Quiz 9 Solutions

MATH 103A
August 28, 2018

(1) (Q) Use the method of residues over the upper half plane to evaluate (*Hint: To complexify the integral replace* $\cos(x)$ *with* e^{iz} *and only at the very end expand this exponential into real and imaginary components using Euler's formula*):

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x+a)^2+b^2}\,\mathrm{d}x \quad \text{where} \quad b \in \mathbb{R}_{>0}$$

(A) Let $\gamma_1 = [-R, R]$ and $\gamma_2 = \{Re^{i\theta} \in \mathbb{C} \mid 0 \le \theta \le \pi\}$ with $\gamma = \gamma_1 \cup \gamma_2$. By definition we have the breakup:

$$\int\limits_{\gamma} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z = \int\limits_{\gamma_1} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z + \int\limits_{\gamma_2} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z$$

Now we evaluate two of these integrals to extract the value of the remaining one:

* Solving for the zeroes of the denominator leads to the values $z = -a \pm bi$. Only one of these lies in the upper half plane and so by the Residue Theorem:

$$\begin{aligned}\int\limits_{\gamma} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z &= 2\pi i \cdot \operatorname*{Res}_{z=-a+bi} \frac{e^{iz}}{(z+a)^2+b^2} \\ &= 2\pi i \cdot \lim_{z\to -a+bi} (z-(-a+bi)) \cdot \frac{e^{iz}}{(z+a)^2+b^2} \\ &= 2\pi i \cdot \frac{e^{i(-a+bi)}}{(-a+bi)-(-a-bi)} = \frac{\pi}{b} e^{-b}(\cos(a) - i\sin(a))\end{aligned}$$

* For the right hand side, along γ_2 we want to use the fact that:

$$\left|\frac{e^{iz}}{(z+a)^2+b^2}\right| = \left|\frac{e^{iRe^{i\theta}}}{(Re^{i\theta}+a)^2+b^2}\right| \le \left|\frac{e^{iR(\cos(\theta)+i\sin(\theta))}}{(Re^{i\theta}+a)^2}\right| \le \left|\frac{e^{-R\sin(\theta)}}{R^2}\right| \le \frac{1}{R^2}$$

which forces:

$$\left|\int\limits_{\gamma_2} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z\right| \le \int\limits_{\gamma_2} \left|\frac{e^{iz}}{(z+a)^2+b^2}\right| \mathrm{d}z \le 2\pi R \cdot \frac{1}{R^2} = \frac{2\pi}{R}$$

With this we can conclude:

$$\lim_{R\to\infty} \int\limits_{\gamma_2} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z = 0$$

Combining the work above leads to:

$$\begin{aligned}\lim_{R\to\infty} \int\limits_{\gamma} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z &= \lim_{R\to\infty} \int\limits_{\gamma_1} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z + \lim_{R\to\infty} \int\limits_{\gamma_2} \frac{e^{iz}}{(z+a)^2+b^2}\,\mathrm{d}z \\ \frac{\pi}{b} e^{-b}(\cos(a) - i\sin(a)) &= \int_{-\infty}^{\infty} \frac{e^{ix}}{(x+a)^2+b^2}\,\mathrm{d}x \\ \frac{\pi}{b} e^{-b}(\cos(a) - i\sin(a)) &= \int_{-\infty}^{\infty} \frac{\cos(x)}{(x+a)^2+b^2}\,\mathrm{d}x + i\int_{-\infty}^{\infty} \frac{\sin(x)}{(x+a)^2+b^2}\,\mathrm{d}x\end{aligned}$$

Comparison of the real and imaginary parts provides:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(x)}{(x+a)^2+b^2}\,\mathrm{d}x &= \frac{\pi}{b} e^{-b}\cos(a) \\ \int_{-\infty}^{\infty} \frac{\sin(x)}{(x+a)^2+b^2}\,\mathrm{d}x &= -\frac{\pi}{b} e^{-b}\sin(a)\end{aligned}$$

(2) (Q) Use the Residue Theorem to evaluate the integral (*Hint: Rewrite the cosine in terms of exponentials and utilize the substitution* $z = e^{i\theta}$):

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a\cos(\theta)} \quad \text{where} \quad a \in (-1, 1)$$

(A) Proceed as suggested:

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a\cos(\theta)} &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + \frac{a}{2}(e^{i\theta} + e^{-i\theta})} \\
&= \int\limits_{\mathbb{S}^1} \frac{\frac{\mathrm{d}z}{iz}}{1 + \frac{a}{2}(z + z^{-1})} \\
&= -\frac{2i}{a} \int\limits_{\mathbb{S}^1} \frac{\mathrm{d}z}{z^2 + \frac{2}{a}z + 1} \\
&= -\frac{2i}{a} \cdot 2\pi i \cdot \operatorname*{Res}_{z = \frac{-\frac{2}{a} + \sqrt{\frac{4}{a^2} - 4}}{2}} \left(\frac{1}{z^2 + \frac{2}{a}z + 1} \right) \\
&= \frac{4\pi}{a} \cdot \frac{1}{\frac{-\frac{2}{a} + \sqrt{\frac{4}{a^2} - 4}}{2} - \frac{-\frac{2}{a} - \sqrt{\frac{4}{a^2} - 4}}{2}} \\
&= \frac{2\pi}{\sqrt{1 - a^2}}
\end{aligned}$$

Notice that during calculation the value $a = 0$ has to be excluded due to a blow-up, however this scenario still matches up with the general pattern nonetheless. Furthermore, when applying the Residue Theorem it is important to note that only one of the roots belongs to the interior of the unit circle.