

Comments on Hypergeometric Differential Equation

Winter 2019

INTRODUCTION

For a homogeneous linear second-order differential equation of the form:

$$y'' + p(x)y' + q(x)y = 0$$

we say that x_0 is a regular singular point if $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at x_0 . For the setup above the functions p and q are restricted to meromorphic functions (functions that have at most countably many isolated singularities).

HYPERGEOMETRIC EQUATION SINGULARITIES

We are interested in looking at an equation of the form:

$$x(1-x)y'' + (\gamma - (1+\alpha+\beta)x)y' - \alpha\beta y = 0$$

for $\alpha, \beta, \gamma \in \mathbb{R}$. In order to identify the singular points we begin by rewriting the above into:

$$y'' + \frac{\gamma - (1+\alpha+\beta)x}{x(1-x)}y' - \frac{\alpha\beta}{x(1-x)}y = 0$$

It is clear from here that $x_0 = 0, 1$ because the functions have singularities there. In order to determine if these are regular singular points we identify $p(x) = \frac{\gamma - (1+\alpha+\beta)x}{x(1-x)}$ and $q(x) = -\frac{\alpha\beta}{x(1-x)}$ to see that:

$$\begin{aligned} x_0 = 0 : \quad xp(x) &= \frac{\gamma - (1+\alpha+\beta)x}{1-x} \quad \text{and} \quad x^2p(x) = -\frac{\alpha\beta x}{1-x} \\ x_0 = 1 : \quad (x-1)p(x) &= \frac{\gamma - (1+\alpha+\beta)x}{x} \quad \text{and} \quad (x-1)^2p(x) = \frac{\alpha\beta(x-1)}{x} \end{aligned}$$

In both cases we see that $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic.

FIRST SINGULAR POINT: $x_0 = 0$

In order to achieve a power series solution here we use the Frobenius Method to write down:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

and directly plug in to obtain:

$$\begin{aligned} 0 &= (x - x^2) \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + (\gamma - (1+\alpha+\beta)x) \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \alpha\beta \sum_{n=0}^{\infty} a_n x^{n+r} \\ 0 &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} \gamma(n+r)a_n x^{n+r-1} \\ &\quad - \sum_{n=0}^{\infty} (1+\alpha+\beta)(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} \alpha\beta a_n x^{n+r} \\ 0 &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r-1)(n+r-2)a_{n-1} x^{n+r-1} + \sum_{n=0}^{\infty} \gamma(n+r)a_n x^{n+r-1} \\ &\quad - \sum_{n=1}^{\infty} (1+\alpha+\beta)(n+r-1)a_{n-1} x^{n+r-1} - \sum_{n=1}^{\infty} \alpha\beta a_{n-1} x^{n+r-1} \\ 0 &= \sum_{n=1}^{\infty} \left[\left[(n+r)(n+r-1) + \gamma(n+r) \right] a_n - \left[(n+r-1)(n+r-2) + (1+\alpha+\beta)(n+r-1) + \alpha\beta \right] a_{n-1} \right] x^{n+r-1} \\ &\quad + \left(r(r-1) + \gamma r \right) a_0 x^{r-1} \end{aligned}$$

To determine the valid values of r we solve the indicial equation:

$$\begin{aligned} 0 &= r(r-1) + \gamma r \\ 0 &= r(r-(1-\gamma)) \\ r &= 0, 1-\gamma \end{aligned}$$

