Calculus 2
Recitation 15

Nathan Marianovsky

Taylor and Maclaurin Series

Definition 1 (Taylor Series)**.** If $f(x)$ has a power series representation centered at $x = a$ then, the coefficients will satisfy:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{s.t.} \quad c_n = \frac{f^{(n)}(a)}{n!}$$

where $f^{(n)}(a)$ represents the n^{th} derivative of the function evaluated at $x = a$. With this, an approximation can now be made towards the function. A k^{th} degree approximation is going to be given by the taylor series capped off at $n = k$. A specific case of interest is the linear approximation of a function:

$$f(x) = f(a) + f^{(1)}(a)(x-a)$$

Definition 2 (Maclaurin Series)**.** A Maclaurin series is just a special case of the taylor series. Whenever a Maclaurin series is used, it merely means that the taylor series is centered at $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 3 (Common Maclaurin Series)**.** Some common Maclaurin series are defined as:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad \text{where} \quad |x| < 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{where} \quad x \in \mathbb{R} \\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{where} \quad x \in \mathbb{R} \\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{where} \quad x \in \mathbb{R} \\
\arctan(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad \text{where} \quad |x| \le 1 \\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \quad \text{where} \quad |x| < 1 \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{where} \quad |x| < 1
\end{aligned}$$

Example 1. *Find the Maclaurin series of $f(x)=\sin(2x)$.*

Solution 1. *Use the known Maclaurin series for $sin(x)$:*

$$\sin(2x)=\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}=(2x)-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\dots}$$

Example 2. *Find the Maclaurin series of $f(x)=\sinh(x)$.*

Solution 2. *Use the known Maclaurin series for e^x:*

$$\sinh(x)=\frac{e^x-e^{-x}}{2}=\frac{1}{2}\Big[\sum_{n=0}^{\infty}\frac{x^n}{n!}-\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\Big]=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1+(-1)^{n+1}}{n!}x^n=\boxed{\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}}$$

Example 3. *Find the Maclaurin series of $f(x)=\cosh(x)$.*

Solution 3. *Use the known Maclaurin series for e^x:*

$$\cosh(x)=\frac{e^x+e^{-x}}{2}=\frac{1}{2}\Big[\sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\Big]=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1+(-1)^{n}}{n!}x^n=\boxed{\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}}$$

Example 4. *Find the Maclaurin series of $f(x)=xe^x$.*

Solution 4. *Use the known Maclaurin series for e^x:*

$$xe^x=x\sum_{n=0}^{\infty}\frac{x^n}{n!}=\boxed{\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}}$$

Example 5. *Find the Maclaurin series of $f(x)=\sqrt{1+x}$.*

Solution 5. *Use the binomial expansion:*

$$\sqrt{1+x}=(1+x)^{\frac{1}{2}}=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n=1+\frac{(\frac{1}{2})!}{1!(-\frac{1}{2})!}x+\frac{(\frac{1}{2})!}{2!(-\frac{3}{2})!}x^2+\frac{(\frac{1}{2})!}{3!(-\frac{5}{2})!}x^3-\frac{(\frac{1}{2})!}{1!(-\frac{7}{2})!}x^4\dots$$
$$=\boxed{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\dots}$$

Example 6. *In classical mechanics the energy of a free particle is defined as the kinetic energy: $E_1=\frac{1}{2}mv^2$ where E_1 is the total classical energy, m is the mass, and v is velocity of the particle. In Einstein's formulation of Special Relativity the following energy relationship is used instead: $E_2^2=(pc)^2+(mc^2)^2$ where p is the momentum of the particle and c is the speed of light. Kinetic energy in Special Relativity is defined as: $K=E_2-(mc^2)$ where E_2 is the total relativistic energy and mc^2 is the rest energy. Show that as $c\to\infty$ this kinetic energy reduces down to the classical limit.* [1]

