Quiz 4 Solutions

MATH 100
October 29, 2018

(1) (Q) Let $A = \{n \in \mathbb{Z} \mid n \equiv 2 \pmod 3\}$ and $B = \{n \in \mathbb{Z} \mid n \equiv 1 \pmod 2\}$.

(a) Describe the elements of the set $A - B$.

(b) Prove that if $n \in A \cap B$, then $n^2 \equiv 1 \pmod{12}$.

(A)

(a) Elements satisfying the condition for A are of the form $n = 2+3k$ for $k \in \mathbb{Z}$ and so $A = \{\dots, -1, 2, 5, 8, 11, \dots\}$. At the same time elements in B are of the form $n = 1+2j$ for $j \in \mathbb{Z}$ implying $B = \{\dots, -1, 1, 3, 5, 7, 9, \dots\}$. By observation B is the set of odd integers letting us know that to find the difference $A - B$ we remove all of the odd elements:

$$\begin{aligned} A - B &= \{\dots, -4, 2, 8, 14, \dots\} \\ &= \{n \in \mathbb{Z} \mid n \equiv 2 \pmod 3 \text{ and } n \not\equiv 1 \pmod 2\} \\ &= \{n \in \mathbb{Z} \mid n \equiv 2 \pmod 3 \text{ and } n \equiv 0 \pmod 2\} \end{aligned}$$

(b) If $n \in A \cap B$, then it must be an odd integer inside the set A. Since $n = 2 + 3k \in A$ includes both even and odd integers, we can reduce down to only the odd ones by setting $n = 5 + 6k \in A \cap B$. It follows directly that:

$$n^2 = (5 + 6k)^2 = 25 + 60k + 36k^2$$

and consequently $n^2 \equiv 1 \pmod{12}$.

(2) (Q) Prove that $\sqrt{2}$ is irrational.

(A) We aim to prove the statement by a contradiction. Assume that $\sqrt{2} \in \mathbb{Q}$ and so $\sqrt{2} = \frac{p}{q}$ where p, q are coprime. Note that p and q cannot be simultaneously even as that would break the coprime condition. Now we proceed by noticing that $2 = \frac{p^2}{q^2}$, or rather $p^2 = 2q^2$. This implies that p^2 is even and consequently p is even. By design $p = 2k$ for some $k \in \mathbb{Z}$ and allows us to rewrite $p^2 = 2q^2$ as $2k^2 = q^2$. It directly follows that q^2 is even and so is q. This is a contradiction as both p and q being even implies that p and q are not coprime, contrary to the initial assumption. Therefore, $\sqrt{2}$ is irrational.