SECOND SINGULAR POINT: $x_0 = 1$

In order to achieve a power series solution here we use the Frobenius Method to write down:

$$y = \sum_{n=0}^{\infty} a_n (x-1)^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-2}$$

and directly plug in to obtain:

$$\begin{aligned} 0 &= (x-x^2) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-2} \\ &\quad + (\gamma - (1+\alpha+\beta)x) \sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r-1} - \alpha\beta \sum_{n=0}^{\infty} a_n (x-1)^{n+r} \\ 0 &= -\left((x-1)^2 + (x-1)\right) \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-2} \\ &\quad + \left((\gamma - (1+\alpha+\beta)) - (1+\alpha+\beta)(x-1)\right) \sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r-1} - \alpha\beta \sum_{n=0}^{\infty} a_n (x-1)^{n+r} \\ 0 &= -\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-1} \\ &\quad + (\gamma - (1+\alpha+\beta)) \sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r-1} - (1+\alpha+\beta) \sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r} \\ &\quad - \alpha\beta \sum_{n=0}^{\infty} a_n (x-1)^{n+r} \\ 0 &= -\sum_{n=1}^{\infty} (n+r-1)(n+r-2) a_{n-1} (x-1)^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-1)^{n+r-1} \\ &\quad + (\gamma - (1+\alpha+\beta)) \sum_{n=0}^{\infty} (n+r) a_n (x-1)^{n+r-1} - (1+\alpha+\beta) \sum_{n=1}^{\infty} (n+r-1) a_{n-1} (x-1)^{n+r-1} \\ &\quad - \alpha\beta \sum_{n=1}^{\infty} a_{n-1} (x-1)^{n+r-1} \\ 0 &= \sum_{n=1}^{\infty} \left[(n+r)(\gamma - n - r - \alpha - \beta) a_n - \left[(n+r-1)(n+r+\alpha+\beta-1) - \alpha\beta \right] a_{n-1} \right] x^{n+r-1} \\ &\quad + \left(-r(r-1) + (\gamma - (1+\alpha+\beta))r \right) a_0 x^{r-1} \end{aligned}$$

To determine the valid values of r we solve the indicial equation:

$$\begin{aligned} 0 &= -r(r-1) + (\gamma - (1+\alpha+\beta))r \\ 0 &= -r^2 + (\gamma - (\alpha+\beta))r \\ 0 &= -r(r - (\gamma - (\alpha+\beta))) \\ r &= 0, \gamma - \alpha - \beta \end{aligned}$$

THIRD SINGULAR POINT: $x_0 = \infty$

To determine if the given differential equation has a singularity at ∞ we aim to use the substitution $w = \frac{1}{x}$ and see if $w = 0$ is a singularity. To rewrite the differential equation using this substitution we need the following:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dw} \cdot \frac{dw}{dx} = -w^2 \frac{dy}{dw} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(-w^2 \cdot \frac{dy}{dw} \right) = w^4 \frac{d^2y}{dw^2} + 2w^3 \frac{dy}{dw}\end{aligned}$$

Plugging this into the given differential equation yields:

$$\begin{aligned}0 &= (x - x^2) \frac{d^2y}{dx^2} + (\gamma - (1 + \alpha + \beta)x) \frac{dy}{dx} - \alpha\beta y \\ 0 &= \left(\frac{1}{w} - \frac{1}{w^2} \right) \left(w^4 \frac{d^2y}{dw^2} + 2w^3 \frac{dy}{dw} \right) + \left(\gamma - (1 + \alpha + \beta) \frac{1}{w} \right) \left(-w^2 \frac{dy}{dw} \right) - \alpha\beta y \\ 0 &= (w^2 - w)(y''w + 2y') + ((1 + \alpha + \beta)w - \gamma w^2)y' - \alpha\beta y \\ 0 &= w^2(w - 1)y'' + ((\alpha + \beta - 1)w + (2 - \gamma)w^2)y' - \alpha\beta y \\ 0 &= y'' + \frac{(\alpha + \beta - 1) + (2 - \gamma)w}{w(w - 1)}y' - \frac{\alpha\beta}{w^2(w - 1)}y\end{aligned}$$