Solution 6. *Use the result of Example 5 to expand the kinetic energy and take the limit:*

$$\begin{aligned}\lim_{c\to\infty}K&=\lim_{c\to\infty}(E-mc^2)\\&=\lim_{c\to\infty}\Big(\sqrt{(pc)^2+(mc^2)^2}-mc^2\Big)\\&=\lim_{c\to\infty}\Big(\sqrt{\Big(\frac{p}{mc}\Big)^2+1}-1\Big)mc^2\\&=\lim_{c\to\infty}\Big(1+\frac{1}{2}\Big(\frac{p}{mc}\Big)^2-\frac{1}{8}\Big(\frac{p}{mc}\Big)^4+\frac{1}{16}\Big(\frac{p}{mc}\Big)^6-\frac{5}{128}\Big(\frac{p}{mc}\Big)^8+\dots-1\Big)mc^2\\&=\lim_{c\to\infty}\Big(\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}-\frac{5p^8}{128m^7c^6}+\dots\Big)=\frac{p^2}{2m}=\frac{(mv)^2}{2m}=\boxed{\frac{1}{2}mv^2}\end{aligned}$$

[1] This essentially shows that if the restriction of a top velocity such as the speed of light is ignored, all of Einstein's results in Special Relativity will reduce to Newton's work of classical mechanics.

Example 7. *Evaluate $\int x\cos(x^3)dx$ as an infinite series.*

Solution 7. *Use the known Maclaurin series for $\cos(x)$:*

$$\int x\cos(x^3)dx = \int x\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n}}{(2n)!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\int x^{6n+1}dx = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+2}}{(2n)!(6n+2)}}$$

Example 8. *Evaluate $\int \frac{sin(x)}{x}dx$ as an infinite series.*

Solution 8. *Use the known Maclaurin series for $\sin(x)$:*

$$\int \frac{sin(x)}{x}dx = \int \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int x^{2n}dx = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)(2n+1)!}}$$

Example 9. *Evaluate $\int \frac{e^x-1}{x}dx$ as an infinite series.*

Solution 9. *Use the known Maclaurin series for e^x:*

$$\int \frac{e^x-1}{x}dx = \int \frac{1}{x}\Big(\sum_{n=0}^{\infty}\frac{x^n}{n!}-1\Big)dx = \sum_{n=1}^{\infty}\frac{1}{n!}\int x^{n-1}dx = \boxed{\sum_{n=1}^{\infty}\frac{x^n}{(n)n!}}$$

Example 10. *Evaluate $\lim_{x\to 0}\frac{x-\arctan(x)}{x^3}$ using a known infinite series.*

Solution 10. *Use the known Maclaurin series for $\arctan(x)$:*

$$\lim_{x\to 0}\frac{x-\arctan(x)}{x^3} = \lim_{x\to 0}\frac{1}{x^3}\Big(x-\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\Big) = \lim_{x\to 0}\frac{1}{x^3}\Big(\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}-\dots\Big) = \lim_{x\to 0}\Big(\frac{1}{3}-\frac{x^2}{5}+\frac{x^4}{7}-\dots\Big) = \boxed{\frac{1}{3}}$$

Example 11. *Evaluate $\lim_{x\to 0}\frac{1-\cos(x)}{1+x-e^x}$ using known infinite series.*

Solution 11. *Use the known Maclaurin series for $\cos(x)$ and e^x:*

$$\lim_{x\to 0}\frac{1-\cos(x)}{1+x-e^x} = \lim_{x\to 0}\frac{1-\Big(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\Big)}{1+x-\Big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\Big)} = \lim_{x\to 0}\frac{x^2\Big(\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\dots\Big)}{x^2\Big(-\frac{1}{2!}-\frac{x}{3!}-\frac{x^2}{4!}-\dots\Big)} = \frac{\frac{1}{2!}}{-\frac{1}{2!}} = \boxed{-1}$$

