Comments on Notation & Functions

Let me layout all of the notation I typically use, its corresponding definition, and some examples of usage:

I. SETS

In this course we strictly deal with the set of real numbers, \mathbb{R} . A subset, \mathcal{S} , of the real numbers is any collection of real numbers and is denoted by $\mathcal{S} \subseteq \mathbb{R}$ which is read as " \mathcal{S} is a *subset* of \mathbb{R} ". In some cases we want to make the relation stricter by writing $\mathcal{S} \subset \mathbb{R}$ which is read as " \mathcal{S} is a *proper subset* of \mathbb{R} " which just implies that $\mathcal{S} \neq \mathbb{R}$. For examples of subsets consider the following:

$$\{1,2\} \subset \mathbb{R}, \quad [1,2) \subseteq \mathbb{R}, \quad \{0\} \cup (\pi,100) \subset \mathbb{R}, \quad (-\infty,\infty) \subseteq \mathbb{R}, \quad \text{and} \quad (0,\infty) \subset \mathbb{R}$$

Of course there are a couple of subsets of the real numbers that have their own notation. So formally we have the following:

$$\emptyset = \{\} \quad \text{(The empty set)}$$

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\} \quad \text{(The set of natural numbers)}$$

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \quad \text{(The set of integers)}$$

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\} \quad \text{(The set of non-negative integers)}$$

$$\mathbb{Q} = \left\{\frac{p}{q} \middle| p, q \in \mathbb{Z}\right\} \quad \text{(The set of rational numbers)}$$

$$\mathbb{R} = \text{(The set of real numbers)}$$

$$\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\} \quad \text{(The set of positive real numbers)}$$

$$\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x \geq 0\} \quad \text{(The set of non-negative real numbers)}$$

Whenever we want to talk about a specific element belonging to a set S, we denote it by $x \in S$ which is read as "x belongs to S". For examples consider the following: $3 \in \mathbb{R}$, $1 \in (0,1)$, $0 \in \{0\}$, $7 \in \mathbb{N}$, and $\frac{9}{10} \in \mathbb{Q}$.

II. FUNCTIONS

For functions, also known as mappings, we use the notation:

$$f:A\to B$$

which is read as "f is the function with domain A and codomain B". For the range of f we use the notation f(A) which naturally satisfies the relation $f(A) \subseteq B$. With this structure you might note that the codomain of a function and its range need not be equal. As an example consider the mapping:

$$q: \mathbb{R} \to \mathbb{R}$$
 where $q(x) = x^2$

In this scenario we have $g(\mathbb{R}) = \mathbb{R}_{\geq 0}$. If it so happens that f(A) = B, then the function is said to be *onto*, or more formally *surjective*. In the example above if we restrict the codomain to be $\mathbb{R}_{\geq 0}$ then the function would be onto. Another important property of function is the idea of a *one-to-one* correspondence, or more formally function *injectivity*. A function is said to be one-to-one if its associated graph passes the *Horizontal Line Test*. In the case where the graph is not easy to draw we need a more formal approach to determining a one-to-one correspondence. So from a technical viewpoint, a function is said to be one-to-one if:

$$f(x) = f(y) \implies x = y$$

To prove such a condition, it is typically approached by utilizing a proof by contradiction. For example take the function g from above. We know this function to not be one-to-one, so here is the proof:

Proof: Assume the opposite, so g(x) is one-to-one. This means that given g(x) = g(y) implies x = y. A simple counterexample is f(1) = f(-1) because $1 \neq -1$. This contradiction implies that g(x) is not one-to-one.

Any function that is both onto, surjective, and one-to-one, injective, is said to be *bijective*. A bijection implies that there exists an *inverse function* denoted by $f^{-1}: B \to A$.