Calculus 1 with Precalculus
Recitation 1

Nathan Marianovsky

MATHEMATICAL NOTATION

Definition 1 (Common Notation)**.** Here is a list of commonly used mathematical symbols which make writing down mathematical expressions easier:

Notation	Meaning
i	$\sqrt{-1}$
$\in$	Belongs to
$\forall$	For all
$\exists$	There exists
s.t.	Such that
iff	If and only if
$\{$ objects in the set $\mid$ conditions $\}$	Set Builder Notation

ALGEBRA REVIEW

Definition 2 (Types of Numbers)**.** All known numbers can be grouped into different sets of numbers that describe their very structure. The simplest group is the set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

which typically does not include zero. A modification to the natural numbers which does include zero is the set of whole numbers:

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, 5, \ldots\}$$

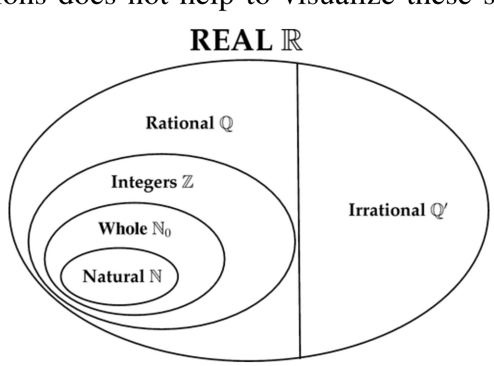
Following the whole numbers, the next largest set consists of the integers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

which is the negative integers added onto the whole numbers. Next are the rational numbers defined as:

$$\mathbb{Q} = \left\{ \left. \frac{p}{q} \;\right|\; p, q \in \mathbb{Z} \right\}$$

where p and q are any two integers giving rise to fractions. Any number that cannot be written down as a ratio of two integers is called an irrational number. Common examples of the irrationals include π and the natural number e. Now combining the rational and irrational numbers gives rise to the real numbers, $\mathbb{R}$. Perhaps writing down pure definitions does not help to visualize these sets, so consider the following:



Definition 3 (Interval Notation)**.** When dealing with intervals of real numbers, there exists four different scenarios that can arise. Each has a notation to simplify the interval of real numbers:

Interval	Notation
$a < x < b$	(a, b)
$a \leq x < b$	$[a, b)$
$a < x \leq b$	$(a, b]$
$a \leq x \leq b$	$[a, b]$

where $a, b \in \mathbb{R}$. Now note the importance of the interval notation. Whenever the boundary is not included, a parenthesis is used. To include the boundary use a square bracket instead.

Definition 4 (Properties of Exponents)**.** Given any real numbers x and y:

$$x^0 = 1$$
$$x^{-n} = \frac{1}{x^n} \text{ and } x^{\frac{1}{n}} = \sqrt[n]{x}$$
$$x^{\frac{m}{n}} = \left(x^m\right)^{\frac{1}{n}} = \left(x^{\frac{1}{n}}\right)^m$$
$$x^m \cdot x^n = x^{m+n} \text{ and } \frac{x^m}{x^n} = x^{m-n} \text{ iff } x \neq 0$$
$$\left(x^m\right)^n = x^{mn} \text{ and } \left(xy\right)^n = x^n \cdot y^n$$

Example 1. *Evaluate 2^{-3} without using a calculator.*

Solution 1. *Using the properties of exponents from above:*

$$2^{-3} = \frac{1}{2^3} = \boxed{\frac{1}{8}}$$

Example 2. *Evaluate $8^{\frac{2}{3}}$ without using a calculator.*

Solution 2. *Using the properties of exponents from above:*

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = \left(2\right)^2 = \boxed{4}$$

Example 3. *Evaluate $\frac{\sqrt{27}\left(\sqrt{3}\right)^3}{9}$ without using a calculator.*

Solution 3. *Using the properties of exponents from above:*

$$\frac{\sqrt{27}\left(\sqrt{3}\right)^3}{9} = \frac{\sqrt{3^3} \cdot \sqrt{3}^3}{3^2} = \frac{3^{\frac{3}{2}} \cdot 3^{\frac{3}{2}}}{3^2} = \frac{3^3}{3^2} = \boxed{3}$$

Example 4. *Simplify $\left(a^4b^2c^{12}\right)^{-\frac{1}{2}}$ as much as possible assuming $a, b,$ and c are positive real numbers.*

