, fields we have Faraday's Law stating:

$$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

Now assuming that we can bring the time-derivative under the integral in combination with Stokes' Theorem we arrive at:

$$-\frac{\partial \Phi_{\overrightarrow{B}}}{\partial t} = -\frac{\partial}{\partial t} \oiint\limits_{S} \overrightarrow{B} \cdot \mathrm{d}\overrightarrow{S} = \oiint\limits_{S} -\frac{\partial \overrightarrow{B}}{\partial t} \cdot \mathrm{d}\overrightarrow{S} = \oiint\limits_{S} (\nabla \times \overrightarrow{E}) \cdot \mathrm{d}\overrightarrow{S} = \oint\limits_{\partial S} \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{s} = \mathcal{E}$$

where $\Phi_{\overrightarrow{B}}$ is the magnetic flux though the closed surface S and $\mathcal{E}$ is the induced electromotive force. Of course you can also start with the integral form and derive the differential form implying that the two are equivalent.

AMPERE'S LAW: DIFFERENTIAL FORM $\implies$ INTEGRAL FORM

Given a vector field for the acting electric current, $\overrightarrow{J}$, and a magnetic field, $\overrightarrow{B}$, Ampere's Law states that:

$$\nabla \times \overrightarrow{B} = \mu_0 \overrightarrow{J}$$

where μ_0 is the permeability of free space. Now by Stokes' Theorem:

$$\oint\limits_{\partial S} \overrightarrow{B} \cdot \mathrm{d}\overrightarrow{s} = \oiint\limits_{S} (\nabla \times \overrightarrow{B}) \cdot \mathrm{d}\overrightarrow{S} = \mu_0 \oiint\limits_{S} \overrightarrow{J} \cdot \mathrm{d}\overrightarrow{S} = \mu_0 I_{\text{enc}}$$

where I_{enc} represents the total current passing through a closed surface S. Of course you can also start with the integral form and derive the differential form implying that the two are equivalent.