The two dimensional heat equation

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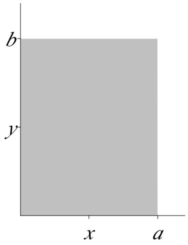
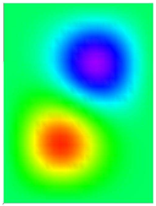
Partial Differential Equations
March 6, 2012

Physical motivation

Consider a thin rectangular plate made of some thermally conductive material. Suppose the dimensions of the plate are $a \times b$.





- The plate is heated in some way, and then insulated along its top and bottom.
- Our goal is to mathematically model the way thermal energy moves through the plate.

We let

$$u(x, y, t) = \text{temperature of plate at position } (x, y) \text{ and time } t.$$

For a fixed t, the height of the surface $z = u(x, y, t)$ gives the temperature of the plate at time t and position (x, y).

Under ideal assumptions (e.g. uniform density, uniform specific heat, perfect insulation, no internal heat sources etc.) one can show that u satisfies the **two dimensional heat equation**

$$\boxed{u_t = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy})} \tag{1}$$

for $0 < x < a$, $0 < y < b$.

At the edges of the plate we impose some sort of **boundary conditions.** The simplest are **homogeneous Dirichlet conditions:**

$$\begin{aligned} u(0,y,t) = u(a,y,t) &= 0, \qquad 0 \le y \le b,\ t \ge 0,\\ u(x,0,t) = u(x,b,t) &= 0, \qquad 0 \le x \le a,\ t \ge 0. \end{aligned} \tag{2}$$

Physically, these correspond to holding the temperature along the edges of the plate at 0.

The way the plate is heated initially is given by the **initial condition**

$$u(x,y,0) = f(x,y),\ (x,y) \in R, \tag{3}$$

where $R = [0,a] \times [0,b]$.

Solving the 2D wave equation: homogeneous Dirichlet boundary conditions

Goal: Write down a solution to the heat equation (1) subject to the boundary conditions (2) and initial conditions (3).

As usual, we

- **separate variables** to produce simple solutions to (1) and (2), and then
- use the **principle of superposition** to build up a solution that satisfies (3) as well.

Separation of variables

Assuming that

$$u(x, y, t) = X(x)Y(y)T(t),$$

plugging into the heat equation (1) and using the boundary conditions (2) yields the separated equations

$$X'' - BX = 0, \qquad X(0) = 0, \qquad X(a) = 0, \tag{4}$$

$$Y'' - CY = 0, \qquad Y(0) = 0, \qquad Y(b) = 0, \tag{5}$$

$$T' - c^2(B + C)T = 0. \tag{6}$$

We have already seen that the solutions to (4) and (5) are

$$X_m(x) = \sin \mu_m x, \qquad \mu_m = \frac{m\pi}{a}, \qquad B = -\mu_m^2$$
$$Y_n(y) = \sin \nu_n y, \qquad \nu_n = \frac{n\pi}{b}, \qquad C = -\nu_n^2,$$

for $m, n \in \mathbb{N}$. Using these values in (6) gives

$$T_{mn}(t) = e^{-\lambda_{mn}^2 t},$$

where

$$\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Superposition

Assembling these results, we find that for any *pair* $m, n \geq 1$ we have the **normal mode**

$$\boxed{u_{mn}(x, y, t) = X_m(x)Y_n(y)T_{mn}(t) = \sin \mu_m x \, \sin \nu_n y \, e^{-\lambda_{mn}^2 t}.}$$

For any choice of constants A_{mn}, by the principle of superposition we then have the **general solution** to (1) and (2)

$$\boxed{u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \mu_m x \, \sin \nu_n y \, e^{-\lambda_{mn}^2 t}.}$$

Initial conditions

We must determine the values of the coefficients A_{mn} so that our solution also satisfies the initial condition (3). We need

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

which is just the **double Fourier series** for $f(x,y)$. We know that if f is a C^2 function then

$$\boxed{A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, dy\, dx.}$$

Conclusion

Theorem

Suppose that $f(x,y)$ is a C^2 function on the rectangle $[0,a]\times[0,b]$. The solution to the heat equation (1) with homogeneous Dirichlet boundary conditions (2) and initial conditions (3) is given by

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\mu_m x\,\sin\nu_n y\,e^{-\lambda_{mn}^2 t},$$

where $\mu_m=\dfrac{m\pi}{a}$, $\nu_n=\dfrac{n\pi}{b}$, $\lambda_{mn}=c\sqrt{\mu_m^2+\nu_n^2}$, *and*

$$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx.$$

Remarks:

- We have not actually verified that this solution is *unique*, i.e. that this is the *only* solution to problem.
- We will prove uniqueness later using **the maximum principle**.