Solution 4. *Using the properties of exponents from above:*

$$\left(a^4b^2c^{12}\right)^{-\frac{1}{2}} = a^{-\frac{4}{2}}b^{-\frac{2}{2}}c^{-\frac{12}{2}} = a^{-2}b^{-1}c^{-6} = \boxed{\frac{1}{a^2bc^6}}$$

Example 5. *Simplify $\sqrt[5]{(a^{24}b^{-8}c^{11})^4}$ as much as possible assuming $a, b,$ and c are positive real numbers.*

Solution 5. *Using the properties of exponents from above:*

$$\sqrt[5]{(a^{24}b^{-8}c^{11})^4} = \left[\left(a^{24}b^{-8}c^{11}\right)^4\right]^{\frac{1}{5}} = \left(a^{24}b^{-8}c^{11}\right)^{\frac{4}{5}} = a^{\frac{96}{5}}b^{-\frac{32}{5}}c^{\frac{44}{5}} = \boxed{\frac{a^{\frac{96}{5}}c^{\frac{44}{5}}}{b^{\frac{32}{5}}}}$$

Definition 5 (Properties of Logarithms)**.** Given any real numbers x, y, and z:

$$\log_x(1) = 0 \quad \text{and} \quad \log_x(x) = 1$$
$$\log_x(yz) = \log_x(y) + \log_x(z)$$
$$\log_x\left(\frac{y}{z}\right) = \log_x(y) - \log_x(z)$$
$$\log_x(y^z) = z\log_x(y)$$
$$\log_x(y) = \frac{\log_z(y)}{\log_z(x)}$$
$$\log_x(y) = z \implies y = x^z$$

In the case where $x = 10$, it is customary to just write $\log$. For the natural base, $x = e$, it is instead written as $\ln$. Another special case is when $x = 2$, which has applications in computer science, is written as $\lg$.

Example 6. *Simplify* $\log(10x^2y^{27})$.

Solution 6.

$$\log(10x^2y^{27}) = \log(10) + \log(x^2) + \log(y^{27}) = \boxed{1 + 2\log(x) + 27\log(y)}$$

Example 7. *Simplify* $\log\left(\left(\frac{x^2-1}{x+1}+1\right)^{42}\right)$.

Solution 7.

$$\log\left(\left(\frac{x^2-1}{x+1}+1\right)^{42}\right) = 42\log\left(\frac{(x+1)(x-1)}{x+1}+1\right) = \boxed{42\log(x)}$$

Example 8. *Solve for x in* $\ln(x^2) = 3$.

Solution 8.

$$\ln(x^2) = 3$$
$$e^{\ln(x^2)} = e^3$$
$$x^2 = e^3$$
$$x = \boxed{\pm e^{\frac{3}{2}}}$$

Example 9. *Solve for x in* $\ln^2(x) + \ln(x) = 3$.

Solution 9. *First to simplify let* $y = \ln(x)$*:*

$$y^2 + y - 3 = 0$$

Now using the quadratic formula:

$$y = -\frac{1}{2} \pm \frac{\sqrt{13}}{2}$$

With y determined, use the original substitution to obtain x:

$$x = e^y = \boxed{e^{-\frac{1}{2}\pm\frac{\sqrt{13}}{2}}}$$

Example 10. *Solve for x in* $\ln^3(x) = \ln(x)$.

Solution 10.

$$\ln^3(x) - \ln(x) = 0$$
$$\ln(x)\Big(\ln^2(x) - 1\Big) = 0$$
$$\ln(x)(\ln(x) - 1)(\ln(x) + 1) = 0$$

Now to solve for x consider each case independently. First consider the fact that there does not exist an x such that $\ln(x) = 0$. Same reasoning for $\ln(x) = -1$. We only care about $\ln(x) = 1$ which provides a solution of $\boxed{x = e}$.

Definition 6 (FOIL)**.** Essentially this is the method of multiplying any two binomials together. Observe:

First (a + b)(c + d) = ac

Outside (a + b)(c + d) = ad

Inside (a + b)(c + d) = bc

Last (a + b)(c + d) = bd

FOIL (a + b)(c + d) = ac + ad + bc + bd

Example 11. *Find the product of* $3x(x-9)$.

Solution 11.

$$3x(x-9) = 3x(x) - 3x(9) = \boxed{3x^2 - 27x}$$

Example 12. *Find the product of* $(x+1)(x+5)$.

Solution 12.

$$(x+1)(x+5) = x(x) + x(5) + 1(x) + 1(5) = \boxed{x^2 + 6x + 5}$$

Example 13. *Find the product of* $(x-1)(x^2+2x-3)$.

