Sample Midterm Solutions

MATH 103A
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[1] (a) Calculate:

$$\begin{aligned}\frac{(4+i)+(3+2i)}{-5-(-2+4i)} &= \frac{7+3i}{-3-4i} \\ &= \frac{7+3i}{-3-4i}\cdot\frac{-3+4i}{-3+4i} \\ &= \frac{(-21-12)+i(28-9)}{9+16} \\ &= \frac{-33+19i}{25}\end{aligned}$$

(b) Write the quotient in polar form:

$$\frac{-\sqrt{2}+\sqrt{2}i}{1+\sqrt{3}i} = \frac{2e^{\frac{3\pi i}{4}}}{2e^{\frac{\pi i}{3}}} = e^{\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)i} = e^{\frac{5\pi i}{12}}$$

(c) Compute:

$$\begin{aligned}(-\sqrt{3}+i)^{20} &= \left(2e^{i\left(\frac{5\pi}{6}+2\pi n\right)}\right)^{20} \\ &= 2^{20}e^{i\left(\frac{50\pi}{3}+40\pi n\right)} \\ &= 2^{20}\cos\left(\frac{50\pi}{3}+40\pi n\right)+i\sin\left(\frac{50\pi}{3}+40\pi n\right) \\ &= 2^{20}\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right) \\ &= 2^{20}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\ &= -2^{19}+2^{19}\sqrt{3}i\end{aligned}$$

(d) List all fourth roots of unity:

$$\begin{aligned}z^4 &= 1 \\ z^4 &= e^{2\pi i n} \\ z &= e^{\frac{\pi i}{2}\cdot n} \\ &= e^{\frac{\pi i}{2}\cdot 0}, e^{\frac{\pi i}{2}\cdot 1}, e^{\frac{\pi i}{2}\cdot 2}, e^{\frac{\pi i}{2}\cdot 3} \\ &= 1, i, -1, -i\end{aligned}$$

[2] (a) Compute each of the following limits:

(1)

$$\lim_{n\to\infty} \frac{(n+2)+(2n+3)i}{n+3} = \lim_{n\to\infty} \frac{n+2}{n+3} + i \lim_{n\to\infty} \frac{2n+3}{n+3} = 1+2i$$

(2)

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{-3+4i}{6}\right)^n &= \lim_{n\to\infty} \left(-\frac{1}{2}+\frac{2}{3}i\right)^n = \lim_{n\to\infty} \left(\frac{5}{6}e^{i\arctan\left(-\frac{4}{3}\right)}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{5}{6}\right)^n \cdot \lim_{n\to\infty} e^{in\arctan\left(-\frac{4}{3}\right)} \\
&= 0 \cdot \lim_{n\to\infty} e^{in\arctan\left(-\frac{4}{3}\right)} \\
&= 0
\end{aligned}$$

(3)

$$\begin{aligned}
\lim_{z\to i} \frac{z^2-(3+i)z+3i}{z^2+1} &= \lim_{z\to i} \frac{(z-i)(z-3)}{(z-i)(z+i)} \\
&= \lim_{z\to i} \frac{z-3}{z+i} \\
&= \frac{i-3}{2i} \\
&= -\frac{1+3i}{2}
\end{aligned}$$

(b) Check where

$$f(z) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$$

is complex differentiable (analytic) and compute $f'(z)$.

In order for a complex-valued function to be differentiable we need to check that the Cauchy-Riemann equations are satisfied for components $u(x,y) = \cosh(x)\cos(y)$ and $v(x,y) = \sinh(x)\sin(y)$:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \sinh(x)\cos(y) = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\cosh(x)\sin(y) = -\frac{\partial v}{\partial x}
\end{aligned}$$

Now the complex derivative is given by:

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$$

(c) Verify that $\cosh(x)\cos(y)$ is harmonic in the complex plane and find its harmonic conjugate.

For a given function to be harmonic we check that it satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\Big(\sinh(x)\cos(y)\Big) + \frac{\partial}{\partial y}\Big(-\cosh(x)\sin(y)\Big) = \cosh(x)\cos(y) - \cosh(x)\cos(y) = 0$$

Now in order to determine the harmonic conjugate we aim to find a function $v(x,y)$ s.t.:

$$\frac{\partial v}{\partial x} = \cosh(x)\sin(y) \quad \text{and} \quad \frac{\partial v}{\partial y} = \sinh(x)\cos(y)$$

Starting with the equation on the left-hand side:

$$v(x,y) = \int \frac{\partial v}{\partial x}\, \mathrm{d}x = \int \cosh(x)\sin(y)\, \mathrm{d}x = \sinh(x)\sin(y) + g(y)$$

To finish up we need to calculate $g(y)$ which can be determined by differentiating with respect to y giving $\frac{\partial v}{\partial y} = \sinh(x)\cos(y) + g'(y)$. This will satisfy the equation on the right-hand side if $g'(y) = 0$ providing directly $g(y) = C$ for $C \in \mathbb{R}$. Thus, the harmonic conjugate is given by $v(x,y) = \sinh(x)\sin(y) + C$ (there are infinitely many choices without some initial condition).