Example 12. *Find the sum of $\sum_{n=0}^{\infty}\frac{x^{4n}}{n!}$ using a known infinite series.*

Solution 12. *Use the known Maclaurin series for e^x:*

$$\sum_{n=0}^{\infty}\frac{x^{4n}}{n!} = \sum_{n=0}^{\infty}\frac{(x^4)^n}{n!} = \boxed{e^{x^4}}$$

Example 13. *Find the sum of $\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{6^{2n}(2n)!}$ using a known infinite series.*

Solution 13. *Use the known Maclaurin series for $\cos(x)$:*

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$$

Example 14. *Find the sum of $\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$ using a known infinite series.*

Solution 14. *Use the known Maclaurin series for $\sin(x)$:*

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}}$$

Example 15. *Find the sum of* $1-\ln(2)+\frac{\ln^2(2)}{2!}-\frac{\ln^3(2)}{3!}+\dots$ *using a known infinite series.*

Solution 15. *Use the known Maclaurin series for* e^x*:*

$$1-\ln(2)+\frac{\ln^2(2)}{2!}-\frac{\ln^3(2)}{3!}+\dots=\sum_{n=0}^{\infty}(-1)^n\frac{\ln^n(2)}{n!}=\sum_{n=0}^{\infty}\frac{(-\ln(2))^n}{n!}=e^{-\ln(2)}=\boxed{\frac{1}{2}}$$

Example 16. *Find the sum of* $\sum_{n=0}^{\infty}\frac{3^n}{5^n n!}$ *using a known infinite series.*

Solution 16. *Use the known Maclaurin series for* $\sin(x)$*:*

$$\sum_{n=0}^{\infty}\frac{3^n}{5^n n!}=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{5}\right)^n}{n!}=\boxed{e^{\frac{3}{5}}}$$

Example 17. *Find the Maclaurin series of* $f(x)=\sin^2(x)$*.*

Solution 17. *Use a double angle formula:*

$$\sin^2(x)=\frac{1}{2}(1-\cos(2x))=\frac{1}{2}\left(1-\left[1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\right]\right)=\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-1}x^{2n}}{(2n)!}}$$

Example 18. *Find the Taylor series of* $f(x)=e^x$ *at* $a=3$*.*

Solution 18. *Create a table of values needed:*

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	e^x	e^3
1	e^x	e^3
2	e^x	e^3
3	e^x	e^3
$\vdots$	$\vdots$	$\vdots$

Using this information gives:

$$e^x=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)(x-a)^n}{n!}=\boxed{e^3\sum_{n=0}^{\infty}\frac{(x-3)^n}{n!}}$$

Example 19. *Find the Taylor series of* $f(x)=\cos(x)$ *at* $a=\pi$*.*

Solution 19. *Create a table of values needed:*

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\cos(x)$	-1
1	$-\sin(x)$	0
2	$-\cos(x)$	1
3	$\sin(x)$	0
4	$\cos(x)$	-1
$\vdots$	$\vdots$	$\vdots$

Using this information gives:

$$\cos(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)(x-a)^n}{n!}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$$

Example 20. *Find the Taylor series of $f(x) = \ln(x)$ at $a = 2$.*

Solution 20. *Create a table of values needed:*

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\ln(x)$	$\ln(2)$
1	$\frac{1}{x}$	$\frac{1}{2}$
2	$-\frac{1}{x^2}$	$-\frac{1}{4}$
3	$\frac{2}{x^3}$	$\frac{1}{4}$
4	$-\frac{6}{x^4}$	$-\frac{3}{8}$
$\vdots$	$\vdots$	$\vdots$

Using this information gives:

$$\ln(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = \boxed{\ln(2) + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64} + \dots}$$

Example 21. *Find the Taylor series of $f(x) = \sin(x)$ at $a = \frac{\pi}{2}$.*

Solution 21. *Create a table of values needed:*

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\sin(x)$	1
1	$\cos(x)$	0
2	$-\sin(x)$	-1
3	$-\cos(x)$	0
4	$\sin(x)$	1
$\vdots$	$\vdots$	$\vdots$