Solution 13.

$$(x-1)(x^2+2x-3) = x(x^2) + x(2x) - x(3) - 1(x^2) - 1(2x) + 1(3) = \boxed{x^3 + x^2 - 5x + 3}$$

Example 14. *Find the product of* $(-x-3)(5-3x)$.

Solution 14.

$$(-x-3)(5-3x) = -x(5) + x(3x) - 3(5) + 3(3x) = \boxed{3x^2 + 4x - 15}$$

Example 15. *Find the product of* $(2x^3 + x^2 - 5)(x^2 - x - 3)$.

Solution 15.

$$\begin{aligned}(2x^3 + x^2 - 5)(x^2 - x - 3) &= 2x^3(x^2) - 2x^3(x) - 2x^3(3) + x^2(x^2) - x^2(x) - x^2(3) - 5(x^2) + 5(x) + 5(3) \\ &= \boxed{2x^5 - x^4 - 7x^3 - 8x^2 + 5x + 15}\end{aligned}$$

Definition 7 (Factoring Quadratic Polynomials)**.** To factor any given quadratic polynomial:

$$ax^2 + bx + c$$

we want to find coefficients that will satisfy:

$$ax^2 + bx + c = (k_1x + k_2)(k_3x + k_4)$$

To determine the k's, expand the right side:

$$(k_1x + k_2)(k_3x + k_4) = k_1k_3x^2 + (k_1k_4 + k_2k_3)x + k_2k_4$$

and now matching both sides gives the requirements that:

$$\begin{aligned} a &= k_1k_3 \\ b &= k_1k_4 + k_2k_3 \\ c &= k_2k_4 \end{aligned}$$

So if such k's can be found, the quadratic can be factored. In some cases, the only way to satisfy the above conditions may require imaginary numbers and so we say that the quadratic term is *irreducible* over the real numbers.

Example 16. *Factor the polynomial* $12x^2 - 11x - 15$ *using integer coefficients.*

Solution 16. *Using the above setup for the* k*'s we must satisfy:*

$$\begin{aligned} 12 &= k_1 k_3 \\ -11 &= k_1 k_4 + k_2 k_3 \\ -15 &= k_2 k_4 \end{aligned}$$

Letting $k_1 = 4$, $k_2 = 3$, $k_3 = 3$ *and* $k_4 = -5$ *happens to work perfectly:*

$$12x^2 - 11x - 15 = \boxed{(4x+3)(3x-5)}$$

Example 17. *Factor the polynomial* $3x^2 - x - 14$ *using integer coefficients.*

Solution 17. *Using the above setup for the* k*'s we must satisfy:*

$$\begin{aligned} 3 &= k_1 k_3 \\ -1 &= k_1 k_4 + k_2 k_3 \\ -14 &= k_2 k_4 \end{aligned}$$

Letting $k_1 = 1$, $k_2 = 2$, $k_3 = 3$ *and* $k_4 = -7$ *happens to work perfectly:*

$$12x^2 - 11x - 15 = \boxed{(x+2)(3x-7)}$$

Example 18. *Factor the polynomial* $x^2 + 8x + 15$ *using integer coefficients.*

Solution 18. *Using the above setup for the* k*'s we must satisfy:*

$$\begin{aligned} 1 &= k_1 k_3 \\ 8 &= k_1 k_4 + k_2 k_3 \\ 15 &= k_2 k_4 \end{aligned}$$

Letting $k_1 = 1$, $k_2 = 3$, $k_3 = 1$ *and* $k_4 = 5$ *happens to work perfectly:*

$$x^2 + 8x + 15 = \boxed{(x+3)(x+5)}$$

Example 19. *Factor the polynomial* $2x^2 - x - 15$ *using integer coefficients.*

Solution 19. *Using the above setup for the* k*'s we must satisfy:*

$$\begin{aligned} 2 &= k_1 k_3 \\ -1 &= k_1 k_4 + k_2 k_3 \\ -15 &= k_2 k_4 \end{aligned}$$

Letting $k_1 = 1$, $k_2 = -3$, $k_3 = 2$ *and* $k_4 = 5$ *happens to work perfectly:*

$$2x^2 - x - 15 = \boxed{(x-3)(2x+5)}$$

Example 20. *Factor the polynomial* $7x^2 + 12x + 5$ *using integer coefficients.*

