Calculus 1 with Precalculus
Recitation 2

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FRACTIONAL PROPERTIES

Definition 1 (Properties)**.** The following rules summarize how to operate with any two given fractions:

(a)
$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

(b)
$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)=\frac{ac}{bd}$$

(c)
$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Example 1. *Simplify* $\left(\frac{x^2-8}{x}\right)\left(\frac{x^2-3x}{x-2}\right)$.

Solution 1. *Using property (b) from above:*

$$\left(\frac{x^2-8}{x}\right)\left(\frac{x^2-3x}{x-2}\right)=\frac{(x^2-8)(x^2-3x)}{x(x-2)}=\frac{x(x^2-8)(x-3)}{x(x-2)}=\boxed{\frac{(x^2-8)(x-3)}{x-2}}$$

Example 2. *Simplify* $\frac{4}{x^2+5x+6}+\frac{x-2}{x+3}$.

Solution 2. *Using property (a) from above:*

$$\frac{4}{x^2+5x+6}+\frac{x-2}{x+3}=\frac{4}{(x+3)(x+2)}+\frac{x-2}{x+3}=\frac{4+(x-2)(x+2)}{(x+3)(x+2)}=\boxed{\frac{x^2}{(x+3)(x+2)}}$$

Example 3. *Simplify* $\frac{\frac{1}{x}}{1+\frac{1}{x}}$.

Solution 3. *This can easily be done by multiplying by a certain factor:*

$$\frac{\frac{1}{x}}{1+\frac{1}{x}}=\frac{x}{x}\cdot\frac{\frac{1}{x}}{1+\frac{1}{x}}=\boxed{\frac{1}{1+x}}$$

Example 4. *Simplify* $\frac{4}{x+2}-\frac{3}{x-1}-\frac{2x}{x^2+x-2}$.

Solution 4. *Using property (a) from above:*

$$\frac{4}{x+2}-\frac{3}{x-1}-\frac{2x}{x^2+x-2}=\frac{4}{x+2}-\frac{3}{x-1}-\frac{2x}{(x+2)(x-1)}=\frac{4(x-1)-3(x+2)-2x}{(x+2)(x-1)}=\boxed{\frac{-x-10}{(x+2)(x-1)}}$$

Example 5. *Simplify* $\frac{\frac{x}{x^2-9}-\frac{1}{x+3}}{\frac{3}{x-3}}$.

Solution 5. *Using property (c) from above:*

$$\frac{\frac{x}{x^2-9}-\frac{1}{x+3}}{\frac{3}{x-3}}=\left(\frac{x}{(x+3)(x-3)}-\frac{1}{x+3}\right)\cdot\frac{x-3}{3}=\frac{x}{3(x+3)}-\frac{x-3}{3(x+3)}=\boxed{\frac{1}{x+3}}$$

Example 6. *Simplify* $\frac{\frac{1}{x}-\frac{1}{3}}{\frac{1}{x}+\frac{1}{3}}$.

Solution 6. *This can easily be done by multiplying by a certain factor:*

$$\frac{\frac{1}{x}-\frac{1}{3}}{\frac{1}{x}+\frac{1}{3}} = \frac{3x}{3x}\cdot\frac{\frac{1}{x}-\frac{1}{3}}{\frac{1}{x}+\frac{1}{3}} = \boxed{\frac{3-x}{3+x}}$$

Example 7. *Solve for* x *in* $1+\frac{4}{x}-\frac{5}{x^2}=0$

Solution 7.

$$\begin{aligned}
0 &= 1+\frac{4}{x}-\frac{5}{x^2}\\
0 &= \frac{x^2+4x-5}{x^2}\\
0 &= x^2+4x-5\\
0 &= (x+5)(x-1)\\
x &= \boxed{-5,1}
\end{aligned}$$

Example 8. *Solve for* x *in* $\frac{x}{x-2}-\frac{4}{x+3}-\frac{10}{x^2+x-6}=0$

Solution 8.

$$\begin{aligned}
0 &= \frac{x}{x-2}-\frac{4}{x+3}-\frac{10}{x^2+x-6}\\
0 &= \frac{x}{x-2}-\frac{4}{x+3}-\frac{10}{(x+3)(x-2)}\\
0 &= \frac{x(x+3)-4(x-2)-10}{(x+3)(x-2)}\\
0 &= x(x+3)-4(x-2)-10\\
0 &= x^2-x-2\\
0 &= (x-2)(x+1)\\
x &= \boxed{-1}
\end{aligned}$$

Observe that we cannot allow $x=2$ *since it is a vertical asymptote.*

Example 9. *Find the interval for* x *which satisfies* $\frac{9}{x^2}-\frac{6}{x}+1\geq 0$

