# Calculus 2
# Recitation 3

Nathan Marianovsky

## INTEGRATION BY PARTS

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**Definition 1** (Integration by Parts)**.** The formula for integration by parts reads as:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \quad \text{and} \quad \int u\,dv = uv - \int v\,du$$

for definite integrals and indefinite integrals, respectively. The trouble with this method is knowing how to tell what $u$ and $dv$ should be. Look at the following:

| L | ogarithmic |
|---|---|
| I | nverse |
| P | olynomial |
| E | xponential |
| T | rigonometric |

Inverse refers to trigonometric functions such as $\arctan(x), \arccos(x)$, and $\arcsin(x)$. Any problem involving integration by parts will usually have any combination of two from the above table. The one that is higher takes precedence on being $u$, the rest is usually bundled with the $dv$.

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**Example 1.** *Evaluate $\int_{-1}^2 xe^{6x}dx$*

**Solution 1.** *Because the integrand is a polynomial and exponential multiplied, the polynomial takes precedence as u:*

$$u = x \quad du = dx$$
$$dv = e^{6x}dx \quad v = \frac{1}{6}e^{6x}$$

*Evaluating the integral by parts gives:*

$$\int_{-1}^2 xe^{6x}dx = \frac{x}{6}e^{6x}\Big|_{-1}^2 - \frac{1}{6}\int_{-1}^2 e^{6x}dx$$
$$= \left(\frac{x}{6}e^{6x} - \frac{1}{36}e^{6x}\right)\Big|_{-1}^2$$
$$= \left(\frac{1}{3}e^{12} - \frac{1}{36}e^{12}\right) - \left(-\frac{1}{6}e^{-6} - \frac{1}{36}e^{-6}\right)$$
$$= \boxed{\frac{11}{36}e^{12} + \frac{7}{36}e^{-6}}$$

**Example 2.** *Evaluate* $\int x\sqrt{x+1}dx$

**Solution 2.** *This can actually be accomplished two ways:*

(i) *First consider integration by parts. The function inside is the product of a polynomial and a term with a non-integer exponent. Identify the parts as:*

$$u = x \quad du = dx$$
$$dv = (x+1)^{\frac{1}{2}}dx \quad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

*The integral now becomes:*

$$\int x\sqrt{x+1}dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}dx$$
$$= \boxed{\frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C_1}$$

(ii) *Now by a simple substitution:*

$$u = x + 1$$
$$du = dx$$

*the integral becomes:*

$$\int x\sqrt{x+1}dx = \int (u-1)\sqrt{u}du$$
$$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du$$
$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}$$
$$= \boxed{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C_2}$$

*These two results have shown something rather interesting. That even though both methods of integration produce solutions, they produce different solutions, or so you would think so. In reality these two answers are the same because there exists a family of antiderivatives for an indefinite integral. Thus these two answers are off of each other just by a constant. The easiest way to check this fact is to find the difference between the two solutions.*

**Example 3.** *Evaluate* $\int \ln(x)dx$

**Solution 3.** *To approach this by parts consider that the function inside the is the multiplication of $\ln(x)$, a logarithmic function, and 1, a polynomial. The parts will take the form:*

$$u = \ln(x) \quad du = \frac{1}{x}dx$$
$$dv = dx \quad v = x$$

*The integral now becomes:*

$$\int \ln(x)dx = x\ln(x) - \int dx$$
$$= \boxed{x\ln(x) - x + C}$$

**Example 4.** *Evaluate $\int x^2 e^x dx$*

**Solution 4.** *Begin by identifying the parts:*

$$u = x^2 \quad du = 2xdx$$
$$dv = e^x dx \quad v = e^x$$

*The integral now becomes:*

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

*Unfortunately this problem will require integration by parts once more. This time the parts are:*

$$u = x \quad du = dx$$
$$dv = e^x dx \quad v = e^x$$

*The solution take the form:*

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$
$$= x^2 e^x - 2\left(xe^x - \int e^x dx\right)$$
$$= x^2 e^x - 2xe^x - 2e^x + C$$
$$= \boxed{(x^2 - 2x - 2)e^x + C}$$

**Example 5.** *Evaluate $\int_{-1}^{1} \arctan(x)dx$*

**Solution 5.** *In this case the function inside is the multiplication of an inverse trigonometric function, $\arctan(x)$, and a polynomial, $1$. Therefore, the parts are:*

$$u = \arctan(x) \quad du = \frac{1}{1+x^2}dx$$
$$dv = dx \quad v = x$$

*The integral now becomes:*

$$\int_{-1}^{1} \arctan(x)dx = x\arctan(x)\Big|_{-1}^{1} - \int_{-1}^{1} \frac{x}{1+x^2}dx$$
$$= \arctan(1) + \arctan(-1) - \int_{-1}^{1} \frac{x}{1+x^2}dx$$
$$= -\int_{-1}^{1} \frac{x}{1+x^2}dx$$

