Calculus 1 with Precalculus
Recitation 5

Nathan Marianovsky

MORE ON FUNCTIONS

Definition 1 (Composition of Functions)**.** Given two functions $f(x)$ and $g(x)$, a composition of the functions is the binary operation defined as:

$$(f \circ g)(x) = f(g(x))$$
$$(g \circ f)(x) = g(f(x))$$

and in general the two above:

$$(f \circ g)(x) \neq (g \circ f)(x)$$

Example 1. *Find the composite function $f(g(x))$ where:*

$$f(u) = 3u^2 + 2u - 6 \text{ and } g(x) = x + 2$$

Solution 1.

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= 3(x+2)^2 + 2(x+2) - 6 \\ &= 3(x^2 + 4x + 4) + 2x - 2 \\ &= \boxed{3x^2 + 14x + 10} \end{aligned}$$

Example 2. *Find $f(x^2 + 3x - 1)$ where:*

$$f(x) = 2x - 20$$

Solution 2.

$$\begin{aligned} f(x^2 + 3x - 1) &= 2(x^2 + 3x - 1) - 20 \\ &= \boxed{2x^2 + 6x - 22} \end{aligned}$$

Example 3. *Find the composite function $f(g(x))$ where:*

$$f(u) = \sqrt{u+1} \text{ and } g(x) = x^2 - 1$$

Solution 3.

$$\begin{aligned} f(g(x)) &= f(x^2 - 1) \\ &= \sqrt{(x^2 - 1) + 1} \\ &= \sqrt{x^2} \\ &= \boxed{|x|} \end{aligned}$$

Example 4. *Find all x s.t. $f(g(x)) = g(f(x))$ where:*

$$f(x) = \frac{1}{x} \text{ and } g(x) = \frac{4-x}{2+x}$$

Solution 4. *The compositions are given as:*

$$\begin{aligned} f(g(x)) &= f\Big(\frac{4-x}{2+x}\Big) \\ &= \frac{1}{\frac{4-x}{2+x}} \\ &= \boxed{\frac{2+x}{4-x}} \\ g(f(x)) &= g\Big(\frac{1}{x}\Big) \\ &= \frac{4-\frac{1}{x}}{2+\frac{1}{x}} \\ &= \boxed{\frac{4x-1}{2x+1}} \end{aligned}$$

Now to determine where they match:

$$\begin{aligned} \frac{2+x}{4-x} &= \frac{4x-1}{2x+1} \\ (2+x)(2x+1) &= (4x-1)(4-x) \\ 2x^2+5x+2 &= -4x^2+17x-4 \\ 6x^2-12x+6 &= 0 \\ x^2-2x+1 &= 0 \\ (x-1)^2 &= 0 \\ x &= \boxed{1} \end{aligned}$$

Example 5. *Find functions $h(x)$ and $g(u)$ such that $f(x) = g(h(x))$ where:*

$$f(x) = (x-1)^2 + 2(x-1) + 3$$

Solution 5. *Notice that if we set* $\boxed{h(x) = (x-1) \text{ and } g(u) = u^2 + 2u + 3}$:

$$\begin{aligned} f(x) &= g(h(x)) \\ &= g(x-1) \\ &= (x-1)^2 + 2(x-1) + 3 \end{aligned}$$

Example 6. *Find functions $h(x)$ and $g(u)$ such that $f(x) = g(h(x))$ where:*

$$f(x) = \sqrt[3]{2-x} + \frac{4}{2-x}$$

Solution 6. *Notice that if we set* $\boxed{h(x) = 2-x \text{ and } g(u) = \sqrt[3]{u} + \frac{4}{u}}$:

$$\begin{aligned} f(x) &= g(h(x)) \\ &= g(2-x) \\ &= \sqrt[3]{x-2} + \frac{4}{x-2} \end{aligned}$$

Example 7. *Find all x s.t. $f(g(x)) = g(f(x))$ where:*

$$f(x) = \sqrt{x} \text{ and } g(x) = 1 - 3x$$

Solution 7. *The compositions are given as:*

$$\begin{aligned} f(g(x)) &= f(1-3x) \\ &= \boxed{\sqrt{1-3x}} \\ g(f(x)) &= g(\sqrt{x}) \\ &= \boxed{1-3\sqrt{x}} \end{aligned}$$

Now to determine where they match:

$$\begin{aligned} \sqrt{1-3x} &= 1 - 3\sqrt{x} \\ 1 - 3x &= (1-3\sqrt{x})^2 \\ 1 - 3x &= 1 - 6\sqrt{x} + 9x \\ -12x &= -6\sqrt{x} \\ 144x^2 &= 36x \\ 144x^2 - 36x &= 0 \\ x(4x-1) &= 0 \\ x &= \boxed{0} \end{aligned}$$