Solution 20. *Using the above setup for the* k*'s we must satisfy:*

$$\begin{aligned} 7 &= k_1 k_3 \\ 12 &= k_1 k_4 + k_2 k_3 \\ 5 &= k_2 k_4 \end{aligned}$$

Letting $k_1 = 1$, $k_2 = 1$, $k_3 = 7$ *and* $k_4 = 5$ *happens to work perfectly:*

$$7x^2 + 12x - 5 = \boxed{(x+1)(7x+5)}$$

Definition 8 (Common Factorization Formulas)**.** Some cases of polynomials are so common that they have their own formulas which are worth memorizing:

$$\begin{aligned}
(A+B)^2 &= A^2 + 2AB + B^2 \\
(A-B)^2 &= A^2 - 2AB + B^2 \\
(A+B)(A-B) &= A^2 - B^2 \\
(A-B)(A^2+AB+B^2) &= A^3 - B^3 \\
(A+B)(A^2-AB+B^2) &= A^3 + B^3
\end{aligned}$$

Example 21. *Factor the polynomial $x^3 - 27$ using integer coefficients.*

Solution 21. *Using the fourth formula from above:*

$$x^3 - 27 = x^3 - 3^3 = \boxed{(x-3)(x^2+3x+9)}$$

Example 22. *Factor the polynomial $x^7 - x^5$ using integer coefficients.*

Solution 22. *Using the third formula from above:*

$$x^7 - x^5 = x^5(x^2-1) = \boxed{x^5(x-1)(x+1)}$$

Example 23. *Factor the polynomial $3(x+2)^3 - 5(x+2)^2$ using integer coefficients.*

Solution 23. *Using the third formula from above:*

$$3(x+2)^3 - 5(x+2)^2 = (x+2)^2\Big[3(x+2) - 5\Big] = \boxed{(x+2)^2(3x+1)}$$

Example 24. *Factor the polynomial $25x^2 - 81$ using integer coefficients.*

Solution 24. *Using the third formula from above:*

$$25x^2 - 81 = (5x)^2 - 9^2 = \boxed{(5x-9)(5x+9)}$$

Example 25. *Factor the polynomial $x^5 + x^2$ using integer coefficients.*

Solution 25. *Using the fifth formula from above:*

$$x^5 + x^2 = x^2(x^3+1) = \boxed{x^2(x+1)(x^2-x+1)}$$

Definition 9 (Completing the Square)**.** Given any quadratic polynomial:

$$ax^2 + bx + c$$

that cannot be factored, completing the square will result in a quadratic of the form:

$$\begin{aligned}
ax^2 + bx + c &= a\Big(x^2 + \frac{b}{a}x + \frac{c}{a}\Big) \\
&= a\Big(x^2 + \frac{b}{a}x + \Big(\frac{b}{2a}\Big)^2 - \Big(\frac{b}{2a}\Big)^2 + \frac{c}{a}\Big) \\
&= a\left(\Big(x + \frac{b}{2a}\Big)^2 + \frac{c}{a} - \Big(\frac{b}{2a}\Big)^2\right) \\
&= a\Big(x + \frac{b}{2a}\Big)^2 + c - \frac{b^2}{4a}
\end{aligned}$$

Example 26. *Solve the quadratic equation* $x^2 + 5x + 4 = 0$ *by completing the square.*

Solution 26.

$$\begin{aligned}
0 &= x^2 + 5x + 4 \\
0 &= x^2 + 5x + \frac{25}{4} + 4 - \frac{25}{4} \\
0 &= \left(x + \frac{5}{2}\right)^2 - \frac{9}{4} \\
\frac{9}{4} &= \left(x + \frac{5}{2}\right)^2 \\
\pm\frac{3}{2} &= x + \frac{5}{2} \\
x &= \boxed{-\frac{5}{2} \pm \frac{3}{2}}
\end{aligned}$$

Example 27. *Solve the quadratic equation* $3x^2 + 5x + 7 = 0$ *by completing the square.*

Solution 27.

$$\begin{aligned}
0 &= 3x^2 + 5x + 7 \\
0 &= x^2 + \frac{5}{3}x + \frac{7}{3} \\
0 &= x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + \frac{7}{3} \\
0 &= \left(x + \frac{5}{6}\right)^2 - \frac{25}{36} + \frac{7}{3} \\
-\frac{59}{36} &= \left(x + \frac{5}{6}\right)^2 \\
\pm\frac{\sqrt{59}}{6}i &= x + \frac{5}{6} \\
x &= \boxed{-\frac{5}{6} \pm \frac{\sqrt{59}}{6}i}
\end{aligned}$$

