Quiz 3 Solutions

MATH 103A
August 7, 2018

(1) (Q) Express the following values in rectangular coordinates where $\log$ references the complex logarithm:

a) $\log(-1)$ b) $\log(i)$
c) $\sin(i)$ d) $\sinh(i)$

(A)

(a) Recall that $-1 = e^{(2n+1)\pi}$ for $n \in \mathbb{Z}$. It follows that:

$$\log(-1) = \ln(1) + i((2n+1)\pi) = (2n+1)\pi i$$

(b) Recall that $i = e^{(4n+1)\frac{\pi}{2}}$ for $n \in \mathbb{Z}$. It follows that:

$$\log(i) = \ln(1) + i\Big((4n+1)\frac{\pi}{2}\Big) = \frac{(4n+1)\pi}{2}i$$

(c) Expressing the sine in terms of exponentials provides:

$$\sin(i) = \frac{e^{i(i)} - e^{-i(i)}}{2i} = \frac{e - e^{-1}}{2}i$$

(d) Expressing the hyperbolic sine in terms of exponentials provides:

$$\sinh(i) = \frac{e^{(i)} - e^{-(i)}}{2} = \frac{1}{2}\Bigg(\Big(\cos(1) + i\sin(1)\Big) - \Big(\cos(1) - i\sin(1)\Big)\Bigg) = \sin(1)i$$

(2) (Q) Using the fact that:

$$z = \tan(w) = \frac{\sin(w)}{\cos(w)} = \frac{\frac{e^{iw}-e^{-iw}}{2i}}{\frac{e^{iw}+e^{iw}}{2}} = \frac{e^{-iw} - e^{iw}}{e^{iw} + e^{-iw}}i$$

Isolate w so as to attain a formula for $w = \arctan(z)$ in terms of the complex logarithm.

(A) Let $\xi = e^{iw}$ and use algebraic manipulation to attain:

$$\begin{aligned}
z &= \frac{e^{-iw} - e^{iw}}{e^{iw} + e^{-iw}}i \\
z &= \frac{\xi^{-1} - \xi}{\xi + \xi^{-1}}i \\
z &= \frac{1 - \xi^2}{1 + \xi^2}i \\
z + z\xi^2 &= i - i\xi^2 \\
(i + z)\xi^2 &= i - z \\
\xi^2 &= \frac{i - z}{i + z} \\
e^{2iw} &= \frac{i - z}{i + z} \\
2iw &= \log\left(\frac{i - z}{i + z}\right) \\
w &= \frac{1}{2i}\log\left(\frac{i - z}{i + z}\right) \\
\arctan(z) &= \frac{i}{2}\log\left(\frac{i + z}{i - z}\right)
\end{aligned}$$