*To evaluate the remaining integral, make the substitution:*

$$u = 1 + x^2$$
$$du = 2xdx \rightarrow \frac{du}{2} = dx$$

*Reevaluating the bounds and plugging in gives:*

$$-\int_{-1}^{1} \frac{x}{1+x^2}dx = -\frac{1}{2}\int_{2}^{2} \frac{du}{u}$$
$$= \boxed{0}$$

*All of this work could have actually been avoided knowing that $\arctan(x)$ is an odd function and the integral of an odd function over a symmetrical region is always 0.*

**Example 6.** *Evaluate* $\int \frac{\ln(x)}{x^2}dx$

**Solution 6.** *Begin by identifying the parts:*

$$u = \ln(x) \quad du = x^{-1}dx$$
$$dv = x^{-2}dx \quad v = -x^{-1}$$

*The integral now becomes:*

$$\int \frac{\ln(x)}{x^2}dx = -\frac{\ln(x)}{x} + \int x^{-2}dx$$
$$= -\frac{\ln(x)}{x} - \frac{1}{x} + C$$
$$= \boxed{-\frac{\ln(x) + 1}{x} + C}$$

**Example 7.** *Evaluate* $\int_1^2 (\ln(x))^2 dx$

**Solution 7.** *Begin by identifying the parts:*

$$u = (\ln(x))^2 \quad du = \frac{2\ln(x)}{x}dx$$
$$dv = dx \quad v = x$$

*The integral now becomes:*

$$\int_1^2 (\ln(x))^2 dx = x(\ln(x))^2 \Big|_1^2 - 2\int_1^2 \ln(x)dx$$
$$= \left( x(\ln(x))^2 - 2x\ln(x) + 2x \right)\Big|_1^2$$
$$= \boxed{2(\ln(2))^2 - 4\ln(2) + 2}$$

**Example 8.** *Evaluate* $\int \cos(x)e^x dx$

**Solution 8.** *The inside composes of trigonometric and exponential functions. Therefore, the parts are:*

$$u = e^x \quad du = e^x dx$$
$$dv = \cos(x)dx \quad v = \sin(x)$$

*The integral now becomes:*

$$\int \cos(x)e^x dx = \sin(x)e^x - \int \sin(x)e^x dx$$

*Integration by parts is needed once more, this time the parts are:*

$$u = e^x \quad du = e^x dx$$
$$dv = \sin(x)dx \quad v = -\cos(x)$$

*The integral becomes:*

$$\int \cos(x)e^x dx = \sin(x)e^x - \int \sin(x)e^x dx$$
$$= \sin e^x - \left( -\cos(x)e^x + \int \cos(x)e^x dx \right) = \sin e^x + \cos(x)e^x - \int \cos(x)e^x dx$$

*Solving for the integral gives:*

$$\int \cos(x)e^x dx = \boxed{\frac{1}{2}(\sin(x) + \cos(x))e^x + C}$$

**Example 9.** *Evaluate* $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx$

**Solution 9.** *The above has a combination of a trigonometric function,* $\arctan$, *and a polynomial,* $1$. *Therefore, the parts are:*

$$u = \arctan\left(\frac{1}{x}\right) \quad du = -\frac{1}{1+x^2}dx$$
$$dv = dx \quad v = x$$

*The integral now becomes:*

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = x\arctan\left(\frac{1}{x}\right)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{1+x^2}dx$$
$$= \left[x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln|1+x^2|\right]\Big|_1^{\sqrt{3}}$$
$$= \left(\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2}\ln(4)\right) - \left(\arctan(1) + \frac{1}{2}\ln(2)\right)$$
$$= \boxed{\left(\frac{1}{2\sqrt{3}} - \frac{1}{4}\right)\pi + \frac{1}{2}\ln(2)}$$

**Example 10.** *Evaluate* $\int_1^4 e^{\sqrt{x}}dx$

**Solution 10.** *First a substitution:*

$$w = x^{\frac{1}{2}}$$
$$dw = \frac{1}{2}x^{-\frac{1}{2}}dx \rightarrow 2x^{\frac{1}{2}}dw = dx \rightarrow 2wdw = dx$$

*Reevaluating the bounds and plugging in gives:*

$$\int_1^4 e^{\sqrt{x}}dx = 2\int_1^2 we^w dw$$

*which is a familiar integration by parts problem. Identify the parts as:*

$$u = w \quad du = dw$$
$$dv = e^w dw \quad v = e^w$$

*Now the integral becomes:*

$$2\int_1^2 we^w dw = 2\left[we^w\Big|_1^2 - \int_1^2 e^w dw\right]$$
$$= 2\left[we^w - e^w\right]\Big|_1^2$$
$$= 2\left[\left(2e^2 - e^2\right) - \left(e - e\right)\right]$$
$$= \boxed{2e^2}$$