Example 28. *Solve the quadratic equation* $6x^2 + 17x - 4 = 0$ *by completing the square.*

Solution 28.

$$\begin{aligned}
0 &= 6x^2 + 17x - 4 \\
0 &= x^2 + \frac{17}{6}x - \frac{2}{3} \\
0 &= x^2 + \frac{17}{6}x + \left(\frac{17}{12}\right)^2 - \left(\frac{17}{12}\right)^2 - \frac{2}{3} \\
0 &= \left(x + \frac{17}{12}\right)^2 - \frac{289}{144} - \frac{2}{3} \\
\frac{385}{144} &= \left(x + \frac{17}{12}\right)^2 \\
\pm\frac{\sqrt{385}}{12} &= x + \frac{17}{12} \\
x &= \boxed{-\frac{17}{12} \pm \frac{\sqrt{385}}{12}}
\end{aligned}$$

Example 29. *Solve the quadratic equation* $4x^2 + 3x + 1 = 0$ *by completing the square.*

Solution 29.

$$\begin{aligned} 0 &= 4x^2 + 3x + 1 \\ 0 &= x^2 + \frac{3}{4}x + \frac{1}{4} \\ 0 &= x^2 + \frac{3}{4}x + \left(\frac{3}{8}\right)^2 - \left(\frac{3}{8}\right)^2 + \frac{1}{4} \\ 0 &= \left(x + \frac{3}{8}\right)^2 - \frac{9}{64} + \frac{1}{4} \\ -\frac{7}{4} &= \left(x + \frac{3}{8}\right)^2 \\ \pm\frac{\sqrt{7}}{2}i &= x + \frac{3}{8} \\ x &= \boxed{-\frac{3}{8} \pm \frac{\sqrt{7}}{2}i} \end{aligned}$$

Example 30. *Solve the quadratic equation* $x^2 + 5x + 11 = 0$ *by completing the square.*

Solution 30.

$$\begin{aligned} 0 &= x^2 + 5x + 11 \\ 0 &= x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 11 \\ &= \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + 11 \\ -\frac{19}{4} &= \left(x + \frac{5}{2}\right)^2 \\ \pm\frac{\sqrt{19}}{2}i &= x + \frac{5}{2} \\ x &= \boxed{-\frac{5}{2} \pm \frac{\sqrt{19}}{2}i} \end{aligned}$$

Definition 10 (Quadratic Formula)**.** Completing the square, while a tedious process, is very useful. Observe how easily the quadratic formula can be derived by just completing the square for the general quadratic polynomial:

$$\begin{aligned} 0 &= ax^2 + bx + c \\ 0 &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\ \frac{b^2}{4a} - c &= a\left(x + \frac{b}{2a}\right)^2 \\ \frac{b^2 - 4ac}{4a^2} &= \left(x + \frac{b}{2a}\right)^2 \\ \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} &= x + \frac{b}{2a} \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Example 31. *Use the quadratic formula to solve* $2x^2 + 3x + 1 = 0$.

Solution 31. *In this case* $a = 2$, $b = 3$, *and* $c = 1$ *giving:*

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \boxed{-\frac{3}{4} \pm \frac{1}{4}}$$

Example 32. *Use the quadratic formula to solve* $21x^2 + 11x - 2 = 0$.

Solution 32. *In this case* $a = 21$, $b = 11$, *and* $c = -2$ *giving:*

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \boxed{-\frac{11}{42} \pm \frac{17}{42}}$$

Example 33. *Use the quadratic formula to solve* $2x^3 + 8x^2 - 10x = 0$.

Solution 33. *First:*

$$0 = 2x^3 + 8x^2 - 10x = x(2x^2 + 8x - 10)$$

So $x = 0$ *is one root, now for the remaining use the quadratic formula. In this case* $a = 2$, $b = 8$, *and* $c = -10$ *giving:*

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \boxed{-2 \pm 3}$$

Example 34. *Use the quadratic formula to solve* $3x^2 - x - 14 = 0$.

Solution 34. *In this case* $a = 3$, $b = -1$, *and* $c = -14$ *giving:*

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \boxed{\frac{1}{6} \pm \frac{13}{6}}$$

Example 35. *Use the quadratic formula to solve* $x^2 + 9 = 0$.

Solution 35. *In this case* $a = 1$, $b = 0$, *and* $c = 9$ *giving:*

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \pm\frac{\sqrt{-36}}{2} = \boxed{\pm 3i}$$