Calculus 2
Recitation 5

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TRIGONOMETRIC INTEGRALS

Definition 1 (Trigonometric Integrals)**.** Integrals of this type are nothing special except that they involve trigonometric functions or even combinations of them. In cases such as these it is useful to remember trigonometric formulas such as:

$$\begin{aligned}
\cos^2(x) + \sin^2(x) &= 1 \\
1 + \tan^2(x) &= \sec^2(x) \\
1 + \cot^2(x) &= \csc^2(x) \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) = \cos^2(x) - \sin^2(x) &= 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \\
\sin(x)\sin(y) &= \frac{1}{2}(\cos(x-y) - \cos(x+y)) \\
\cos(x)\cos(y) &= \frac{1}{2}(\cos(x-y) + \cos(x+y)) \\
\sin(x)\cos(y) &= \frac{1}{2}(\sin(x+y) + \sin(x-y)) \\
\cos(x)\sin(y) &= \frac{1}{2}(\sin(x+y) - \sin(x-y))
\end{aligned}$$

Example 1. *Evaluate* $\int \sin^5(x)\cos^2(x)dx$

Solution 1. *Usually when dealing with power cases, it is wiser to reduce the odd power:*

$$\begin{aligned}
\int \sin^5(x)\cos^2(x)dx &= \int \sin^4(x)\cos^2(x)\sin(x)dx \\
&= \int (\sin^2(x))^2\cos^2(x)\sin(x)dx \\
&= \int (1-\cos^2(x))^2\cos^2(x)\sin(x)dx \\
&= \int (1 - 2\cos^2(x) + \cos^4(x))\cos^2(x)\sin(x)dx \\
&= \int (\cos^2(x) - 2\cos^4(x) + \cos^6(x))\sin(x)dx \\
&= \int -(u^2 - 2u^4 + u^6)du \\
&= -\frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^7}{7} + C \\
&= \boxed{-\frac{\cos^3(x)}{3} + \frac{2\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C}
\end{aligned}$$

Example 2. *Evaluate* $\int \tan^6(x)\sec^4(x)dx$

Solution 2. *Knowing that* $\sec^2(x)$ *is the derivative of* $\tan(x)$*, it is better to just reduce the secant once:*

$$\begin{aligned}
\int \tan^6(x)\sec^4(x)dx &= \int \tan^6(x)\sec^2(x)\sec^2(x)dx \\
&= \int \tan^6(x)\sec^2(x)(1+\tan^2(x))dx \\
&= \int (\tan^6(x)+\tan^8(x))\sec^2(x)dx \\
&= \int (u^6+u^8)du \\
&= \frac{u^7}{7}+\frac{u^9}{9}+C \\
&= \boxed{\frac{\tan^7(x)}{7}+\frac{\tan^9(x)}{9}+C}
\end{aligned}$$

Example 3. *Evaluate* $\int \tan(x)dx$

Solution 3.

$$\begin{aligned}
\int \tan(x)dx &= \int \frac{\sin(x)}{\cos(x)}dx \\
&= \int -\frac{du}{u} \\
&= -\ln|u|+C \\
&= -\ln|\cos(x)|+C \\
&= \boxed{\ln|\sec(x)|+C}
\end{aligned}$$

Example 4. *Evaluate* $\int \tan^3(x)dx$

Solution 4. *Try reducing the power of the tangent:*

$$\begin{aligned}
\int \tan^3(x)dx &= \int \tan^2(x)\tan(x)dx \\
&= \int (\sec^2(x)-1)\tan(x)dx \\
&= \int \tan(x)\sec^2(x)dx - \int \tan(x)dx \\
&= \int udu - \ln|\sec(x)|+C \\
&= \frac{u^2}{2} - \ln|\sec(x)|+C \\
&= \boxed{\frac{\tan^2(x)}{2} - \ln|\sec(x)|+C}
\end{aligned}$$

Example 5. *Evaluate* $\int \cos^3(x)dx$

Solution 5. *Reduce the cosine:*

$$\begin{aligned}
\int \cos^3(x)dx &= \int \cos^2(x)\cos(x)dx \\
&= \int (1-\sin^2(x))\cos(x)dx \\
&= \int \cos(x)dx - \int \sin^2(x)\cos(x)dx \\
&= \sin(x) - \int u^2 du \\
&= \sin(x) - \frac{u^3}{3} + C \\
&= \boxed{\sin(x) - \frac{\sin^3(x)}{3} + C}
\end{aligned}$$

Example 6. *Evaluate* $\int \sec(x)dx$

Solution 6. *The only way to integrate this integral for now is through a trick. Multiply the inside by "1":*

$$\begin{aligned}
\int \sec(x)dx &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx \\
&= \int \frac{\sec^2(x)+\tan(x)\sec(x)}{\sec(x)+\tan(x)}dx \\
&= \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \boxed{\ln|\sec(x)+\tan(x)| + C}
\end{aligned}$$