Example

A 2×2 *square plate with* $c = 1/3$ *is heated in such a way that the temperature in the lower half is 50, while the temperature in the upper half is 0. After that, it is insulated laterally, and the temperature at its edges is held at 0. Find an expression that gives the temperature in the plate for* $t > 0$.

We must solve the heat equation problem (1) - (3) with

$$f(x,y) = \begin{cases} 50 & \text{if } y \le 1, \\ 0 & \text{if } y > 1. \end{cases}$$

The coefficients in the solution are

$$\begin{aligned} A_{mn} &= \frac{4}{2 \cdot 2} \int_0^2 \int_0^2 f(x,y) \sin \frac{m\pi}{2} x \sin \frac{n\pi}{2} y \, dy \, dx \\ &= 50 \int_0^2 \sin \frac{m\pi}{2} x \, dx \int_0^1 \sin \frac{n\pi}{2} y \, dy \\ &= 50 \left(\frac{2(1 + (-1)^{m+1})}{\pi m} \right) \left(\frac{2(1 - \cos \frac{n\pi}{2})}{\pi n} \right) \\ &= \frac{200}{\pi^2} \frac{(1 + (-1)^{m+1})(1 - \cos \frac{n\pi}{2})}{mn}. \end{aligned}$$

Since

$$\lambda_{mn} = \frac{\pi}{3}\sqrt{\frac{m^2}{4} + \frac{n^2}{4}} = \frac{\pi}{6}\sqrt{m^2 + n^2}$$

the solution is

$$\boxed{\begin{aligned} u(x, y, t) \quad = \quad & \frac{200}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{(1 + (-1)^{m+1})(1 - \cos\frac{n\pi}{2})}{mn} \sin\frac{m\pi}{2}x \right. \\ & \left. \times \sin\frac{n\pi}{2} y\, e^{-\pi^2(m^2+n^2)t/36} \right). \end{aligned}}$$

Steady state solutions

- To deal with inhomogeneous boundary conditions in heat problems, one must study the solutions of the heat equation that do not vary with time.
- These are the **steady state solutions.** They satisfy

$$u_t = 0.$$

- In the 1D case, the heat equation for steady states becomes

$$u_{xx} = 0.$$

The solutions are simply straight lines.

Laplace's equation

In the 2D case, we see that steady states must solve

$$\boxed{\nabla^2 u = u_{xx} + u_{yy} = 0.} \tag{7}$$

- This is **Laplace's equation.**
- Solutions to Laplace's equation are called **harmonic functions.**
- See assignment 1 for examples of harmonic functions.

We want to find steady state solutions to (7) that satisfy the **Dirichlet boundary conditions**

$$u(x,0) = f_1(x), \qquad u(x,b) = f_2(x), \qquad 0 < x < a \tag{8}$$
$$u(0,y) = g_1(x), \qquad u(a,y) = g_2(y), \qquad 0 < y < b \tag{9}$$

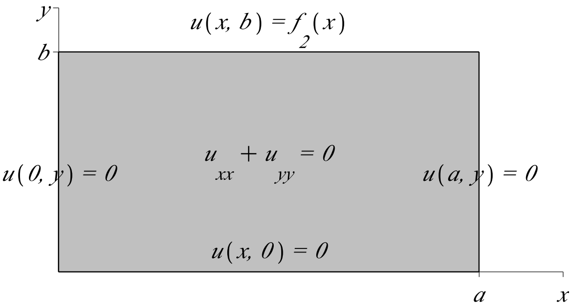
- The problem of finding a solution to (7) - (9) is known as the **Dirichlet problem**.
- We will begin by assuming $f_1 = g_1 = g_2 = 0$.
- The general solution to the Dirichlet problem will be obtained by superposition.

