Midterm I Solutions

MATH 100

[1] Let P, Q, R be statements. Prove the following logical equivalences. (You can use truth tables, or you can use basic logical equivalences including De Morgan's laws, distributive laws, etc.)

(1) $P \implies (Q \vee R) \equiv (\sim Q) \implies (P \implies R)$

(2) $((P \wedge Q) \implies R) \equiv ((P \wedge (\sim R)) \implies (\sim Q))$

Proof.

(1)

$$\begin{aligned} (\sim Q) \implies (P \implies R) &\equiv (\sim Q) \implies ((\sim P) \vee R) \\ &\equiv Q \vee ((\sim P) \vee R) \\ &\equiv (\sim P) \vee (Q \vee R) \\ &\equiv P \implies (Q \vee R) \end{aligned}$$

(2)

$$\begin{aligned} (P \wedge (\sim R)) \implies (\sim Q) &\equiv \sim (P \wedge (\sim R)) \vee (\sim Q) \\ &\equiv ((\sim P) \vee R) \vee (\sim Q) \\ &\equiv \sim (P \wedge Q) \vee R \\ &\equiv (P \wedge Q) \implies R \end{aligned}$$

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[2] Fully negate the following statements, and express your statements in plain and natural English and in positive terms as much as possible. Note that the domain of variables in (2) is the set of positive integers.

(1) For every integer m, there exists an integer n such that $m + n$ is a cube.

(2) There exist positive integers a and b such that both $a \mid (3b + 2)$ and $a^2 + 3b^2 < 15$ are satisfied.

(3) For all integers a and b, $a > 3b - 2$ implies that $a^2 + 3b^2 > 2(a + 3b)^2$.

Proof.

(1) In terms of quantifiers the given statement is written as: $\forall\, m \in \mathbb{Z}$, $\exists\, n \in \mathbb{Z}$ such that $m + n$ is a cube. The negation provides:

$$\sim (\forall\, m \in \mathbb{Z}, \exists\, n \in \mathbb{Z} \quad \text{such that} \quad m+n \quad \text{is a cube.}) \equiv \exists\, m \in \mathbb{Z}, \forall\, n \in \mathbb{Z} \quad \text{such that} \quad m+n \quad \text{is not a cube.}$$

In plain English this is read as: There exists an integer m for all integers n such that $m + n$ is not a cube.

(2) In terms of quantifiers the given statement is written as: $\exists\, a, b \in \mathbb{N}$ such that $a \mid (3b + 2)$ and $a^2 + 3b^2 < 15$ are satisfied. The negation provides:

$$\begin{aligned} &\sim (\exists\, a, b \in \mathbb{N} \quad \text{such that} \quad a \mid (3b+2) \quad \text{and} \quad a^2 + 3b^2 < 15 \quad \text{are satisfied.}) \\ &\equiv \forall\, a, b \in \mathbb{N} \quad a \nmid (3b+2) \quad \text{or} \quad a^2 + 3b^2 \geq 15 \quad \text{is satisfied.} \end{aligned}$$

In plain English this is read as: For all positive integers a and b it must be that $a \nmid (3b + 2)$ or $a^2 + 3b^2 \geq 15$ is satisfied.

(3) In terms of quantifiers the given statement is written as: $\forall\, a, b \in \mathbb{Z}\ a > 3b - 2 \implies a^2 + 3b^2 > 2(a + 3b)^2$. The negation provides:

$$\begin{aligned} &\sim (\forall\, a, b \in \mathbb{Z} \quad a > 3b - 2 \implies a^2 + 3b^2 > 2(a+3b)^2) \\ &\equiv \exists\, a, b \in \mathbb{Z} \quad \sim (a > 3b - 2 \implies a^2 + 3b^2 > 2(a+3b)^2) \\ &\equiv \exists\, a, b \in \mathbb{Z} \quad \sim (a \leq 3b - 2 \quad \vee \quad a^2 + 3b^2 > 2(a+3b)^2) \\ &\equiv \exists\, a, b \in \mathbb{Z} \quad a > 3b - 2 \quad \wedge \quad a^2 + 3b^2 \leq 2(a+3b)^2 \end{aligned}$$

In plain English this is read as: There exist integers a and b satisfying $a > 3b - 2$ and $a^2 + 3b^2 \leq 2(a + 3b)^2$.

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[3] Let x be a non-zero real number. Prove that if $x > 0$, then $x + \frac{1}{x} \geq 2$, by the following methods:

(1) a direct proof
(2) A proof by contrapositive
(3) a proof by contradiction

Proof.