Using this information gives:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}}$$

Example 22. *Find the Taylor series of $f(x) = x^{-2}$ at $a = 1$.*

Solution 22. *Create a table of values needed:*

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120e^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

Using this information gives:

$$x^{-2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)!(x-1)^n}{n!} = \boxed{\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n}$$

Example 23. *The force due to gravity on an object is defined as $F = \frac{mgR^2}{(R+h)^2}$, where m is the mass, h is the height above the surface of the Earth, R is the radius of the Earth and g is the acceleration due to gravity. Express the force as a series in powers of $\frac{h}{R}$ and show that as $R \to \infty$ the expression for the force of gravity reduces down to a form of Newton's second law of motion.*

Solution 23. *Use Example 6 from Recitation 14 to get the power series form:*

$$\begin{aligned}\lim_{R\to\infty} F &= \lim_{R\to\infty} \frac{mgR^2}{(R+h)^2} \\ &= \lim_{R\to\infty} \frac{mg}{\left(1+\frac{h}{R}\right)^2} \\ &= \lim_{R\to\infty} mg \sum_{m=0}^{\infty} \binom{m+1}{1}\left(\frac{h}{R}\right)^m \\ &= \lim_{R\to\infty} mg\left(1 + 2\left(\frac{h}{R}\right) + 3\left(\frac{h}{R}\right)^2 + \dots\right) \\ &= \boxed{mg}\end{aligned}$$

Example 24. *The electric potential, V, at a distance R along the axis perpendicular to the center of a charged disc with radius a and constant charge density σ, is given by $V = 2\pi\sigma\left(\sqrt{R^2+a^2} - R\right)$. Show that for large R, $V \approx \frac{\pi a^2 \sigma}{R}$.*

Solution 24. *Use the result of Example 5 to expand:*

$$\begin{aligned}V &= 2\pi\sigma\left(\sqrt{R^2+a^2} - R\right) \\ &= 2\pi\sigma R\left(\sqrt{1+\left(\frac{a}{R}\right)^2} - 1\right) \\ &= 2\pi\sigma R\left(1 + \frac{1}{2}\left(\frac{a}{R}\right)^2 - \frac{1}{8}\left(\frac{a}{R}\right)^4 + \frac{1}{16}\left(\frac{a}{R}\right)^6 - \frac{5}{128}\left(\frac{a}{R}\right)^8 + \dots - 1\right) \\ &= 2\pi\sigma R\left(\frac{1}{2}\left(\frac{a}{R}\right)^2 - \frac{1}{8}\left(\frac{a}{R}\right)^4 + \frac{1}{16}\left(\frac{a}{R}\right)^6 - \frac{5}{128}\left(\frac{a}{R}\right)^8 + \dots\right)\end{aligned}$$

Now for large R, any term beyond the first one will be very insignificant giving:

$$\boxed{V \approx \frac{\pi a^2 \sigma}{R}}$$

Example 25. *A hydrogen atom consists of an electron, of mass m, orbiting a proton, of mass M, where $m \ll M$. The reduced mass, μ, of the hydrogen atom is defined by $\mu = \frac{mM}{m+M}$. Show that $\mu \approx m$.*

Solution 25. *Use a geometric power series to expand:*

$$\begin{aligned}\mu &= \frac{mM}{m+M} \\ &= \frac{m}{1+\frac{m}{M}} \\ &= \frac{m}{1-\left(-\frac{m}{M}\right)} \\ &= m\sum_{n=0}^{\infty}\left(-\frac{m}{M}\right)^n \\ &= m\left(1 - \left(\frac{m}{M}\right) + \left(\frac{m}{M}\right)^2 - \left(\frac{m}{M}\right)^3 + \dots\right) \approx \boxed{m}\end{aligned}$$