Solution of the Dirichlet problem on a rectangle

Goal: Solve the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0, && 0 < x < a,\ 0 < y < b, \\ u(x,0) = 0,\ u(x,b) &= f_2(x), && 0 < x < a, \\ u(0,y) = u(a,y) &= 0, && 0 < y < b. \end{aligned}$$

Picture:



Separation of variables

Setting $u(x,y) = X(x)Y(y)$ leads to

$$X'' + kX = 0 \quad , \quad Y'' - kY = 0,$$
$$X(0) = X(a) = 0 \quad , \quad Y(0) = 0.$$

We know the nontrivial solutions for X:

$$X(x) = X_n(x) = \sin \mu_n x,\ \mu_n = \frac{n\pi}{a},\ k = \mu_n^2.$$

for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. The corresponding solutions for Y are

$$Y(y) = Y_n(y) = A_n e^{\mu_n y} + B_n e^{-\mu_n y}.$$

The condition $Y(0) = 0$ yields $A_n = -B_n$. Choosing $A_n = 1/2$ we find that

$$Y_n(y) = \sinh \mu_n y.$$

This gives the separated solutions

$$\boxed{u_n(x, y) = X_n(x) Y_n(y) = \sin \mu_n x \sinh \mu_n y,}$$

and superposition gives the general solution

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n \sin \mu_n x \sinh \mu_n y.}$$

The general solution satisfies the Laplace equation (7) inside the rectangle, as well as the three homogeneous boundary conditions on three of its sides (left, right and bottom).

We now determine the values of B_n to get the boundary condition on the top of the rectangle. This requires

$$f_2(x) = u(x, b) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi}{a} x,$$

which is the **Fourier sine series for** $f_2(x)$ on $0 < x < a$.

Appealing to previous results, we can now summarize our findings.

Conclusion

Theorem

If $f_2(x)$ is piecewise smooth, the solution to the Dirichlet problem

$$\begin{aligned} \nabla^2 u &= 0, && 0 < x < a,\ 0 < y < b, \\ u(x,0) = 0,\ u(x,b) &= f_2(x), && 0 < x < a, \\ u(0,y) = u(a,y) &= 0, && 0 < y < b. \end{aligned}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \mu_n x \sinh \mu_n y,$$

where $\mu_n = \dfrac{n\pi}{a}$ *and* $B_n = \dfrac{2}{a \sinh \frac{n\pi b}{a}} \displaystyle\int_0^a f_2(x) \sin \frac{n\pi}{a} x \, dx.$

Remarks: As we have noted previously:

- We have not proven this solution is unique. This requires the **maximum principle**.

- If we know the sine series expansion for $f_2(x)$ then we can use the relationship

$$B_n = \frac{1}{\sinh \frac{n\pi b}{a}} \left(n\text{th sine coefficient of } f_2(x)\right)$$

to avoid integral computations.

Example

Example

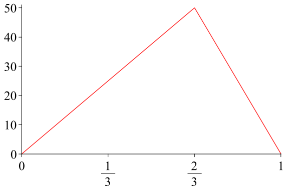
Solve the Dirichlet problem on the square $[0,1] \times [0,1]$*, subject to the boundary conditions*

$$\begin{aligned} u(x,0) = 0,\ u(x,b) = f_2(x), & \qquad 0 < x < a, \\ u(0,y) = u(a,y) = 0, & \qquad 0 < y < b. \end{aligned}$$

where

$$f_2(x) = \begin{cases} 75x & \text{if } 0 \leq x \leq \frac{2}{3}, \\ 150(1-x) & \text{if } \frac{2}{3} < x \leq 1. \end{cases}$$

We have $a = b = 1$ and we require the sine series of $f_2(x)$. The graph of $f_2(x)$ is:



According to exercise 2.4.17 (with $p = 1$, $a = 2/3$ and $h = 50$):

$$f_2(x) = \frac{450}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{2n\pi}{3}}{n^2} \sin n\pi x.$$

Thus,

$$B_n = \frac{1}{\sinh n\pi}\left(\frac{450}{\pi^2}\frac{\sin\frac{2n\pi}{3}}{n^2}\right) = \frac{450}{\pi^2}\frac{\sin\frac{2n\pi}{3}}{n^2\sinh n\pi},$$

and

$$\boxed{u(x,y) = \frac{450}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{2n\pi}{3}}{n^2\sinh n\pi}\sin n\pi x\,\sinh n\pi y.}$$