Example 7. *Evaluate* $\int \sin(x)\sin(4x)dx$

Solution 7. *Simplify the inside using a product to sum formula:*

$$\begin{aligned}
\int \sin(x)\sin(4x)dx &= \int \frac{1}{2}(\cos(-3x) - \cos(5x))dx \\
&= \frac{1}{2}\int (\cos(3x) - \cos(5x))dx \\
&= \frac{1}{2}\left[\frac{\sin(3x)}{3} - \frac{\sin(5x)}{5}\right] \\
&= \boxed{\frac{5\sin(3x) - 3\sin(5x)}{30}}
\end{aligned}$$

Definition 2 (Trigonometric Substitutions)**.** Evaluating integrals with trigonometric substitutions only comes in useful for certain scenarios. Specifically when the integrand contains $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, where $a \in \mathbb{R}$, in a fraction, a trigonometric substitution is ideal. Here is a table to illustrate what substitution is to be used:

Expression	Substitution	Identity Related to Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\sec^2(x) - 1 = \tan^2(\theta)$

Example 8. *Evaluate* $\int \frac{\sqrt{25x^2-4}}{x}dx$

Solution 8. *First:*

$$\int \frac{\sqrt{25x^2 - 4}}{x}dx = \int \frac{\sqrt{25\left(x^2 - \frac{4}{25}\right)}}{x}dx = 5\int \frac{\sqrt{x^2 - \frac{4}{25}}}{x}dx$$

Now this integral fits the third form:

$$x = \frac{2}{5}\sec(\theta)$$
$$dx = \frac{2}{5}\sec(\theta)\tan(\theta)d\theta$$

Plugging in yields:

$$\begin{aligned}
5\int \frac{\sqrt{x^2 - \frac{4}{25}}}{x}dx &= 5\int \frac{\sqrt{\frac{4}{25}\sec^2(\theta) - \frac{4}{25}}}{\frac{2}{5}\sec(\theta)}\frac{2}{5}\sec(\theta)\tan(\theta)d\theta \\
&= 2\int \sqrt{\sec^2(\theta) - 1}\tan(\theta)d\theta \\
&= 2\int \tan^2(\theta)d\theta \\
&= 2\int \frac{\sin^2(\theta)}{\cos^2(\theta)}d\theta \\
&= 2\int \frac{1 - \cos^2(\theta)}{\cos^2(\theta)}d\theta \\
&= 2\int (\sec^2(\theta) - 1)d\theta \\
&= 2(\tan(\theta) - \theta) + C
\end{aligned}$$

Now converting back to terms involving x is going to take a little bit of work. The original substitution was $x = \frac{2}{5}\sec(\theta)$. This could be rewritten as $\cos(\theta) = \frac{2}{5x}$. Now imagine a right triangle where one of the angles is θ and the adjacent side has length 2 with hypotenuse $5x$. From this triangle $\tan(\theta)$ can be obtained. The final solution is:

$$5\int \frac{\sqrt{x^2 - \frac{4}{25}}}{x}dx = 2\Big(\frac{\sqrt{25x^2 - 4}}{2} - \arccos\left(\frac{2}{5x}\right)\Big) + C = \boxed{\sqrt{25x^2 - 4} - 2\arccos\left(\frac{2}{5x}\right) + C}$$

Example 9. *Evaluate* $\int \frac{dx}{x^2\sqrt{x^2+4}}$

Solution 9. *This integral fits the second form:*

$$\begin{aligned} x &= 2\tan(\theta) \\ dx &= 2\sec^2(\theta)d\theta \end{aligned}$$

Plugging in yields:

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2(\theta)}{4\tan^2(\theta)\sqrt{4\tan^2(\theta)+4}}d\theta \\ &= \frac{1}{4}\int \frac{\sec^2(\theta)}{\tan^2(\theta)\sqrt{\tan^2(\theta)+1}}d\theta \\ &= \frac{1}{4}\int \frac{\sec(\theta)}{\tan^2(\theta)}d\theta \\ &= \frac{1}{4}\int \frac{\cos(\theta)}{\sin^(\theta)}d\theta \\ &= \frac{1}{4}\int \frac{du}{u^2} \\ &= -\frac{1}{4u}+C \\ &= -\frac{1}{4\sin(\theta)}+C \end{aligned}$$

The original substitution was $x = 2\tan(\theta)$*. Better rewritten as* $\tan(\theta) = \frac{x}{2}$*. Now imagine a right triangle with an angle* θ *that has an opposite side of length* x *and adjacent side of length 2. From this* $\sin(\theta)$ *can be obtained. The final answer is:*

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)}+C = \boxed{-\frac{\sqrt{x^2+4}}{4x}+C}$$

Example 10. *Evaluate* $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution 10. *This integral fits the first form:*

$$\begin{aligned} x &= 3\sin(\theta) \\ dx &= 3\cos(\theta)d\theta \end{aligned}$$

