Weirdness of Quantum Mechanics

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July 30, 2020

Abstract

What makes the theory of quantum mechanics so weird for people to understand from an intuitive point of view? We will begin by delving into the workings of the no-cloning theorem and quantum teleportation, both of which have deep philosophical impacts on the interpretation of the physical world with the latter having been experimentally verified. All of these results will make it clear that a formulation of quantum logic will have to be intrinsically distinct from its classical counterpart, with the statement of the no-teleportation theorem making this evidently clear via an explanation that a quantum state cannot be reconstructed from a collection (even infintie) of classical bits.

1. No-Cloning Theorem

What does it mean to clone a particle? In the most basic terms this literally translates into taking one particle and recreating an exact copy which is indistinguishable from the original and in the context of classical mechanics cloning is not forbidden as all identifying characteristics of a particle can be identified simultaneously. Thus, the natural question to ask is why is it forbidden to perform the same action in quantum mechanics? Suppose to the contrary that we have such a cloning device which can take a particle represented by $|\psi\rangle$ and copy its information onto a blank slate $|X\rangle$. Mathematically, this would take the form:

$$|\psi\rangle|X\rangle \mapsto |\psi\rangle|\psi\rangle$$

So far nothing is wrong, but since we can do this for any particle suppose that we specifically have two such copies:

$$|\psi_1\rangle|X\rangle \mapsto |\psi_1\rangle|\psi_1\rangle \quad \text{and} \quad |\psi_2\rangle|X\rangle \mapsto |\psi_2\rangle|\psi_2\rangle$$

for $|\psi_1\rangle$ and $|\psi_2\rangle$ representing the states of distinct particles. Clearly the cloning machine has no issues with pure states but what happens if we instead consider a mixed state, or rather a superposition?

$$\begin{aligned}(a|\psi_1\rangle + b|\psi_2\rangle)|X\rangle &= (a|\psi_1\rangle + b|\psi_2\rangle)(a|\psi_1\rangle + b|\psi_2\rangle)\\ a|\psi_1\rangle|X\rangle + b|\psi_2\rangle|X\rangle &= a^2|\psi_1\rangle|\psi_1\rangle + b^2|\psi_2\rangle|\psi_2\rangle + 2ab|\psi_1\rangle|\psi_2\rangle\\ a|\psi_1\rangle|\psi_1\rangle + b|\psi_2\rangle|\psi_2\rangle &= a^2|\psi_1\rangle|\psi_1\rangle + b^2|\psi_2\rangle|\psi_2\rangle + 2ab|\psi_1\rangle|\psi_2\rangle\end{aligned}$$

Clearly the left and right-hand sides above do not match up as expected and so it must be that the cloning device cannot clone non-pure states. Being able to copy pure states is analogous to cloning in the classical sense and so if superimposed states cannot be cloned then there is truly no cloning in quantum mechanics.

2. Quantum Teleportation

The idea of moving an object from one point in space to another instantaneously is known as teleportation, and has been subject of much controversy as to whether it is possible. The closest analogy to teleportation classical mechanics can provide is driving from point A to point B, essentially stating that there are no shortcuts. Thus, explaining teleportation classically is impossible, but using quantum teleportation data can be sent instantly. To perform this teleportation first suppose that there exists a device called the Stern-Gerlach machine that measures the spin of an electron with respect to a given axis. Electrons can only take one of two values for spin for a given axis, up or down. The mathematics behind the nature of the teleportation ensures that all possible cases are handled such that the data can always be received.

2.1 Mathematics of One Quantum Spin

Given a single electron, its state with respect to the spin of the z-axis can be written as:

$$|\psi\rangle = a|\uparrow\langle +b|\downarrow\rangle \quad \text{where} \quad |a|^2 + |b|^2 = 1$$

But besides the z-axis, there also x-axis spins. The spins along the x direction can form a basis for the z direction via:

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle = a\left(\frac{1}{\sqrt{2}}\Big[|\rightarrow\rangle + |\leftarrow\rangle\Big]\right) + b\left(\frac{1}{\sqrt{2}}\Big[|\rightarrow\rangle - |\leftarrow\rangle\Big]\right) = \frac{a+b}{\sqrt{2}}|\rightarrow\rangle + \frac{a-b}{\sqrt{2}}|\leftarrow\rangle$$

and so a measurement in the x-axis will yield a $\left|\frac{a+b}{\sqrt{2}}\right|^2$ probability of having spin right and $\left|\frac{a-b}{\sqrt{2}}\right|^2$ probability of having spin left. The same can be done for the y direction, but we will not be covering it here since it will not be needed for the formulation of teleportation.

