# Calculus 1 with Precalculus
# Recitation 11

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## Limits

**Definition 1** (Left and Right Limits)**.** The left and right limits of a function at a point are defined as:

(a)
$$\lim_{x \to c^-} f(x) = L_1$$

(b)
$$\lim_{x \to c^+} f(x) = L_2$$

where $L_1$ and $L_2$ represent the values that the function approaches from each respective side. If it happens that $L_1 = L_2$, then:
$$\lim_{x \to c} f(x) = L_1 = L_2$$

otherwise, the limit does not exist.

**Definition 2** (Properties of Limits)**.** Given $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist:

(a)
$$\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

(b)
$$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) \quad \text{where} \quad k \in \mathbb{R}$$

(c)
$$\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$$

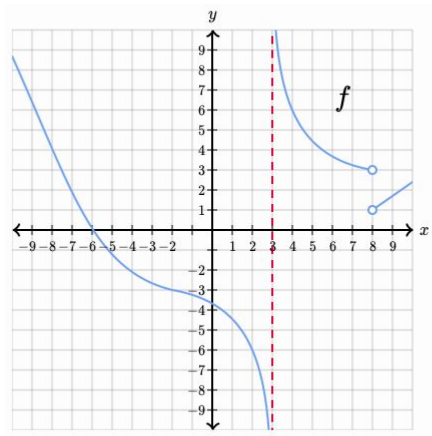
(d)
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

(e)
$$\lim_{x \to c}(f(x))^p = (\lim_{x \to c} f(x))^p \quad \text{where} \quad p \in \mathbb{R}$$

**Example 1.** *Determine the value of:*

$$\lim_{x \to 3} f(x)$$

*given:*




**Solution 1.** *The limits for each side are given as:*

$$\lim_{x \to 3^-} f(x) = -\infty$$
$$\lim_{x \to 3^+} f(x) = \infty$$

*Since the limits are not equal, the limit* $\boxed{\textit{does not exist}}$.

**Example 2.** *Determine the value of:*

$$\lim_{x \to 2} f(x)$$

*given the same graph as example one.*

**Solution 2.** *The limits for each side are given as:*

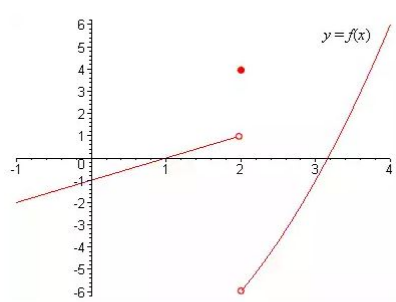
$$\lim_{x \to 2^-} f(x) = -6$$
$$\lim_{x \to 2^+} f(x) = -6$$

*Since the limits are equal, the limit is* $\boxed{-6}$.

**Example 3.** *Determine the value of:*

$$\lim_{x \to 2} f(x)$$

*given:*




**Solution 3.** *The limits for each side are given as:*

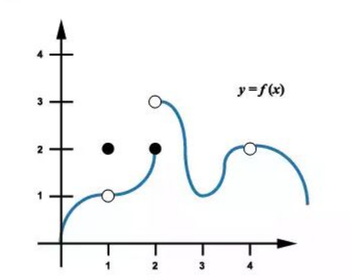
$$\lim_{x \to 2^-} f(x) = 1$$
$$\lim_{x \to 2^+} f(x) = -6$$

*Since the limits are not equal, the limit* $\boxed{\textit{does not exist}}$.

**Example 4.** *Determine the value of:*

$$\lim_{x\to 1} f(x)$$

*given:*




**Solution 4.** *The limits for each side are given as:*

$$\lim_{x\to 1^-} f(x) = 1$$
$$\lim_{x\to 1^+} f(x) = 1$$

*Since the limits are equal, the limit is* $\boxed{1}$.

**Example 5.** *Determine the value of:*

$$\lim_{x\to 4} f(x)$$

*given the same graph as example four.*

**Solution 5.** *The limits for each side are given as:*

$$\lim_{x\to 4^-} f(x) = 2$$
$$\lim_{x\to 4^+} f(x) = 2$$

*Since the limits are equal, the limit is* $\boxed{2}$.

**Example 6.** *Determine the value of:*

$$\lim_{x\to 1} \frac{x^3+1}{x-1}$$

*using a table of values.*

**Solution 6.**

| $x$ | 0.9 | 0.99 | 0.999 | 1.001 | 1.01 | 1.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | -17.29 | -197.03 | -1997.00 | 2003.00 | 203.03 | 23.31 |

*With this it seems that from the left side the values keep getting smaller while on the right side the values keep getting bigger. Since they do not approach the same value, the limit* $\boxed{\text{does not exist}}$.

**Example 7.** *Determine the value of:*

$$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

*using a table of values.*

**Solution 7.**

| $x$ | 0.9 | 0.99 | 0.999 | 1.001 | 1.01 | 1.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 2.71 | 2.97 | 3.00 | 3.00 | 3.03 | 3.31 |

*With this it seems that both from the left and right sides the function approaches* $\boxed{3}$ *as* $x$ *approaches* 1.

**Example 8.** *Determine the value of:*

$$\lim_{x \to 2} x^2 - x$$

*using a table of values.*

**Solution 8.**

| $x$ | *1.9* | *1.99* | *1.999* | *2.001* | *2.01* | *2.1* |
|---|---|---|---|---|---|---|
| $f(x)$ | *1.71* | *1.97* | *2.00* | *2.00* | *2.03* | *2.31* |

*With this it seems that both from the left and right sides the function approaches* $\boxed{2}$ *as* $x$ *approaches* $2$.

**Example 9.** *Given:*

$$\lim_{x \to c} f(x) = 5$$
$$\lim_{x \to c} g(x) = -2$$

*determine the value of:*

$$\lim_{x \to c} \sqrt{f(x) + g(x)}$$

**Solution 9.**

$$\lim_{x \to c} \sqrt{f(x) + g(x)} = \sqrt{\lim_{x \to c}(f(x) + g(x))} = \sqrt{\lim_{x \to c} f(x) + \lim_{x \to c} g(x)} = \sqrt{5 - 2} = \boxed{\sqrt{3}}$$

**Example 10.** *Given:*

$$\lim_{x \to \infty} f(x) = -3$$
$$\lim_{x \to \infty} g(x) = 4$$

*determine the value of:*

$$\lim_{x \to \infty} \frac{f(x)}{x + g(x)}$$

**Solution 10.**

$$\lim_{x \to \infty} \frac{f(x)}{x + g(x)} = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty}(x + g(x))} = \frac{-3}{\infty + 4} = \boxed{0}$$