Plugging in and reevaluating the bounds gives:

$$\begin{aligned} \int_0^3 \frac{dx}{\sqrt{9-x^2}} &= \int_0^{\frac{\pi}{2}} \frac{3\cos(\theta)}{\sqrt{9-9\sin^2(\theta)}}d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos(\theta)}{\sqrt{1-\sin^2(\theta)}}d\theta \\ &= \int_0^{\frac{\pi}{2}} d\theta \\ &= \theta\Big|_0^{\frac{\pi}{2}} \\ &= \boxed{\frac{\pi}{2}} \end{aligned}$$

Example 11. *Evaluate* $\int_0^2 x\sqrt{x^2+4}dx$

Solution 11. *This can actually be done two ways:*

(i) *This integral fits the second form:*

$$x = 2\tan(\theta)$$
$$dx = 2\sec^2(\theta)d\theta$$

Plugging in and reevaluating the bounds gives:

$$\begin{aligned}
\int_0^2 x\sqrt{x^2+4}dx &= \int_0^{\frac{\pi}{4}} 4\tan(\theta)\sec^2(\theta)\sqrt{4\tan^2(\theta)+4}d\theta \\
&= 8\int_0^{\frac{\pi}{4}} \tan(\theta)\sec^2(\theta)\sqrt{\tan^2(\theta)+1}d\theta \\
&= 8\int_0^{\frac{\pi}{4}} \tan(\theta)\sec^3(\theta)d\theta \\
&= 8\int_0^{\frac{\pi}{4}} \frac{\sin(\theta)}{\cos(\theta)}\frac{1}{\cos^3(\theta)}d\theta \\
&= 8\int_0^{\frac{\pi}{4}} \frac{\sin(\theta)}{\cos^4(\theta)}d\theta \\
&= 8\int_1^{\frac{1}{\sqrt{2}}} -\frac{du}{u^4} \\
&= \frac{8}{3u^3}\Big|_1^{\frac{1}{\sqrt{2}}} \\
&= \boxed{\frac{8}{3}\left[2^{\frac{3}{2}}-1\right]}
\end{aligned}$$

(ii) *By substitution:*

$$\begin{aligned}
\int_0^2 x\sqrt{x^2+4}dx &= \frac{1}{2}\int_4^8 u^{\frac{1}{2}}du \\
&= \frac{1}{3}u^{\frac{3}{2}}\Big|_4^8 \\
&= \boxed{\frac{1}{3}\left[8^{\frac{3}{2}}-2^3\right]}
\end{aligned}$$

Example 12. *Evaluate* $\int_0^{\frac{1}{6}} \frac{x}{(36x^2+1)^{\frac{3}{2}}} dx$

Solution 12. *This integral fits the second form:*

$$x = \frac{1}{6}\tan(\theta)$$
$$dx = \frac{1}{6}\sec^2(\theta)d\theta$$

Plugging in and reevaluating the bounds gives:

$$\begin{aligned}
\int_0^{\frac{1}{6}} \frac{x}{(36x^2+1)^{\frac{3}{2}}} dx &= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{6}\tan(\theta)}{(\tan^2(\theta)+1)^{\frac{3}{2}}} \frac{1}{6}\sec^2(\theta)d\theta \\
&= \frac{1}{36}\int_0^{\frac{\pi}{4}} \frac{\tan(\theta)\sec^2(\theta)}{\sec^3(\theta)} d\theta \\
&= \frac{1}{36}\int_0^{\frac{\pi}{4}} \frac{\tan(\theta)}{\sec(\theta)} d\theta \\
&= \frac{1}{36}\int_0^{\frac{\pi}{4}} \sin(\theta)d\theta \\
&= -\frac{1}{36}\cos(\theta)\Big|_0^{\frac{\pi}{4}} \\
&= -\frac{1}{36}\left[\frac{1}{\sqrt{2}} - 1\right] \\
&= \boxed{\frac{1}{36}\left[1 - \frac{1}{\sqrt{2}}\right]}
\end{aligned}$$

Example 13. *Evaluate* $\int_2^4 \frac{dx}{x\sqrt{16-x^2}}$

Solution 13. *This integral fits the first form:*

$$x = 4\sin(\theta)$$
$$dx = 4\cos(\theta)d\theta$$

Plugging in and reevaluating the bounds gives:

$$\begin{aligned}
\int_2^4 \frac{dx}{x\sqrt{16-x^2}} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4\cos(\theta)}{4\sin(\theta)\sqrt{16-16\sin^2(\theta)}} d\theta \\
&= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos(\theta)}{\sin(\theta)\sqrt{1-\sin^2(\theta)}} d\theta \\
&= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc(\theta)d\theta \\
&= -\frac{1}{4}\ln|\cot(\theta)+\csc(\theta)|\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \boxed{\frac{1}{4}\ln|\sqrt{3}+2|}
\end{aligned}$$