2.2 Mathematics of Two Quantum Spins

Now given two spins, the state with respect to the spin of the z-axis can be written as:

$$|\theta\rangle = a|\uparrow,\uparrow\rangle + b|\uparrow,\downarrow\rangle + c|\downarrow,\uparrow\rangle + d|\downarrow,\downarrow\rangle \quad \text{where} \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$

Unlike the single spin, if one of the spins is measured then the state still remains in a superposition with respect to the second spin. For example, if the spin of the first is measured to be up, then the state collapses down to:

$$|\theta\rangle = \frac{a}{\sqrt{|a|^2 + |b|^2}}|\uparrow,\uparrow\rangle + \frac{b}{\sqrt{|a|^2 + |b|^2}}|\uparrow,\downarrow\rangle$$

where the constants are readjusted so as to renormalize the state to ensure total probability of 1. Furthermore, just like the single spin a change of basis can be defined:

$$|\uparrow,\uparrow\rangle = \frac{1}{\sqrt{2}}\Big[|\rightarrow,\uparrow\rangle + |\leftarrow,\uparrow\rangle\Big]$$
$$|\uparrow,\downarrow\rangle = \frac{1}{\sqrt{2}}\Big[|\rightarrow,\downarrow\rangle + |\leftarrow,\downarrow\rangle\Big]$$

where the above only changed the basis of the first spin, but the same can be done to the second spin.

2.3 The Singlet State

The singlet state is defined as:

$$|\phi\rangle = \frac{1}{\sqrt{2}}\Big[|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\Big]$$

The truth behind fermions is that they obey the Pauli Exclusion Principle and as a result can never occupy the same quantum state. It is exactly for this reason why the minus sign is chosen and why the up-up and down-down configurations are dismissed since quantum states of the same spin are not allowed. In this entanglement once the spin of one of the particles is known, the other spin is also known. This is the true state that two entangled electrons will obey with a probability of $\frac{1}{2}$ of being either in the up-down or down-up configuration. This will be used in the formulation of quantum teleportation because the singlet state will be used to represent the entangled state of two locations.

2.4 Unitary Operators

Now we want to define some useful operators that are going to be utilized in the teleportation process. These operators will be unitary by design since they reveal nothing about the quantum state and hence do not change the quantum states. First we have a π rotation about the x-axis given by:

$$X\Big(a|\uparrow\rangle + b|\downarrow\rangle\Big) = a|\downarrow\rangle + b|\uparrow\rangle$$

Similarly, we have a rotation about the z-axis given by:

$$Z\Big(a|\uparrow\rangle + b|\downarrow\rangle\Big) = a|\uparrow\rangle - b|\downarrow\rangle$$

Lastly, the *controlled-X* operator operates on a two spin system by flipping the second spin if the first is down:

$$C_X\Big(a|\uparrow,\uparrow\rangle + b|\uparrow,\downarrow\rangle + c|\downarrow,\uparrow\rangle + d|\downarrow,\downarrow\rangle\Big) = a|\uparrow,\uparrow\rangle + b|\uparrow,\downarrow\rangle + c|\downarrow,\downarrow\rangle + d|\downarrow,\uparrow\rangle$$

2.5 *Setup for Teleportation*

We now want to layout the setup for what we are trying to accomplish. It is our goal to teleport a state of unknown spin:

$$|\psi\rangle = a|\uparrow\rangle - b|\downarrow\rangle$$

At the same time we also start with an entangled singlet state that represents the connection between our starting point A and ending point B:

$$|\phi\rangle = \frac{1}{\sqrt{2}}\Big[|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\Big]$$

We find that the combined state is therefore given as:

$$\begin{aligned}
|\psi,\phi\rangle &= |\psi\rangle|\phi\rangle \\
&= \Big(a|\uparrow\rangle - b|\downarrow\rangle\Big)\left(\frac{1}{\sqrt{2}}\Big[|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\Big]\right) \\
&= \frac{a}{\sqrt{2}}|\uparrow\rangle|\uparrow,\downarrow\rangle - \frac{b}{\sqrt{2}}|\downarrow\rangle|\uparrow,\downarrow\rangle - \frac{a}{\sqrt{2}}|\uparrow\rangle|\downarrow,\uparrow\rangle + \frac{b}{\sqrt{2}}|\downarrow\rangle|\downarrow,\uparrow\rangle \\
&= \frac{a}{\sqrt{2}}|\uparrow,\uparrow,\downarrow\rangle - \frac{b}{\sqrt{2}}|\downarrow,\uparrow,\downarrow\rangle - \frac{a}{\sqrt{2}}|\uparrow,\downarrow,\uparrow\rangle + \frac{b}{\sqrt{2}}|\downarrow,\downarrow,\uparrow\rangle
\end{aligned}$$

The components of the three spin system are:

(1) The unknown spin that is going to be teleported, but for now is located at point A.

(2) The first spin of the entangled pair and is located at point A.

(3) The second spin of the entangled pair and is located at point B.

2.6 *Recipe for Teleportation*

We now want to list out the exact steps needed to be taken so as to perform the teleportation:

(1) Apply the C_X operator to spins 1 and 2.

(2) Measure spin 2 in the z basis.

(3) Measure spin 1 in the x basis.

