Quiz 5 Solutions

MATH 103A
August 14, 2018

(1) (Q) Evaluate the following integrals where $\mathbb{S}^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ orientated counter-clockwise:

a) $\int\limits_{\mathbb{S}^1} \frac{e^z + e^{-z}}{z - \frac{\pi i}{2}}\, \mathrm{d}z$ b) $\int\limits_{\mathbb{S}^1} \frac{\sinh(z)}{z^4}\, \mathrm{d}z$

(A) For the second part we aim to use the Cauchy Integral formula.

(a) For this integral recognize that the integrand is holomorphic on the whole unit disc, thereby implying that by the Cauchy-Goursat Theorem:

$$\int\limits_{\mathbb{S}^1} \frac{e^z + e^{-z}}{z - \frac{\pi i}{2}}\, \mathrm{d}z = 0$$

(b) For the formula we identify $f(z) = \sinh(z)$ so that:

$$\int\limits_{\mathbb{S}^1} \frac{f(z)}{z^4}\, \mathrm{d}z = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{\pi i}{3} \frac{\mathrm{d}^3}{\mathrm{d}z^3} \sinh(z)\Big|_{z=0} = \frac{\pi i}{3} \cosh(z)\Big|_{z=0} = \frac{\pi i}{3} \cdot \frac{e^0 + e^0}{2} = \frac{\pi i}{3}$$

(2) (Q) Let $C_R = \{z \in \mathbb{C} \mid \|z\| = R\}$ oriented counter-clockwise. Show that:

$$\left| \int\limits_{C_R} \frac{\mathrm{Log}(z)}{z^2}\, \mathrm{d}z \right| \leq 2\pi\Big(\frac{\pi + \ln(R)}{R}\Big)$$

What is the value of the integral as $R \to \infty$?

(A) When looking along the circle of radius R we can use the fact that:

$$\left| \frac{\mathrm{Log}(z)}{z^2} \right| = \left| \frac{\ln(\|z\|) + i\mathrm{Arg}(z)}{z^2} \right| = \frac{|\ln(R) + i\mathrm{Arg}(z)|}{R^2} \leq \frac{|\ln(R)| + |i\mathrm{Arg}(z)|}{R^2} \leq \frac{\ln(R) + \pi}{R^2}$$

It follows that:

$$\left| \int\limits_{C_R} \frac{\mathrm{Log}(z)}{z^2}\, \mathrm{d}z \right| \leq \int\limits_{C_R} \left| \frac{\mathrm{Log}(z)}{z^2} \right| \mathrm{d}z \leq \int\limits_{C_R} \frac{\ln(R) + \pi}{R^2}\, \mathrm{d}z = 2\pi R \cdot \frac{\ln(R) + \pi}{R^2} = 2\pi \left(\frac{\ln(R) + \pi}{R} \right)$$

For the long-term behavior we take a limit:

$$\lim_{R \to \infty} 2\pi \left(\frac{\ln(R) + \pi}{R} \right) = 2\pi \lim_{R \to \infty} \frac{\frac{1}{R}}{1} = 0$$

and see that the integral satisfies:

$$\lim_{R \to \infty} \left| \int\limits_{C_R} \frac{\mathrm{Log}(z)}{z^2}\, \mathrm{d}z \right| \leq 0$$

With the zero on the right-hand side we can drop the absolute value to conclude:

$$\lim_{R \to \infty} \int\limits_{C_R} \frac{\mathrm{Log}(z)}{z^2}\, \mathrm{d}z = 0$$