Solution 9. *Begin by simplifying the left side:*

$$\frac{9}{x^2}-\frac{6}{x}+1=\frac{9-6x+x^2}{x^2}$$

Now we have to check where the function may change sign. Fortunately, the denominator is always positive, so the sign behavior will only depend on the numerator. Now to determine where the numerator changes sign:

$$\begin{aligned}
0 &= x^2-6x+9\\
0 &= (x-3)^2\\
x &= 3
\end{aligned}$$

So the change in sign will occur at $x=3$*. To see which side satisfies the question, check any value belonging to the side. For the left check at* $x=1$ *which gives a positive value. To the right check at* $x=5$ *which gives a positive value. So it seems the numerator will satisfy the inequality for all real numbers. With all of this, the final solution is:*

$$\boxed{(-\infty,0)\cup(0,\infty)}$$

since at $x=0$ *the function becomes undefined.*

Example 10. *Find the interval for x which satisfies* $\frac{x}{x+1}+\frac{3}{2x+3}-\frac{11x+10}{2x^2+5x+3}<0$

Solution 10. *Begin by simplifying the left side:*

$$\frac{x}{x+1}+\frac{3}{2x+3}-\frac{11x+10}{2x^2+5x+3}=\frac{x(2x+3)+3(x+1)-11x-10}{(x+1)(2x+3)}=\frac{2x^2-5x-7}{(x+1)(2x+3)}$$

Now we have to check where the function may change sign. The denominator changes sign at $x=-\frac{3}{2},-1$, so there are three intervals to consider:

(a) *For the interval $\Big(-\infty,-\frac{3}{2}\Big)$ check with $x=-10$:*

$$(x+1)(2x+3)=(-10+1)(2(-10)+3)=(-9)(-17)=153$$

(b) *For the interval $\Big(-\frac{3}{2},-1\Big)$ check with $x=-\frac{5}{4}$:*

$$(x+1)(2x+3)=\Big(-\frac{5}{4}+1\Big)\Big(2\cdot-\frac{5}{4}+3\Big)=-\frac{1}{4}\cdot\frac{1}{2}=-\frac{1}{8}$$

(c) *For the interval $\Big(-1,\infty\Big)$ check with $x=0$:*

$$(x+1)(2x+3)=(1)(3)=3$$

Now to determine where the numerator changes sign:

$$\begin{aligned}0&=2x^2-5x-7\\0&=(2x-7)(x+1)\\x&=-1,\frac{7}{2}\end{aligned}$$

The numerator changes sign at $x=-1,\frac{7}{2}$, so there are three intervals to consider:

(a) *For the interval $\Big(-\infty,-1\Big)$ check with $x=-2$:*

$$(2x-7)(x+1)=(2(-2)-7)((-2)+1)=(-11)(-1)=11$$

(b) *For the interval $\Big(-1,\frac{7}{2}\Big)$ check with $x=0$:*

$$(2x-7)(x+1)=(-7)(1)=-7$$

(c) *For the interval $\Big(\frac{7}{2},\infty\Big)$ check with $x=5$:*

$$(2x-7)(x+1)=(2(5)-7)((5)+1)=(3)(6)=18$$

With behaviors determined for both the numerator and denominator, we have to check how they behave together. Since they do not change signs at the same points, there are more than just three intervals to consider:

(a) *For the interval $\Big(-\infty,-\frac{3}{2}\Big)$, both the numerator and denominator are positive, thus not satisfying the inequality.*

(b) *For the interval $\Big(-\frac{3}{2},-1\Big)$, the numerator is positive and denominator negative, thus satisfying the inequality.*

(c) *For the interval $\Big(-1,\frac{7}{2}\Big)$, the numerator is negative and denominator positive, thus satisfying the inequality.*

(d) *For the interval $\Big(\frac{7}{2},\infty\Big)$, both the numerator and denominator are positive, thus not satisfying the inequality.*

With all of this, the final solution is:

$$\boxed{\Big(-\frac{3}{2},-1\Big)\cup\Big(-1,\frac{7}{2}\Big)}$$

Example 11. *Find the interval for x which satisfies $x^4 - x^2 < 0$*

Solution 11. *First to find the points where the function changes signs:*

$$\begin{aligned} 0 &= x^4 - x^2 \\ 0 &= x^2(x^2 - 1) \\ 0 &= x^2(x-1)(x+1) \\ x &= -1, 0, 1 \end{aligned}$$

Now check the four intervals:

