

# Quiz 7 Solutions

MATH 103A  
August 21, 2018

- (1) (Q) For each of the following functions determine the type of singularity (*removable, pole of order  $n$ , or essential*) at  $z = 0$ :

a)  $f(z) = \frac{\sinh(z)}{z}$

b)  $f(z) = z^3 e^{\frac{1}{z}}$

c)  $f(z) = \frac{\cos(z) \sin(z)}{z^4}$

(A)

- (a) First we need the Laurent series:

$$f(z) = \frac{\sinh(z)}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)!}$$

Since the expansion does not have any terms with  $z$  in the denominator, it follows that  $z = 0$  is a removable singularity.

- (b) First we need the Laurent series:

$$f(z) = z^3 e^{\frac{1}{z}} = z^3 \sum_{n=0}^{\infty} \frac{1}{n! z^n} = \sum_{n=0}^{\infty} \frac{1}{n! z^{n-3}}$$

Since the expansion has infinitely many terms with  $z$  in the denominator, it follows that  $z = 0$  is an essential singularity.

- (c) First we need the Laurent series:

$$f(z) = \frac{\cos(z) \sin(z)}{z^4} = \frac{\sin(2z)}{2z^4} = \frac{1}{2z^4} \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} z^{2n-3}$$

Since the expansion has finitely many terms with  $z$  in the denominator, it follows that  $z = 0$  is a pole of order 3.

- (2) (Q) Let  $C = \{z \in \mathbb{C} \mid \|z\| = 2\}$  oriented counter-clockwise. Evaluate the following integral using the Residue Theorem:

$$\int_C \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} dz$$

- (A) The poles of the integrand correspond to  $z = \pm i, -1$  whose residues are:

$$\begin{aligned} \operatorname{Res}_{z=i} \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} &= \lim_{z \rightarrow i} (z - i) \cdot \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} = \lim_{z \rightarrow i} \frac{\cosh(\pi z)}{(z + i)(z + 1)} \\ &= \frac{\cosh(\pi i)}{2i(i + 1)} = \frac{1}{2i - 2} \cdot \frac{e^{\pi i} + e^{-\pi i}}{2} = -\frac{i + 1}{4} \\ \operatorname{Res}_{z=-i} \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} &= \lim_{z \rightarrow -i} (z + i) \cdot \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} = \lim_{z \rightarrow -i} \frac{\cosh(\pi z)}{(z - i)(z + 1)} \\ &= \frac{\cosh(-\pi i)}{2i(i - 1)} = -\frac{1}{2i + 2} \cdot \frac{e^{\pi i} + e^{-\pi i}}{2} = \frac{-1 + i}{4} \\ \operatorname{Res}_{z=-1} \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} &= \lim_{z \rightarrow -1} (z + 1) \cdot \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} = \lim_{z \rightarrow -1} \frac{\cosh(\pi z)}{z^2 + 1} \\ &= \frac{\cosh(-\pi)}{2} = \frac{\cosh(\pi)}{2} \end{aligned}$$

By the Residue Theorem it follows that:

$$\int_C \frac{\cosh(\pi z)}{(z^2 + 1)(z + 1)} \, dz = 2\pi i \cdot \frac{\cosh(\pi) - 1}{2} = \pi i (\cosh(\pi) - 1)$$