From the above we see that $w_0 = 0$ is a regular singular point as $wp(w) = \frac{(\alpha + \beta - 1) + (2 - \gamma)w}{w - 1}$ and $w^2q(w) = -\frac{\alpha\beta}{w - 1}$ are both analytic at the origin. In order to achieve a power series solution here we use the Frobenius Method to write down:

$$y = \sum_{n=0}^{\infty} a_n w^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r)a_n w^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n w^{n+r-2}$$

and directly plug in to obtain:

$$\begin{aligned}0 &= (w^2 - w) \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n w^{n+r-2} + ((\alpha + \beta - 1)w + (2 - \gamma)w^2) \sum_{n=0}^{\infty} (n+r)a_n w^{n+r-1} - \alpha\beta \sum_{n=0}^{\infty} a_n w^{n+r} \\ 0 &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n w^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n w^{n+r-1} + \sum_{n=0}^{\infty} (\alpha + \beta - 1)\gamma(n+r)a_n w^{n+r} \\ &\quad + \sum_{n=0}^{\infty} (2 - \gamma)(n+r)a_n w^{n+r+1} - \sum_{n=0}^{\infty} \alpha\beta a_n w^{n+r} \\ 0 &= \sum_{n=1}^{\infty} (n+r-1)(n+r-2)a_{n-1} w^{n+r-1} - \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n w^{n+r-1} + \sum_{n=1}^{\infty} (\alpha + \beta - 1)\gamma(n+r-1)a_{n-1} w^{n+r-1} \\ &\quad + \sum_{n=2}^{\infty} (2 - \gamma)(n+r-2)a_{n-2} w^{n+r-1} - \sum_{n=1}^{\infty} \alpha\beta a_{n-1} w^{n+r-1} \\ 0 &= \left(-r(r-1) \right) a_0 w^{r-1} + \left(\left[r(r-1) + (\alpha + \beta - 1)\gamma r - \alpha\beta \right] a_0 - (r+1)ra_1 \right) w^r \\ &\quad + \sum_{n=2}^{\infty} \left[(2 - \gamma)(n+r-2)a_{n-2} + \left[(n+r-1)((n+r-2) + (\alpha + \beta - 1)\gamma) - \alpha\beta \right] a_{n-1} - (n+r)(n+r-1)a_n \right] w^{n+r-1}\end{aligned}$$

To determine the valid values of r we solve the indicial equation:

$$\begin{aligned}0 - r(r-1) \\ r = 0, 1\end{aligned}$$

RADIUS OF CONVERGENCE FOR $x_0 = 0, 1$

In order to determine the radius of convergence for the first two regular singular points we observe the following setups:

- For $x_0 = 0$:

$$0 = \left[(n+r)(n+r-1) + \gamma(n+r) \right] a_n - \left[(n+r-1)(n+r-2) + (1+\alpha+\beta)(n+r-1) + \alpha\beta \right] a_{n-1}$$
$$\frac{a_n}{a_{n-1}} = \frac{(n+r-1)(n+r-2) + (1+\alpha+\beta)(n+r-1) + \alpha\beta}{(n+r)(n+r-1) + \gamma(n+r)}$$

- For $x_0 = 1$:

$$0 = (n+r)(\gamma - n - r - \alpha - \beta) a_n - \left[(n+r-1)(n+r+\alpha+\beta-1) - \alpha\beta \right] a_{n-1}$$
$$\frac{a_n}{a_{n-1}} = \frac{(n+r-1)(n+r+\alpha+\beta-1) - \alpha\beta}{(n+r)(\gamma - n - r - \alpha - \beta)}$$

In both cases we achieve $\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right| = 1$. Recall that the Ratio Test states that given a series $\sum b_n$ it will converge absolutely if $\lim_{n \rightarrow \infty} \left| \frac{b_n}{b_{n-1}} \right| < 1$. For both of the cases above we take $b_n = a_n x^{n+r-1}$ so that:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{b_n}{b_{n-1}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{a_n x^{n+r}}{a_{n-1} x^{n+r-1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} x \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right| \\ &= |x| \end{aligned}$$

Thus, to guarantee convergence we must have $|x| < 1$, i.e. the radius of convergence is 1.