(a) *For the interval $(-\infty, -1)$ check with $x = -2$:*

$$x^2(x-1)(x+1) = (-2)^2((-2)-1)((-2)+1) = 12$$

(b) *For the interval $(-1, 0)$ check with $x = -\frac{1}{2}$:*

$$x^2(x-1)(x+1) = \frac{1}{4} \cdot -\frac{3}{2} \cdot \frac{1}{2} = -\frac{3}{16}$$

(c) *For the interval $(0, 1)$ check with $x = \frac{1}{2}$:*

$$x^2(x-1)(x+1) = \frac{1}{4} \cdot -\frac{1}{2} \cdot \frac{3}{2} = -\frac{3}{16}$$

(d) *For the interval $(1, \infty)$ check with $x = 2$:*

$$x^2(x-1)(x+1) = (2)^2((2)-1)((2)+1) = 12$$

With all of this, the final solution is:

$$\boxed{(-1, 0) \cup (0, 1)}$$

Absolute Value Function

Definition 2 (Properties of Absolute Value)**.** The absolute value function can be properly defined as:

$$|x| = \sqrt{x^2}$$

and satisfies the following properties:

(a)

$$|x| > 0 \;\; \forall x \in \mathbb{R} \text{ and } |x| = 0 \text{ iff } x = 0$$

(b)

$$|xy| = |x||y| \;\; \forall x, y \in \mathbb{R}$$

(c)

$$|x + y| \leq |x| + |y| \;\; \forall x, y \in \mathbb{R} \text{ with equality when } x = y$$

Some common applications of absolute value include an inequality. The two useful scenarios are:

(d)

$$|x| \leq y \implies -y \leq x \leq y$$

(e)

$$|x| \geq y \implies x \leq -y \text{ or } y \leq x$$

Example 12. *Find the exact solution to* $|3x - 7| + 10 = 0$.

Solution 12.

$$\begin{aligned} |3x - 7| + 10 &= 0 \\ |3x - 7| &= -10 \end{aligned}$$

Now according to the first property of absolute value functions, makes the problem have $\boxed{\textit{no solutions}}$.

Example 13. *Find the exact solution to* $|2x + 4| < 8$.

Solution 13.

$$\begin{aligned} |2x + 4| &< 8 \\ -8 <& 2x + 4 < 8 \\ -12 <& 2x < 4 \\ -6 <& x < 2 \end{aligned}$$

In interval notation, the solution is $\boxed{(-6, 2)}$.

Example 14. *Find the exact solution to* $|x - 3| \geq 10$.

Solution 14.

$$\begin{gathered} |x - 3| \geq 10 \\ x - 3 \leq -10 \;\; \textit{or} \;\; 10 \leq x - 3 \\ x \leq -7 \;\; \textit{or} \;\; 13 \leq x \end{gathered}$$

In interval notation, the solution is $\boxed{(-\infty, -7) \cup (13, \infty)}$.

Example 15. *Find the exact solution to* $|70x - 99| \geq -3$.

Solution 15. *Due to the first property of absolute value functions, the answer is all real solutions. In interval notation, the solution is* $\boxed{(-\infty, \infty)}$.

Example 16. *Find the exact solution to* $|x^2 + 2x - 3| \geq 0$.

Solution 16. *Due to the first property of absolute value functions, the answer is all real solutions. In interval notation, the solution is* $\boxed{(-\infty, \infty)}$.

Example 17. *Find the exact solution to* $|x^2 + 2x - 3| = 2$.

Solution 17. *There are two different cases to consider:*

(a)

$$\begin{aligned} 2 &= x^2 + 2x - 3 \\ 0 &= x^2 + 2x - 5 \\ x &= -1 \pm \sqrt{6} \end{aligned}$$

(b)

$$\begin{aligned} -2 &= x^2 + 2x - 3 \\ 0 &= x^2 + 2x - 1 \\ x &= -1 \pm \sqrt{2} \end{aligned}$$

So the final solution is $\boxed{x = -1 \pm \sqrt{6}, -1 \pm \sqrt{2}}$.

Example 18. *Find the exact solution to* $|x^2| = x$.

Solution 18. *Since squaring a number always outputs a positive result, the absolute value can be ignored. So just solve:*

$$\begin{aligned} x^2 &= x \\ 0 &= x^2 - x \\ 0 &= x(x-1) \\ x &= \boxed{0, 1} \end{aligned}$$

Example 19. *Find the exact solution to* $\left|\frac{x-2}{3}\right| < \frac{1}{3}$.

Solution 19.

$$\begin{aligned} \left|\frac{x-2}{3}\right| &< \frac{1}{3} \\ |x-2| &< 1 \\ -1 < x - 2 &< 1 \\ 1 < x &< 3 \end{aligned}$$

So the final solution in interval solution is $\boxed{(1, 3)}$.