(4) Now have someone at point A call a person at point B and inform them of the outcome of the two measurements.

(5) The person at point B does the following depending on the situation:

- If the measurement outcomes were $\downarrow$ for spin 2 and $\rightarrow$ for spin 1: Apply the X, Z, and X operators to spin 3 in that order.
- If the measurement outcomes were $\downarrow$ for spin 2 and $\leftarrow$ for spin 1: Apply the X, Z, X, and Z operators to spin 3 in that order.
- If the measurement outcomes were $\uparrow$ for spin 2 and $\rightarrow$ for spin 1: Apply the X and Z operators to spin 3 in that order.
- If the measurement outcomes were $\uparrow$ for spin 2 and $\leftarrow$ for spin 1: Apply the X operator to spin 3.

(6) The person at point B reads off the components of spin 3 as the original state that was teleported.

2.7 Verification of Teleportation

Let us take the unknown spin $|\psi\rangle$ to have the same form as described in section 2.5. Now taking the combined three spin state:

$$|\psi, \phi\rangle = \frac{a}{\sqrt{2}}|\uparrow, \uparrow, \downarrow\rangle - \frac{b}{\sqrt{2}}|\downarrow, \uparrow, \downarrow\rangle - \frac{a}{\sqrt{2}}|\uparrow, \downarrow, \uparrow\rangle + \frac{b}{\sqrt{2}}|\downarrow, \downarrow, \uparrow\rangle$$

we start by applying the C_X operator to spins 1 and 2:

$$C_X|\psi, \phi\rangle = \frac{a}{\sqrt{2}}|\uparrow, \uparrow, \downarrow\rangle - \frac{b}{\sqrt{2}}|\downarrow, \downarrow, \downarrow\rangle - \frac{a}{\sqrt{2}}|\uparrow, \downarrow, \uparrow\rangle + \frac{b}{\sqrt{2}}|\downarrow, \uparrow, \uparrow\rangle$$

Measuring the second spin in the z basis can produce either a result of spin down or spin up with probability of spin up given by $P[\uparrow_2] = \frac{|a|^2+|b|^2}{2} = \frac{1}{2}$. Without loss of generality assume that the measurement produces a spin up, thereby giving us:

$$a|\uparrow, \uparrow, \downarrow\rangle + b|\downarrow, \uparrow, \uparrow\rangle$$

Now prior to measuring the first spin in the x basis we must rewrite the result above in the x basis for the first spin:

$$a|\uparrow, \uparrow, \downarrow\rangle + b|\downarrow, \uparrow, \uparrow\rangle = \frac{a}{\sqrt{2}}\Big(|\rightarrow, \uparrow, \downarrow\rangle + |\leftarrow, \uparrow, \downarrow\rangle\Big) + \frac{b}{\sqrt{2}}\Big(|\rightarrow, \uparrow, \uparrow\rangle - |\leftarrow, \uparrow, \uparrow\rangle\Big)$$

Measuring the first spin in the x basis can produce either a result of spin right or spin left. Without loss of generality assume the measurement produces a spin right, thereby giving us:

$$a|\rightarrow, \uparrow, \downarrow\rangle + b|\rightarrow, \uparrow, \uparrow\rangle$$

Following the recipe for teleportation we see that we now must apply the X and Z operators to the above in order to attain:

$$a|\rightarrow, \uparrow, \uparrow\rangle - b|\rightarrow, \uparrow, \downarrow\rangle$$

where the components of the third spin are exactly the components of the original state that was supposed to be teleported.

2.8 Philosophical Remark

Having discussed the no-cloning theorem prior to quantum teleportation it is quite natural to ask whether it plays any significance on our process above. As it turns out, whenever we decide to teleport some particle we cannot simply make a clone of it as it is forbidden. Therefore, whenever a particle is teleported it must be the case that we destroy the original particle and recreate one that has indistinguishable properties. What might be more familiar is the situation in Star Trek where Spock beams down from the enterprise. If we were to think of this as teleportation, which it is depicted to be, then it has to be that every time Spock beams down he is killed while at the same time there is another Spock who is created down on the planet, who is indistinguishable from the original one. From a philosophical perspective this introduces the very question as to whether the recreated Spock is truly the same Spock as before?

3. No-Teleportation Theorem

As a final touch on the presentation we now want to mention the No-Teleportation theorem (not to be confused with quantum teleportation) which states that an arbitrary quantum state cannot be converted in a sequence of classical bits (or even an infinite number of such bits); nor can such bits be used to reconstruct the original state, thus ”teleporting” it by merely moving classical bits around. This result immediately follows from the no-cloning theorem first discussed because if were to assume to the contrary that this can be accomplished, i.e. a qubit can be convered into classical bits, then a qubit would be able to be cloned as classical bits can be cloned. Clearly this presents a contradiction to the no-cloning theorem and so a qubit cannot be approximated by classical bits ensuring us that quantum logic supersedes classical logic.