(1) Given that $x > 0$ it follows that $(x-1)^2 \geq 0$ because the square of any real number is always non-negative. Expanding the square provides $x^2 + 1 \geq 2x$. Now division by a factor of a positive x will keep the inequality direction invariant and establish the desired result $x + \frac{1}{x} \geq 2$.
(2) The contrapositive of the given statement reads: If $x + \frac{1}{x} < 2$, then $x < 0$. The given information tells us that $x - 2 + \frac{1}{x} < 0$ and as a result $\frac{(x-1)^2}{x} < 0$. By design the square of any real number is always non-negative, thereby implying that $x < 0$ if this statement is true.
(3) For a proof by contradiction we assume that $x + \frac{1}{x} < 2$. This tells us $x - 2 + \frac{1}{x} < 0$ and as a result $\frac{(x-1)^2}{x} < 0$. Since the numerator is strictly non-negative, it follows that the denominator must be negative giving $x < 0$. $\Rightarrow\!\Leftarrow$ This result contradicts the initial assumption $x > 0$, hence it must be that $x + \frac{1}{x} \geq 2$.

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[4] For any integer a, prove that $a^3 \equiv a \pmod 3$.

Proof. By algebraic manipulation:

$$\begin{aligned} a^3 &\equiv a \pmod 3 \\ a^3 - a &\equiv 0 \pmod 3 \\ a(a^2-1) &\equiv 0 \pmod 3 \\ a(a-1)(a+1) &\equiv 0 \pmod 3 \end{aligned}$$

The last line provides $a \equiv 0 \pmod 3$, $a \equiv 1 \pmod 3$, or $a \equiv -1 \pmod 3 \equiv 2 \pmod 3$. Since any integer must fall into exactly one of these cases, it follows that the given statement holds true for all integers. ■

[5] Let $P(x)$ and $Q(x)$ be open sentences with $x \in T$. Which of the following statements implies that $P(x) \implies Q(x)$ is true for some $x \in T$? Justify your answer for full credits for each statement.

(1) $P(x) \wedge Q(x)$ is false for all $x \in T$
(2) $P(x) \vee Q(x)$ is true for some $x \in T$
(3) $P(x) \wedge Q(x)$ is true for some $x \in T$
(4) $Q(x)$ is true for all $x \in T$
(5) $(\sim P(x)) \implies (\sim Q(x))$ is false for some $x \in T$
(6) $(\sim P(x)) \wedge (\sim Q(x))$ is false for all $x \in T$
(7) $(\sim P(x)) \wedge Q(x)$ is true for some $x \in T$

Proof. Options (3), (4), (5), and (7) satisfy the desired result:

(1) The given information tells us that $P(x)$ or $Q(x)$ is false for all $x \in T$. This is not enough to guarantee that $P(x) \implies Q(x) \equiv \mathbf{T}$ because a scenario exists in which $P(x) \equiv \mathbf{T}$ and $Q(x) \equiv \mathbf{F}$ for all $x \in T$.
(2) The given information tells us that $P(x)$ or $Q(x)$ is true for some $x \in T$. This is not enough to guarantee that $P(x) \implies Q(x) \equiv \mathbf{T}$ because $P(x) \equiv \mathbf{T}$ and $Q(x) \equiv \mathbf{F}$ is a possibility for this value $x \in T$.
(3) The given information tells us that $P(x)$ and $Q(x)$ are both true for some $x \in T$. For this single value we have that $P(x) \implies Q(x) \equiv \mathbf{T}$ thereby satisfying the desired condition.
(4) If $Q(x)$ is always true, then we right away arrive at $P(x) \implies \mathbf{T} \equiv \mathbf{T}$ for all $x \in T$.
(5) If $(\sim P(x)) \implies (\sim Q(x))$ is false for some $x \in T$, then $\sim P(x) \equiv \mathbf{T}$ and $\sim Q(x) \equiv \mathbf{F}$. It directly follows that $P(x) \equiv \mathbf{F}$ and $Q(x) \equiv \mathbf{T}$ thereby giving $P(x) \implies Q(x) \equiv \mathbf{T}$ for some $x \in T$.
(6) The given statement $(\sim P(x)) \wedge (\sim Q(x)) \equiv \mathbf{F}$ for all $x \in T$ implies that $P(x) \vee Q(x) \equiv \mathbf{T}$ for all $x \in T$. This allows us to infer that $P(x)$ or $Q(x)$ is true for all $x \in T$. This is not enough to guarantee that $P(x) \implies Q(x) \equiv \mathbf{T}$ because a scenario exists in which $P(x) \equiv \mathbf{T}$ and $Q(x) \equiv \mathbf{F}$ for all $x \in T$.
(7) The given information tells us that $\sim P(x)$ and $Q(x)$ are both true for some $x \in T$. It follows that for this same value $P(x)$ is false and $Q(x)$ is true. Thus, we arrive at a situation where $\mathbf{F} \implies \mathbf{T} \equiv \mathbf{T}$ for some $x \in T$.

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