Quiz 6 Solutions

MATH 103A
August 16, 2018

(1) (Q) Find the Laurent series centered about $z = 0$ for the following functions with $\alpha \in \mathbb{R} \setminus \{0\}$:

(a) $f(z) = \frac{1}{\alpha+z}$ on the intervals $|z| < |\alpha|$ and $|\alpha| < |z|$

(b) $f(z) = \frac{z^2}{\alpha^2+z^2}$ on the intervals $|z| < |\alpha|$ and $|\alpha| < |z|$

(A)

(a) · First we have:

$$f(z) = \frac{1}{\alpha + z} = \frac{1}{\alpha} \cdot \frac{1}{1 - \left(- \frac{z}{\alpha} \right)} = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(- \frac{z}{\alpha} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{\alpha^{n+1}}$$

For the region of convergence we aim to use the Ratio Test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ \lim_{n\to\infty} \left| \frac{z^{n+1}}{\alpha^{n+2}} \cdot \frac{\alpha^{n+1}}{z^n} \right| &< 1 \\ \frac{|z|}{|\alpha|} &< 1 \\ |z| &< |\alpha| \end{aligned}$$

· Now we have:

$$f(z) = \frac{1}{\alpha + z} = \frac{1}{z} \cdot \frac{1}{1 - \left(- \frac{\alpha}{z} \right)} = \frac{1}{z} \sum_{n=0}^{\infty} \left(- \frac{\alpha}{z} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^n}{z^{n+1}}$$

For the region of convergence we aim to use the Ratio Test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ \lim_{n\to\infty} \left| \frac{\alpha^{n+1}}{z^{n+2}} \cdot \frac{z^{n+1}}{\alpha^n} \right| &< 1 \\ \frac{|\alpha|}{|z|} &< 1 \\ |\alpha| &< |z| \end{aligned}$$

(b) · First we have:

$$f(z) = \frac{z^2}{\alpha^2 + z^2} = \frac{1}{\alpha^2} \cdot \frac{z^2}{1 - \left(- \frac{z^2}{\alpha^2} \right)} = \frac{z^2}{\alpha^2} \sum_{n=0}^{\infty} \left(- \frac{z^2}{\alpha^2} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+2}}{\alpha^{2n+2}}$$

For the region of convergence we aim to use the Ratio Test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ \lim_{n\to\infty} \left| \frac{z^{2n+4}}{\alpha^{2n+4}} \cdot \frac{\alpha^{2n+2}}{z^{2n+2}} \right| &< 1 \\ \frac{|z|^2}{|\alpha|^2} &< 1 \\ |z| &< |\alpha| \end{aligned}$$

· Now we have:

$$f(z) = \frac{z^2}{\alpha^2 + z^2} = \frac{1}{1 - \left(- \frac{\alpha^2}{z^2} \right)} = \sum_{n=0}^{\infty} \left(- \frac{\alpha^2}{z^2} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{z^{2n}}$$

For the region of convergence we aim to use the Ratio Test:

$$\begin{aligned} \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ \lim_{n \to \infty} \left| \frac{\alpha^{2n+2}}{z^{2n+2}} \cdot \frac{z^{2n}}{\alpha^{2n}} \right| &< 1 \\ \frac{|\alpha|^2}{|z|^2} &< 1 \\ |\alpha| &< |z| \end{aligned}$$

(2) (Q) For $\mathbb{S}^1 = \{ z \in \mathbb{C} \mid \|z\| = 1 \}$ orientated counter-clockwise use a Laurent series to evaluate the following integral with $k \in \mathbb{N}$ (*the answer will be in terms of* k):

$$\int\limits_{\mathbb{S}^1} z^k e^{\frac{1}{z}} \, \mathrm{d}z$$

(A) First we need to make the identification:

$$z^k e^{\frac{1}{z}} = z^k \left(1 + \frac{1}{z} + \frac{1}{2! z^2} + \ldots \right) = z^k \sum_{n=0}^{\infty} \frac{1}{n! z^n} = \sum_{n=0}^{\infty} \frac{1}{n! z^{n-k}}$$

Now note that power functions are holomorphic outside of a single exception which when combined with the Cauchy-Goursat Theorem provides:

$$\int\limits_{\mathbb{S}^1} z^k e^{\frac{1}{z}} \, \mathrm{d}z = \int\limits_{\mathbb{S}^1} \left(\sum_{n=0}^{\infty} \frac{1}{n! z^{n-k}} \right) \mathrm{d}z = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int\limits_{\mathbb{S}^1} \frac{1}{z^{n-k}} \, \mathrm{d}z \right) = \frac{2\pi i}{(k+1)!}$$