Example 8. *Find all x s.t. $f(g(x)) = g(f(x))$ where:*

$$f(x) = x^2 + 1 \text{ and } g(x) = 1 - x$$

Solution 8. *The compositions are given as:*

$$\begin{aligned} f(g(x)) &= f(1-x) \\ &= (1-x)^2 + 1 \\ &= x^2 - 2x + 1 + 1 \\ &= \boxed{x^2 - 2x + 2} \\ g(f(x)) &= g(x^2+1) \\ &= 1 - (x^2+1) \\ &= \boxed{-x^2} \end{aligned}$$

Now to determine where they match:

$$\begin{aligned} x^2 - 2x + 2 &= -x^2 \\ 2x^2 - 2x + 2 &= 0 \\ x^2 - x + 1 &= 0 \\ x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} &= \boxed{\frac{1}{2} \pm \frac{\sqrt{3}}{2}i} \end{aligned}$$

Example 9. *Find all x s.t. $f(g(x)) = g(f(x))$ where:*

$$f(x) = \frac{2x+3}{x-1} \text{ and } g(x) = \frac{x+3}{x-2}$$

Solution 9. *The compositions are given as:*

$$\begin{aligned}
f(g(x)) &= f\left(\frac{x+3}{x-2}\right) \\
&= \frac{2\left(\frac{x+3}{x-2}\right)+3}{\left(\frac{x+3}{x-2}\right)-1} \\
&= \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)} \\
&= \boxed{x} \\
g(f(x)) &= g\left(\frac{2x+3}{x-1}\right) \\
&= \frac{\left(\frac{2x+3}{x-1}\right)+3}{\left(\frac{2x+3}{x-1}\right)-2} \\
&= \frac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)} \\
&= \boxed{x}
\end{aligned}$$

Now to determine where they match:

$$\begin{aligned}
x &= x \\
0 &= 0
\end{aligned}$$

which is known as a tautology, statement that is always true, giving the solution for x as $\boxed{(\infty, \infty)}$.

Example 10. *Find functions $h(x)$ and $g(u)$ such that $f(x) = g(h(x))$ where:*

$$f(x) = \frac{1}{x^2+1}$$

Solution 10. *Notice that if we set $\boxed{h(x) = x^2 + 1 \text{ and } g(u) = \frac{1}{u}}$:*

$$\begin{aligned}
f(x) &= g(h(x)) \\
&= g(x^2+1) \\
&= \frac{1}{x^2+1}
\end{aligned}$$

Definition 2 (Difference Quotient)**.** The difference quotient of a function $f(x)$ is the composite function defined as:

$$\frac{f(x+h)-f(x)}{h} \quad \text{where } h \in \mathbb{R}\setminus\{0\}$$

Example 11. *Determine the difference quotient of:*

$$f(x) = 4 - 5x$$

Solution 11.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\Big(4-5(x+h)\Big)-\Big(4-5x\Big)}{h} \\
&= \frac{(4-5x-5h)-(4-5x)}{h} \\
&= \frac{-5h}{h} \\
&= \boxed{-5}
\end{aligned}$$

Example 12. *Determine the difference quotient of:*

$$f(x) = 2x + 3$$

Solution 12.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\Big(2(x+h)+3\Big)-\Big(2x+3\Big)}{h} \\
&= \frac{(2x+2h+3)-(2x+3)}{h} \\
&= \frac{2h}{h} \\
&= \boxed{2}
\end{aligned}$$

Example 13. *Determine the difference quotient of:*

$$f(x) = 4x - x^2$$

Solution 13.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\Big(4(x+h)-(x+h)^2\Big)-\Big(4x-x^2\Big)}{h} \\
&= \frac{(4x+4h-x^2-h^2+2xh)-(4x-x^2)}{h} \\
&= \frac{4h+2xh-h^2}{h} \\
&= \boxed{4+2x-h}
\end{aligned}$$

Example 14. *Determine the difference quotient of:*

$$f(x) = x^2$$

Solution 14.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\Big((x+h)^2\Big)-\Big(x^2\Big)}{h} \\
&= \frac{(x^2+2xh+h^2)-(x^2)}{h} \\
&= \frac{2xh+h^2}{h} \\
&= \boxed{2x+h}
\end{aligned}$$

Example 15. *Determine the difference quotient of:*

$$f(x) = \frac{x}{x+1}$$

Solution 15.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{\left(\frac{(x+h)}{(x+h)+1}\right) - \left(\frac{x}{x+1}\right)}{h} \\
&= \frac{\frac{(x+h)(x+1) - x(x+h+1)}{(x+1)(x+h+1)}}{h} \\
&= \frac{(x^2 + (h+1)x + h) - (x^2 + (h+1)x)}{h(x+1)(x+h+1)} \\
&= \boxed{\frac{1}{(x+1)(x+h+1)}}
\end{aligned}$$