[3] (a) Find the partial fraction decomposition of:

$$\frac{4+i}{(z+1)(z+2i)(z+i)^2}$$

Over the complex field any rational function can be expanded as:

$$R(z) = \frac{p(z)}{(z-\zeta_1)^{k_1}(z-\zeta_2)^{k_2}\dots(z-\zeta_n)^{k_n}} = \sum_{q=1}^{n}\sum_{p=0}^{k_q-1}\frac{A_p^q}{(z-\zeta_q)^{k_q-p}}$$

where:

$$A_p^q = \frac{1}{p!}\frac{\mathrm{d}^p}{\mathrm{d}z^p}(z-\zeta_q)^{k_q}R(z)\Big|_{z=\zeta_q}$$

Thus, for the function we are given we want to expand it as:

$$R(z) = \frac{4+i}{(z+1)(z+2i)(z+i)^2} = \frac{A_0^1}{z+1} + \frac{A_0^2}{z+2i} + \frac{A_0^3}{(z+i)^2} + \frac{A_1^3}{z+i}$$

with the coefficients being given by:

$$\begin{aligned}
A_0^1 &= \frac{1}{0!}\frac{\mathrm{d}^0}{\mathrm{d}z^0}(z+1)R(z)\Big|_{z=-1}\\
&= \frac{4+i}{(z+2i)(z+i)^2}\Big|_{z=-1}\\
&= \frac{4+i}{(-1+2i)(-1+i)^2}\\
A_0^2 &= \frac{1}{0!}\frac{\mathrm{d}^0}{\mathrm{d}z^0}(z+2i)R(z)\Big|_{z=-2i}\\
&= \frac{4+i}{(z+1)(z+i)^2}\Big|_{z=-2i}\\
&= \frac{4+i}{(1-2i)(-i)^2}\\
&= \frac{4+i}{2i-1}\\
A_0^3 &= \frac{1}{0!}\frac{\mathrm{d}^0}{\mathrm{d}z^0}(z+i)^2R(z)\Big|_{z=-i}\\
&= \frac{4+i}{(z+1)(z+2i)}\Big|_{z=-i}\\
&= \frac{4+i}{(1-i)(i)}\\
&= \frac{4+i}{1+i}\\
A_1^3 &= \frac{1}{1!}\frac{\mathrm{d}^1}{\mathrm{d}z^1}(z+i)^2R(z)\Big|_{z=-i}\\
&= \frac{\mathrm{d}}{\mathrm{d}z}\frac{4+i}{(z+1)(z+2i)}\Big|_{z=-i}\\
&= -\frac{(4+i)(2z+(1+2i))}{(z+1)^2(z+2i)^2}\Big|_{z=-i}\\
&= -\frac{4+i}{(1-i)^2(i)^2}\\
&= \frac{4+i}{(1-i)^2}
\end{aligned}$$

(b) Evaluate $\sin(i)$ and $\cos(1-i)$:

$$\sin(i) = \frac{e^{i(i)} - e^{-i(i)}}{2i} = \frac{e^{-1} - e}{2i} = i \cdot \frac{e - e^{-1}}{2} = i\sinh(1)$$

$$\begin{aligned}
\cos(1-i) &= \frac{e^{i(1-i)} + e^{-i(1-i)}}{2} \\
&= \frac{e^{1+i} + e^{-1-i}}{2} \\
&= \frac{e}{2} \cdot e^{i} + \frac{e^{-1}}{2} \cdot e^{-i} \\
&= \frac{e}{2}\Big(\cos(1) + i\sin(1) \Big) + \frac{e^{-1}}{2}\Big(\cos(1) - i\sin(1) \Big) \\
&= \left(\frac{e + e^{-1}}{2} \right) \cos(1) + i \left(\frac{e - e^{-1}}{2} \right) \sin(1) \\
&= \cosh(1)\cos(1) + i\sinh(1)\sin(1)
\end{aligned}$$

(c) Find the principal value of $(1+i)^{i}$:

$$\begin{aligned}
\text{p.v.}\Big((1+i)^{i} \Big) &= e^{\operatorname{Log}\left((1+i)^{i} \right)} \\
&= e^{i \operatorname{Log}(1+i)} \\
&= e^{i\left(\ln(\sqrt{2}) + \frac{\pi i}{4} \right)} \\
&= e^{-\frac{\pi}{4} + i\ln(\sqrt{2})} \\
&= e^{-\frac{\pi}{4}}\Big(\cos(\ln(\sqrt{2})) + i\sin(\ln(\sqrt{2})) \Big)
\end{aligned}$$

and all values of $\arccos(i)$:

$$\begin{aligned}
i &= \cos(z) \\
i &= \frac{e^{iz} + e^{-iz}}{2} \\
0 &= (e^{iz})^2 - 2i(e^{iz}) + 1 \\
e^{iz} &= i \pm \sqrt{2}i \\
iz &= \log((1 \pm \sqrt{2})i) \\
iz &= \ln(1+\sqrt{2}) + \Big(\frac{\pi}{2} + 2\pi n \Big)i, \ln(\sqrt{2}-1) + \Big(-\frac{\pi}{2} + 2\pi n \Big)i \\
z &= \Big(\frac{\pi}{2} + 2\pi n \Big) - i\ln(\sqrt{2}+1), \Big(-\frac{\pi}{2} + 2\pi n \Big) - i\ln(\sqrt{2}-1)
\end{aligned}$$